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AN ISOMETRIC DYNAMICS FOR A CAUSAL SET APPROACH TO DISCRETE QUANTUM GRAVITY

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Abstract

We consider a covariant causal set approach to discrete quantum gravity. We first review the microscopic picture of this approach. In this picture a universe grows one element at a time and its geometry is determined by a sequence of integers called the shell sequence. We next present the macroscopic picture which is described by a sequential growth process. We introduce a model in which the dynamics is governed by a quantum transition amplitude. The amplitude satisfies a stochastic and unitary condition and the resulting dynamics becomes isometric. We show that the dynamics preserves stochastic states. By "doubling down" on the dynamics we obtain a unitary group representation and a natural energy operator. These unitary operators are employed to define canonical position and momentum operators.

1 Microscopic Picture

We call a finite poset $(x,<)$ a *causet* and interpret $a<b$ in x to mean that b is in the causal future of a. If x and y are causets with cardinality $|y|=|x|+1$, then x *produces* y (denoted $x\mapsto y$) if y is obtained from x by adjoining a

single maximal element to x. If $x \to y$ we call y an *offspring* of x. A *labeling* for a causet x is a bijection

$$\ell\colon x \to \{1, 2, \ldots, |x|\}$$

such that $a, b \in x$ with $a < b$ implies $\ell(a) < \ell(b)$. We then call $x = (x, \ell)$ a *labeled* causet. A labeling of x corresponds to a "birth order" for the elements of x. Two labeled causets x, y are *isomorphic* if there is a bijection $\phi\colon x \to y$ such that $a < b$ in x if and only if $\phi(a) < \phi(b)$ in y and $\ell\left[\phi(a)\right] = \ell(a)$ for all $a \in x$. A causet is *covariant* if it has a unique labeling (up to isomorphisms) and we call a covariant causet a *c-causet*. Covariance corresponds to the properties of a manifold being independent of the coordinate system used to describe it. Denote the set of c-causets with cardinality n by $\mathcal{P}_n$ and the set of all c-causets by $\mathcal{P}$. It is shown in [3] that any $x \in \mathcal{P}$ with $x \neq \emptyset$ has a unique producer in $\mathcal{P}$ and precisely two offspring in $\mathcal{P}$. It follows that $|\mathcal{P}_n| = 2^{n-1}$, $n = 1, 2, \ldots$. For more background concerning the causet approach to discrete quantum gravity we refer the reader to [4, 5, 7]. For more information about c-causets the reader can refer to [1, 2, 3].

Two elements $a, b \in x$ are *comparable* if $a < b$ or $b < a$. We say that a is a *parent* of b and b is a *child* of a if $a < b$ and there is no $c \in x$ with $a < c < b$. A *path* from a to b in x is a sequence $a_1 = a, a_2, \ldots, a_{n-1}, a_n = b$ where a_i is a parent of a_{i+1}, $i = 1, \ldots, n-1$. The *height* $h(a)$ of $a \in x$ is the cardinality minus one of a longest path in x that ends with a. If there is no such path, we set $h(a) = 0$. It is shown in [3] that a causet x is covariant if and only if $a, b \in x$ are comparable whenever $h(a) \neq h(b)$.

If $x \in \mathcal{P}$, we call the sets

$$S_j(x) = \{a \in x\colon h(a) = j\}, j = 0, 1, 2, \ldots$$

shells and the sequence of integers $s_j(x) = |S_j(x)|$, $j = 0, 1, 2, \ldots$ is the *shell sequence* for x [1]. A c-causet is uniquely determined by its shell sequence and we think of $\{s_j(x)\}$ as describing the "shape" or geometry of x. The tree $(\mathcal{P}, \to)$ can be thought of as a growth model and an $x \in \mathcal{P}_n$ is a possible universe at step (time) n. An instantaneous universe $x \in \mathcal{P}_n$ grows one element at a time in one of two ways. If $x \in \mathcal{P}_n$ has shell sequence $(s_0(x), s_1(x), \ldots, s_m(x))$, then $x \to x_0$ or $x \to x_1$ where x_0, x_1 have shell sequence $(s_0(x), s_1(x), \ldots, s_m(x)+1)$ and $(s_0(x), s_1(x), \ldots, s_m(x), 1)$, respectively. In this way, we recursively order the c-causets in $\mathcal{P}$ using the notation

$x_{n,j}$, $n = 1, 2, \ldots$, $j = 0, 1, 2, \ldots, 2^{n-1} - 1$, where $n = |x_{n,j}|$. For example, in terms of their shell sequences we have:

$x_{1,0} = (1), x_{2,0} = (2), x_{2,1} = (1,1), x_{3,0} = (3), x_{3,1} = (2,1), x_{3,2} = (1,2), x_{3,3} = (1,1,1)$
$x_{4,0} = (4), x_{4,1} = (3,1), x_{4,2} = (2,2), x_{4,3} = (2,1,1), x_{4,4} = (1,3), x_{4,5} = (1,2,1)$
$x_{4,6} = (1,1,2), x_{4,7} = (1,1,1,1)$

In the microscopic picture, we view a c-causet as a framework or scaffolding for a possible universe. The vertices of x represent small cells that can be empty or occupied by a particle. The shell sequence for x gives the geometry of the framework. In [1] we have shown how to construct a metric or distance function on x. This metric has simple and useful properties. However, the present paper is mainly devoted to the macroscopic picture and the quantum dynamics that can be developed in that picture. Figure 1 illustrates the first four steps of the sequential growth process $(\mathcal{P}, \to)$. Notice that this is a multiverse model in which infinite paths represent the histories of "completed" universes [4].

2 Macroscopic Picture

We now study the macroscopic picture which describes the evolution of a universe as a quantum sequential growth process. In such a process, the probabilities and propensities of competing geometries are determined by quantum amplitudes. These amplitudes provide interferences that are characteristic of quantum systems. A *transition amplitude* is a map $\widetilde{a}\colon \mathcal{P}\times\mathcal{P} \to \mathbb{C}$ satisfying $\widetilde{a}(x,y) = 0$ if $x \not\to y$ and $\sum_{y\in\mathcal{P}} \widetilde{a}(x,y) = 1$ for every $x \in \mathcal{P}$. Since $x_{n,j}$ only has the offspring $x_{n+1,2j}$ and $x_{n+1,2j+1}$ we have that

$$\sum_{k=0}^{1} \widetilde{a}(x_{n,j}, x_{n+1,2j+k}) = 1 \tag{2.1}$$

for all $n = 1, 2, \ldots$, $j = 0, 1, 2, \ldots, 2^{n-1} - 1$. We call $\widetilde{a}$ a *unitary transition amplitude* (uta) if $\widetilde{a}$ also satisfies $\sum_{y\in\mathcal{P}} |\widetilde{a}(x,y)|^2 = 1$ or as in (2.1) we have

$$\sum_{k=0}^{1} |\widetilde{a}(x_{n,j}, x_{n+1,2j+k})|^2 = 1 \tag{2.2}$$

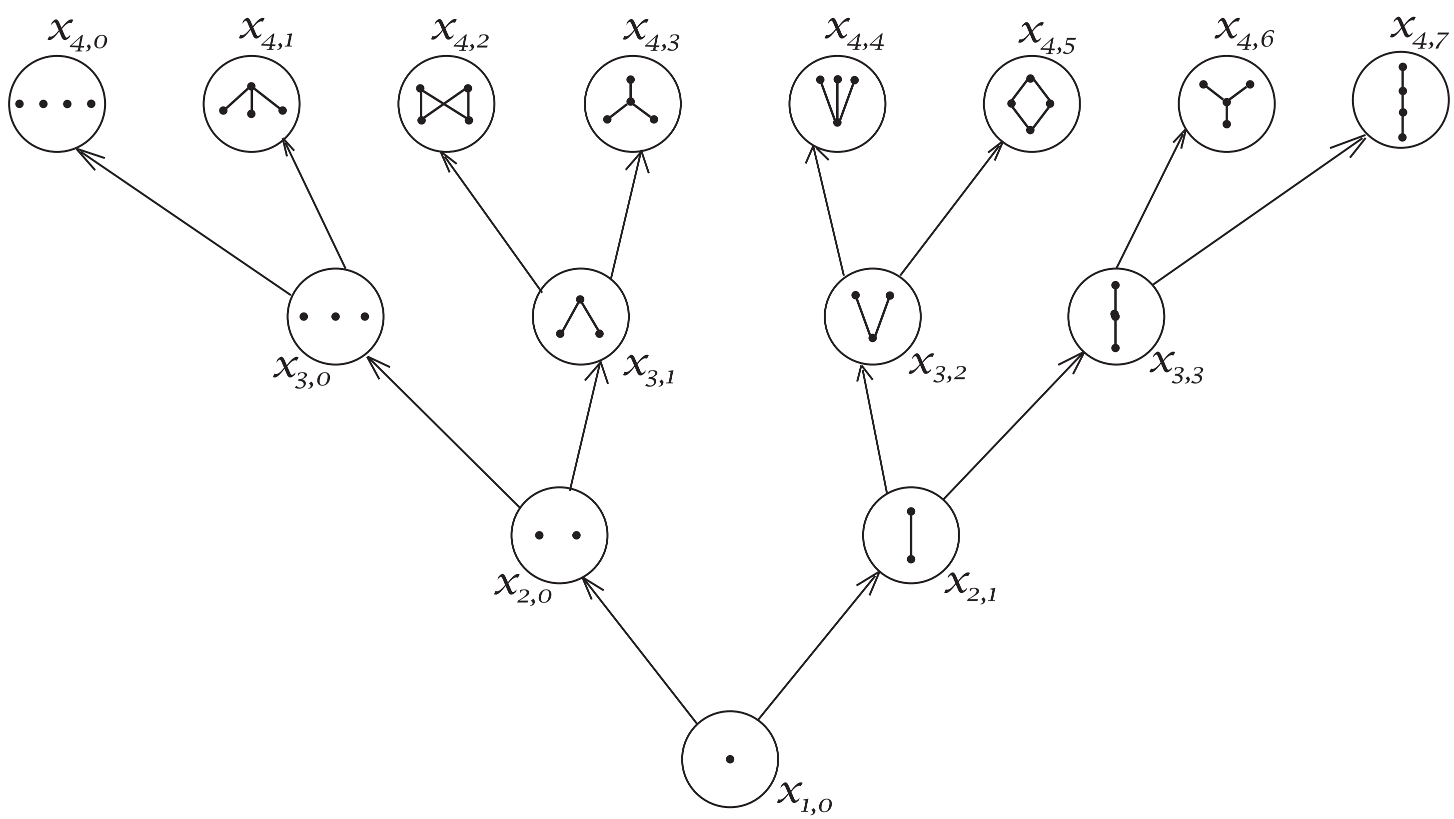

Figure 1 (Ordering for first four steps.)

One might suspect that these restrictions on a uta are so strong that the possibilities are very limited. This would be true if $\widetilde{a}$ were real valued. In this case, $\widetilde{a}(x,y)=1$ for one y with $x\to y$ and $\widetilde{a}(x,y)=0$, otherwise. However, in the complex case, the next result shows that there are a continuum of possibilities.

Theorem 2.1. *Two complex numbers a,b satisfy $a+b=\left|a\right|^2+\left|b\right|^2=1$ if and only if there exists a $\theta\in[0,\pi)$ such that $a=\cos\theta e^{i\theta}$ and $b=-i\sin\theta e^{i\theta}$. Moreover, θ is unique.*

Proof. Necessity is clear. For sufficiency, suppose the conditions $a+b=\left|a\right|^2+\left|b\right|^2=1$ hold. Then

$$1=\left|a\right|^2+\left|b\right|^2=\left|a\right|^2+\left|1-a\right|^2=\left|a\right|^2+(1-a)(1-\overline{a})=1-2\,Re\,a+2\left|a\right|^2$$

Hence, $\left|a\right|^2=Re\,a$. Letting $a=\left|a\right|e^{i\theta}$ we have that $\left|a\right|^2=\left|a\right|\cos\theta$. If $a=0$, the result holds with $\theta=\pi/2$. If $a\neq 0$, we have that $\left|a\right|=\cos\theta$ and $Re\,a=\left|a\right|\cos\theta$. Hence, $a=\cos\theta e^{i\theta}$ and

$$\begin{aligned}b&=1-\cos\theta e^{i\theta}=1-\cos^2\theta-i\cos\theta\sin\theta=\sin\theta(\sin\theta-i\cos\theta)\\&=-i\sin\theta e^{i\theta}\end{aligned}$$

Uniqueness follows from the fact that $\cos\theta$ is injective on $[0,\pi)$. □

If $\widetilde{a}\colon\mathcal{P}\times\mathcal{P}\to\mathbb{C}$ is a uta, we call

$$c_{n,j}^k=\widetilde{a}(x_{n,j},x_{n+1,2j+k}),\quad k=0,1$$

the coupling constants for $\widetilde{a}$. It follows from Theorem 2.1 that there exist $\theta_{n,j}\in[0,\pi)$ such that

$$c_{n,j}^0=\cos\theta_{n,j}e^{i\theta_{n,j}},\quad c_{n,j}^1=-i\sin\theta_{n,j}e^{i\theta_{n,j}}$$

It follows that $c_{n,j}^0+c_{n,j}^1=\left|c_{n,j}^0\right|^2+\left|c_{n,j}^1\right|^2=1$ for all $n=1,2,\ldots$, $j=0,1,2\ldots,2^{n-1}-1$. Let H_n be the Hilbert space

$$H_n=L_2(\mathcal{P}_n)=\{f\colon\mathcal{P}_n\to\mathbb{C}\}$$

with the standard inner product

$$\langle f,g\rangle=\sum_{x\in\mathcal{P}_n}\overline{f(x)}g(x)$$

A *path* in $\mathcal{P}$ is a sequence $\omega = \omega_1\omega_2\cdots$ where $\omega_i \in \mathcal{P}_i$ and $\omega_i \to \omega_{i+1}$ Similarly, an *n-path* has the form $\omega = \omega_1\omega_2\cdots\omega_n$ where again $\omega_i \in \mathcal{P}_i$ and $\omega_i \to \omega_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n. Since every $x \in \mathcal{P}_n$ has a unique n-path terminating at x, we can identify $\mathcal{P}_n$ with Ω_n and we write $\mathcal{P} \approx \Omega_n$. Similarly, we identify H_n with $L_2(\Omega_n)$. If $\widetilde{a}$ is a uta and $\omega = \omega_1\omega_2\cdots\omega_n \in \Omega_n$, we define the *amplitude* of ω to be

$$a(\omega) = \widetilde{a}(\omega_1,\omega_2)\widetilde{a}(\omega_2,\omega_3)\cdots\widetilde{a}(\omega_{n-1},\omega_n)$$

Moreover, we define the *amplitude* of $x \in \mathcal{P}_n$ to be $a(\omega)$ where $\omega \in \Omega_n$ terminates at x.

Let $\widehat{x}_{n,}$ be the unit vector in H_n given by the characteristic function $\chi_{x_{n,j}}$. Then clearly, $\{\widehat{x}_{n,j}\colon j = 0,1,\ldots,2^{n-1}-1\}$ forms an orthonormal basis for H_n. Define the operators $U_n\colon H_n \to H_{n+1}$ by

$$U_n\widehat{x}_{n,j} = \sum_{k=0}^{1} c_{n,j}^k \widehat{x}_{n+1,2j+k}$$

and extend U_n to H_n by linearity.

Theorem 2.2. (i) *The adjoint of U_n is given by $U_n^*\colon H_{n+1} \to H_n$ where*

$$U_n^*\widehat{x}_{n+1,2j+k} = \overline{c}_{n,j}^k \widehat{x}_{n,j}, \quad k = 0,1 \tag{2.3}$$

(ii) *U_n is a partial isometry with $U_n^*U_n = I_n$ and*

$$U_nU_n^* = \sum_{j=0}^{2^{n-1}-1} \left| \sum_{k=0}^{1} c_{n,j}^k \widehat{x}_{n+1,2j+k} \right\rangle \left\langle \sum_{k=0}^{1} c_{n,j}^k \widehat{x}_{n+1,2j+k} \right| \tag{2.4}$$

Proof. (i) To show that (2.3) holds, we have

$$\begin{aligned}
\langle U_n^*\widehat{x}_{n+1,2j'+k'}, \widehat{x}_{n,j}\rangle &= \langle \widehat{x}_{n+1,2j'+k'}, U_n\widehat{x}_{n,j}\rangle \\
&= \left\langle \widehat{x}_{n+1,2j'+k'}, \sum_{k=0}^{1} c_{n,k}^k \widehat{x}_{n+1,2j+k} \right\rangle \\
&= c_{n,j}^k \delta_{jj'}\delta_{kk'} = \left\langle \overline{c}_{n,j'}^{k'} \widehat{x}_{n,j'} \widehat{x}_{n,j} \right\rangle
\end{aligned}$$

(ii) To show that $U_n^*U_n = I_n$ we have by (i) that

$$U_n^*U_n\widehat{x}_{n,j} = \sum_{k=0}^{1} c_{n,j}^k U_n^*\widehat{x}_{n+1,2j+k} = \sum_{k=0}^{1} \left| c_{n,j}^k \right|^2 \widehat{x}_{n,j} = \widehat{x}_{n,j}$$

Since $\{\widehat{x}_{n,j}\colon j=0,1,\ldots,2^{n-1}-1\}$ forms an orthonormal basis for H_n, the result follows. Equation (2.4) holds because it is well-known that $U_nU_n^*$ is the projection onto the range of $\mathcal{R}(U_n)$. We can also show this directly as follows

$$\begin{aligned}\sum_{j=0}^{2^{n-1}-1}&\left|\sum_{k=0}^{1}c_{n,j}^k\widehat{x}_{n+1,2j+k}\right\rangle\left\langle\sum_{k=0}^{1}c_{n,j}^k\widehat{x}_{n+1,2j+k}\right|\widehat{x}_{n+1,2j'+k'}\\&=\sum_{j=0}^{2^{n-1}-1}\sum_{k=0}^{1}c_{n,j}^k\widehat{x}_{n+1,2j+k}\overline{c}_{n,j'}^{k'}\delta_{jj'}=\overline{c}_{n,j'}^{k'}\sum_{k=0}^{1}c_{n,j'}^kx_{n+1,2j'+k}\\&=\overline{c}_{n,j'}^{k'}U_n\widehat{x}_{n,j'}=U_nU_n^*\widehat{x}_{n+1,2j'+k'}\end{aligned}$$ □

It follows from Theorem 2.2 that the dynamics $U_n\colon H_n\to H_{n+1}$ for a uta $\widetilde{a}$ is an isometric operator. As usual a *state* on H_n is a positive operator ρ on H_n with $\mathrm{tr}(\rho)=1$. A *stochastic state* on H_n is a state ρ that satisfies $\langle\rho 1_n,1_n\rangle=1$ where $1_n=\chi_{\mathcal{P}_n}$; that is, $1_n(x)=1$ for every $x\in\mathcal{P}_n$. Notice that $U_n^*1_{n+1}=1_n$.

Lemma 2.3. (i) *If ρ is a state on H_n, then $U_n\rho U_n^*$ is a state on H_{n+1}.* (i) *If ρ is a stochastic state on H_n, then $U_n\rho U_n^*$ is a stochastic state on H_{n+1}.*

Proof. (i) To show that $U_n\rho_nU_n^*$ is positive, we have

$$\langle U_n\rho U_n^*\phi,\phi\rangle=\langle\rho U_n^*\phi,U_n^*\phi\rangle\geq 0$$

for all $\phi\in H_{n+1}$. Moreover, by Theorem 2.2(ii) we have

$$\mathrm{tr}(U_n\rho U_n^*)=\mathrm{tr}(U_n^*U_n\rho)=\mathrm{tr}(\rho)=1$$

(ii) Since $U_n^*1_{n+1}=1_n$ we have

$$\langle U_n\rho U_n^*1_{n+1},1_{n+1}\rangle=\langle\rho U_n^*1_{n+1},U_n^*1_{n+1}\rangle=\langle\rho 1_n1_n\rangle=1$$ □

Denoting the time evolution of states by $\rho_n\to\rho_{n+1}$, Lemma 2.3 shows that $\rho\to U_n\rho U_n^*$ gives a quantum dynamics for states. We now show this explicitly for the transition amplitude. Since

$$\langle\widehat{x}_{n+1,2j+k},U_n\widehat{x}_{n,j}\rangle=c_{n,j}^k=\widetilde{a}(x_{n,j},x_{n+1,2j+k})$$

we have for every $\omega=\omega_1\omega_2\cdots\omega_n\in\Omega_n$ that

$$a(\omega)=\langle\widehat{\omega}_2,U_1\widehat{\omega}_1\rangle\langle\widehat{\omega}_3,U_2\widehat{\omega}_2\rangle\cdots\langle\widehat{\omega}_n,U_{n-1}\widehat{\omega}_{n-1}\rangle$$

Define the operator ρ_n on H_n by $\langle\widehat{\omega},\rho_n\widehat{\omega}'\rangle=\overline{a(\omega)}a(\omega')$ where $\widehat{\omega}=\chi_{\{\omega\}}\in H_n$ for any $\omega\in\Omega_n$.

Theorem 2.4. *The operator ρ_n is a stochastic state on H_n.*

Proof. To show that ρ_n is positive we have

$$\begin{aligned}
\langle f, \rho_n f\rangle &= \left\langle \sum \langle \widehat{\gamma}_i, f\rangle \widehat{\gamma}_i, \rho_n \sum \langle \widehat{\gamma}_j, f\rangle \widehat{\gamma}_j \right\rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f\rangle} \sum \langle \widehat{\gamma}_j, f\rangle \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j\rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f\rangle} \sum \langle \widehat{\gamma}_j, f\rangle \overline{a(\gamma_i)} a(\gamma_i) \\
&= \left| \sum a(\gamma_i) \langle \widehat{\gamma}_i, f\rangle \right|^2 \geq 0
\end{aligned}$$

To show that ρ_n is a state on H_n we have that

$$\begin{aligned}
\operatorname{tr}(\rho_n) &= \sum \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_i\rangle = \sum \overline{a(\gamma_i)} a(\gamma_i) = \sum |a(\gamma_i)|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_n} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1\rangle|^2 \, |\langle \widehat{\omega}_3, U_2 \widehat{\omega}_2\rangle|^2 \cdots |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1}\rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1\rangle|^2 \\
&\qquad \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2}\rangle|^2 \sum_{\omega_n} |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1}\rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1\rangle|^2 \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2}\rangle|^2 \\
&\;\;\vdots \\
&= \sum_{\omega_2} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1\rangle|^2 = 1
\end{aligned}$$

Finally, ρ_n is stochastic on H_n because

$$\begin{aligned}
\langle 1_n, \rho_n 1_n\rangle &= \left\langle \sum \widehat{\gamma}_i, \rho \sum \widehat{\gamma}_j \right\rangle = \sum_{i,j} \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j\rangle \\
&= \sum_{i,j} \overline{a(\gamma_i)} a(\gamma_j) = \left| \sum a(\gamma_i) \right|^2
\end{aligned}$$

As before, we obtain

$$\begin{aligned}\sum_{\omega\in\Omega_n} a(\omega) &= \sum_{\omega_2}\sum_{\omega_3}\cdots\sum_{\omega_n}\langle\widehat{\omega}_2, U_1\widehat{\omega}_1\rangle\langle\widehat{\omega}_3, U_2\widehat{\omega}_2\rangle\cdots\langle\widehat{\omega}_n, U_{n-1}\widehat{\omega}_{n-1}\rangle\\ &= \sum_{\omega_2}\sum_{\omega_3}\cdots\sum_{\omega_{n-1}}\langle\widehat{\omega}_2, U_1\widehat{\omega}_1\rangle\langle\widehat{\omega}_3, U_2, \widehat{\omega}_2\rangle\cdots\langle\widehat{\omega}_{n-1}U_{n-2}\widehat{\omega}_{n-2}\rangle\\ &\ \ \vdots\\ &= \sum_{\omega_2}\langle\widehat{\omega}_2, U_1\widehat{\omega}_1\rangle = 1 \qquad \square\end{aligned}$$

If $\omega = \omega_1\omega_2\cdots\omega_n \in \Omega_n$, we have seen that ω_n produces two offspring $\omega_{n,0}, \omega_{n,1} \in \mathcal{P}_{n+1}$. We call the set

$$(\omega \rightarrow) = \{\omega_1\omega_2\cdots\omega_n\omega_{n,0}, \omega_1\omega_2\cdots\omega_n\omega_{n,1}\} \subseteq \Omega_{n+1}$$

the *one-step causal future* of ω. We say that the sequence ρ_n is *consistent* if

$$\langle(\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge\rangle = \langle\widehat{\omega}, \rho_n\widehat{\omega}'\rangle$$

for every $\omega, \omega' \in \Omega_n$ where $(\omega \rightarrow)^\wedge = \chi_{(\omega\rightarrow)}$. Consistency is important because it follows that the probabilities and propensities given by the dynamics ρ_n are conserved in time [2, 3].

Theorem 2.5. *The sequence ρ_n is consistent.*

Proof. Let $\omega = \omega_1\omega_2\cdots\omega_n$, $\omega' = \omega'_1, \omega'_2\cdots\omega'_n \in \Omega_n$ and suppose that $\omega_n \rightarrow \omega_{n,0}, \omega_{n,1}$ and $\omega'_n \rightarrow \omega'_{n,0}, \omega'_{n,1}$. We then have

$$\begin{aligned}&\langle(\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge\rangle\\ &\quad= \left\langle(\omega\omega_{n,0})^\wedge + (\omega\omega_{n,1})^\wedge, \rho_{n+1}\ \left[(\omega'\omega'_{n,0})^\wedge + (\omega'\omega'_{n,1})^\wedge\right]\right\rangle\\ &\quad= \left\langle(\omega\omega_{n,0})^\wedge, \rho_{n+1}(\omega'\omega'_{n,0})^\wedge\right\rangle + \left\langle(\omega\omega_{n,0})^\wedge, \rho_{n+1}(\omega'\omega'_{n,1})^\wedge\right\rangle\\ &\qquad+ \left\langle(\omega\omega_{n,1})^\wedge, \rho_{n+1}(\omega'\omega'_{n,0})^\wedge\right\rangle\left\langle(\omega\omega_{n,1})^\wedge, \rho_{n+1}(\omega'\omega'_{n,1})^\wedge\right\rangle\\ &\quad= \overline{a(\omega)\widetilde{a}(\omega_n\omega_{n,0})}a(\omega')\left[\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})\right]\\ &\qquad+ \overline{a(\omega)\widetilde{a}(\omega_n, \omega_{n,1})}a(\omega')\left[\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})\right]\\ &\quad= \overline{a(\omega)}a(\omega') = \langle\widehat{\omega}, \rho_n\widehat{\omega}'\rangle \qquad \square\end{aligned}$$

The *n-decoherence functional* is the map $D_n\colon 2^{\Omega_n}\times 2^{\Omega_n}\to\mathbb{C}$ defined by [4, 5, 7]

$$D_n(A,B)=\langle\chi_B,\rho_n\chi_A\rangle$$

The functional $D_n(A,B)$ gives a measure of the interference between A and B when the system is in the state ρ_n. Clearly $D_n(\Omega_n,\Omega_n)=1$, $D_n(A,B)=\overline{D_n(A,B)}$ and $A\mapsto D_n(A,B)$ is a complex measure for every $B\in 2^{\Omega_n}$. It is also well-known that if $A_1,\ldots,A_n\in 2^{\Omega_n}$, then the matrix with entries $D_n(A_j,A_k)$ is positive semidefinite [5]. Notice that

$$D_n\left(\{\omega\},\{\omega'\}\right)=\overline{a(\omega)}a(\omega')$$

for every $\omega,\omega'\in\Omega_n$ and

$$D(A,B)=\sum\left\{\overline{a(\omega)}a(\omega')\colon\omega\in A,\omega'\in B\right\}$$

Since ρ_n is consistent, we have that

$$D_{n+1}\left((A\to),(B\to)\right)=D_n(A,B)$$

for every $A,B\in 2^{\Omega_n}$ where $(A\to)=\cup\{(\omega\to)\colon\omega\in A\}$. The corresponding *q-measure* [2, 5, 6] is the map $\mu_n\colon 2^{\Omega}\to\mathbb{R}^+$ defined by

$$\mu_n(A)=D_n(A,A)=\langle\chi_A,\rho_n\chi_A\rangle$$

It follows that $\mu_n(\Omega_n)=1$ and $\mu_{n+1}\left((A\to)\right)=\mu_n(A)$ for all $A\in 2^{\Omega_n}$. Although μ_n is not additive, it satisfies the *grade* 2-*additive condition*: if $A,B,C\in 2^{\Omega_n}$ are mutually disjoint then [4, 5, 6, 7]

$$\mu_n(A\cup B\cup C)=\mu_n(A\cup B)+\mu_n(A\cup C)+\mu_n(B\cup C)-\mu_n(A)-\mu_n(B)-\mu_n(C)$$

Since μ_n is not a measure we do call it a probability but we interpret $\mu_n(A)$ as the quantum propensity for the occurrence of A. We have discussed in [2, 3] ways of extending the μ_ns to a q-measure μ on suitable subsets of Ω.

A uta is *completely stationary* (cs) *with parameter* $\theta\in[0,\pi)$ if $\theta_{n,j}=\theta$ for all n,j. For example, let $\widetilde{a}$ be cs with parameter 0. Then the path $x_{1,0}x_{2,0}x_{3,0}\cdots$ has q-measure 1 and all other paths have q-measure 0. Now consider a general cs uta $\widetilde{a}$ with parameter $\theta\in(0,\pi)$. When a path "turns left" $\widetilde{a}$ has the value $\cos\theta e^{i\theta}$ and when it "turns right" $\widetilde{a}$ has the value $-i\sin\theta e^{i\theta}$. Hence if $\omega\in\Omega_n$ turns left ℓ times and right r times we have

$$a(\omega)=(\cos\theta)^\ell(-i)^r(\sin\theta)^r e^{in\theta}$$

We then have

$$\mu_n(\{\omega\}) = |a(\omega)|^2 = |\cos\theta|^{2\ell}\,|\sin\theta|^{2r}$$

Hence, $\lim_{n\to\infty}\mu_n(\{\omega\}) = 0$ and is is natural to define $\mu(\{\omega\}) = 0$.

A vector

$$v = \sum_{j=0}^{2^{n-1}-1} v_j\widehat{x}_{n,j} = (v_o, v_1, \ldots, v_{2^{n-1}-1}) \in H_n$$

is called a *stochastic state vector* if $\|v\| = 1$ and $\langle v, 1_n\rangle = 1$. we call the vector

$$\widehat{a}_n = (a(x_{n,o}), a(x_{n,1}), \ldots, a(x_{n,2^{n-1}-1})) \in H_n$$

an *amplitude vector*. Of course, $\widehat{a}_n$ is a stochastic state vector.

Theorem 2.6. (i) *If $v \in H_n$ is a stochastic state vector, then $U_nv \in H_{n+1}$ is also.* (ii) $U_n\widehat{a}_n = \widehat{a}_{n+1}$. (iii) $U_n^*\widehat{a}_{n+1} = \widehat{a}_n$.

Proof. (i) This follows from the fact that U_n is isometric and $U_n^*1_{n+1} = 1_n$.
(ii) This holds because

$$U_n\widehat{a}_n = U_n\sum_{j=0}^{2^{n-1}-1} a(x_{n,j})\widehat{x}_{n,j} = \sum_{j=0}^{2^{n-1}-1}\left[a(x_{n,j})c_{n,j}^0\widehat{x}_{n+1,2j} + a(x_{n,j})c_{n,j}^1\widehat{x}_{n+1,2j+1}\right]$$
$$= \sum_{j+0}^{2^n-1} a(x_{n+1,j})\widehat{x}_{n+1,j} = \widehat{a}_{n+1}$$

(iii) This is obtained from

$$\begin{aligned}
U_n^*\widehat{a}_{n+1} &= \sum\left[a(x_{n+1,2j})\widehat{x}_{n+1,2j} + a(x_{n+1,2j+1})\widehat{x}_{n+1,2j+1}\right]\\
&= \sum\left[a(x_{n+1,2j})\overline{c}_{n,j}^0\widehat{x}_{n,j} + a(x_{n+1,2j+1})\overline{c}_{n,j}^1\widehat{x}_{n,j}\right]\\
&= \sum\left[c_{n,j}^0 a(x_{n,j})\overline{c}_{n,j}^0 + c_{n,j}^1 a(x_{n,j})\overline{c}_{n,j}^1\right]\widehat{x}_{n,j}\\
&= \sum a(x_{n,j})\widehat{x}_{n,j} = \widehat{a}_n
\end{aligned}$$

□

Actually, (iii) follows from (ii) in Theorem 2.6 because $\widehat{a}_{n+1} = U_n\widehat{a}_n \in \mathcal{R}(U_n)$ so $U_n^*\widehat{a}_{n+1} = U_n^*U_n\widehat{a}_n = \widehat{a}_n$. Interference in $\mathcal{P}_n$ or Ω_n can be described by the nonadditivity of the q-measure μ_n. We say that $x, y \in \mathcal{P}_n$ *do not interfere* if

$$\mu_n(\{x,y\}) = \mu_n(\{x\}) + \mu_n(\{y\})$$

The next result gives an application of this concept.

Theorem 2.7. *If $x, y \in \mathcal{P}$ have the same producer, then x and y do not interfere.*

Proof. Suppose $x = x_{n+1,2y}, y = x_{n+1,2j+1}$ so x, y have the same producer $x_{n,j}$. Then

$$\begin{aligned}\mu_{n+1}\left(\{x,y\}\right) &= \left|a(x)+a(y)\right|^2 = \left|a(x_{n,j})c_{n,j}^0 + a(x_{n,j})c_{n,j}^1\right|^2 \\ &= \left|a(x_{n,j})\right|^2\left|c_{n,j}^0 + c_{n,j}^1\right|^2 = \left|a(x_{n,j})\right|^2 \\ &= \left|a(x_{n,j})\right|^2\left[\left|c_{n,j}^0\right|^2 + \left|c_{n,j}^1\right|^2\right] \\ &= \left|a(x_{n,j})c_{n,j}^0\right|^2 + \left|a(x_{n,j})c_{n,j}^1\right|^2 \\ &= \left|a(x_{n+1,2j})\right|^2 + \left|a(x_{n+1,2j+1})\right|^2 = \mu_{n+1}\left(\{x\}\right) + \mu_{n+1}\left(\{y\}\right)\end{aligned}$$

Hence, x and y do not interfere. □

In general, the noninterference result in Theorem 2.7 does not hold if x and y have different producers. This is shown in the next two examples.

Example 1. For simplicity, suppose the uta is cs so we have just two coupling constants c^0, c^1. We have seen in Theorem 2.7 that $x_{3,0}$ and $x_{3,1}$ do not interfere. In a similar way, we see that $x_{3,0}$ and $x_{3,2}$ do not interfere. We also have that $x_{3,1}$ does not interfere with $x_{3,j}$, $j = 0, 1, 2$ and $x_{3,2}$ does not interfere with $x_{3,j}$, $j = 0, 1, 3$. Let us now consider $x_{3,0}$ and $x_{3,3}$. We have that

$$\begin{aligned}\mu_3\left(\{x_{3,0}, x_{3,3}\}\right) &= \left|a(x_{3,0}) + a(x_{3,3})\right|^2 = \left|(c^0)^2 + (c^1)^2\right|^2 \\ &= \left|\cos^2\theta - \sin^2\theta\right| = \cos^2 2\theta\end{aligned}$$

On the other hand

$$\begin{aligned}\mu_3\left(\{x_{3,0}\}\right) + \mu_2\left(\{x_{3,3}\}\right) &= \left|a(x_{3,0})\right|^2 + \left|a(x_{3,3}\right|^2 \\ &= \cos^4\theta + \sin^4\theta = \tfrac{1}{2}(1 + \cos^2 2\theta)\end{aligned}$$

so $x_{3,0}$ and $x_{3,3}$ interfere, in general.

Example 2. If the uta is not cs, the situation is more complicated and we incur more interference. In the cs case, we saw in Example 1 that $x_{3,0}$ and $x_{3,2}$ do not interfere. However, in this more general case we have

$$\mu_3\left(\{x_{3,0}, x_{3,2}\}\right) = \left|a(x_{3,0}) + a(x_{3,2})\right| = \left|c_{1,0}^0 c_{2,0}^0 + c_{1,0}^1 c_{2,1}^0\right|^2$$

On the other hand

$$\mu_3(\{x_{3,0}\}) + \mu_3(\{x_{3,2}\}) = |a(x_{3,0})|^2 + |a(x_{3,2})|^2 = \left|c_{1,0}^0\right|^2 \left|c_{2,0}^0\right|^2 + \left|c_{1,0}^1\right|^2 \left|c_{2,1}^0\right|^2$$

But these two quantities do not agree unless

$$Re(c_{1,0}^0 c_{2,0}^0 \overline{c}_{1,0}^1 \overline{c}_{2,1}^0) = 0$$

so $x_{3,0}$ and $x_{3,3}$ interfere, in general.

3 Double-Down To Unitary

We have seen that corresponding to a uta with coupling constants $c_{n,j}^k$ there are isometries $U_n \colon H_n \to H_{n+1}$ that describe the dynamics for a quantum sequential growth process on $(\mathcal{P}, \to)$. The operators U_n cannot be unitary because H_n and H_{n+1} are different dimensional Hilbert spaces. However, we can "double-down" the U_n to form operators $V_{n+1} \colon H_{n+1} \to H_{n+1}$ by

$$\begin{aligned} V_{n+1}\widehat{x}_{n+1,2j} &= c_{n,j}^0 \widehat{x}_{n+1,2j} + c_{n,j}^1 \widehat{x}_{n+1,2j+1} \\ V_{n+1}\widehat{x}_{n+1,2j+1} &= c_{n,j}^1 \widehat{x}_{n+1,2j} + c_{n,j}^0 \widehat{x}_{n+1,2j+1} \end{aligned}$$

Theorem 3.1. *The operators V_{n+1} are unitary and $V_{n+1}1_{n+1} = 1_{n+1}$, $n = 1, 2, \ldots$.*

Proof. Since $\|V_{n+1}\widehat{x}_{n+1,2j}\| = \|V_{n+1}\widehat{x}_{n+1,2j+1}\| = 1$ and

$$\langle V_{n+1}\widehat{x}_{n+1,2j}, V_{n+1}\widehat{x}_{n+1,2j+1}\rangle = \overline{c}_{n,j}^0 c_{n,j}^1 + \overline{c}_{n,j}^1 c_{n,j}^0 = 0$$

we conclude that V_{n+1} sends an orthonormal basis to an orthonormal basis. Hence, V_{n+1} is unitary. To show that $V_{n+1}1_{n+1} = 1_{n+1}$ we have

$$\begin{aligned} V_{n+1}1_{n+1} &= \sum_j (V_{n+1}\widehat{x}_{n+1,2j} + V_{n+1}\widehat{x}_{n+1,2j+1}) \\ &= \sum_j \left[(c_{n,j}^0 + c_{n,j}^1)\widehat{x}_{n+1,2j} + (c_{n,j}^1 + c_{n,j}^0)\widehat{x}_{n+1,2j+1}\right] \\ &= \sum_j (\widehat{x}_{n+1,2j} + \widehat{x}_{n+1,2j+1}) = 1_{n+1} \end{aligned}$$

□

The unitary operator V_2 corresponds to the coupling constants $c_{1,0}^0$, $c_{1,0}^1$ and relative to the basis $\{\widehat{x}_{2,0}, \widehat{x}_{2,1}\}$ has the form

$$V_2 = \begin{bmatrix} c_{1,0}^0 & c_{1,0}^1 \\ c_{1,0}^1 & c_{1,0}^0 \end{bmatrix}$$

Besides being unitary, V_2 is doubly stochastic (row and column sums are one). Of course, this is also true of V_n. By Theorem 2.1, there exists a unique $\theta \in [0, \pi)$ such that $c_{1,0}^0 = \cos\theta e^{i\theta}$, $c_{1,0}^1 = -i\sin\theta e^{i\theta}$. To make θ explicit, we write $V_2 = V_2(\theta)$.

Lemma 3.2. *The operator $V_2(\theta)$ has eigenvalues $1, e^{2i\theta}$ with corresponding unit eigenvectors $2^{-1/2}(1,1)$, $2^{-1/2}(1,-1)$.*

Proof. By direct verification we have

$$V_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_{1,0}^0 + c_{1,0}^1 \\ c_{1,0}^1 + c_{1,0}^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_{1,0}^0 - c_{1,0}^1 \\ c_{1,0}^1 - c_{1,0}^0 \end{bmatrix} = (c_{1,0}^0 - c_{1,0}^1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

But

$$\begin{aligned} c_{1,0}^0 - c_{1,0}^1 &= c_{1,0}^0 - (1 - c_{1,0}^0) = 2c_{1,0}^0 - 1 = 2\cos\theta e^{i\theta} - 1 \\ &= 2\cos^2\theta + 2i\cos\theta\sin\theta - 1 = \cos^2\theta - \sin^2\theta + i\sin 2\theta \\ &= \cos 2\theta + i\sin 2\theta = e^{2i\theta} \end{aligned}$$

□

We can write the 2^n-dimensional Hilbert space H_{n+1} as

$$H_{n+1} = H_2 \oplus H_2 \oplus \cdots \oplus H_2$$

where there are 2^{n-1} summands and the jth summand has the basis $\{\widehat{x}_{n+1,2j}, \widehat{x}_{n+1,2j+1}\}$. In general, V_{n+1} has the form

$$V_{n+1}(\theta_1, \theta_2, \ldots, \theta_{2^{n-1}}) = V_2(\theta_1) \oplus V_2(\theta_2) \oplus \cdots \oplus V(\theta_{2^{n-1}})$$

It follows from Lemma 3.2 that $V_{n+1}(\theta_1, \theta_2, \ldots, \theta_{2^{n-1}})$ has eigenvalues 1 (with multiplicity 2^{n-1}) and $e^{2i\theta_1}, e^{2i\theta_2}, \ldots, e^{2i\theta_{2^{n-1}}}$. The unit eigenvectors corresponding to 1 are

$$2^{-1/2}(\widehat{x}_{n+1,2j} + \widehat{x}_{n+1,2j+1}), \quad j = 0, 1, \ldots, 2^{n-1} - 1$$

and the unit eigenvector corresponding to $e^{2i\theta j}$ is

$$2^{-1/2}(\widehat{x}_{n+1,2j} - \widehat{x}_{n+1,2j+1})$$

Let $\mathcal{S}(H_{n+1})$ be the set of operators on H_{n+1} of the form

$$\mathcal{S}(H_{n+1}) = \{V_{n+1}(\theta_1, \theta_2, \ldots, \theta_{2^{n-1}})\colon \theta_n \in [0,\pi)\}$$

Now $[0,\pi)$ forms an abelian group with operations $a \oplus b = a + b \pmod{\pi}$.

Lemma 3.3. *For $\theta_1, \theta_2 \in [0,\pi)$ we have $V_2(\theta_1)V_2(\theta_2) = V_2(\theta_1 + \theta_2)$.*

Proof. Since $V_2(\theta_1)$ and $V_2(\theta_2)$ have the same eigenvectors, they commute and can be simultaneously diagonalized as

$$V_2(\theta_1) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_1} \end{bmatrix} \quad V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_2} \end{bmatrix}$$

Hence, if $\theta_1 + \theta_2 < \pi$ then

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2)} \end{bmatrix} = V_2(\theta_1 \oplus \theta_2)$$

and if $\theta_1 + \theta_2 \geq \pi$ then $\theta_1 \oplus \theta_2 = \theta_1 + \theta_2 - \pi$ and we have

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2-\pi)} \end{bmatrix} = V_2(\theta_1 + \theta_2 - \pi) = V(\theta_1 \oplus \theta_2) \qquad \square$$

We now form the product group $[0,\pi)^{2^{n-1}} = [0,\pi) \times \cdots \times [0,\pi)$ to obtain the following result.

Corollary 3.4. *Under operator multiplication, $\mathcal{S}(H_{n+1})$ is an abelian group and $(\theta_1, \ldots, \theta_{2^{n-1}}) \mapsto V_{n+1}(\theta_1, \ldots, \theta_{2^{n-1}})$ is a unitary representation of the group $[0,\pi)^{2^{n-1}}$.*

Since V_{n+1} is unitary, there exists a unique self-adjoint operator K_{n+1} on H_{n+1} such that $V_{n+1} = e^{iK_{n+1}}$. We call K_{n+1} a *Hamiltonian operator.* For $V_{n+1}(\theta_1, \ldots, \theta_{2^{n-1}}$ the eigenvalues of K_{n+1} are 0 (with multiplicity 2^{n-1}) and $2\theta_1, \ldots, 2\theta_{2^{n-1}}$. Hence, $\theta_j = 2^{-1}\lambda_j$ where λ_j is the jth energy value,

$j = 1, \ldots, 2^{n-1}$. This gives a physical significance for the angles θ_j. The corresponding eigenvectors are the same as those given for V_{n+1}.

It is natural to define the *position operator* Q_{n+1} on H_{n+1} by $Q_{n+1} f(\widehat{x}_{n+1,k}) = k$. Thus, $Q_{n+1}\widehat{x}_{n+1,2j} = 2j$ and $Q_{n+1}\widehat{x}_{n+1,2j+1} = 2j+1$. Since Q_{n+1} is diagonal, we immediately see that its eigenvalues are $0, 1, \ldots, 2^n - 1$ with corresponding eigenvector $\widehat{x}_{n+1,k}$. It also seems natural to define the *canonical momentum operator* P_{n+1} on the subspace generated by $\{\widehat{x}_{n+1,2j}, \widehat{x}_{n+1,2j+1}\}$ as

$$\begin{aligned}
P_2(\theta_j) &= V_2(\theta_j)^* Q_2(\theta_j) V_2(\theta_j) \\
&= \begin{bmatrix} \overline{c}^0_{n,j} & \overline{c}^1_{n,j} \\ \overline{c}^1_{n,j} & \overline{c}^0_{n,j} \end{bmatrix} \quad \begin{bmatrix} 2j & 0 \\ 0 & 2j+1 \end{bmatrix} \quad \begin{bmatrix} c^0_{n,j} & c^1_{n,j} \\ c^1_{n,j} & c^0_{n,j} \end{bmatrix} \\
&= \begin{bmatrix} 2j + \left| c^1_{n,j} \right|^2 & c^0_{n,j}\overline{c}^1_{n,j} \\ \overline{c}^0_{n,j} c^1_{n,j} & 2j + \left| c^0_{n,j} \right|^2 \end{bmatrix} = \begin{bmatrix} 2j + \sin^2\theta_{n,j} & \frac{i}{2}\sin 2\theta_{n,j} \\ -\frac{i}{2}\sin 2\theta_{n,j} & 2j + \cos^2\theta_{n,j} \end{bmatrix}
\end{aligned}$$

The eigenvalues of $P_2(\theta_j)$ are $2j$ and $2j+1$ with corresponding unit eigenvectors

$$V_2(\theta_j)^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{c}^0_{n,j} \\ \overline{c}^1_{n,j} \end{bmatrix}$$

$$V_2(\theta_j)^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{c}^1_{n,j} \\ \overline{c}^0_{n,j} \end{bmatrix}$$

The complete momentum operator P_{n+1} is given by

$$P_{n+1}(\theta_1, \ldots, \theta_{2^{n-1}}) = P_2(\theta_1) \oplus P_2(\theta_2) \oplus \cdots \oplus P_2(\theta_{2n-1})$$

We now compute the commutator

$$[P_2(\theta_j), Q_2(\theta_j)] = P_2(\theta_j)Q_2(\theta_j) - Q_2(\theta_j)P_2(\theta_n) = c^0_{n,j}\overline{c}^1_{n,j} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{i}{2}\sin 2\theta_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The complete commutation relation is

$$\begin{aligned}&[P_{n+1}(\theta_1,\ldots,\theta_{2^{n-1}}),Q_{n+1}(\theta_1,\ldots,\theta_{2^{n-1}})]\\&\qquad=[P_2(\theta_1),Q_2(\theta_1)]\oplus\cdots\oplus[P_2(\theta_{2^{n-1}})Q_2(\theta_{2^{n-1}})]\end{aligned}$$

As in the Heisenberg uncertainty relation, the number $|\langle\phi,[P_{n+1},Q_{n+1}]\,\phi\rangle|$ gives a lower bound for the product of the variances of P_{n+1} and Q_{n+1}. We now compute this number for an amplitude state $\widehat{a}_{n+1}$. We have that

$$\begin{aligned}&\langle\widehat{a}_{n+1},[P_{n+1},Q_{n+1}]\,\widehat{a}_{n+1}\rangle\\&\quad=\sum_j\left\langle\begin{bmatrix}a(x_{n+1,2j})\\a(x_{n+1,2j+1})\end{bmatrix},\ c_{n,j}^0c_{n,j}^1\begin{bmatrix}0&1\\1&0\end{bmatrix}\ \begin{bmatrix}a(x_{n+1,2j})\\a(x_{n+1,2j+1})\end{bmatrix}\right\rangle\\&\quad=\sum_j c_{n,j}^0\overline{c}_{n,j}^1\left[\overline{a}(x_{n+1,2j})a(x_{n+1,2j+1})+\overline{a}(x_{n+1,2j+1})a(x_{n+1,2j})\right]\\&\quad=\sum_j c_{n,j}^0\overline{c}_{n,j}^1\,|a(x_{n,j})|^2\left[\overline{c}_{n,}^0c_{n,j}^1+\overline{c}_{n,j}^1c_{n,j}^0\right]=0\end{aligned}$$

This shows that even though P_{n+1} and Q_{n+1} do not commute, there is no lower bound for the product of their variances when the system is in an amplitude state.

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