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Abstract

This paper is based on a covariant causal set (c-causet) approach to discrete quantum gravity. A c-causet is a partially ordered set $(x, <)$ that is invariant under labeling. We first consider the microscopic picture which describes the detailed structure of c-causets. The unique labeling of a c-causet x enables us to define a natural metric $d(a,b)$ between comparable vertices a, b of x. The metric is then employed to define geodesics and curvatures on x. We next consider the macroscopic picture which describes the growth process $x \to y$ of c-causets. We propose that this process is governed by a quantum dynamics given by complex amplitudes. Denoting the set of c-causets by $\mathcal{P}$ we show that the growth process $(\mathcal{P}, \to)$ can be structured into a discrete 4-manifold. This 4-manifold presents a unified approach to a discrete quantum gravity for which we define discrete analogues of Einstein's field equations and Dirac's equation.

1 Microscopic Picture

In this article we continue our work on a covariant causal set approach to discrete quantum gravity [4, 6, 7]. For background and more details, we refer the reader to [2, 3]. We call a finite partially ordered set a *causet* and interpret the order $a < b$ in a causet x to mean that b is in the causal future of a. We denote the cardinality of a causet x by $|x|$. If x and y are causets with $|y| = |x| + 1$ then x *produces* y (written $x \to y$) if y is obtained from x by adjoining a single maximal element a to x. If $x \to y$ we call y an *offspring* of x.

A *labeling* for a causet x is a bijection $\ell\colon x \to \{1, 2, \ldots, |x|\}$ such that $a, b \in x$ with $a < b$ implies that $\ell(a) < \ell(b)$. If ℓ is labeling for x, we call $x = (x, \ell)$ an *ℓ-causet*. Two ℓ-causets x and y are *isomorphic* if there exists a bijection $\phi\colon x \to y$ such that $a < b$ in x if and only if $\phi(a) < \phi(b)$ in y and $\ell\left[\phi(a)\right] = \ell(a)$ for every $a \in x$. Isomorphic ℓ-causets are considered identical as ℓ-causets. We say that a causet is *covariant* if it has a unique labeling (up to ℓ-causet isomorphism) and call a covariant causet a *c-causet*. We denote the set of a c-causets with cardinality n by $\mathcal{P}_n$ and the set of all c-causets by $\mathcal{P}$. It is easy to show that any $x \in \mathcal{P}$ with $|x| > 1$ has a unique producer and that any $x \in \mathcal{P}$ has precisely two offspring [3]. It follows that $|\mathcal{P}_n| = 2^{n-1}$, $n = 1, 2, \ldots$.

Two elements $a, b \in x$ are *comparable* if $a < b$ or $b < a$. We say that a is a *parent* of b and b is a *child* of a if $a < b$ and there is no $c \in x$ with $a < c < b$. A *path from a to b* in x is a sequence $a_1 = a$, $a_2, \ldots a_{n-1}$, $a_n = b$ where a_i is a parent of a_{i+1}, $i = 1, \ldots, n-1$. The *height* $h(a)$ of $a \in x$ is the cardinality minus one of the longest path in x that ends with a. If there are no such paths, then $h(a) = 0$ by convention. It is shown in [3] that a causet x is covariant if and only if $a, b \in x$ are comparable whenever a and b have different heights. Notice that in any c-causet, two elements with the same height cannot be comparable. If $x \in \mathcal{P}$ we call the sets

$$S_j(x) = \left\{a \in x\colon h(a) = j\right\}, j = 0, 1, 2, \ldots$$

shells and the sequence of integers $s_j(x) = |S_j(x)|$, $j = 0, 1, 2, \ldots$, is the *shell sequence*. A c-causet is uniquely determined by its shell sequence and we think of $\{s_j(x)\}$ as describing the "shape" or geometry of x [2, 3].

The tree $(\mathcal{P}, \to)$ can be thought of as a growth model and an $x \in \mathcal{P}_n$ is a possible universe at step (time) n. An instantaneous universe $x \in \mathcal{P}_n$ grows one element at a time in one of two ways. To be specific, if $x \in \mathcal{P}_n$ has shell sequence $(s_0(x), s_1(x), \ldots, s_m(x))$, then x will grow to one of its two offspring $x \to x_0$, $x \to x_1$ where x_0 and x_1 have shell sequences

$$\begin{gathered}(s_0(x), s_1(x), \ldots, s_m(x) + 1)\\ (s_0(x), s_1(x), \ldots, s_m(x), 1)\end{gathered}$$

respectively. In this way, we can recursively order the c-causets in $\mathcal{P}$ by using the notation $x_{n,j}$, $n=1,2,\ldots$, $j=0,1,2,\ldots,2^{n-1}-1$ where $n=|x_{n,j}|$. For example, in terms of their shell sequences we have:

$$
\begin{aligned}
&x_{1,0}=(1), x_{2,0}=(2), x_{2,1}=(1,1)\\
&x_{3,0}=(3), x_{3,1}=(2,1), x_{3,2}=(1,2), x_{3,3}=(1,1,1)\\
&x_{4,0}=(4), x_{4,1}=(3,1), x_{4,2}=(2,2), x_{4,3}=(2,1,1), x_{4,4}=(1,3)\\
&x_{4,5}=(1,2,1), x_{4,6}=(1,1,2), x_{4,7}=(1,1,1,1)
\end{aligned}
$$

In general, the c-causet $x_{n,j}$ has the two offspring $x_{n,j}\to x_{n+1,2j}$ and $x_{n,j}\to x_{n+1,2j+1}$, $n=1,2,\ldots$, $j=0,1,2,\ldots,2^{n-1}-1$. For example, $x_{3,2}\to x_{4,4}$ and $x_{3,2}\to x_{4,5}$ while $x_{3,3}\to x_{4,6}$ and $x_{3,3}\to x_{4,7}$. Conversely, for $n=2,3,\ldots$, $x_{n,j}$ has the unique producer $x_{n-1,\lfloor j/2\rfloor}$ where $\lfloor j/2\rfloor$ is the integer part of $j/2$. For example, $x_{5,14}$ had the producer $x_{4,7}$ and $x_{5,13}$ has the producer $x_{4,6}$. With the previously notation $\mathcal{P}=\{x_{n,j}\}$ in place, we call $\{\mathcal{P},\to\}$ a *sequential growth process* (SGP).

In the microscopic picture, we view a c-causet x as a framework or scaffolding for a possible universe. The vertices of x represent small cells that can be empty or occupied by a particle. The shell sequence that determines x gives the geometry of the framework. In order to describe the universe, we would like to find out how particles move and which vertices they are likely to occupy. We accomplish this by introducing a distance or metric on x.

Let $x=\{a_1,a_2,\ldots,a_n\}\in\mathcal{P}_n$, where the subscript i of a_i is the label of the vertex. We can think of a path

$$\gamma=a_{i_1}a_{i_2}\cdots a_{i_m} \tag{1.1}$$

as a sequence in x starting with a_{i_1} and moving along successive shells until a_{i_m} is reached. We define the *length* of γ by

$$\ell(\gamma)=\left[\sum_{j=2}^{m}(i_j-i_{j-1})^2\right]^{1/2} \tag{1.2}$$

Of course, there are a variety of definitions that one can give for the length of a path, but this is one of the simplest nontrivial choices. We shall compare (1.2) with another possible choice later in order to illustrate it's advantages. For $a,b\in x$ with $a<b$, a *geodesic* from a to b is a path from a to b that has the shortest length. Clearly, if $a<b$, then there is at least one geodesic from

a to b. If $a, b \in x$ are comparable and $a < b$ say, then the *distance* $d(a,b)$ is the length of a geodesic from a to b. The next result shows that the triangle inequality holds when applicable so $d(a,b)$ has the most important property of a metric. A *subpath* of a path $a_{i_1}a_{i_2}\cdots a_{i_m}$ is a subset of $\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$ that is again a path. The next result also shows that once we have a geodesic we can take subpaths to form other geodesics.

Theorem 1.1. [2] (i) *If $a < c < b$, then $d(a,b) \le d(a,c) + d(c,b)$.* (ii) *A subpath of a geodesic is a geodesic.*

As we shall see, there may be more than one geodesic from a to b when $a < b$. In the remainder of this section, we shall refer to a vertex by its label. If there are j geodesics from vertex 1 to vertex n, we define the *curvature* $K(n)$ at n to be $K(n) = j - 1$. One might argue that the curvature should be a local property and should not depend so heavily on vertex 1 which could be a considerable distance away. However, if there are a lot of geodesics from 1 to n, then by Theorem 1.1(ii), there are also a lot of geodesics from other vertices to n. Thus, the definition of curvature is not so dependent on the initial vertex 1 as it first appears. Assuming that particles tend to move along geodesics, we see that $K(n)$ gives a measure of the tendency for vertex n to be occupied.

We now give an example that appeared in [2] and we refer the reader to that reference for details and other examples. Let x be the c-causet with shell sequence (1,2,3,4,5,4,3,2,1). Separating shells by semicolons, the labels of the vertices become

$$(1; 2, 3; 4, 5, 6; 7, 8, 9, 10; 11, 12, 13, 14, 15; 16, 17, 18, 19; 20, 21, 22; 23, 24; 25)$$

The c-causet x represents a toy universe that expands uniformly and then contracts uniformly. The following table summarizes the curvatures for vertices of x.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$K(i)$	-1	0	0	1	0	0	0	1	0	0	2	0	1	0

i	15	16	17	18	19	20	21	22	23	24	25
$K(i)$	0	2	2	0	1	5	3	1	5	9	5

Table 1

For illustrative purposes, we now give an example of a different distance function. For a path γ given by (1.1) define the *length* ℓ_1 by

$$\ell_1(\gamma) = \max\{|i_j - i_{j-1}| : j = 2, \ldots, m\} \tag{1.3}$$

We now define 1-*geodesics*, *distance* $d_1(a, b)$ and *curvature* K_1 as before with $\ell(\gamma)$ replaced by $\ell_1(\gamma)$. Although d_1 satisfies the triangle inequality of Theorem 1.1(i), it gives a rather course distance measure compared to d. One way of seeing this is that Theorem 1.1(ii) does not hold for d_1 as the next example shows.

Let x be the c-causet with shell sequence (1,2,3,4) and vertex labels (1;2,3;4,5,6;7,8,9,10). The following two tables summarize the distances and curvatures for the vertices of x.

i	2	3	4	5	6	7	8	9	10
$d_1(1, i)$	1	2	2	2	3	2	3	3	4

Table 2

i	1	2	3	4	5	6	7	8	9	10
$K_1(i)$	-1	0	0	1	0	0	0	2	0	1

Table 3

Notice that the path 1-2-5-8 is a 1-geodesic, but the path 1-2-5 is not which shows that Theorem 1.1(ii) does not hold for d_1. Another example is that 1-2-6-10 is a 1-geodesic, but 1-2-6 is not.

2 Macroscopic Picture

In general relativity theory it is postulated that the mass-energy distribution determines the curvature, while in our microscopic picture we assume that it is the other way around. That is, the curvature determines the mass distribution and the curvature is given by the geometry (shell sequence). We are now confronted with the question: What determines the shell sequence of our particular universe? To study this question, the present section studies the macroscopic picture. This picture describes the evolution of a universe as a quantum sequential growth process. In such a process, the probabilities of competing evolutions are determined by quantum amplitudes. Moreover, we shall see the emergence of a discrete 4-manifold.

In [2] we gave a method for constructing a discrete 4-manifold from the SGP $(\mathcal{P},\to)$. This method was based on forming "twin c-causets." In this article we find it convenient to employ a related but different method based on incident edge pairs. If $x\to y$ we call xy an *edge* in $\mathcal{P}$. Two edges of the form $e=xy$ and $f=yz$ are said to be *incident* and we call the pair (e,f) of incident edges a *direction*. The direction (e,f) can also be described by the vertices (x,y,z) where $x\to y\to z$. Starting at any $x_{n,j}\in\mathcal{P}$ we have the four directions given by

$$\begin{aligned}
d_{n,j}^1 &= (x_{n,j},x_{n+1,2j},x_{n+2,4j})\\
d_{n,j}^2 &= (x_{n,j},x_{n+1,2j},x_{n+2,4j+1})\\
d_{n,j}^3 &= (x_{n,j},x_{n+1,2j+1},x_{n+2,4j+2})\\
d_{n,j}^4 &= (x_{n,j},x_{n+1,2j+1},x_{n+2,4j+3})
\end{aligned}$$

We denote the set of directions by

$$D=\left\{d_{n,j}^k\colon n=1,2,\ldots,j=0,1,\ldots,2^{n-1}-1,k=1,2,3,4\right\}$$

Two directions (e,f), (e_1,f_1) are *incident* if f and e_1 are incident. A *direction path* ω in $\mathcal{P}$ is a sequence of successively incident directions and a *direction n-path* in $\mathcal{P}$ is a finite sequence of n successively incident directions beginning at one of the *initial vertices* $x_{1,0}$, $x_{2,0}$ or $x_{2,1}$. Specifically, a direction n-path has one of the forms

$$d_{1,0}^{k_1}d_{3,j_3}^{k_3}\cdots d_{2n-1,j_{2n-1}}^{k_{2n-1}} \tag{2.1}$$

$$d_{2,0}^{k_2}d_{4,j_4}^{k_3}\cdots d_{2n,j_{2n}}^{k_{2n}} \tag{2.2}$$

$$d_{2,1}^{k_2}d_{4,j_4}^{k_4}\cdots d_{2n,j_{2n}}^{k_{2n}} \tag{2.3}$$

where $k_i\in\{1,2,3,4\}$ and each direction is incident to the direction that follows it. We say that (2.1), (2.2), (2.3) have *initial vertices* $x_{1,0},x_{2,0},x_{2,1}$, respectively and have *final vertices* given by the last vertices of their directions, respectively. For example, the unique direction 2-path with initial vertex $x_{1,0}$ and final vertex $x_{5,7}$ is $d_{1,0}^2d_{3,1}^4$. Notice that every *odd vertex* $x_{2n-1,j_{2n-1}}$ is the final vertex of a unique direction n-path with initial vertex $x_{1,0}$ and every *even vertex* $x_{2n,j_{2n}}$ is the final vertex of a unique direction n-path with initial vertex $x_{2,0}$ or $x_{2,1}$. A direction path is like one of the direction n-paths (2.1), (2.2), (2.3) except it does not terminate.

We denote the set of direction paths in $\mathcal{P}$ by Ω and the set of direction n-paths by Ω_n. We interpret a direction path as a completed universe or history of an evolved universe. A direction path or n-path ω *contains* $x \in \mathcal{P}$ if there is a direction of ω that has the form (x, y, z). We then write $x \in \omega$.

A *transition amplitude* is a map $\widetilde{a}\colon D \to \mathbb{C}$ satisfying $\sum_{k=1}^{4} \widetilde{a}(d_{n,j}^k) = 1$ for all n, j. Corresponding to $\widetilde{a}$ we define the *amplitude* of a direction n-path $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$ to be $a(\omega) = \widetilde{a}(\omega_1)\widetilde{a}(\omega_2) \cdots \widetilde{a}(\omega_n)$ if the initial vertex of ω is $x_{1,0}$ and $a(\omega) = (1/2)\widetilde{a}(\omega_1)\widetilde{a}(\omega_2) \cdots \widetilde{a}(\omega_n)$ if the initial vertex of ω is $x_{2,0}$ or $x_{2,1}$. The *amplitude* of a set $A \subseteq \Omega_n$ is $a(Q) = \sum \{a(\omega)\colon \omega \in A\}$. Notice that $a(\Omega_n) = 1$. The *amplitude* $a(x_{n,j})$ for $x_{n,j} \in \mathcal{P}_n$, $n \geq 3$ is $a(\omega)$ where ω is the unique finite direction path that has $x_{n,j}$ as its final vertex. It follows that $\sum \{a(x)\colon x \in \mathcal{P}_n\} = 1$. We call $c_{n,j}^k = \widetilde{a}(d_{n,j}^k)$ *coupling constants* and note that $\sum_{k=1}^{4} c_{n,j}^k = 1$ for all n, j. We define $a(x_{n,0}) = 1$, $a(x_{2,0}) = a(x_{2,1}) = 1/2$.

The *q-measure* of a set $A \subseteq \Omega_n$ corresponding to $\widetilde{a}$ is defined by $\mu_n(A) = |a(A)|^2$ [5]. In particular, $\mu_n(\omega) = |a(\omega)|^2$, $\omega \in \Omega_n$ and $\mu_n(x_{n,j}) = |a(x_{n,j})|^2$. Letting $\mathcal{A}_n = 2^{\Omega_n}$ be the power set on Ω_n we have that $(\Omega_n, \mathcal{A}_n)$ is a measurable space and we call $(\Omega_n, \mathcal{A}_n, \mu_n)$ a *q-measure space*. We interpret $\mu_n(A)$ as the quantum propensity of the event $A \in \mathcal{A}_n$. It is believed that once the coupling constants and hence the q-measures μ_n are known, then certain direction n-paths and c-causets $x_{n,j}$ will have dominate propensities. In this way we will determine dominate geometries for the microscopic picture of our particular universe. Because of quantum interference the q-measure μ_n is not a measure on the σ-algebra $\mathcal{A}_n$, in general. This is because the additivity condition $\mu_n(A \cup B) = \mu_n(A) + \mu_n(B)$ whenever $A \cap B = \emptyset$ need not hold. Of course, we do have that $\mu_n(\Omega_n) = 1$ and $\mu_n(A) \geq 0$ for all $A \in \mathcal{A}$.

We refer the reader to [2] for an example of a transition amplitude $\widetilde{a}\colon D \to \mathbb{C}$ that may have physical significance. In this situation, the c-causets and direction paths with highest propensity lie toward the "middle" of the process $(\mathcal{P}, \to)$. For example, the c-causets of highest propensity are those $x_{n,j}$ with $j = 2^{n-2}$ and the propensities decrease to zero for large n as j gets smaller and larger than 2^{n-2}

3 Covariant Difference Operators

Viewing the directions $d_{n,j}^k$, $k = 1, 2, 3, 4$ as "tangent vectors" at the vertex $x_{n,j}$ we see the emergence of a discrete 4-manifold. We now carry this analogy further by defining covariant difference operators.

Let $H = L_2(\mathcal{P})$ be the Hilbert space of square summable complex-valued functions on $\mathcal{P}$ with the usual inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}} \overline{f(x)} g(x)$$

We define the *covariant difference operators* ∇_k, $k = 1, 2, 3, 4$ by

$$\nabla_k f(x_{n,j}) = f(x_{n+2,4j+k-1}) - c^k_{n+2,4j+k-1} f(x_{n,j}) \tag{3.1}$$

The operator ∇_k can be considered to be the difference operator in the direction k, $k = 1, 2, 3, 4$. This is a slight variation of the difference operators considered in [1, 2].

For $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4$, define the function $\omega \colon \mathbb{R}^4 \times \mathcal{P} \to \mathbb{C}$ recursively by

$$\omega(p, x_{n+2,4j+k-1}) = \begin{cases} (c^k_{n+2,4j+k-1} + ip_k)\omega(p, x_{n,j}) & \text{if } k = 1, 2, 3 \\ (c^k_{n+2,4j+k-1} - ip_k)\omega(p, x_{n,j}) & \text{if } k = 4 \end{cases} \tag{3.2}$$

where $i = \sqrt{-1}$. The values $\omega(p, x_{1,0})$, $\omega(p, x_{2,0})$, $\omega(p, x_{2,1})$ are arbitrary and are the *initial conditions*. For fixed p, $\omega(p, x_{n,j})$ corresponds to a *discrete plane wave* in the "direction" p. In general, $\omega(p, \cdot) \notin H$ except for certain values of p depending on the coupling constants.

Example. If we define the initial conditions

$$\omega(p, x_{1,0}) = \omega(p, x_{2,0}) = \omega(p, x_{2,1}) = 1$$

we have that

$$\begin{aligned}
\omega(p, x_{3,j}) &= c^{j+1}_{3,j} + ip_{j+1}, \ \ j = 0, 1, 2 \\
\omega(p, x_{4,j}) &= c^{j \pmod 4 + 1}_{4,j} + ip_{j \pmod 4 + 1}, \ \ j = 0, 1, 2, 4, 5, 6 \\
\omega(p, x_{5,j}) &= (c^{j+1}_{5,j} + ip_1)(c^1_{3,0} + ip_1), \ \ j = 0, 1, 2 \\
\omega(p, x_{5,j}) &= c^{j \pmod 4 + 1}_{5,j} + ip_{j \pmod 4 + 1}(c^2_{3,1} + ip_2), \ \ j = 4, 5, 6 \\
\omega(p, x_{5,j}) &= c^{j \pmod 4 + 1}_{4,j} + ip_{j \pmod 4 + 1}(c^3_{3,2} + ip_3), \ \ j = 8, 9, 10 \\
\omega(p, x_{5,j}) &= c^{j \pmod 4 + 1}_{4,j} + ip_{j \pmod 4 + 1}(c^4_{3,3} + ip_4), \ \ j = 12, 13, 14
\end{aligned}$$

We have omitted the values of $\omega(p, x_{n,j})$ for $j = 3 \pmod 4$. These are given by

$$\begin{aligned}
\omega(p, x_{3,3}) &= c^4_{3,3} - ip_4 \\
\omega(p, x_{4,3}) &= c^4_{34,3} - ip_4 \\
\omega(p, x_{4,7}) &= c^4_{4,7} - ip_4 \\
\omega(p, x_{5,3}) &= (c^4_{5,3} - ip_4)(c^1_{3,0} + ip_2) \\
\omega(p, x_{5,7}) &= (c^4_{5,7} - ip_4)(c^2_{3,1} + ip_2) \\
\omega(p, x_{5,11}) &= (c^4_{5,11} - ip_4)(c^3_{3,2} + ip_3) \\
\omega(p, x_{5,15}) &= (c^4_{5,15} - ip_4)(c^4_{3,3} + ip_4)
\end{aligned}$$

In general, ∇_k is an unbounded operator and we denote its domain by $\mathcal{D}(\nabla_k)$ $k = 1, 2, 3, 4$. The next result shows that if $\nabla_k \omega(p, \cdot)$ is defined, then $\omega(p, \cdot)$ is a simultaneous eigenvector of ∇_k, $k = 1, 2, 3, 4$.

Lemma 3.1. *If $\omega(p, \cdot) \in \bigcap_{k=1}^{4} \mathcal{D}(\nabla_k)$, then*

$$\nabla_k \omega(p, x_{n,j}) = \begin{cases} ip_k \omega(p, x_{n,j}) & \text{if } k = 1, 2, 3 \\ -ip_k \omega(p, x_{n,j}) & \text{if } k = 4 \end{cases}$$

Proof. By (3.1) and (3.2) we have

$$\begin{aligned}
\nabla_k \omega(p, x_{n,j}) &= \omega(p, x_{n+2,4j+k-1}) - c^k_{n+2,4j+k-1} \omega(p, x_{n,j}) \\
&= (c^k_{n+2,4j+k-1} \pm ip_k) - c_{n+2,4j+k-1} \omega(p, x_{n,j}) \\
&= \begin{cases} ip_k \omega(p, x_{n,j}) & \text{if } k = 1, 2, 3 \\ -ip_k \omega(p, x_{n,j}) & \text{if } k = 4 \end{cases} \qquad \square
\end{aligned}$$

If $\omega = \omega_1 \omega_2 \cdots \in \Omega$ and $x_{n,j} \in \omega$, then $\omega_m = (x_{n,j}, y, z)$ for some integer m. Now for some m', j', k' we have $\omega_m = d^{k'}_{m',j'}$, and we write $k(\omega, x_{n,j}) = k'$. We think of $k(\omega, x_{n,j})$ as specifying the direction of ω at the vertex $x_{n,j}$. Corresponding to the amplitude a and an arbitrary $x_{n,j} \in \mathcal{P}$ we define

$$a_\omega(x_{n,j}) = \begin{cases} a(x_{n,j}) & \text{if } x_{n,j} \in \omega \\ 0 & \text{if } x_{n,j} \notin \omega \end{cases}$$

The ω-*covariant difference operator* is the operator ∇_ω on H given by

$$\nabla_\omega f(x_{n,j}) = a_\omega(x_{n,j})\nabla_{k(\omega,x_{n,j})}f(x_{n,j}) \tag{3.3}$$

We then have by (3.1) that

$$\begin{aligned}\nabla_\omega f(x_{n,j}) = a_\omega(x_{n,j})&f(x_{n+2,4j+k(\omega,x_{n,j})-1}) \\ &- a_\omega(x_{n+2,4j+k(\omega,x_{n,j})-1})f(x_{n,j})\end{aligned} \tag{3.4}$$

If $a_\omega \in H$, then it follows from (3.4) that $\nabla_\omega a_\omega = 0$ which is why ∇_k and ∇_ω are called covariant.

We now extend this formalism to functions of two variables. Let $K = H \otimes H$ which we can identify with $L_2(\mathcal{P} \times \mathcal{P})$. For $k, k' = 1,2,3,4$, we define the *covariant bidifference operators* $\nabla_{k,k'}$ with domains in K by

$$\begin{aligned}\nabla_{k,k'}f(x_{n,j},x_{n'j'}) = f(&x_{n+2,4j+k-1},x_{n'+2,4j'+k'-1}) \\ &- \overline{c}^{k}_{n+2,4j+k-1}c^{k'}_{n'+2,4j'+k'-1}f(x_{n,j},x_{n',j'})\end{aligned}$$

For $\omega, \omega' \in \Omega$, the $\omega\omega'$-*covariant bidifference operator* is given by

$$\nabla_{\omega,\omega'}f(x_{n,j},x_{n'j'}) = \overline{a}_\omega(x_{n,j})a_{\omega'}(x_{n',j'})\nabla_{k(\omega,x_{n,j}),k(\omega',x_{n',j'})}f(x_{n,j},x_{n'j'})$$

Again, $\nabla_{\omega,\omega'}$ is called covariant because we have $\nabla_{\omega,\omega'}\overline{a}_\omega a_{\omega'} = 0$

4 Discrete Dirac and Einstein Equations

This section shows that we can employ the covariant difference operators presented in Section 3 to construct discrete analogues of Dirac's and Einstein's equations. Letting c be the speed of light in a vacuum, we use units in which $c = \hbar = 1$. In $\mathbb{R}^4$ we employ the indefinite inner product

$$p \cdot q = -p_1q_1 - p_2q_2 - p_3q_3 + p_4q_4$$

and we let σ_k, $k = 1,2,3,4$, be the Pauli matrices

$$\sigma_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad \sigma_2 = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \quad \sigma_3 = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \quad \sigma_4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$

Defining $\sigma = (\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ and $\nabla = (\nabla_1,\nabla_2,\nabla_3,\nabla_4)$ we have that

$$\sigma \cdot \nabla = -\sigma_1\nabla_1 - \sigma_2\nabla_2 - \sigma_3\nabla_3 + \sigma_4\nabla_4$$

The *discrete Weyl equation* is defined by

$$\sigma \cdot \nabla\phi(x_{n,j}) = 0 \tag{4.1}$$

where $\phi\colon \mathcal{P} \to \mathbb{C}^2$ is a two-component function $\phi = (\phi_1, \phi_2)$. Writing (4.1) in full gives

$$\begin{bmatrix} -\nabla_3 + \nabla_4 & -\nabla_1 + i\nabla_2 \\ -\nabla_1 - i\nabla_2 & \nabla_3 + \nabla_4 \end{bmatrix} \begin{bmatrix} \phi_1(x_{n,j}) \\ \phi_2(x_{n,j}) \end{bmatrix} = 0 \tag{4.2}$$

For $p \in \mathbb{R}^4$, let $u(p)$ be a two-component function $u\colon \mathbb{R}^4 \to \mathbb{C}^2$ satisfying

$$\sigma \cdot p \; u(p) = 0 \tag{4.3}$$

Writing (4.3) in full gives

$$\begin{bmatrix} -p_3 + p_4 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_3 + p_4 \end{bmatrix} \begin{bmatrix} u_1(p) \\ u_2(p) \end{bmatrix} = 0 \tag{4.4}$$

If there is a nonzero solution of (4.3), the determinate of the matrix in (4.4) must be zero. Hence,

$$p \cdot p = -p_1^2 - p_2^2 - p_3^2 + p_4^2 = 0 \tag{4.5}$$

Moreover, solutions of (4.3) have the form $u_1(p) = 1$

$$u_2(p) = (p_3 - p_4)/(-p_1 + ip_2)$$

Lemma 4.1. *Discrete plane wave solutions of* (4.1) *have the form*

$$\phi(x_{n,j}) = u(p)w(p, x_{n,j}) \tag{4.6}$$

when p satisfies (4.5) *and* $w(p, \cdot) \in \bigcap_{k=1}^{4} \mathcal{D}(\nabla_k)$.

Proof. Letting $\phi(x_{n,j})$ be defined by (4.6) we have by (4.3) that

$$\begin{aligned} \sigma \cdot \nabla\phi(x_{n,j}) &= \sigma \cdot \nabla u(p)w(p,x_{n,j}) \\ &= \begin{bmatrix} -\nabla_3 + \nabla_4 & -\nabla_1 + i\nabla_2 \\ -\nabla_1 - i\nabla_2 & \nabla_3 + \nabla_4 \end{bmatrix} \begin{bmatrix} u_1(p)w(p, x_{n,j}) \\ u_2(p)w(p, x_{n,j}) \end{bmatrix} \\ &= \begin{bmatrix} (-p_3 + p_4)w(p, x_{n,j})u_1(p) + (-p_1 + ip_2)w(p, x_{n,j})u_2(p) \\ (-p_1 - ip_2)w(p, x_{n,j})u_1(p) + (p_3 + p_4)w(p, x_{n,j})u_2(p) \end{bmatrix} \\ &= w(p, x_{n,j})\sigma \cdot pu(p) = 0 \end{aligned}$$

□

Define the 4×4 *gamma matrices* by

$$\gamma_4 = \begin{bmatrix} \sigma_4 & 0 \\ 0 & -\sigma_4 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad k = 1,2,3$$

We define the *discrete free Dirac equation* by

$$(i\gamma \cdot \nabla - m)\phi(x_{n,j}) = 0 \tag{4.7}$$

where $m > 0$ and $\phi \colon \mathcal{P} \to \mathbb{C}^4$ is a four-component function $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ called a *spinor*. Notice that when $m = 0$, (4.7) essentially reduces to (4.1). Equation (4.7) says that ϕ is an eigenvector of the *discrete free Dirac operator* $i\gamma \cdot \nabla$ with eigenvalue m. We now proceed in the usual way to find solutions of (4.7). Although the method is standard, for completeness we give some details.

For $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ we use the usual inner product $\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3$ and norm $\|\mathbf{p}\| = (\mathbf{p} \cdot \mathbf{p})^{1/2}$. For $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$ define $p_4 = \sqrt{\|\mathbf{p}\|^2 + m^2}$ and let $|0\rangle$, $|1\rangle$ be the qubits

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $s = 0, 1$, $\mathbf{p} \in \mathbb{R}^3$, $p_4 = \sqrt{\|\mathbf{p}\|^2 + m^2}$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ define the four-vectors

$$u(\mathbf{p}, s) = \begin{bmatrix} |s\rangle \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_4 + m}|s\rangle \end{bmatrix}, \quad v(\mathbf{p}, s) = \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_4 + m}(-i\sigma_2|s\rangle) \\ -i\sigma_2|s\rangle \end{bmatrix}$$

and note that $-i\sigma_2|0\rangle = |1\rangle$ and $-i\sigma_2|1\rangle = -|0\rangle$

Theorem 4.2. *Letting $p = (p_1, p_2, p_3, p_4) = (\mathbf{p}, p_4)$ where $p_4 = \sqrt{\|\mathbf{p}\|^2 + m^2}$, discrete plane wave solutions of (4.7) are*

$$\psi_{p,s}^{(+)}(x_{n,j}) = u(\mathbf{p}, s)w(p, x_{n,j}), \quad s = 0, 1$$

and

$$\psi_{-p,-s}^{(-)}(x_{n,j}) = v(\mathbf{p}, s)w(-p, x_{n,j}), \quad s = 0, 1$$

Proof. Defining $\boldsymbol{\nabla} = (\nabla_1, \nabla_2, \nabla_3)$ we can write the Dirac operator as

$$\begin{aligned} i\gamma \cdot \nabla &= -i\gamma_1\nabla_1 - i\gamma_2\nabla_2 - i\gamma_3\nabla_3 + i\gamma_4\nabla_4 \\ &= i\begin{bmatrix} \nabla_4 & \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & -\nabla_4 \end{bmatrix} \end{aligned} \tag{4.8}$$

Consider the case $s = 0$. We have that

$$
\begin{aligned}
\boldsymbol{\sigma} \cdot \mathbf{p}|0\rangle &= (p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3)|0\rangle \\
&= \begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_3 \\ p_1 + ip_2 \end{bmatrix}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_4 + m}|0\rangle w(p, x_{n,j}) &= \frac{1}{p_4 + m} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \begin{bmatrix} p_3 \\ p_1 + ip_2 \end{bmatrix} w(p, x_{n,j}) \\
&= \frac{1}{p_4 + m} \begin{bmatrix} \nabla_3 & \nabla_1 - i\nabla_2 \\ \nabla_1 + i\nabla_2 & -\nabla_3 \end{bmatrix} \begin{bmatrix} p_3 \\ p_1 + ip_2 \end{bmatrix} w(p, x_{n,j}) \\
&= \frac{1}{p_4 + m} \begin{bmatrix} ip_3^2 & (p_1 + ip_2)(ip_1 + p_2) \\ p_3(ip_1 - p_2) & -(p_1 + ip_2)p_3 \end{bmatrix} w(p, x_{n,j}) \\
&= \frac{1}{p_4 + m} \begin{bmatrix} i\|\mathbf{p}\|^2 \\ 0 \end{bmatrix} w(p, x_{n,j}) \qquad (4.9)
\end{aligned}
$$

By (4.8) and (4.9) we have that

$$
\begin{aligned}
i\gamma \cdot \nabla \psi_{p,0}^{(+)}(x_{n,j}) &= i \begin{bmatrix} \nabla_4 & \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & -\nabla_4 \end{bmatrix} u(\mathbf{p}, 0) w(p, x_{n,j}) \\
&= i \begin{bmatrix} \nabla_4 & \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & -\nabla_4 \end{bmatrix} \begin{bmatrix} |0\rangle \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_4 + m}|0\rangle \end{bmatrix} w(p, x_{n,j}) \\
&= i \begin{bmatrix} -ip_4|0\rangle + \frac{i}{p_4 + m}\|\mathbf{p}\|^2|0\rangle \\ \begin{bmatrix} -ip_3 \\ -ip_1 + p_2 \end{bmatrix} + \frac{ip_4}{p_4 + m} \begin{bmatrix} p_3 \\ p_1 + ip_2 \end{bmatrix} \end{bmatrix} w(p, x_{n,j}) \\
&= i \begin{bmatrix} \left(p_4 + \frac{\|\mathbf{p}\|^2}{p_4 + m}\right)|0\rangle \\ \left(1 - \frac{p_4}{p_4 + m}\right) \boldsymbol{\sigma} \cdot \mathbf{p}|0\rangle \end{bmatrix} w(p, x_{n,j}) \\
&= \begin{bmatrix} \frac{p_4^2 + mp_4 - \|\mathbf{p}\|^2}{p_4 + m}|0\rangle \\ \frac{m}{p_4 + m} \boldsymbol{\sigma} \cdot \mathbf{p}|0\rangle \end{bmatrix} w(p, x_{n,j}) \\
&= m \begin{bmatrix} |0\rangle \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p_4 + m}|0\rangle \end{bmatrix} w(p, x_{n,j}) = m\psi_{p,0}^{(+)}(x_{n,j})
\end{aligned}
$$

the other cases are similar. □

Finally, we consider a discrete analogue of Einstein's field equations. The *global curvature operator* is defined by $\mathcal{R}_{\omega,\omega'} = \nabla_{\omega,\omega'} - \nabla_{\omega',\omega}$. The term global is used to distinguish this from the local curvature we defined in the microscopic picture. We then have that

$$\begin{aligned}\mathcal{R}_{\omega,\omega'} f(x_{n,j}, x_{n',j'}) &= \overline{a}_\omega(x_{n,j}) a_{\omega'}(x_{n',j'}) \nabla_{k(\omega,x_{n,j}),k(\omega',x_{n,j})} f(x_{n,j}, x_{n',j'}) \\ &\quad - \overline{a}_{\omega'}(x_{n,j}) a_\omega(x_{n',j'}) \nabla_{k(\omega',x_{n',j'}),k(\omega,x_{n,j})} f(x_{n,j}, x_{n',j'})\end{aligned}$$

Define the operators $\mathcal{D}_{k,k'}$ with domains in K by

$$\begin{aligned}\mathcal{D}_{k,k'} f(x_{n,j}, x_{n',j'}) &= \overline{a}_\omega(x_{n,j}) a_{\omega'}(x_{n',j'}) f(x_{n+2,4j+k-1}, x_{n'+2,4j'+k'-1}) \\ &\quad - \overline{a}_{\omega'}(x_{n,j}) a_\omega(x_{n',j'}) f(x_{n'+2,4j'+k'-1}, x_{n+2,4j+k-1})\end{aligned}$$

and the operators $\mathcal{T}_{k,k'}$ with domains in K by

$$\begin{aligned}\mathcal{T}_{k,k'} f(x_{n,j}, x_{n',j'}) = &\left[\overline{a}_{\omega'}(x_{n+2,4j+k-1}) a_\omega(x_{n'+2,4j'+k'-1})\right. \\ &\left. -\overline{a}_\omega(x_{n+2,4j+k-1}) a_{\omega'}(x_{n'+2,4j'+k'-1})\right] f(x_{n,j}, x_{n',j'})\end{aligned}$$

If we define $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{T}_{\omega,\omega'}$ by

$$\mathcal{D}_{\omega,\omega'} f(x_{n,j}, x_{n',j'}) = \mathcal{D}_{k(\omega,x_{n,j}),k(\omega',x_{n',j'})} f(x_{n,j}, x_{n',j'})$$

and

$$\mathcal{T}_{\omega,\omega'} f(x_{n,j}, x_{n',j'}) = \mathcal{T}_{k(\omega,x_{n,j}),k(\omega',x_{n',j'})} f(x_{n,j}, x_{n',j'})$$

then it is easy to check that

$$\mathcal{R}_{\omega,\omega'} = \mathcal{D}_{\omega,\omega'} + \mathcal{T}_{\omega,\omega'} \tag{4.10}$$

We call (4.10) the *discrete Einstein equations*. For a further discussion of these equations, we refer the reader to [1].

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