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Discrete Quantum Gravity is not Isometric

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DISCRETE QUANTUM GRAVITY IS NOT ISOMETRIC

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Abstract

We show that if a discrete quantum gravity is not classical, then it cannot be generated by an isometric dynamics. In particular, we show that if the quantum measure μ (or equivalently the decoherence functional) is generated by an isometric dynamics, then there is no interference between events so the system describing evolving universes is classical. The result follows from a forbidden configuration in the path space of causal sets.

1 Introduction

This introduction presents an overview of the article and precise definitions will be given in Section 2. We denote the collection of causal sets of cardinality i by $\mathcal{P}_i$, $i = 1, 2, \ldots$. If $x_i \in \mathcal{P}_i$, $x_{i+1} \in \mathcal{P}_{i+1}$ satisfy a certain growth relationship, we write $x_i \to x_{i+1}$. A path is a sequence $x_1 x_2 \cdots$, $x_i \in \mathcal{P}_i$ with $x_i \to x_{i+1}$ and an *n-path* is a sequence of length n, $x_1 x_2 \cdots x_n$, $x_i \in \mathcal{P}_i$ with $x_i \to x_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n. For $\omega \in \Omega_n$, $\mathrm{cyl}(\omega)$ is the collection of all paths whose initial n-path is ω and $\mathcal{A}_n$ is the algebra generated by $\mathrm{cyl}(\omega)$ for all $\omega \in \Omega_n$. Letting $\mathcal{A}$ be the σ-algebra generated by $\mathcal{A}_n$, $n = 1, 2, \ldots$, we have that $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$. The theory of classical sequential growth process [3, 4, 7] provides us with a probability measure ν on $\mathcal{A}$ so $(\Omega, \mathcal{A}, \nu)$ becomes a probability space. Letting $\nu_n = \nu \mid \mathcal{A}_n$ be the restriction of ν to $\mathcal{A}_n$ we obtain the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu)$ together with the increasing sequence of closed subspaces $H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)$. The dynamics of a discrete quantum gravity is described by a sequence of positive operators ρ_n on H_n, $n = 1, 2, \ldots$, satisfying a normalization and consistency condition [2].

We call $\mathcal{P}_n$ the *n-site space* and the associated Hilbert space K_n is the *n-site Hilbert space*. Let $U_n \colon K_n \to K_{n+1}$ be an isometric operator (isometry), $n = 1, 2, \ldots$, that is compatible with the growth relation $x \to y$. When U_n describes the evolution of the system, there is a standard prescription [5, 6] for defining the amplitude $a_n(\omega)$ in terms of U_n for every $\omega \in \Omega_n$. Also, for $\omega = x_1 x_2 \cdots x_n$, $\omega' = x_1' x_2' \cdots x_n'$ one defines the *decoherence*

$$D_n(\omega, \omega') = a(\omega)\overline{a(\omega')}\delta_{x_n, x_n'}$$

Moreover, the *decoherence functional* $D_n \colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ is given by

$$D_n(A, B) = \sum \{D_n(\omega, \omega') \colon \omega \in A, \omega' \in B\}$$

and the *quantum measure* $\mu_n \colon \mathcal{A}_n \to \mathbb{R}^+$ is defined as $\mu_n(A) = D_n(A, A)$. It can be shown that the matrix with components $D_n(\omega, \omega')$ defines a positive operator ρ_n on H_n satisfying the conditions of the previous paragraph. In this case, we say that ρ_n is *generated* by the isometry U_n.

Our main result follows from a forbidden configuration (FC) theorem for the path space Ω. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$ with $x \to y$ we say that x *produces* y and y is an *offspring* of x. The FC theorem states that two different producers cannot have two distinct offspring in common. The FC theorem greatly restricts the allowed isometrics U_n which in turn restricts the possible generated operators ρ_n. In fact, if ρ_n is generated by an isometry, then its matrix representation $D_n(\omega, \omega')$ is diagonal. This implies that there is no interference between paths and that μ_n is a classical probability measure. We conclude that if a discrete quantum gravity is not classical, then it cannot be generated by an isometric dynamics. Of course, almost by definition, a discrete quantum gravity is not classical, hence the title of this paper. Since ρ_n is not generated by an isometry, we must obtain ρ_n in other ways. We refer the reader to [2] for a study of this problem.

2 Discrete Quantum Gravity

A *partially ordered set* (poset) is a set x together with an irreflexive, transitive relation $<$ on x. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $\mathcal{P}_n$ be the collection of all posets with cardinality n, $n = 1, 2, \ldots$, and let $\mathcal{P} = \cup\mathcal{P}_n$. An element of $\mathcal{P}$ is called a *causal set* and if $a < b$ for $a, b \in x$ where $x \in \mathcal{P}_n$, then b is in the causal future of a. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x *produces* y if y is obtained from x by adjoining a single new element a to x that is maximal in y. Thus, $a \in y$ and there is no $b \in y$ such that $a < b$. In this case, we write $y = x \uparrow a$. We also say that x is a *producer* of y and y is an *offspring* of x. If x produces y we write $x \to y$. We denote the set of offspring of x by $x \to$ and for $A \in \mathcal{P}_n$ we use the notation

$$A \to = \{y \in \mathcal{P}_{n+1} \colon x \to y, x \in A\}$$

The transitive closure of $\to$ makes $\mathcal{P}$ itself a poset [1, 3, 5].

A *path* in $\mathcal{P}$ is a string (sequence) $x_1x_2\cdots$ where $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$, $i = 1, 2, \ldots$. An *n-path* in $\mathcal{P}$ is a finite string $x_1x_2\cdots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n. The set of paths whose initial n-path is $\omega_n \in \Omega_n$ is denoted by $\omega_n \Rightarrow$. Thus, if $\omega_n = x_1x_2\cdots x_n$ then

$$\omega_n \Rightarrow = \{\omega \in \Omega \colon x_1x_2\cdots x_ny_{n+1}y_{n+2}\cdots\}$$

For $A \subseteq \Omega_n$ we use the notation

$$A \Rightarrow = \cup\{\omega \Rightarrow \colon \omega \in A\}$$

Thus, $A \Rightarrow$ is the set of paths whose initial n-paths are elements of A. We call $A \Rightarrow$ a *cylinder set* and define

$$\mathcal{A}_n = \{A \Rightarrow \colon A \subseteq \Omega_n\}$$

In particular, if $\omega_n \in \Omega_n$ then the *elementary cylinder set* $\mathrm{cyl}(\omega_n)$ is given by $\mathrm{cyl}(\omega_n) = \omega_n \Rightarrow$. It is easy to check that $\mathcal{A}_n$ forms a increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on Ω and hence $\mathcal{C}(\Omega) = \cup\mathcal{A}_n$ is an algebra of subsets of Ω. We denote the σ-algebra generated by $\mathcal{C}(\Omega)$ by $\mathcal{A}$.

It is shown in [4, 7] that a classical sequential growth process (CSGP) on $\mathcal{P}$ that satisfies natural causality and covariance conditions is determined by a sequence of nonnegative numbers $c = (c_1, c_2, \ldots)$ called *coupling constants*. The coupling constants determine a unique probability measure ν_c on $\mathcal{A}$ making $(\Omega, \mathcal{A}, \nu_c)$ a probability space. The *path Hilbert space* is given by $H = L_2(\Omega, \mathcal{A}, \nu_c)$. If $\nu_c^n = \nu_c \mid \mathcal{A}_n$ is the restriction of ν_c to $\mathcal{A}_n$, then $H_n = L_2(\Omega, \mathcal{A}, \nu_c^n)$ is an increasing sequence of closed subspaces of H. Assuming that $\nu_c^n\left(\mathrm{cyl}(\omega)\right) \neq 0$, an orthonormal basis for H_n is

$$e_\omega^n = \nu_c^n\left(\mathrm{cyl}(\omega)\right)^{-1/2} \chi_{\mathrm{cyl}(\omega)}, \omega \in \Omega_n$$

where χ_A denotes the characteristic function of a set A.

A bounded operator T on H_n will also be considered as a bounded operator on H by defining $Tf = 0$ for all $f \in H_n^\perp$. We employ the notation $\chi_\Omega = 1$. A *q-probability operator* is a positive operator ρ_n on H_n that satisfies $\langle \rho_n 1, 1\rangle = 1$. Denote the set of q-probability operators on H_n by $\mathcal{Q}(H_n)$. For $\rho_n \in \mathcal{Q}(H_n)$ we define the *n-decoherence functional* [1, 2, 3] $D_n \colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

The functional $D_n(A, B)$ gives a measure of the interference between A and B when the system is described by ρ_n. It is clear that $D_n(\Omega_n, \Omega_n) = 1$, $D_n(A, B) = \overline{D_n(B, A)}$ and $A \mapsto D_n(A, B)$ is a complex measure for every $B \in \mathcal{A}_n$. It is also well known that if $A_1, \ldots, A_n \in \mathcal{A}_n$ then the matrix with entries $D_n(A_j, A_k)$ is positive semidefinite. We define the map $\mu_n \colon \mathcal{A}_n \to \mathbb{R}^+$ by

$$\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle$$

Notice that $\mu_n(\Omega_n) = 1$. Although μ_n is not additive, it satisfies the *grade* 2-*additive condition*: if $A, B, C \in \mathcal{A}_n$ are mutually disjoint, then

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

We call μ_n the *q-measure* corresponding to ρ_n [1, 5, 6].

We call a sequence $\rho_n \in \mathcal{Q}(H_n)$, $n = 1, 2, \ldots$, *consistent* if $D_{n+1}(A, B) = D_n(A, B)$ for all $A, B \in \mathcal{A}_n$. Of course, if the sequence ρ_n, $n = 1, 2, \ldots$, is consistent, then $\mu_{n+1}(A) = \mu_n(A)$ for every $A \in \mathcal{A}_n$. In the present context, a *quantum sequential growth process* (QSGP) is a consistent sequence $\rho_n \in \mathcal{Q}(H_n)$. We consider a QSGP as a model for discrete quantum gravity. It is hoped that additional theoretical principles or experimental data will help determine the coupling constants and the $\rho_n \in \mathcal{Q}(H_n)$. We will then know ν_c which is the classical part of the process and ρ_n, $n = 1, 2, \ldots$, which is the quantum part.

3 Isometric Generation

Let K_n be the Hilbert space of complex-valued functions on $\mathcal{P}_n$ with the usual inner product

$$\langle f, g\rangle = \sum_{x\in\mathcal{P}_n} \overline{f(x)}g(x)$$

We call K_n the *n-site Hilbert space* and we denote the standard basis $\chi_{\{x\}}$ of K_n by e_x^n, $x \in \mathcal{P}_n$. The projection operators $P_n(x) = |e_x^n\rangle\langle e_x^n|$, $x \in \mathcal{P}_n$, describe the site at step n of the process. Let $U_n\colon K_n \to K_{n+1}$ be an operator satisfying the two conditions

(1) $U_n^*U_n = I_n$ (isometry).

(2) If $x_n \not\to x_{n+1}$, then $\left\langle e_{x_{n+1}^{n+1}}, U_n e_{x_n}^n\right\rangle = 0$ (compatibility).

Condition (1) implies that U_n is an isometry; that is,

$$\langle U_n f, U_n g\rangle = \langle f, g\rangle$$

for all $f, g \in K_n$. The compatibility condition (2) ensures that U_n preserves the growth relation $x_n \to x_{n+1}$; that is, when $e_{x_n}^n$ corresponds to site x_n, then $U_n e_{x_n}^n$ corresponds to sites in $x_n \to$. Notice that $Q_n = U_n U_n^*$ is the projection from K_{n+1} onto Range (U_n). We call

$$a(x_n \to x_{n+1}) = \left\langle e_{x_{n+1}}^{n+1}, U_n e_{x_n}^n\right\rangle$$

the *transition amplitude* from x_n to x_{n+1}. Of course, by (2) $a(x_n \to x_{n+1}) = 0$ if $x_n \not\to x_{n+1}$. The corresponding *transition probability* is $|a(x_n \to x_{n+1}|^2$. Since

$$U_n e_{x_n}^n = \sum_{x_{n+1}\in\mathcal{P}_{n+1}} a(x_n \to x_{n+1}) e_{x_{n+1}}^{n+1}$$

we conclude that $|a(x_n \to x_{n+1})|^2$ can be interpreted as a probability because

$$\sum_{x_{n+1}\in\mathcal{P}_{n+1}} |a(x_n \to x_{n+1}|^2 = \left\|U_n e_{x_n}^n\right\|^2 = 1 \tag{3.1}$$

For $r \le s \in \mathbb{N}$, define $U(s,r)\colon K_r \to K_s$ by $U(r,r) = I_r$ if $r = s$ and if $r < s$, then

$$U(s,r) = U_r U_{r+1} \cdots U_{s-1}$$

Then $U(s,r)$ is an isometry and $U(t,r) = U(t,s)U(s,r)$ for all $r \le s \le t \in \mathbb{N}$. We call $U(s,r)$ $r \le s \in \mathbb{N}$ a *discrete isometric system*. Such systems frequently describe the dynamics (evolution) in quantum mechanics [1, 5, 6].

We can assume that all paths or n-paths begin at the poset x_1 that has one element. We describe the n-path $\omega = x_1x_2\cdots x_n$ quantum mechanically by the operator $C_n(\omega)\colon K_1 \to K_n$ given as

$$C_n(\omega) = P_n(x_n)U_{n-1}P_{n-1}(x_{n-1})U_{n-2}\cdots P_2(x_2)U_1 \tag{3.2}$$

Defining the *amplitude* $a(\omega)$ of ω by

$$a(\omega) = a(x_{n-1} \to x_n)a(x_{n-2} \to x_{n-1})\cdots a(x_1 \to x_2) \tag{3.3}$$

we can write (3.2) as

$$C_n(\omega) = a(\omega)\left|e_{x_n}^n\right\rangle\left\langle e_{x_1}^1\right| \tag{3.4}$$

We interpret $|a(\omega)|^2$ as the probability of the n-path ω according to the dynamics $U(s,r)$. It follows from (3.1) that

$$\sum_{\omega\in\Omega_n} |a(\omega)|^2 = 1$$

so $|a(\omega)|^2$ is indeed a probability distribution on Ω_n.

The operator $C_n(\omega')^*C_n(\omega)$ describes the interference between the two n-paths $\omega, \omega' \in \Omega_n$. Applying (3.4) we conclude that

$$C_n(\omega')^*C_n(\omega) = \overline{a(\omega')}a(\omega)\delta_{x_n,x_n'}I_1$$

which we can identify with the complex number

$$D_n(\omega,\omega') = \overline{a(\omega')}a(\omega)\delta_{x_n,x_n'} \tag{3.5}$$

The matrix D_n with entries $D_n(\omega,\omega')$ is called the *decoherence matrix*. We say that a QSGP ρ_n, $n = 1, 2, \ldots$, is *isometrically generated* if there exists a discrete isometric system given by $U_n\colon K_n \to K_{n+1}$ such that ρ_n is the operator corresponding to the matrix D_n; that is,

$$\langle \rho_n e_\omega^n, e_{\omega'}^n\rangle = D_n(\omega',\omega) \tag{3.6}$$

for every $\omega, \omega' \in \Omega_n$. At first sight, isometric generation appears to be a natural way to construct a QSGP. However, the next section shows that this does not work unless the QSGP is classical. For methods of constructing such processes that are truly quantum, we refer the reader to [2].

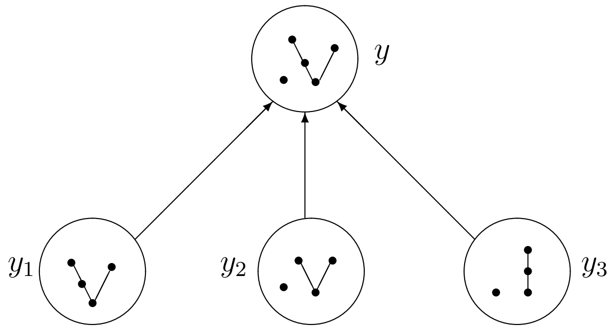

Figure 1

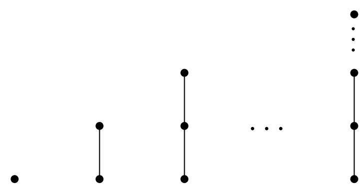

Figure 2

4 Forbidden Configurations

Various configurations can occur in the poset $(\mathcal{P}, \to)$. For instance, it is quite common for two different producers to share a common offspring. The next example discusses the case in which more than two producers share a common offspring.

Example 1. Figure 1 illustrates a case in which three producers share a common offspring. In this figure, a rising line called a *link* from vertex a to vertex b designates that $a < b$ and there is no c such that $a < c < b$. In this figure, y_1, y_2 and y_3 produce the offspring y. This is the smallest cardinality example of this configuration. Indeed, if y has four elements then y would need three nonisomorphic maximal elements to have three producers and this is impossible. Figure 2 illustrates a poset that is the offspring of n producers.

We call the next result the *forbidden configuration* (FC) theorem. The proof of the theorem is illustrated in Figure 3.

Theorem 4.1. *Two different producers cannot have two distinct offspring in common.*

Proof. Suppose $x_1 \neq x_2$ both produce $y_1 \neq y_2$ where $=$ means isomorphic. Then there exist a_1, a_2, b_1, b_2 such that $y_1 = x_1 \uparrow a_1$, $y_2 = x_1 \uparrow a_2$, $y_1 = x_2 \uparrow$

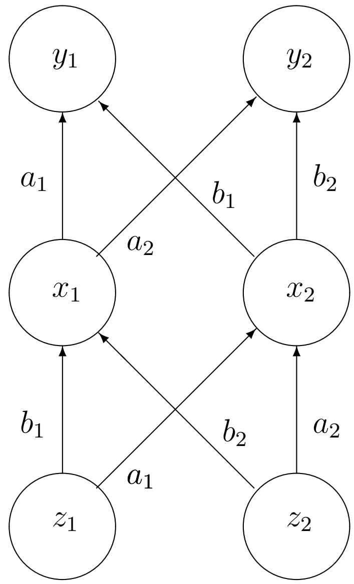

Figure 3

b_1, $y_2 = x_2 \uparrow b_2$. Since $y_1 \neq y_2$, $a_1 \neq a_2$ in the sense that the links of a_1 are not the same as the links of a_2. Similarly, $b_1 \neq b_2$. Since $x_1 \neq x_2$, we have that $a_1 \neq b_1$. Similarly, $a_2 \neq b_2$. Since $b_1 \in y_1$ we have that $b_1 \in x_1$. Since $b_2 \in y_2$ we have that $b_2 \in x_1$. Hence, $\{b_1, b_2\} \subseteq x_1$ and similarly $\{a_1, a_2\} \subseteq x_2$. Since $a_1 \notin x_1$ we conclude that $a_1 \neq b_1$. Hence, $\{a_1, b_1, b_2\} \subseteq y_1$ and similarly $\{a_1, a_2, b_2\} \subseteq y_2$. Let $z_1 = y_1 \setminus \{a_1, b_1\}$ and $z_2 = y_2 \setminus \{a_2, b_2\}$. Then $z_1 \neq z_2$ because $b_2 \in z_1$ and $b_2 \notin z_2$. Now $x_1 = z_1 \uparrow b_1$ and $x_1 = z_2 \uparrow b_2$. Similarly, $x_2 = z_1 \uparrow a_1$ and $x_2 = z_2 \uparrow a_2$. We conclude that x_1 and x_2 are common offspring of distinct producers z_1 and z_2. Of course,

$$\operatorname{card}(z_1) = \operatorname{card}(z_2) = \operatorname{card}(x_1) - 1$$

We can continue this process until we obtain distinct producers of cardinality 2 at which point we have a contradiction. □

We now present our main result.

Theorem 4.2. *If a QSGP ρ_n is generated by isometries $U_n \colon K_n \to K_{n+1}$ then the corresponding q-measures μ_n are classical probability measures, $n = 1, 2, \ldots$.*

Proof. Suppose ρ_n is generated by isometries $U_n \colon K_n \to K_{n+1}$. Letting $\omega = x_1 x_2 \cdots x_n$, $\omega = x'_1, x'_2 \cdots x'_n$ be n-paths in Ω_n with $\omega \neq \omega'$ we shall show that $D_n(\omega, \omega') = 0$. If $x_n \neq x'_n$, then by (3.5) we have $D_n(\omega, \omega') = 0$ so suppose that $x_n = x'_n$. Assume that $x_{n-1} \neq x'_{n-1}$ so x_n is a common offspring of the distinct producers x_{n-1}, x'_{n-1}. By Theorem 4.1, x_n is the only common offspring of x_{n-1}, x'_{n-1} so by the compatibility condition we have

$$\begin{aligned}\overline{a(x_{n-1} \to x_n)}a(x'_{n-1} \to x_n) &= \left\langle U_{n-1}e^{n-1}_{x_{n-1}}, e^n_{x_n}\right\rangle\left\langle e^n_{x_n}, U_{n-1}e^{n-1}_{x'_{n-1}}\right\rangle \\ &= \sum_{y\in\mathcal{P}_n}\left\langle U_{n-1}e^{n-1}_{x_{n-1}}, e^n_y\right\rangle\left\langle e^n_y, U_{n-1}e^{n-1}_{x'_{n-1}}\right\rangle \\ &= \left\langle U_{n-1}e^{n-1}_{x_{n-1}}, U_{n-1}e^{n-1}_{x'_{n-1}}\right\rangle = 0\end{aligned}$$

It follows that $a(x_{n-1} \to x_n) = 0$ or $a(x'_{n-1} \to x_n) = 0$. Applying (3.3) we conclude that $a(\omega) = 0$ or $a(\omega') = 0$ and hence, by (3.5) $D_n(\omega, \omega') = 0$. If $x_{n-1} = x'_{n-1}$, since $\omega \neq \omega'$ we will eventually have a largest $r \in \mathbb{N}$ such that $x_r \neq x'_r$, $2 \leq r \leq n-2$. We now proceed as before to obtain $D_n(\omega, \omega') = 0$. It follows from (3.6) that $\langle \rho_n e^n_\omega, e^n_{\omega'}\rangle = 0$. Hence, if $A \in \mathcal{A}_n$ we have

$$\begin{aligned}\mu_n(A) &= \sum\left\{\langle \rho_n e^n_\omega, e^n_{\omega'}\rangle : \omega \Rightarrow, \omega' \Rightarrow \subseteq A\right\} \\ &= \sum\left\{\langle \rho_\omega e^n_\omega, e^n_\omega\rangle : \omega \Rightarrow \subseteq A\right\} \\ &= \sum\left\{\mu\left(\{\omega\}\right) : \omega \Rightarrow \subseteq A\right\}\end{aligned}$$

We conclude that μ_n is a classical probability measure, $n = 1, 2, \ldots$. □

If μ_n is a classical probability measure, then there is no interference between events and the QSGP is classical. We conclude that if a QSGP is isometrically generated, then it is classical.

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