Models for Discrete Quantum Gravity

S. Gudder

Follow this and additional works at: https://digitalcommons.du.edu/math_preprints

Part of the Mathematics Commons

Recommended Citation

This Article is brought to you for free and open access by the Department of Mathematics at Digital Commons @ DU. It has been accepted for inclusion in Mathematics Preprint Series by an authorized administrator of Digital Commons @ DU. For more information, please contact jennifer.cox@du.edu,dig-commons@du.edu.
Models for Discrete Quantum Gravity

Comments
The final version of this article published in Reports on Mathematical Physics is available online at:
https://doi.org/10.1016/S0034-4877(13)60010-5
MODELS FOR
DISCRETE QUANTUM GRAVITY

S. Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208, U.S.A.
sgudder@du.edu

Abstract
We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of q-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is “quantized” to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

1 Introduction
In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) \( \rho_n, n = 1, 2, \ldots \). Instead of requiring that \( \rho_n \) be a state on a Hilbert space \( H_n \), we require that \( \rho_n \) be a q-probability operator on \( H_n \). This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring \( \text{tr}(\rho_n) = 1 \), this condition normalizes the
corresponding quantum measure. By superimposing a concrete DQP on a classical sequential growth process we obtain a model for discrete quantum gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of $q$-probability operators is a convex set and find its set of extreme elements. We discuss the property of consistency for a DQP and introduce the so-called quadratic algebra of suitable sets. The suitable sets are those on which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process (CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets are discussed. In Section 4 we show how to “quantize” a CSGP to obtain a quantum sequential growth process (QSGP). Some results concerning the consistency of a DQP are given. Finally, Section 5 provides two methods for constructing examples of QSGP.

2 Discrete Quantum Processes

Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and let

$$H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f : \Omega \to \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\}$$

be the corresponding Hilbert space. Let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$ be an increasing sequence of sub $\sigma$-algebras of $\mathcal{A}$ that generate $\mathcal{A}$ and let $\nu_n = \nu | \mathcal{A}_n$ be the restriction of $\nu$ to $\mathcal{A}_n$, $n = 1, 2, \ldots$. Then $H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)$ forms an increasing sequence of closed subspaces of $H$ called a filtration of $H$. A bounded operator $T$ on $H_n$ will also be considered as a bounded operator on $H$ by defining $Tf = 0$ for all $f \in H_n^\perp$. We denote the characteristic function $\chi_\Omega$ of $\Omega$ by 1. Of course, $\|1\| = 1$ and $\langle 1, f \rangle = \int f \, d\nu$ for every $f \in H$. A $q$-probability operator is a bounded positive operator $\rho$ on $H$ that satisfies $\langle \rho 1, 1 \rangle = 1$. Denote the set of $q$-probability operators on $H$ and $H_n$ by $\mathcal{Q}(H)$ and $\mathcal{Q}(H_n)$, respectively. Since $1 \in H_n$, if $\rho \in \mathcal{Q}(H_n)$ by our previous convention, $\rho \in \mathcal{Q}(H)$. Notice that a positive operator $\rho \in \mathcal{Q}(H)$ if and only if $\|\rho^{1/2} 1\| = 1$ where $\rho^{1/2}$ is the unique positive square root of $\rho$.

A rank 1 element of $\mathcal{Q}(H)$ is called a pure $q$-probability operator. Thus $\rho \in \mathcal{Q}(H)$ is pure if and only if $\rho$ has the form $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in H$ satisfying

$$\|\langle 1, \psi \rangle\| = \left| \int \psi \, d\nu \right| = 1$$
We then call \( \psi \) a *q-probability vector* and we denote the set of *q*-probability vectors by \( \mathcal{V}(H) \) and the set of pure *q*-probability operators by \( \mathcal{Q}_p(H) \). Notice that if \( \psi \in \mathcal{V}(H) \), then \( \| \psi \| \geq 1 \) and \( \| \psi \| = 1 \) if and only if \( \psi = \alpha 1 \) for some \( \alpha \in \mathbb{C} \) with \( |c| = 1 \). Two operators \( \rho_1, \rho_2 \in \mathcal{Q}(H) \) are orthogonal if \( \rho_1 \rho_2 = 0 \).

**Theorem 2.1.** (i) \( \mathcal{Q}(A) \) is a convex set and \( \mathcal{Q}_p(H) \) is its set of extreme elements. (ii) \( \rho \in \mathcal{Q}(H) \) is of trace class if and only if there exists a sequence of mutually orthogonal \( \rho_i \in \mathcal{Q}_p(H) \) and \( \alpha_i > 0 \) with \( \sum \alpha_i = 1 \) such that \( \rho = \sum \alpha_i \rho_i \) in the strong operator topology. The \( \rho_i \) are unique if and only if the corresponding \( \alpha_i \) are distinct.

**Proof.** (i) If \( 0 < \lambda < 1 \) and \( \rho_1, \rho_2 \in \mathcal{Q}(H) \), then \( \rho = \lambda \rho_1 + (1 - \lambda) \rho_2 \) is a positive operator and

\[
\langle \rho_1, 1 \rangle = \langle (\lambda \rho_1 + (1 - \lambda) \rho_2), 1 \rangle = \lambda \langle \rho_1, 1 \rangle + (1 - \lambda) \langle \rho_2, 1 \rangle = 1
\]

Hence, \( \rho \in \mathcal{Q}(H) \) so \( \mathcal{Q}(H) \) is a convex set. Suppose \( \rho \in \mathcal{Q}_p(H) \) and \( \rho = \lambda \rho_1 + (1 - \lambda) \rho_2 \) where \( 0 < \lambda < 1 \) and \( \rho_1, \rho_2 \in \mathcal{Q}(H) \). If \( \rho_1 \neq \rho_2 \), then \( \text{rank}(\rho) \neq 1 \) which is a contradiction. Hence, \( \rho_1 = \rho_2 \) so \( \rho \) is an extreme element of \( \mathcal{Q}(H) \). Conversely, suppose \( \rho \in \mathcal{Q}(H) \) is an extreme element. If the cardinality of the spectrum \( |\sigma(\rho)| > 1 \), then by the spectral theorem \( \rho = \rho_1 + \rho_2 \) where \( \rho_1, \rho_2 \neq 0 \) are positive and \( \rho_1 \neq \alpha \rho_2 \) for \( \alpha \in \mathbb{C} \). If \( \rho_1 \rho_2 = 0 \), then \( \langle \rho_1, 1 \rangle, \langle \rho_2, 1 \rangle = 0 \) and we can write

\[
\rho = \frac{\langle \rho_1, 1 \rangle \rho_1}{\langle \rho_1, 1 \rangle} + \frac{\langle \rho_2, 1 \rangle \rho_2}{\langle \rho_2, 1 \rangle}
\]

Now \( \langle \rho_1, 1 \rangle^{-1} \rho_1, \langle \rho_2, 1 \rangle^{-1} \rho_2 \in \mathcal{Q}(H) \) and

\[
\langle \rho_1, 1 \rangle + \langle \rho_2, 1 \rangle = \langle \rho_1, 1 \rangle = 1
\]

which is a contradiction. Hence, \( \rho_1 = 0 \) or \( \rho_2 = 0 \). Without loss of generality suppose that \( \rho_2 = 0 \). We can now write

\[
\rho = \frac{1}{2} \rho_1 + \frac{1}{2} (\rho_1 + 2 \rho_2)
\]

Now \( \rho_1 \neq 0 \), \( \rho_1 + 2 \rho_2 \neq 0 \) and as before we get a contradiction. We conclude that \( |\sigma(\rho)| = 1 \). Hence, \( \rho = \alpha P \) where \( P \) is a projection and \( \alpha > 0 \). If \( \text{rank}(P) > 1 \), then \( P = P_1 + P_2 \) where \( P_1 \) and \( P_2 \) are orthogonal nonzero projections so \( \rho = \alpha P_1 + \alpha P_2 \). Proceeding as before we obtain a contradiction. Hence, \( \text{rank}(P) = 1 \) so \( \rho = \alpha P \) is pure. (ii) This follows from the spectral theorem. \( \square \)
Let \( \{H_n : n = 1, 2, \ldots \} \) be a filtration of \( H \) and let \( \rho_n \in \mathcal{Q}(H_n), \ n = 1, 2, \ldots \) The \( n \)-decoherence functional \( D_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C} \) defined by

\[
D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle
\]
gives a measure of the interference between \( A \) and \( B \) when the system is described by \( \rho_n \). It is clear that \( D_n(\Omega_n, \Omega_n) = 1 \), \( D_n(A, B) = D_n(B, A) \) and \( A \mapsto D_n(A, B) \) is a complex measure for all \( B \in \mathcal{A}_n \). It is also well-known that if \( A_1, \ldots, A_r \in \mathcal{A}_n \) then the matrix with entries \( D_n(A_j, A_k) \) is positive semidefinite. We define the map \( \mu_n : \mathcal{A}_n \to \mathbb{R}^+ \) by

\[
\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle
\]
Notice that \( \mu_n(\Omega_n) = 1 \). Although \( \mu_n \) is not additive, it does satisfy the \textit{grade-2 additivity condition}: if \( A, B, C \in \mathcal{A}_n \) are mutually disjoint, then

\[
\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)
\]
(2.1)

We say that \( \rho_{n+1} \) is \textit{consistent} with \( \rho_n \) if \( D_{n+1}(A, B) = D_n(A, B) \) for all \( A, B \in \mathcal{A}_n \). We call the sequence \( \rho_n, \ n = 1, 2, \ldots, \) consistent if \( \rho_{n+1} \) is consistent with \( \rho_n \) for \( n = 1, 2, \ldots \). Of course, if the sequence \( \rho_n, \ n = 1, 2, \ldots, \) is consistent, then \( \mu_{n+1}(A) = \mu_n(A) \ \forall A \in \mathcal{A}_n, \ n = 1, 2, \ldots \). A \textit{discrete quantum process} (DQP) is a consistent sequence \( \rho_n \in \mathcal{Q}(H_n) \) for a filtration \( H_n, n = 1, 2, \ldots \). A DQP \( \rho_n \) is \textit{pure} if \( \rho_n \in \mathcal{Q}_p(H_n), n = 1, 2, \ldots \).

If \( \rho_n \) is a DQP, then the corresponding maps \( \mu_n : \mathcal{A}_n \to \mathbb{R}^+ \) have the form

\[
\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \| \rho_n^{1/2} \chi_A \|^2
\]

Now \( A \mapsto \rho_n^{1/2} \chi_A \) is a vector-valued measure on \( \mathcal{A}_n \). We conclude that \( \mu_n \) is the squared norm of a vector-valued measure. In particular, if \( \rho_n = |\psi_n\rangle \langle \psi_n| \) is a pure DQP, then \( \mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2 \) so \( \mu_n \) is the squared modulus of the complex-valued measure \( A \mapsto \langle \psi_n, \chi_A \rangle \).

For a DQP \( \rho_n \in \mathcal{Q}(H_n) \), we say that a set \( A \in \mathcal{A} \) is \textit{suitable} if \( \lim \langle \rho_j \chi_A, \chi_A \rangle \) exists and is finite and in this case we define \( \mu(A) \) to be the limit. We denote the set of suitable sets by \( \mathcal{S}(\rho_n) \). If \( A \in \mathcal{A}_n \) then

\[
\lim \langle \rho_j \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle
\]
so \( A \in \mathcal{S}(\rho_n) \) and \( \mu(A) = \mu_n(A) \). This shows that the algebra \( \mathcal{A}_0 = \cup \mathcal{A}_n \subseteq \mathcal{S}(\rho_n) \). In particular, \( \Omega \in \mathcal{S}(\rho_n) \) and \( \mu(\Omega) = 1 \). In general, \( \mathcal{S}(\rho_n) \neq \mathcal{A} \) and \( \mu \)
may not have a well-behaved extension from $A_0$ to all of $A$ [2, 7]. A subset $B$ of $A$ is a \textit{quadratic algebra} if $\emptyset, \Omega \in B$ and whenever $A, B, C \in B$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in B$, we have $A \cup B \cup C \in B$. For a quadratic algebra $B$, a \textit{q-measure} is a map $\mu_0 : B \rightarrow \mathbb{R}^+$ that satisfies the grade-2 additivity condition (2.1). Of course, an algebra of sets is a quadratic algebra and we conclude that $\mu_n : A_n \rightarrow \mathbb{R}^+$ is a \textit{q-measure}. It is not hard to show that $S(\rho_n)$ is a quadratic algebra and $\mu : S(\rho_n) \rightarrow \mathbb{R}^+$ is a \textit{q-measure} on $S(\rho_n)$ [3].

3 Classical Sequential Growth Processes

A \textit{partially ordered set} (poset) is a set $x$ together with an irreflexive, transitive relation $<$ on $x$. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $P_n$ be the collection of all posets with cardinality $n$, $n = 1, 2, \ldots$. If $x \in P_n$, $y \in P_{n+1}$, then $x$ \textit{produces} $y$ if $y$ is obtained from $x$ by adjoining a single new element to $x$ that is maximal in $y$. We also say that $x$ is a \textit{producer} of $y$ and $y$ is an \textit{offspring} of $x$. If $x$ produces $y$ we write $x \rightarrow y$. We denote the set of offspring of $x$ by $x \rightarrow$ and for $A \subseteq P_n$ we use the notation

$$A \rightarrow = \{y \in P_{n+1} : x \rightarrow y, x \in A\}$$

The transitive closure of $\rightarrow$ makes the set of all finite posets $\mathcal{P} = \bigcup P_n$ into a poset.

A \textit{path} in $\mathcal{P}$ is a string (sequence) $x_1, x_2, \ldots$ where $x_i \in P_i$ and $x_i \rightarrow x_{i+1}$, $i = 1, 2, \ldots$. An $n$-\textit{path} in $\mathcal{P}$ is a finite string $x_1x_2\cdots x_n$ where again $x_i \in P_i$ and $x_i \rightarrow x_{i+1}$. We denote the set of paths by $\Omega$ and the set of $n$-paths by $\Omega_n$. The set of paths whose initial $n$-path is $\omega_0 \in \Omega_n$ is denoted by $\omega_0 \Rightarrow$.

Thus, if $\omega_0 = x_1x_2\cdots x_n$ then

$$\omega_0 \Rightarrow = \{\omega \in \Omega : \omega = x_1, x_2\cdots x_n y_{n+1}y_{n+2} \cdots\}$$

If $x$ produces $y$ in $r$ isomorphic ways, we say that the \textit{multiplicity} of $x \rightarrow y$ is $r$ and write $m(x \rightarrow y) = r$. For example, in Figure 1, $m(x \rightarrow y) = 3$. (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)
If $x \in \mathcal{P}$ and $a, b \in x$ we say that $a$ is an ancestor of $b$ and $b$ is a successor of $a$ if $a < b$. We say that $a$ is a parent of $b$ and $b$ is a child of $a$ if $a < b$ and there is no $c \in x$ such that $a < c < b$. Let $c = (c_0, c_1, \ldots)$ be a sequence of nonnegative numbers called coupling constants [5, 9]. For $r, s \in \mathbb{N}$ with $r \leq s$, we define

$$
\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}
$$

For $x \in \mathcal{P}_n$ $y \in \mathcal{P}_{n+1}$ with $x \rightarrow y$ we define the transition probability

$$
p_c(x \rightarrow y) = m(x \rightarrow y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}
$$

where $\alpha$ is the number of ancestors and $\pi$ the number of parents of the adjoined maximal element in $y$ that produces $y$ from $x$. It is shown in [5, 9] that $p_c(x \rightarrow y)$ is a probability distribution in that it satisfies the Markov-sum rule

$$
\sum \{p_c(x \rightarrow y) : y \in \mathcal{P}_{n+1}, x \rightarrow y\} = 1
$$

In discrete quantum gravity, the elements of $\mathcal{P}$ are thought of as causal sets and $a < b$ is interpreted as $b$ being in the causal future of $a$. The distribution $y \mapsto p_c(x \rightarrow y)$ is essentially the most general that is consistent with principles of causality and covariance [5, 9]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take $c_k = 1/k!$ [6, 7]. The case $c_k = c^k$ for some $c > 0$ has been previously studied and is called a percolation dynamics [5, 6, 8].

We call an element $x \in \mathcal{P}$ a site and we sometimes call an $n$-path an $n$-universe and a path a universe. The set $\mathcal{P}$ together with the set of transition probabilities $p_c(x \rightarrow y)$ forms a classical sequential growth process (CSGP).
which we denote by \((\mathcal{P}, p_c)\) [4, 5, 6, 8, 9]. It is clear that \((\mathcal{P}, p_c)\) is a Markov chain and as usual we define the probability of an \(n\)-path \(\omega = y_1y_2 \cdots y_n\) by

\[ p_c^n(\omega) = p_c(y_1 \rightarrow y_2)p_c(y_2 \rightarrow y_3) \cdots p_c(y_{n-1} \rightarrow y_n) \]

Denoting the power set of \(\Omega_n\) by \(2^{\Omega_n}\), \((\Omega_n, 2^{\Omega_n}, p_c^n)\) becomes a probability space where

\[ p_c^n(A) = \sum \{p_c^n(\omega) : \omega \in A\} \]

for all \(A \in 2^{\Omega_n}\). The probability of a site \(x \in \mathcal{P}_n\) is

\[ p_c^n(x) = \sum \{p_c^n(\omega) : \omega \in \Omega_n, \omega \text{ ends at } x\} \]

Of course, \(x \mapsto p_c^n(x)\) is a probability measure on \(\mathcal{P}_n\) and we have

\[ \sum_{x \in \mathcal{P}_n} p_c^n(x) = 1 \]

**Example 1.** Figure 2 illustrates the first two steps of a CSGP where the 2 indicates the multiplicity \(m(x_3 \rightarrow x_6) = 2\). Table 1 lists the probabilities of the various sites for the general coupling constants \(c_k\) and the particular coupling constants \(c'_k = 1/k!\) where \(d = (c_0 + c_1)(c_0 + 2c_1 + c_2)\).
Table 1

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_c^n(x_i)$</td>
<td>1</td>
<td>$\frac{c_1}{c_0+c_1}$</td>
<td>$\frac{c_0}{c_0+c_1}$</td>
<td>$\frac{c_1(c_1+c_2)}{d}$</td>
<td>$\frac{c^2}{d}$</td>
<td>$\frac{3c_0c_1}{d}$</td>
<td>$\frac{c_0c_2}{d}$</td>
</tr>
<tr>
<td>$p_c^n(x_i)$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{14}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{3}{14}$</td>
<td>$\frac{1}{7}$</td>
</tr>
</tbody>
</table>

For $A \subseteq \Omega_n$ we use the notation

$$A \Rightarrow = \cup \{ \omega \Rightarrow : \omega \in A \}$$

Thus, $A \Rightarrow$ is the set of paths whose initial $n$-paths are elements of $A$. We call $A \Rightarrow$ a cylinder set and define

$$\mathcal{A}_n = \{ A \Rightarrow : A \subseteq \Omega_n \}$$

In particular, if $\omega \in \Omega_n$ then the elementary cylinder set $\text{cyl}(\omega)$ is given by $\text{cyl}(\omega) = \omega \Rightarrow$. It is easy to check that the $\mathcal{A}_n$ form an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on $\Omega$ and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of $\Omega$. Also for $A \in \mathcal{C}(\Omega)$ of the form $A = A_1 \Rightarrow$, $A_1 \subseteq \Omega_n$, we define $p_c(A) = p_c(A_1)$. It is easy to check that $p_c$ is a well-defined probability measure on $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that $p_c$ has a unique extension to a probability measure $\nu_c$ on the $\sigma$-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space, the increasing sequence of subalgebras $\mathcal{A}_n$ generates $\mathcal{A}$ and that the restriction $\nu_c | \mathcal{A}_n = p_c^n$. Hence, the subspaces $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ form a filtration of the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$.

4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP ($\mathcal{P}, p_c$) developed in Section 3. We have seen that the $n$-path Hilbert space $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ forms a filtration of the path Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$. In the sequel, we assume that $p_c^n(\omega) \neq 0$ for every $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then the set of vectors

$$e^n_\omega = p_c^n(\omega)^{1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n$$

8
form an orthonormal basis for \( H_n, n = 1, 2, \ldots \). For \( A \in \mathcal{A}_n \), notice that \( \chi_A \in H \) with \( \| \chi_A \| = p_n(A)^{1/2} \).

We call a DQP \( \rho_n \in \mathcal{Q}(H_n) \) a quantum sequential growth process (QSGP). We call \( \rho_n \) the local operators and \( \mu_n(A) = D_n(A, A) \) the local q-measures for the process. If \( \rho = \lim \rho_n \) exists in the strong operator topology, then \( \rho \) is a \( q \)-probability operator on \( H \) called the global operator for the process. If the global operator \( \rho \) exists, then \( \hat{\mu}(A) = \langle \rho \chi_A, \chi_A \rangle \) is a (continuous) \( q \)-measure on \( \mathcal{A} \) that extends \( \mu_n, n = 1, 2, \ldots \). Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators [2, 3, 7]. In this case, we still have the \( q \)-measure \( \mu \) on the quadratic algebra \( \mathcal{S}(\rho_n) \subseteq \mathcal{A} \) that extends \( \mu_n, n = 1, 2, \ldots \). We frequently identify a set \( A \subseteq \Omega_n \) with the corresponding cylinder set \( (A \Rightarrow) \in \mathcal{A}_n \). We then have the \( q \)-measure, also denoted by \( \mu_n \), on \( 2^{\Omega_n} \) defined by \( \mu_n(A) = \mu_n(A \Rightarrow) \). Moreover, we define the \( q \)-measure, again denoted by \( \mu_n \), on \( \mathcal{P}_n \) by

\[
\mu_n(A) = \mu_n(\{ \omega \in \Omega_n : \omega \text{ end in } A \})
\]

for all \( A \subseteq \mathcal{P}_n \). In particular, for \( x \in \mathcal{P}_n \) we have

\[
\mu_n(\{x\}) = \mu_n(\{\omega \in \Omega_n : \omega \text{ ends with } x\})
\]

If \( A \in \mathcal{A}_n \) has the form \( A_1 \Rightarrow \) for \( A_1 \subseteq \Omega_n \) then \( A \in \mathcal{A}_{n+1} \) and \( A = (A_1 \Rightarrow) \Rightarrow \) where \( A_1 \Rightarrow \subseteq \Omega_{n+1} \). Let \( \rho_n \in \mathcal{Q}(H_n) \), \( \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) and let \( D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle \), \( D_{n+1}(A, B) = \langle \rho_{n+1} \chi_B, \chi_A \rangle \) be the corresponding decoherence functionals. Then \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for all \( A, B \subseteq \Omega_n \) we have

\[
D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = D_n(A \Rightarrow, B \Rightarrow) \tag{4.1}
\]

**Lemma 4.1.** For \( \rho_n \in \mathcal{Q}(H_n) \), \( \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for all \( \omega, \omega' \in \Omega_n \) we have

\[
D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow] = D_n(\omega \Rightarrow, \omega' \Rightarrow) \tag{4.2}
\]

**Proof.** Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every \( A, B \subseteq \Omega_n \) we have

\[
D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = \sum_{\omega \in A} \sum_{\omega' \in B} D_{n+1}D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow] = \sum_{\omega \in A} \sum_{\omega' \in B} D_n(\omega \Rightarrow, \omega' \Rightarrow) = D_n(A \Rightarrow, B \Rightarrow)
\]

and the result follows from (4.1). \( \square \)
For \( \omega = x_1 x_2 \cdots x_n \in \Omega_n \) and \( y \in \mathcal{P}_{n+1} \) with \( x_n \to y \) we use the notation \( \omega y \in \Omega_{n+1} \) where \( \omega y = x_1 x_2 \cdots x_n y \). We also define \( p_c(\omega \to y) = p_c(x_n \to y) \) and write \( \omega \to y \) whenever \( x_n \to y \).

**Theorem 4.2.** For \( \rho_n \in \mathcal{Q}(H_n) \), \( \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for every \( \omega, \omega' \in \Omega_n \) we have

\[
\langle \rho_n e_{\omega'}^n, e_{\omega}^n \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p_c(\omega' \to x)^{1/2} p_c(\omega \to y)^{1/2} \langle \rho_{n+1} e_{\omega'x}^{n+1}, e_{\omega y}^{n+1} \rangle \quad (4.3)
\]

**Proof.** By Lemma 4.1, \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if (4.2) holds. But

\[
D_n(\omega \Rightarrow, \omega' \Rightarrow) = \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle = \langle \rho_n \chi_{\cyl(\omega')}, \chi_{\cyl(\omega)} \rangle = p_c^n(\omega')^{1/2} p_c^n(\omega)^{1/2} \langle \rho_n e_{\omega'}^n, e_{\omega}^n \rangle
\]

Moreover, we have

\[
D_{n+1}[(\omega \to) \Rightarrow, (\omega' \to) \Rightarrow] = \langle \rho_{n+1} \chi_{(\omega' \to) \Rightarrow}, \chi_{(\omega \to) \Rightarrow} \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\omega'x \Rightarrow}, \chi_{\omega y \Rightarrow} \rangle
\]

\[
= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\cyl(\omega'x)}, \chi_{\cyl(\omega y)} \rangle
\]

\[
= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} e_{\omega'x}^{n+1}, e_{\omega y}^{n+1} \rangle
\]

\[
= p_c^n(\omega')^{1/2} p_c^n(\omega)^{1/2} \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p_c(\omega' \to x) p_c(\omega \to y)^{1/2} \langle \rho_{n+1} e_{\omega'x}^{n+1}, e_{\omega y}^{n+1} \rangle
\]

The result now follows. \( \square \)

Viewing \( H_n \) as \( L_2(\Omega_n, 2^{\Omega_n}, p_c^n) \) we can write (4.3) in the simple form

\[
\langle \rho_n \chi_{\omega'} \rangle, \chi_{\omega} \rangle = \langle \rho_{n+1} \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle \quad (4.4)
\]

**Corollary 4.3.** A sequence \( \rho_n \in \mathcal{Q}(H_n) \) is a \( \mathcal{QSGP} \) if and only if (4.3) or (4.4) hold for every \( \omega, \omega' \in \Omega_n \), \( n = 1, 2, \ldots \).
We now consider pure $q$-probability operators. In the following results we again view $H_n$ as $L^2(\Omega_n, \mathcal{G}_n, p_n^c)$.

**Corollary 4.4.** If $\rho_n \in Q_p(H_n)$, $\rho_{n+1} \in Q_p(H_{n+1})$ with $p_n = |\psi_n\rangle\langle\psi_n|$, $\rho_{n+1} = |\psi_{n+1}\rangle\langle\psi_{n+1}|$, then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have

$$\langle\psi_n, \chi_{\omega}\rangle\langle\chi_{\omega'}, \psi_n\rangle = \langle\psi_{n+1}, \chi_{\omega'\rightarrow}\rangle\langle\chi_{\omega\rightarrow}, \psi_{n+1}\rangle$$  \hspace{1cm} (4.5)

**Corollary 4.5.** A sequence $|\psi_n\rangle\langle\psi_n| \in Q_p(H_n)$ is a QSGP if and only if (4.5) holds for every $\omega, \omega' \in \Omega_n$.

We say that $\psi_{n+1} \in V(H_{n+1})$ is strongly consistent with $\psi_n \in V(H_n)$ if for every $\omega \in \Omega_n$ we have

$$\langle\psi_n, \chi_{\omega}\rangle = \langle\psi_{n+1}, \chi_{\omega'\rightarrow}\rangle$$  \hspace{1cm} (4.6)

By (4.5) strong consistency implies the consistency of the corresponding $q$-probability operators.

**Corollary 4.6.** If $\psi_{n+1} \in V(H_{n+1})$ is strongly consistent with $\psi_n \in V(H_n)$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n| \in Q_p(H_n)$ is a QSGP.

**Lemma 4.7.** If $\psi_n \in V(H_n)$ and $\psi_{n+1} \in H_{n+1}$ satisfies (4.6) for every $\omega \in \Omega_n$, then $\psi_{n+1} \in V(H_{n+1})$.

**Proof.** Since $\psi_n \in V(H_n)$ we have by (4.6) that

$$|\langle\psi_{n+1}, 1\rangle| = \left| \sum_{\omega \in \Omega_n} \langle\psi_{n+1}, \chi_{\omega'\rightarrow}\rangle \right| = \left| \sum_{\omega \in \Omega_n} \langle\psi_n, \chi_{\omega}\rangle \right| = |\langle\psi_n, 1\rangle| = 1$$

The result now follows. \hfill $\Box$

**Corollary 4.8.** If $||\psi_1|| = 1$ and $\psi_n \in H_n$ satisfies (4.6) for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n|$ is a QSGP.

**Proof.** Since $||\psi_1|| = 1$, it follows that $\psi_1 \in V(H_1)$. By Lemma 4.7, $\psi_n \in V(H_n)$, $n = 1, 2, \ldots$. Since (4.6) holds, the result follows from Corollary 4.6. \hfill $\Box$

Another way of writing (4.6) is

$$\sum_{\omega' \rightarrow x} p_{c}^{n+1}(\omega x)\psi_{n+1}(\omega x) = p_{c}^{n}(\omega)\psi_n(x)$$  \hspace{1cm} (4.7)

for every $\omega \in \Omega_n$. 

---

11
5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for discrete quantum gravity. The simplest way to construct a QSGP is to form the constant pure DQP $\rho_n = |1\rangle\langle 1|$, $n = 1, 2, \ldots$. To show that $\rho_n$ is indeed consistent, we have for $\omega \in \Omega_n$ that

$$\sum_{\omega \rightarrow x} p^{n+1}_c(\omega x) = \sum_{\omega \rightarrow x} p^n_c(\omega) p_c(\omega \rightarrow x) = p^n_c(\omega) \sum_{\omega \rightarrow x} p_c(\omega \rightarrow x) = p^n_c(\omega)$$

so consistency follows from (4.7). The corresponding $q$-measures are given by

$$\mu_n(A) = \left| \langle 1, \chi_A \rangle \right|^2 = p^n_c(A)^2$$

for every $A \in \mathcal{A}_n$. Hence, $\mu_n$ is the square of the classical measure. Of course, $|1\rangle\langle 1|$ is the global $q$-probability operator for this QSGP and in this case $S(\rho_n) = \mathcal{A}$. Moreover, we have the global $q$-measure $\mu(A) = \nu_c(A)^2$ for $A \in \mathcal{A}$.

Another simple way to construct a QSGP is to employ Corollary 4.8. In this way we can let $\psi_1 = 1$, $\psi_2$ any vector in $L^2(\Omega_2, 2^{\Omega_2}, p^2_c)$ satisfying

$$\langle \psi_2, \chi_{\{x_1x_2\}} \rangle + \langle \psi_2, \chi_{\{x_1x_3\}} \rangle = \langle \psi_1, \chi_{\{x_1\}} \rangle = 1$$

and so on, where $x_1, x_2, x_3$ are given in Figure 2. As a concrete example, let $\psi_1 = 1$,

$$\psi_2 = \frac{1}{2} \left[ p^2_c(x_1x_2)^{-1} \chi_{\{x_1x_2\}} + p^2_c(x_1x_3) \chi_{\{x_1x_3\}} \right]$$

and in general

$$\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p^n_c(\omega)^{-1} \chi(\omega)$$

The $q$-measure $\mu_1$ is $\mu_1(\{x_1\}) = 1$ and $\mu_2$ is given by

$$\mu_2(\{x_1x_2\}) = \left| \langle \psi_2, \chi_{\{x_1x_2\}} \rangle \right|^2 = \frac{1}{4}$$

$$\mu_2(\{x_1x_3\}) = \left| \langle \psi_2, \chi_{\{x_1x_3\}} \rangle \right|^2 = \frac{1}{4}$$

$$\mu_2(\Omega_2) = \left| \langle \psi_2, 1 \rangle \right|^2 = 1$$

In general, we have $\mu_n(A) = |A|^2 / |\Omega_n|^2$ for all $A \in \Omega_n$. Thus $\mu_n$ is the square of the uniform distribution. The global operator does not exist because there is no $q$-measure on $\mathcal{A}$ that extends $\mu_n$ for all $n \in \mathbb{N}$. For $A \in \mathcal{A}$ we have

$$\langle \psi_n, \chi_A \rangle = \int \psi_n \chi_A d\nu_c = \frac{|A \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}|}{|\Omega_n|}$$
Letting $\rho_n = |\psi_n\rangle \langle \psi_n|$ we conclude that $A \in \mathcal{S}(\rho_n)$ if and only if
\[
\lim_{n \to \infty} \frac{|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}
\]
eexists. For example, if $|A| < \infty$ then for $n$ sufficiently large we have
\[
|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |A|
\]
so $A \in \mathcal{S}(\rho_n)$ and $\mu(A) = 0$. In a similar way if $|A| < \infty$ then for the complement $A'$, if $n$ is sufficiently large we have
\[
|A' \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |\Omega_n| - |A|
\]
so $A' \in \mathcal{S}(\rho_n)$ with $\mu(A') = 1$.

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let $\alpha_\omega \in \mathbb{C}$, $\omega \in \Omega_n$ satisfy
\[
\left| \sum_{\omega \in \Omega_n} \alpha_\omega p^n_c(\omega)^{1/2} \right| = 1 \quad (5.1)
\]
and let $\rho_n$ be the operator on $H_n$ satisfying
\[
\langle \rho_n e_\omega^n, e_\omega'^n \rangle = \alpha_\omega \overline{\alpha_\omega} \quad (5.2)
\]
Then $\rho_n$ is a positive operator and by (5.1), (5.2) we have
\[
\langle \rho_n 1, 1 \rangle = \left\langle \rho_n \sum_\omega p^n_c(\omega)^{1/2} e_\omega^n, \sum_{\omega'} p^n_c(\omega')^{1/2} e_\omega'^n \right\rangle = \sum_{\omega, \omega'} p^n_c(\omega)^{1/2} p^n_c(\omega')^{1/2} \langle \rho_n e_\omega^n, e_\omega'^n \rangle = \left| \sum_\omega p^n_c(\omega)^{1/2} \alpha_\omega \right|^2 = 1
\]
Hence, $\rho_n \in \mathcal{Q}(H_n)$. Now
\[
\Omega_{n+1} = \{ \omega x : \omega \in \Omega_n, x \in \mathcal{P}_{n+1}, \omega \to x \}
\]
and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy

$$\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} p_{c}^{n+1}(\omega x)^{1/2} \right| = 1$$

Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying

$$\langle \rho_{n+1} e_{\omega x}^{n+1}, e_{\omega' x'}^{n+1} \rangle = \beta_{\omega' x'} \beta_{\omega x} (5.3)$$

As before, we have that $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** The operator $\rho_{n+1}$ is consistent with $\rho_{n}$ if and only if for every $\omega, \omega' \in \Omega_{n}$ we have

$$\alpha_{\omega} \alpha_{\omega'} = \sum_{x' \in P_{n+1}^{\omega}} \beta_{\omega' x'} p_{c}(\omega' \rightarrow x')^{1/2} \sum_{x \in P_{n+1}^{\omega' \rightarrow x}} \beta_{\omega x} p_{c}(\omega \rightarrow x)^{1/2} (5.4)$$

A sufficient condition for (5.4) to hold is

$$\sum_{x \in P_{n+1}^{\omega \rightarrow x}} \beta_{\omega x} p_{c}(\omega \rightarrow x)^{1/2} = \alpha_{\omega} (5.5)$$

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** Let $\rho_{n} \in \mathcal{Q}(H_{n})$ be defined by (5.2) and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by (5.3). If (5.5) holds, then $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_{n}$.

The next result gives the general construction.

**Corollary 5.3.** Let $\rho_{1} = I \in \mathcal{Q}(H_{1})$ and define $\rho_{n} \in \mathcal{Q}(H_{n})$ inductively by (5.3). Then $\rho_{n}$ is a QSGP.

**References**


