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# DISCRETE QUANTUM PROCESSES

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## Abstract

A discrete quantum process is defined as a sequence of local states  $\rho_t$ ,  $t = 0, 1, 2, \dots$ , satisfying certain conditions on an  $L_2$  Hilbert space  $H$ . If  $\rho = \lim \rho_t$  exists, then  $\rho$  is called a global state for the system. In important cases, the global state does not exist and we must then work with the local states. In a natural way, the local states generate a sequence of quantum measures which in turn define a single quantum measure  $\mu$  on the algebra of cylinder sets  $\mathcal{C}$ . We consider the problem of extending  $\mu$  to other physically relevant sets in a systematic way. To this end we show that  $\mu$  can be properly extended to a quantum measure  $\tilde{\mu}$  on a “quadratic algebra” containing  $\mathcal{C}$ . We also show that a random variable  $f$  can be “quantized” to form a self-adjoint operator  $\hat{f}$  on  $H$ . We then employ  $\hat{f}$  to define a quantum integral  $\int f d\tilde{\mu}$ . Various examples are given

## 1 Introduction

This section presents an overview of the paper. Detailed definitions will be given in Sections 2, 3 and 4. The main arena for this study is a Hilbert space  $H = L_2(\Omega, \mathcal{A}, \nu)$  where  $(\Omega, \mathcal{A}, \nu)$  is a probability space. We think of  $\Omega$  as the set of paths or trajectories or histories of a physical system. It is unusual to consider paths for a quantum system because such systems are not supposed to have well-defined trajectories, so paths are considered to be meaningless. However, paths are the basic ingredients of the histories

approach to quantum mechanics [1, 3, 4, 9, 10] and they appear in Feynman integrals and quantum gravity studies [1, 4, 14]. Our attitude is that we are not abandoning the usual quantum formalism, but we are gleaning more information from this formalism by allowing the consideration of paths.

If  $\rho$  is a density operator (state) on  $H$  we define the *decoherence functional*  $D_\rho: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$  by

$$D_\rho(A, B) = \langle \rho \chi_B, \chi_A \rangle$$

where  $\chi_A$  is the characteristic function for  $A \in \mathcal{A}$ . We then define the *quantum measure*  $\mu_\rho: \mathcal{A} \rightarrow \mathbb{R}^+$  by  $\mu_\rho(A) = D_\rho(A, A)$ . If  $f: \Omega \rightarrow \mathbb{R}$  is a random variable with  $f \in H$  we define the “quantization” of  $f$  to be a certain self-adjoint operator  $\widehat{f}$  on  $H$ . The *quantum integral* of  $f$  is defined as

$$\int f d\mu_\rho = \text{tr}(\rho \widehat{f})$$

Properties of  $D_\rho$ ,  $\mu_\rho$  and  $\int f d\mu_\rho$  are reviewed in Section 2.

A sequence of states  $\rho_t$ ,  $t = 0, 1, 2, \dots$ , on  $H$  that have certain properties is called a discrete quantum process and we call  $\rho_t$  the local states for the process. If  $\rho = \lim \rho_t$  exists, we call  $\rho$  the global state for the process. For important cases, the global state does not exist and we must then work with the local states. In a natural way, the local states generate a sequence of quantum measures which in turn define a single quantum measure  $\mu$  on the algebra of cylinder sets  $\mathcal{C}$ . It appears to be impossible to extend  $\mu$  to a quantum measure on  $\mathcal{A}$ . An important problem is to extend  $\mu$  to other physically relevant sets in a systematic way. We say that a set  $A \in \mathcal{A}$  is *suitable* if  $\lim \langle \rho_t \chi_A, \chi_A \rangle$  exists and is finite. We denote the collection of suitable sets by  $\mathcal{S}$  and for  $A \in \mathcal{S}$  we define

$$\tilde{\mu}(A) = \lim \langle \rho_t \chi_A, \chi_A \rangle$$

It is shown in Section 3 that  $\mathcal{S}$  is a “quadratic algebra” that properly contains  $\mathcal{C}$  and that  $\tilde{\mu}$  is a quantum measure on  $\mathcal{S}$  that extends  $\mu$ . It is also shown that the quantum integral extends in a natural way.

Section 4 considers finite unitary systems. Such a system is a set of unitary operators  $U(s, r)$ ,  $r \leq s \in \mathbb{N}$  on a position Hilbert space  $\mathbb{C}^m$ . The operator  $U(s, r)$  describes the evolution of a finite-dimensional quantum system in discrete time-steps from time  $r$  to time  $s$ . We call the elements of  $S = \{0, 1, \dots, m-1\}$  *sites* and we call infinite strings  $\gamma = \gamma_0 \gamma_1 \dots$ ,  $\gamma_i \in S$  *paths*. The *path space*  $\Omega$  is the set of all paths and the *n-path space*  $\Omega_n$  is

the set of all  $n$ -paths  $\gamma = \gamma_0\gamma_1 \cdots \gamma_n$ . The  $n$ -events are sets in the power set  $\mathcal{A}_n = 2^{\Omega_n}$ . Given an initial state  $\psi \in \mathbb{C}^m$ , the operators  $U(s, r)$  define a decoherence functional  $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$  in a natural way. The *decoherence matrix* is the  $m^{n+1} \times m^{n+1}$  matrix with components

$$D_n(\gamma, \gamma') = D_n(\{\gamma\}, \{\gamma'\}), \quad \gamma, \gamma' \in \Omega_n$$

We can think of this matrix as an operator  $\widehat{D}_n$  on the  $n$ -path Hilbert space  $H_n = (\mathbb{C}^m)^{\otimes(n+1)}$ . It is shown that  $\widehat{D}_n$  is a state on  $H_n$  and the eigenvalues and eigenvectors of  $\widehat{D}_n$  are computed.

Section 5 shows how a finite unitary system can be employed to construct a discrete quantum process. Place the uniform probability distribution on  $S$  and form the product measure on  $\Omega = S \times S \times \cdots$  to obtain a probability space  $(\Omega, \mathcal{A}, \nu)$ . The path Hilbert space becomes  $H = L_2(\Omega, \mathcal{A}, \nu)$ . It is shown that the states  $\widehat{D}_t$ ,  $t = 0, 1, 2, \dots$ , generate a discrete quantum process  $\rho_t$ . We demonstrate that the event  $A =$ “the particle visits the origin” as well as its complement  $A'$  are elements of  $\mathcal{S} \setminus \mathcal{C}$ . An example of a two-site quantum random walk is explored and it is shown that the particle executes a periodic motion with period 4.

Section 6 considers quantum integrals. The operator  $\widehat{f}$  can be complicated and the expression  $(\rho\widehat{f})$  can be difficult to evaluate. The eigenvalues and eigenvectors of  $\widehat{f}$  are found for a two-valued simple function  $f$ . These are then employed to treat arbitrary simple functions. The quantum integral of an arbitrary random variable may then be computed by a limit process.

## 2 Quantum Measures and Integrals

In a certain sense a quantum process is a generalization of a classical stochastic process. Moreover, quantum measures and integrals are generalizations of classical probability measures and classical expectations. For these reasons we begin with a short review of classical probability theory and then present a method of “quantizing” this structure.

A *probability space* is a triple  $(\Omega, \mathcal{A}, \nu)$  where  $\Omega$  is a *sample space* whose elements are *sample points* or *outcomes*,  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\Omega$  whose elements are *events* and  $\nu$  is a measure on  $\mathcal{A}$  satisfying  $\nu(\Omega) = 1$ . For  $A \in \mathcal{A}$ ,  $\nu(A)$  is interpreted as the probability that event  $A$  occurs. We denote the set of measurable functions  $f: \Omega \rightarrow \mathbb{C}$  by  $\mathcal{M}(\mathcal{A})$ . The first quantization

step is to form the Hilbert space

$$H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f \in \mathcal{M}(\mathcal{A}) : \int |f|^2 d\nu < \infty \right\}$$

with inner product  $\langle f, g \rangle = \int \bar{f}g d\nu$  and norm  $\|f\| = \langle f, f \rangle^{1/2}$ . We call real-valued functions  $f \in H$  *random variables*. If  $f$  is a random variable, then by Schwarz's inequality we have  $\int |f| d\nu \leq \|f\|$  so the expectation  $E(f) = \int f d\nu$  exists and is finite. In general probability theory, random variables whose expectations are infinite or do not exist are considered, but for our purposes this more restricted concept is convenient.

The characteristic function  $\chi_A$  of  $A \in \mathcal{A}$  is a random variable with  $\|\chi_A\| = \nu(A)^{1/2}$  and we write  $\chi_\Omega = 1$ . For  $A, B \in \mathcal{A}$  we define the *decoherence operator*  $D(A, B)$  as the operator on  $H$  defined by  $D(A, B) = |\chi_B\rangle\langle\chi_A|$ . Thus, for  $f \in H$  we have

$$D(A, B)f = \langle\chi_A, f\rangle\chi_B = \int_A f d\nu\chi_B$$

Of course, if  $\nu(A)\nu(B) = 0$  then  $D(A, B) = 0$ . If  $\nu(A)\nu(B) \neq 0$ , it is easy to show that  $D(A, B)$  is a rank 1 operator with  $\|D(A, B)\| = \nu(A)^{1/2}\nu(B)^{1/2}$ . For  $A \in \mathcal{A}$  we define the *q-measure operator*  $\hat{\mu}(A)$  on  $H$  by  $\hat{\mu}(A) = D(A, A)$ . Hence, for  $f \in H$  we have

$$\hat{\mu}(A)f = \langle\chi_A, f\rangle\chi_A = \int_A f d\nu\chi_A$$

In particular,  $\hat{\mu}(\Omega)f = E(f)1$ . If  $\nu(A) = 0$ , then  $\hat{\mu}(A) = 0$  and if  $\nu(A) \neq 0$ , then  $\hat{\mu}(A)$  is a positive (and hence, self-adjoint) rank 1 operator with  $\|\hat{\mu}(A)\| = \nu(A)$ . Moreover, if  $\nu(A) \neq 0$ , then

$$\frac{1}{\nu(A)}\hat{\mu}(A) = \frac{1}{\nu(A)}|\chi_A\rangle\langle\chi_A|$$

is an orthogonal projection.

The map  $D$  from  $\mathcal{A} \times \mathcal{A}$  into the set of bounded operators  $\mathcal{B}(H)$  on  $H$  has some obvious properties:

- (1) If  $A \cap B = \emptyset$ , then  $D(A \cup B, C) = D(A, C) + D(B, C)$  for all  $C \in \mathcal{A}$  (additivity)

$$(2) D(A, B)^* = D(B, A) \text{ (conjugate symmetry)}$$

$$(3) D(A, B)^2 = \nu(A \cap B)D(A, B)$$

$$(4) D(A, B)D(A, B)^* = \nu(A)\widehat{\mu}(B), D(A, B)^*D(A, B) = \nu(B)\widehat{\mu}(A)$$

Less obvious properties are given in the following theorem proved in [8].

**Theorem 2.1.** (a)  $D: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}(H)$  is positive semidefinite in the sense that if  $A_i \in \mathcal{A}$ ,  $c_i \in \mathbb{C}$ ,  $i = 1, \dots, n$ , then

$$\sum_{i,j=1}^n D(A_i, A_j)c_i\bar{c}_j$$

is a positive operator. (b) If  $A_1 \subseteq A_2 \subseteq \dots$  is an increasing sequence in  $\mathcal{A}$ , then the continuity condition

$$\lim D(A_i, B) = D(\cup A_i, B)$$

holds for every  $B \in \mathcal{A}$  where the limit is in the operator norm topology.

It follows from (1), (2) and Theorem 2.1(b) that  $A \mapsto D(A, B)$  and  $B \mapsto D(A, B)$  are operator-valued measures from  $\mathcal{A}$  to  $\mathcal{B}(H)$ .

The map  $\widehat{\mu}: \mathcal{A} \rightarrow \mathcal{B}(H)$  need not be additive. For example, if  $A, B \in \mathcal{A}$  are disjoint, then

$$\begin{aligned} \widehat{\mu}(A \cup B) &= D(A \cup B, A \cup B) = |\chi_{A \cup B}\rangle\langle\chi_{A \cup B}| = |\chi_A + \chi_B\rangle\langle\chi_A + \chi_B| \\ &= |\chi_A\rangle\langle\chi_A| + |\chi_B\rangle\langle\chi_B| + |\chi_A\rangle\langle\chi_B| + |\chi_B\rangle\langle\chi_A| \\ &= \widehat{\mu}(A) + \widehat{\mu}(B) + 2\text{Re } D(A, B) \end{aligned}$$

Thus, additivity is spoiled by the *interference term*  $2\text{Re } D(A, B)$ . Because of this nonadditivity, we have that  $\widehat{\mu}(A') \neq \widehat{\mu}(\Omega) - \widehat{\mu}(A)$  in general, where  $A'$  is the complement of  $A$ . Moreover,  $A \subseteq B$  need not imply  $\widehat{\mu}(A) \leq \widehat{\mu}(B)$  in the usual order of self-adjoint operators. However,  $\widehat{\mu}$  does satisfy the *grade-2 additivity* condition given in the next theorem which is proved in [8].

**Theorem 2.2.** (a)  $\widehat{\mu}$  satisfies grade-2 additivity:

$$\widehat{\mu}(A \cup B \cup C) = \widehat{\mu}(A \cup B) + \widehat{\mu}(A \cup C) + \widehat{\mu}(B \cup C) - \widehat{\mu}(A) - \widehat{\mu}(B) - \widehat{\mu}(C)$$

whenever  $A, B, C \in \mathcal{A}$  are mutually disjoint. (b)  $\widehat{\mu}$  satisfies the continuity conditions

$$\begin{aligned}\lim \widehat{\mu}(A_i) &= \widehat{\mu}(\cup A_i) \\ \lim \widehat{\mu}(B_i) &= \widehat{\mu}(\cap A_i)\end{aligned}$$

in the operator norm topology for any increasing sequence  $A_i \in \mathcal{A}$  or decreasing sequence  $B_i \in \mathcal{A}$ .

If  $\rho$  is a density operator (or state) on  $H$  we define the *decoherence functional*  $D_\rho: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$  by

$$D_\rho(A, B) = \text{tr} [\rho D(A, B)] = \langle \rho \chi_B, \chi_A \rangle$$

We interpret  $D_\rho(A, B)$  as a measure of the interference between  $A$  and  $B$ . The next result follows from Theorem 2.1.

**Corollary 2.3.** (a)  $A \mapsto D_\rho(A, B)$  is a complex measure on  $\mathcal{A}$ . (b) If  $A_1, \dots, A_n \in \mathcal{A}$ , then the  $n \times n$  matrix  $D_\rho(A_i, A_j)$  is positive semidefinite.

For the density operator  $\rho$  on  $H$  we define the  $q$ -measure  $\mu_\rho: \mathcal{A} \rightarrow \mathbb{R}^+$  by

$$\mu_\rho(A) = \text{tr} [\rho \widehat{\mu}(A)] = \langle \rho \chi_A, \chi_A \rangle$$

It can be shown that  $\mu_\rho(\Omega) \leq 1$  and that Theorem 2.2 holds with  $\widehat{\mu}$  replaced by  $\mu_\rho$  [8]. We interpret  $\mu_\rho(A)$  as the  $q$ -probability or propensity of the event  $A$  in the state  $\rho$  [6, 11, 12, 13].

We next introduce the second quantization step. Let  $\widehat{f}$  be a nonnegative random variable. The *quantization* of  $f$  is the operator  $\widehat{f}$  on  $H$  defined by

$$(\widehat{f}g)(y) = \int \min [f(x), f(y)] g(x) d\nu(x)$$

It easily follows that  $\|\widehat{f}\| \leq \|f\|$  so  $\widehat{f}$  is a bounded self-adjoint operator on  $H$ .

**Lemma 2.4.** If  $\widehat{f}_1, \widehat{f}_2$  are nonnegative random variables with disjoint support, then  $\widehat{f}_1 \widehat{f}_2 = \widehat{f}_2 \widehat{f}_1 = 0$  and

$$\|\widehat{f}_1 + \widehat{f}_2\| = \max [\|\widehat{f}_1\|, \|\widehat{f}_2\|]$$

*Proof.* For every  $g \in H$  we have by Fubini's theorem that

$$\begin{aligned}
(\widehat{f}_1 \widehat{f}_2 g)(y) &= \widehat{f}_1 \left[ \int \min [f_2(x), f_2(z)] g(x) d\nu(x) \right] (y) \\
&= \int \min [f_1(z), f_1(y)] \left\{ \int \min [f_2(x), f_2(z)] g(x) d\nu(x) \right\} d\nu(z) \\
&= \int \left\{ \int \min [f_1(z), f_1(y)] \min [f_2(x), f_2(z)] d\nu(z) \right\} g(x) d\nu(x) \\
&= 0
\end{aligned}$$

where the last equality follows from  $f_1(z)f_2(z) = 0$ . The second statement now follows.  $\square$

If  $f$  is an arbitrary random variable, we have that  $f = f^+ - f^-$  where  $f^+(x) = \max [f(x), 0]$  and  $f^-(x) = -\min [f(x), 0]$ . Then  $f^+, f^- \geq 0$ ,  $f^+ f^- = 0$  and we define the bounded self-adjoint operator  $\widehat{f}$  by  $\widehat{f} = f^{+\wedge} - f^{-\wedge}$ . It follows from Lemma 2.4 that  $\|\widehat{f}\| = \max [\|f^+\|, \|f^-\|]$ . The next result summarizes some of the important properties of  $\widehat{f}$  [8].

**Theorem 2.5.** (a) For any  $A \in \mathcal{A}$ ,  $\widehat{\chi}_A = |\chi_A\rangle\langle\chi_A| = \widehat{\mu}(A)$ . (b) For any  $\alpha \in \mathbb{R}$ ,  $(\alpha f)^\wedge = \alpha \widehat{f}$ . (c) If  $f \geq 0$ , then  $\widehat{f}$  is a positive operator. (d) If  $0 \leq f_1 \leq f_2 \leq \dots$  is an increasing sequence of random variables converging in norm to a random variable  $f$ , then  $\lim \widehat{f}_i = \widehat{f}$  in the operator norm topology. (e) If  $f, g, h$  are random variables with disjoint supports, then

$$(f + g + h)^\wedge = (f + g)^\wedge + (f + h)^\wedge + (g + h)^\wedge - \widehat{f} - \widehat{g} - \widehat{h}$$

Let  $\rho$  be a density operator on  $H$  and let  $\mu_\rho(A) = \text{tr}(\rho \widehat{\mu}(A))$  be the corresponding  $q$ -measure. If  $f$  is a random variable we define the  $q$ -integral (or  $q$ -expectation) of  $f$  with respect to  $\mu_\rho$  as

$$\int f d\mu_\rho = \text{tr}(\rho \widehat{f})$$

As usual, for  $A \in \mathcal{A}$  we define

$$\int_A f d\mu_\rho = \int \chi_A f d\mu_\rho$$

The next result follows from Theorem 2.5.



**Corollary 2.6.** (a) For every  $A \in \mathcal{A}$ ,  $\int \chi_A d\mu_\rho = \mu_\rho(A)$ . (b) For every  $\alpha \in \mathbb{R}$ ,  $\int \alpha f d\mu_\rho = \alpha \int f d\mu_\rho$ . (c) If  $f \geq 0$ , then  $\int f d\mu_\rho \geq 0$ . (d) If  $f_i \geq 0$  is an increasing sequence of random variables converging in norm to a random variable  $f$ , then  $\lim \int f_i d\mu_\rho = \int f d\mu_\rho$ . (e) If  $f, g, h$  are random variables with disjoint supports, then

$$\begin{aligned} \int (f + g + h) d\mu_\rho &= \int (f + g) d\mu_\rho + \int (f + h) d\mu_\rho + \int (g + h) d\mu_\rho \\ &\quad - \int f d\mu_\rho - \int g d\mu_\rho - \int h d\mu_\rho \end{aligned}$$

The next result is called the *tail-sum formula* and gives a justification for calling  $\int f d\mu_\rho$  a *q-integral* [5, 8]. The classical tail-sum formula is quite useful in traditional probability theory [5].

**Theorem 2.7.** If  $f \geq 0$  is a random variable, then

$$\int f d\mu_\rho = \int_0^\infty \mu_\rho \{x: f(x) > \lambda\} d\lambda$$

where  $d\lambda$  denotes Lebesgue measure on  $\mathbb{R}$ .

It follows from Theorem 2.7 that if  $f$  is an arbitrary random variable, then

$$\int f d\mu_\rho = \int_0^\infty \mu_\rho \{x: f(x) > \lambda\} d\lambda - \int_0^\infty \mu_\rho \{x: f(x) < -\lambda\} d\lambda$$

### 3 Discrete Quantum Processes

Let  $(\Omega, \mathcal{A}, \nu)$  be a probability space and let  $\mathcal{A}_t \subseteq \mathcal{A}$ ,  $t = 0, 1, 2, \dots$ , be an increasing sequence of  $\sigma$ -algebras such that  $\mathcal{A}$  is the smallest  $\sigma$ -algebra containing  $\cup \mathcal{A}_t$ . We then say that  $\mathcal{A}_t = 0, 1, 2, \dots$ , *generates*  $\mathcal{A}$ . Let  $\nu_t$  be the restriction of  $\nu$  to  $\mathcal{A}_t$ . We think of the probability space  $(\Omega, \mathcal{A}_t, \nu_t)$  as a classical description of a physical system until a discrete time  $t$ . A corresponding quantum description takes place in the closed subspace  $H_t = L_2(\Omega, \mathcal{A}_t, \nu_t)$  of  $H = L_2(\Omega, \mathcal{A}, \nu)$   $t = 0, 1, 2, \dots$ . A sequence of density operators  $\rho_t$  on  $H_t$ ,  $t = 0, 1, 2, \dots$ , is *consistent* if  $D_{\rho_{t+1}}(A, B) = D_{\rho_t}(A, B)$  for every  $A, B \in \mathcal{A}_t$ . In particular, we then have that  $\mu_{\rho_{t+1}}(A) = \mu_{\rho_t}(A)$  for all  $A \in \mathcal{A}_t$ ,  $t = 0, 1, 2, \dots$ . We call a consistent sequence  $\rho_t$ ,  $t = 0, 1, 2, \dots$ , a *discrete*

$q$ -process and we call  $\rho_t$  the *local states* for the process. If each  $\mathcal{A}_t$  has finite cardinality, we call a consistent sequence  $\rho_t$  a *finite  $q$ -process*. Notice for a finite  $q$ -process that each of the subspaces  $H_t$  is finite-dimensional.

Let  $\rho_t, t = 0, 1, 2, \dots$ , be a discrete  $q$ -process. We then have an increasing sequence of closed subspaces  $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq H$ . We now extend  $\rho_t$  from  $H_t$  to a state on  $H$  by defining  $\rho_t f = 0$  for all  $f \in H_t^\perp$ . If  $\lim \rho_t$  exists in the strong operator topology, then the limit  $\rho$  is a state on  $H$ . In this case we call  $\rho$  the *global state* for the  $q$ -process  $\rho_t$ . If the global state  $\rho$  exists, then for every  $A, B \in \mathcal{A}_t$  we have

$$D_\rho(A, B) = \lim_{n \rightarrow \infty} D_{\rho_n}(A, B) = D_{\rho_t}(A, B) \quad (3.1)$$

We call  $D_\rho$  and  $\mu_\rho$  the *global decoherence functional* and *global  $q$ -measure* for the  $q$ -process, respectively. Equation (3.1) shows that  $D_\rho$  and  $\mu_\rho$  extend all the  $D_{\rho_t}$  and  $\mu_{\rho_t}$  from  $\mathcal{A}_t$  to  $\mathcal{A}, t = 0, 1, 2, \dots$ , respectively. It follows from the work in [2, 7] that  $\lim \rho_t$  may not exist in which case we would not have a global state and hence no global decoherence functional or global  $q$ -measure. Moreover, we would not have a global integral  $\int f d\mu_\rho$ . In this case we are forced to work with the local states  $\rho_t, t = 0, 1, 2, \dots$ , and this is what we now explore.

A collection  $\mathcal{Q}$  of subsets of a set  $X$  is a *quadratic algebra* if  $\emptyset, X \in \mathcal{Q}$  and if  $A, B, C \in \mathcal{Q}$  are mutually disjoint and  $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$ , then  $A \cup B \cup C \in \mathcal{Q}$ . Of course, an algebra of subsets of  $X$  is a quadratic algebra. However, there are examples of quadratic algebras that are not closed under complementation, union or intersection [7]. A general  $q$ -measure is a nonnegative grade-2 set function  $\mu$  on a quadratic algebra  $\mathcal{Q}$ . That is,  $\mu: \mathcal{Q} \rightarrow \mathbb{R}^+$  and if  $A, B, C \in \mathcal{Q}$  are mutually disjoint with  $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$ , then

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$$

Let  $\rho_t, t = 0, 1, 2, \dots$ , be a discrete  $q$ -process. Defining  $\mathcal{C}(\Omega) = \cup \mathcal{A}_t$ , it is clear that  $\mathcal{C}(\Omega)$  is an algebra of subsets of  $\Omega$ . For  $A \in \mathcal{C}(\Omega)$  we have that  $A \in \mathcal{A}_t$  for some  $t \in \mathbb{N}$  and we define  $\mu(A) = \mu_{\rho_t}(A)$ . To show that  $\mu$  is well-defined, suppose that  $A \in \mathcal{A}_t \cap \mathcal{A}_{t'}$ . We can assume without loss of generality that  $\mathcal{A}_t \subseteq \mathcal{A}_{t'}$ . Hence,  $\mu_{\rho_{t'}}(A) = \mu_{\rho_t}(A)$  so  $\mu$  is well-defined. It easily follows that  $\mu: \mathcal{C}(\Omega) \rightarrow \mathbb{R}^+$  is a  $q$ -measure. It appears to be impossible to extend  $\mu$  to a  $q$ -measure on  $\mathcal{A}$  in general. In fact, it is shown in [2, 7] that in general  $\mu$  cannot be extended to a continuous  $q$ -measure on  $\mathcal{A}$ . However, we can extend

$\mu$  to a  $q$ -measure on a larger quadratic algebra than  $\mathcal{C}(\Omega)$  and this quadratic algebra contains physically relevant sets that are not in  $\mathcal{C}(\Omega)$ . A set  $A \in \mathcal{A}$  is *suitable* if  $\lim \text{tr}(\rho_t \widehat{\chi}_A)$  exists and is finite. We denote the collection of suitable sets by  $\mathcal{S}(\Omega)$  and for  $A \in \mathcal{S}(\Omega)$  we define  $\tilde{\mu}(A) = \lim \text{tr}(\rho_t \widehat{\chi}_A)$ .

**Theorem 3.1.**  $\mathcal{S}(\Omega)$  is a quadratic algebra that contains  $\mathcal{C}(\Omega)$  and  $\tilde{\mu}$  is a  $q$ -measure on  $\mathcal{S}(\Omega)$  that extends  $\mu$ .

*Proof.* If  $A \in \mathcal{C}(\Omega)$  then  $A \in \mathcal{A}_t$  for some  $t \in \mathbb{N}$ . Since  $\mu_{\rho_{t'}}(A) = \mu_{\rho_t}(A)$  for all  $t' \geq t$  we have

$$\lim \text{tr}(\rho_{t'} \widehat{\chi}_A) = \text{tr}(\rho_t \widehat{\chi}_A) = \mu_{\rho_t}(A) = \mu(A)$$

Hence,  $A \in \mathcal{S}(\Omega)$  and  $\tilde{\mu}(A) = \mu(A)$ . We conclude that  $\mathcal{C}(\Omega) \subseteq \mathcal{S}(\Omega)$  and  $\tilde{\mu}$  extends  $\mu$  to  $\mathcal{S}(\Omega)$ . To show that  $\mathcal{S}(\Omega)$  is a quadratic algebra, suppose  $A, B, C \in \mathcal{S}(\Omega)$  are mutually disjoint with  $A \cup B, A \cup C, B \cup C \in \mathcal{S}(\Omega)$ . Applying Theorem 2.5(e) we obtain

$$\begin{aligned} \lim \text{tr}(\rho_t \widehat{\chi}_{A \cup B \cup C}) &= \lim \text{tr}[\rho_t(\chi_A + \chi_B + \chi_C)^\wedge] \\ &= \lim \text{tr}(\rho_t \widehat{\chi}_{A \cup B}) + \lim \text{tr}(\rho_t \widehat{\chi}_{A \cup C}) + \lim \text{tr}(\rho_t \widehat{\chi}_{B \cup C}) \\ &\quad - \lim \text{tr}(\rho_t \widehat{\chi}_A) - \lim \text{tr}(\rho_t \widehat{\chi}_B) - \lim \text{tr}(\rho_t \widehat{\chi}_C) \end{aligned}$$

We conclude that  $A \cup B \cup C \in \mathcal{S}(\Omega)$  and that  $\tilde{\mu}$  is a  $q$ -measure on  $\mathcal{S}(\Omega)$ .  $\square$

A random variable  $f \in H$  is *integrable* for  $\rho_t$  if  $\lim \text{tr}(\rho_t \widehat{f})$  exists and is finite. If  $f$  is integrable we define

$$\int f d\tilde{\mu} = \lim \text{tr}(\rho_t \widehat{f})$$

Notice that if  $A \in \mathcal{S}(\Omega)$  then  $\chi_A$  is integrable and

$$\int \chi_A d\tilde{\mu} = \tilde{\mu}(A)$$

The proof of the next result is similar to the proof of Theorem 3.1 and follows from Corollary 2.6.

**Theorem 3.2.** (a) If  $f$  is integrable and  $\alpha \in \mathbb{R}$ , then  $\alpha f$  is integrable and  $\int \alpha f d\tilde{\mu} = \alpha \int f d\tilde{\mu}$  (b) If  $f$  is integrable with  $f \geq 0$ , then  $\int f d\tilde{\mu} \geq 0$ . (c) If

$f, g, h$  are integrable with disjoint support and  $f+g, f+h, g+h$  are integrable, then  $f+g+h$  is integrable and

$$\begin{aligned} \int (f+g+h)d\tilde{\mu} &= \int (f+g)d\tilde{\mu} + \int (f+h)d\tilde{\mu} + \int (g+h)d\tilde{\mu} \\ &\quad - \int f d\tilde{\mu} - \int g d\tilde{\mu} - \int h d\tilde{\mu} \end{aligned}$$

## 4 Finite Unitary Systems

Finite unitary systems and their relationship to finite  $q$ -processes have been studied in the past [2, 7]. They are closely related to the histories approach to quantum mechanics [9, 10, 12, 14]. In this section we study finite unitary systems and in Section 5 we employ them to construct finite  $q$ -processes.

Let  $\mathbb{C}^m$  be the  $m$ -dimensional Hilbert space with elements

$$f: \{0, 1, \dots, m-1\} \rightarrow \mathbb{C}$$

and inner product

$$\langle f, g \rangle = \sum_{j=0}^{m-1} \overline{f(j)} g(j)$$

We call  $\mathbb{C}^m$  the *position Hilbert space* and denote the standard basis on  $\mathbb{C}^m$  by  $e_0, e_1, \dots, e_{m-1}$ . A *finite unitary system* is a collection of unitary operators  $U(s, r), r \leq s \in \mathbb{N}$  on  $\mathbb{C}^m$  such that  $U(r, r) = I$  and

$$U(t, r) = U(t, s)U(s, r)$$

for  $r \leq s \leq t \in \mathbb{N}$ . If  $U(s, r), r \leq s \in \mathbb{N}$ , is a finite unitary system, then we have the unitary operators  $U(n+1, n), n \in \mathbb{N}$  such that

$$U(s, r) = U(s, s-1)U(s-1, s-2) \cdots U(r+1, r) \quad (4.1)$$

Conversely, if  $U(n+1, n), n \in \mathbb{N}$ , are unitary operators on  $\mathbb{C}^m$ , then defining  $U(r, r) = I$  and for  $r < s$  defining  $U(s, r)$  by (4.1) we have that  $U(s, r), r \leq s \in \mathbb{N}$ , is a finite unitary system. A finite unitary system  $U(s, r), r \leq s \in \mathbb{N}$  is *stationary* if there is a unitary operator  $U$  on  $\mathbb{C}^m$  such that  $U(s, r) = U^{s-r}$  for all  $r \leq s \in \mathbb{N}$ . In this case  $U(n+1, n) = U$  for all  $n \in \mathbb{N}$ .

We call the elements of  $S = \{0, 1, \dots, m-1\}$  *sites* or *positions* and we call infinite strings  $\gamma = \gamma_0\gamma_1\gamma_2 \cdots$ ,  $\gamma_i \in S$ , *paths* or *trajectories* or *histories*. The *path* or *sample space* is

$$\Omega = \{\gamma: \gamma \text{ a path}\}$$

We also call finite strings  $\gamma_0\gamma_1 \cdots \gamma_n$  *n-paths* and

$$\Omega_n = \{\gamma: \gamma \text{ an } n\text{-path}\}$$

is the *n-path space* on *n-sample space*. Notice that the cardinality  $|\Omega_n| = m^{n+1}$ . We call the elements of  $\mathcal{A}_n = 2^{\Omega_n}$  *n-events*.

A finite unitary system  $U(s, r)$  describes the evolution of a finite-dimensional quantum system and the projections  $P(i) = |e_i\rangle\langle e_i|$ ,  $i = 0, 1, \dots, m-1$ , describe the position. The *n-path*  $\gamma \in \Omega_n$  is described by the operator  $C_n(\gamma)$  on  $\mathbb{C}^m$  given by

$$C_n(\gamma) = P(\gamma_n)U(n, n-1)P(\gamma_{n-1})U(n-1, n-2) \cdots P(\gamma_1)U(1, 0)P(\gamma_0) \quad (4.2)$$

Defining  $b(\gamma)$  by

$$b(\gamma) = \langle e_{\gamma_n}, U(n, n-1)e_{\gamma_{n-1}} \rangle \langle e_{\gamma_{n-1}}, U(n-1, n-2)e_{\gamma_{n-2}} \rangle \cdots \langle e_{\gamma_1}, U(1, 0)e_{\gamma_0} \rangle \quad (4.3)$$

Equation (4.2) becomes

$$C_n(\gamma) = b(\gamma)|e_{\gamma_n}\rangle\langle e_{\gamma_0}| \quad (4.4)$$

**Lemma 4.1.** *For  $i = 0, 1, \dots, m-1$  we have*

$$\sum_{\gamma \in \Omega_n} \{|b(\gamma)|^2 : \gamma_0 = i\} = \sum_{\gamma \in \Omega_n} \{|b(\gamma)|^2 : \gamma_n = i\} = 1$$

*Proof.* The result follows from

$$|b(\gamma)|^2 = |\langle e_{\gamma_n}, U(n, n-1)e_{\gamma_{n-1}} \rangle|^2 |\langle e_{\gamma_{n-1}}, U(n-1, n-2)e_{\gamma_{n-2}} \rangle|^2 \cdots |\langle e_{\gamma_1}, U(1, 0)e_{\gamma_0} \rangle|^2$$

and calculating the designated sums. □

If  $\psi \in \mathbb{C}^m$ ,  $\|\psi\| = 1$ , we define the *amplitude* of  $\gamma \in \Omega_n$  by  $a_\psi(\gamma) = b(\gamma)\psi(\gamma_0)$ . We interpret  $|a_\psi(\gamma)|^2$  as the probability of the path  $\gamma$  with initial distribution  $\psi$ . The next result shows that these probabilities sum to 1.

**Corollary 4.2.** *For the path space  $\Omega_n$  we have*

$$\sum_{\gamma \in \Omega_n} |a_\psi(\gamma)|^2 = 1$$

*Proof.* By Lemma 4.1 we have

$$\begin{aligned} \sum_{\gamma \in \Omega_n} |a_\psi(\gamma)|^2 &= \sum_{\gamma_0} \sum_{\gamma_n, \dots, \gamma_1} |a_\psi(\gamma)|^2 = \sum_{\gamma_0} |\psi(\gamma_0)|^2 \sum_{\gamma_n, \dots, \gamma_1} |b(\gamma)|^2 \\ &= \sum_{\gamma_0} |\psi(\gamma_0)|^2 = 1 \end{aligned} \quad \square$$

It is interesting to note that

$$\begin{aligned} a_\psi(\gamma) &= \langle e_{\gamma_n} \otimes \dots \otimes e_{\gamma_0}, U(n, n-1) \otimes \dots \otimes U(1, 0) \otimes I e_{\gamma_{n-1}} \otimes \dots \otimes e_{\gamma_0} \otimes \psi \rangle \end{aligned}$$

and when the system is stationary with evolution operator  $U$  we have

$$a_\psi(\gamma) = \langle e_{\gamma_n} \otimes \dots \otimes e_{\gamma_0}, U^{\otimes n} \otimes I e_{\gamma_{n-1}} \otimes \dots \otimes e_{\gamma_0} \otimes \psi \rangle$$

The operator  $C_n(\gamma')^* C_n(\gamma)$  describes the interference between the two paths  $\gamma, \gamma' \in \Omega_n$ . More precisely,  $C_n(\gamma')^* C_n(\gamma)$  describes the interference between two particles, one moving along path  $\gamma$  and the other along path  $\gamma'$ . Applying (4.4) we see that

$$C_n(\gamma')^* C_n(\gamma) = \overline{b(\gamma')} b(\gamma) |e_{\gamma'_0}\rangle \langle e_{\gamma_0}| \delta_{\gamma_n, \gamma'_n} \quad (4.5)$$

For  $A \in \mathcal{A}_n$  the *class operator*  $C_n(A)$  is

$$C_n(A) = \sum_{\gamma \in A} C_n(\gamma)$$

It is clear that  $A \mapsto C_n(A)$  is an operator-valued measure on  $\mathcal{A}_n$  satisfying  $C_n(\Omega_n) = U(n, 0)$ . Indeed, by (4.3) and (4.4) we have

$$C_n(\Omega_n) = \sum_{\gamma \in \Omega_n} C_n(\gamma) = \sum_{\gamma \in \Omega_n} \langle e_{\gamma_n}, U(n, 0) e_{\gamma_0} \rangle |e_{\gamma_n}\rangle \langle e_{\gamma_0}| = U(n, 0)$$

The *decoherence functional*  $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$  is defined by

$$D_n(A, B) = \langle C_n(A)^* C_n(B) \psi, \psi \rangle$$

where  $\psi \in \mathbb{C}^m$ ,  $\|\psi\| = 1$ , is the initial state. It is clear that  $A \mapsto D_n(A, B)$  is a complex-valued measure on  $\mathcal{A}_n$  with  $D_n(\Omega_n, \Omega_n) = 1$ . It is well known that for any  $A_1, \dots, A_k \in \mathcal{A}_n$ ,  $D_n(A_i, A_j)$  is a positive semidefinite matrix [4, 11, 12].

The  $q$ -measure  $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$  is defined by  $\mu_n(A) = D_n(A, A)$ . It is well known that  $\mu_n$  indeed satisfies the grade-2 additivity condition required for a  $q$ -measure [4, 11, 12].

The  $n$ -distribution given by

$$p_n(i) = \mu_n(\{\gamma \in \Omega_n: \gamma_n = i\})$$

is interpreted as the probability that the system is at site  $i$  at time  $n$ . The next result shows that  $p_n(i)$  gives the usual quantum distribution.

**Theorem 4.3.** *For  $i = 0, 1, \dots, m$  we have*

$$p_n(i) = \left| \sum_{\gamma_n=i} a_\psi(\gamma) \right|^2 = |\langle e_i, U(n, 0)\psi \rangle|^2$$

*Proof.* Letting  $A = \{\gamma \in \Omega_n: \gamma_n = i\}$  we have by (4.5) that

$$\begin{aligned} p_n(i) &= D_n(A, A) = \langle C_n(A)^* C_n(A) \psi, \psi \rangle \\ &= \sum \{ \langle C(\gamma')^* C(\gamma) \psi, \psi \rangle : \gamma'_n = \gamma_n = i \} \\ &= \sum \{ \overline{b(\gamma')} b(\gamma) \overline{\psi(\gamma_0)} \psi(\gamma'_0) : \gamma'_n = \gamma_n = i \} \\ &= \left| \sum_{\gamma_n=i} b(\gamma) \psi(\gamma_0) \right|^2 = \left| \sum_{\gamma_n=i} a_\psi(\gamma) \right|^2 \end{aligned}$$

Applying (4.3) gives

$$\begin{aligned} \sum_{\gamma_n=i} b(\gamma) \psi(\gamma_0) &= \sum_{\gamma_0} \langle e_i, U(n, 0) e_{\gamma_0} \rangle \psi(\gamma_0) \\ &= \sum_{\gamma_0} \langle U(n, 0)^* e_i, e_{\gamma_0} \rangle \langle e_{\gamma_0}, \psi \rangle \\ &= \langle e_i, U(n, 0) \psi \rangle \end{aligned}$$

The result now follows. □

Corresponding to an initial state  $\psi \in \mathbb{C}^m$ ,  $\|\psi\| = 1$ , the  $n$ -decoherence matrix is  $D_n(\gamma, \gamma') = D_n(\{\gamma\}, \{\gamma'\})$ . We have by (4.5) that

$$\begin{aligned} D_n(\gamma, \gamma') &= \langle C_n(\gamma')^* C_n(\gamma) \psi, \psi \rangle = b(\gamma) \overline{b(\gamma')} \psi(\gamma_0) \overline{\psi(\gamma'_0)} \delta_{\gamma_n, \gamma'_n} \\ &= a_\psi(\gamma) \overline{a_\psi(\gamma')} \delta_{\gamma_n, \gamma'_n} \end{aligned} \quad (4.6)$$

Notice that

$$\mu_n(\gamma) = D_n(\gamma, \gamma) = |a_\psi(\gamma)|^2$$

and by Corollary 4.2,

$$\sum_{\gamma \in \Omega_n} \mu_n(\gamma) = 1$$

We also have

$$\mu_n(\{\gamma, \gamma'\}) = \mu_n(\gamma) + \mu_n(\gamma') + 2\operatorname{Re} D_n(\gamma, \gamma')$$

Finally, notice that

$$\begin{aligned} D_n(A, B) &= \sum \{D_n(\gamma, \gamma') : \gamma \in A, \gamma' \in B\} \\ &= \sum \{a_\psi(\gamma) \overline{a_\psi(\gamma')} \delta_{\gamma_n, \gamma'_n} : \gamma \in A, \gamma' \in B\} \end{aligned}$$

and hence,

$$\mu_n(A) = D_n(A, A) = \sum_{\gamma, \gamma' \in A} D_n(\gamma, \gamma') = \sum_{\gamma, \gamma' \in A} a_\psi(\gamma) \overline{a_\psi(\gamma')} \delta_{\gamma_n, \gamma'_n} \quad (4.7)$$

Define the  $n$ -path Hilbert space  $H_n = (\mathbb{C}^m)^{\otimes(n+1)}$ . We associate  $\gamma \in \Omega_n$  with the unit vector in  $H_n$  given by

$$e_{\gamma_n} \otimes e_{\gamma_{n-1}} \otimes \cdots \otimes e_{\gamma_0}$$

We can think of  $H_n$  as the set  $\{\phi: \Omega_n \rightarrow \mathbb{C}\}$  with the usual inner product. Then  $\gamma \in \Omega_n$  corresponds to  $\chi_{\{\gamma\}}$  and the matrix with components  $D_n(\gamma, \gamma')$  corresponds to the operator

$$\left(\widehat{D}_n \phi\right)(\gamma) = \sum_{\gamma'} D_n(\gamma, \gamma') \phi(\gamma')$$

**Theorem 4.4.** *The operator  $\widehat{D}_n$  is a state on  $H_n$ .*



*Proof.* It follows from (4.6) that  $\widehat{D}_n$  is a positive operator [7]. By Corollary 4.2 we have

$$\text{tr}(\widehat{D}_n) = \sum_{\gamma \in \Omega_n} D_n(\gamma, \gamma) = \sum_{\gamma \in \Omega_n} |a_\psi(\gamma)|^2 = 1$$

Hence,  $\widehat{D}_n$  is a trace 1 positive operator so  $\widehat{D}_n$  is a state on  $H_n$ .  $\square$

For every  $A \in \mathcal{A}_n$  we have the vector  $|\chi_A\rangle = \sum \{\gamma : \gamma \in A\}$ .

**Lemma 4.5.** *The decoherence functional satisfies*

$$D_n(A, B) = \text{tr}(|\chi_B\rangle\langle\chi_A|\widehat{D}_n)$$

*Proof.* For  $A, B \in \mathcal{A}_n$  we have

$$\begin{aligned} \text{tr}(|\chi_B\rangle\langle\chi_A|\widehat{D}_n) &= \sum_{\gamma \in \Omega_n} \langle |\chi_B\rangle\langle\chi_A|\widehat{D}_n\gamma, \gamma \rangle \\ &= \sum_{\gamma \in \Omega_n} \langle \widehat{D}_n\gamma, |\chi_A\rangle\langle\chi_B|\gamma \rangle = \sum_{\gamma \in B} \langle \widehat{D}_n\gamma, \chi_A \rangle \\ &= \sum \{ \langle \widehat{D}_n\gamma, \gamma' \rangle : \gamma \in B, \gamma' \in A \} \\ &= \sum \{ D_n(\gamma', \gamma) : \gamma' \in A, \gamma \in B \} = D_n(A, B) \quad \square \end{aligned}$$

The next result characterizes the eigenvalues and eigenvectors of the operator  $\widehat{D}_n$ . Let  $\gamma^0, \gamma^1, \dots, \gamma^{m-1} \in \Omega_n$  be the  $n$ -paths given by  $\gamma^i = 00 \cdots 0i$ ,  $i = 0, 1, \dots, m-1$ .

**Theorem 4.6.** *The nonzero eigenvalues of  $\widehat{D}_n$  have the form*

$$\lambda_i = \sum \{ \mu_n(\gamma) : \gamma_n = i \}$$

for  $i = 0, 1, \dots, m-1$  and corresponding eigenvectors  $u^i \in H_n$  have entries

$$u^i(\gamma) = \overline{a_\psi(\gamma^i)} a_\psi(\gamma) \delta_{i, \gamma_n}$$

*Proof.* For  $i = 0, 1, \dots, m - 1$  we have

$$\begin{aligned}
(\widehat{D}_n u^i)(\gamma) &= \sum_{\gamma' \in \Omega_n} D_n(\gamma, \gamma') u^i(\gamma') \\
&= \sum_{\gamma' \in \Omega_n} a_\psi(\gamma) \overline{a_\psi(\gamma')} \delta_{\gamma_n, \gamma'_n} \overline{a_\psi(\gamma^i)} a_\psi(\gamma) \delta_{i, \gamma'_n} \\
&= \overline{a_\psi(\gamma^i)} a_\psi(\gamma) \delta_{i, \gamma_n} \sum_{\gamma' \in \Omega_n} |a_\psi(\gamma')|^2 \delta_{i, \gamma'_n} \\
&= \left[ \sum \{ |a_\psi(\gamma)|^2 : \gamma_n = i \} \right] u^i(\gamma) \\
&= \left[ \sum \{ \mu_n(\gamma) : \gamma_n = i \} \right] u^i(\gamma)
\end{aligned}$$

This shows that  $u^i$ ,  $i = 0, 1, \dots, m - 1$  are eigenvectors of  $D_n$  with corresponding eigenvalues  $\lambda_i$ . By Corollary 4.2 and the fact that  $\mu_n(\Omega_n) = 1$  we have that  $\sum \lambda_i = 1 = \text{tr}(\widehat{D}_n)$ . Hence,  $\lambda_0, \dots, \lambda_{m-1}$  include all nonzero eigenvalues of  $\widehat{D}_n$ .  $\square$

Assuming that  $u^i \neq 0$  in Theorem 4.6,  $i = 0, \dots, m - 1$ , let  $v_i = u^i / \|u^i\|$  be the corresponding unit eigenvectors. We then have the spectral resolution  $\widehat{D}_n = \sum \lambda_i P_{v_i}$  where  $P_{v_i}$  is the projection onto the subspace spanned by  $v_i$ ,  $i = 0, 1, \dots, m - 1$ .

**Corollary 4.7.** *For the eigenvalues  $\lambda_i$  of  $\widehat{D}_n$  we have for every  $A \in \mathcal{A}_n$  that*

$$\mu_n(A) = \sum_{i=0}^{m-1} \lambda_i \left| \sum_{\gamma \in A} \langle \chi_{\{\gamma\}}, v_i \rangle \right|^2$$

*Proof.* By Lemma 4.5 and Theorem 4.6 we have

$$\begin{aligned}
\mu_n(A) &= \text{tr}(|\chi_A\rangle \langle \chi_A| \widehat{D}_n) = \sum_{i=0}^{m-1} \lambda_i \text{tr}(|\chi_A\rangle \langle \chi_A| P_{v_i}) \\
&= \sum_{i=0}^{m-1} \lambda_i |\langle \chi_A, v_i \rangle|^2 = \sum_{i=0}^{m-1} \lambda_i \left| \sum_{\gamma \in A} \langle \chi_{\{\gamma\}}, v_i \rangle \right|^2 \quad \square
\end{aligned}$$

## 5 Finite Unitary Processes

This section shows how a finite unitary system can be employed to construct a finite unitary  $q$ -process. As in Section 4, let  $S = \{0, 1, \dots, m-1\}$  be a set of sites and let  $\Omega = S \times S \times \dots$  be the set of all paths. Place the discrete topology on  $S$  and endow  $\Omega$  with the product topology. Then  $\Omega$  is a compact Hausdorff space. Let  $\mathcal{A}$  be the  $\sigma$ -algebra generated by the open sets in  $\Omega$  and let  $\mathcal{C}$  be the algebra of cylinder sets

$$A_0 \times A_1 \times \dots \times A_n \times S \times S \times \dots$$

$A_i \subseteq S$ ,  $i = 0, 1, \dots, n$ . Let  $p_0$  be the probability measure on  $S$  given by  $p_0(A) = |A|/m$ ,  $A \subseteq S$  and  $p_1$  be the probability measure on  $\mathcal{C}$  given by

$$p_1(A_0 \times A_1 \times \dots \times A_n \times S \times S \times \dots) = p_0(A_0)p_0(A_1) \dots p_0(A_n)$$

Then  $p_1$  is countably additive on  $\mathcal{C}$  so by the Kolmogorov extension theorem  $p_1$  has a unique extension to a probability measure  $\nu$  on  $\mathcal{A}$ . We define the *path Hilbert space* by  $H = L_2(\Omega, \mathcal{A}, \nu)$ .

**Example 1.** Let  $A \in \mathcal{A}$  be the event “the system visits the origin.” Then we have that

$$A = \{\omega \in \Omega: \omega = \omega_0\omega_1 \dots, \omega_i = 0 \text{ for some } i = 0, 1, \dots\}$$

Define  $B = \{1, 2, \dots, m-1\}$ ,  $B_1 = B \times S \times S \times \dots$ ,  $B_2 = B \times B \times S \times S \times \dots$ . Then  $B_i \in \mathcal{C}$ ,  $B_1 \supseteq B_2 \supseteq \dots$  and  $A' = \bigcap B_i \in \mathcal{A}$  is the event “the system never visits the origin.” Now for  $i = 1, 2, \dots$ , we have

$$\nu(B_i) = \left(\frac{m-1}{m}\right)^i$$

Hence,

$$\nu(A') = \nu(\bigcap B_i) = \lim \nu(B_i) = \lim \left(\frac{m-1}{m}\right)^i = 0$$

Hence,  $\nu(A) = 1$ . We will show later that the  $q$ -measure of  $A$  is also 1.

Let  $U(s, r)$ ,  $r \leq s \in \mathbb{N}$  be a finite unitary system on  $\mathcal{C}^m$  and as in Section 4, let  $H_n = (\mathcal{C}^m)^{\otimes(n+1)}$  be the  $n$ -path space. According to Theorem 4.4, the operators  $\widehat{D}_n$  are states on  $H_n$ ,  $n = 1, 2, \dots$ , where we identify  $H_n$  with  $\{\phi: \Omega_n \rightarrow \mathbb{C}\}$ . Then  $\{\chi_{\{\gamma\}}: \gamma \in \Omega_n\}$  becomes an orthonormal basis for  $H_n$ .

For  $\gamma = \gamma_0\gamma_1\cdots\gamma_n \in \Omega_n$  define the cylinder set  $\text{cyl}(\gamma)$  as the subset of  $\Omega$  given by

$$\text{cyl}(\gamma) = \{\gamma_0\} \times \{\gamma_1\} \times \cdots \times \{\gamma_n\} \times S \times S \times \cdots$$

Then

$$\widehat{\gamma} = m^{(n+1)/2} \chi_{\text{cyl}(\gamma)}$$

is a unit vector in  $H$  and we define  $U_n \chi_{\{\gamma\}} = \widehat{\gamma}$ . Extending  $U_n$  by linearity,  $U_n: H_n \rightarrow H$  becomes a unitary operator from  $H_n$  into  $H$ . Letting  $P_n$  be the projection of  $H$  onto the subspace  $U_n H_n$  we have

$$P_n f = \sum_{\widehat{\gamma}} \langle \widehat{\gamma}, f \rangle \widehat{\gamma} = m^{(n+1)} \sum_{\gamma \in \Omega_n} \int f \chi_{\text{cyl}(\gamma)} d\nu \chi_{\text{cyl}(\gamma)}$$

In particular, for  $A \in \mathcal{A}$  we have

$$P_n \chi_A = m^{(n+1)} \sum_{\gamma \in \Omega_n} \nu[A \cap \text{cyl}(\gamma)] \chi_{\text{cyl}(\gamma)}$$

Hence,

$$P_n 1 = \sum_{\gamma \in \Omega_n} \chi_{\text{cyl}(\gamma)} = 1$$

It is also clear that  $\rho_n = U_n \widehat{D}_n U_n^* P_n$  is a state on  $H$  and also on  $U_n H_n$ .

Let  $\mathcal{A}_t$  be the algebra of all time- $t$  cylinder sets

$$A = A_0 \times \cdots \times A_t \times S \times S \times \cdots$$

where  $A_i \subseteq S$ ,  $i = 0, 1, \dots, t$ . Then  $\mathcal{A}_t \subseteq \mathcal{A}$ ,  $t = 0, 1, \dots$ , is an increasing sequence of  $\sigma$ -algebras generating  $\mathcal{A}$ . As in Section 3, let  $\nu_t$  be the restriction of  $\nu$  to  $\mathcal{A}_t$ ,  $t = 0, 1, \dots$ . Then  $U_t H_t$  is isomorphic to  $L_2(\Omega, \mathcal{A}_t, \nu_t)$ ,  $t = 0, 1, \dots$ , and forms an increasing sequence of subspaces on  $H$ .

**Theorem 5.1.** *The sequence of states  $\rho_t$ ,  $t = 0, 1, \dots$ , is consistent.*

*Proof.* Let  $D_n$  be the decoherence functional on  $H_n$  given by  $D_{\rho_n}$ . To show that  $\rho_t$  is consistent we must show that

$$D_{n+1}(A \times S, B \times S) = D_n(A, B) \quad (5.1)$$

for every  $A, B \subseteq \Omega_n$ . Using the notation  $\gamma j = \gamma_0\gamma_1\cdots\gamma_n j$ , (5.1) is equivalent to

$$D_n(\gamma, \gamma') = \sum_{j,k=0}^{m-1} D_{n+1}(\gamma j, \gamma' k) = \sum_{j=0}^{m-1} D_n(\gamma j, \gamma' j) \quad (5.2)$$

for all  $\gamma, \gamma' \in \Omega_n$ . Since

$$\sum_j \langle U(n+1, n)e_{\gamma'_n}, e_j \rangle \langle e_n, U(n+1, n)e_{\gamma_n} \rangle = \delta_{\gamma_n, \gamma'_n}$$

it follows that

$$\sum_j b(\gamma j) \overline{b(\gamma' j)} = b(\gamma) \overline{b(\gamma')} \delta_{\gamma_n, \gamma'_n}$$

for all  $\gamma, \gamma' \in \Omega_n$ . Hence,

$$\begin{aligned} \sum_j D_{n+1}(\gamma j, \gamma' j) &= \sum_j a_\psi(\gamma j) \overline{a_\psi(\gamma' j)} = \sum_j b(\gamma j) \overline{b(\gamma' j)} \psi(\gamma_0) \overline{\psi(\gamma'_0)} \\ &= b(\gamma) \overline{b(\gamma')} \psi(\gamma_0) \overline{\psi(\gamma'_0)} = a_\psi \overline{a_\psi(\gamma')} \delta_{\gamma_n, \gamma'_n} \\ &= D_n(\gamma, \gamma') \end{aligned}$$

so (5.2) holds.  $\square$

We call the consistent sequence  $\rho_t$ ,  $t = 0, 1, \dots$ , a *finite unitary process*. It follows from Theorem 3.1 that the quadratic algebra  $\mathcal{S}$  of suitable sets contains  $\mathcal{C}$  and  $\tilde{\mu}$  is a  $q$ -measure on  $\mathcal{S}$  that extends the natural  $q$ -measure  $\mu$  on  $\mathcal{C}$ . We now consider some sets in  $\mathcal{S} \setminus \mathcal{C}$ .

If  $\gamma = \gamma_0 \gamma_1 \dots \in \Omega$ , letting  $A_n = \text{cyl}(\gamma_0 \gamma_1 \dots \gamma_n)$  we have that  $\{\gamma\} = \bigcap A_n$ . Hence,

$$\nu(\{\gamma\}) = \lim_{n \rightarrow \infty} \nu(A_n) = \lim_{n \rightarrow \infty} \frac{1}{m^{(n+1)}} = 0$$

It follows that  $\chi_{\{\gamma\}} = 0$  as an element of  $H$  so

$$\lim \langle e_n \chi_{\{\gamma\}}, \chi_{\{\gamma\}} \rangle = 0$$

We conclude that  $\{\gamma\} \in \mathcal{S}$  and  $\tilde{\mu}(\{\gamma\}) = 0$ . Since  $\{\gamma\} \notin \mathcal{C}$  we have that  $\mathcal{S}$  is a proper extension of  $\mathcal{C}$ . More generally, if  $B$  is a countable subset of  $\Omega$ , then  $B \in \mathcal{A}$  and  $\nu(B) = 0$ . Hence,  $B \in \mathcal{S}$  and  $\tilde{\mu}(B) = 0$ . Moreover, since  $\nu(B') = 1$ ,  $\chi_{B'} = 1$  a.e. $[\nu]$ . Since  $\mu_n(\Omega_n) = 1$  we have that

$$\begin{aligned} \langle e_n \chi_{B'}, \chi_{B'} \rangle &= \langle e_n 1, 1 \rangle = \langle U_n \widehat{D}_n U_n^* P_n 1, 1 \rangle \\ &= \langle \widehat{D}_n U_n^* 1, U_n 1 \rangle = \langle \widehat{D}_n \chi_{\Omega_n}, \chi_{\Omega_n} \rangle \\ &= \mu_n(\Omega_n) = 1 \end{aligned}$$

Hence,  $B' \in \mathcal{S}$  and  $\tilde{\mu}(B') = 1$ .

Let  $A \in \mathcal{A}$  be the event “the system visits the origin” of Example 1. Then  $A'$  is the event “the system never visits the origin” and we showed in Example 1 that  $\nu(A') = 0$ . Hence  $A' \in \mathcal{S} \setminus \mathcal{C}$  and  $\tilde{\mu}(A') = 0$ . Thus, the  $q$ -propensity that the system never visits the origin is 0. Since  $\nu(A) = 1$ ,  $\chi_A = 1$  a.e. $[\nu]$ . As before we have that  $A \in \mathcal{S}$  and  $\tilde{\mu}(A) = 1$ . Of course, the origin can be replaced by any of the sites  $1, 2, \dots, m-1$ .

As another example, let  $B_t$  be the event “the system visits site  $j$  for the first time at  $t$ .” Letting

$$C = \{0, 1, \dots, j-1, j+1, \dots, m-1\}$$

we see that  $B_t \in \mathcal{C}$  and

$$B_t = C \times C \times \dots \times C \times \{j\} \times S \times S \times \dots$$

where there are  $t$  factors of  $C$ . We have that

$$\nu(B_t) = \left(\frac{m-1}{m}\right)^t \frac{1}{m} = \frac{(m-1)^t}{m^{t+1}}$$

Since  $B_t \in \mathcal{C}$  we have that

$$\tilde{\mu}(B_t) = \mu_t(B_t) = \left\langle \widehat{D}_t \chi_{B_t}, \chi_{B_t} \right\rangle$$

which depends on  $U$ .

We close this section with an example of a two-site quantum random walk or a two-hopper [7, 13]. The unitary operator

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

generates a stationary unitary system on  $\mathbb{C}^2$ . We think of this system as a particle that is located at one of the two-sites  $S = \{0, 1\}$ . We assume that the initial state is  $e_0$  so the particle always begins at site 0. In this case, we can let  $\Omega = \{0\} \times S \times S \times \dots$  and we obtain a finite unitary process on  $H = L_2(\Omega, \mathcal{A}, \nu)$ . Let  $\mu_n$  be the  $q$ -measure on  $\mathcal{A}_n = 2^{\Omega_n}$  given by  $\mu_n(A) = \left\langle \widehat{D}_n \chi_A, \chi_A \right\rangle$ . Let  $C_t$  be the event “the particle is at site 1 at time  $t$ .” Then  $C_t \in \mathcal{C}$  and we have

$$C_t = \{0\} \times S \times \dots \times S \times \{1\} \times S \times S \dots$$

Letting  $E_t = \{0\} \times S \times \cdots \times S \times \{1\}$  we have that  $E_t \in \mathcal{A}_t$  and  $\tilde{\mu}(C_t) = \mu_t(E_t)$ . We now compute  $\mu_t(E_t)$  for  $t = 1, 2, \dots$ .

Of course,  $\nu_t(E_t) = 1/2$  for  $t = 1, 2, \dots$ , which is the usual classical result. As we shall see, the quantum result is somewhat surprising. This result suggests a periodic motion with period 4. To compute  $\mu_t(E_t)$  we shall employ Corollary 4.7. We see from the form of  $U$  that  $\mu_n(\{0\}) = 1/2^n$  for every  $\gamma \in \Omega_n$ . Hence, by Theorem 4.6,  $\widehat{D}_n$  has eigenvalues  $1/2$  with multiplicity 2 and eigenvalues 0 with multiplicity  $2^n - 2$ . For  $\gamma \in \Omega_n$ ,  $\gamma$  is the binary representation for a unique integer  $a \in \{0, 1, \dots, 2^n - 1\}$ . We then identify  $\Omega_n$  with  $\{0, 1, \dots, 2^n - 1\}$ . Let  $c(\gamma)$  be the number of position changes (bit flips) in  $\gamma$ . Applying Theorem 4.7 of [7], the unit eigenvectors corresponding to  $1/2$  are  $\psi_0^n, \psi_1^n$  given by

$$\psi_0^n = \frac{1}{2^{(n-1)/2}} \begin{bmatrix} i^{c_n(0)} \\ 0 \\ i^{c_n(2)} \\ 0 \\ \vdots \\ i^{c_n(2^n-2)} \\ 0 \end{bmatrix}, \quad \psi_1^n = \frac{1}{2^{(n-1)/2}} \begin{bmatrix} 0 \\ i^{c_n(1)} \\ 0 \\ i^{c_n(3)} \\ 0 \\ \vdots \\ 0 \\ i^{c_n(2^n-1)} \end{bmatrix}$$

In order to compute  $\psi_0^n, \psi_1^n$  the following lemma is useful.

**Lemma 5.2.** [7]. *For  $n \in \mathbb{N}$ ,  $j = 0, 1, \dots, 2^n - 1$ , the function  $c_n(j)$  satisfies*

$$c_{n+1}(2^{n+1} - 1 - j) = c_n(j) + 1$$

We employ the vector notation  $c_n = (c_n(0), c_n(1), \dots, c_n(2^n - 1))$ . Applying Lemma 5.2, since  $c_0 = (0)$  it follows that  $c_1 = (0, 1)$ ,  $c_2 = (0, 1, 2, 1)$  and  $c_3 = (0, 1, 2, 1, 2, 3, 2, 1)$ . Hence,

$$\psi_0^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_1^1 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

$$\psi_0^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \psi_1^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 0 \\ i \end{bmatrix}$$

$$\psi_0^3 = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \psi_0^3 = \frac{1}{2} \begin{bmatrix} 0 \\ i \\ 0 \\ i \\ 0 \\ -i \end{bmatrix},$$

Applying Corollary 4.7 we have that  $\mu_0(E_0) = 0$  and

$$\begin{aligned} \mu_1(E_1) &= \mu_1(\{1\}) = \frac{1}{2} |\langle \chi_{\{1\}}, \psi_0^1 \rangle|^2 + \frac{1}{2} |\langle \chi_{\{1\}}, \psi_1^1 \rangle|^2 \\ &= 1/2 \\ \mu_2(E_2) &= \mu_2(\{1, 3\}) = \frac{1}{2} |\langle \chi_{\{1\}}, \psi_0^2 \rangle + \langle \chi_{\{3\}}, \psi_0^2 \rangle|^2 \\ &\quad + \frac{1}{2} |\langle \chi_{\{1\}}, \psi_1^2 \rangle + \langle \chi_{\{3\}}, \psi_1^2 \rangle|^2 \\ &= \frac{1}{2} \left| \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} i \right|^2 = 1 \\ \mu_3(E_3) &= \mu_3(\{1, 3, 5, 7\}) = \frac{1}{2} \left| \frac{1}{2} i + \frac{1}{2} i - \frac{1}{2} i + \frac{1}{2} i \right|^2 = 1/2 \end{aligned}$$

In general, we have

$$\begin{aligned} \mu_t(E_t) &= \mu_t(\{1, 3, \dots, 2^t - 1\}) = \frac{1}{2} \left| \sum_{j=1}^{2^t-1} \langle \chi_{\{j\}}, \psi_1^t \rangle \right|^2 \\ &= \frac{1}{2^t} \left| \sum_{\substack{j=1 \\ j \text{ odd}}}^{2^t-1} i^{c_t(j)} \right|^2 \end{aligned}$$

Letting

$$F(t) = \sum_{\substack{j=0 \\ j \text{ even}}}^{2^t-2} i^{c_t(j)}, \quad G(t) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{2^t-1} i^{c_t(j)}$$

we have

$$\mu_t(E_t) = \frac{1}{2^t} |G(t)|^2 \tag{5.3}$$

Notice we have shown that  $G(0) = 0$ ,  $G(1) = i$ ,  $G(2) = G(3) = 2i$ .



**Lemma 5.3.** For  $j = 0, 1, 2, 3$  and  $m \in \mathbb{N}$  we have  $G(4m + j) = (-4)^m G(j)$ .

*Proof.* Applying Lemma 5.2 we have that  $G(t) = G(t - 1) + iF(t - 1)$  and  $F(t) = F(t - 1) + iG(t - 1)$ . Hence,

$$\begin{aligned} G(t) &= G(t - 1) + i[F(t - 2) + iG(t - 2)] = G(t - 1) - G(t - 2) + iF(t - 2) \\ &= G(t - 1) - G(t - 2) + i[F(t - 3) + iG(t - 3)] \\ &= G(t - 1) - G(t - 2) - G(t - 3) + iF(t - 3) \end{aligned}$$

Continuing this process we have

$$\begin{aligned} G(t) &= G(t - 1) - G(t - 2) - \cdots - G(1) + iF(1) \\ &= G(t - 1) - G(t - 2) - \cdots - G(1) + i \end{aligned}$$

Hence,

$$\begin{aligned} G(t) &= G(t - 2) - G(t - 3) - \cdots - G(1) + i \\ &\quad - G(t - 2) - G(t - 3) - \cdots - G(i) + i \\ &= -2[G(t - 3) - G(t - 4) - \cdots - G(1) + i] \\ &= -2[G(t - 4) - G(t - 5) - \cdots - G(1) + i] \\ &\quad - 2[G(t - 4) + G(t - 5) + \cdots + G(1) - i] \\ &= -4G(t - 4) \end{aligned}$$

Letting  $t = 4m + j$ ,  $j = 0, 1, 2, 3$  we have

$$\begin{aligned} G(t) &= -4G(4(m - 1) + j) = -4[-4G(4(m - 2) + j)] \\ &= (-4)^2 G(4(m - 2) + j) \end{aligned}$$

Continuing this process gives our result □

Applying (5.3) and Lemma 5.3 we have for  $t = 4m + j$ ,  $j = 0, 1, 2, 3$  and  $m \in \mathbb{N}$  that

$$\mu_t(E_t) = \frac{1}{2^{4m+j}} 4^{2m} |G(j)|^2 = \frac{1}{2^j} |G(j)|^2$$

This shows that  $\mu_t(E_t)$  is periodic with period 4. The first few values of  $\mu_t(E_t)$  are  $0, 1/2, 1, 1/2, 0, 1/2, 1, 1/2, 0, \dots$

Let  $F_t$  be the event “the particle is at site 0 at time  $t$ .” Then  $F_t \in \mathcal{C}$  and we have

$$F_t = \{0\} \times S \times \cdots \times S \times \{0\} \times S \times S \cdots$$

Again, setting

$$G_t = \{0\} \times S \times \cdots \times S \times \{0\}$$

we have that  $G_t \in \mathcal{A}_t$  and  $\tilde{\mu}(F_t) = \mu_t(G_t)$ . Of course,  $G_t = E'_t$  and we have computed  $\mu_t(E_t)$ . It does not immediately follow that  $\mu_t(G_t) = 1 - \mu_t(E_t)$  because  $\mu_t$  is not additive. However, as in (5.3) we have that  $\mu_t(G_t) = 2^{-t} |F(t)|^2$  where  $F(0) = 1$ ,  $F(1) = 1$ ,  $F(2) = 0$ ,  $F(3) = -2$ . As in Lemma 5.3, for  $j = 0, 1, 2, 3$  and  $m \in \mathbb{N}$  we have that  $F(4m + j) = (-4)^m F(j)$ . We conclude that in this case,  $\mu_t(G_t) = 1 - \mu_t(E_t)$ . Thus,  $\mu_t(G_t)$  is periodic with period 4. The first few values of  $\mu_t(G_t)$  are  $1, 1/2, 0, 1/2, 1, 1/2, 0, 1/2, \dots$ .

Finally, let  $f_t$  be the random variable that gives the position of the particle at time  $t$ . Thus, for  $\gamma \in \Omega$ ,  $\gamma = \gamma_0 \gamma_1 \cdots$ ,  $f_t(\gamma) = \gamma_t$ . Then  $f = \chi_{C_t}$  and we have

$$\int f_t d\tilde{\mu} = \tilde{\mu}(C_t) = \mu_t(E_t) = \frac{1}{2^t} |G(t)|^2$$

which we have already computed.

## 6 Quantum Integrals

Let  $\rho_t$ ,  $t = 0, 1, 2, \dots$ , be a discrete  $q$ -process on  $H = L_2(\Omega, \mathcal{A}, \nu)$ . As in Section 3, a random variable  $f \in H$  is integrable if  $\lim(\rho_t \hat{f})$  exists and is finite, in which case  $\int f d\tilde{\mu}$  is this limit. If a global state  $\rho$  exists, it is also of interest to compute the integral  $\int f d\mu_\rho = \text{tr}(\rho \hat{f})$ . The operator  $\hat{f}$  can be complicated and the expression  $\text{tr}(\rho \hat{f})$  difficult to evaluate. This section considers the case in which  $f$  is a simple function. A general random variable can be treated using Corollary 2.6(d). We first find the eigenvectors and eigenvalues for two-valued random variables.

If  $f$  has the form  $f = \alpha \chi_A$ ,  $\alpha \in \mathbb{R}$ , then  $\hat{f} = \alpha \hat{\chi}_A$  and

$$\int f d\mu_t = \text{tr}(\rho \hat{f}) = \alpha \langle \rho \chi_A, \chi_A \rangle$$

The next result treats nonnegative random variables with two nonzero values

**Theorem 6.1.** *If  $f = \alpha \chi_A + \gamma \chi_B$  where  $A \cap B = \emptyset$ ,  $\nu(A)\nu(B) \neq 0$  and  $0 < \alpha < \beta$  then  $\hat{f}$  has two nonzero eigenvalues*

$$\lambda_{\pm} = \frac{\alpha\nu(A) + \gamma\nu(B) \pm \sqrt{[\alpha\nu(A) - \gamma\nu(B)]^2 + 2\alpha^2\nu(A)\nu(B)}}{2\nu(B)}$$

with corresponding eigenvectors

$$g_{\pm} = \chi_A + b_{\pm}\chi_B$$

where  $b_{\pm} = [\lambda_{\pm} - \nu(A)] / \nu(B)$ .

*Proof.* We first treat the case in which  $f$  has the form

$$f = \chi_A + \beta\chi_B$$

where  $A \cap B = \emptyset$ ,  $\nu(A)\nu(B) \neq 0$  and  $\beta > 1$ . For  $g \in H$  we have

$$\begin{aligned} \widehat{f}g(x) &= \int \min(f(x), f(y)) g(y) d\nu(y) \\ &= \int_{\{y: f(y) \geq f(x)\}} f(x)g(y) d\nu(y) + \int_{\{y: f(y) < f(x)\}} f(y)g(y) d\nu(y) \end{aligned} \quad (6.1)$$

Letting  $C = (A \cup B)' = A' \cap B'$  it follows from (6.1) that:

If  $x \in C$ , then  $(\widehat{f}g)(x) = 0$

If  $x \in A$ , then  $(\widehat{f}g)(x) = \int_{A \cup B} g(y) d\nu(y)$

If  $x \in B$  then  $(\widehat{f}g)(x) = \beta \int_B g(x) d\nu(y) + \int_A g(y) d\nu(y)$

We conclude that

$$(\widehat{f}g)(x) = \int_{A \cup B} g(y) d\nu(y) \chi_A + \left[ \beta \int_B g(y) d\nu(y) + \int_A g(y) d\nu(y) \right] \chi_B \quad (6.2)$$

If  $g \perp \chi_A$  and  $g \perp \chi_B$  then  $\widehat{f}g = 0$ . Thus,

$$\text{Range}(\widehat{f}) = \text{span} \{ \chi_A, \chi_B \}$$

It follows that eigenvectors corresponding to nonzero eigenvalues have the form  $g = a\chi_A + b\chi_B$ ,  $a, b \in \mathbb{C}$ . Assuming that  $g$  has this form and  $\widehat{f}g = \lambda g$ , applying (6.2) gives

$$\begin{aligned} (\widehat{f}g)(x) &= [a\nu(A) + b\nu(B)] \chi_A + [\beta b\nu(B) + a\nu(A)] \chi_B \\ &= a\lambda\chi_A + b\lambda\chi_B \end{aligned}$$

Hence,

$$a\lambda = a\nu(A) + b\nu(B), \quad b\lambda = \beta b\nu(B) + a\nu(A)$$

Notice that  $a \neq 0$  since otherwise  $b = 0$  which contradicts  $g \neq 0$ . We can therefore assume that  $a = 1$ . Then  $b = [\lambda - \nu(A)] / \nu(B)$  and  $b[\lambda - \beta\nu(B)] = \nu(A)$ . Eliminating  $b$  from these two equations gives

$$\lambda^2 - [\nu(A) + \beta\nu(B)]\lambda + (\beta - 1)\nu(A)\nu(B) = 0$$

Applying the quadratic formula we have the two solutions

$$\lambda_{\pm} = \frac{\nu(A) + \beta\nu(B) \pm \sqrt{[\nu(A) - \beta\nu(B)]^2 + 2\nu(A)\nu(B)}}{2\nu(B)}$$

and corresponding eigenvectors are

$$g_{\pm} = \chi_A + b_{\pm}\chi_B$$

where  $b_{\pm} = \frac{\lambda_{\pm} - \nu(A)}{\nu(B)}$ . Since the original  $f$  satisfies

$$f = \alpha \left[ \chi_A + \frac{\gamma}{\alpha} \chi_B \right]$$

letting  $\beta = \gamma/\alpha$  and multiplying  $\lambda_{\pm}$  by  $\alpha$  gives the result.  $\square$

The eigenvectors  $g_{\pm}$  need not be normalized but this can easily be done. Theorem 6.1 treats random variables with two positive values. The remaining case for two-valued random variables is one of the form  $f = \alpha\chi_A + \beta\chi_B$  where  $A \cap B = \emptyset$ ,  $\nu(A)\nu(B) \neq 0$  and  $\alpha > 0$ ,  $\beta < 0$ . This case is easy to treat because we can write  $f = \alpha\chi_A - (-\beta)\chi_B$  so that

$$\widehat{f} = \alpha\widehat{\chi}_A - (-\beta)\widehat{\chi}_B = \alpha|\chi_A\rangle\langle\chi_A| + \beta|\chi_B\rangle\langle\chi_B|$$

Hence, the nonzero eigenvalues of  $\widehat{f}$  are  $\alpha\nu(A)$ ,  $\beta\nu(B)$  with corresponding unit eigenvectors  $\nu(A)^{-1/2}\chi_A$ ,  $\nu(B)^{-1/2}\chi_B$ . We can thus find the eigenvalues  $\lambda_i$  and normalized eigenvectors  $v_i$ ,  $i = 1, 2$ , for an arbitrary two-valued random variable  $f$ . If  $\rho$  is a state and  $\mu_{\rho}$  the corresponding  $q$ -measure, we have

$$\begin{aligned} \int f d\mu_{\rho} &= \text{tr}(\rho\widehat{f}) = \langle \rho\widehat{f}v_1, v_1 \rangle + \langle \rho\widehat{f}v_2, v_2 \rangle \\ &= \lambda_1 \langle \rho v_1, v_1 \rangle + \lambda_2 \langle \rho v_2, v_2 \rangle \end{aligned}$$

Now suppose  $f = \alpha\chi_A + \beta\chi_B + \gamma\chi_C$  is a three-valued random variable where  $\alpha, \beta, \gamma \in \mathbb{R}$  are distinct and  $A, B, C$  are mutually disjoint. By Theorem 2.5(e) we have

$$\begin{aligned}\widehat{f} &= (\alpha\chi_A + \beta\chi_B + \gamma\chi_C)^\wedge \\ &= (\alpha\chi_A + \beta\chi_B)^\wedge + (\alpha\chi_A + \gamma\chi_C)^\wedge + (\beta\chi_B + \gamma\chi_C)^\wedge - \alpha\widehat{\chi}_A - \beta\widehat{\chi}_B - \gamma\widehat{\chi}_C\end{aligned}\tag{6.3}$$

The right side of (6.3) contains the quantization operators for three two-valued random variables. Letting  $\lambda_1^i, \lambda_2^i$  be the eigenvalues of the  $i$ th operator with corresponding unit eigenvectors  $v_1^i, v_2^i$ ,  $i = 1, 2, 3$  we have

$$\begin{aligned}\int f d\mu_\rho &= \text{tr}(\rho\widehat{f}) \\ &= \sum_{i=1}^3 \sum_{j=1}^2 \lambda_j^i \langle \rho v_j^i, v_j^i \rangle - \alpha \langle \rho\chi_A, \chi_A \rangle - \beta \langle \rho\chi_B, \chi_B \rangle - \gamma \langle \rho\chi_C, \chi_C \rangle\end{aligned}$$

Continuing by induction we have for  $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$  that

$$\widehat{f} = \sum_{i < j=1}^n (\alpha_i \chi_{A_i} + \alpha_j \chi_{A_j})^\wedge - (n-1) \sum_{i=1}^n \alpha_i \widehat{\chi}_{A_i}$$

Using a similar notation as before gives

$$\begin{aligned}\int f d\mu_\rho &= \text{tr}(\rho\widehat{f}) \\ &= \sum_{i=1}^{n(n-1)/2} \sum_{j=1}^2 \lambda_j^i \langle \rho v_j^i, v_j^i \rangle - (n-1) \sum_{i=1}^n \alpha_i \langle \rho\chi_{A_i}, \chi_{A_i} \rangle\end{aligned}$$

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