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$\mathbb{Z}_2$-Orbifolds of Affine Vertex Algebras and $\mathcal{W}$-Algebras

A Dissertation

Presented to

the Faculty of Natural Sciences and Mathematics

University of Denver

in Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Masoumah Al-Ali

June 2017

Advisor: Andrew Linshaw

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Abstract

Vertex algebras arose in conformal field theory and were first defined axiomatically by Borcherds in his famous proof of the Moonshine Conjecture in 1986. The orbifold construction is a standard way to construct new vertex algebras from old ones. Starting with a vertex algebra $\mathcal{V}$ and a group G of automorphisms, one considers the invariant subalgebra $\mathcal{V}^G$ (called G-orbifold of $\mathcal{V}$), and its extensions. For example, the Moonshine vertex algebra arises as an extension of the $\mathbb{Z}_2$-orbifold of the lattice vertex algebra associated to the Leech lattice.

In this thesis we consider two problems. First, given a simple, finite-dimensional Lie algebra $\mathfrak{g}$, there is an involution on $\mathfrak{g}$ called the *Cartan involution*, which lifts to a $\mathbb{Z}_2$-action on the universal affine vertex algebra $V^k(\mathfrak{g})$ at level k. For any $\mathfrak{g}$, we shall find an explicit minimal strong generating set for the orbifold $V^k(\mathfrak{g})^{\mathbb{Z}_2}$, for generic values of k. Let $l = \mathrm{rank}(\mathfrak{g})$ and let m be the number of positive roots of $\mathfrak{g}$, so that $\dim(\mathfrak{g}) = 2m + l$. We will prove that for $\mathfrak{g} \neq \mathfrak{s}l_2$, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is of type

$$\mathcal{W}\big(1^m, 2^{d+\binom{d}{2}}, 3^{\binom{d}{2}}, 4\big), \qquad d = m + l,$$

for generic values of k. In this notation, a vertex algebra is said to be of type $\mathcal{W}((d_1)^{n_1}, \dots (d_r)^{n_r})$ if it has a minimal strong generating set consisting of n_i fields in weight d_i, for $i = 1, \dots, r$. In the case $\mathfrak{g} = \mathfrak{s}l_2$, there is one extra field in weight 4, so that $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is of type $\mathcal{W}(1, 2^3, 3, 4^2)$ for generic value of k. In the case $\mathfrak{g} = \mathfrak{s}l_2$, we explicitly determine the set of *nongeneric* values of k where this set

does not strongly generate the orbifold; it consists only of $\{0, \pm\frac{32}{3}, 16, 48\}$. Second, we consider the $\mathbb{Z}_2$-orbifold of the Zamolodchikov $\mathcal{W}_3$-algebra with central charge c, which we denote by $\mathcal{W}_3^c$. It was conjectured over 20 years ago in the physics literature that $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$ should be of type $\mathcal{W}(2,4,6,8,10)$ for generic values of c. We prove this conjecture for all values of $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$, and we show that for these two values of c, $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2,6,8,10,12,14)$. The method introduced to study $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$ involves ideas from algebraic geometry and is applicable to a broad range of problems of this kind.

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Chapter 1

Introduction

Vertex algebra has its roots in physics literature including string theory and conformal field theory. More recently, this notion attracted the mathematicians and so many applications have appeared including finite group theory and representations of infinite dimensional Lie algebras. The notion of vertex algebra first was introduced by Borcherds in 1986 in his astonishing proof of the Moonshine Conjecture. Thereby, it has been developed emphasizing beautifully the overlapping between physics and mathematics [B, FLM, K].

Given a vertex algebra $\mathcal{V}$ and a group G of automorphisms of $\mathcal{V}$, the invariant subalgebra $\mathcal{V}^G$ is called a *G-orbifold* of $\mathcal{V}$. Many interesting vertex algebras can be constructed either as orbifolds or as extensions of orbifolds. A spectacular example is the Moonshine vertex algebra $V^\natural$, which is an extension of the $\mathbb{Z}_2$-orbifold of the lattice vertex algebra associated to the Leech lattice [B, FLM]. Its full automorphism group is the Monster finite simple group of order

$$2^{45}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71 \approx 8.10^{53},$$

and its graded character is the modular invariant j-function from number theory up to a constant. There is a substantial literature on the structure and representation theory of orbifolds under finite group actions; see for example [DVVV, DHVW, DM, DLMI, DLMII, DRX]. It is widely believed that nice properties of $\mathcal{V}$ such as C_2-cofiniteness and rationality will be inherited by $\mathcal{V}^G$ when G is finite. In the case where G is cyclic, the C_2-cofiniteness of $\mathcal{V}^G$ was proven by Miyamoto in [M], and the rationality was recently established by Carnahan and Miyamoto in [CM].

Many vertex algebras depend continuously on a parameter k. Examples include the universal affine vertex algebra $V^k(\mathfrak{g})$ associated to a simple, finite-dimensional Lie algebra $\mathfrak{g}$, and the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ associated to $\mathfrak{g}$ together with a nilpotent element $f \in \mathfrak{g}$. Typically, if $\mathcal{V}^k$ is such a vertex algebra depending on k, it is simple for generic values of k but has a nontrivial maximal proper ideal $\mathcal{I}_k$ for special values. Often, one is interested in the structure and representation theory of the simple quotient $\mathcal{V}^k/\mathcal{I}_k$ at these points. For example, the C_2-cofiniteness and rationality of simple affine vertex algebras at positive integer level was proven by Frenkel and Zhu in [FZ], and the C_2-cofiniteness and rationality of several families of $\mathcal{W}$-algebras is due to Arakawa [A].

A vertex algebra $\mathcal{V}$ is called a strongly finitely generated (shortly SFG) if there is a finite set of generators, such that the collection of iterated Wick products of the generators and their derivatives spans $\mathcal{V}$. Many well-known vertex algebras have this property, and finding a minimal strong finite generating set for a vertex algebra is a very useful step towards understanding its structure and representation theory. Since a strong generating set for a vertex algebra gives rise to a generating set for Zhu's associative algebra $\mathrm{Zhu}(\mathcal{V})$ [Zh], the SFG property implies that $\mathrm{Zhu}(\mathcal{V})$ is finitely generated. This is important because the irreducible, positive energy representations of $\mathcal{V}$ are in one-to-one correspondence with the irreducible modules over $\mathrm{Zhu}(\mathcal{V})$. Also, the SFG property of $\mathcal{V}$ is equivalent to the finite generation of

Zhu's commutative algebra, which implies that the associated variety of $\mathcal{V}$ is if finite type.

In general, it is a difficult problem to determine if a vertex algebra $\mathcal{V}$ has the SFG property. In a series of papers [LI, LII, LIII, LIV, LV, CLII], Linshaw has established that if $\mathcal{V}$ is a free field algebra (i.e., either a Heisenberg algebra, a $\beta\gamma$-system, a free fermion algebra, or a symplectic fermion algebra of finite rank), and G is a reductive group of automorphisms of $\mathcal{V}$, the orbifold $\mathcal{V}^G$ has the SFG property. This also holds if $\mathcal{V}$ is an affine vertex algebra $V^k(\mathfrak{g})$ for any Lie (super)algebra $\mathfrak{g}$ equipped with a nondegenerate supersymmetric bilinear form, for generic values of k [LIV, CLIII]. However, there are very few nontrivial examples where an explicit minimal strong generating set can be written down. Also, for vertex algebras like $V^k(\mathfrak{g})$ which depend on a parameter k, it is important to determine the *nongeneric* set, where the strong generating set does not work. By a result of [CLIII], it is known that this set contains at most the poles of the structure constants appearing in the operator product expansions (OPEs) of the generators, and in particular is a finite set. Unfortunately there are few examples where it is practical to work this out explicitly, even with the help of a computer.

The importance of finding the nongeneric points is that it allows us to study orbifolds of the simple quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$, provided that k is generic in the above sense. The projection $V^k(\mathfrak{g}) \to L_k(\mathfrak{g})$ always restricts to a surjective homomorphism

$$V^k(\mathfrak{g})^G \to L_k(\mathfrak{g})^G,$$

so a strong generating set for $V^k(\mathfrak{g})^G$ descends to a strong generating set for $L_k(\mathfrak{g})^G$. In the examples we consider, most of the interesting values of k for which $V^k(\mathfrak{g})$ is highly reducible turn out to be generic, so we obtain strong generators for $L_k(\mathfrak{g})^G$.

1.1 Outline

The goal of this dissertation is to study the explicit structure of the $\mathbb{Z}_2$-orbifold of the universal affine vertex algebra $V^k(\mathfrak{g})$ associated to a simple, finite-dimensional Lie algebra $\mathfrak{g}$ at level k, and the Zamolodchikov $\mathcal{W}_3$ algebra $\mathcal{W}_3^c$ with central charge c. Our calculations are done using the Mathematica package of Thielemans [T]. This dissertation is organized as follows:

Chapter 2, we give some foundations, and review basic notations towards defining a vertex algebra. We then give a precise definition of a vertex algebra. A vertex algebra is a (super) vector space V with a vacuum vector $\mathbf{1}$, and a translation operator T together with a linear map:

$$Y(.,z) : V \to End(V)[[z, z^{-1}]];$$

assigns to each $a \in V$ a field

$$a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in End(V)[[z, z^{-1}]],$$

satisfying that for $v \in V$, $a(n).v = 0$ for n sufficiently large enough.

This data is required to satisfy some axioms. Loosely speaking, the vertex algebra can be viewed as a generalization of an associative commutative algebra with a unit. We provide extensive examples and explore the algebraic structure of some well-known vertex algebras associated with infinite-dimensional Lie algebras, like the Heisenberg, affine Kac-Moody, Virasoro vertex algebras and the fermionic ghost system.

Chapter 3 is devoted to introduce the classical invariant theory. We present highlighted theorems such as Hilbert Finiteness Theorem, and Weyl First and Second Fundamental Theorems. In the Weyl Theorem, all invariants of many copies of

a given representation V can be obtained from only n copies where $\dim(V) = n$ by a process called *polarization*. In chapter 4, we give an introduction to the orbifold theory, and a motivation to the strong finite generating set.

The main body is presented in Chapters 5 and 6. First, we study $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ in Chapter 5. Here $\mathfrak{g}$ is a simple, finite-dimensional Lie algebra and $V^k(\mathfrak{g})$ denotes the corresponding universal affine vertex algebra at level k. There is an involution of $\mathfrak{g}$ known as the *Cartan involution*, and it gives rise to the action of $\mathbb{Z}_2$ on $V^k(\mathfrak{g})^{\mathbb{Z}_2}$. Let $l = \mathrm{rank}(\mathfrak{g})$ and let m be the number of positive roots, so that $\dim(\mathfrak{g}) = 2m + l$. Our main result is that for any $\mathfrak{g}$ with $dim(\mathfrak{g}) > 3$, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is of type

$$\mathcal{W}\big(1^m, 2^{d+\binom{d}{2}}, 3^{\binom{d}{2}}, 4\big),$$

for generic values of k. Here $d = m+l$. In this notation, we say that a vertex algebra is of type $\mathcal{W}((d_1)^{n_1}, \dots (d_r)^{n_r})$ if it has a minimal strong generating set consisting of n_i fields in weight d_i, for $i = 1, \dots, r$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, there is one extra field in weight 4, so that $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is of type $\mathcal{W}(1, 2^3, 3, 4^2)$ for generic value of k.

To prove this result, we use a deformation argument [LIV] that says that in an appropriate sense,

$$\lim_{k\to\infty} V^k(\mathfrak{g})^{\mathbb{Z}_2} \cong \mathcal{H}(m) \otimes \big(\mathcal{H}(d)^{\mathbb{Z}_2}\big).$$

Here $\mathcal{H}(k)$ denotes the rank k Heisenberg vertex algebra, and the action of $\mathbb{Z}_2$ is given on the generators by multiplication by -1. Moreover, the limiting structure has a minimal strong generating set of the same type as $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ for generic values of k. So the problem of finding a minimal strong generating set for $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is reduced to finding the minimal strong generating set for $\mathcal{H}(d)^{\mathbb{Z}_2}$ for all d. In the case $d = 1$, $\mathcal{H}(1)^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2, 4)$ by a celebrated result of Dong and Nagatomo [DNI], and this is the starting point for the study of $\mathbb{Z}_2$-orbifolds of rank one lat-

tice vertex algebras. Much is also known about the structure and representation theory of $\mathcal{H}(n)^{\mathbb{Z}_2}$ (see [DNII]), although a minimal strong generating set was not previously determined in the literature. We will show that for $d = 2$, $\mathcal{H}(2)^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2^3, 3, 4^2)$ and for $d \geq 3$, $\mathcal{H}(d)^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2^{d+\binom{d}{2}}, 3^{\binom{d}{2}}, 4)$. This is the key technical ingredient in the above description of $V^k(\mathfrak{g})^{\mathbb{Z}_2}$.

It is also of interest to explicitly describe the set of *nongeneric* values of k where the above strong generating sets do not work. By a general result of [CLIII], this set is always finite and consists at most of the poles of the structure constants of the OPE algebra among the generators. In practice, it is very difficult to explicitly compute the generators and these structure constants. In the case of $\mathfrak{s}l_2$ we carry this out and give the complete set of nongeneric values; it consists only of $\{0, \pm\frac{32}{3}, 16, 48\}$. It follows that for all other values of k, the strong generating set for $V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2}$ will descend to a strong generating set for the simple orbifold $L_k(\mathfrak{s}l_2)^{\mathbb{Z}_2}$.

In Chapter 6, we study the $\mathbb{Z}_2$-orbifold of the Zamolodchikov algebra $\mathcal{W}_3^c$. This is the simplest nontrivial example of a $\mathcal{W}$-algebra, and the first to appear in the literature [Za] in the context of two-dimensional conformal field theories with extensions of Virasoro symmetry. It is of type $\mathcal{W}(2, 3)$ is generated by the Virasoro field L and a weight 3 primary field W. Unlike free field and affine vertex algebras, the OPE relations among L and W are nonlinear in the sense that normally ordered products of the generators and their derivatives appear. This makes their study much more difficult. It was conjectured over 20 years ago in the physics literature [BS, B-H] that $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$ should be of type $\mathcal{W}(2, 6, 8, 10, 12)$ for generic values of c. Our main result in this chapter is a proof of this conjecture for all values of c except for $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$. Additionally, we show that for these two values of c, $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2, 6, 8, 10, 12, 14)$.

The proof of this result is more difficult than our description of $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ in Chapter 5, and requires a different approach since it is not practical to calculate the

full OPE algebra among the generators and determined the poles of all structure constants. We first construct a natural infinite strong generating set

$$\{L, U_{2n,0} |\ n \geq 0\}$$

for $(\mathcal{W}_3^c)^{\mathbb{Z}_2}$, where $U_{2n,0} = :(\partial^{2n}W)W:$, which has weight $2n+6$. This generating set comes from classical invariant theory, and there are infinitely many nontrivial normally ordered relations among these generators. The relation of minimal weight 14 is unique up to scalar multiples, and has the form

$$\frac{181248 + 5590c - 475c^2}{60480(22+5c)} U_{8,0} = P(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}),$$

where P is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$ and their derivatives. The pole at $c = -\frac{22}{5}$ is inessential and can be removed; it is a consequence of the choice of normalization of W. Therefore $U_{8,0}$ can be eliminated if and only if $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$. Similarly, we construct decoupling relations for all c expressing $U_{10,0}, U_{12,0}$ and $U_{14,0}$ as normally ordered polynomials in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}$, and their derivatives. To construct decoupling relations for $U_{n,0}$ for all even integers $n \geq 16$, we apply the operators $U_{0,0} \circ_1$ and $U_{2,0} \circ_1$ to the above relations. This yields two families of relations

$$F(n,c) U_{n+4,0} = A_n(L, U_{0,0}, U_{2,0}, \ldots, U_{n+2,0}), \tag{1.1.1}$$

$$G(n,c) U_{n+6,0} = B_n(L, U_{0,0}, U_{2,0}, \ldots, U_{n+4,0}), \tag{1.1.2}$$

where A_n and B_n are normally ordered polynomials as above. The key observation is that $F(n,c)$ and $G(n,c)$ are rational functions of c and n which have no poles for $c \neq -\frac{22}{5}$ and $n \geq 10$. So if (c,n) does not lie on the affine variety $V \subset \mathbb{C}^2$ determined by $F(n-4,c) = 0$ and $G(n-6,c) = 0$, we can use either (1.1.1) or

(1.1.2) to eliminate $U_{n,0}$ for all $n \geq 16$. The main technical result in Chapter 6 is finding the explicit form of $F(n,c)$ and $G(n,c)$. It is then straightforward to prove that V has no such points (c,n) where $n \geq 16$ is an even positive integer.

This method provides a general algorithmic approach to determining the non-generic set for orbifolds of the form $(\mathcal{V}^k)^G$, where $\mathcal{V}^k$ is a vertex algebra depending on a parameter k. Typically, there is a natural infinite strong generating set for $(\mathcal{V}^k)^G$ coming from classical invariant theory. There are also infinitely many nontrivial normally ordered relations among these generators. These relations allow certain generators to be eliminated, and for generic values of k, all but finitely many can be eliminated. If we eliminate as many generators as possible, the remaining ones will form a minimal strong generating set S that works for generic k. We expect that families of relations can be constructed such that the coefficients of the generators to be eliminated are rational functions in finitely many variables

$$F_i(k, n_1, \ldots, n_r), \qquad i = 1, \ldots, s.$$

Here $n_1, \ldots, n_r$ must be positive integers, and are related to the weights of the generators to be eliminated. Corresponding to such a system of relations is the variety $V \subset \mathbb{C}^{r+1}$ determined by $F_i(k, n_1, \ldots, n_r) = 0$ for $i = 1, \ldots, s$. A value of k will be generic if there is no point $(k, n_1, \ldots, n_r) \in V$ such that the remaining coordinates $n_1, \ldots, n_t$ are positive integers. Points with such strong integrality constraints are expected to be rare, and in principle can be found.

Chapter 2

Introduction to Vertex Algebras

In this chapter, we review some basic definitions and important lemmas in order to give a precise definition of vertex algebras. We provide well-known examples of commutative vertex algebras as well as non-commutative ones associated to infinite dimensional Lie algebras along with their algebraic structure.

2.1 Preliminaries

Let V be a vector space (generally over $\mathbb{C}$).

Definition 2.1.1. *A V-valued formal distribution in formal variables $z_1, z_2, ..., z_n$ is a finite or infinite series*

$$f(z_1, z_2, ..., z_n) = \sum_{i_1 \in \mathbb{Z}} ... \sum_{i_n \in \mathbb{Z}} f_{i_1,...,i_n} z_1^{i_1} ... z_n^{i_n}, \tag{2.1.1}$$

where each $f_{i_1,...,i_n} \in V$. The space of all series of this form defines a vector space denoted by $V[[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_n^{\pm 1}]]$.

Multiplying a formal distribution $f(z_1, z_2, ..., z_n) \in V[[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_n^{\pm 1}]]$ by a formal distribution $g(w_1, w_2, ..., w_m) \in V[[w_1^{\pm 1}, w_2^{\pm 1}, ..., w_m^{\pm 1}]]$ is well-defined element in $V[[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_n^{\pm 1}, w_1^{\pm 1}, w_2^{\pm 1}, ..., w_m^{\pm 1}]]$. Nevertheless, the product of two elements of $V[[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_n^{\pm 1}]]$ in general does not make sense since it gives an infinite series which might not converge. However, the product of a formal power series by a *Laurent series*, that is a series of the form (2.1.1) satisfying that $f_{i_1,...,i_n} = 0$ for $n << 0$ is well-defined. The space of all Laurent series with coefficients in V in one formal variable z is denoted by $V((z))$, while $V((z))((w))$ indicates the space of Laurent series in w whose coefficients are Laurent series in z. In several variables, the space $V((z_1, z_2, ..., z_n))$ will denote the field of fractions of $V[[z_1, z_2, ..., z_n]]$. The space of all Laurent polynomials with coefficients in V in formal variables $z_1, z_2, ..., z_n$ is denoted by $V[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_n, z_n^{-1}]$.

An interesting example of the formal distribution in two variables is the delta-function as it will be defined in the next section.

Definition 2.1.2. *Let $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n \in V[[z^{\pm 1}]]$ be a formal distribution in one formal variable. The linear map $Res_z : V[[z^{\pm 1}]] \to V$ is the formal residue of $f(z)$ defined to be the coefficient of z^{-1}. That is,*

$$Res_z f(z) = f_{-1}.$$

The formal derivative $\partial_z : V[[z^{\pm 1}]] \to V[[z^{\pm 1}]]$ is defined by

$$\partial_z(\sum_{n\in\mathbb{Z}} f_n z^n) = \sum_{n\in\mathbb{Z}} n f_n z^{n-1}.$$

Clearly, $Res_z \partial_z f(z) = 0$.

Definition 2.1.3. *Given a $\mathbb{Z}_{\geq 0}$-graded vector space V over $\mathbb{C}$*

$$V = \bigoplus_{n=0}^{\infty} V_n.$$

A linear map (an endomorphism) $\varphi : V \to V$ is called homogenous of degree m, if $\varphi(V_n) \subset V_{n+m}$ for all n.

Conventions 2.1.4. *For any two formal variables z, w, and an arbitrary complex number m, the binomial series $(z+w)^m$ can be given by the following expansion:*

$$(z+w)^m = \sum_{n \geq 0} \binom{m}{n} z^{m-n} w^n,$$

where

$$\binom{m}{n} = \frac{m(m-1)...(m-n+1)}{n!}.$$

A special case gives

$$(1-z)^{-1} = \sum_{n \geq 0} z^n, \qquad for \ \ |z| < 1.$$

2.2 Delta-function

The delta-function is a formal distribution $\delta \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ in formal two variables z, w, defined as follows:

$$\delta(z,w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}. \tag{2.2.1}$$

Remark 2.2.1. *It is easy seen from the definition of Delta-function that $\delta(z,w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1} = \sum_{m \in \mathbb{Z}} z^{-m-1} w^m = \delta(w,z)$. Moreover, since*

$$\delta(z,w) = \sum_{m \in \mathbb{Z}} z^{-m-1} w^m = \sum_{m \in \mathbb{Z}} z^{-m-2} w^{m+1},$$

so $\partial_z \delta(z,w) = -\partial_w \delta(w,z)$.

Proposition 2.2.2. *Let* $a(z)$ *be a formal distribution in one formal variable. The Delta-function satisfies The following:*

1. $a(z)\delta(z,w) = a(w)\delta(z,w)$.
2. $Res_z(a(z)\delta(z,w)) = a(w)$.
3. $(z-w)^{k+1}\partial_w^k \delta(z,w) = 0$.

2.3 The power series expansion of a rational function

Given a rational function in two formal variables $f(z,w)$ with poles at $z = 0, w = 0, |z| = |w|$ only. *The power series expansion of* $f(z,w)$ in the domain $|z| > |w|$ (respectively $|w| > |z|$) is denoted by $i_{|z|>|w|}f$ (respectively $i_{|w|>|z|}f$).

Example 2.3.1. *For* $k \geq 0$, *we have*

$$i_{|z|>|w|}\frac{1}{(z-w)^{k+1}} = \sum_{j=0}^{\infty}\binom{j}{k} z^{-j-1}w^{j-k},$$
$$i_{|w|>|z|}\frac{1}{(z-w)^{k+1}} = -\sum_{j=-1}^{-\infty}\binom{j}{k} z^{-j-1}w^{j-k}.$$

Remark 2.3.2. *For* $k \in \mathbb{Z}$, *we have*

$$i_{|z|>|w|}(z-w)^k = \sum_{j=0}^{\infty}\binom{k}{j}(-w)^j z^{k-j} \in \mathbb{C}[z][[z^{-1},w]],$$

$$i_{|w|>|z|}(z-w)^k = \sum_{j=0}^{\infty}\binom{k}{j} z^j(-w)^{k-j} \in \mathbb{C}[w][[w^{-1},z]].$$

For $k > 0$, *the above two expansions are equal, but for* $k < 0$, $i_{|z|>|w|}(z-w)^k \neq i_{|w|>|z|}(z-w)^k$.

Remark 2.3.3. *Using the power series expansion, Delta-function can be interpreted as follows:*

$$\delta(z,w) = i_{|z|>|w|}\frac{1}{z-w} - i_{|w|>|z|}\frac{1}{z-w}. \tag{2.3.1}$$

2.4 Vertex algebras

Let V be a (super) vector space, that is, a $\mathbb{Z}_2$-graded vector space (generally infinite dimensional over $\mathbb{C}$). Let z be a formal variable. Recall, $V((z))$ is the space of Laurent series with coefficients in V, which is defined earlier in this chapter,

$$V((z)) := \{\sum_{n\in\mathbb{Z}} v(n)z^n |\ \ v(n) \in V,\ \ v(n) = 0\ \ for\ \ n << 0\}.$$

Let $a : V \to V((z))$ be a linear map that takes $v \mapsto a(v) = \sum_{n\in\mathbb{Z}} v(n)z^n$ such that $v(n) = 0$ for $n << 0$. We shall rewrite $a(v) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1},\ \ v(n) \in V,\ \ v(n) = 0$ for $n >> 0$.

Convention 2.4.1. *Another point of view, the following are two equivalent definitions of the linear map:*

$$a : V \to V((z)) \Leftrightarrow a(z) : V \to End(V)[[z, z^{-1}]]$$

where $a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1},\ \ a(n) \in End(V)$ *and for any* $v \in V$, $a(n).v = 0$ *for* $n >> 0$.

The vector space of all such linear maps is denoted $QO(V)$. Each $a \in QO(V)$ has a unique representation by the following formal distribution:

$$a = a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in End(V)[[z, z^{-1}]],$$

where $a(n)$ is the nth Fourier mode of $a(z)$, and for any $v \in V, a(n).v = 0$ for $n >> 0$. We call $a(z)$ a *field* on $QO(V)$. Since V is a (super) vector space, each a in a (super) $QO(V)$ has the form $a = a_0 + a_1$ (even, respectively odd) where $a_i : V_j \to V_{i+j}((z))$ for $i, j \in \mathbb{Z}_2$ and $|a_i| = i$.

The space $End(V)$ could be viewed as a subspace of $QO(V)$ if we consider the linear map a as the constant series $\sum_{n\in\mathbb{Z}} a\delta_{n,-1}z^{-n-1}$. Here $\delta_{n,-1}$ is the Kronecker delta. The space $End(V)$ is an associative algebra with a unit but $QO(V)$ is not quite algebra. The product of two elements of $QO(V)$ in general does not make sense, since the product gives an infinite series which need not converge. Indeed, the product of $a(z), b(z) \in QO(V)$ gives

$$\begin{aligned} a(z).b(z) &= \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \sum_{m\in\mathbb{Z}} b(m)z^{-m-1} \\ &= \sum_{n,m\in\mathbb{Z}} a(n)b(m)z^{-(n+m)-2} \\ &= \sum_{k\in\mathbb{Z}} \Big(\sum_{n+m=k} a(n)b(m) \Big) z^{-k-2}. \end{aligned}$$

However, we can fix this and extend the product in $End(V)$ to all $QO(V)$. This extension is called *The Wick product.*

Definition 2.4.2. *Let $a(z), b(w) \in QO(V)$. The normal-ordered product of $a(z)$ and $b(w)$ is called the Wick product, and is defined as follows*

$$: a(z)b(w) := \sum_{n<0} a(n)z^{-n-1}b(w) + (-1)^{|a||b|}b(w) \sum_{n\geq 0} a(n)z^{-n-1}.$$

$: a(z)b(w) :$ is well-defined element in $QO(V)$.

For $a_1(z), ..., a_n(z) \in QO(V)$, we define the k-fold iterated Wick product inductively as follows:

$$: a_1(z)...a_n(z) :=: a_1(z)(: a_2(z)...a_n(z) :) : .$$

Conventions 2.4.3. *For any two fields*

$$a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \qquad b(w) = \sum_{m\in\mathbb{Z}} b(m)w^{-m-1}$$

in $QO(V)$, their commutator is defined by

$$[\sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \sum_{m\in\mathbb{Z}} b(m)w^{-m-1}] = \sum_{n,m\in\mathbb{Z}} [a(n), b(m)]z^{-n-1}w^{-m-1} \in End[[z^{\pm 1}, w^{\pm 1}]].$$

We shall write $a(z)_- =: \sum_{n<0} a(n)z^{-n-1}$, and $a(z)_+ =: \sum_{n\geq 0} a(n)z^{-n-1}$. As we noted before, the coefficient of z^{-1} defines the $Res_z a(z)$. Recall, $\partial_z a(z)$ is the formal derivative $\partial_z = \frac{d}{dz}$. We can check that $\partial_z(a(z)_\pm) = (\partial_z a(z))_\pm$, and so ∂_z acts as a derivation with respect to the Wick product:

$$\partial_z(: a(z)b(z) :) =: (\partial_z a(z))b(z) : + : a(z)(\partial_z b(z)) : .$$

From now on, we will write a instead of $a(z)$, and ∂ instead of ∂_z if it is clear from the context.

Definition 2.4.4. *Let $a, b \in QO(V)$, and let n be an integer. Define the nth circle product on $QO(V)$ by*

$$a(w) \circ_n b(w) = Res_z a(z)b(w)i_{|z|>|w|}(z-w)^n - (-1)^{|a||b|} Res_z b(w)a(z)i_{|w|>|z|}(z-w)^n.$$

Remark 2.4.5. *1. $a(w) \circ_n b(w)$ is well-defined element in $QO(V)$.*

2. The negative circle product can be obtained by

$$n!a(z) \circ_{-n-1} b(z) =: (\partial^n a(z))b(z) : . \tag{2.4.1}$$

3. $a(z) \circ_{-1} b(z) =: a(z)b(z) : .$

4. *The* $QO(V)$ *is a nonassociative algebra with the operations* $\circ_n$ *and a unit* $1 \in End(V)$ *satisfying* $1 \circ_n a = \delta_{n,-1} a \ \ for \ \ all \ \ n$, *and* $a \circ_n 1 = \delta_{n,-1} a \ \ for \ \ n \geq -1$.

Lemma 2.4.6. ***(OPE) The operator product expansion****: Let* $a, b \in QO(V)$. *Then*

$$a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} + : a(z)b(w) : . \tag{2.4.2}$$

We often write $a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w)(z-w)^{-n-1}$, *where* $\sim$ *means equal modulo the regular term* $: a(z)b(w) : .$

Definition 2.4.7. *Let* $\mathcal{A}$ *be a QOA. A subset* $S = \{a_i | i \in I\}$ *of* $\mathcal{A}$ *is said to generate* $\mathcal{A}$ *if every element* $a \in \mathcal{A}$ *can be written as a linear combination of nonassociative words in the letters* $a_i, \circ_n, \ \ for \ \ i \in I \ \ and \ \ n \in \mathbb{Z}$.

Definition 2.4.8. *Two fields* $a, b \in QO(V)$ *are said to circle commute (or quantum commute) if*

$$(z-w)^N [a(z), b(w)] = 0, \tag{2.4.3}$$

for some an integer $N \geq 0$. *In addition, if every two elements in QOA pairwise circle commute (quantum commute), then QOA is a circle commutative algebra (or commutative quantum operator algebra* $(CQOA)$*).*

Note that since we are working over a super vector space V, so $[.,.]$ denotes the super bracket, that is $[a, b] = ab - (-1)^{|a||b|} ba$, for $a, b \in QO(V)$. It follows from (2.4.3) that $a \circ_n b = 0$ for $n \geq N$. The above definition is referred as a definition of the locality of $a, b \in QO(V)$ on some books in vertex algebras.

Lemma 2.4.9. ***(Dong's Lemma)*** *Let* $a, b, c \in QO(V)$. *If* a, b, c *are three pairwise quantum commuting fields, then* $a \circ_n b$ *quantum commute with* c *as well for all* $n \in \mathbb{Z}$.

Remark 2.4.10. *It follows from Dong's Lemma that if $a(z), b(z), c(z)$ are three pairwise quantum commuting fields, then $:a(z)b(z):$ quantum commutes with $c(z)$.*

We shall now give the definition of a vertex algebra.

Definition 2.4.11. *[B], [FLM] A vertex algebra $(V, \mathbf{1}, T, Y)$ is a collection of data:*

- *(space of states) a super vector space $V = V_0 + V_1$;*
- *(vacuum vector) a vector $\mathbf{1} \in V_0$;*
- *(translation operator) a linear operator $T : V \to V$ acting on V;*
- *(fields) a linear map:*

$$Y(., z) : V \to End(V)[[z, z^{-1}]];$$

taking each $a \in V$ to a field acting on V

$$a \mapsto a(z) := \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in End(V)[[z, z^{-1}]].$$

This data is required to satisfy the following axioms:

- *(vacuum axiom) $\mathbf{1}(z) = 1$. Furthermore, for any $a \in V$ we have*

$$a(z)\mathbf{1} \in V[[z]],$$

and $a(z)\mathbf{1}|_{z=0} = a$, i.e. $a(n)\mathbf{1} = 0, n \geq 0$, and $a(-1)\mathbf{1} = a$.

- *(translation axiom) For any $a \in V$,*

$$[T, a(z)] = \partial_z a(z)$$

and $T\mathbf{1} = 0$.

- *(quantum commute axiom) All fields* $a(z)$ *pairwise quantum commute.*

  Borcherds identities: For $a, b, c \in V$, and $k, m, n \in \mathbb{Z}$, we have

$$\sum_{j=0}^{\infty} \binom{n}{j} (a \circ_{m+j} b) \circ_{n+k-j} c = \sum_{j=0}^{\infty} \binom{m}{j} ((-1)^j a \circ_{n+m-j} (b \circ_{k+j} c) - (-1)^{j+m} b \circ_{k+m-j} (a \circ_{n+j} c)). \tag{2.4.4}$$

Note that the above sums are finite since $a(n).b = 0$ for $n >> 0$.

Definition 2.4.12. *(The Tensor Product of Vertex Algebras)*
Given two vertex algebras $(V^1, \mathbf{1}_{V^1}, T_1, Y_1)$ *and* $(V^2, \mathbf{1}_{V^2}, T_2, Y_2)$. *Their tensor product* $V^1 \otimes_{\mathbb{C}} V^2$ *is a well-defined vertex algebra where* $Y(a^1 \otimes a^2, z) = Y(a^1, z) \otimes Y(a^2, z)$ *for* $a^i \in V^i$, $\mathbf{1} = \mathbf{1}_{V^1} \otimes \mathbf{1}_{V^2}$ *and* $T = T_1 \otimes 1 + 1 \otimes T_2$.

Vertex algebras homomorphism, vertex subalgebras, ideals, and modules are defined similarly in a usual way.

Definition 2.4.13. *Given two vertex algebras* $(V^1, \mathbf{1}_{V^1}, T_1, Y_1)$ *and* $(V^2, \mathbf{1}_{V^2}, T_2, Y_2)$. *A vertex algebra homomorphism* f *from* $(V^1, \mathbf{1}_{V^1}, T_1, Y_1)$ *to* $(V^2, \mathbf{1}_{V^2}, T_2, Y_2)$ *is a linear map* $f : V^1 \to V^2$ *that maps* $\mathbf{1}_{V^1}$ *to* $\mathbf{1}_{V^2}$, *and preserves all circle products.*

Definition 2.4.14. *Let* $(V, \mathbf{1}, T, Y)$ *be a vertex algebra, a vertex subalgebra* $V' \subset V$ *is a* T*-invariant subspace which contains the vacuum vector and satisfies that* $Y(a, z)b \in V'((z))$ *for* $a, b \in V'$ *with the induced vertex algebra.*
A vertex algebra ideal $I \subset V$ *is a* T*-invariant subspace which satisfies that* $\mathbf{1} \notin I$, *and* $Y(a, z)b \in I((z))$ *for* $a \in I$ *and* $b \in V$.

Definition 2.4.15. *For a vertex algebra $(V, \mathbf{1}, T, Y)$, a module over a vertex algebra V is a vector space M equipped with a linear map*

$$V \to End(M)[[z, z^{-1}]], \qquad a \mapsto a^M(z) = \sum_{n\in\mathbb{Z}} a(n)^M z^{-n-1},$$

satisfying the following:

1. $a^M(n) \in End(M)$ *such that for all* $a \in V$, *and* $m \in M, a^M(n).m = 0$ *for* $n >> 0$,

2. $\mathbf{1}(z) = 1_M$,

3. *Borcherds identity holds:*

$$\sum_{j=0}^{\infty} \binom{n}{j} (a^M \circ_{m+j} b^M) \circ_{n+k-j} c^M = \sum_{j=0}^{\infty} \binom{m}{j} ((-1)^j a^M \circ_{n+m-j} (b^M \circ_{k+j} c^M) - (-1)^{j+m} b^M \circ_{k+m-j} (a^M \circ_{n+j} c^M)).$$

Proposition 2.4.16. *Let a, b, c be three fields in some vertex algebra $\mathcal{A}$, and let $n > 0$. Then*

$$:(:ab:)c: - :abc: := \sum_{k\geq 0} \frac{1}{(k+1)!} (:(\partial^{k+1}a)(b \circ_k c): + (-1)^{|a||b|} :(\partial^{k+1}b)(a \circ_k c):), \tag{2.4.5}$$

$$:ab: - (-1)^{|a||b|} :ba: := \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} \partial^{k+1}(a \circ_k b), \tag{2.4.6}$$

$$a \circ_n (:bc:) - :(a \circ_n b)c: - (-1)^{|a||b|} :b(a \circ_n c): := \sum_{k=1}^{n} \binom{n}{k} (a \circ_{n-k} b) \circ_{k-1} c, \tag{2.4.7}$$

$$(:ab:) \circ_n c = \sum_{k\geq 0} \frac{1}{k!} (\partial^k a)(b \circ_{n+k} c) + (-1)^{|a||b|} \sum_{k\geq 0} b \circ_{n-k-1} (a \circ_k c). \tag{2.4.8}$$

To illustrate the notion of vertex algebras, we give examples and we check the quantum commute axiom given in the Definition (2.4.11) for each since it is not obvious.

2.5 Examples of vertex algebras

2.5.1 Commutative vertex algebras

A commutative vertex algebra V is a vertex algebra whose all fields $a(z), b(z)$, $a, b \in V$ pairwise commute. This is stronger than the notion of quantum commute. Here, we have $N = 0$ in the definition (2.4.8).

Example 2.5.1. *Given an associative, commutative subalgebra with a unit V furnished with an even derivation T of degree 1. Set*

$$\mathbf{1} = 1, \qquad a(z)b = (e^{zT}a)b = \sum_{n\geq 0} \frac{z^n}{n!}(T^n a)b,$$

where 1 is the unit element of V, and $a, b \in V$. Moreover, For $a, b \in V$, we have

$$[T, a(z)]b = (\partial_z a(z))b.$$

Conversely, given a commutative vertex algebra V, then by the vacuum axiom we have for $a, b \in V$

$$a(z)b = a(z)b(w)\mathbf{1}|_{w=0} = b(w)a(z)\mathbf{1}|_{w=0}.$$

Therefore $a(z)b \in V[[z]]$, and so $a(z) \in End[[z]]$ for all $a \in V$. Clearly, the commutative vertex algebra is a commutative algebra with a unit, and a product $a.b =: ab :$. The unit is given by the vacuum vector $\mathbf{1}$, and the translation operator T of V acts

as derivation on V with respect to this product, that is

$$T(a.b) = (Ta).b + a.T(b).$$

This example shows that the definition of the commutative vertex algebra is an analogue to the definition of the commutative algebra with a unit and a derivation.

2.5.2 Non-commutative vertex algebras

Before we introduce the examples, we need the following definitions.

Definition 2.5.2. *A Lie algebra $\mathfrak{g}$ is a vector space over some field $\mathbb{F}$ together with a bilinear operation $[.,.] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket satisfying the following axioms:*

- *(skew-symmetry)* $[x, y] = -[y, x]$,
- *(Jacobi identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 2.5.3. *Given a Lie algebra $\mathfrak{g}$, a central extension of $\mathfrak{g}$ is an exact sequence*

$$0 \to \mathfrak{a} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras. Here $\mathfrak{a} \subset \mathcal{Z}(\hat{\mathfrak{g}})$ is central, that is $\mathcal{Z}(\hat{\mathfrak{g}}) = \{x \in \hat{\mathfrak{g}} | [x, \hat{\mathfrak{g}}] = 0\}$.

We are interested in one-dimensional central extensions, where $\mathfrak{a} \simeq \mathbb{C}_\kappa$ and κ is the generator as we will see in the up coming examples.

Definition 2.5.4. *The tensor algebra of a vector space V is a pair $(\mathcal{T}(V), i)$, where $\mathcal{T}(V) = \bigoplus_{k \geq 0} \mathcal{T}^k V$ is an associative algebra of tensors on V (of any rank) over $\mathbb{C}$, with multiplication being the tensor product, and $i : V \to \mathcal{T}(V)$ is a linear injection map satisfying the universal property: if B is an associative algebra and $\varphi : V \to B$ is any linear map, then there exists a unique algebra homomorphism $\tilde{\varphi} : \mathcal{T}(V) \to B$ such that $\varphi = \tilde{\varphi} \circ i$.*

In the above definition, the tensor algebra $\mathcal{T}(V) = \bigoplus_{k\geq 0} \mathcal{T}^k V = \mathbb{C} \oplus V \oplus (V \oplus V) \oplus (V \oplus V \oplus V) \oplus ....$ The tensor algebra is very useful since many other algebras can be constructed as quotient algebras of $\mathcal{T}(V)$. Some examples of this, the exterior algebra, the symmetric algebra, Clifford algebras, and universal enveloping algebras.

Definition 2.5.5. *Let $\mathfrak{g}$ be a Lie algebra, and let I be a two sided ideal in $\mathcal{T}(\mathfrak{g})$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y], \ \ x, \ \ y \in \mathfrak{g}$. The universal enveloping algebra for $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is a tensor algebra $\mathcal{T}(\mathfrak{g})$ modulo the ideal I, i.e.*

$$U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I.$$

Note that $U(\mathfrak{g})$ is an associative algebra with a unit, and that any representation of $\mathfrak{g}$ is a $U(\mathfrak{g})$-module.

Theorem 2.5.6. *(**Poincaré-Birkhoff-Witt**, PBW) Let $\mathfrak{g}$ be a Lie algebra, and let $\{x_i\}_{i\in I}$ its basis. Then the universal enveloping algebra $U\mathfrak{g}$ has the following monomial basis called PBW basis:*

$$x_{i_1}^{n_1} ... \ x_{i_k}^{n_k}, \qquad i_1 < ... < i_k, \qquad n_i \geq 0 \ \ integers.$$

Most well-known vertex algebras $\mathcal{V}$ are the universal enveloping vertex algebras corresponds to some Lie conformal algebras. Many of them have the virasoro element $L(z)$, (it will be introduced later), and often it is required that L_0 be diagonalizable and L_{-1} acts on $\mathcal{V}$ by formal differentiation, and so we call the pair $(\mathcal{V}, L(z))$ a conformal vertex algebra of central charge c.

Definition 2.5.7. *Let $(\mathcal{V}, L(z))$ be a conformal vertex algebra. A conformal weight $\triangle$ of an element $a(z) \in \mathcal{V}$ is an eigenvalue $\triangle$ of the eigenvector of L_0. Furthermore, we denote the subspace of conformal weight $\triangle$ by $\mathcal{V}_\triangle$.*

Remark 2.5.8. *In any conformal vertex algebra $\mathcal{V}$, the operator $\circ_n$ is homogeneous of weight $-n-1$. In particular, the Wick product $\circ_{-1}$ is homogeneous of weight 0.*

Remark 2.5.9. *Let V be a $\mathbb{Z}_2$-graded vector space. A field $\varphi(z)$ of conformal weight $\triangle \in \mathbb{Z}_+$ can be defined by the following formal distribution:*

$$\varphi(z) := \sum_{n\in\mathbb{Z}} \varphi(n) z^{-n-\triangle},$$

where $\varphi(n)$ is homogenous in $End(V)$ of degree $-n$.

Definition 2.5.10. *Let $(\mathcal{V}, L(z))$ be a conformal vertex algebra. An element $a(z) \in \mathcal{V}_\triangle$ of conformal weight $\triangle$ is called primary if it satisfies the OPE relation*

$$L(z)a(w) \sim \triangle a(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1},$$

i.e. there are no 3 poles and higher. Moreover, we call a vector $a(z) \in \mathcal{V}_\triangle$ is quasi-primary if it satisfies the OPE relation

$$L(z)a(w) \sim C(z-w)^{\triangle-2} + \triangle a(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1},$$

for some constant C.

2.6 The Heisenberg vertex algebra

The Heisenberg Lie algebra $\mathfrak{h} = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}_\kappa$ is the one-dimensional central extension of the commutative Lie algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$, (the loop algebra of one-dimensional commutative Lie algebra $\mathbb{C}$), that is

$$0 \to \mathbb{C}_\kappa \to \mathfrak{h} \to \mathbb{C}[t, t^{-1}] \to 0,$$

where a generator κ is called the central charge. Sometimes, the central charge is also called the rank of a vertex algebra. The Heisenberg Lie algebra $\mathfrak{h}$ is spanned by $\langle t^n, n \in \mathbb{Z}, \kappa \rangle$, and these generators satisfy the following Lie bracket:

$$[t^n, t^m] = n\delta_{n+m,0}\kappa, \qquad [\kappa, t^n] = 0,$$

with $\mathbb{Z}$-gradation $deg(t^n) = n$, and $deg(\kappa) = 0$.

Let $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+$, where $\mathfrak{h}_- = \oplus_{n<0}\mathfrak{h}_n$ while $\mathfrak{h}_+ = \oplus_{n\geq 0}\mathfrak{h}_n$, and both are commutative Lie subalgebras of $\mathfrak{h}$. Here $\mathfrak{h}_n$ denotes the subspace of degree n. By the PBW theorem,

$$V \cong \mathfrak{h}_- = \mathbb{C}[t^{-1}, t^{-2}, ...], \tag{2.6.1}$$

where $t^{-1}, t^{-2}, ...$ are algebraically independent variables. Define a representation ρ as follows:

$$\mathfrak{h} \xrightarrow{\rho} End(V)$$

$$\rho(t^n) = \begin{cases} n\frac{\partial}{\partial t^{-n}} & \text{if } n \geq 0, \\ t^n & \text{if } n > 0, \end{cases}$$

and κ acts as a multiplication by scalar 1, that is, κ acts by id. Indeed, ρ defines a $\mathfrak{h}$-module on V.

Recall, we have constructed a $\mathfrak{h}$-module on V, where $\mathfrak{h} = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}_\kappa$. Due to the commutativity of the Lie algebra $\mathfrak{h}_+$, it has a trivial one-dimensional representation. We shall define an induced representation of $\mathfrak{h}$ from the representation of $\mathfrak{h}_+$. Let C be the one dimensional $\mathfrak{h}_+$-module on which $t^n, n \geq 0$ acts trivially, and κ acts as a multiplication by scalar 1. Therefore,

$$V = U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_+)} C,$$

where $U(\mathfrak{h})$ denotes the universal enveloping algebra of $\mathfrak{h}$. This is called the Fock representation of $\mathfrak{h}$. Under the isomorphism (2.6.1), the action of $t^n, n \geq 0$ on V is just the multiplication. On the other hand, the action of $t^n, n \geq 0$ can be obtained by moving $t^n, n \geq 0$ through $t^n, n < 0$ by the relation $[t^n, t^{-n}] = n$, and $t^n.1 = 0, n \geq 0$ where 1 is the unit element of $U(\mathfrak{h})$. By induction, $t^n, n \geq 0$ acts by derivation $n\frac{\partial}{\partial t^{-n}}$ and t^0 acts trivially on V just like the action of $\mathfrak{h}$ on V via the representation ρ. We call the operators $t^n, n < 0$ creation operators, while $t^n, n \geq 0$ annihilation operators.

To construct the generating field $\alpha(z)$, let $\alpha(n) \in End(V)$ be a linear operator representing t^n on V. Define $\alpha(z)$:

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1},$$

to be an even generator of weight 1 lying in $QO(V)$, satisfying the OPE relation:

$$\alpha(z)\alpha(w) \sim (z-w)^{-2}. \tag{2.6.2}$$

To check that $\alpha(z)$ quantum commutes with itself:

Recall the delta-function $\delta(z, w)$, and so

$$\begin{aligned}
[\alpha(z), \alpha(w)] &= [\sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1}, \sum_{m\in\mathbb{Z}} \alpha(m) w^{-m-1}] \\
&= \sum_{n\in\mathbb{Z}} [\alpha(n), \alpha(-n)] z^{-n-1} w^{n-1} \\
&= \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} \\
&= \partial_w \delta(z, w).
\end{aligned}$$

By Proposition 2.2.2, item (3), we have

$$(z-w)^2[\alpha(z),\alpha(w)]=0. \tag{2.6.3}$$

The Heisenberg vertex algebra $\mathcal{H}=\langle\alpha(z)\rangle$ on V has a PBW basis as follows:

$$:\partial^{k_1}\alpha...\partial^{k_s}\alpha:, \qquad s\geq 0,\ \ k_1\geq ...\geq k_s\geq 0. \tag{2.6.4}$$

To demonstrate how the OPE relation is carried out, we shall calculate the OPE relation for $L(z)=\frac{1}{2}:\alpha(z)\alpha(z):$ using the creation and annihilation operators as follows:

We have

$$L(z)=\frac{1}{2}:\alpha(z)\alpha(z):=\frac{1}{2}:\alpha\alpha:=\frac{1}{2}:(\alpha_-\alpha+\alpha\alpha_+):,$$
$$L(w)=\frac{1}{2}:\alpha(w)\alpha(w):=\frac{1}{2}:\bar{\alpha}\bar{\alpha}:=\frac{1}{2}:(\bar{\alpha_-}\bar{\alpha}+\bar{\alpha}\bar{\alpha_+}):.$$

Then

$$L(z)L(w)=\frac{1}{4}(:\alpha_-\alpha\bar{\alpha_-}\bar{\alpha}:+:\alpha_-\alpha\bar{\alpha}\bar{\alpha_+}:+:\alpha\alpha_+\bar{\alpha_-}\bar{\alpha}:+:\alpha\alpha_+\bar{\alpha}\bar{\alpha_+}:).$$

Using the creation and annihilation operators, as well as (2.3.1) once can write

$$\begin{aligned}:\alpha_-\alpha\bar{\alpha_-}\bar{\alpha}:&=:\alpha_-\bar{\alpha}_-\alpha\bar{\alpha}:+:\alpha_-[\alpha,\bar{\alpha}_-]\bar{\alpha}:,\\ &=2:\alpha_-\bar{\alpha}_-:\frac{1}{(z-w)^2}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
:\alpha\alpha_+\bar{\alpha_-}\bar{\alpha}: &=: \alpha\bar{\alpha}_-\alpha_+\bar{\alpha}_- :+: \alpha[\alpha_+,\bar{\alpha}_-]\bar{\alpha}_- :,\\
&=: \bar{\alpha}_-\alpha\alpha_+\bar{\alpha}_- :+: [\alpha,\bar{\alpha}_-]\alpha_+\bar{\alpha}_- :+[\alpha_+\bar{\alpha}_-]\frac{1}{(z-w)^2}+: \alpha\bar{\alpha}_- : \frac{1}{(z-w)^2},\\
&=: \bar{\alpha}_-\alpha_- : \frac{1}{(z-w)^2}+\frac{1}{(z-w)^4}+: \alpha_-\bar{\alpha}_- : \frac{1}{(z-w)^2}+\frac{1}{(z-w)^4},\\
&= 2: \bar{\alpha}_-\alpha_- : \frac{1}{(z-w)^2}+\frac{2}{(z-w)^4}.
\end{aligned}
$$

On the other hand, we have

$$
: \bar{\alpha}_-\alpha_- :=: \bar{\alpha}_-\alpha_- :+\partial_z(: \bar{\alpha}_-\alpha_- :)|_{z=w}(z-w)+\partial_z^2(: \bar{\alpha}_-\alpha_- :)|_{z=w}(z-w)^2+....
$$

So,

$$
: \bar{\alpha}_-\alpha_- : \frac{1}{(z-w)^2} =: \bar{\alpha}_-\bar{\alpha}_- : \frac{1}{(z-w)^2}+\partial_w : \bar{\alpha}\bar{\alpha}_- : \frac{1}{(z-w)}.
$$

Therefore,

$$
\begin{aligned}
L(z)L(w) &\sim \frac{1}{2}\frac{1}{(z-w)^4}+: \bar{\alpha}_-\bar{\alpha}_- : \frac{1}{(z-w)^2}+\partial_w : \bar{\alpha}\bar{\alpha}_- : \frac{1}{(z-w)},\\
&\sim \frac{1}{2}\frac{1}{(z-w)^4}+2L(w)\frac{1}{(z-w)^2}+\partial_w L(w)\frac{1}{(z-w)}.
\end{aligned}
$$

This is exactly the OPE relation for the virasoro field as we shall see later.

2.7 The rank n Heisenberg vertex algebra $\mathcal{H}(n)$

The rank n Heisenberg vertex algebra $\mathcal{H}(n)$ is the tensor product of n copies of rank 1 Heisenberg vertex algebra $\mathcal{H}$, i.e. $\mathcal{H}(n) = \underbrace{\mathcal{H}\otimes ... \otimes \mathcal{H}}_{n \text{ copies}}$ with even generators $\alpha^1, ..., \alpha^n$. For $i = 1, ..., n$, the OPE relation (2.6.2) holds for each generator α^i, that is

$$
\alpha^i(z)\alpha^j(w) \sim \delta_{i,j}(z-w)^{-2}. \tag{2.7.1}
$$

There is a natural conformal structure of central charge n on $\mathcal{H}(n)$ with the Virasoro element $L(z)$, that is

$$L(z) = \frac{1}{2}\sum_{i=1}^{n} : \alpha^i(z)\alpha^i(z) :, \tag{2.7.2}$$

under which each α^i is of weight 1.

The rank n Heisenberg vertex algebra $\mathcal{H}(n) = \langle \alpha^i(z) | i = 1, ..., n\rangle$ has a PBW basis as follows:

$$: \partial^{k_1^1}\alpha^1 \dots \partial^{k_{s_1}^1}\alpha^1 \dots \partial^{k_1^n}\alpha^n \dots \partial^{k_{s_n}^n}\alpha^n :, \quad s_i \geq 0, \;\; k_1^i \geq \dots k_{s_i}^i \geq 0.$$

2.8 The universal affine vertex algebras $V^k(\mathfrak{g}, B)$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, equipped with a nondegenerate, symmetric, invariant bilinear form B. *The Affine Kac-Moody algebra* $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}_\kappa$, determined by B, is the one-dimensional central extension of the loop algebra $\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, that is

$$0 \to \mathbb{C}_\kappa \to \hat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \to 0,$$

where a generator κ is the central charge. The Lie algebra $\hat{\mathfrak{g}}$ is spanned by $\langle \zeta \otimes t^n, \zeta \in \mathfrak{g}, n \in \mathbb{Z}, \kappa\rangle$, where these generators satisfy the following Lie bracket:

$$[\zeta \otimes t^n, \eta \otimes t^m] = [\zeta, \eta] \otimes t^{n+m} + nB(\zeta, \eta)\delta_{n+m,0}\kappa, \qquad [\kappa, \zeta \otimes t^n] = 0, \tag{2.8.1}$$

for $\zeta, \eta \in \mathfrak{g}, n, m \in \mathbb{Z}$, and $\mathbb{Z}$-gradation $deg(\zeta \otimes t^n) = n$, and $deg(\kappa) = 0$.

Definition 2.8.1. *For a Lie algebra $\mathfrak{g}$, a representation M over $\mathfrak{g}$ has a level $k \in \mathbb{C}$ if κ acts as a multiplication by scalar k.*

To construct generating fields, let $X^{\zeta_i}(n) \in End(V^k)$ be the linear operator representing $\zeta_i \otimes t^n$ on V^k. Define

$$X^{\zeta_i}(z) = \sum_{n\in\mathbb{Z}} X^{\zeta_i}(n) z^{-n-1},$$

to be an even generating field of conformal weight 1. It is easy to see that it lies in $QO(V^k)$, and satisfies the OPE relation

$$X^{\zeta}(z)X^{\eta}(w) \sim kB(\zeta,\eta)(z-w)^{-2} + X^{[\zeta,\eta]}(w)(z-w)^{-1}. \tag{2.8.2}$$

To check that $X^{\zeta}(z), X^{\eta}(z)$ quantum commute:
Recall the delta-function $\delta(z,w)$, and so

$$\begin{aligned}
[X^{\zeta}(z), X^{\eta}(w)] &= [\sum_{n\in\mathbb{Z}} X^{\zeta}(n) z^{-n-1}, \sum_{m\in\mathbb{Z}} X^{\eta}(m) w^{-m-1}] \\
&= \sum_{n,m\in\mathbb{Z}} [X^{\zeta}, X^{\eta}]_{n+m} z^{-n-1} w^{-m-1} + \sum_{n\in\mathbb{Z}} nB(X^{\zeta}, X^{\eta}) k z^{-n-1} w^{n-1} \\
&= \sum_{l\in\mathbb{Z}} [X^{\zeta}, X^{\eta}]_l (\sum_{n\in\mathbb{Z}} z^{-n-1} w^n) w^{-l-1} + nB(X^{\zeta}, X^{\eta}) k \sum_{n\in\mathbb{Z}} z^{-n-1} w^{n-1} \\
&= [X^{\zeta}, X^{\eta}]_l(w)\delta(z,w) + B(X^{\zeta}, X^{\eta}) k \partial_w \delta(z,w).
\end{aligned}$$

By (3), we have

$$(z-w)^2 [X^{\zeta}(z), X^{\eta}(w)] = 0. \tag{2.8.3}$$

The Affine vertex algebra $V^k(\mathfrak{g}, B) = \langle X^{\zeta_i} | i = 1, ..., dim(\mathfrak{g}) \rangle$ on V^k has a PBW basis as follows:

$$: \partial^{k_1^1} X^{\zeta_1} \dots \partial^{k_{s_1}^1} X^{\zeta_1} \dots \partial^{k_1^m} X^{\zeta_m} \dots \partial^{k_{s_m}^m} X^{\zeta_m} :, \quad s_i \geq 0, \;\; k_1^i \geq \dots k_{s_i}^i \geq 0. \tag{2.8.4}$$

The vertex algebra $V^k(\mathfrak{g}, B)$ generated by $\{X^{\zeta_i} | \zeta_i \in \mathfrak{g}\}$ is called the universal affine vertex algebra associated to $\mathfrak{g}$ and B at level k.

A special case is when $\mathfrak{g}$ is *a simple Lie algebra*, that is a non abelian Lie algebra which has no nontrivial ideals. The bilinear form B is then defined to be the normalized Killing form as follows.

Definition 2.8.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. The Killing form on $\mathfrak{g}$ is the bilinear form $\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by*

$$(x, y)_{\kappa_{\mathfrak{g}}} = tr(adx.ady).$$

Here adx is the adjoint representation of $x \in \mathfrak{g}$ defined by $adx(y) = [x, y]$ for $y \in \mathfrak{g}$.

So, (2.8.1) can be defined in this case as follows:

$$[\zeta \otimes t^n, \eta \otimes t^m] = [\zeta, \eta] \otimes t^{n+m} + n(\zeta|\eta)\delta_{n+m,0}\kappa, \qquad [\kappa, \zeta \otimes t^n] = 0,$$

for $\zeta, \eta \in \mathfrak{g}, n, m \in \mathbb{Z}$. Here $(.|.)$ is *the normalized Killing form*, and is defined as

$$(.|.) = \frac{1}{2h^\vee}(.,.)_{\kappa_{\mathfrak{g}}},$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. In this case, we denote $V^k(\mathfrak{g}, B)$ by $V^k(\mathfrak{g})$.

Let $\{\zeta_1, ..., \zeta_n\}$ be an orthonormal basis for $\mathfrak{g}$ relative to $(.|.)$. There is a natural conformal structure of central charge $\frac{k.dim(\mathfrak{g})}{k+h^\vee}$ on $V^k(\mathfrak{g})$ with the Virasoro element $L(z)$, that is

$$L(z) = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{n} : X^{\zeta_i}(z) X^{\zeta_i}(z) :, \tag{2.8.5}$$

where $k \neq -h^\vee$. In this case, the Virasoro element is called the *Sugawara conformal vector.* For $k = -h^\vee$, the Virasoro element $L(z)$ does not exist.

For the case where $\mathfrak{g}$ is an abelian Lie algebra. Since B is nondegenerate, $V^k(\mathfrak{g}, B)$ is just the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. If we choose an orthonormal basis $\{\zeta_1, ..., \zeta_n\}$ for $\mathfrak{g}$, then $\mathcal{H}(n)$ is generated by $\{\alpha^i = X^{\zeta_i} | i = 1, ..., n\}$.

2.9 Affine vertex algebras $V^k(\mathfrak{s}l_2)$

The ordered basis of $\mathfrak{s}l_2$ is $\{x, y, h\}$, and satisfies the following commutation relations:

$$[x, y] = h, \;\; [h, x] = 2x, \;\; [h, y] = -2y.$$

If we let $X^x(n) \in End(V)$ be the linear operator representing $X^x \otimes t^n$ on V^k. Then, we define

$$X^x(z) = \sum_{n \in \mathbb{Z}} X^x(n) z^{-n-1}.$$

Similarly,

$$X^y(z) = \sum_{n \in \mathbb{Z}} X^y(n) z^{-n-1},$$

$$X^h(z) = \sum_{n \in \mathbb{Z}} X^h(n) z^{-n-1},$$

to be even generating fields each of conformal weight 1. It is easy to see that they lie in $QO(V^k)$, and satisfy the OPE relations

$$X^x(z) X^y(w) \sim k(z-w)^{-2} + X^h(w)(z-w)^{-1}, \tag{2.9.1}$$

$$X^h(z) X^x(w) \sim 2X^x(w)(z-w)^{-1}, \tag{2.9.2}$$

$$X^h(z) X^y(w) \sim -2X^y(w)(z-w)^{-1}, \tag{2.9.3}$$

$$X^h(z)X^h(w) \sim 2k(z-w)^{-2}. \tag{2.9.4}$$

The Affine vertex algebra $V^k(\mathfrak{sl}_2)$ has a PBW basis as follows:

$$: \partial^{k_1^1} X^x ... \partial^{k_{s_1}^1} X^x \partial^{k_1^2} X^y ... \partial^{k_{s_2}^2} X^y \partial^{k_1^3} X^h ... \partial^{k_{s_3}^3} X^h :, \tag{2.9.5}$$

$s_i \geq 0, \;\; k_1^i \geq ... \geq k_{s_i}^i \geq 0, \;\; for \;\; i = 1, 2, 3.$

2.10 Virasoro vertex algebra

Let $\mathcal{K} = \mathbb{C}[t, t^{-1}]$ be the loop algebra. Consider the Witt algebra $Der\mathcal{K} = \{f(t)\partial t : f \in \mathcal{K}\}$ of (continuous) derivations of $\mathcal{K}$. It has a basis given by $\{L_j = -t^{j+1}\partial_t : j \in \mathbb{Z}\}$, and a Lie bracket satisfies:

$$\begin{aligned}
[L_n, L_m]f &= [t^{n+1}\partial t, t^{m+1}\partial t]f \\
&= (m-n)t^{n+m+1}\partial f \\
&= (n-m)L_{n+m}f.
\end{aligned}$$

The Virasoro algebra, denoted by Vir, is the one-dimensional central extension of the Witt algebra

$$0 \to \mathbb{C}_C \to Vir \to \mathrm{D}er\mathcal{K} \to 0,$$

where the generator C is the central charge. The Virasoro Lie algebra Vir is spanned by $\langle L_n, n \in \mathbb{Z}, C\rangle$, where these generators satisfy the following Lie bracket:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}C, \qquad [C, L_n] = 0,$$

and $\mathbb{Z}$-gradation $deg(L_n) = n$, and $deg(C) = 0$.

To construct the generating field, let $L_n \in End(V^k)$ be a linear operator. Define

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

to be an even generating field of conformal weight 2. It is easy to see that it lies in $QO(V_c)$, and satisfies the OPE relation

$$L(z)L(w) \sim \frac{C}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}. \tag{2.10.1}$$

To check that $L(z)$ is quantum commute with itself:

$$\begin{aligned}
[L(z), L(w)] &= [\sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \sum_{m\in\mathbb{Z}} L_m w^{-m-2}] \\
&= \sum_{n,m\in\mathbb{Z}} [L_n, L_m] z^{-n-2} w^{-m-2} \\
&= \sum_{n,m\in\mathbb{Z}} (n-m) L_{n+m} z^{-n-2} w^{-m-2} \\
&+ \frac{C}{12} \sum_{n,m\in\mathbb{Z}} (n^3 - n) L_{n+m} z^{-n-2} w^{-m-2}.
\end{aligned} \tag{2.10.2}$$

The first term in the right hand side of (2.10.2) can be rewritten as:

$$\begin{aligned}
\sum_{n,m\in\mathbb{Z}} (n-m) L_{n+m} z^{-n-2} w^{-m-2} &= \sum_{n,m\in\mathbb{Z}} 2(n+1) L_{n+m} z^{-n-2} w^{-m-2} \\
&+ \sum_{n,m\in\mathbb{Z}} (-n-m-2) L_{n+m} z^{-n-2} w^{-m-2}.
\end{aligned} \tag{2.10.3}$$

Recall the delta-function $\delta(z,w)$, and compute the right hand side of (2.10.3). Thus,

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}} 2(n+1)L_{n+m}z^{-n-2}w^{-m-2} &= \sum_{n,m\in\mathbb{Z}} 2(n+1)L_{n+m}z^{-n-2}w^{-n-m-2}w^{n}\\ &= \sum_{k,n\in\mathbb{Z}} 2L_k w^{-k-2}(n+1)z^{-n-2}w^{n}\\ &= 2L(w)\partial_w\delta(z,w).\end{aligned} \tag{2.10.4}$$

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}} (-n-m-2)L_{n+m}z^{-n-2}w^{-m-2} &=\\ &\sum_{n,m\in\mathbb{Z}} (-n-m-2)L_{n+m}z^{-n-2}w^{-n-m-3}w^{n+1}\\ &= \sum_{k,n\in\mathbb{Z}} (-k-2)L_k w^{-k-3}z^{-n-2}w^{n+1}\\ &= \partial_w L(w)\partial_w\delta(z,w).\end{aligned} \tag{2.10.5}$$

$$\frac{C}{12}\sum_{n,m\in\mathbb{Z}} (n^3-n)L_{n+m}z^{-n-2}w^{-m-2} = \frac{C}{12}\partial_w^3\delta(z,w). \tag{2.10.6}$$

Combining (2.10.4), (2.10.5), (2.10.6), and so by (3), we have

$$(z-w)^4[L(z),L(w)] = 0. \tag{2.10.7}$$

The Virasoro vertex algebra $\langle L(z)\rangle$ on V_c has a PBW basis as follows:

$$:\partial^{k_1}L...\partial^{k_s}L:, \qquad s\geq 0,\ \ k_1\geq ...\geq k_s\geq 0. \tag{2.10.8}$$

Remark 2.10.1. *Note That* $L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-1}$, *but we usually use the previous definition so that* $L(n) = L_{n-1}$. *Recall that* L_0 *is diagonalizable since*

$[L_0, L_n] = -nL_n$, *and* $L_0 v_c = 0$. *The eigenvalues of* L_0 *correspond to a highest weight vector* $v \neq 0$, *satisfying* $L_0 v = hv$ *for some* $h \in \mathbb{C}$.

2.11 The fermionic ghost system (bc-system $\mathcal{E}(V)$)

Let Cl be the Clifford algebra associated to the vector space $\mathbb{C}((t)) \oplus \mathbb{C}((t))dt$ and the non-degenerate bilinear form $B(.,.)$. It is generated by $\psi_n = t^n$, and $\psi_n^* = t^{n-1}dt$ which satisfy the following Lie bracket:

$$[\psi_n, \psi_m^*]_+ = \delta_{n,-m}, \qquad [\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0, \tag{2.11.1}$$

where the notation $[A, B]_+$ means $AB + BA$.

To construct generating fields, let $\psi_n \in End(V)$, and $\psi_n^* \in End(V^*)$ be linear operators. Define

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \qquad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}$$

to be odd generating fields of conformal weights 1 and 0, respectively.

To check that $\psi(z), \psi^*(z)$ quantum commute :

From (2.11.1), we have

$$[\psi(z), \psi(w)]_+ = [\psi^*(z), \psi^*(w)]_+ = 0, \qquad [\psi(z), \psi^*(w)]_+ = \delta(z, w), \tag{2.11.2}$$

and this implies that $(z - w)[\psi(z), \psi^*(w)]_+ = 0$.

The fermionic ghost system $\langle \psi(z), \psi^*(z) \rangle$ on V has a basis as follows:

$$: \partial^{k_1^1} \psi ... \partial^{k_{s_1}^1} \psi \partial^{k_1^2} \psi^* ... \partial^{k_{s_2}^2} \psi^* :, \qquad s_i \geq 0, \;\; k_1^i \geq ... \geq k_{s_i}^i \geq 0 \;\; for \;\; i = 1, 2. \tag{2.11.3}$$

The fields $\psi(z)$ and $\psi^*(z)$ are denoted by $b(z)$ and $c(z)$, respectively in the physics literature, and the corresponding conformal vertex algebra is called the bc-system $\mathcal{E}(V)$. This vertex algebra is the unique vertex algebra with odd generators, was introduced by Friedan-Martinec-Shenker. For a vector space V of dimension n over $\mathbb{C}$, the fields $b^x(z), c^{x^*}(z)$ for $x \in V$ and $x^* \in V^*$ satisfy the OPE relations

$$b^x(z)c^{x^*}(w) \sim \langle x^*, x\rangle (z-w)^{-1}, \tag{2.11.4}$$

$$c^{x^*}(z)b^x(w) \sim \langle x^*, x\rangle (z-w)^{-1}, \tag{2.11.5}$$

$$b^x(z)b^y(w) \sim 0, \qquad c^{x^*}(z)c^{y^*}(w) \sim 0, \tag{2.11.6}$$

where $\langle , \rangle$ demotes the natural pairing between V^* and V.

The rank n fermionic ghost system is the tensor product of n copies of rank 1 fermionic ghost system with odd generators b^i, c^i for $i = 1, ..., n$. The OPE relations above still hold for each generator, that is

$$b^i(z)c^j(w) \sim \delta_{i,j}(z-w)^{-1}. \tag{2.11.7}$$

2.12 The fermionic vertex superalgebra

For a finite dimensional vector space V, let Cl_V be the Clifford algebra associated to the vector space $V((t)) \oplus V^*((t))dt$ and the non-degenerate bilinear form $B(.,.)$. If we consider $\{x_i | i \in I\}$ as a basis of V, and $\{x_i^* | i \in I\}$ as the dual basis of V^*, then Cl_V is generated by $\psi_{i,n} = x_i \otimes t^n,\ \ \psi^*_{i,n} = x_i^* \otimes t^{n-1}dt$ for $i \in I,\ \ n \in \mathbb{Z}$,

and these generators satisfy the following Lie bracket:

$$[\psi_{i,n}, \psi^*_{j,m}]_+ = \delta_{i,j}\delta_{n,-m}, \qquad [\psi_{i,n}, \psi_{j,m}]_+ = [\psi^*_{i,n}, \psi^*_{j,m}]_+ = 0, \tag{2.12.1}$$

with $\mathbb{Z}$-gradation $deg(\psi^*_{i,n}) = -deg(\psi_{i,n}) = 1$, and $deg(\mathbf{1}) = 0$.

Let $\bigwedge_V$ be the Fock representation of Cl, generated by a vector $\mathbf{1}$ satisfying:

$$\psi_{i,n}\mathbf{1} = 0, \;\; n \geq 0, \qquad \psi^*_{i,n}\mathbf{1} = 0, \;\; n > 0.$$

Given a Lie algebra $\mathfrak{g}$. Let $\hat{\mathfrak{g}}$ be the one dimensional central extension of $\mathfrak{g}((t))$

$$0 \to \mathbb{C}_\kappa \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0,$$

where a generator κ is the central charge. The Lie algebra $\hat{\mathfrak{g}}$ is spanned by $\langle A \otimes f(t), A \in \mathfrak{g}, \kappa\rangle$, where these generators satisfy the following Lie bracket:

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - nB(A, B)\delta_{n,-m}\kappa, \qquad [\kappa, A \otimes f(t)] = 0, \tag{2.12.2}$$

for $A, B \in \mathfrak{g}$.

The vertex subalgebra $\bigwedge_V$ is isomorphic to the tensor product of $|I|$ copies of the fermonic ghost system $\bigwedge$. The vertex algebra structure is defined similarly as in section (2.11), and in particular, we have generating fields

$$\psi_i(z) = \sum_{n\in\mathbb{Z}} \psi_{i,n} z^{-n-1}, \qquad \psi^*_i(z) = \sum_{n\in\mathbb{Z}} \psi^*_{i,n} z^{-n}.$$

Chapter 3

A Glance at Classical Invariant Theory

In this chapter, we shall provide a brief glimpse into the Classical Invariant Theory. The term "invariant" in mathematics refers to a quantity that does not change under certain classes of transformations. A precise definition for this notion and instructive examples are given. Some important theorems in this branch of mathematics are stated too, such as the Hilbert Finiteness Theorem, and the Weyl First and Second Fundamental Theorem of the Invariant Theory.

3.1 Invariants

Let V be a finite-dimensional vector space. The notation $GL(V)$ stands for the group of all invertible linear maps $V \to V$. If we choose a basis $\{v_1, ..., v_n\}$ of V, then $GL(V)$ is identified with the group of invertible $n \times n$ matrices $GL_n(\mathbb{C})$. More precisely, for an automorphism $g \in GL(V)$, we have

$$g(v_k) = \sum_{i=1}^{n} a_{ik} v_i$$

for constants $a_{ik} \in \mathbb{C}$. Then, the matrix corresponding to g is just the matrix whose entries are a_{ik} based on our choice of basis.

Definition 3.1.1. *Given a group G, a representation of G on a vector space V is a group homomorphism*

$$\rho : G \longrightarrow GL(V).$$

We also call V a G-module.

If V is a G-module, there is an induced action of G on the ring $\mathbb{C}[V]$ of polynomial functions on V. If $f \in \mathbb{C}[V]$ is a function and $g \in G$, we define $g(f) \in \mathbb{C}[V]$ by

$$g(f)(v) = f(g^{-1}(v)).$$

Definition 3.1.2. *Let V be a finite-dimensional G-module, a polynomial function $f \in \mathbb{C}[V]$ is called G-invariant if $g(f) = f$ for all $g \in G$. Equivalently, $f(gv) = f(v)$ for $g \in G$ and $v \in V$. Evidently, the set of G-invariant polynomials $\mathbb{C}[V]^G$ is a subring of $\mathbb{C}[V]$ called the invariant ring.*

Natural questions might arise:

- Is $\mathbb{C}[V]^G$ a finitely generated over $\mathbb{C}$? If yes,
- What are all invariants of the ring $\mathbb{C}[V]$ under a group action G? Equivalently, what is the invariant ring $\mathbb{C}[V]^G$?

Example 3.1.3. *Let $\mathbb{Z}_2$ act on the polynomial ring $\mathbb{C}[x_1, x_2]$ via the action $x_i \mapsto -x_i$ for $i = 1, 2$. The invariant ring is generated by the polynomials $p = x_1^2$, $q = x_2^2$, and $r = x_1x_2$. Also, the ideal of relations among p, q, r is generated by $r^2 - pq$, so $\mathbb{C}[x_1, x_2]^{\mathbb{Z}_2} \cong \mathbb{C}[p, q, r]/\langle r^2 - pq\rangle$.*

Example 3.1.4. *Let $SL_n(\mathbb{C})$ act on the polynomial ring $\mathbb{C}[x_1, x_2, ..., x_n]$, where the special linear group $SL_n(\mathbb{C})$ is the subgroup of $GL_n(\mathbb{C})$ of matrices with determinant*

1. *Consider the representation of* $SL_n(\mathbb{C})$ *on the space of* $n \times n$ *matrices* $M_n(\mathbb{C})$ *given by left multiplication*

$$(g, A) \longmapsto gA, \qquad for \;\; g \in SL_n(\mathbb{C}) \;\; and \;\; A \in M_n(\mathbb{C}).$$

The invariant ring is generated by the determinant polynomial, that is

$$\mathbb{C}[x_1, ..., x_n]^{SL_n(\mathbb{C})} = \mathbb{C}[det].$$

In all examples we have met so far, the invariant ring $\mathbb{C}[V]^G$ under action of G was finitely generated, but this is not always the case. One of highlighted results in describing the invariant ring $\mathbb{C}[V]^G$ is due to David Hilbert (1893). He investigated the finiteness of the invariant ring $\mathbb{C}[V]^G$ under a group action.

Theorem 3.1.5. ***Hilbert Finiteness Theorem*** *For any finite dimensional, reductive group* $G \subset GL_n(\mathbb{C})$, *the invariant ring* $\mathbb{C}[V]^G$ *is finitely generated.*

Throughout this dissertation, we will consider the case where G is a finite-dimensional, reductive subgroup of $GL_n(\mathbb{C})$.

3.2 The classical invariant theory (CIT)

The preceding section leads to a fundamental problem in the classical invariant theory (shortly CIT), which is to find generators and relations for the invariant ring. A significant contributions have been made by famous mathematicians during the 19th century.

Given a finite-dimensional G-module V, let V^* denote the dual representation, consider the invariant ring

$$R = \mathbb{C}[\bigoplus_{j \geq 0} V_j \oplus V_j^*]^G,$$

where each V_j is isomorphic to V, and each V_j^* is isomorphic to V^*. In Weyl's terminology, a *first fundamental theorem of invariant theory* for the pair (G, V) is a set of generators for R, and a *second fundamental theorem of invariant theory* for (G, V) is a set of generators for the ideal of relations among the generators of R. It is very difficult to describe these rings in general, and even for $G = SL_2$, first and second fundamental theorems are known only for the first few finite-dimensional SL_2-modules. An important theorem of Weyl [W] are the explicit first and second fundamental theorems when G is a classical group and V is the standard module ($\mathbb{C}^n$ for $G = GL_n$, SL_n, or SO_n or O_n, and $\mathbb{C}^{2n}$ when $G = Sp_{2n}$). In the case of the adjoint representations of the classical groups, this was proven by Procesi [P], and for the 7-dimensional representation of G_2 and the 8-dimensional representation of $Spin_7$ it was proved by Schwarz [Sch].

Let V be a finite-dimensional representation of G and let $V_j \cong V$ for $j \geq 0$, as above. For all $p \geq 0$, there is an action of GL_p on $\oplus_{j=0}^{p-1} V_j$ which commutes with the action of G. The inclusions $GL_p \hookrightarrow GL_q$ for $p < q$ sending

$$M \to \begin{bmatrix} M & 0 \\ 0 & I_{q-p} \end{bmatrix}$$

induce an action of $GL_\infty = lim_{p\to\infty} GL_p$ on $\oplus_{j\geq 0} V_j$. We obtain an action of GL_∞ on $\mathbb{C}[\oplus_{j\geq 0} V_j]$ which commutes with the action of G, so GL_∞ acts on R as well. The elements $\sigma \in GL_\infty$ are called *polarization operators*, and for a given $f \in R, \sigma f$ is known as a polarization of f.

Theorem 3.2.1. *(Weyl) R is generated by the set of polarizations of any set of generators for $\mathbb{C}[\oplus_{j\geq 0}^{n-1} V_j]^G$, where $n = dim(V)$. Since G is reductive, $\mathbb{C}[\oplus_{j\geq 0} V_j^{n-1}]^G$ is finitely generated, so there exists a finite subset $\{f_1, ..., f_r\} \subset R$ whose polarizations generate R.*

In this dissertation, we need the explicit first and second fundamental theorems in the special case $G = \mathbb{Z}_2$ and V the nontrivial one-dimensional module on which the generator of $\mathbb{Z}_2$ acts by -1. This can be viewed as the special case of Weyl's first and second fundamental theorems for the orthogonal group O_n for $n = 1$.

Theorem 3.2.2. *(Weyl) For $k \geq 0$, let V_k be the copy of the standard $\mathbb{Z}_2$-module $\mathbb{C}$ with a basis $\{x_k\}$. The invariant ring $\mathbb{C}[\oplus_{k\geq 0} V_k]^{\mathbb{Z}_2}$ is generated by the quadratics*

$$q_{a,b} = x_a x_b, \qquad 0 \leq a \leq b. \tag{3.2.1}$$

For $a > b$, define $q_{a,b} = q_{b,a}$ and let $\{Q_{a,b} | a, b \geq 0\}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$ and no other algebraic relations. The kernel I_1 of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to \mathbb{C}[\oplus_{k\geq 0} V^k]^{\mathbb{Z}_2}, \qquad Q_{a,b} \mapsto q_{a,b} \tag{3.2.2}$$

is generated by the 2×2 determinates

$$d_{I,J} = det \begin{bmatrix} Q_{i_0,j_0} & Q_{i_0,j_1} \\ Q_{i_1,j_0} & Q_{i_1,j_1} \end{bmatrix}. \tag{3.2.3}$$

In this notation, $I = (i_0, i_1)$ and $J = (j_0, j_1)$ are lists of integers satisfying

$$0 \leq i_0 < i_1, \qquad 0 \leq j_0 < j_1. \tag{3.2.4}$$

Since $Q_{a,b} = Q_{b,a}$, so it is clear that $d_{I,J} = d_{J,I}$.

Chapter 4

Orbifolds and Strong Generated Vertex Algebras

4.1 Orbifolds

In this section, we shall define the invariant subalgebras, and the orbifolds which is a standard method to construct a new vertex algebra from an old one. Many interesting vertex algebras are built as orbifolds or as extensions of orbifolds. Before we dive into this, we introduce the notion of filtrations in order to make connection between vertex algebras and commutative algebras. This allows us to apply Weyl's first and second fundamental theorems of invariant theory. Next, we define strong finite generated vertex algebras. We give a procedure to detect a strong finite generating set for an orbifold, and further determine the minimal strong finite generators have to be considered. We do not require the Classical Invariant Theory here.

Definition 4.1.1. *Given a vertex algebra $\mathcal{V}$, and a group G of automorphisms of $\mathcal{V}$ acting on $\mathcal{V}$. The invariant vertex algebra of $\mathcal{V}$ is a subalgebra $\mathcal{V}^G \subset \mathcal{V}$ of all*

G-invariants in $\mathcal{V}$ defined by

$$\mathcal{V}^G = \{a \in \mathcal{V} | ga = a \ \ for \ \ g \in G\}.$$

4.2 Weak and good increasing filtrations

Definition 4.2.1. *A good increasing filtration [LiII] on a vertex algebra $\mathcal{V}$ is a $\mathbb{Z}_{\geq 0}$-filtration*

$$\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \mathcal{V}_{(2)} ..., \qquad \mathcal{V} = \bigcup_{d \geq 0} \mathcal{V}_{(d)} \tag{4.2.1}$$

satisfying that $\mathcal{V}_{(0)} = \mathbb{C}$, , and for all $a \in \mathcal{V}_{(r)}, b \in \mathcal{V}_{(s)}$ we have

$$a \circ_n b \in \mathcal{V}_{(r+s)}, \quad for \ \ n < 0, \tag{4.2.2}$$

$$a \circ_n b \in \mathcal{V}_{(r+s-1)}, \quad for \ \ n \geq 0. \tag{4.2.3}$$

Definition 4.2.2. *An element $a(z) \in \mathcal{V}$ has at most degree d if $a(z) \in \mathcal{V}_{(d)}$. We then write deg $a(z) = d$.*

Definition 4.2.3. *A vertex algebra $\mathcal{V}$ is called graded by degree if it is equipped with a $\mathbb{Z}_{\geq 0}$-grading*

$$\mathcal{V} = \bigoplus_{k \geq 0} \mathcal{V}^{(k)},$$

where $\mathcal{V}_{(k)} = \oplus_{i=0}^{k} \mathcal{V}^{(i)}$.

Definition 4.2.4. *For a vertex subalgebra $\mathcal{A} \subset \mathcal{V}$, a good increasing filtration on $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}$-filtration*

$$\mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} ...,$$

induced by a filtration (4.2.1) on $\mathcal{V}$ after restriction, where $\mathcal{A}_{(k)} = \mathcal{A} \cap \mathcal{V}_{(k)}$.

Definition 4.2.5. *The associated graded algebra* $gr(\mathcal{V}) = \bigoplus_{d\geq 0} \mathcal{V}_{(d)}/\mathcal{V}_{(d-1)}$ *is a* $\mathbb{Z}_{\geq 0}$*-graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on* $\mathcal{V}$*, where* $\mathcal{V}_{-1} = \{0\}$*. It has a derivation* ∂ *of degree zero (induced by the operator* $\partial_z = \frac{d}{dz}$ *on* $\mathcal{V}$*).*

Remark 4.2.6. *1. There is no natural linear map* $\mathcal{V} \to gr(\mathcal{V})$ *in general, but for each* $r \geq 1$ *we have the projections*

$$\varphi_r : \mathcal{V}_{(r)} \to \mathcal{V}_{(r)}/\mathcal{V}_{(r-1)} \subset gr(\mathcal{V}). \tag{4.2.4}$$

2. If $a, b \in gr(\mathcal{V})$ *are homogeneous of degrees* r, s *respectively, and*

$$a(z) \in \mathcal{V}_{(r)}, b(z) \in \mathcal{V}_{(s)}, \ \textit{such that} \ \varphi_r(a(z)) = a, \ \ \varphi_s(b(z)) = b,$$

then $\varphi_{r+s}(: a(z)b(z) :) = ab$*.*

Let $\mathcal{R}$ be the category of pairs $(\mathcal{V}, deg)$, where $\mathcal{V}$ is a vertex algebra equipped with a $\mathbb{Z}_{\geq 0}$-filtration. Morphisms in $\mathcal{R}$ are morphisms of vertex algebras which preserve the filtration. The assignment $\mathcal{V} \mapsto gr(\mathcal{V})$ is a functor from the category of vertex algebras $\mathcal{R}$ to the category of $\mathbb{Z}_{\geq 0}$-graded (super)commutative rings with a differential ∂ called ∂-ring.

Definition 4.2.7. *A* ∂*-ring* R *is said to be generated by a subset* $\{a_i | i \in I\}$ *if* $\{\partial^k a_i | i \in I, k \geq 0\}$ *generates* R *as a graded ring.*

Definition 4.2.8. *[ACL] A weak increasing filtration on a vertex algebra* $\mathcal{V}$ *is a* $\mathbb{Z}_{\geq 0}$*-filtration*

$$\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \mathcal{V}_{(2)} \ldots, \qquad \mathcal{V} = \bigcup_{d\geq 0} \mathcal{V}_{(d)} \tag{4.2.5}$$

satisfying that for all $a \in \mathcal{V}_{(r)}, b \in \mathcal{V}_{(s)}$ *we have*

$$a \circ_n b \in \mathcal{V}_{(r+s)}, \quad for \ \ n \in \mathbb{Z}, \tag{4.2.6}$$

Remark 4.2.9. *The associated graded algebra* $gr(\mathcal{V}) = \bigoplus_{d\geq 0} \mathcal{V}_{(d)}/\mathcal{V}_{(d-1)}$, *obtained by the weak increasing filtration, is a vertex algebra, but it is no longer abelian in general. By this, the filtration we have here is weaker than the previous one.*

4.3 Strong finite generating set

Definition 4.3.1. *Given a vertex algebra* $\mathcal{V}$, *and* $S = \{a_i | i \in I\} \subset \mathcal{V}$. *We say that* S *generates* $\mathcal{V}$ *if every* $a \in \mathcal{V}$ *is a linear combination of words in* $a_i, \circ_n$ *for* $i \in I$ *and* $n \in \mathbb{Z}$.

We say that S *strongly generates* $\mathcal{V}$ *if every* $a \in \mathcal{V}$ *is a linear combination of of words in* a_i, *for* $i \in I$ *and the negative circle products, i.e.* $\circ_n$ *for* $n < 0$. *Equivalently,* $\mathcal{V}$ *is spanned by the ordered monomials of vertex operators*

$$\{: \partial^{k_1} a_{i_1} ... \partial^{k_m} a_{i_m} : | i_1, ..., i_m \in I, \ \ 0 \leq k_1 \leq ... \leq k_m\}. \tag{4.3.1}$$

If I *is finite, then the vertex algebra* $\mathcal{V}$ *is strongly finitely generated, (shortly SFG). Moreover, we say that a vertex algebra* $\mathcal{V}$ *is of type* $\mathcal{W}((d_1)^{n_1}, ..., (d_r)^{n_r})$ *if it has a minimal strong generating set consisting of* n_i *fields in each weight* d_i *for* $i = 1, ..., r$.

Example 4.3.2. *All examples we introduced in the first chapter, Heisenberg, Affine, Virasoro vertex algebras, the fermionic ghost system, as well as the* $\mathcal{W}$*-algebras are SFG.*

The category $\mathcal{R}$ mentioned in the previous section is closed under taking subalgebras. This category together with the functor connecting vertex algebras $\mathcal{V}$ equipped with a $\mathbb{Z}_{\geq 0}$-filtration to $\mathbb{Z}_{\geq 0}$- graded commutative algebras with derivation allow us to find some of the algebraic structure of the vertex algebras. For example, for any vertex algebra $\mathcal{V} \in \mathcal{R}$, the ring structure of ∂-ring, namely $gr(\mathcal{V})$ allows us to construct the strongly generating set for $\mathcal{V}$ as a vertex algebra as well by the following reconstruction property:

Lemma 4.3.3. *[LI] Let $\mathcal{V}$ be a vertex algebra in $\mathcal{R}$. Consider the collection $\{a_i | i \in I\}$ that generates $gr(\mathcal{V})$ as a ∂-ring, where a_i is homogenous of degree d_i. Then $\mathcal{V}$ is strongly generated by the collection $\{a_i(z) | i \in I\}$, where $a_i(z) \in \mathcal{V}_{(d_i)}$ such that $\varphi_{d_i}(a_i(z)) = a_i$.*

Similarly, the alternative reconstruction property for the weak increasing filtration is given by the following lemma:

Lemma 4.3.4. *[AL] Given a vertex algebra $\mathcal{V}$ with weak increasing filtration. Consider the collection $\{a_i | i \in I\}$ that generates $gr(\mathcal{V})$ as a ∂-ring, where a_i is homogenous of degree d_i. Then $\mathcal{V}$ is strongly generated by the collection $\{a_i(z) | i \in I\}$, where $a_i(z) \in \mathcal{V}_{(d_i)}$ such that $\varphi_{d_i}(a_i(z)) = a_i$.*

One of an interesting problems arising is that if we have a SFG vertex algebra $\mathcal{V}$ with a group G of automorphisms acting on $\mathcal{V}$. Does the orbifold $\mathcal{V}^G$ inherit this property, i.e. is it SFG? This is an analogous question to the Hilbert problem 3.1.5. To answer this, we need to demonstrate an explicit strong generating set for $\mathcal{V}^G$. We shall follow the following procedure given in [LIV]:

Consider a ring $R = \mathbb{C}[\oplus_{j \geq 0} V_j]^G$, as it was defined in (3.2), graded by degree and weight equipped with the natural derivation ∂. Each $x_j \in V_j$ has degree 1 and weight $j + 1$.

On the other hand, let $\mathcal{V}$ be a vertex algebra with weight grading $\mathcal{V} = \bigoplus_{m \geq 0} \mathcal{V}[m]$. Consider a group G acting on $\mathcal{V}$ by automorphisms. Recall, we will be considering a reductive, finite-dimensional group G.

Our task is to find a strong finite generating set for an orbifold of $\mathcal{V}$. In order to get this, define a $\mathbb{Z}_{\geq 0}$ good increasing filtration for $\mathcal{V}^G$ so that the pair $(\mathcal{V}^G, deg)$ lies in $\mathcal{R}$. This filtration satisfies that $gr(\mathcal{V}^G) \cong gr(\mathcal{V})^G \cong R$ as ∂-rings graded by degree and weight. Construct a natural infinite strong generating set for $\mathcal{V}^G$ using classical invariant theory, and then choose a generating set $S = \{s_i | i \in I\}$ for R as a

∂-ring consisting a finite number of elements in each weight where s_i has degree d_i, and weight w_i. Lemma 4.3.3 allows us to get a strong generating set $T = \{t_i | i \in I\}$ for $\mathcal{V}^G$ such that $s_i = \varphi_{d_i}(t_i)$, where

$$\varphi_{d_i} : (\mathcal{V}^G)_{(d_i)} \to (\mathcal{V}^G)_{(d_i)}/(\mathcal{V}^G)_{(d_i-1)} \subset gr(\mathcal{V}^G)$$

is the usual projection.

Our goal is to find a minimal strong finite generating set. To do this, find a normally ordered polynomial relation (it will be explained later) of minimal weight among the generators to eliminate as much as you can of the generators, and so the remaining ones will form a minimal strong generating set.

Definition 4.3.5. *Let $\mathcal{V}$ be a vertex algebra, we say that $\omega \in \mathcal{V}$ is in normal form if it has been expressed as a linear combination of monomials of the form* (4.3.1).

Definition 4.3.6. *Let $p \in R$ be a homogeneous polynomial of degree d, and S, T are two generating sets defined as above. A normal ordering of p will be a choice of normally ordered polynomial $P \in (\mathcal{V}^G)_{(d)}$ obtained by replacing each $s_i \in S$ by $t_i \in T$, and replacing ordinary products with iterated Wick products. The normally ordered polynomial P is not unique, but for any choice of P, we have $\varphi_d(P) = p$.*

Now, we shall see how the normally ordered polynomial relation in $\mathcal{V}^G$ among the elements of T and their derivatives is formed [LIV]:

Let p be a relation among the generators of R (from the second fundamental theorem for (G, V)). We may assume it to be a homogeneous of degree d. Consider some normal ordering of p, namely $P^d \in \mathcal{V}^G$. Since $gr(\mathcal{V}^G) \cong R$ as graded rings, it follows that $P^d \in (\mathcal{V}^G)_{(d-1)}$. The polynomial $\varphi_{d-1}(P^d) \in R$ is homogenous of degree $d-1$; if it is nonzero, it could be expressed as a polynomial in the variables $s_i \in S$ and their derivatives. Let $-P^{d-1}$ be some normal ordering of this polynomial.

Then, $P^d + P^{d-1}$ has the property

$$\varphi_d(P^d + P^{d-1}) = p, \qquad P^d + P^{d-1} \in (\mathcal{V}^G)_{(d-2)}.$$

Continuing in this manner yields

$$P = \sum_{k=1}^{d} P^k \in \mathcal{V}^G, \tag{4.3.2}$$

which is exactly zero. We could think of P as a correction of the relation p. By induction on d, we could see that all normally ordered polynomial relations in $\mathcal{V}^G$ among the elements of T and their derivatives are consequences of such relations.

In order to examine the SFG of $\mathcal{V}^G$, each element $t \in T \setminus T'$ must admit a "decoupling relation" as it is defined as follows:

Definition 4.3.7. *Let S, T be two generating sets defined as above. A decoupling relation is a relation expressed as a normally ordered polynomial in the elements of a finite subset $T' \subset T$ and their derivatives.*

Remark 4.3.8. *In general, R is not finitely generated as a ∂-ring, but $\mathcal{V}^G$ is more likely to be strongly generated as a vertex algebra by a finite subset $T' \subset T$. Consider a relation in $\mathcal{V}^G$ of the form* (4.3.2)*, and suppose that for some $t \in T \setminus T'$ which appears in P^k for some $k < d$, with nonzero coefficient. If the remaining terms in* (4.3.2) *depend on the elements of T' only and their derivatives, then t could be expressed as a linear combination of such elements, and so we get a decoupling relation.*

We shall include concrete examples of the procedure of finding minimal strong generating set for an orbifold next sections.

Definition 4.3.9. *Let $\mathcal{V}$ be a vertex algebra, and $S = \{a_i | i \in I\} \subset \mathcal{V}$. We say that $\mathcal{V}$ is freely generated by S if there are no nontrivial normally ordered polynomial relations among the generators and their derivatives.*

Chapter 5

The $\mathbb{Z}_2$-Orbifold of the Universal Affine Vertex Algebra

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra equipped with the standard normalized Killing form $(.|.)$. In this chapter, we shall give an explict minimal strong finite generating set for the $\mathbb{Z}_2$-orbifold of the universal affine vertex algebra $V^k(\mathfrak{g})$ associated to $\mathfrak{g}$ at generic level k. We study this by reducing the problem into studying a simpler object, the $\mathbb{Z}_2$-orbifold of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$ using the deformation argument. The case where $\dim(\mathfrak{g}) = 3$ (i.e., $\mathfrak{g} = \mathfrak{s}l_2$) is studied separately. Also, the set of nongeneric values of k is determined in this case by computing all poles of the structure constants appearing in the OPE algebra of the generators.

5.1 The $\mathbb{Z}_2$-orbifold of $\mathcal{H}(n)$

5.1.1 Filtrations

The rank n Heisenberg vertex algebra $\mathcal{H}(n)$ is freely generated by α^i for $i = 1, ..., n$, and spanned by all normally ordered monomials of the form

$$: \partial^{k_1^1}\alpha^1 \dots \partial^{k_{s_1}^1}\alpha^1 \dots \partial^{k_1^n}\alpha^n \dots \partial^{k_{s_n}^n}\alpha^n :, \quad s_i \geq 0, \;\; k_1^i \geq \dots k_{s_i}^i \geq 0. \tag{5.1.1}$$

Therefore, (5.1.1) forms a PBW basis for $\mathcal{H}(n)$.

Filtrations on $\mathcal{H}(n)$. Define an increasing filtration on $\mathcal{H}(n)$ as follows:

$$\mathcal{H}(n)_{(0)} \subset \mathcal{H}(n)_{(1)} \subset \mathcal{H}(n)_{(2)} \subset ..., \qquad \mathcal{H}(n) = \bigcup_{d\geq 0} \mathcal{H}(n)_{(d)}, \tag{5.1.2}$$

where $\mathcal{H}(n)_{(-1)} = \{0\}$, and $\mathcal{H}(n)_{(r)}$ is spanned by the iterated Wick products of the generators α^i and their derivatives such that at most r of α^i and their derivatives appear. That is, $\mathcal{H}(n)_{(r)}$ is spanned by all normally ordered monomials of the form (5.1.1) such that the sum $s_1 + \dots + s_n \leq r$. In particular, each α^i and its derivatives have degree 1.

We have a $\mathbb{Z}_{\geq 0}$-grading

$$\mathcal{H}(n) = \bigoplus_{d\geq 0} \mathcal{H}(n)^{(d)}, \tag{5.1.3}$$

where $\mathcal{H}(n)_{(d)} = \oplus_{k=0}^{d} \mathcal{H}(n)^{(k)}$. From defining the OPE relation (2.7.1), this is actually a good increasing filtration, and so, $\mathcal{H}(n)$ equipped with such a good filtration lies in the category $\mathcal{R}$. The OPE relation will be replaced with $\alpha^i(z)\alpha^j(w) \sim 0$, and so the $\mathbb{Z}_{\geq 0}$-associated graded algebra

$$gr(\mathcal{H}(n)) = \bigoplus_{d\geq 0} \mathcal{H}(n)_{(d)}/\mathcal{H}(n)_{(d-1)}$$

is now an abelian vertex algebra freely generated by α^i. The rank n Heisenberg vertex algebra $\mathcal{H}(n)$ equipped with such filtrations lies in the category $\mathcal{R}$. Then, $\mathcal{H}(n) \cong gr(\mathcal{H}(n))$ as linear spaces, and as commutative algebras, we have

$$gr(\mathcal{H}(n)) \cong \mathbb{C}[\partial^a \alpha^i | a \geq 0, i = 1, ..., n].$$

Therefore, $gr(\mathcal{H}(n))$ is regarded as a commutative algebra of all polynomials in $\partial^a \alpha^i, a \geq 0$ with a differential ∂ of degree zero acting on the generators as follows:

$$\partial(\alpha^i_a) = \alpha^i_{a+1}. \tag{5.1.4}$$

Here α^i_a is the image of $\partial^a \alpha^i(z)$ in $gr(\mathcal{H}(n))$ under the projection

$$\varphi_1 : \mathcal{H}(n)_{(1)} \to \mathcal{H}(n)_{(1)}/\mathcal{H}(n)_{(0)} \subset gr(\mathcal{H}(n)).$$

For any reductive group $G \subset O(n)$, $\mathcal{H}(n)^G$ will inherit the filtration, where $O(n)$ is the full automorphism group of $\mathcal{H}(n)$.

Consider the subgroup $\mathbb{Z}_2$ of the automorphism group of $\mathcal{H}(n)$, which is generated by the nontrivial involution θ. The action of θ on the generators will be as follows:

$$\theta(\alpha^i) = -\alpha^i. \tag{5.1.5}$$

The OPE relations (2.7.1) will be preserved by this action on $\mathcal{H}(n)$, that is

$$\alpha^i \circ_m \alpha^j = \theta(\alpha^i) \circ_m \theta(\alpha^j)$$

for all m as well as the filtration (5.1.2), and induces an action of $\mathbb{Z}_2$ on $gr(\mathcal{H}(n))$.

Going back to $\mathcal{H}(n)^{\mathbb{Z}_2}$, it is also spanned by all normally ordered monomials of the form (5.1.1), where the length $s_1 + \cdots s_n$ is even. Since $\mathcal{H}(n)$ is freely generated by α^i, these monomials form a basis for $\mathcal{H}(n)^{\mathbb{Z}_2}$, and the normal form is unique.

The filtration on $\mathcal{H}(n)^{\mathbb{Z}_2}$ is obtained from the filtration (5.1.2) after restriction as follows:

$$(\mathcal{H}(n)^{\mathbb{Z}_2})_{(0)} \subset (\mathcal{H}(n)^{\mathbb{Z}_2})_{(1)} \subset ..., \qquad \mathcal{H}(n)^{\mathbb{Z}_2} = \bigcup_{d \geq 0} (\mathcal{H}(n)^{\mathbb{Z}_2})_{(d)},$$

where $(\mathcal{H}(n)^{\mathbb{Z}_2})_{(r)} = \mathcal{H}(n)^{\mathbb{Z}_2} \cap \mathcal{H}(n)_{(r)}$.

The action of $\mathbb{Z}_2$ on $\mathcal{H}(n)$ descends to an action on $gr(\mathcal{H}(n))$, and so we have a linear isomorphism $\mathcal{H}(n)^{\mathbb{Z}_2} \cong gr(\mathcal{H}(n)^{\mathbb{Z}_2})$ as linear spaces. Similarly, $\mathbb{Z}_2$ acts on $gr(\mathcal{H}(n)) \cong \mathbb{C}[\partial^a \alpha^i | a \geq 0, i = 1, ..., n]$, and so we have a linear isomorphism

$$gr(\mathcal{H}(n)^{\mathbb{Z}_2}) \cong gr(\mathcal{H}(n))^{\mathbb{Z}_2} \cong \mathbb{C}[\partial^a \alpha^i | a \geq 0, i = 1, ..., n]^{\mathbb{Z}_2} \tag{5.1.6}$$

as commutative algebras. The weight and degree are preserved by (5.1.6) where $wt(\partial^a \alpha^i) = a + 1$.

To describe the generators and relations for the invariant ring $\mathbb{C}[\oplus_{k \geq 0} V^k]^{\mathbb{Z}_2} \cong gr(\mathcal{H}(n))^{\mathbb{Z}_2}$, where each $V_k \cong \mathbb{C}$ for all k with basis α^i_k, we need the classical theorem of Weyl (Weyl's First and Second Fundamental Theorem of Invariant Theory for the standard representation of $\mathbb{Z}_2$) that we introduced in the last chapter.

Generators for the invariant ring. Recall, the $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ is a commutative algebra of even degree with a differential ∂ of degree zero, and it extends to $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ by the product rule, that is

$$\begin{aligned} \partial(\alpha^i_a \alpha^i_b) &= (\partial \alpha^i_a)\alpha^i_b + \alpha^i_a(\partial \alpha^i_b), \\ &= \alpha^i_{a+1}\alpha^i_b + \alpha^i_a \alpha^i_{b+1}. \end{aligned}$$

Specializing this, for $i, j = 1, ..., n$, we have $\partial(\alpha_a^i \alpha_b^j) = \alpha_{a+1}^i \alpha_b^j + \alpha_a^i \alpha_{b+1}^j$.

Define

$$q_{a,b}^{i,i} = \alpha_a^i \alpha_b^i, \qquad q_{a,b}^{i,j} = \alpha_a^i \alpha_b^j,$$

as generators for $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$. The action of ∂ on these generators is defined as follows:

$$\partial(q_{a,b}^{i,i}) = q_{a+1,b}^{i,i} + q_{a,b+1}^{i,i}, \qquad \partial(q_{a,b}^{i,j}) = q_{a+1,b}^{i,j} + q_{a,b+1}^{i,j}. \tag{5.1.7}$$

Remark 5.1.1. *1. The action of $\mathbb{Z}_2$ on the $gr(\mathcal{H}(n))$ which is given by $\theta(\alpha_a^i) = -\alpha_a^i$ guarantees that $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ is generated by the subset $\{q_{a,b}^{i,i}, q_{a,b}^{i,j} | a, b \geq 0, \ 1 \leq i, j \leq n\}$.*

2. Since $q_{a,b}^{i,i} = q_{b,a}^{i,i}$, and $q_{a,b}^{i,j} = q_{b,a}^{j,i}$, so $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ is generated by the subset

$$\{q_{a,b}^{i,i} | 0 \leq a \leq b, \ \ i = 1, ..., n\} \bigcup \{q_{a,b}^{i,j} | 0 \leq a, b, \ \ i, j = 1, ..., n\}. \tag{5.1.8}$$

Avoiding the repetition, we will let $i, j = 1, ..., n$ to be fixed.

The ideal of relations. Among these generators, the ideal of relations is generated by

$$q_{r,s}^{i,j} q_{t,u}^{k,l} - q_{r,u}^{i,l} q_{s,t}^{j,k}, \ \ i, j, k, l = 1, ..., n, \ \ 0 \leq r, s, t, u. \tag{5.1.9}$$

Under the projection

$$\varphi_2 : (\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)} \to (\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)} / (\mathcal{H}(n)^{\mathbb{Z}_2})_{(1)} \subset gr(\mathcal{H}(n)^{\mathbb{Z}_2}) \cong \mathbb{C}[\oplus_{k \geq 0} V^k]^{\mathbb{Z}_2},$$

the generators $q_{a,b}^{i,i}, q_{a,b}^{i,j}$ of $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ correspond to fields $\omega_{a,b}^{i,i}, \omega_{a,b}^{i,j}$, respectively defined by

$$\omega_{a,b}^{i,i} =: \partial^a \alpha^i(z) \partial^b \alpha^i(z) :\in (\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}, \qquad 0 \leq a \leq b, \tag{5.1.10}$$

$$\omega_{a,b}^{i,j} =: \partial^a \alpha^i(z) \partial^b \alpha^j(z) :\in (\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}, \qquad 0 \leq a \leq b. \tag{5.1.11}$$

Both fields $\omega^{i,i}_{a,b}, \omega^{i,j}_{a,b}$ satisfy $\varphi_2(\omega^{i,i}_{a,b}) = q^{i,i}_{a,b}$, $\varphi_2(\omega^{i,j}_{a,b}) = q^{i,j}_{a,b}$, respectively and have weight $a+b+2$. Note that $\sum_{i=1}^{n} \omega^{i,i}_{0,0} = 2L$, where L is the Virasoro field.

Remark 5.1.2. *The subspace $(\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}$ has degree at most 2, and has a basis $\{1\} \cup \{\omega^{i,i}_{a,b}, \omega^{i,j}_{a,b}\}$. Moreover, for $m \geq 0$, the operators $\omega^{i,j}_{a,b} \circ_m$ preserve this vector space.*

The following proposition for the case where $i = j$ could be found in [LIII].

Proposition 5.1.3. *For $a, b, c \geq 0$, $0 \leq m \leq a+b+c+1$, and $i < j$*

$$\omega^{i,j}_{a,b} \circ_m \partial^c \alpha^i = (-1)^a \frac{(a+c+1)!}{(a+c+1-m)!} \partial^{a+b+c+1-m} \alpha^j, \tag{5.1.12}$$

$$\omega^{i,j}_{a,b} \circ_m \partial^c \alpha^j = (-1)^b \frac{(b+c+1)!}{(b+c+1-m)!} \partial^{a+b+c+1-m} \alpha^i. \tag{5.1.13}$$

$$\omega^{i,i}_{a,b} \circ_m \partial^c \alpha^i = \lambda_{a,b,c,m} \partial^{a+b+c+1-m} \alpha^i, \tag{5.1.14}$$

where

$$\lambda_{a,b,c,m} = (-1)^b \frac{(b+c+1)!}{(b+c+1-m)!} + (-1)^a \frac{(a+c+1)!}{(a+c+1-m)!}.$$

It follows that for $m \leq a+b+c+1$, and $i < j < k$ we have

$$\begin{aligned} \omega^{i,j}_{a,b} \circ_m \omega^{i,j}_{c,d} = (-1)^a & \frac{(a+c+1)!}{(a+c+1-m)!} \omega^{j,j}_{a+b+c+1-m,d} \\ + (-1)^b & \frac{(b+d+1)!}{(b+d+1-m)!} \omega^{i,i}_{a+b+d+1-m,c}, \end{aligned} \tag{5.1.15}$$

$$\omega^{i,j}_{a,b} \circ_m \omega^{j,k}_{c,d} = (-1)^b \frac{(b+c+1)!}{(b+c+1-m)!} \omega^{i,k}_{a+b+c+1-m,d}, \tag{5.1.16}$$

$$\omega^{i,i}_{a,b} \circ_m \omega^{i,j}_{c,d} = \lambda_{a,b,c,m} \omega^{i,j}_{a+b+c+1-m,d}, \tag{5.1.17}$$

$$\omega^{j,j}_{a,b} \circ_m \omega^{i,j}_{c,d} = \lambda_{a,b,d,m} \omega^{i,j}_{c,a+b+d+1-m}, \tag{5.1.18}$$

$$\omega^{i,i}_{a,b} \circ_m \omega^{i,i}_{c,d} = \lambda_{a,b,c,m} \omega^{i,i}_{a+b+c+1-m,d} + \lambda_{a,b,d,m} \omega^{i,i}_{c,a+b+d+1-m}. \tag{5.1.19}$$

As a differential algebra with derivation ∂, some of the generators in the generating set (5.1.8) for $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ could be eliminated due to (5.1.7).

Example 5.1.4. *Case $i = j$: By the product rule, it is easily seen that*

$$q_{0,1}^{i,i} = \frac{1}{2}\partial q_{0,0}^{i,i}.$$

Furthermore, we have

$$q_{1,1}^{i,i} = -q_{0,2}^{i,i} + \frac{1}{2}\partial^2 q_{0,0}^{i,i},$$

$$q_{0,3}^{i,i} = \frac{3}{2}\partial q_{0,2}^{i,i} - \frac{1}{4}\partial^3 q_{0,0}^{i,i},$$

$$q_{1,3}^{i,i} = -q_{0,4}^{i,i} + \frac{3}{2}\partial^2 q_{0,2}^{i,i} - \frac{1}{4}\partial^4 q_{0,0}^{i,i}, \ldots.$$

Case $i < j$: By the product rule, once get

$$q_{r,m-r}^{i,j} = \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} \partial^k q_{0,m-k}^{i,j}, \tag{5.1.20}$$

for $r = 0, ..., m$.

For $m \geq 0$, let $A_m = span\{\omega_{a,b}^{i,i} | a + b = m\}$ be a vector space which is homogenous of weight $m + 2$. Use the relation $\partial\omega_{a,b}^{i,i} = \omega_{a+1,b}^{i,i} + \omega_{a,b+1}^{i,i}$. We have $dim(A_{2m}) = m + 1 = dim(A_{2m+1})$, for $m \geq 0$. Moreover, $\partial(A_m) \subset A_{m+1}$, and

$$dim(A_{2m}/\partial(A_{2m-1})) = 1, \qquad dim(A_{2m+1}/\partial(A_{2m})) = 0. \tag{5.1.21}$$

Thus, A_{2m} has a decomposition

$$A_{2m} = \partial(A_{2m-1}) \oplus \langle\omega_{0,2m}^{i,i}\rangle = \partial^2(A_{2m-2}) \oplus \langle\omega_{0,2m}^{i,i}\rangle, \tag{5.1.22}$$

where $\langle \omega^{i,i}_{0,2m} \rangle$ is the linear span of $\omega^{i,i}_{0,2m}$. Similarly, A_{2m+1} has a decomposition

$$A_{2m+1} = \partial^2(A_{2m-1}) \oplus \langle \partial\omega^{i,i}_{0,2m} \rangle = \partial^3(A_{2m-2}) \oplus \langle \partial\omega^{i,i}_{0,2m} \rangle. \tag{5.1.23}$$

Therefore,

$$span\{\omega^{i,i}_{a,b} | a+b = 2m\} = span\{\partial^{2k}\omega^{i,i}_{0,2m-2k} | 0 \leq k \leq m\}$$

and

$$span\{\omega^{i,i}_{a,b} | a+b = 2m+1\} = span\{\partial^{2k+1}\omega^{i,i}_{0,2m-2k} | 0 \leq k \leq m\}$$

are bases of A_{2m} and A_{2m+1}, respectively and so for each $\omega^{i,i}_{a,b} \in A_{2m}$ and $\omega^{i,i}_{c,d} \in A_{2m+1}$ can be written uniquely in the form

$$\omega^{i,i}_{a,b} = \sum_{k=0}^{m} \lambda_k \partial^{2k}\omega^{i,i}_{0,2m-2k}, \quad \omega^{i,i}_{c,d} = \sum_{k=0}^{m} \mu_k \partial^{2k+1}\omega^{i,i}_{0,2m-2k}, \tag{5.1.24}$$

for constants λ_k, μ_k.

Specializing this, for $m \geq 0$, let $A'_m = span\{\omega^{i,j}_{a,b} | a+b = m\}$, and use the relation $\partial\omega^{i,j}_{a,b} = \omega^{i,j}_{a+1,b} + \omega^{i,j}_{a,b+1}$. We have $dim(A'_m) = m+1$, for $m \geq 0$. Moreover, $\partial(A'_m) \subset A'_{m+1}$, and

$$dim(A'_m/\partial(A'_{m-1})) = 1. \tag{5.1.25}$$

Hence, A'_m has a decomposition

$$A'_m = \partial(A'_{m-1}) \oplus \langle \omega^{i,j}_{0,m} \rangle \tag{5.1.26}$$

where $\langle \omega^{i,j}_{0,m} \rangle$ is the linear span of $\omega^{i,j}_{0,m}$. Therefore,

$$span\{\omega^{i,i}_{a,b} | a+b = m\} = span\{\partial^{k}\omega^{i,j}_{0,m-k} | 0 \leq k \leq m\}$$

is a basis of A'_m. It follows that for each $\omega^{i,j}_{r,m-r} \in A'_m$ could be written uniquely in the form

$$\omega^{i,j}_{r,m-r} = \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} \partial^k \omega^{i,j}_{0,m-k}, \tag{5.1.27}$$

where $r = 0, \ldots, m$.

The minimal strong generating set for $\mathcal{H}(n)^{\mathbb{Z}_2}$ is emphasized by the following lemma.

Lemma 5.1.5. *$\mathcal{H}(n)^{\mathbb{Z}_2}$ is strongly generated as a vertex algebra by the subset*

$$\{\omega^{i,i}_{0,2m} | m \geq 0, \ \ and \ \ i = 1, ..., n\} \bigcup \{\omega^{i,j}_{0,m} | m \geq 0, \ \ and \ \ 1 \leq i < j \leq n\}. \tag{5.1.28}$$

Proof. Since $gr(\mathcal{H}(n))^{\mathbb{Z}_2} = gr(\mathcal{H}(n)^{\mathbb{Z}_2})$ is generated by the subset $\{q^{i,i}_{0,2m} | m \geq 0, \ \ i = 1, ..., n\} \bigcup \{q^{i,j}_{0,m} | m \geq 0, \ \ and \ \ 1 \leq i < j \leq n\}$ as a ∂-ring, Lemma 4.3.3 shows that the corresponding set strongly generates $\mathcal{H}(n)^{\mathbb{Z}_2}$ as a vertex algebra. □

5.1.2 Decoupling relations and higher decoupling relations

We shall distinguish three cases for $\mathcal{H}(n)^{\mathbb{Z}_2}$. Notice first that $\mathcal{H}(n)^{\mathbb{Z}_2}$ is not freely generated by (5.1.28).

Case 1: For $n = 1$. The first relation of the form (5.1.9) among the generators $\{q_{0,2m} | m \geq 0\}$ of $gr(\mathcal{H}(1))^{\mathbb{Z}_2}$, and their derivatives occurs of minimal weight 6, corresponds to $I = (0, 1), J = (0, 1)$ in (3.2.2) and has the form

$$q_{0,0}q_{1,1} - q_{0,1}q_{0,1} = 0, \tag{5.1.29}$$

where $q_{r,s} = q^{1,1}_{r,s}$. This relation is unique. The corresponding element $: \omega_{0,0}\omega_{1,1} : - : \omega_{0,1}\omega_{0,1} :$ lies in $(\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}$. Similarly, we shall use the notation $\omega_{a,b}$ for $\omega^{1,1}_{a,b}$ when no confusion may arise. This element element has a correction. By computer

calculations, it has the form

$$\begin{aligned} :\omega_{0,0}\omega_{1,1}: - :\omega_{0,1}\omega_{0,1}: &= -\frac{5}{4}\omega_{0,4} + \frac{7}{4}\partial^2\omega_{0,2} - \frac{7}{24}\partial^4\omega_{0,0} \\ &= -\frac{5}{4}\omega_{0,4} + P(\omega_{0,0}, \omega_{0,2}), \end{aligned} \tag{5.1.30}$$

where P is a normally ordered polynomial in $\omega_{0,0}, \omega_{0,2}$, and their derivatives.

On the other hand, once can check that

$$\omega_{0,1} = \frac{1}{2}\partial\omega_{0,0}, \tag{5.1.31}$$

$$\omega_{1,1} = -\omega_{0,2} + \frac{1}{2}\partial^2\omega_{0,0}. \tag{5.1.32}$$

Thus, the left hand side of (5.1.30) can be written as a normally ordered polynomial in $\omega_{0,0}, \omega_{0,2}$, and their derivatives, and so we can rewrite (5.1.30) as follows:

$$-\frac{5}{4}\omega_{0,4} = P_2(\omega_{0,0}, \omega_{0,2}). \tag{5.1.33}$$

This decoupling relation allows $\omega_{0,4}$ to be expressed as a normally ordered polynomial in $\omega_{0,0}, \omega_{0,2}$ and their derivatives. No decoupling relations for $\omega_{0,0}, \omega_{0,2}$ could be found since there are no relations of weight less than 6 in $gr(\mathcal{H}(1))^{\mathbb{Z}_2}$ due to Weyl First Fundamental Theorem of Invariant Theory for $\mathbb{Z}_2$. Furthermore, (5.1.33) is unique up to scalar multiples due to the uniqueness of (5.1.38).

In order to construct more decoupling relations of the form (5.1.33), we apply the operator $\omega_{0,2}\circ_1$ repeatedly to the relation we have already constructed.

The following theorem gives the minimal strong generating set for $\mathcal{H}(1)^{\mathbb{Z}_2}$, and it is due to (Dong-Nagatomo).

Theorem 5.1.6. *[DNI] $\mathcal{H}(1)^{\mathbb{Z}_2}$ has a minimal strong generating set $\{\omega_{0,0}, \omega_{0,2}\}$ and is of type $\mathcal{W}(2,4)$.*

Case 2: For n=2. Specializing the above case, once get:

We shall have from the previous case a strong generating set consisting $\omega^{1,1}_{0,0}, \omega^{1,1}_{0,2}, \omega^{2,2}_{0,0}$, $\omega^{2,2}_{0,2}$. In addition, we have $\omega^{1,2}_{0,m}, m \geq 0$ as strong generators for $\mathcal{H}(2)^{\mathbb{Z}_2}$ but not all of them are needed.

The first relation of the form (5.1.9) among the generators $\{q^{1,2}_{0,m} | m \geq 0, \ \textit{and} \ 1 \leq i < j \leq 2\}$ of $gr(\mathcal{H}(2)^{\mathbb{Z}_2})$ and their derivatives occurs of minimal weight 5, and has the form

$$q^{1,2}_{0,0} q^{2,2}_{0,1} - q^{1,2}_{0,1} q^{2,2}_{0,0} = 0. \tag{5.1.34}$$

The corresponding element $: \omega^{1,2}_{0,0}\omega^{2,2}_{0,1} : - : \omega^{1,2}_{0,1}\omega^{2,2}_{0,0} :$ in $(\mathcal{H}(2)^{\mathbb{Z}_2})_{(2)}$, and has a correction. By computer calculations, it has the form

$$\begin{aligned} : \omega^{1,2}_{0,0}\omega^{2,2}_{0,1} : - : \omega^{1,2}_{0,1}\omega^{2,2}_{0,0} : &= -\frac{1}{2}\omega^{1,2}_{0,3} + 2\partial\omega^{1,2}_{0,2} - \frac{5}{2}\partial^2\omega^{1,2}_{0,1} + \partial^3\omega^{1,2}_{0,0} \\ &= -\frac{1}{2}\omega^{1,2}_{0,3} + Q(\omega^{1,2}_{0,0}, \omega^{1,2}_{0,1}, \omega^{1,2}_{0,2}) \end{aligned} \tag{5.1.35}$$

where Q is a normally ordered polynomial in $\omega^{1,2}_{0,0}, \omega^{1,2}_{0,1}, \omega^{1,2}_{0,2}$, and their derivatives, and it could be rewritten as follows:

$$-\frac{1}{2}\omega^{1,2}_{0,3} = Q_3(\omega^{2,2}_{0,0}, \omega^{1,2}_{0,0}, \omega^{1,2}_{0,1}, \omega^{1,2}_{0,2}). \tag{5.1.36}$$

No decoupling relations for $\omega^{2,2}_{0,0}, \omega^{1,2}_{0,0}, \omega^{1,2}_{0,1}, \omega^{1,2}_{0,2}$ could be found since there are no relations of weight less than 5 in $gr(\mathcal{H}(2))^{\mathbb{Z}_2}$ due to Weyl First Fundamental Theorem of Invariant Theory for $\mathbb{Z}_2$.

On the other hand, We have the following relation

$$: \omega^{1,2}_{0,0}\omega^{1,2}_{0,0} := \frac{1}{2}\omega^{1,1}_{0,2} + \frac{1}{2}\omega^{2,2}_{0,2} + : \omega^{1,1}_{0,0}\omega^{2,2}_{0,0} : . \tag{5.1.37}$$

In order to construct more decoupling relations of the form (5.1.36), we apply the operator $\omega^{2,2}_{0,1}\circ_1$ repeatedly to the relation we have already constructed.

Case 3: For $n \geq 3$. Specializing the first case, once get:

We shall have from the first case a strong generating set consisting $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}, i = 1, ..., n$. In addition, we have $\omega^{i,j}_{0,m}, m \geq 0 \;\; 1 \leq i < j \leq n$ as strong generators for $\mathcal{H}(n)^{\mathbb{Z}_2}$. This generating set is not optimized.

The first relation of the form (5.1.9) among the generators $\{q^{i,i}_{0,2m} | m \geq 0\}$ of $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$, and their derivatives occurs of minimal weight 6, corresponds to $I = (0,1), J = (0,1)$ in (3.2.2) and has the form

$$q^{i,i}_{0,0} q^{i,i}_{1,1} - q^{i,i}_{0,1} q^{i,i}_{0,1} = 0. \tag{5.1.38}$$

For each i, this relation is unique. The corresponding element $: \omega^{i,i}_{0,0}\omega^{i,i}_{1,1} : - : \omega^{i,i}_{0,1}\omega^{i,i}_{0,1} :$ lies in $(\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}$, and has a correction. By computer calculations, it has the form

$$\begin{aligned} : \omega^{i,i}_{0,0}\omega^{i,i}_{1,1} : - : \omega^{i,i}_{0,1}\omega^{i,i}_{0,1} : &= -\frac{5}{4}\omega^{i,i}_{0,4} + \frac{7}{4}\partial^2\omega^{i,i}_{0,2} - \frac{7}{24}\partial^4\omega^{i,i}_{0,0} \\ &= -\frac{5}{4}\omega^{i,i}_{0,4} + R(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}), \end{aligned} \tag{5.1.39}$$

where R is a normally ordered polynomial in $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}$, and their derivatives, and this can be rewritten as follows:

$$-\frac{5}{4}\omega^{i,i}_{0,4} = R_2(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}). \tag{5.1.40}$$

In order to construct more decoupling relations of the form (5.1.40), we apply the operator $\omega^{i,i}_{0,2}\circ_1$ repeatedly to the relation we have already constructed.

Remark 5.1.7. *Applying the operator $\omega^{i,i}_{0,2}\circ_1$ to the both sides of* (5.1.39) *yields*

$$16\omega^{i,i}_{0,6} = \frac{192}{25}(: \omega^{i,i}_{0,0}\omega^{i,i}_{0,0}\omega^{i,i}_{1,1} : + : \omega^{i,i}_{0,0}\omega^{i,i}_{0,1}\omega^{i,i}_{0,1} :) + S(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}), \tag{5.1.41}$$

where S is a normally ordered polynomial in all linear and quadratic in $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}$, and their derivatives. The monomials of degree 3 in $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,1}, \omega^{i,i}_{1,1}$ that appear in (5.1.41) *are of weight* 8, *and can be rewritten as follows*

$$
\begin{aligned}
:\omega^{i,i}_{0,0}\omega^{i,i}_{0,0}\omega^{i,i}_{1,1}: - :\omega^{i,i}_{0,0}\omega^{i,i}_{0,1}\omega^{i,i}_{0,1}: = \frac{203}{552}\omega^{i,i}_{0,6} + \frac{143}{1104}\partial^4\omega^{i,i}_{0,2} + \frac{41}{5520}\partial^6\omega^{i,i}_{0,0} \\
-\frac{1}{12}:\omega^{i,i}_{0,0}(\partial^2\omega^{i,i}_{0,2}): - \frac{97}{276}:(\partial\omega^{i,i}_{0,0})(\partial^3\omega^{i,i}_{0,0}): + \frac{63}{46}:(\partial\omega^{i,i}_{0,0})(\partial\omega^{i,i}_{0,2}) \\
+\frac{17}{23}:(\partial^2\omega^{i,i}_{0,0})\omega^{i,i}_{0,2}: - \frac{385}{276}:\omega^{i,i}_{0,2}\omega^{i,i}_{0,2}: .
\end{aligned}
$$

The first relation of the form (5.1.9) among the generators $\{q^{i,j}_{0,m} | m \geq 0,\ 3 \leq i < j \leq n\}$ of $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ and their derivatives occurs of minimal weight 4, and has the form

$$
q^{i,j}_{0,0}q^{j,k}_{0,0} - q^{i,k}_{0,0}q^{j,j}_{0,0} = 0, \quad for \ \ i < j < k. \tag{5.1.42}
$$

The corresponding element $:\omega^{i,j}_{0,0}\omega^{j,k}_{0,0}: - :\omega^{i,k}_{0,0}\omega^{j,j}_{0,0}:$ lies in $(\mathcal{H}(n)^{\mathbb{Z}_2})_{(2)}$, and has a correction. By computer calculations, it has the form

$$
\begin{aligned}
:\omega^{i,j}_{0,0}\omega^{j,k}_{0,0}: - :\omega^{i,k}_{0,0}\omega^{j,j}_{0,0}: &= \frac{1}{2}\omega^{i,k}_{0,2} - \partial\omega^{i,k}_{0,1} + \frac{1}{2}\partial^2\omega^{i,k}_{0,0} \\
&= -\frac{1}{2}\omega^{i,k}_{0,2} + T(\omega^{i,k}_{0,0}, \omega^{i,k}_{0,1}),
\end{aligned} \tag{5.1.43}
$$

where T is a normally ordered polynomial in $\omega^{i,k}_{0,0}, \omega^{i,k}_{0,1}$, and their derivatives. So, (5.1.43) can be rewritten as follows:

$$
\frac{1}{2}\omega^{i,k}_{0,2} = T_2(\omega^{j,j}_{0,0}, \omega^{i,j}_{0,0}, \omega^{j,k}_{0,0}, \omega^{i,k}_{0,0}, \omega^{i,k}_{0,1}). \tag{5.1.44}
$$

No decoupling relations for $\omega^{j,j}_{0,0}, \omega^{i,j}_{0,0}, \omega^{j,k}_{0,0}, \omega^{i,k}_{0,0}, \omega^{i,k}_{0,1}$ could be found since there are no relations of weight less than 4 in $gr(\mathcal{H}(n))^{\mathbb{Z}_2}$ due to Weyl First Fundamental Theorem of Invariant Theory for $\mathbb{Z}_2$. Furthermore, (5.1.44) is unique up to scalar multiples due to the uniqueness of (5.1.42).

In order to construct more decoupling relations of the form (5.1.44), we apply the operator $\omega^{k,k}_{0,1}\circ_1$ repeatedly to the relation we have already constructed.

Lemma 5.1.8. *For $n \geq 3$, and $1 \leq i < j \leq n$, $\mathcal{H}(n)^{\mathbb{Z}_2}$ is generated by $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}, \omega^{i,j}_{0,0}$, $\omega^{i,j}_{0,1}$ as a vertex algebra.*

Proof. First, for $i = 1, ..., n$, it suffices to show that each $\omega^{i,i}_{0,2k}$ is generated by $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}$ for all $k \geq 2$. This follows from

$$\omega^{i,i}_{0,2} \circ_1 \omega^{i,i}_{0,2k} = (4+2k)\omega^{i,i}_{0,2k+2} + \partial^2\mu, \tag{5.1.45}$$

where μ is a linear combination of $\partial^{2r}\omega^{i,i}_{0,2k-2r}$ for $r = 0, ..., k$.
Second, for $i < j$, it suffices to show that each $\omega^{i,j}_{0,k}$ is generated by $\omega^{i,j}_{0,0}, \omega^{i,j}_{0,1}$for all $k \geq 2$. This follows from

$$\omega^{j,j}_{0,1} \circ_1 \omega^{i,j}_{0,k} = -\omega^{i,j}_{0,k+1}. \tag{5.1.46}$$

□

Lemma 5.1.9. *[LV] Let R_0 denote the remainder of the element P_0. The condition $R_0 \neq 0$ is equivalent to the existence of a decoupling relation in $\mathcal{H}(n)^{\mathbb{Z}_2}$ of the form*

$$\omega^{i,i}_{0,2m} = P_m(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}), \tag{5.1.47}$$

where P is a normally ordered polynomial in $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}$ and their derivatives.

Lemma 5.1.10. *[LV] Suppose that $R_0 \neq 0$. Then for all $m \geq 2$, there exists a decoupling relation*

$$\omega^{i,i}_{0,2m} = P_m(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}), \tag{5.1.48}$$

where P_m is a normally ordered polynomial in $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}$ and their derivatives.

Similarly, let R_0, denote the remainder of the element D_0. The condition $R_0 \neq 0$, is equivalent to the existence of a decoupling relation in $\mathcal{H}(n)^{\mathbb{Z}_2}$ of the form

$$\omega^{i,j}_{0,m} = Q(\omega^{i,j}_{0,0}, \omega^{i,j}_{0,1}, \omega^{i,j}_{0,2}), \tag{5.1.49}$$

where Q is a normally ordered polynomial in $\omega^{i,j}_{0,0}, \omega^{i,j}_{0,1}, \omega^{i,j}_{0,2}$ and their derivatives, and Q' is a normally ordered polynomial in h^0, h^1 and their derivatives.

Theorem 5.1.11. *1. For $n = 2$, $\mathcal{H}(n)^{\mathbb{Z}_2}$ has a minimal strong generating set*

$$\{\omega^{1,1}_{0,0}, \omega^{1,2}_{0,0}, \omega^{1,2}_{0,1}, \omega^{1,2}_{0,2}, \omega^{2,2}_{0,0}, \omega^{2,2}_{0,2}, \}, \tag{5.1.50}$$

and is of type $\mathcal{W}(2^3, 3, 4^2)$.

2. For $n \geq 3$, $\mathcal{H}(n)^{\mathbb{Z}_2}$ has a minimal strong generating set

$$\{\omega^{i,i}_{0,0}, \omega^{1,1}_{0,2} | i = 1, ..., n\} \bigcup \{\omega^{i,j}_{0,0}, \omega^{i,j}_{0,1} | 1 \leq i < j \leq n\}, \tag{5.1.51}$$

and is of type $\mathcal{W}(2^{n+\binom{n}{2}}, 3^{\binom{n}{2}}, 4)$.

Proof. We shall prove the case where $n \geq 3$, and the proof of the other case is similar and so it is omitted.

First, Lemma 5.1.5 asserts that $\mathcal{H}(n)^{\mathbb{Z}_2}$ is strongly generated by the natural infinite set

$$\{\omega^{i,i}_{0,2m} | m \geq 0, \ \textit{and} \ i = 1, ..., n\} \bigcup \{\omega^{i,j}_{0,m} | m \geq 0, \ \textit{and} \ 1 \leq i < j \leq n\}.$$

Next, It suffices to construct decoupling relations of the form

$$\omega^{i,i}_{0,2m} = P_m(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}), \tag{5.1.52}$$

for $m \geq 4$ since we already have such relations for $m = 2, 3$, where $i = 1, ..., n$. Applying the operator $\omega^{i,i}_{0,2} \circ_1$, which raises the weight by 2 to the relations we have constructed, and so Lemma 5.1.8 and Lemma 5.1.10 yield the result.

Moreover, we need to construct decoupling relations of the form

$$\omega^{i,j}_{0,m} = Q_m(\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}, \omega^{i,j}_{0,0}, \omega^{i,j}_{0,1}),$$

for $m \geq 2$. Applying the operator $\omega^{j,j}_{0,1} \circ_1$, which raises the weight by 2 to the relations we have constructed. Lemma 5.1.8, and Lemma 5.1.10 with the relations

$$: \omega^{i,j}_{0,0} \omega^{i,j}_{0,0} := \frac{1}{2}\omega^{i,i}_{0,2} + \frac{1}{2}\omega^{j,j}_{0,2} + : \omega^{i,i}_{0,0} \omega^{j,j}_{0,0} :,$$

$$: \omega^{i,j}_{0,0} \omega^{j,k}_{0,0} := \frac{1}{2}\omega^{i,k}_{0,2} - (\partial \omega^{i,k}_{0,1}) + \frac{1}{2}(\partial^2 \omega^{i,k}_{0,0}) + : \omega^{i,k}_{0,0} \omega^{j,j}_{0,0} :$$

yield the result. Thus, (5.1.51) is a strong generating set for $\mathcal{H}(n)^{\mathbb{Z}_2}$.

The fact that this set is minimal is a consequence of Weyl's First Fundamental Theorem of invariant theory for $\mathbb{Z}_2$; there are no normally ordered relations of weight less than 4. $\square$

Notice that $\omega^{i,i}_{0,0}, \omega^{i,i}_{0,2}, \omega^{i,j}_{0,0}, \omega^{i,j}_{0,1}, \omega^{i,j}_{0,2}$ are not primary fields with respect to L. It is easy to correct them to be a primary ones by adding a normally ordered polynomial in the previous set and their derivatives. Using the computer calculations, we obtain

the following primary fields:

$$
\begin{aligned}
L(z) &= \frac{1}{2}\sum_{i=1}^{n} : a^i(z)a^i(z) :, \\
C^k &= \frac{1}{2}(\omega_{0,0}^{1,1} - \omega_{0,0}^{k,k}), \ \ where \ \ k = 2, ..., n. \\
C_{0,2}^{i,i} &= \omega_{0,2}^{i,i} - \frac{2}{9} : \omega_{0,0}^{i,i}\omega_{0,0}^{i,i} : - \frac{1}{6}\partial^2\omega_{0,0}^{i,i}, \ \ where \ \ i = 1, ..., n, \\
C_{0,0}^{i,j} &= \omega_{0,0}^{i,j}, \\
C_{0,1}^{i,j} &= \omega_{0,1}^{i,j} - \frac{1}{2}\partial\omega_{0,0}^{i,j}, \\
C_{0,2}^{i,j} &= \omega_{0,2}^{i,j} - \frac{4}{9} : \omega_{0,0}^{i,j}\omega_{0,0}^{j,j} : + \frac{5}{9}\partial^2\omega_{0,0}^{i,j} - \frac{13}{9}\partial\omega_{0,1}^{i,j}.
\end{aligned}
\tag{5.1.53}
$$

5.2 Deformations

Let $K \subset \mathbb{C}$ be at most countable. Consider the $\mathbb{C}$-algebra $F_K = \{\frac{p(\kappa)}{q(\kappa)} | deg(p) \leq deg(q) \ \ and \ \ r(q) \in K\}$, where $r(q)$ denotes the toots of q.

Definition 5.2.1. *A F_K-module $\mathcal{B}$ with the vertex algebra structure, and coefficients in F_K is called a deformable family.*

Note that vertex algebras over F_K are defined similarly as vertex algebras over $\mathbb{C}$. Let $\mathcal{B}$ be a vertex algebra, defined in the above definition, with basis $\{a_i | i \in I\}$. For $k \notin K$, we have a vertex algebra

$$\mathcal{B}_k = \mathcal{B}/(\kappa - k),$$

where $(\kappa - k)$ is the ideal generated by $\kappa - k$. The limit

$$\mathcal{B}_\infty = lim_{\kappa \to \infty}\mathcal{B}$$

is a vertex algebra with basis $\{\alpha_i | i \in I\}$, such that $\alpha_i = lim_{\kappa \to \infty} a_i$.

Remark 5.2.2. *The algebraic structure for the vertex algebra $\mathcal{B}_\infty$ can be described as follows:*

1. $\alpha_i \circ_n \alpha_j = lim_{\kappa\to\infty} a_i \circ_n a_j,\ i,j \in I,\ \ n \in \mathbb{Z}$.

2. *The F_K-linear map $\varphi : \mathcal{B} \to \mathcal{B}_\infty$, $a_i \mapsto \alpha_i$ satisfies*

$$\varphi(\omega \circ_n \nu) = \varphi(\omega) \circ_n \varphi(\nu), \qquad \omega, \nu \in \mathcal{B},\ \ n \in \mathbb{Z}. \tag{5.2.1}$$

Example 5.2.3. *[LIV] Let $\mathfrak{g}$ be a simple, finite dimensional Lie algebra equipped with the normalized Killing form $(.|.)$, and a related an orthonormal basis $\{\xi_1, ..., \xi_n\}$. The generators $X^{\xi_i} \in V^k(\mathfrak{g})$ satisfy the OPE relations*

$$X^{\xi_i}(z)X^{\xi_j}(w) \sim k\delta_{i,j}(z-w)^{-2} + X^{[\xi_j,\xi_j]}(w)(z-w)^{-1}. \tag{5.2.2}$$

Let κ be formal variable satisfying $\kappa^2 = k$. Let $F = F_K$ for $K = \{0\}$. Let $\mathcal{V}$ be the vertex algebra with coefficients in F, freely generated by $\{a^{\xi_i} | i = 1, ..., n\}$, which satisfy the OPE relations

$$a^{\xi_i}(z)a^{\xi_j}(w) \sim \delta_{i,j}(z-w)^{-2} + \frac{1}{\kappa}a^{[\xi_i,\xi_j]}(w)(z-w)^{-1}.$$

For $k \neq 0$, we have a surjective vertex algebra homomorphism

$$\mathcal{V} \to V^k(\mathfrak{g}), \qquad a^{\xi_i} \mapsto \frac{1}{\sqrt{k}}X^{\xi_i},$$

with the kernel $(\kappa - \sqrt{k})$. It follows that $V^k(\mathfrak{g}) \cong \mathcal{V}/(\kappa - \sqrt{k})$, and the limit $\mathcal{V}_\infty = lim_{\kappa\to\infty}\mathcal{V}$ is a vertex algebra over $\mathbb{C}$ with generators $\{\alpha^{\xi_i} | i = 1, ..., n\}$, which satisfy the OPE relations

$$\alpha^{\xi_i}(z)\alpha^{\xi_j}(w) \sim \delta_{i,j}(z-w)^{-2}.$$

Therefore, $\mathcal{V}_\infty \cong \mathcal{H}(n)$.

Corollary 1. *[LIV]* $(\mathcal{V}^G)_\infty = (\mathcal{V}_\infty)^G = \mathcal{H}(n)^G$.

5.3 The universal Affine vertex algebra, revisited

Filtrations. We define an increasing filtration on $V^k(\mathfrak{g})$ for any simple, finite dimensional Lie algebra $\mathfrak{g}$ as follows:

$$V^k(\mathfrak{g})_{(0)} \subset V^k(\mathfrak{g})_{(1)} \subset V^k(\mathfrak{g}) \subset ..., \qquad V^k(\mathfrak{g}) = \bigcup_{d\geq 0} V^k(\mathfrak{g})_{(d)}, \tag{5.3.1}$$

where $V^k(\mathfrak{g})_{(-1)} = \{0\}$, and $V^k(\mathfrak{g})_{(r)}$ is spanned by the iterated Wick products of the generators X^{ζ_i} and their derivatives, such that at most r of the generators and their derivatives appear, that is, $V^k(\mathfrak{g})$ is spanned by all normally ordered monomials of the form (2.8.4), such that the total length $s_1 + \cdots + s_m \leq r$. In particular, each X^{ζ_i} and their derivatives have degree 1.

We have a $\mathbb{Z}_{\geq 0}$-grading

$$V^k(\mathfrak{g}) = \bigoplus_{r\geq 0} V^k(\mathfrak{g})^{(r)}, \tag{5.3.2}$$

where $V^k(\mathfrak{g})_{(d)} = \oplus_{r=0}^{d} V^k(\mathfrak{g})^{(r)}$. So, $V^k(\mathfrak{g})$ equipped with such a good filtration lies in the category $\mathcal{R}$. The $\mathbb{Z}_{\geq 0}$-associated graded algebra

$$gr(V^k(\mathfrak{g})) = \bigoplus_{d\geq 0} V^k(\mathfrak{g})_{(d)}/V^k(\mathfrak{g})_{(d-1)}$$

is now an abelian vertex algebra freely generated by X^{ζ_i}. Then, $V^k(\mathfrak{g}) \cong gr(V^k(\mathfrak{g}))$ as linear spaces, and as commutative algebras we have

$$gr(V^k(\mathfrak{g})) \cong \mathbb{C}[X^{\zeta_i}, \partial X^{\zeta_i}, \partial^2 X^{\zeta_i}, ...].$$

According to the Cartan Killing classification, we have the following list for the simple Lie algebras.

Theorem 5.3.1. *[BH] Any complex finite dimensional simple Lie algebra is isomorphic to exactly one of the following:*

Classical Lie Algebras:

1. $\mathfrak{sl}_{n+1}, n \geq 1$, *and it has a Cartan notation* A_n.

2. $\mathfrak{so}_{2n+1}, n \geq 2$, *and it has a Cartan notation* B_n.

3. $\mathfrak{sp}_{2n}, n \geq 3$, *and it has a Cartan notation* C_n.

4. $\mathfrak{so}_{2n}, n \geq 4$, *and it has a Cartan notation* D_n.

Exceptional Lie Algebras:

1. G_2.

2. F_4.

3. E_6, E_7, *or* E_8.

The table below shows the dimension, rank l, and the number of positive roots m for each simple Lie algebra from the previous list:

Lie algebra	Dimension	Rank l	The number of positive roots m
$\mathfrak{sl}_{n+1}$	$(n+1)^2-1$	n	$\frac{(n^2+n)}{2}$
$\mathfrak{so}_{2n+1}$	$\frac{2n(2n+1)}{2}$	n	n^2
$\mathfrak{sp}_{2n}$	$n(2n+1)$	n	n^2
$\mathfrak{so}_{2n}$	$\frac{2n(2n-1)}{2}$	n	n^2-n
G_2	14	2	6
F_4	52	4	24
E_6	78	6	36
E_7	133	7	63
E_8	248	8	120

5.3.1 The Cartan involution and its extension to $V^k(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra as above, and let $l = \mathrm{rank}(\mathfrak{g})$ and m be the number of positive roots. With respect to a choice of base for the root system Φ, we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-,$$

where $\mathfrak{h}$ is the Cartan subalgebra with basis h_r, $r = 1, \ldots, l$, and $\mathfrak{n}_+$ has basis x_{β_i} for $i = 1, \ldots, m$, and $\mathfrak{n}_-$ has basis y_{β_i} for $i = 1, \ldots, m$. The Cartan involution θ of $\mathfrak{g}$ is then given by

$$\theta(x_{\beta_i}) = -y_{\beta_i}, \qquad \theta(y_{\beta_i}) = -x_{\beta_i}, \qquad \theta(h_r) = -h_r.$$

Since θ preserves the Lie bracket as well as the normalized Killing form, it is immediate that it extends to an automorphism of the vertex algebra $V^k(\mathfrak{g})$ given by

the same formula, where $h_r, x_{\beta_i}, y_{\beta_i}$ are now interpreted as the generating fields for $V^k(\mathfrak{g})$.

It is convenient to apply a linear change of variables to replace the generators $h_r, x_{\beta_i}, y_{\beta_i}$ of $V^k(\mathfrak{g})$ with a set of eigenvectors for θ as follows:

$$E_{\beta_i} = x_{\beta_i} + y_{\beta_i}, \qquad F_{\beta_i} = x_{\beta_i} - y_{\beta_i} \qquad h_r.$$

The action of θ on the new generators will be as follows:

$$\theta(E_{\beta_i}) = -E_{\beta_i}, \qquad \theta(F_{\beta_i}) = F_{\beta_i}, \qquad \theta(h_r) = -h_r.$$

Since the monomials (2.8.4) form a basis for $V^k(\mathfrak{g})$, and the new generators are related to the old ones by a linear change of variables, there is also a PBW basis consisting of normally ordered monomials of the new generators and their derivatives.

Note that the fields F_{β_i} lie in the orbifold $V^k(\mathfrak{g})^{\mathbb{Z}_2}$. Define additional elements of $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ as follows:

$$\begin{aligned} Q_{a,b}^{\beta_i,\beta_j} &=: \partial^a E_{\beta_i}(z)\partial^b E_{\beta_j}(z) :, \\ Q_{a,b}^{h_r,\beta_i} &=: \partial^a h_r(z)\partial^b E_{\beta_i}(z) :, \\ Q_{a,b}^{h_r,h_s} &=: \partial^a h_r(z)\partial^b h_s(z) :, \end{aligned}$$

which each have weight $a + b + 2$.

Note that in the case $\mathfrak{g} = \mathfrak{s}l_2$, there is only one positive root β, so there is one element F_β, one element E_β, and one basis vector h for $\mathfrak{h}$. In this case, the above elements are

$$\begin{aligned} Q_{a,b}^{\beta,\beta} &=: \partial^a E_\beta(z)\partial^b E_\beta(z) :, \\ Q_{a,b}^{h,\beta} &=: \partial^a h(z)\partial^b E_\beta(z) :, \\ Q_{a,b}^{h,h} &=: \partial^a h(z)\partial^b h(z) :, \end{aligned}$$

5.3.2 The structure of $V^k(\mathfrak{g})^{\mathbb{Z}_2}$

The following theorem describes $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ for generic values of k.

Theorem 5.3.2. *Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra, and let $l = rank(\mathfrak{g})$, m the number of positive roots, and set $d = m + l$.*

1. *For $\mathfrak{g} \neq \mathfrak{sl}_2$, and k generic, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ has a minimal strong generating set*

$$\{F_{\beta_i}, Q^{\beta_a,\beta_b}_{0,0}, Q^{\beta_1,\beta_1}_{0,2}, Q^{h_r,h_s}_{0,0}, Q^{h_t,\beta_u}_{0,0}, Q^{h_t,\beta_u}_{0,1}\},$$

   for $1 \leq i \leq m$, $1 \leq a \leq b \leq m$, $1 \leq r \leq s \leq l$, $1 \leq t \leq l$, and $1 \leq u \leq m$. In particular, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ is of type $\mathcal{W}(1^m, 2^{d+\binom{d}{2}}, 3^{\binom{d}{2}}, 4)$.

2. *For $\mathfrak{g} = \mathfrak{sl}_2$ and k generic, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$ has a minimal strong generating set*

$$\{F, Q^{\beta,\beta}_{0,0}, Q^{h,h}_{0,0}, Q^{h,h}_{0,2}, Q^{\beta,h}_{0,0}, Q^{\beta,h}_{0,1}, Q^{\beta,h}_{0,2}\},$$

   and in particular, is of type $\mathcal{W}(1, 2^3, 3, 4^2)$.

Proof. Let $n = 2m + l = \dim(\mathfrak{g})$. By deformation argument, we have

$$\lim_{k\to\infty} V^k(\mathfrak{g}) \cong \mathcal{H}(n).$$

Here $\mathcal{H}(n)$ is the rank n Heisenberg algebra with generators F_{β_i}, E_{β_i} and h_r. (By abuse of notation, we use the same symbols for the limits of these fields). Moreover, the action of $\mathbb{Z}_2$ is trivial on the rank m Heisenberg subalgebra generated by $\{F_{\beta_i} |\ i = 1, \ldots, m$, and it acts by -1 on the rank $d = m+l$ Heisenberg algbera with generators $\{E_{\beta_i}, h_r |\ i = 1, \ldots, m, \quad r = 1, \ldots, l\}$. It is immediate that

$$\lim_{k\to\infty} V^k(\mathfrak{g})^{\mathbb{Z}_2} \cong \mathcal{H}(m) \otimes \left(\mathcal{H}(d)^{\mathbb{Z}_2}\right).$$

In the limit $k \to \infty$, the fields F_{β_i} are the generators of $\mathcal{H}(m)$, and the remaining quadratic fields are precisely the generators for $\mathcal{H}(d)^{\mathbb{Z}_2}$. The claim in both cases then follows by Theorem 5.1.11. $\square$

5.3.3 The nongeneric set for $V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2}$

Here we determine the set of nongeneric values of k where the strong finite generating set for $V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2}$ does not work. For convenience, we change our notation in the case of $\mathfrak{s}l_2$. We work in the usual root basis for $\{x, y, h\}$ for $\mathfrak{s}l_2$, and the corresponding generators $\{X^x, X^y, X^h\}$ for $V^k(\mathfrak{s}l_2)$. The action of θ is then given by

$$\theta(X^x) = -X^y, \qquad \theta(X^y) = -X^x, \qquad \theta(X^h) = -X^h.$$

We change the basis to the basis of eigenvectors as follows:

$$G = X^x + X^y, \qquad F = X^x - X^y, \qquad H = X^h. \tag{5.3.3}$$

The nontrivial involution θ acts on the new generators as follows:

$$\theta(G) = -G, \qquad \theta(F) = F, \qquad \theta(H) = -H.$$

Define the new generators for $V^k(\mathfrak{s}l_2)$ as follows:

$$\begin{aligned} Q_{i,j} &=: \partial^i G(z)\partial^j G(z) :\in (V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2})_{(2)}, \\ U_{i,j} &=: \partial^i H(z)\partial^j H(z) :\in (V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2})_{(2)}, \\ V_{i,j} &=: \partial^i H(z)\partial^j G(z) :\in (V^k(\mathfrak{s}l_2)^{\mathbb{Z}_2})_{(2)}. \end{aligned}$$

which each have weight $i + j + 2$.

In this notation, the strong generators for $V^k(\mathfrak{s}l_2)$ given by Theorem 5.3.2 are $\{F, Q_{0,0}, U_{0,0}, U_{0,2}, V_{0,0}, V_{0,1}, V_{0,2}\}$. In particular, these fields close under OPE, so

for α_1, α_2 in the above set, each term in the OPE of $\alpha_1(z)\alpha_2(w)$ can be expressed as a linear combination of normally ordered monomials in these generators. The coefficients of these monomials are called the *structure constants* of the OPE algebra, and they are all rational functions of k. By Theorem 5.3 of [CLIII], the only nongeneric values of k are the poles of these structure constants. Clearly there are at most finitely many such poles. By computing the remaining OPE relations, we find that all poles of structure constant lie in the set $\{0, \frac{16}{51}, \frac{16}{9}, \pm\frac{32}{3}, 16, 32, 48\}$.

An immediate consequence is the following:

Corollary 2. *For at most* $k \neq 0, \frac{16}{51}, \frac{16}{9}, \pm\frac{32}{3}, 16, 32, 48$, $V^k(\mathfrak{sl}_2)^{\mathbb{Z}_2}$ *is of type* $\mathcal{W}(1, 2^3, 3, 4^2)$.

It is important to point out that in general, it is difficult to describe the nongeneric set using the structure constants method especially with generators that have higher weights than they are here. In this case, we use the decoupling relations to describe the generic behavior.

Chapter 6

$\mathcal{W}$-Algebras

In this chapter, we introduce the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}) := \mathcal{W}^k(\mathfrak{g}, f_{\text{prin}})$ associated to a simple Lie algebra $\mathfrak{g}$ and its principal nilpotent element f_{prin}. There are several ways to construct $\mathcal{W}^k(\mathfrak{g})$, and here we briefly review a standard construction known as the *quantum Drinfeld-Sokolov reduction*. It is a certain semi-infinite cohomology of a BRST complex (refers to the physicists Becchi, Rouet, Stora and Tyutin) associated to $\mathfrak{g}$, and we follow the presentation on the book [FBZ] of Frenkel and Ben-Zvi. In the case $\mathfrak{g} = \mathfrak{s}l_3$, $\mathcal{W}^k(\mathfrak{s}l_3)$ is known as the Zamolodchikov $\mathcal{W}_3$-algebra [Za]. Our main result is a complete description of the $\mathbb{Z}_2$-orbifold of the $\mathcal{W}_3$-algebra, which is a joint work with Linshaw [AL].

6.1 The BRST complex and the quantum Drinfeld-Sokolov reduction

Let $\mathfrak{g}$ be a simple Lie algebra of rank l with the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_+$ (resp., $\mathfrak{n}_+$) is the upper (resp., lower) nilpotent subalgebra, and $\mathfrak{h}$ is the Cartan subalgebra.

Let $Cl\mathfrak{n}_+$ be the Clifford algebra associated to the vector space $\mathfrak{n}_+((t)) \oplus \mathfrak{n}_+^*((t))dt$ and the non-degenerate bilinear form $B(.,.)$. Let $\{e_\alpha\}_{\alpha\in\triangle_+}$ be a basis of $\mathfrak{n}_+$, where $\triangle_+$ is the set of positive roots of $\mathfrak{g}$. Then, $Cl\mathfrak{n}_+$ is generated by $\psi_{\alpha,n} = e_\alpha \otimes t^n$, $\psi^*_{\alpha,n} = e^*_\alpha \otimes t^{n-1}dt$ for $\alpha \in \triangle_+$, $n \in \mathbb{Z}$, and these generators satisfy the Lie bracket (2.12.1), with the corresponding Fock representation $\bigwedge_{\mathfrak{n}_+}$. Let $c^\gamma_{\alpha,\beta}$ be the structure constants in $\mathfrak{n}_+$, that is

$$[e_\alpha, e_\beta] = \sum_{\gamma\in\triangle_+} c^\gamma_{\alpha,\beta} e_\gamma.$$

Consider the vertex subalgebra

$$C^\bullet_k(\mathfrak{g}) = V^k(\mathfrak{g}) \otimes \bigwedge^\bullet_{\mathfrak{n}_+}, \tag{6.1.1}$$

where $V^k(\mathfrak{g})$ is the universal Affine vertex algebra. It is equipped with a the standard differential d_{st} of degree 1. The complex pair $(C^\bullet_k(\mathfrak{g}), d_{st})$ is the standard differential of *semi-infinite cohomology* of the $\mathfrak{n}_+((t))$ with coefficients in $V^k(\mathfrak{g})$.

Define a Lie algebra homomorphisim

$$\begin{aligned} \mathfrak{n}_+ &\xrightarrow{\rho} Cl_{\mathfrak{n}_+} \\ e_\alpha &\longmapsto \sum_{\gamma\in\triangle_+} c^\gamma_{\alpha,\beta} e_\gamma e^*_\beta. \end{aligned}$$

For $e_\alpha, e_\beta \in \mathfrak{n}_+$, we have

$$[\rho(e_\alpha), e_\beta] = [e_\alpha, e_\beta] \in \mathfrak{n}_+ \subset Cl_{\mathfrak{n}_+}.$$

Let $Q(z)$ be an odd field corresponds to Q of degree 1 in $C_k^1(\mathfrak{g})$, defined as

$$Q(z) = \sum_{\alpha\in\triangle_+} e_\alpha(z)\psi_\alpha^*(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\triangle_+} c_{\alpha,\beta}^\gamma : \psi_\alpha^*(z)\psi_\beta^*(z)\psi_\gamma(z) :,$$

and satisfies the OPE relation

$$Q(z)Q(w) \sim 0,$$

that is $Q(z)Q(w)$ is regular at $z = w$.

Consider a linear functional χ on $\mathfrak{n}_+((t))$ as follows

$$\chi(e_\alpha^n) = \begin{cases} 1 & \text{if } \alpha \textit{ is simple}, \ \ n = -1, \\ 0 & \textit{otherwise}. \end{cases}$$

Note that $\chi([x, y]) = 0$ for $x, y \in \mathfrak{n}_+((t))$, that is χ is a character of $\mathfrak{n}_+((t))$ called the Drinfeld-Sokolov character. Then,

$$d = d_{st} + \chi$$

is a differential on $C_k^\bullet(\mathfrak{g})$ satisfies $d^2 = 0$. The complex $(C_k^\bullet(\mathfrak{g}), d)$ with cohomological gradation is called the *BRST complex of the quantum Drinfeld-Sokolov reduction.* We refer to its cohomology by $H_k^\bullet(\mathfrak{g})$, and it is a vertex algebra. In particular, $H_k^0(\mathfrak{g})$ is a vertex algebra called the $\mathcal{W}$-algebra associated to $\hat{\mathfrak{g}}$ at level k and denoted by $\mathcal{W}^k(\mathfrak{g})$.

The following theorem gives the vertex algebra structure. For the proof, see [FBZ].

Theorem 6.1.1. *$H_k^0(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g})$ is a vertex algebra generated by W_i of degree d_i+1 for $i = 1, ..., l$, d_i is the ith exponent of $\mathfrak{g}$, and $H_k^i(\mathfrak{g}) = 0$ for $i \neq 0$.*

Example 6.1.2. *1. For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{W}^k(\mathfrak{sl}_2) = Vir_{c(k)}$, where the central charge $c(k) = 1 - \frac{6(k+1)^2}{k+2}$. By Theorem 6.1.1, it is finitely generated by the Virasoro field $W_1(z) = L(z)$.*

2. For $\mathfrak{g} = \mathfrak{sl}_3$, $\mathcal{W}^k(\mathfrak{sl}_3) = \mathcal{W}_3^c$ the $\mathcal{W}$-algebra which was first constructed by Zamolodchikov [Za], and the central charge $c(k) = 2 - \frac{24(k+2)^2}{k+3}$. It is finitely generated by the Virasoro field $W_1(z) = L(z)$, and $W_2(z)$ of conformal weight 3.

6.2 The $\mathcal{W}_3$-algebra

The $\mathcal{W}_3$-algebra $\mathcal{W}_3^c$ with central charge c is an extension of the Virasoro algebra. It is categorized of type $\mathcal{W}(2,3)$ strongly generated by a Virasoro field L and a weight 3 primary field W satisfying the OPE relations

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}, \tag{6.2.1}$$

$$L(z)W(w) \sim 3W(w)(z-w)^{-2} + \partial W(w)(z-w)^{-1}, \tag{6.2.2}$$

$$\begin{aligned} W(z)W(w) \sim \frac{c}{3}(z-w)^{-6} &+ 2L(w)(z-w)^{-4} + \partial L(w)(z-w)^{-3} \\ &+ \left(\frac{32}{22+5c} : LL : + \frac{3(-2+c)}{2(22+5c)}\partial^2 L\right)(z-w)^{-2} \\ &+ \left(\frac{32}{22+5c} : (\partial L)L : + \frac{-2+c}{3(22+5c)}\partial^3 L\right)(z-w)^{-1}. \end{aligned} \tag{6.2.3}$$

It is the first and simplest example of nonlinear vertex algebra. The OPE relation (6.2.3) is special in sense that it has a field of conformal weight 4 indicated by $\frac{16}{22+5c}(: LL : - \frac{3}{10}\partial^2 L)$, and so $\mathcal{W}_3^c$ is viewed as a non-closed vertex algebra.

Remark 6.2.1. *The OPE relation (6.2.3) has a pole at $c = -\frac{22}{5}$, and this pole is inessential and can be removed by multiplying W by a factor of $\sqrt{22+5c}$, and then taking the limit as $c \to -\frac{22}{5}$. The correspondent $\mathcal{W}_3$-algebra is $\mathcal{W}_3^{-22/5}$, with the*

rescaled generator, will be denoted by W, satisfies

$$W(z)W(w) \sim \left(32 : LL : -\frac{48}{5}\partial^2 L\right)(z-w)^{-2} + \left(32 : (\partial L)L : -\frac{32}{15}\partial^3 L\right)(z-w)^{-1}. \tag{6.2.4}$$

For all $c \in \mathbb{C}$, $\mathcal{W}_3^c$ is freely generated by L, W, and has a PBW basis as follows

$$: (\partial^{k_1^1} L)...(\partial^{k_{s_1}^1} L)(\partial^{k_1^2} W)...(\partial^{k_{s_2}^2} W) :, \;\; s_i \geq 0, \;\; k_1^i \geq ... \geq k_{s_i}^i \geq 0 \;\; for \;\; i = 1, 2. \tag{6.2.5}$$

From now and on, we will shortly use the notation $\mathcal{W}$ for $\mathcal{W}_3^c$.

6.2.1 Filtrations

Define a filtration on $\mathcal{W}$ as follows:

$$\mathcal{W}_{(0)} \subset \mathcal{W}_{(1)} \subset ... \qquad \mathcal{W} = \bigcup_{d \geq 0} \mathcal{W}_{(d)}, \tag{6.2.6}$$

where $\mathcal{W}_{(-1)} = \{0\}$, and $\mathcal{W}_{(r)}$ is spanned by iterated Wick products of the generators L, W and their derivatives, such that at most r copies of W and its derivatives appear. That is, $\mathcal{W}_{(r)}$ is spanned by all normally ordered monomials of the form (6.2.5) such that $s_2 \leq r$. In particular, each L and its derivatives have degree 0, while each W and its derivatives have degree 1.

Remark 6.2.2. *Checking the filtrations axioms' on the OPE relations* (6.2.1)-(6.2.3)*, once see that the above filtrations is a weak increasing filtration. Note that the associated graded algebra*

$$\mathcal{V} = gr(\mathcal{W}) = \bigoplus_{d \geq 0} \mathcal{W}_{(d)}/\mathcal{W}_{(d-1)}$$

is not abelian since it contains $\mathcal{W}_{(0)}$ as a subalgebra which it is just the Virasoro algebra with generator L. Also, the associated graded algebra $\mathcal{V}$ is freely generated

by L, W. The vertex algebra $\mathcal{V}$ will still have the same OPE relations (6.2.1)-(6.2.2) *while* (6.2.3) *will be replaced with $W(z)W(w) \sim 0$.*

Since the associated graded algebra $\mathcal{V}^c$ is nonabelian vertex algebra, so we will need to define another filtration on $\mathcal{V}$ as follows:

$$\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset ..., \qquad \mathcal{V} = \bigcup_{d \geq 0} \mathcal{V}_{(d)}, \tag{6.2.7}$$

where $\mathcal{V}_{(-1)} = \{0\}$, and $\mathcal{V}_{(r)}$ is spanned by iterated Wick products of the generators L, W and their derivatives, of length at most r. That is, $\mathcal{V}_{(r)}$ is spanned by all normally ordered monomials of the form (6.2.5) such that $s_1 + s_2 \leq r$. In particular, both L and W, and its derivatives have degree 1. From defining the OPE relations, this is actually a good increasing filtration, and therefore the associated graded algebra

$$gr(\mathcal{V}) = \bigoplus_{d \geq 0} \mathcal{V}_{(d)}/\mathcal{V}_{(d-1)}$$

is now abelian. Hence, $gr(\mathcal{V})$ is considered as an abelian vertex algebra generated by the virasoro element L of weight 2 and W of weight 3 satisfying the OPE relations

$$L(z)L(w) \sim 0,$$

$$L(z)W(w) \sim 0,$$

$$W(z)W(w) \sim 0.$$

The $\mathcal{W}$-algebra equipped with such two consequent filtrations lies in the category $\mathcal{R}$, and so $\mathcal{W} \cong gr(\mathcal{V})$ as linear spaces. In particular, $gr(\mathcal{V})$ is isomorphic to the polynomial algebra

$$\mathbb{C}[L, \partial L, \partial^2 L, \ldots, W, \partial W, \partial^2 W, \ldots].$$

6.2.2 The $\mathbb{Z}_2$-orbifold of $\mathcal{W}$

The full automorphism group of $\mathcal{W}$ is $\mathbb{Z}_2$, generated by the nontrivial involution θ. The action of θ on the generators will be as follows:

$$\theta(L) = L, \qquad \theta(W) = -W. \tag{6.2.8}$$

The OPE relations (6.2.1)-(6.2.3) will be preserved by this action on $\mathcal{W}$, that is

$$L \circ_n L = \theta(L) \circ_n \theta(L),$$

$$L \circ_n W = \theta(L) \circ_n \theta(W),$$

$$W \circ_n W = \theta(W) \circ_n \theta(W),$$

for all n as well as the filtration (6.2.6), (6.2.7) and induces an action of $\mathbb{Z}_2$ on $gr(\mathcal{V})$.

As we noted before in chapter 3, the invariant ring $\mathbb{C}[\partial^k L, \partial^l W | k, l \geq 0]^{\mathbb{Z}_2}$ is generated by all polynomials in L, and W of even degree. Going back to $\mathcal{W}^{\mathbb{Z}_2}$, is also spanned by all normally ordered monomials of the form (6.2.5) such that s_2 is even. In addition to this fact, $\mathcal{W}$ is freely generated by L, W, and so these monomials form a basis for $\mathcal{W}^{\mathbb{Z}_2}$, and the normal form is unique.

The filtration on $\mathcal{W}^{\mathbb{Z}_2}$ is obtained from the filtration (6.2.6) after restriction as follows:

$$\mathcal{W}^{\mathbb{Z}_2}_{(0)} \subset \mathcal{W}^{\mathbb{Z}_2}_{(1)} \subset \cdots, \;\; \mathcal{W}^{\mathbb{Z}_2}_{(r)} = \mathcal{W}^{\mathbb{Z}_2} \cap \mathcal{W}_{(r)}.$$

The action of $\mathbb{Z}_2$ on $\mathcal{W}$ descends to an action on $\mathcal{V} = gr(\mathcal{W})$, and

$$gr(\mathcal{W}^{\mathbb{Z}_2}) \cong \mathcal{V}^{\mathbb{Z}_2}.$$

Similarly, $\mathbb{Z}_2$ acts on $gr(\mathcal{V}) \cong \mathbb{C}[L, \partial L, \partial^2 L, ..., W, \partial W, \partial^2 W, ...]$, and

$$gr(\mathcal{V}^{\mathbb{Z}_2}) \cong gr(\mathcal{V})^{\mathbb{Z}_2} \cong \mathbb{C}[L, \partial L, \partial^2 L, \ldots, W, \partial W, \partial^2 W, \ldots]^{\mathbb{Z}_2}.$$

To describe the generators and relations for the invariant ring $\mathbb{C}[\partial^k L, \partial^l W | k, l \geq 0]^{\mathbb{Z}_2}$, we need the classical theorem of Weyl (Weyl's First and Second Fundamental Theorem of Invariant Theory for the standard representation of $\mathbb{Z}_2$).

Remark 6.2.3. *1. The action of $\mathbb{Z}_2$ on the $gr(\mathcal{V})$, which is given by $\theta(\partial^k L) = \partial^k L$ and $\theta(\partial^l W) = -\partial^l W$ guarantees that $gr(\mathcal{V})^{\mathbb{Z}_2}$ is generated by $\{L, u_{i,j} |\ i, j \geq 0\}$, where $u_{i,j} = (\partial^i W)(\partial^j W)$.*

2. The definition of $u_{i,j}$, and $gr(\mathcal{V})^{\mathbb{Z}_2}$ being an abelian imply that $u_{i,j} = u_{j,i}$, and as a consequence we only need $\{L, u_{i,j} |\ i \geq j \geq 0\}$.

The generators: Define

$$U_{i,j} = :(\partial^i W)(\partial^j W): \in \mathcal{W}^{\mathbb{Z}_2}_{(2)}, \tag{6.2.9}$$

which has filtered degree 2, weight $i + j + 6$ and corresponds to $u_{i,j} \in gr(\mathcal{V})^{\mathbb{Z}_2}$. Hence, $\mathcal{W}^{\mathbb{Z}_2}$ is strongly generated by

$$\{L, U_{i,j} |\ i \geq j \geq 0\}.$$

In $gr(\mathcal{V})^{\mathbb{Z}_2}$, as it is considered as differential algebra with derivation ∂, some of the generators could be eliminated since

$$\partial u_{i,j} = u_{i+1,j} + u_{i,j+1}.$$

To see this, let A_n be the vector space spanned by $\{u_{i,j} |\ i + j = n\}$. If n is an odd, once can see that $A_n = \partial(A_{n-1})$, and if n is an even, then $A_n = \partial(A_{n-1}) \oplus \langle u_{n,0} \rangle$. Therefore, $\{\partial^m u_{2n,0} |\ m, n \geq 0\}$ spans A_n as well as $\{u_{i,j} |\ i, j \geq 0\}$ does, and so $\{L, u_{2n,0} |\ n \geq 0\}$ is a minimal generating set for $gr(\mathcal{V})^{\mathbb{Z}_2}$ as a differential algebra.

The ideal of relations: Among these generators, the ideal of relations is generated by

$$u_{i,j}u_{k,l} - u_{i,l}u_{k,j} = 0, \;\; 0 \leq i < k, \;\; 0 \leq j < l. \tag{6.2.10}$$

Lemma 6.2.4. *$\mathcal{W}^{\mathbb{Z}_2}$ is strongly generated by*

$$\{L, U_{2n,0} |\ n \geq 0\}. \tag{6.2.11}$$

Proof. As a differential algebra, $gr(\mathcal{V})^{\mathbb{Z}_2} \cong gr(\mathcal{V}^{\mathbb{Z}_2})$ is generated by $\{L, u_{2n,0} |\ n \geq 0\}$ and so Lemma 3.6 of [LL] shows that the corresponding set strongly generates $\mathcal{V}^{\mathbb{Z}_2}$ as a vertex algebra. The claim then follows by Lemma (4.3.4). In particular, in $\mathcal{W}^{\mathbb{Z}_2}$, we can express both $U_{i,j} - U_{j,i}$ and $\partial U_{i,j} - U_{i+1,j} - U_{i,j+1}$ as normally ordered polynomials in L and its derivatives. So, $\{U_{i,j} |\ i, j \geq 0\}$ and $\{\partial^m U_{2n,0} |\ m, n \geq 0\}$ span the same vector space modulo the Virasoro algebra generated by L. □

Remark 6.2.5. *Regarding the generating set* (6.2.11)*, the filtered piece $\mathcal{W}^{\mathbb{Z}_2}_{(2r)}$ is spanned by elements with at most r of the fields $U_{2n,0}$, and $\mathcal{W}^{\mathbb{Z}_2}_{(2r)} = \mathcal{W}^{\mathbb{Z}_2}_{(2r+1)}$.*

6.2.3 Decoupling relations

The $\mathbb{Z}_2$-orbifold of the $\mathcal{W}$ is not freely generated by (6.2.11). The first relation of the form (6.2.10) among the generators (6.2.11) has the form

$$u_{0,0}u_{1,1} - u_{1,0}u_{1,0} = 0, \tag{6.2.12}$$

and is the unique relation in $gr(\mathcal{V})^{\mathbb{Z}_2}$. It corresponds to the element $:U_{0,0}U_{1,1}: - :U_{1,0}U_{1,0}: \in \mathcal{W}^{\mathbb{Z}_2}$ of minimal weight 14. The OPE relation (6.2.3) emphasizes that the above element has some corrections. However, it lies in the degree 2 filtered

piece $\mathcal{W}^{\mathbb{Z}_2}_{(2)}$ and by Computer calculations, it has the form

$$: U_{0,0}U_{1,1} : - : U_{1,0}U_{1,0} := \frac{181248 + 5590c - 475c^2}{60480(22+5c)} U_{8,0} + P(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \tag{6.2.13}$$

where P is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$, and their derivatives. To be more accurate, this relation is written more explicitly in the Appendix.

Remark 6.2.6. *Since the set* (6.2.11) *strongly generates* $\mathcal{W}^{\mathbb{Z}_2}$, *so all* $U_{i,j}$ *can be expressed in term of this generating set. For our purpose, we can write both* $U_{1,0}$, *and* $U_{1,1}$ *as follows*

$$U_{1,0} = \frac{1}{2}\partial U_{0,0} - \frac{8}{3(22+5c)} : (\partial^3 L)L : - \frac{8}{22+5c} : (\partial^2 L)\partial L : - \frac{-2+c}{48(22+5c)}\partial^5 L,$$

$$U_{1,1} = -U_{2,0} + \frac{1}{2}\partial^2 U_{0,0} - \frac{8}{3(22+5c)} : (\partial^4 L)L : - \frac{32}{3(22+5c)} : (\partial^3 L)(\partial L) :$$
$$- \frac{8}{22+5c} : (\partial^2 L)(\partial^2 L) : + \frac{2-c}{48(22+5c)}\partial^6 L : .$$

The left side of (6.2.13) is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}$ due to the Remark 6.2.6, and so (6.2.13) can be written in the form

$$\frac{181248 + 5590c - 475c^2}{60480(22+5c)} U_{8,0} = P_8(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}). \tag{6.2.14}$$

For $c \neq -\frac{22}{5}, \frac{559 \pm 7\sqrt{76657}}{95}$, the above decoupling relation allows $U_{8,0}$ to be expressed as a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$ and their derivatives. Recall, the pole at $c = -\frac{22}{5}$ is inessential. From now on, we shall assume that $c \neq -\frac{22}{5}$ for the rest of Subsections, and we deal with the case $c = -\frac{22}{5}$ separately in Subsection (6.2.7).

No decoupling relations for $U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$ could be found since there are no relations of weight less than 14 in $gr(\mathcal{V})^{\mathbb{Z}_2}$ due to Weyl First Fundamental Theorem of Invariant Theory for $\mathbb{Z}_2$. Furthermore, (6.2.14) is unique up to scalar

multiples due to the uniqueness of (6.2.14), and so no decoupling relation for $U_{8,0}$ could be found for $c = \frac{559 \pm 7\sqrt{76657}}{95}$.

Weight 16 relations: This relation could be obtained by correcting the relation $u_{0,0}u_{2,2} - u_{2,0}u_{2,0} = 0$ in $gr(\mathcal{V})^{\mathbb{Z}_2}$ and similarly as above, the corresponding element in $\mathcal{W}^{\mathbb{Z}_2}$ of weight 16:

$$:U_{0,0}U_{2,2}: - :U_{2,0}U_{2,0}: = -\frac{434176 - 20326c + 35c^2}{151200(22+5c)}U_{10,0} + Q(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.15}$$

where Q is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}$ and their derivatives. As in Remark 6.2.6, $U_{2,2}$ can be written as a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}$, and their derivatives as follows

$$\begin{aligned} U_{2,2} = U_{4,0} - 2\partial^2 U_{2,0} + \frac{1}{2}\partial^4 U_{0,0} - \frac{32}{22+5c} :(\partial^3 L)(\partial^3 L): - \frac{48}{22+5c} :(\partial^4 L)(\partial^2 L): \\ - \frac{96}{5(22+5c)} :(\partial^5 L)(\partial L): - \frac{16}{5(22+5c)} :(\partial^6 L)L: + \frac{50-41c}{1680(22+5c)}\partial^8 L. \end{aligned} \tag{6.2.16}$$

So (6.2.15) can be written in the form

$$-\frac{434176 - 20326c + 35c^2}{151200(22+5c)}U_{10,0} = Q_{10}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}). \tag{6.2.17}$$

Here $U_{10,0}$ can be eliminated whenever $c \neq \frac{10163 \pm \sqrt{88090409}}{35}$.

Likewise, correcting the relation $u_{0,0}u_{3,1} - u_{3,0}u_{1,0} = 0$ yields a relation in the same weight

$$:U_{0,0}U_{3,1}: - :U_{3,0}U_{1,0}: = -\frac{13(-1920 - 42c + 5c^2)}{9450(22+5c)}U_{10,0} + Q'(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.18}$$

which can be rewritten as

$$-\frac{13(-1920-42c+5c^2)}{9450(22+5c)}U_{10,0}=Q'_{10}(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0},U_{8,0}). \tag{6.2.19}$$

Here $U_{10,0}$ can be eliminated whenever $c\neq\frac{21\pm\sqrt{10041}}{5}$. For all c, either (6.2.17) or (6.2.19) can be used to express $U_{10,0}$ as a normally ordered polynomial in $L,U_{0,0},U_{2,0}$, $U_{4,0},U_{6,0},U_{8,0}$, and their derivatives.

Remark 6.2.7. *For $c\neq\frac{559\pm7\sqrt{76657}}{95}$, $U_{8,0}$ can be eliminated from either* (6.2.17) *or* (6.2.19) *by using* (6.2.14)*, and so the following relations are obtained*

$$-\frac{434176-20326c+35c^2}{151200(22+5c)}U_{10,0}=P_{10}(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0}), \tag{6.2.20}$$

$$-\frac{13(-1920-42c+5c^2)}{9450(22+5c)}U_{10,0}=P'_{10}(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0}). \tag{6.2.21}$$

Weight 18 **relations:** This relation could be obtained by correcting the relation $u_{0,0}u_{3,3}-u_{3,0}u_{3,0}=0$, and the corresponding element in $\mathcal{W}^{\mathbb{Z}_2}$ has the form $:U_{0,0}U_{3,3}:-:U_{3,0}U_{3,0}:=\frac{4012032+28306c-9625c^2}{1663200(22+5c)}U_{12,0}+R(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0},$ $U_{8,0},U_{10,0})$.

Either (6.2.17) or (6.2.19) can be used to eliminate $U_{10,0}$ and so

$$\frac{4012032+28306c-9625c^2}{1663200(22+5c)}U_{12,0}=Q_{12}(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0},U_{8,0}). \tag{6.2.22}$$

Here $U_{12,0}$ can be eliminated whenever $c\neq\frac{14153\pm\sqrt{38816115409}}{9625}$.

Similarly, correcting the relation $u_{0,0}u_{4,2}-u_{4,0}u_{2,0}=0$, and eliminating $U_{10,0}$ yields

$$\frac{-2785280+145762c-385c^2}{1108800(22+5c)}U_{12,0}=Q'_{12}(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0},U_{8,0}), \tag{6.2.23}$$

Here $U_{12,0}$ can be eliminated whenever $c \neq \frac{72881 \pm \sqrt{4239307361}}{385}$. Therefore using either (6.2.22) or (6.2.23), we can eliminate $U_{12,0}$ for all c.

Remark 6.2.8. *For $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$, $U_{8,0}$ can be eliminated from these equations by using* (6.2.14)*, so the following relations are obtained*

$$\frac{4012032 + 28306c - 9625c^2}{1663200(22+5c)} U_{12,0} = P_{12}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \tag{6.2.24}$$

$$\frac{-2785280 + 145762c - 385c^2}{1108800(22+5c)} U_{12,0} = P'_{12}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}). \tag{6.2.25}$$

Weight 20 **relations:** This relation could be obtained by correcting the relation $u_{0,0}u_{4,4} - u_{4,0}u_{4,0} = 0$, and the corresponding element in $\mathcal{W}^{\mathbb{Z}_2}$ has the form

$$: U_{0,0}U_{4,4} : - : U_{4,0}U_{4,0} := \frac{-20559360 + 1209594c - 5005c^2}{9459450(22+5c)} U_{14,0}$$

$$+ S(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}, U_{10,0}, U_{12,0}).$$

Once we eliminate $U_{10,0}$ and $U_{12,0}$, the following relation is obtained

$$\frac{-20559360 + 1209594c - 5005c^2}{9459450(22+5c)} U_{14,0} = Q_{14}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}). \tag{6.2.26}$$

so $U_{14,0}$ can be eliminated whenever $c \neq \frac{604797 \pm \sqrt{262879814409}}{5005}$.
Similarly, correcting the relation $u_{0,0}u_{6,2} - u_{6,0}u_{2,0} = 0$, and eliminating $U_{10,0}$ and $U_{12,0}$ yields

$$\frac{-26284032 + 1487354c - 5005c^2}{12108096(22+5c)} U_{14,0} = Q'_{14}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.27}$$

so $U_{14,0}$ can be eliminated for $c \neq \frac{743677 \pm \sqrt{421503900169}}{5005}$. Either (6.2.26) or (6.2.27) can be used to eliminate $U_{14,0}$ for all c.

Remark 6.2.9. *For* $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$, $U_{8,0}$ *can be eliminated from these equations by using* (6.2.14)*, and so the following relations are obtained*

$$\frac{-20559360 + 1209594c - 5005c^2}{9459450(22+5c)} U_{14,0} = P_{14}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \tag{6.2.28}$$

$$\frac{-26284032 + 1487354c - 5005c^2}{12108096(22+5c)} U_{14,0} = P'_{14}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}). \tag{6.2.29}$$

6.2.4 Higher decoupling relations

The calculations we have seen last subsection guarantee that whenever $c \neq \frac{559 \pm 7\sqrt{76657}}{95}$, there exist higher decoupling relations

$$U_{n,0} = P_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \;\; n = 16, 18, 20, \ldots, \tag{6.2.30}$$

and whenever $c = \frac{559 \pm 7\sqrt{76657}}{95}$, there exist relations

$$U_{n,0} = Q_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \;\; n = 16, 18, 20, \ldots. \tag{6.2.31}$$

The questions may arise:

- Are those higher decoupling relations existed?
- What if we change the normally ordered polynomial, will the coefficient of $U_{2n,0}$ be changed?

To answer these questions, first we need a certain invariant of elements of $\mathcal{W}_{(2)}^{\mathbb{Z}_2}$ of even weight. Let $\omega \in \mathcal{W}_{(2)}^{\mathbb{Z}_2}$ of weight $n+6$ be given, where n is an even integer. Write ω in normal form. For $i = 0, 1, \ldots, \frac{n}{2}$, in the normal form let

$$C_{n,i}(\omega) \tag{6.2.32}$$

denote the coefficient of $:(\partial^{n-i}W)(\partial^i W):$, which is well-defined by uniqueness of (6.2.5). Let

$$C_n(\omega) = \sum_{i=0}^{n/2} (-1)^i C_{n,i}(\omega), \tag{6.2.33}$$

that is $C_n(\omega)$ is an alternate sum of the coefficients of $:(\partial^{n-i}W)(\partial^i W):$. Recall, $\{L, U_{n,0} |\ n = 0, 2, 4, \dots\}$ strongly generates $\mathcal{W}^{\mathbb{Z}_2}$ and since $U_{n,0}$ has weight $n+6$, so we can write

$$\omega = P_\omega(L, U_{0,0}, U_{2,0}, \dots, U_{n,0}),$$

where P_ω is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, \dots, U_{n,0}$, and their derivatives. This expression for ω is not unique due to existence of normally ordered relations among these generators, as well as different choices of normal ordering. In particular, the coefficients of $\partial^i U_{n-i,0}$ for $i = 2, 4, \dots, n$ will depend on the choice of P_ω.

Example 6.2.10. *Recall the decoupling relation* (6.2.14) *of weight* 14. *For a given* ω *of filtered degree* 2, *and weight* 14, *the coefficient* $C_8(\omega)$ *can be given by*

$$\begin{aligned}
C_8(\omega) &= \sum_{i=0}^{4} (-1)^i C_{n,i}(\omega) \\
&= \frac{12556 + 772c - 5c^2}{20160(22+5c)} - \frac{6504 + 3014c + 115c^2}{15120(22+5c)} + \frac{450 + 31c}{2640 + 600c} - \frac{454 + 29c}{3960 + 900c} \\
&+ \left(\frac{1}{32} + \frac{8}{9(22+5c)}\right) \\
&= \frac{181248 + 5590c - 475c^2}{60480(22+5c)}.
\end{aligned}$$

This is actually the coefficient of $U_{8,0}$ *in* P_ω, *and it is canonical, in the sense it is independent of all choices of normal ordering as the following Lemma proves.*

Lemma 6.2.11. *For any* $\omega \in \mathcal{W}^{\mathbb{Z}_2}_{(2)}$ *of weight* $n+6$, *the coefficient of* $U_{n,0}$ *in* P_ω *is canonical, and coincides with* $C_n(\omega)$.

Proof. Consider a subspace $\mathcal{J} \subset \mathcal{W}^{\mathbb{Z}_2}$ spanned by elements of the form $:a\partial b:$ with $a, b \in \mathcal{W}^{\mathbb{Z}_2}$. It is well known that Zhu's commutative algebra $C(\mathcal{W}^{\mathbb{Z}_2}) = \mathcal{W}^{\mathbb{Z}_2}/\mathcal{J}$ is a commutative, associative algebra with generators corresponding to the strong generators $\{L, U_{2n,0} |\ n \geq 0\}$. In particular, consider two expressions for $\omega \in \mathcal{W}^{\mathbb{Z}_2}_{(2)}$ of degree 2 filtered piece and even weight $n+6$, that is

$$\omega = P_\omega(L, U_{0,0}, U_{2,0}, \ldots, U_{n,0}) = Q_\omega(L, U_{0,0}, U_{2,0}, \ldots, U_{n,0}).$$

Let $\tilde{P}_\omega$ and $\tilde{Q}_\omega$ denote the components of P_ω, Q_ω which are linear combinations of $\partial^i U_{n-i,0}$ for $i = 0, 2, \ldots n$. Then $\tilde{P}_\omega - \tilde{Q}_\omega$ lies in $\mathcal{J}$, and hence must be a total derivative.

Recall now that for $i = 0, 1, \ldots, \frac{n}{2}$,

$$u_{n-i,i} = (\partial^{n-i} W)(\partial^i W) \in gr(\mathcal{V})^{\mathbb{Z}_2} \cong \mathbb{C}[L, \partial L, \partial^2 L, \ldots, W, \partial W, \partial^2 W, \ldots]^{\mathbb{Z}_2}.$$

We claim that

$$u_{n-i,i} = (-1)^i u_{n,0} + \nu,$$

where ν is a linear combination of $\partial^j u_{n-j,0}$ for $j = 2, 4, \ldots, n$, and hence is a total derivative. For $i = 0$, take $\nu = 0$, and since $\partial(u_{n-i,i-1}) = u_{n+1-i,i-1} + u_{n-i,i}$, which is a total derivative, it holds by induction on i. It follows from (6.2.3) that for $i = 0, 1, \ldots, \frac{n}{2}$,

$$U_{n-i,i} = (-1)^i U_{n,0} + \omega,$$

where ω is a linear combination of $\partial^j U_{n-j,0}$ for $j = 2, 4, \ldots, n$ modulo terms in the Virasoro algebra generated by L. This proves the claim. $\square$

Corollary 3. *The coefficient of* $U_{8,0}$ *in* (6.2.14) *coincides with*

$$C_8(: U_{0,0} U_{1,1} : - : U_{1,0} U_{1,0} :),$$

and is independent of all choices of normal ordering in P_8. Similarly, the coefficient of $U_{10,0}$ in (6.2.17)-(6.2.21)*, the coefficient of $U_{12,0}$ in* (6.2.22)-(6.2.25)*, and the coefficient of $U_{14,0}$ in* (6.2.26)-(6.2.29)*, are independent of all choices of normally ordering in these expressions.*

As indicated in the Example 6.2.10, ω is given by the expression $:U_{0,0}U_{1,1}: - :U_{1,0}U_{1,0}:$. Similarly, the coefficient of $U_{10,0}$ in (6.2.17) coincides with

$$\begin{aligned}
C_{10}(:U_{0,0}U_{2,2}: - :U_{2,0}U_{2,0}:) &= \sum_{i=0}^{5}(-1)^i C_{n,i}(:U_{0,0}U_{2,2}: - :U_{2,0}U_{2,0}:) \\
&= \frac{65806-3573c-945c^2}{302400(22+5c)} - \frac{814-207c}{3780(22+5c)} \\
&+ \frac{-25842+12107c+175c^2}{60480(22+5c)} - \frac{4958+289c}{41580+9450c} \\
&+ \frac{2386+263c}{4320(22+5c)} - \frac{1}{900}\left(3+\frac{272}{22+5c}\right) \\
&= -\frac{434176-20326c+35c^2}{151200(22+5c)}.
\end{aligned}$$

The coefficient of $U_{10,0}$ in (6.2.21) coincides with

$$\begin{aligned}
C_{10}(:U_{0,0}U_{3,1}: - :U_{3,0}U_{1,0}:) &= \sum_{i=0}^{5}(-1)^i C_{n,i}(:U_{0,0}U_{3,1}: - :U_{3,0}U_{1,0}:) \\
&= \frac{13082+85c-45c^2}{75600(22+5c)} - \frac{-3134+761c+35c^2}{15120(22+5c)} \\
&+ \frac{1874+143c}{36960+8400c} - \frac{51+20c-\frac{8512}{22+5c}}{25200} \\
&+ \frac{98-17c}{7920+1800c} - -\frac{29(6+c)}{300(22+5c)} \\
&= -\frac{13(-1920-42c+5c^2)}{9450(22+5c)}.
\end{aligned}$$

Remark 6.2.12. *The operator $U_{0,0}\circ_1$ raises the weight by 4, and so applying this to $U_{n,0}$ yields*

$$U_{0,0} \circ_1 U_{n,0} = F(n,c)U_{n+4,0} + R_n(L, U_{0,0}, U_{2,0}, \dots, U_{n+2,0}),$$

where $F(n,c)$ denotes the coefficient of $U_{n+4,0}$ and R_n is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, \dots, U_{n+2,0}$ and their derivatives. From the OPE relations (6.2.2) *and* (6.2.3)*, we deduce that $U_{0,0} \circ_1 U_{n,0}$ lies in $\mathcal{W}^{\mathbb{Z}_2}_{(2)}$, and so by Lemma 6.2.11, we have*

$$F(n,c) = C_{n+4}(U_{0,0} \circ_1 U_{n,0}). \tag{6.2.34}$$

Likewise, the operator $U_{2,0}\circ_1$ raises the weight by 6, and so applying this to $U_{n,0}$ yields

$$U_{2,0} \circ_1 U_{n,0} = G(n,c)U_{n+6,0} + S_n(L, U_{0,0}, U_{2,0}, \dots, U_{n+4,0}),$$

where $G(n,c)$ denotes the coefficient of $U_{n+6,0}$ and S_n is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, \dots, U_{n+4,0}$ and their derivatives. Then $U_{2,0} \circ_1 U_{n,0}$ lies in $\mathcal{W}^{\mathbb{Z}_2}_{(2)}$, and

$$G(n,c) = C_{n+6}(U_{2,0} \circ_1 U_{n,0}). \tag{6.2.35}$$

The following theorems give an explicit formulas for $F(n,c)$ and $G(n,c)$. It turns out they are rational functions of n and c. The proof of this essential observation will be given in next subsection.

Theorem 6.2.13. *For all even integers $n \geq 0$,*

$$F(n,c) = -\frac{(10+n)(p_0(c) + p_1(c)n + p_2(c)n^2 + p_3(c)n^3)}{36(22+5c)(1+n)(3+n)(4+n)},$$

where

$$p_0(c) = 720 + 384c + 12c^2, \;\; p_1(c) = -5286 + 125c + 19c^2,$$

$$p_2(c) = -2160 + 40c + 8c^2, \;\; p_3(c) = -186 + 11c + c^2$$

Theorem 6.2.14. *For all even integers $n \geq 0$,*

$$G(n,c) = -\frac{(12+n)(q_0(c) + q_1(c)n + q_2(c)n^2 + q_3(c)n^3 + q_4(c)n^4)}{1260(22+5c)(1+n)(3+n)(4+n)(5+n)},$$

where

$$q_0(c) = -466200 + 20580c + 2100c^2, \;\; q_1(c) = -183780 - 46096c + 3745c^2,$$

$$q_2(c) = -74076 - 31732c + 2065c^2, \;\; q_3(c) = -19116 - 5624c + 455c^2,$$

$$q_4(c) = -1308 - 248c + 35c^2.$$

6.2.5 Proof of theorem 6.2.13

For all $n \geq 0$, setting $a = U_{0,0}, \;\; b = \partial^n W, \;\; c = W$ in the Proposition 2.4.16-(2.4.7) yields

$$U_{0,0} \circ_1 U_{n,0} = :(U_{0,0} \circ_1 \partial^n W)W: + (U_{0,0} \circ_0 \partial^n W) \circ_0 W + :(\partial^n W)(U_{0,0} \circ_1 W):.$$

So in order to calculate $F(n,c) = C_{n+4}(U_{0,0} \circ_1 U_{n,0})$, it is sufficient to calculate the following three expressions:

$$C_{n+4}\Big(:(U_{0,0} \circ_1 \partial^n W)W: \Big), \tag{6.2.36}$$

$$C_{n+4}\Big((U_{0,0} \circ_0 \partial^n W) \circ_0 W \Big), \tag{6.2.37}$$

$$C_{n+4}\Big(: (\partial^n W)(U_{0,0} \circ_1 W) : \Big). \tag{6.2.38}$$

Lemma 6.2.15. *For all $n \geq 1$, $C_{n+4}\Big((U_{0,0} \circ_0 \partial^n W) \circ_0 W\Big) = 0$.*

Proof. We have $\partial\big(U_{0,0} \circ_0 \partial^{n-1} W\big) = \partial(U_{0,0}) \circ_0 \partial^{n-1} W + U_{0,0} \circ_0 \partial^n W$. The first term in the right is always 0, so $\partial\big(U_{0,0} \circ_0 \partial^{n-1} W\big) = U_{0,0} \circ_0 \partial^n W$. Therefore,

$$(U_{0,0} \circ_0 \partial^n W) \circ_0 W = \partial\big(U_{0,0} \circ_0 \partial^{n-1} W\big) \circ_0 W = 0,$$

and so the expression (6.2.37) always vanish. □

To calculate (6.2.36), we need the following lemma.

Lemma 6.2.16. *For all $n \geq 1$,*

$$U_{0,0} \circ_0 \partial^{n-1} W - \frac{64}{22+5c}\partial^n\big(: LLW : \big) + \frac{64}{22+5c}\partial^{n-1}\big(: (\partial L)LW : \big)$$

$$-\frac{10(14+c)}{3(22+5c)}\partial^{n+2}\big(: LW : \big) + \frac{86+5c}{22+5c}\partial^{n+1}\big(: (\partial L)W : \big)$$

$$-\frac{26+3c}{22+5c}\partial^n\big(: (\partial^2 L)W : \big) + \frac{2(-2+c)}{3(22+5c)}\partial^{n-1}\big(: (\partial^3 L)W : \big)$$

$$-\frac{-186+11c+c^2}{36(22+5c)}\partial^{n+4} W = 0.$$

Proof. For $n = 1$ follows from the fact that

$$\partial\big(U_{0,0} \circ_0 \partial^{n-1} W\big) = U_{0,0} \circ_0 \partial^n W.$$

The claim then follows by induction on n. □

Lemma 6.2.17. *For all $n \geq 1$,*

$$U_{0,0} \circ_1 \partial^n W - \frac{64(1+n)}{22+5c}\partial^n\big(: LLW : \big) + \frac{64n}{22+5c}\partial^{n-1}\big(: (\partial L)LW : \big)$$

$$-\frac{2(258+15c+70n+5cn)}{3(22+5c)}\partial^{n+2}\big(:LW:\big)+\frac{236+10c+86n+5cn}{22+5c}\partial^{n+1}\big(:(\partial L)W:\big)$$

$$-\frac{58+3c+26n+3cn}{22+5c}\partial^{n}\big(:(\partial^2 L)W:\big)+\frac{2(-2+c)}{3(22+5c)}\partial^{n-1}\big(:(\partial^3 L)W:\big)$$

$$-\frac{-426+91c+5c^2-186n+11cn+c^2n}{36(22+5c)}\partial^{n+4}W=0.$$

Proof. For $n=1$, it is obvious. The proof then follows by induction on n using the previous lemma and the formula

$$\partial\big(U_{0,0}\circ_1\partial^{n-1}W\big)=-U_{0,0}\circ_0\partial^{n-1}W+U_{0,0}\circ_1\partial^{n}W.$$

□

Corollary 4. *For all $n\geq 1$,*

$$C_{n+4}\Big((:U_{0,0}\circ_1\partial^n W)W:\Big)-\frac{64(1+n)}{22+5c}C_{n+4}\Big(:(\partial^n(:LLW:))W:\Big)$$

$$+\frac{64n}{22+5c}C_{n+4}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW:\big)\big)W:\Big)$$

$$-\frac{2(258+15c+70n+5cn)}{3(22+5c)}C_{n+4}\Big(:(\partial^{n+2}(:LW:))W:\Big)$$

$$+\frac{236+10c+86n+5cn}{22+5c}C_{n+4}\Big(:\partial^{n+1}\big(:(\partial L)W:\big)W:\Big)$$

$$-\frac{58+3c+26n+3cn}{22+5c}C_{n+4}\Big(:\partial^{n}\big(:(\partial^2 L)W:\big)W:\Big)$$

$$+\frac{2(-2+c)}{3(22+5c)}C_{n+4}\Big(:\partial^{n-1}\big(:(\partial^3 L)W:\big)W:\Big)$$

$$-\frac{-426+91c+5c^2-186n+11cn+c^2n}{36(22+5c)}=0.$$

We shall calculate C_{n+4} to each term in the Corollary 4 for our purpose. So, first we have

$$C_{n+4,0}\Big(:(\partial^n(:LLW:))W: \Big) = \frac{15}{(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(:(\partial^n(:LLW:))W: \Big) = \frac{7}{(1+n)(2+n)(3+n)},$$

$$C_{n+4,2}\Big(:(\partial^n(:LLW:))W: \Big) = \frac{1}{(1+n)(2+n)},$$

$$C_{n+4,i}\Big(:(\partial^n(:LLW:))W: \Big) = 0, \qquad 3 \le i \le \frac{n+4}{2}.$$

Using (6.2.33) yields

$$C_{n+4}\Big(:(\partial^n(:LLW:))W: \Big) = \frac{-1+n}{(2+n)(3+n)(4+n)}. \tag{6.2.39}$$

Next, we have

$$C_{n+4,0}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW: \big)\big)W: \Big) = -\frac{24}{n(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW: \big)\big)W: \Big) = -\frac{10}{n(1+n)(2+n)(3+n)},$$

$$C_{n+4,2}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW: \big)\big)W: \Big) = -\frac{1}{n(1+n)(2+n)},$$

$$C_{n+4,i}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW: \big)\big)W: \Big) = 0, \;\; 3 \le i \le \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(:\big(\partial^{n-1}\big(:(\partial L)LW: \big)\big)W: \Big) = -\frac{-4+n}{n(2+n)(3+n)(4+n)}. \tag{6.2.40}$$

Next, we have

$$C_{n+4,0}\Big(: (\partial^{n+2}(: LW :))W : \Big) = \frac{3}{(3+n)(4+n)},$$

$$C_{n+4,1}\Big(: (\partial^{n+2}(: LW :))W : \Big) = \frac{1}{3+n},$$

$$C_{n+4,i}\Big(: (\partial^{n+2}(: LW :))W : \Big) = 0, \;\; 2 \leq i \leq \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(: (\partial^{n+2}(: LW :))W : \Big) = -\frac{1+n}{(3+n)(4+n)}. \tag{6.2.41}$$

Next, we have

$$C_{n+4,0}\Big(: \partial^{n+1}\big(: (\partial L)W : \big)W : \Big) = -\frac{6}{(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(: \partial^{n+1}\big(: (\partial L)W : \big)W : \Big) = -\frac{1}{(2+n)(3+n)},$$

$$C_{n+4,i}\Big(: \partial^{n+1}\big(: (\partial L)W : \big)W : \Big) = 0, \;\; 2 \leq i \leq \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(: \partial^{n+1}\big(: (\partial L)W : \big)W : \Big) = \frac{-2+n}{(2+n)(3+n)(4+n)}. \tag{6.2.42}$$

Next, we have

$$C_{n+4,0}\Big(: \partial^{n}\big(: (\partial^2 L)W : \big)W : \Big) = \frac{18}{(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(: \partial^{n}\big(: (\partial^2 L)W : \big)W : \Big) = \frac{2}{(1+n)(2+n)(3+n)},$$

$$C_{n+4,i}\Big(: \partial^{n}\big(: (\partial^2 L)W : \big)W : \Big) = 0, \;\; 2 \leq i \leq \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(: \partial^n\big(: (\partial^2 L)W : \big)W : \Big) = -\frac{2(-5+n)}{(1+n)(2+n)(3+n)(4+n)}. \tag{6.2.43}$$

Next, we have

$$C_{n+4,0}\Big(: \partial^{n-1}\big(: (\partial^3 L)W : \big)W : \Big) = -\frac{72}{n(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(: \partial^{n-1}\big(: (\partial^3 L)W : \big)W : \Big) = -\frac{6}{n(1+n)(2+n)(3+n)},$$

$$C_{n+4,i}\Big(: \partial^{n-1}\big(: (\partial^3 L)W : \big)W : \Big) = 0, \;\; 2 \le i \le \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(: \partial^{n-1}\big(: (\partial^3 L)W : \big)W : \Big) = \frac{6(-8+n)}{n(1+n)(2+n)(3+n)(4+n)}. \tag{6.2.44}$$

Combining (6.2.39)-(6.2.44) yields (6.2.36).

The following calculation is needed to find (6.2.38):

$$\begin{aligned} U_{0,0} \circ_1 W - \frac{64}{22+5c} : LLW : - \frac{2(258+15c)}{3(22+5c)}\partial^2(: LW :) + \frac{236+10c}{22+5c}\partial\big(: (\partial L)W : \big) \\ - \frac{58+3c}{22+5c} : (\partial^2 L)W : - \frac{-426+91c+5c^2}{36(22+5c)}\partial^4 W. \end{aligned} \tag{6.2.45}$$

Since $C_{n+4}\big(: (\partial^n W)(\partial^4 W) : \big) = 1$ when n is even, this implies the following corollary

Corollary 5. *We have*

$$C_{n+4}\Big(: (\partial^n W)(U_{0,0} \circ_1 W) : \Big) - \frac{64}{22+5c} C_{n+4}\Big(: (\partial^n W)(: LLW :) : \Big)$$

$$- \frac{2(258+15c)}{3(22+5c)} C_{n+4}\Big(: (\partial^n W)\big(\partial^2(: LW :)\big) : \Big)$$

$$+\frac{236+10c}{22+5c}C_{n+4}\Big(:(\partial^n W)\big(\partial(:(\partial L)W:)\big): \Big)$$

$$-\frac{58+3c}{22+5c}C_{n+4}\Big(:(\partial^n W)\big(:(\partial^2 L)W: \big): \Big)$$

$$-\frac{-426+91c+5c^2}{36(22+5c)}=0.$$

We shall calculate C_{n+4} to each term in the Corollary 5 for our purpose. So, first we have

$$C_{n+4,0}\Big(:(\partial^n W)(:LLW:): \Big)=\frac{-1+n}{(2+n)(3+n)(4+n)},$$

$$C_{n+4,i}\Big(:(\partial^n W)(:LLW:): \Big)=0, \qquad 1\le i\le \frac{n+4}{2}.$$

Using (6.2.33) yields

$$C_{n+4}\Big(:(\partial^n W)(:LLW:): \Big)=\frac{-1+n}{(2+n)(3+n)(4+n)}. \tag{6.2.46}$$

Next, we calculate

$$C_{n+4,0}\Big(:(\partial^n W)\big(\partial^2(:LW:)\big): \Big)=-\frac{2(-5+n)}{(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(:(\partial^n W)\big(\partial^2(:LW:)\big): \Big)=-\frac{2(-3+n)}{(1+n)(2+n)(3+n)},$$

$$C_{n+4,2}\Big(:(\partial^n W)\big(\partial^2(:LW:)\big): \Big)=-\frac{-1+n}{(1+n)(2+n)},$$

$$C_{n+4,i}\Big(:(\partial^n W)\big(\partial^2(:LW:)\big): \Big)=0,\ \ 3\le i\le \frac{n+4}{2}.$$

Hence

$$C_{n+4}\Big(:(\partial^n W)\big(\partial^2(:LW:)\big): \Big)=-\frac{1+n}{(3+n)(4+n)}. \tag{6.2.47}$$

Next, we have

$$C_{n+4,0}\Big(:(\partial^n W)\big(\partial(:(\partial L)W:)\big): \Big) = -\frac{2(-5+n)}{(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,1}\Big(:(\partial^n W)\big(\partial(:(\partial L)W:)\big): \Big) = -\frac{-3+n}{(1+n)(2+n)(3+n)},$$

$$C_{n+4,i}\Big(:(\partial^n W)\big(\partial(:(\partial L)W:)\big): \Big) = 0, \;\; 2 \leq i \leq \frac{n+4}{2}.$$

Hence

$$C_{n+4}\Big(:(\partial^n W)\big(\partial(:(\partial L)W:)\big): \Big) = \frac{-2+n}{(2+n)(3+n)(4+n)}. \tag{6.2.48}$$

Next, we have

$$C_{n+4,0}\Big(:(\partial^n W)\big(:(\partial^2 L)W:\big): \Big) = -\frac{2(-5+n)}{(1+n)(2+n)(3+n)(4+n)},$$

$$C_{n+4,i}\Big(:(\partial^n W)\big(:(\partial^2 L)W:\big): \Big) = 0, \;\; 1 \leq i \leq \frac{n+4}{2}.$$

Therefore

$$C_{n+4}\Big(:(\partial^n W)\big(:(\partial^2 L)W:\big): \Big) = -\frac{2(-5+n)}{(1+n)(2+n)(3+n)(4+n)}. \tag{6.2.49}$$

Combining (6.2.46)-(6.2.49) yields (6.2.38). Finally, the proof of Theorem 6.2.13 follows by combining the formulas for (6.2.36) and (6.2.38). The proof of Theorem 6.2.14 is similar and so it is omitted.

6.2.6 The minimal strong generating set for the orbifold of $\mathcal{W}$

The main body is presented in this section which gives a complete description to the $\mathbb{Z}_2$-orbifold of the $\mathcal{W}$-algebra. Before we state the main theorem, we introduce some useful observations.

Theorem 6.2.18. *For all $c \neq -\frac{22}{5}$ and all even integers $n \geq 16$, we have either $F(n-4,c) \neq 0$ or $G(n-6,c) \neq 0$. In other words, the variety $V \subset \mathbb{C}^2$ determined by $F(n-4,c) = 0$ and $G(n-6,c) = 0$, has no points (c,n) with $n \geq 16$ an even integer.*

Proof. Let

$$f(n,c) = p_0(c) + p_1(c)n + p_2(c)n^2 + p_3(c)n^3,$$

$$g(n,c) = q_0(c) + q_1(c)n + q_2(c)n^2 + q_3(c)n^3 + q_4(c)n^4,$$

where $p_i(c)$ and $q_i(z)$ are as in Theorems 6.2.13 and 6.2.14. For a positive integer n, once get

$$F(n,c) = 0 \Leftrightarrow f(n,c) = 0, \qquad G(n,c) = 0 \Leftrightarrow g(n,c) = 0.$$

We may regard $f(n,c)$ as a family of quadratics in c parametrized by n, and so

$$\begin{aligned} f(n,c) = (720 - 5286n - 2160n^2 - 186n^3) + (384 + 125n + 40n^2 + 11n^3)c \\ + (12 + 19n + 8n^2 + n^3)c^2. \end{aligned}$$

As $n \to \infty$, the quadratic formula $p_3(c) = -186 + 11c + c^2 = 0$ can be used to express the roots $r_1(n)$ and $r_2(n)$ as functions of n, that is

$$\lim_{n\to\infty} \frac{1}{n^3} f(n,c) = p_3(c),$$

and so

$$\lim_{n\to\infty} r_1(n) = \frac{-11-\sqrt{865}}{2} \sim -20.2054, \quad \lim_{n\to\infty} r_2(n) = \frac{-11+\sqrt{865}}{2} \sim 9.20544.$$

Similarly, we regard $g(n,c)$ as a family of quadratics in c parametrized by n, and so

$$g(n,c) = (-466200 - 183780n - 74076n^2 - 19116n^3 - 1308n^4)$$

$$+(20580 - 46096n - 31732n^2 - 5624n^3 - 248n^4)c$$

$$+(2100 + 3745n + 2065n^2 + 455n^3 + 35n^4)c^2,$$

and we can express the roots $s_1(n)$ and $s_2(n)$ as functions of n. Since

$$\lim_{n\to\infty} \frac{1}{n^4} g(n,c) = q_4(c),$$

we have

$$\lim_{n\to\infty} s_1(n) = \frac{2(62-\sqrt{15289})}{35} \sim -3.52278, \quad \lim_{n\to\infty} s_2(n) = \frac{2(62+\sqrt{15289})}{35} \sim 10.6085.$$

Consider n as a positive real variable, and so for $i = 1, 2$, $r_i(n)$ and $s_i(n)$ are differentiable functions of n. Using the derivatives test for $r_1(n)$ and $r_2(n)$, we see both are decreasing functions on $(9, \infty)$. We have

$$r_1(22) = \frac{-139622 - 2\sqrt{51839598721}}{29900} \cong -19.8993,$$

so $-20.2054 < r_1(n) < -19.8993$ for all $n > 22$. Similarly,

$$r_2(22) = \frac{-139622 + 2\sqrt{51839598721}}{29900} \cong 10.56,$$

so $9.20544 < r_2(n) < 10.56$ for all $n > 22$. This implies that if $n > 26$ is an even positive integer and $F(n-4, c) = 0$, c is a real number that lies either in $(-20.2054, -19.8993)$ or $(9.20544, 10.56)$.

Likewise, both $s_1(n)$ and $s_2(n)$ are decreasing functions on $(7, \infty)$. Note that

$$s_1(20) = \frac{24566535 - 945\sqrt{1800197569}}{5071500} \cong -3.06194,$$

so $-3.52278 < s_1(n) < -3.06194$ for all $n > 20$. We have,

$$s_2(20) = \frac{24566535 + 945\sqrt{1800197569}}{5071500} \cong 12.75,$$

so $10.6085 < s_2(n) < 12.75$ for all $n > 20$. Therefore if $n > 26$ is a positive integer and $G(n-6, c) = 0$, then c is a real number lying either in $(-3.52278, -3.06194)$ or $(10.6085, 12.75)$. This shows that Theorem 6.2.18 holds for all $n > 26$. It is easy to demonstrate the Theorem for $16 \leq n \leq 26$ by constructing decoupling relations in the form (6.2.30), (6.2.31), and so the claim follows. $\square$

The following corollary is an immediate consequence of the previous Theorem, which can confirm the existence of the decoupling relations for $U_{n,0}$ for all $n \geq 8$.

Corollary 6. *1. For all $c \neq -\frac{22}{5}, \frac{559 \pm 7\sqrt{76657}}{95}$ and all even integers $n \geq 8$, there exists a decoupling relation*

$$U_{n,0} = P_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \tag{6.2.50}$$

where P_n is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$ and their derivatives.

2. *For* $c = \frac{559 \pm 7\sqrt{76657}}{95}$ *and all even integers* $n \geq 10$*, there exists a decoupling relation*

$$U_{n,0} = Q_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.51}$$

where Q_n *is a normally ordered polynomial in* $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}$ *and their derivatives.*

Proof. First, suppose that $c \neq -\frac{22}{5}, \frac{559 \pm 7\sqrt{76657}}{95}$. For $n = 8, 10, 12, 14$, we have introduced the relations (6.2.50) in the subsection (6.2.3), so let $n \geq 16$ and assume the result for all even integers $8 \leq m < n$. First, suppose that $F(n-4, c) \neq 0$. Apply the operator $U_{0,0} \circ_1$ to both sides of

$$U_{n-4,0} = P_{n-4}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0})$$

and use the Remark 6.2.12 we obtain

$$F(n-4, c)U_{n,0} + R_{n-4}(L, U_{0,0}, U_{2,0}, \ldots, U_{n-2,0}) = U_{0,0} \circ_1 P_{n-4}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}).$$

The term $U_{0,0} \circ_1 P_{n-4}$ is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, \ldots, U_{10,0}$ and their derivatives. All $U_{8,0}, U_{10,0}, \ldots, U_{n-2,0}$, and their derivatives can be eliminated by using the previous decoupling relations, and so we get the desired relation.

Suppose now $F(n-4, c) = 0$, and so $G(n-6, c) \neq 0$ by assumption. Apply the operator $U_{2,0} \circ_1$ to both sides of

$$U_{n-6,0} = P_{n-6}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}),$$

and use the Remark 6.2.12 we obtain

$$G(n-6, c)U_{n,0} + S_{n-6}(L, U_{0,0}, U_{2,0}, \ldots, U_{n-2,0}) = U_{2,0} \circ_1 P_{n-6}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}).$$

Since the right hand side depends only on $L, U_{0,0}, U_{2,0}, \ldots, U_{12,0}$, so all $U_{8,0}, U_{10,0}, \ldots,$ $U_{n-2,0}$ and their derivatives can eliminated by using the previous relations.

Next, suppose that $c = \frac{559 \pm 7\sqrt{76657}}{95}$. For $n = 10, 12, 14$, we have the desired relations (6.2.51) in the subsection (6.2.3), so let $n \geq 16$ and assume the result for all even integers $10 \leq m < n$. The rest of the proof follows similarly as above. $\square$

Our main theorem is the following:

Theorem 6.2.19. *1. For all $c \neq -\frac{22}{5}, \frac{559 \pm 7\sqrt{76657}}{95}$, $\mathcal{W}^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2, 6, 8, 10, 12)$ with minimal strong generating set $\{L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}\}$.*

2. For $c = \frac{559 \pm 7\sqrt{76657}}{95}$, $\mathcal{W}^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2, 6, 8, 10, 12, 14)$ with minimal strong generating set $\{L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}\}$.

Proof. First, Lemma 6.2.4 asserts that $\mathcal{W}^{\mathbb{Z}_2}$ is strongly generated by the natural infinite set $\{L, U_{2n,0} | n \geq 0\}$. Next, it suffices to construct decoupling relations of the form

$$U_{n,0} = P_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0})$$

for $c \neq -\frac{22}{5}, \frac{559 \pm 7\sqrt{76657}}{95}$, and all even integers $n \geq 16$ since we already have such relations for $n = 8, 10, 12, 14$. Here P_n is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$, and their derivatives. For $c = \frac{559 \pm 7\sqrt{76657}}{95}$, it suffices to construct decoupling relations of the form

$$U_{n,0} = Q_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0})$$

for all even integers $n \geq 16$ since we already have such relations for $n = 10, 12, 14$ where Q_n is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}$, and their derivatives. Applying the operators $U_{0,0} \circ_1$ and $U_{2,0} \circ_1$ to the relations we have

constructed yields two families of relations

$$F(n,c)U_{n+4,0} = A_n(L, U_{0,0}, U_{2,0}, ..., U_{n+2,0}), \tag{6.2.52}$$

$$G(n,c)U_{n+6,0} = B_n(L, U_{0,0}, U_{2,0}, ..., U_{n+4,0}) \tag{6.2.53}$$

where A_n, B_n are normally ordered polynomials as above. Both $F(n,c)$, and $G(n,c)$ have no poles for $c \neq -\frac{22}{5}$ and $n \geq 10$. For $n \geq 16$, $U_{n,0}$ can be eliminated via (6.2.52) or (6.2.53) only if (c,n) does not lie on the affine variety $V \subset \mathbb{C}^2$ determined by $F(n-4,c) = 0$ and $G(n-6,c) = 0$, and this follows by Theorem 6.2.18.

□

6.2.7 The case $c = -\frac{22}{5}$

Recall, the rescaled generator W satisfies (6.2.4), and the generators for the orbifold $\mathcal{W}^{\mathbb{Z}_2}$ are still $\{L, U_{2n,0} |\ n \geq 0\}$.

Decoupling relations Similarly as in (6.2.3), the first relation among the generators has the form

$$u_{0,0}u_{1,1} - u_{1,0}u_{1,0} = 0,$$

and is the unique relation in $gr(\mathcal{V})^{\mathbb{Z}_2}$, of minimal weight 14. It corresponds to the element $: U_{0,0}U_{1,1} : - : U_{1,0}U_{1,0} :$ of $\mathcal{W}^{\mathbb{Z}_2}$. This element has some corrections, and by Computer calculations, it has the form

$$: U_{0,0}U_{1,1} : - : U_{1,0}U_{1,0} : = \frac{256}{105}U_{8,0} + P(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \tag{6.2.54}$$

where P is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}$, and their derivatives. The left side of (6.2.54) is a normally ordered polynomial in $L, U_{0,0}, L_{2,0}$ due

to (6.2.6), and so (6.2.54) can be written in the form

$$\frac{256}{105}U_{8,0} = P_8(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}). \tag{6.2.55}$$

Weight 16 **relations:** This relation could be obtained by correcting the relation $u_{0,0}u_{2,2} - u_{2,0}u_{2,0} = 0$ in $gr(\mathcal{V})^{\mathbb{Z}_2}$ and similarly as above, the corresponding relation in $\mathcal{W}^{\mathbb{Z}_2}$ in weight 16:

$$:U_{0,0}U_{2,2}: - :U_{2,0}U_{2,0}: = -\frac{16384}{4725}U_{10,0} + Q(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.56}$$

where Q is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}$ and their derivatives. Since $U_{2,2}$ can be written as a normally ordered polynomial in $L, U_{0,0}$, $U_{2,0}, U_{4,0}$ and their derivatives, so (6.2.56) can be written in the form

$$-\frac{16384}{4725}U_{10,0} = Q_{10}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}). \tag{6.2.57}$$

Likewise, correcting the relation $u_{0,0}u_{3,1} - u_{3,0}u_{1,0} = 0$ yields a relation in the same weight

$$:U_{0,0}U_{3,1}: - :U_{3,0}U_{1,0}: = \frac{53248}{23625}U_{10,0} + Q'(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}), \tag{6.2.58}$$

which can be rewritten as

$$\frac{53248}{23625}U_{10,0} = Q'_{10}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}, U_{8,0}). \tag{6.2.59}$$

So, both the following relations exist

$$U_{8,0} = P_8(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \;\; U_{10,0} = P_{10}(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}).$$

Higher decoupling relations Applying the operator $U_{0,0}\circ_1$ repeatedly to the relations we have already constructed yields

$$U_{0,0} \circ_1 U_{n,0} = F(n)U_{n+4,0} + P,$$

where P is a normally ordered polynomial in $L, U_{0,0}, U_{2,0}, \ldots, U_{n+2,0}$ and their derivatives. It follows that it is possible to construct higher decoupling relations

$$U_{n,0} = P_n(L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}), \;\; n = 12, 14, \ldots.$$

Theorem 6.2.20. *For all $c = -\frac{22}{5}$, and all even integers $n \geq 0$,*

$$F(n) = -\frac{64(6+n)(10+n)(1+7n)}{75(1+n)(3+n)}, \tag{6.2.60}$$

which is exactly

$$\lim_{c \to -\frac{22}{5}} (22 + 5c)F(n, c).$$

Theorem 6.2.21. *For $c = -\frac{22}{5}$, $\mathcal{W}^{\mathbb{Z}_2}$ is of type $\mathcal{W}(2, 6, 8, 10, 12)$ with minimal strong generating set $\{L, U_{0,0}, U_{2,0}, U_{4,0}, U_{6,0}\}$.*

Chapter 7

Appendix

In this Appendix, an explicit normally ordered polynomial relation in weight 14 for the following relation

$$:U_{0,0}U_{1,1}: - :U_{1,0}U_{1,0}:= \frac{181248+5590c-475c^2}{60480(22+5c)}U_{8,0}+P(L,U_{0,0},U_{2,0},U_{4,0},U_{6,0})$$

in $\mathcal{W}^{\mathbb{Z}_2}_{(2)}$ is written down for the reader convenience. This relation is unique up to scalar multiples.

$$:U_{0,0}U_{1,1}: - :U_{1,0}U_{1,0}: + \frac{40}{22+5c}:LLU_{4,0}: + \frac{144}{22+5c}:(\partial^2 L)LU_{2,0}:$$

$$+\frac{144}{22+5c}:(\partial L)(\partial L)U_{2,0}: + \frac{112}{22+5c}:(\partial L)L\partial U_{2,0}: - \frac{56}{22+5c}:LL\partial^2 U_{2,0}:$$

$$-\frac{8}{22+5c}:(\partial^3 L)L\partial U_{0,0}: - \frac{24}{22+5c}:(\partial^2 L)(\partial L)\partial U_{0,0}: - \frac{64}{22+5c}:(\partial^2 L)L\partial^2 U_{0,0}:$$

$$-\frac{64}{22+5c}:(\partial L)(\partial L)\partial^2 U_{0,0}: - \frac{48}{22+5c}:(\partial L)L\partial^3 U_{0,0}: + \frac{28}{3(22+5c)}:LL\partial^4 U_{0,0}:$$

$$+\frac{47}{180}:LU_{6,0}: + \frac{5(54+13c)}{12(22+5c)}:(\partial^2 L)U_{4,0}: + \frac{1962+155c}{24(22+5c)}:(\partial L)\partial U_{4,0}:$$

$$-\frac{-182+75c}{24(22+5c)}:L\partial^2 U_{4,0}: + \frac{3(-2+c)}{2(22+5c)}:(\partial^4 L)U_{2_0}: - \frac{170-53c}{24(22+5c)}:(\partial^3 L)\partial U_{2,0}:$$

$$-\frac{486+61c}{8(22+5c)}:(\partial^2L)\partial^2U_{2,0}:-\frac{1878+125c}{12(22+5c)}:(\partial L)\partial^3U_{2,0}:$$

$$-\frac{662-75c}{24(22+5c)}:L\partial^4U_{2,0}:-\frac{8}{15(22+5c)}:(\partial^6L)U_{0,0}:$$

$$-\frac{218+3c}{48(22+5c)}:(\partial^5L)\partial U_{0,0}:-\frac{2(15+c)}{3(22+5c)}:(\partial^4L)\partial^2U_{0,0}:$$

$$-\frac{1966+137c}{(144(22+5c)}:(\partial^3L)\partial^3U_{0,0}:+\frac{102+61c}{48(22+5c)}:(\partial^2L)\partial^4U_{0,0}:$$

$$+\frac{25(14+c)}{12(22+5c)}:(\partial L)\partial^5U_{0,0}:+\frac{662-75c}{120(22+5c)}:L\partial^6U_{0,0}:$$

$$-\frac{896}{15(22+5c)^2)}:(\partial^6L)LLL:-\frac{256}{5(22+5c)^2}:(\partial^5L)(\partial L)LL:$$

$$-\frac{1664}{3(22+5c)^2}:(\partial^4L)(\partial^2L)LL:-\frac{5504}{9(22+5c)^2}:(\partial^3L)(\partial^3L)LL:$$

$$+\frac{5632}{3(22+5c)^2}:(\partial^4L)(\partial L)(\partial L)L:+\frac{4352}{(22+5c)^2}:(\partial^3L)(\partial^2L)(\partial L)L:$$

$$+\frac{1024}{(22+5c)^2}:(\partial^2L)(\partial^2L)(\partial^2L)L:+\frac{4096}{3(22+5c)^2}:(\partial^3L)(\partial L)(\partial L)(\partial L):$$

$$+\frac{896}{(22+5c)^2}:(\partial^2L)(\partial^2L)(\partial L)(\partial L):-\frac{29486-2263c}{630(22+5c)^2}:(\partial^8L)LL:$$

$$+\frac{32(-5174+209c)}{315(22+5c)^2}:(\partial^7L)(\partial L)L:+\frac{2(-28198+2427c)}{45(22+5c)^2}:(\partial^6L)(\partial^2L)L:$$

$$+\frac{32(-1174+109c)}{15(22+5c)^2}:(\partial^5L)(\partial^3L)L:-\frac{14486-1307c}{9(22+5c)^2}:(\partial^4L)(\partial^4L)L:$$

$$-\frac{2(25518+2065c)}{45(22+5c)^2}:(\partial^6L)(\partial L)(\partial L):-\frac{32(541+52c)}{5(22+5c)^2}:(\partial^5L)(\partial^2L)(\partial L):$$

$$-\frac{104(482+37c)}{9(22+5c)^2}:(\partial^4L)(\partial^3L)(\partial L):-\frac{2(-286+167c)}{3(22+5c)^2}:(\partial^4L)(\partial^2L)(\partial^2L):$$

$$-\frac{8(-886+23c)}{9(22+5c)^2}:(\partial^3L)(\partial^3L)(\partial^2L):$$

$$-\frac{-342897348-25407820c+402775c^2}{5443200(22+5c)^2}:(\partial^{10}L)L:$$

$$-\frac{-345995076-26686756c+626275c^2}{544320(22+5c)^2}:(\partial^9L)(\partial L):$$

$$-\frac{-349360452-27205180c+577903c^2}{120960(22+5c)^2}:(\partial^8L)(\partial^2L):$$

$$-\frac{-2804245644-218591252c+4546349c^2}{362880(22+5c)^2}:(\partial^7L)(\partial^3L):$$

$$-\frac{-21995034-1714285c+35605c^2}{1620(22+5c)^2}:(\partial^6L)(\partial^4L):$$

$$-\frac{-140780292-10970908c+228175c^2}{17280(22+5c)^2}:(\partial^5L)(\partial^5L):$$

$$-\frac{93733420-225352108c-18450565c^2+381800c^3}{479001600(22+5c)^2}\partial^{12}L$$

$$-\frac{181248+5590c-475c^2}{60480(22+5c)}U_{8,0}-\frac{-63456-3862c+115c^2}{4320(22+5c)}\partial^2U_{6,0}$$

$$+\frac{-74208-5206c+115c^2}{1728(22+5c)}\partial^4U_{4,0}-\frac{-74208-5270c+115c^2}{1440(22+5c)}\partial^6U_{2,0}$$

$$-\frac{1264260+89924c-1955c^2}{120960(22+5c)}\partial^8U_{0,0}=0.$$

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