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### Comments

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# AN ISOMETRIC DYNAMICS FOR A CAUSAL SET APPROACH TO DISCRETE QUANTUM GRAVITY

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## Abstract

We consider a covariant causal set approach to discrete quantum gravity. We first review the microscopic picture of this approach. In this picture a universe grows one element at a time and its geometry is determined by a sequence of integers called the shell sequence. We next present the macroscopic picture which is described by a sequential growth process. We introduce a model in which the dynamics is governed by a quantum transition amplitude. The amplitude satisfies a stochastic and unitary condition and the resulting dynamics becomes isometric. We show that the dynamics preserves stochastic states. By “doubling down” on the dynamics we obtain a unitary group representation and a natural energy operator. These unitary operators are employed to define canonical position and momentum operators.

## 1 Microscopic Picture

We call a finite poset  $(x, <)$  a *causet* and interpret  $a < b$  in  $x$  to mean that  $b$  is in the causal future of  $a$ . If  $x$  and  $y$  are causets with cardinality  $|y| = |x| + 1$ , then  $x$  *produces*  $y$  (denoted  $x \mapsto y$ ) if  $y$  is obtained from  $x$  by adjoining a

single maximal element to  $x$ . If  $x \rightarrow y$  we call  $y$  an *offspring* of  $x$ . A *labeling* for a causet  $x$  is a bijection

$$\ell: x \rightarrow \{1, 2, \dots, |x|\}$$

such that  $a, b \in x$  with  $a < b$  implies  $\ell(a) < \ell(b)$ . We then call  $x = (x, \ell)$  a *labeled causet*. A labeling of  $x$  corresponds to a “birth order” for the elements of  $x$ . Two labeled causets  $x, y$  are *isomorphic* if there is a bijection  $\phi: x \rightarrow y$  such that  $a < b$  in  $x$  if and only if  $\phi(a) < \phi(b)$  in  $y$  and  $\ell[\phi(a)] = \ell(a)$  for all  $a \in x$ . A causet is *covariant* if it has a unique labeling (up to isomorphisms) and we call a covariant causet a *c-causet*. Covariance corresponds to the properties of a manifold being independent of the coordinate system used to describe it. Denote the set of *c-causets* with cardinality  $n$  by  $\mathcal{P}_n$  and the set of all *c-causets* by  $\mathcal{P}$ . It is shown in [3] that any  $x \in \mathcal{P}$  with  $x \neq \emptyset$  has a unique producer in  $\mathcal{P}$  and precisely two offspring in  $\mathcal{P}$ . It follows that  $|\mathcal{P}_n| = 2^{n-1}$ ,  $n = 1, 2, \dots$ . For more background concerning the causet approach to discrete quantum gravity we refer the reader to [4, 5, 7]. For more information about *c-causets* the reader can refer to [1, 2, 3].

Two elements  $a, b \in x$  are *comparable* if  $a < b$  or  $b < a$ . We say that  $a$  is a *parent* of  $b$  and  $b$  is a *child* of  $a$  if  $a < b$  and there is no  $c \in x$  with  $a < c < b$ . A *path* from  $a$  to  $b$  in  $x$  is a sequence  $a_1 = a, a_2, \dots, a_{n-1}, a_n = b$  where  $a_i$  is a parent of  $a_{i+1}$ ,  $i = 1, \dots, n-1$ . The *height*  $h(a)$  of  $a \in x$  is the cardinality minus one of a longest path in  $x$  that ends with  $a$ . If there is no such path, we set  $h(a) = 0$ . It is shown in [3] that a causet  $x$  is covariant if and only if  $a, b \in x$  are comparable whenever  $h(a) \neq h(b)$ .

If  $x \in \mathcal{P}$ , we call the sets

$$S_j(x) = \{a \in x: h(a) = j\}, j = 0, 1, 2, \dots$$

*shells* and the sequence of integers  $s_j(x) = |S_j(x)|$ ,  $j = 0, 1, 2, \dots$  is the *shell sequence* for  $x$  [1]. A *c-causet* is uniquely determined by its shell sequence and we think of  $\{s_j(x)\}$  as describing the “shape” or geometry of  $x$ . The tree  $(\mathcal{P}, \rightarrow)$  can be thought of as a growth model and an  $x \in \mathcal{P}_n$  is a possible universe at step (time)  $n$ . An instantaneous universe  $x \in \mathcal{P}_n$  grows one element at a time in one of two ways. If  $x \in \mathcal{P}_n$  has shell sequence  $(s_0(x), s_1(x), \dots, s_m(x))$ , then  $x \rightarrow x_0$  or  $x \rightarrow x_1$  where  $x_0, x_1$  have shell sequence  $(s_0(x), s_1(x), \dots, s_m(x) + 1)$  and  $(s_0(x), s_1(x), \dots, s_m(x), 1)$ , respectively. In this way, we recursively order the *c-causets* in  $\mathcal{P}$  using the notation

$x_{n,j}$ ,  $n = 1, 2, \dots$ ,  $j = 0, 1, 2, \dots, 2^{n-1} - 1$ , where  $n = |x_{n,j}|$ . For example, in terms of their shell sequences we have:

$$\begin{aligned} x_{1,0} &= (1), x_{2,0} = (2), x_{2,1} = (1, 1), x_{3,0} = (3), x_{3,1} = (2, 1), x_{3,2} = (1, 2), x_{3,3} = (1, 1, 1) \\ x_{4,0} &= (4), x_{4,1} = (3, 1), x_{4,2} = (2, 2), x_{4,3} = (2, 1, 1), x_{4,4} = (1, 3), x_{4,5} = (1, 2, 1) \\ x_{4,6} &= (1, 1, 2), x_{4,7} = (1, 1, 1, 1) \end{aligned}$$

In the microscopic picture, we view a  $c$ -causet as a framework or scaffolding for a possible universe. The vertices of  $x$  represent small cells that can be empty or occupied by a particle. The shell sequence for  $x$  gives the geometry of the framework. In [1] we have shown how to construct a metric or distance function on  $x$ . This metric has simple and useful properties. However, the present paper is mainly devoted to the macroscopic picture and the quantum dynamics that can be developed in that picture. Figure 1 illustrates the first four steps of the sequential growth process  $(\mathcal{P}, \rightarrow)$ . Notice that this is a multiverse model in which infinite paths represent the histories of “completed” universes [4].

## 2 Macroscopic Picture

We now study the macroscopic picture which describes the evolution of a universe as a quantum sequential growth process. In such a process, the probabilities and propensities of competing geometries are determined by quantum amplitudes. These amplitudes provide interferences that are characteristic of quantum systems. A *transition amplitude* is a map  $\tilde{a}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$  satisfying  $\tilde{a}(x, y) = 0$  if  $x \not\rightarrow y$  and  $\sum_{y \in \mathcal{P}} \tilde{a}(x, y) = 1$  for every  $x \in \mathcal{P}$ . Since  $x_{n,j}$  only has the offspring  $x_{n+1,2j}$  and  $x_{n+1,2j+1}$  we have that

$$\sum_{k=0}^1 \tilde{a}(x_{n,j}, x_{n+1,2j+k}) = 1 \quad (2.1)$$

for all  $n = 1, 2, \dots$ ,  $j = 0, 1, 2, \dots, 2^{n-1} - 1$ . We call  $\tilde{a}$  a *unitary transition amplitude* (uta) if  $\tilde{a}$  also satisfies  $\sum_{y \in \mathcal{P}} |\tilde{a}(x, y)|^2 = 1$  or as in (2.1) we have

$$\sum_{k=0}^1 |\tilde{a}(x_{n,j}, x_{n+1,2j+k})|^2 = 1 \quad (2.2)$$

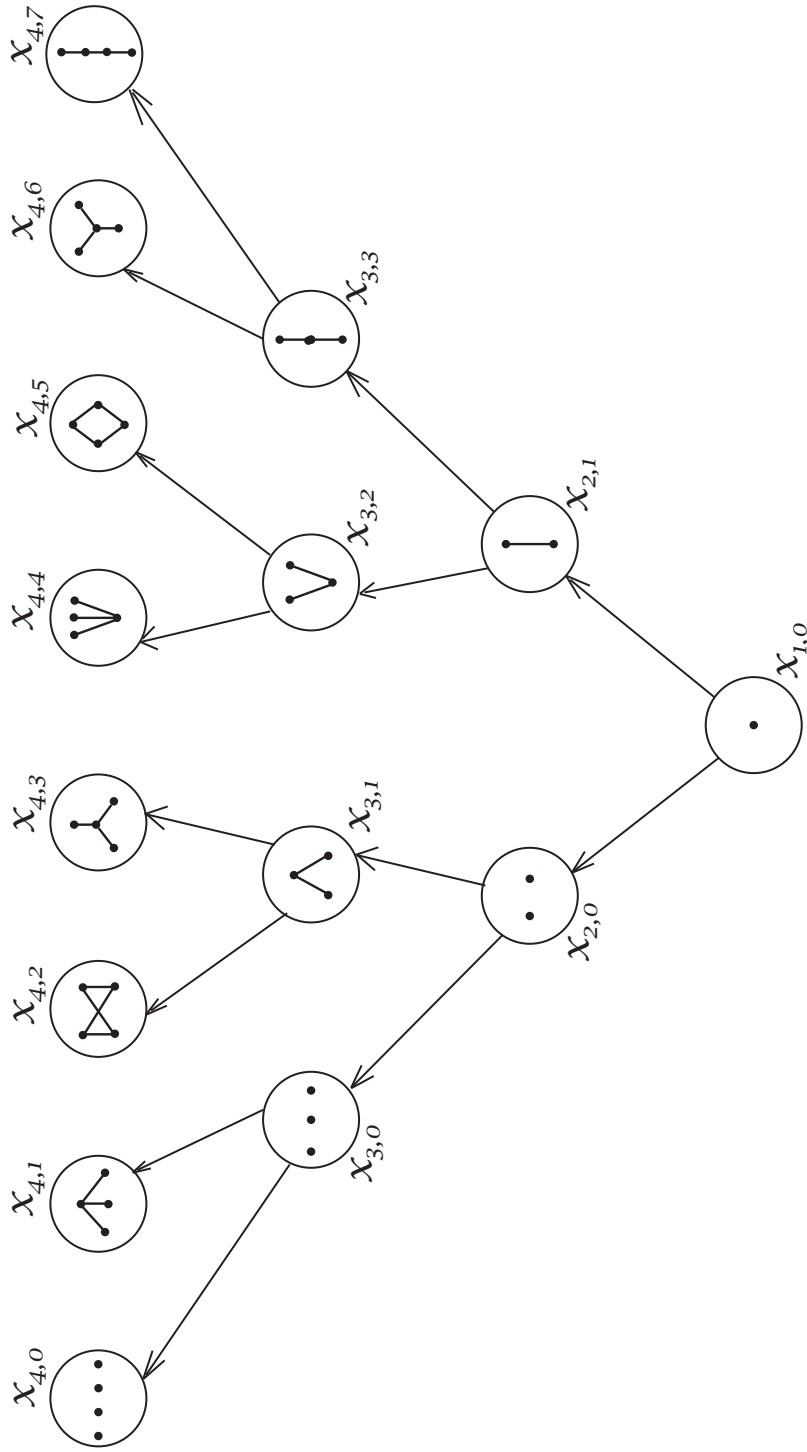


Figure 1 (Ordering for first four steps.)

One might suspect that these restrictions on a uta are so strong that the possibilities are very limited. This would be true if  $\tilde{a}$  were real valued. In this case,  $\tilde{a}(x, y) = 1$  for one  $y$  with  $x \rightarrow y$  and  $\tilde{a}(x, y) = 0$ , otherwise. However, in the complex case, the next result shows that there are a continuum of possibilities.

**Theorem 2.1.** *Two complex numbers  $a, b$  satisfy  $a + b = |a|^2 + |b|^2 = 1$  if and only if there exists a  $\theta \in [0, \pi)$  such that  $a = \cos \theta e^{i\theta}$  and  $b = -i \sin \theta e^{i\theta}$ . Moreover,  $\theta$  is unique.*

*Proof.* Necessity is clear. For sufficiency, suppose the conditions  $a + b = |a|^2 + |b|^2 = 1$  hold. Then

$$1 = |a|^2 + |b|^2 = |a|^2 + |1 - a|^2 = |a|^2 + (1 - a)(1 - \bar{a}) = 1 - 2 \operatorname{Re} a + 2 |a|^2$$

Hence,  $|a|^2 = \operatorname{Re} a$ . Letting  $a = |a| e^{i\theta}$  we have that  $|a|^2 = |a| \cos \theta$ . If  $a = 0$ , the result holds with  $\theta = \pi/2$ . If  $a \neq 0$ , we have that  $|a| = \cos \theta$  and  $\operatorname{Re} a = |a| \cos \theta$ . Hence,  $a = \cos \theta e^{i\theta}$  and

$$\begin{aligned} b &= 1 - \cos \theta e^{i\theta} = 1 - \cos^2 \theta - i \cos \theta \sin \theta = \sin \theta (\sin \theta - i \cos \theta) \\ &= -i \sin \theta e^{i\theta} \end{aligned}$$

Uniqueness follows from the fact that  $\cos \theta$  is injective on  $[0, \pi)$ . □

If  $\tilde{a}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$  is a uta, we call

$$c_{n,j}^k = \tilde{a}(x_{n,j}, x_{n+1,2j+k}), \quad k = 0, 1$$

the coupling constants for  $\tilde{a}$ . It follows from Theorem 2.1 that there exist  $\theta_{n,j} \in [0, \pi)$  such that

$$c_{n,j}^0 = \cos \theta_{n,j} e^{i\theta_{n,j}}, \quad c_{n,j}^1 = -i \sin \theta_{n,j} e^{i\theta_{n,j}}$$

It follows that  $c_{n,j}^0 + c_{n,j}^1 = |c_{n,j}^0|^2 + |c_{n,j}^1|^2 = 1$  for all  $n = 1, 2, \dots, j = 0, 1, 2, \dots, 2^{n-1} - 1$ . Let  $H_n$  be the Hilbert space

$$H_n = L_2(\mathcal{P}_n) = \{f: \mathcal{P}_n \rightarrow \mathbb{C}\}$$

with the standard inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}_n} \overline{f(x)} g(x)$$

A *path* in  $\mathcal{P}$  is a sequence  $\omega = \omega_1\omega_2\cdots$  where  $\omega_i \in \mathcal{P}_i$  and  $\omega_i \rightarrow \omega_{i+1}$ . Similarly, an *n-path* has the form  $\omega = \omega_1\omega_2\cdots\omega_n$  where again  $\omega_i \in \mathcal{P}_i$  and  $\omega_i \rightarrow \omega_{i+1}$ . We denote the set of paths by  $\Omega$  and the set of *n*-paths by  $\Omega_n$ . Since every  $x \in \mathcal{P}_n$  has a unique *n*-path terminating at  $x$ , we can identify  $\mathcal{P}_n$  with  $\Omega_n$  and we write  $\mathcal{P} \approx \Omega_n$ . Similarly, we identify  $H_n$  with  $L_2(\Omega_n)$ . If  $\tilde{a}$  is a uta and  $\omega = \omega_1\omega_2\cdots\omega_n \in \Omega_n$ , we define the *amplitude* of  $\omega$  to be

$$a(\omega) = \tilde{a}(\omega_1, \omega_2)\tilde{a}(\omega_2, \omega_3)\cdots\tilde{a}(\omega_{n-1}, \omega_n)$$

Moreover, we define the *amplitude* of  $x \in \mathcal{P}_n$  to be  $a(\omega)$  where  $\omega \in \Omega_n$  terminates at  $x$ .

Let  $\hat{x}_n$  be the unit vector in  $H_n$  given by the characteristic function  $\chi_{x_{n,j}}$ . Then clearly,  $\{\hat{x}_{n,j}: j = 0, 1, \dots, 2^{n-1} - 1\}$  forms an orthonormal basis for  $H_n$ . Define the operators  $U_n: H_n \rightarrow H_{n+1}$  by

$$U_n\hat{x}_{n,j} = \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k}$$

and extend  $U_n$  to  $H_n$  by linearity.

**Theorem 2.2.** (i) *The adjoint of  $U_n$  is given by  $U_n^*: H_{n+1} \rightarrow H_n$  where*

$$U_n^*\hat{x}_{n+1,2j+k} = \bar{c}_{n,j}^k \hat{x}_{n,j}, \quad k = 0, 1 \quad (2.3)$$

(ii)  *$U_n$  is a partial isometry with  $U_n^*U_n = I_n$  and*

$$U_nU_n^* = \sum_{j=0}^{2^{n-1}-1} \left| \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k} \right\rangle \left\langle \sum_{k=0}^1 \bar{c}_{n,j}^k \hat{x}_{n+1,2j+k} \right| \quad (2.4)$$

*Proof.* (i) To show that (2.3) holds, we have

$$\begin{aligned} \langle U_n^*\hat{x}_{n+1,2j'+k'}, \hat{x}_{n,j} \rangle &= \langle \hat{x}_{n+1,2j'+k'}, U_n\hat{x}_{n,j} \rangle \\ &= \left\langle \hat{x}_{n+1,2j'+k'}, \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k} \right\rangle \\ &= c_{n,j}^k \delta_{jj'} \delta_{kk'} = \left\langle \bar{c}_{n,j}^{k'} \hat{x}_{n,j'}, \hat{x}_{n,j} \right\rangle \end{aligned}$$

(ii) To show that  $U_n^*U_n = I_n$  we have by (i) that

$$U_n^*U_n\hat{x}_{n,j} = \sum_{k=0}^1 c_{n,j}^k U_n^*\hat{x}_{n+1,2j+k} = \sum_{k=0}^1 |c_{n,j}^k|^2 \hat{x}_{n,j} = \hat{x}_{n,j}$$



Since  $\{\widehat{x}_{n,j}: j = 0, 1, \dots, 2^{n-1} - 1\}$  forms an orthonormal basis for  $H_n$ , the result follows. Equation (2.4) holds because it is well-known that  $U_n U_n^*$  is the projection onto the range of  $\mathcal{R}(U_n)$ . We can also show this directly as follows

$$\begin{aligned}
& \sum_{j=0}^{2^{n-1}-1} \left| \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \right\rangle \left\langle \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \right| \widehat{x}_{n+1,2j'+k'} \\
&= \sum_{j=0}^{2^{n-1}-1} \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \overline{c_{n,j'}^{k'}} \delta_{jj'} = \overline{c_{n,j'}^{k'}} \sum_{k=0}^1 c_{n,j'}^k \widehat{x}_{n+1,2j'+k} \\
&= \overline{c_{n,j'}^{k'}} U_n \widehat{x}_{n,j'} = U_n U_n^* \widehat{x}_{n+1,2j'+k'} \quad \square
\end{aligned}$$

It follows from Theorem 2.2 that the dynamics  $U_n: H_n \rightarrow H_{n+1}$  for a utu  $\tilde{a}$  is an isometric operator. As usual a *state* on  $H_n$  is a positive operator  $\rho$  on  $H_n$  with  $\text{tr}(\rho) = 1$ . A *stochastic state* on  $H_n$  is a state  $\rho$  that satisfies  $\langle \rho 1_n, 1_n \rangle = 1$  where  $1_n = \chi_{\mathcal{P}_n}$ ; that is,  $1_n(x) = 1$  for every  $x \in \mathcal{P}_n$ . Notice that  $U_n^* 1_{n+1} = 1_n$ .

**Lemma 2.3.** (i) *If  $\rho$  is a state on  $H_n$ , then  $U_n \rho U_n^*$  is a state on  $H_{n+1}$ .* (ii) *If  $\rho$  is a stochastic state on  $H_n$ , then  $U_n \rho U_n^*$  is a stochastic state on  $H_{n+1}$ .*

*Proof.* (i) To show that  $U_n \rho U_n^*$  is positive, we have

$$\langle U_n \rho U_n^* \phi, \phi \rangle = \langle \rho U_n^* \phi, U_n^* \phi \rangle \geq 0$$

for all  $\phi \in H_{n+1}$ . Moreover, by Theorem 2.2(ii) we have

$$\text{tr}(U_n \rho U_n^*) = \text{tr}(U_n^* U_n \rho) = \text{tr}(\rho) = 1$$

(ii) Since  $U_n^* 1_{n+1} = 1_n$  we have

$$\langle U_n \rho U_n^* 1_{n+1}, 1_{n+1} \rangle = \langle \rho U_n^* 1_{n+1}, U_n^* 1_{n+1} \rangle = \langle \rho 1_n, 1_n \rangle = 1 \quad \square$$

Denoting the time evolution of states by  $\rho_n \rightarrow \rho_{n+1}$ , Lemma 2.3 shows that  $\rho \rightarrow U_n \rho U_n^*$  gives a quantum dynamics for states. We now show this explicitly for the transition amplitude. Since

$$\langle \widehat{x}_{n+1,2j+k}, U_n \widehat{x}_{n,j} \rangle = c_{n,j}^k = \tilde{a}(x_{n,j}, x_{n+1,2j+k})$$

we have for every  $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$  that

$$a(\omega) = \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle$$

Define the operator  $\rho_n$  on  $H_n$  by  $\langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle = \overline{a(\omega)} a(\omega')$  where  $\widehat{\omega} = \chi_{\{\omega\}} \in H_n$  for any  $\omega \in \Omega_n$ .

**Theorem 2.4.** *The operator  $\rho_n$  is a stochastic state on  $H_n$ .*

*Proof.* To show that  $\rho_n$  is positive we have

$$\begin{aligned}
\langle f, \rho_n f \rangle &= \left\langle \sum \langle \widehat{\gamma}_i, f \rangle \widehat{\gamma}_i, \rho_n \sum \langle \widehat{\gamma}_j, f \rangle \widehat{\gamma}_j \right\rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f \rangle} \sum \langle \widehat{\gamma}_j, f \rangle \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j \rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f \rangle} \sum \langle \widehat{\gamma}_j, f \rangle \overline{a(\gamma_i)} a(\gamma_j) \\
&= \left| \sum a(\gamma_i) \langle \widehat{\gamma}_i, f \rangle \right|^2 \geq 0
\end{aligned}$$

To show that  $\rho_n$  is a state on  $H_n$  we have that

$$\begin{aligned}
\text{tr}(\rho_n) &= \sum \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_i \rangle = \sum \overline{a(\gamma_i)} a(\gamma_i) = \sum |a(\gamma_i)|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_n} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 |\langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle|^2 \cdots |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 \\
&\quad \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2} \rangle|^2 \sum_{\omega_n} |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2} \rangle|^2 \\
&\quad \vdots \\
&= \sum_{\omega_2} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 = 1
\end{aligned}$$

Finally,  $\rho_n$  is stochastic on  $H_n$  because

$$\begin{aligned}
\langle 1_n, \rho_n 1_n \rangle &= \left\langle \sum \widehat{\gamma}_i, \rho_n \sum \widehat{\gamma}_j \right\rangle = \sum_{i,j} \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j \rangle \\
&= \sum_{i,j} \overline{a(\gamma_i)} a(\gamma_j) = \left| \sum a(\gamma_i) \right|^2
\end{aligned}$$

As before, we obtain

$$\begin{aligned}
\sum_{\omega \in \Omega_n} a(\omega) &= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_n} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2, \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_{n-1} U_{n-2} \widehat{\omega}_{n-2} \rangle \\
&\quad \vdots \\
&= \sum_{\omega_2} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle = 1 \quad \square
\end{aligned}$$

If  $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$ , we have seen that  $\omega_n$  produces two offspring  $\omega_{n,0}, \omega_{n,1} \in \mathcal{P}_{n+1}$ . We call the set

$$(\omega \rightarrow) = \{\omega_1 \omega_2 \cdots \omega_n \omega_{n,0}, \omega_1 \omega_2 \cdots \omega_n \omega_{n,1}\} \subseteq \Omega_{n+1}$$

the *one-step causal future* of  $\omega$ . We say that the sequence  $\rho_n$  is *consistent* if

$$\langle (\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge \rangle = \langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle$$

for every  $\omega, \omega' \in \Omega_n$  where  $(\omega \rightarrow)^\wedge = \chi_{(\omega \rightarrow)}$ . Consistency is important because it follows that the probabilities and propensities given by the dynamics  $\rho_n$  are conserved in time [2, 3].

**Theorem 2.5.** *The sequence  $\rho_n$  is consistent.*

*Proof.* Let  $\omega = \omega_1 \omega_2 \cdots \omega_n$ ,  $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n \in \Omega_n$  and suppose that  $\omega_n \rightarrow \omega_{n,0}, \omega_{n,1}$  and  $\omega'_n \rightarrow \omega'_{n,0}, \omega'_{n,1}$ . We then have

$$\begin{aligned}
&\langle (\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge \rangle \\
&= \langle (\omega \omega_{n,0})^\wedge + (\omega \omega_{n,1})^\wedge, \rho_{n+1} [(\omega' \omega'_{n,0})^\wedge + (\omega' \omega'_{n,1})^\wedge] \rangle \\
&= \langle (\omega \omega_{n,0})^\wedge, \rho_{n+1}(\omega' \omega'_{n,0})^\wedge \rangle + \langle (\omega \omega_{n,0})^\wedge, \rho_{n+1}(\omega' \omega'_{n,1})^\wedge \rangle \\
&\quad + \langle (\omega \omega_{n,1})^\wedge, \rho_{n+1}(\omega' \omega'_{n,0})^\wedge \rangle + \langle (\omega \omega_{n,1})^\wedge, \rho_{n+1}(\omega' \omega'_{n,1})^\wedge \rangle \\
&= \overline{a(\omega) \widetilde{a}(\omega_n \omega_{n,0})} a(\omega') [\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})] \\
&\quad + \overline{a(\omega) \widetilde{a}(\omega_n, \omega_{n,1})} a(\omega') [\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})] \\
&= \overline{a(\omega)} a(\omega') = \langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle \quad \square
\end{aligned}$$

The *n*-decoherence functional is the map  $D_n: 2^{\Omega_n} \times 2^{\Omega_n} \rightarrow \mathbb{C}$  defined by [4, 5, 7]

$$D_n(A, B) = \langle \chi_B, \rho_n \chi_A \rangle$$

The functional  $D_n(A, B)$  gives a measure of the interference between  $A$  and  $B$  when the system is in the state  $\rho_n$ . Clearly  $D_n(\Omega_n, \Omega_n) = 1$ ,  $D_n(A, B) = \overline{D_n(A, B)}$  and  $A \mapsto D_n(A, B)$  is a complex measure for every  $B \in 2^{\Omega_n}$ . It is also well-known that if  $A_1, \dots, A_n \in 2^{\Omega_n}$ , then the matrix with entries  $D_n(A_j, A_k)$  is positive semidefinite [5]. Notice that

$$D_n(\{\omega\}, \{\omega'\}) = \overline{a(\omega)}a(\omega')$$

for every  $\omega, \omega' \in \Omega_n$  and

$$D(A, B) = \sum \left\{ \overline{a(\omega)}a(\omega') : \omega \in A, \omega' \in B \right\}$$

Since  $\rho_n$  is consistent, we have that

$$D_{n+1}((A \rightarrow), (B \rightarrow)) = D_n(A, B)$$

for every  $A, B \in 2^{\Omega_n}$  where  $(A \rightarrow) = \cup \{(\omega \rightarrow) : \omega \in A\}$ . The corresponding *q*-measure [2, 5, 6] is the map  $\mu_n: 2^\Omega \rightarrow \mathbb{R}^+$  defined by

$$\mu_n(A) = D_n(A, A) = \langle \chi_A, \rho_n \chi_A \rangle$$

It follows that  $\mu_n(\Omega_n) = 1$  and  $\mu_{n+1}((A \rightarrow)) = \mu_n(A)$  for all  $A \in 2^{\Omega_n}$ . Although  $\mu_n$  is not additive, it satisfies the *grade 2-additive condition*: if  $A, B, C \in 2^{\Omega_n}$  are mutually disjoint then [4, 5, 6, 7]

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

Since  $\mu_n$  is not a measure we do call it a probability but we interpret  $\mu_n(A)$  as the quantum propensity for the occurrence of  $A$ . We have discussed in [2, 3] ways of extending the  $\mu_n$ s to a *q*-measure  $\mu$  on suitable subsets of  $\Omega$ .

A uta is *completely stationary* (cs) with parameter  $\theta \in [0, \pi)$  if  $\theta_{n,j} = \theta$  for all  $n, j$ . For example, let  $\tilde{a}$  be cs with parameter 0. Then the path  $x_{1,0}x_{2,0}x_{3,0} \dots$  has *q*-measure 1 and all other paths have *q*-measure 0. Now consider a general cs uta  $\tilde{a}$  with parameter  $\theta \in (0, \pi)$ . When a path “turns left”  $\tilde{a}$  has the value  $\cos \theta e^{i\theta}$  and when it “turns right”  $\tilde{a}$  has the value  $-i \sin \theta e^{i\theta}$ . Hence if  $\omega \in \Omega_n$  turns left  $\ell$  times and right  $r$  times we have

$$a(\omega) = (\cos \theta)^\ell (-i)^r (\sin \theta)^r e^{in\theta}$$

We then have

$$\mu_n(\{\omega\}) = |a(\omega)|^2 = |\cos \theta|^{2\ell} |\sin \theta|^{2r}$$

Hence,  $\lim_{n \rightarrow \infty} \mu_n(\{\omega\}) = 0$  and it is natural to define  $\mu(\{\omega\}) = 0$ .

A vector

$$v = \sum_{j=0}^{2^n-1} v_j \hat{x}_{n,j} = (v_0, v_1, \dots, v_{2^n-1}) \in H_n$$

is called a *stochastic state vector* if  $\|v\| = 1$  and  $\langle v, 1_n \rangle = 1$ . We call the vector

$$\hat{a}_n = (a(x_{n,0}), a(x_{n,1}), \dots, a(x_{n,2^n-1})) \in H_n$$

an *amplitude vector*. Of course,  $\hat{a}_n$  is a stochastic state vector.

**Theorem 2.6.** (i) If  $v \in H_n$  is a stochastic state vector, then  $U_n v \in H_{n+1}$  is also. (ii)  $U_n \hat{a}_n = \hat{a}_{n+1}$ . (iii)  $U_n^* \hat{a}_{n+1} = \hat{a}_n$ .

*Proof.* (i) This follows from the fact that  $U_n$  is isometric and  $U_n^* 1_{n+1} = 1_n$ .

(ii) This holds because

$$\begin{aligned} U_n \hat{a}_n &= U_n \sum_{j=0}^{2^n-1} a(x_{n,j}) \hat{x}_{n,j} = \sum_{j=0}^{2^n-1} [a(x_{n,j}) c_{n,j}^0 \hat{x}_{n+1,2j} + a(x_{n,j}) c_{n,j}^1 \hat{x}_{n+1,2j+1}] \\ &= \sum_{j=0}^{2^n-1} a(x_{n+1,j}) \hat{x}_{n+1,j} = \hat{a}_{n+1} \end{aligned}$$

(iii) This is obtained from

$$\begin{aligned} U_n^* \hat{a}_{n+1} &= \sum [a(x_{n+1,2j}) \hat{x}_{n+1,2j} + a(x_{n+1,2j+1}) \hat{x}_{n+1,2j+1}] \\ &= \sum [a(x_{n+1,2j}) \bar{c}_{n,j}^0 \hat{x}_{n,j} + a(x_{n+1,2j+1}) \bar{c}_{n,j}^1 \hat{x}_{n,j}] \\ &= \sum [c_{n,j}^0 a(x_{n,j}) \bar{c}_{n,j}^0 + c_{n,j}^1 a(x_{n,j}) \bar{c}_{n,j}^1] \hat{x}_{n,j} \\ &= \sum a(x_{n,j}) \hat{x}_{n,j} = \hat{a}_n \quad \square \end{aligned}$$

Actually, (iii) follows from (ii) in Theorem 2.6 because  $\hat{a}_{n+1} = U_n \hat{a}_n \in \mathcal{R}(U_n)$  so  $U_n^* \hat{a}_{n+1} = U_n^* U_n \hat{a}_n = \hat{a}_n$ . Interference in  $\mathcal{P}_n$  or  $\Omega_n$  can be described by the nonadditivity of the  $q$ -measure  $\mu_n$ . We say that  $x, y \in \mathcal{P}_n$  do not interfere if

$$\mu_n(\{x, y\}) = \mu_n(\{x\}) + \mu_n(\{y\})$$

The next result gives an application of this concept.

**Theorem 2.7.** *If  $x, y \in \mathcal{P}$  have the same producer, then  $x$  and  $y$  do not interfere.*

*Proof.* Suppose  $x = x_{n+1,2j}, y = x_{n+1,2j+1}$  so  $x, y$  have the same producer  $x_{n,j}$ . Then

$$\begin{aligned}
\mu_{n+1}(\{x, y\}) &= |a(x) + a(y)|^2 = |a(x_{n,j})c_{n,j}^0 + a(x_{n,j})c_{n,j}^1|^2 \\
&= |a(x_{n,j})|^2 |c_{n,j}^0 + c_{n,j}^1|^2 = |a(x_{n,j})|^2 \\
&= |a(x_{n,j})|^2 \left[ |c_{n,j}^0|^2 + |c_{n,j}^1|^2 \right] \\
&= |a(x_{n,j})c_{n,j}^0|^2 + |a(x_{n,j})c_{n,j}^1|^2 \\
&= |a(x_{n+1,2j})|^2 + |a(x_{n+1,2j+1})|^2 = \mu_{n+1}(\{x\}) + \mu_{n+1}(\{y\})
\end{aligned}$$

Hence,  $x$  and  $y$  do not interfere.  $\square$

In general, the noninterference result in Theorem 2.7 does not hold if  $x$  and  $y$  have different producers. This is shown in the next two examples.

**Example 1.** For simplicity, suppose the uta is cs so we have just two coupling constants  $c^0, c^1$ . We have seen in Theorem 2.7 that  $x_{3,0}$  and  $x_{3,1}$  do not interfere. In a similar way, we see that  $x_{3,0}$  and  $x_{3,2}$  do not interfere. We also have that  $x_{3,1}$  does not interfere with  $x_{3,j}, j = 0, 1, 2$  and  $x_{3,2}$  does not interfere with  $x_{3,j}, j = 0, 1, 3$ . Let us now consider  $x_{3,0}$  and  $x_{3,3}$ . We have that

$$\begin{aligned}
\mu_3(\{x_{3,0}, x_{3,3}\}) &= |a(x_{3,0}) + a(x_{3,3})|^2 = |(c^0)^2 + (c^1)^2|^2 \\
&= |\cos^2 \theta - \sin^2 \theta| = \cos^2 2\theta
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu_3(\{x_{3,0}\}) + \mu_2(\{x_{3,3}\}) &= |a(x_{3,0})|^2 + |a(x_{3,3})|^2 \\
&= \cos^4 \theta + \sin^4 \theta = \frac{1}{2}(1 + \cos^2 2\theta)
\end{aligned}$$

so  $x_{3,0}$  and  $x_{3,3}$  interfere, in general.

**Example 2.** If the uta is not cs, the situation is more complicated and we incur more interference. In the cs case, we saw in Example 1 that  $x_{3,0}$  and  $x_{3,2}$  do not interfere. However, in this more general case we have

$$\mu_3(\{x_{3,0}, x_{3,2}\}) = |a(x_{3,0}) + a(x_{3,2})| = |c_{1,0}^0 c_{2,0}^0 + c_{1,0}^1 c_{2,1}^0|^2$$

On the other hand

$$\mu_3(\{x_{3,0}\}) + \mu_3(\{x_{3,2}\}) = |a(x_{3,0})|^2 + |a(x_{3,2})|^2 = |c_{1,0}^0|^2 |c_{2,0}^0|^2 + |c_{1,0}^1|^2 |c_{2,1}^0|^2$$

But these two quantities do not agree unless

$$\operatorname{Re}(c_{1,0}^0 c_{2,0}^0 \bar{c}_{1,0}^1 \bar{c}_{2,1}^0) = 0$$

so  $x_{3,0}$  and  $x_{3,2}$  interfere, in general.

### 3 Double-Down To Unitary

We have seen that corresponding to a uta with coupling constants  $c_{n,j}^k$ , there are isometries  $U_n: H_n \rightarrow H_{n+1}$  that describe the dynamics for a quantum sequential growth process on  $(\mathcal{P}, \rightarrow)$ . The operators  $U_n$  cannot be unitary because  $H_n$  and  $H_{n+1}$  are different dimensional Hilbert spaces. However, we can “double-down” the  $U_n$  to form operators  $V_{n+1}: H_{n+1} \rightarrow H_{n+1}$  by

$$\begin{aligned} V_{n+1} \hat{x}_{n+1,2j} &= c_{n,j}^0 \hat{x}_{n+1,2j} + c_{n,j}^1 \hat{x}_{n+1,2j+1} \\ V_{n+1} \hat{x}_{n+1,2j+1} &= c_{n,j}^1 \hat{x}_{n+1,2j} + c_{n,j}^0 \hat{x}_{n+1,2j+1} \end{aligned}$$

**Theorem 3.1.** *The operators  $V_{n+1}$  are unitary and  $V_{n+1} 1_{n+1} = 1_{n+1}$ ,  $n = 1, 2, \dots$*

*Proof.* Since  $\|V_{n+1} \hat{x}_{n+1,2j}\| = \|V_{n+1} \hat{x}_{n+1,2j+1}\| = 1$  and

$$\langle V_{n+1} \hat{x}_{n+1,2j}, V_{n+1} \hat{x}_{n+1,2j+1} \rangle = \bar{c}_{n,j}^0 c_{n,j}^1 + \bar{c}_{n,j}^1 c_{n,j}^0 = 0$$

we conclude that  $V_{n+1}$  sends an orthonormal basis to an orthonormal basis. Hence,  $V_{n+1}$  is unitary. To show that  $V_{n+1} 1_{n+1} = 1_{n+1}$  we have

$$\begin{aligned} V_{n+1} 1_{n+1} &= \sum_j (V_{n+1} \hat{x}_{n+1,2j} + V_{n+1} \hat{x}_{n+1,2j+1}) \\ &= \sum_j [(c_{n,j}^0 + c_{n,j}^1) \hat{x}_{n+1,2j} + (c_{n,j}^1 + c_{n,j}^0) \hat{x}_{n+1,2j+1}] \\ &= \sum_j (\hat{x}_{n+1,2j} + \hat{x}_{n+1,2j+1}) = 1_{n+1} \quad \square \end{aligned}$$

The unitary operator  $V_2$  corresponds to the coupling constants  $c_{1,0}^0, c_{1,0}^1$  and relative to the basis  $\{\widehat{x}_{2,0}, \widehat{x}_{2,1}\}$  has the form

$$V_2 = \begin{bmatrix} c_{1,0}^0 & c_{1,0}^1 \\ c_{1,0}^1 & c_{1,0}^0 \end{bmatrix}$$

Besides being unitary,  $V_2$  is doubly stochastic (row and column sums are one). Of course, this is also true of  $V_n$ . By Theorem 2.1, there exists a unique  $\theta \in [0, \pi)$  such that  $c_{1,0}^0 = \cos \theta e^{i\theta}$ ,  $c_{1,0}^1 = -i \sin \theta e^{i\theta}$ . To make  $\theta$  explicit, we write  $V_2 = V_2(\theta)$ .

**Lemma 3.2.** *The operator  $V_2(\theta)$  has eigenvalues  $1, e^{2i\theta}$  with corresponding unit eigenvectors  $2^{-1/2}(1, 1), 2^{-1/2}(1, -1)$ .*

*Proof.* By direct verification we have

$$\begin{aligned} V_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} c_{1,0}^0 + c_{1,0}^1 \\ c_{1,0}^1 + c_{1,0}^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ V_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} c_{1,0}^0 - c_{1,0}^1 \\ c_{1,0}^1 - c_{1,0}^0 \end{bmatrix} = (c_{1,0}^0 - c_{1,0}^1) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} c_{1,0}^0 - c_{1,0}^1 &= c_{1,0}^0 - (1 - c_{1,0}^0) = 2c_{1,0}^0 - 1 = 2 \cos \theta e^{i\theta} - 1 \\ &= 2 \cos^2 \theta + 2i \cos \theta \sin \theta - 1 = \cos^2 \theta - \sin^2 \theta + i \sin 2\theta \\ &= \cos 2\theta + i \sin 2\theta = e^{2i\theta} \end{aligned} \quad \square$$

We can write the  $2^n$ -dimensional Hilbert space  $H_{n+1}$  as

$$H_{n+1} = H_2 \oplus H_2 \oplus \cdots \oplus H_2$$

where there are  $2^{n-1}$  summands and the  $j$ th summand has the basis  $\{\widehat{x}_{n+1,2j}, \widehat{x}_{n+1,2j+1}\}$ . In general,  $V_{n+1}$  has the form

$$V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}}) = V_2(\theta_1) \oplus V_2(\theta_2) \oplus \cdots \oplus V_2(\theta_{2^{n-1}})$$

It follows from Lemma 3.2 that  $V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}})$  has eigenvalues 1 (with multiplicity  $2^{n-1}$ ) and  $e^{2i\theta_1}, e^{2i\theta_2}, \dots, e^{2i\theta_{2^{n-1}}}$ . The unit eigenvectors corresponding to 1 are

$$2^{-1/2}(\widehat{x}_{n+1,2j} + \widehat{x}_{n+1,2j+1}), \quad j = 0, 1, \dots, 2^{n-1} - 1$$



and the unit eigenvector corresponding to  $e^{2i\theta_j}$  is

$$2^{-1/2}(\widehat{x}_{n+1,2j} - \widehat{x}_{n+1,2j+1})$$

Let  $\mathcal{S}(H_{n+1})$  be the set of operators on  $H_{n+1}$  of the form

$$\mathcal{S}(H_{n+1}) = \{V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}}): \theta_n \in [0, \pi)\}$$

Now  $[0, \pi)$  forms an abelian group with operations  $a \oplus b = a + b \pmod{\pi}$ .

**Lemma 3.3.** *For  $\theta_1, \theta_2 \in [0, \pi)$  we have  $V_2(\theta_1)V_2(\theta_2) = V_2(\theta_1 + \theta_2)$ .*

*Proof.* Since  $V_2(\theta_1)$  and  $V_2(\theta_2)$  have the same eigenvectors, they commute and can be simultaneously diagonalized as

$$V_2(\theta_1) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_1} \end{bmatrix} \quad V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_2} \end{bmatrix}$$

Hence, if  $\theta_1 + \theta_2 < \pi$  then

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2)} \end{bmatrix} = V_2(\theta_1 \oplus \theta_2)$$

and if  $\theta_1 + \theta_2 \geq \pi$  then  $\theta_1 \oplus \theta_2 = \theta_1 + \theta_2 - \pi$  and we have

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2-\pi)} \end{bmatrix} = V_2(\theta_1 + \theta_2 - \pi) = V_2(\theta_1 \oplus \theta_2) \quad \square$$

We now form the product group  $[0, \pi)^{2^{n-1}} = [0, \pi) \times \dots \times [0, \pi)$  to obtain the following result.

**Corollary 3.4.** *Under operator multiplication,  $\mathcal{S}(H_{n+1})$  is an abelian group and  $(\theta_1, \dots, \theta_{2^{n-1}}) \mapsto V_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})$  is a unitary representation of the group  $[0, \pi)^{2^{n-1}}$ .*

Since  $V_{n+1}$  is unitary, there exists a unique self-adjoint operator  $K_{n+1}$  on  $H_{n+1}$  such that  $V_{n+1} = e^{iK_{n+1}}$ . We call  $K_{n+1}$  a *Hamiltonian operator*. For  $V_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})$  the eigenvalues of  $K_{n+1}$  are 0 (with multiplicity  $2^{n-1}$ ) and  $2\theta_1, \dots, 2\theta_{2^{n-1}}$ . Hence,  $\theta_j = 2^{-1}\lambda_j$  where  $\lambda_j$  is the  $j$ th energy value,

$j = 1, \dots, 2^{n-1}$ . This gives a physical significance for the angles  $\theta_j$ . The corresponding eigenvectors are the same as those given for  $V_{n+1}$ .

It is natural to define the *position operator*  $Q_{n+1}$  on  $H_{n+1}$  by  $Q_{n+1}f(\hat{x}_{n+1,k}) = k$ . Thus,  $Q_{n+1}\hat{x}_{n+1,2j} = 2j$  and  $Q_{n+1}\hat{x}_{n+1,2j+1} = 2j + 1$ . Since  $Q_{n+1}$  is diagonal, we immediately see that its eigenvalues are  $0, 1, \dots, 2^n - 1$  with corresponding eigenvector  $\hat{x}_{n+1,k}$ . It also seems natural to define the *canonical momentum operator*  $P_{n+1}$  on the subspace generated by  $\{\hat{x}_{n+1,2j}, \hat{x}_{n+1,2j+1}\}$  as

$$\begin{aligned} P_2(\theta_j) &= V_2(\theta_j)^* Q_2(\theta_j) V_2(\theta_j) \\ &= \begin{bmatrix} \bar{c}_{n,j}^0 & \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^1 & \bar{c}_{n,j}^0 \end{bmatrix} \begin{bmatrix} 2j & 0 \\ 0 & 2j+1 \end{bmatrix} \begin{bmatrix} c_{n,j}^0 & c_{n,j}^1 \\ c_{n,j}^1 & c_{n,j}^0 \end{bmatrix} \\ &= \begin{bmatrix} 2j + |c_{n,j}^1|^2 & c_{n,j}^0 \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^0 c_{n,j}^1 & 2j + |c_{n,j}^0|^2 \end{bmatrix} = \begin{bmatrix} 2j + \sin^2 \theta_{n,j} & \frac{i}{2} \sin 2\theta_{n,j} \\ -\frac{i}{2} \sin 2\theta_{n,j} & 2j + \cos^2 \theta_{n,j} \end{bmatrix} \end{aligned}$$

The eigenvalues of  $P_2(\theta_j)$  are  $2j$  and  $2j + 1$  with corresponding unit eigenvectors

$$\begin{aligned} V_2(\theta_j)^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \bar{c}_{n,j}^0 \\ \bar{c}_{n,j}^1 \end{bmatrix} \\ V_2(\theta_j)^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^0 \end{bmatrix} \end{aligned}$$

The complete momentum operator  $P_{n+1}$  is given by

$$P_{n+1}(\theta_1, \dots, \theta_{2^{n-1}}) = P_2(\theta_1) \oplus P_2(\theta_2) \oplus \dots \oplus P_2(\theta_{2^{n-1}})$$

We now compute the commutator

$$\begin{aligned} [P_2(\theta_j), Q_2(\theta_j)] &= P_2(\theta_j)Q_2(\theta_j) - Q_2(\theta_j)P_2(\theta_j) = c_{n,j}^0 \bar{c}_{n,j}^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{i}{2} \sin 2\theta_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The complete commutation relation is

$$\begin{aligned} & [P_{n+1}(\theta_1, \dots, \theta_{2^{n-1}}), Q_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})] \\ &= [P_2(\theta_1), Q_2(\theta_1)] \oplus \dots \oplus [P_2(\theta_{2^{n-1}}), Q_2(\theta_{2^{n-1}})] \end{aligned}$$

As in the Heisenberg uncertainty relation, the number  $|\langle \phi, [P_{n+1}, Q_{n+1}] \phi \rangle|$  gives a lower bound for the product of the variances of  $P_{n+1}$  and  $Q_{n+1}$ . We now compute this number for an amplitude state  $\hat{a}_{n+1}$ . We have that

$$\begin{aligned} & \langle \hat{a}_{n+1}, [P_{n+1}, Q_{n+1}] \hat{a}_{n+1} \rangle \\ &= \sum_j \left\langle \begin{bmatrix} a(x_{n+1,2j}) \\ a(x_{n+1,2j+1}) \end{bmatrix}, c_{n,j}^0 c_{n,j}^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a(x_{n+1,2j}) \\ a(x_{n+1,2j+1}) \end{bmatrix} \right\rangle \\ &= \sum_j c_{n,j}^0 \bar{c}_{n,j}^1 [\bar{a}(x_{n+1,2j})a(x_{n+1,2j+1}) + \bar{a}(x_{n+1,2j+1})a(x_{n+1,2j})] \\ &= \sum_j c_{n,j}^0 \bar{c}_{n,j}^1 |a(x_{n,j})|^2 [\bar{c}_{n,j}^0 c_{n,j}^1 + \bar{c}_{n,j}^1 c_{n,j}^0] = 0 \end{aligned}$$

This shows that even though  $P_{n+1}$  and  $Q_{n+1}$  do not commute, there is no lower bound for the product of their variances when the system is in an amplitude state.

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