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## Extender Sets and Multidimensional Subshifts

### Comments

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# EXTENDER SETS AND MULTIDIMENSIONAL SUBSHIFTS

NIC ORMES AND RONNIE PAVLOV

ABSTRACT. In this paper, we consider a  $\mathbb{Z}^d$  extension of the well-known fact that subshifts with only finitely many follower sets are sofic. As in [4], we adopt a natural  $\mathbb{Z}^d$  analogue of a follower set called an extender set. The extender set of a finite word  $w$  in a  $\mathbb{Z}^d$  subshift  $X$  is the set of all configurations of symbols on the rest of  $\mathbb{Z}^d$  which form a point of  $X$  when concatenated with  $w$ . As our main result, we show that for any  $d \geq 1$  and any  $\mathbb{Z}^d$  subshift  $X$ , if there exists  $n$  so that the number of extender sets of words on a  $d$ -dimensional hypercube of side length  $n$  is less than or equal to  $n$ , then  $X$  is sofic. We also give an example of a non-sofic system for which this number of extender sets is  $n + 1$  for every  $n$ .

We prove this theorem in two parts. First we show that if the number of extender sets of words on a  $d$ -dimensional hypercube of side length  $n$  is less than or equal to  $n$  for some  $n$ , then there is a uniform bound on the number of extender sets for words on any sufficiently large rectangular prisms; to our knowledge, this result is new even for  $d = 1$ . We then show that such a uniform bound implies soficity.

Our main result is reminiscent of the classical Morse-Hedlund theorem, which says that if  $X$  is a  $\mathbb{Z}$  subshift and there exists an  $n$  such that the number of words of length  $n$  is less than or equal to  $n$ , then  $X$  consists entirely of periodic points. However, most proofs of that result use the fact that the number of words of length  $n$  in a  $\mathbb{Z}$  subshift is nondecreasing in  $n$ , and we present an example (due to Martin Delacourt) which shows that this monotonicity does not hold for numbers of extender sets (or follower sets) of words of length  $n$ .

## 1. INTRODUCTION

For any  $\mathbb{Z}$  subshift  $X$  and finite word  $w$  appearing in some point of  $X$ , the **follower set** of  $w$ , written  $F_X(w)$ , is defined as the set of all one-sided infinite sequences  $s$  such that the infinite word  $ws$  occurs in some point of  $X$ . (In some sources, the follower set is defined as the set of all finite words which can legally follow  $w$ , but the former definition may be obtained by taking limits of the latter.) It is well-known that for a  $\mathbb{Z}$  subshift  $X$ , finiteness of  $\{F_X(w) : w \text{ in the language of } X\}$  is equivalent to  $X$  being sofic, i.e. the image of a shift of finite type under a continuous shift-commuting map. (For instance, see [5].)

In [4], extender sets were defined and introduced as a natural extension of follower sets to  $\mathbb{Z}^d$  subshifts with  $d > 1$ . The **extender set** of any finite word  $w$  in the language of  $X$  with shape  $S \subset \mathbb{Z}^d$ , written  $E_X(w)$ , is the set of all configurations on  $\mathbb{Z}^d \setminus S$  which, when concatenated with  $w$ , form a legal point of  $X$ . We can no longer speak of a subshift having only finitely many extender sets, since extender sets of patterns with different shapes cannot be compared as in the one-dimensional case. One way to deal with this is examine the growth rate of the number of distinct

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extender sets for words in  $X$  with a given shape  $S$  (which we denote by  $N_S(X)$ ), as the size of  $S$  approaches infinity. This works nicely in the one-dimensional case; our Lemma 3.4 (which is routine) shows that soficity of a one-dimensional subshift is equivalent to boundedness of the number of extender sets of  $n$ -letter words as  $n \rightarrow \infty$ . Interestingly, this sequence need not stabilize; Example 3.5, due to Martin Delacourt ([1]), demonstrates a  $\mathbb{Z}$  sofic shift  $X$  where  $N_{[1,n]}(X)$  oscillates between two values as  $n$  increases (In this paper,  $[a, b]$  for  $a, b \in \mathbb{Z}$  will always represent the set  $\{a, a + 1, \dots, b\}$ ).

There are many relations between properties of  $X$  and the behavior of  $N_S(X)$ . For instance, it is easy to see that when  $X$  is a nearest-neighbor shift of finite type, the extender set of a word with shape  $[1, n]^d$  is determined entirely by the letters on the boundary. This implies that for such  $X$ ,  $N_{[1,n]^d}(X)$  is bounded from above by  $|A_X|^{\partial[1,n]^d} \leq |A_X|^{2dn^{d-1}}$ , where  $A_X$  denotes the alphabet of  $X$ . It was conjectured in [4] that  $X$  sofic implies that  $\frac{\log N_{[1,n]^d}(X)}{n^d} \rightarrow 0$ , but this remains open.

A partial answer was proven in [4], using an argument basically present in [7]. A finite sequence  $S_n$  of sets,  $1 \leq n \leq N$ , was defined to be a union increasing chain if  $S_n \not\subseteq \bigcup_{i=1}^{n-1} S_i$  for all  $1 \leq n \leq N$ . Theorem 2.3 of [4] states that if there exist union increasing chains of size  $e^{\omega(n^{d-1})}$  of extender sets of words with shape  $[1, n]^d$ , then  $X$  is not sofic. These results can, broadly speaking, be thought of as showing that a very fast growth rate for extender sets implies that a subshift is not an SFT or sofic. Our main result is in the opposite direction, namely it demonstrates that a slow enough growth rate implies soficity.

**Theorem 1.1.** *For any  $d \geq 1$  and any  $\mathbb{Z}^d$  subshift  $X$ , if there exists  $n$  so that  $N_{[1,n]^d}(X) \leq n$ , then  $X$  is sofic.*

The proof of Theorem 1.1 is broken into two mostly disjoint parts:

**Theorem 1.2.** *For any  $d \geq 1$  and any  $\mathbb{Z}^d$  subshift  $X$ , if there exists  $n$  so that  $N_{[1,n]^d}(X) \leq n$ , then there exist  $K, N$  such that for any rectangular prism  $R$  with dimensions at least  $K$ ,  $N_R(X) \leq N$ .*

**Theorem 1.3.** *For any  $d \geq 1$  and any  $\mathbb{Z}^d$  subshift  $X$ , if there exist  $K, N$  so that  $N_R(X) \leq N$  for all rectangular prisms  $R$  with all dimensions at least  $K$ , then  $X$  is sofic.*

Theorem 1.3 can be thought of as a generalization of the previously mentioned fact that one-dimensional shifts with only finitely many follower sets are sofic. We also show that the upper bound in Theorem 1.1 cannot be improved.

**Theorem 1.4.** *For any  $d \geq 1$ , there exists a nonsofic  $\mathbb{Z}^d$  subshift  $X$  for which  $N_{[1,n]^d}(X) = n + 1$  for all  $n$ .*

Our results have similarities to the famous Morse-Hedlund theorem.

**Theorem 1.5.** ([6]) *If  $X$  is a  $\mathbb{Z}$  subshift and there exists an  $n$  such that the number of words of length  $n$  is less than or equal to  $n$ , then  $X$  consists entirely of periodic points. Equivalently, there is a uniform upper bound on the number of words of length  $n$ .*

It is well-known that the bound in Theorem 1.5 is also tight. Sturmian subshifts have no periodic points and have so-called minimal complexity, i.e. any Sturmian

subshift has  $n + 1$  words of length  $n$  for all  $n$ . (For an introduction to Sturmian subshifts, see [2].)

There are similarities between Theorems 1.1 and 1.5; in fact Theorem 1.5 is used in our proof of Theorem 1.2. However, there are also some interesting differences. In the usual proof of Theorem 1.5, a key component is that the number of  $n$ -letter words is nondecreasing in  $n$ . However, Example 3.5 shows that  $N_{[1,n]}(X)$  is not necessarily nondecreasing.

## 2. DEFINITIONS AND PRELIMINARIES

Let  $A$  denote a finite set, which we will refer to as our alphabet.

**Definition 2.1.** A **pattern** over  $A$  is a member of  $A^S$  for some  $S \subset \mathbb{Z}^d$ , which is said to have **shape**  $S$ . For  $d = 1$ , patterns are generally called **words**, especially in the case where  $S$  is an interval.

For any patterns  $v \in A^S$  and  $w \in A^T$  with  $S \cap T = \emptyset$ , we define the concatenation  $vw$  to be the pattern in  $A^{S \cup T}$  defined by  $(vw)|_S = v$  and  $(vw)|_T = w$ .

**Definition 2.2.** For any finite alphabet  $A$ , the  $\mathbb{Z}^d$ -**shift action** on  $A^{\mathbb{Z}^d}$ , denoted by  $\{\sigma_t\}_{t \in \mathbb{Z}^d}$ , is defined by  $(\sigma_t x)(s) = x(s + t)$  for  $s, t \in \mathbb{Z}^d$ .

We always think of  $A^{\mathbb{Z}^d}$  as being endowed with the product discrete topology, with respect to which it is obviously compact.

**Definition 2.3.** A  $\mathbb{Z}^d$  **subshift** is a closed subset of  $A^{\mathbb{Z}^d}$  which is invariant under the  $\mathbb{Z}^d$ -shift action.

**Definition 2.4.** The **language** of a  $\mathbb{Z}^d$  subshift  $X$ , denoted by  $L(X)$ , is the set of all patterns which appear in points of  $X$ . For any finite  $S \subset \mathbb{Z}^d$ ,  $L_S(X) := L(X) \cap A^S$ , the set of patterns in the language of  $X$  with shape  $S$ .

Any subshift inherits a topology from  $A^{\mathbb{Z}^d}$ , and is compact. Each  $\sigma_t$  is a homeomorphism on any  $\mathbb{Z}^d$  subshift, and so any  $\mathbb{Z}^d$  subshift, when paired with the  $\mathbb{Z}^d$ -shift action, is a topological dynamical system. Any  $\mathbb{Z}^d$  subshift can also be defined in terms of disallowed patterns: for any set  $\mathcal{F}$  of patterns over  $A$ , one can define the set  $X(\mathcal{F}) := \{x \in A^{\mathbb{Z}^d} : x|_S \notin \mathcal{F} \text{ for all finite } S \subset \mathbb{Z}^d\}$ . It is well known that any  $X(\mathcal{F})$  is a  $\mathbb{Z}^d$  subshift, and all  $\mathbb{Z}^d$  subshifts are representable in this way. All  $\mathbb{Z}^d$  subshifts are assumed to be nonempty in this paper.

**Definition 2.5.** A  $\mathbb{Z}^d$  **shift of finite type (SFT)** is a  $\mathbb{Z}^d$  subshift equal to  $X(\mathcal{F})$  for some finite  $\mathcal{F}$ . If  $\mathcal{F}$  consists only of patterns consisting of pairs of adjacent letters, then  $X(\mathcal{F})$  is called **nearest-neighbor**.

**Definition 2.6.** A (topological) **factor map** is any continuous shift-commuting map  $\phi$  from a  $\mathbb{Z}^d$  subshift  $X$  onto a  $\mathbb{Z}^d$  subshift  $Y$ . A factor map  $\phi$  is **1-block** if  $(\phi x)(v)$  depends only on  $x(v)$  for  $v \in \mathbb{Z}^d$ .

**Definition 2.7.** A  $\mathbb{Z}^d$  **sofic shift** is the image of a  $\mathbb{Z}^d$  SFT under a factor map. It is well-known that for any  $\mathbb{Z}^d$  sofic shift  $Y$ , there exists a nearest-neighbor  $\mathbb{Z}^d$  SFT  $X$  and 1-block factor map  $\phi$  so that  $Y = \phi(X)$ .

For  $d = 1$ , any  $\mathbb{Z}$  sofic shift can also be defined using graphs; a  $\mathbb{Z}$  subshift is sofic if and only if it is the set of labels of biinfinite paths for some (edge-)labeled graph  $\mathcal{G}$  (see [5] for a proof.)

**Definition 2.8.** For any  $\mathbb{Z}^d$  subshift  $X$  and rectangular prism  $R = \prod_{i=1}^d [0, n_i - 1]$ , the  $R$ -**higher power shift** of  $X$ , denoted  $X^R$ , is a  $\mathbb{Z}^d$  subshift with alphabet  $L_R(X)$  defined by the following rule:  $x \in (L_R(X))^{\mathbb{Z}^d} \in X^R$  if and only if the point  $y$  defined by concatenating the  $x(v)$ , viewed themselves as patterns with shape  $X$ , is in  $X$ . Formally,

$$\forall v \in \mathbb{Z}^d, y(v) := (x(\lfloor v_1 n_1^{-1} \rfloor, \dots, \lfloor v_d n_d^{-1} \rfloor)) (v_1 \pmod{n_1}, \dots, v_d \pmod{n_d}).$$

**Definition 2.9.** For any  $\mathbb{Z}$ -subshift  $X$  and word  $w \in L_{[1,n]}(X)$ , the **follower set** of  $w$  is  $F_X(w) = \{x \in A^{\{n+1, n+2, \dots\}} : wx \in L(X)\}$ . For any  $n$ , we use  $M_{[1,n]}(X)$  to denote  $|\{F_X(w) : w \in L_{[1,n]}(X)\}|$ , the number of distinct follower sets of words of length  $n$ .

**Definition 2.10.** For any  $\mathbb{Z}^d$ -subshift  $X$  and pattern  $w \in L_S(X)$ , the **extender set** of  $w$  is  $E_X(w) = \{x \in A^{\mathbb{Z}^d \setminus S} : wx \in X\}$ . For any  $S$ , we use  $N_S(X)$  to denote  $|\{E_X(w) : w \in L_S(X)\}|$ , the number of distinct extender sets of patterns with shape  $S$ .

### 3. PROOFS

For the proof of Theorem 1.2 we need the following finite version of the Morse-Hedlund theorem. We include a proof for completeness, though it is essentially the same proof as that of the original theorem.

**Lemma 3.1.** *For any word  $w \in A^N$  and  $n \leq \frac{N}{4}$  so that the number of  $n$ -letter subwords of  $w$  is less than or equal to  $n$ , we can write  $w = tuv$  where  $|t| = |v| = n$  and  $u$  is periodic with some period less than or equal to  $n$ .*

*Proof.* Since there are less than or equal to  $n$  subwords of  $w$  of length  $n$  and there are  $N - n + 1 > n$  values of  $i$  for which  $w(i)w(i+1) \dots w(i+n-1)$  is a subword of  $w$ , there exists an  $n$ -letter subword of  $w$  which appears twice. In fact, by the pigeonhole principle we may fix indices  $i < k \in [1, n+1]$  such that  $w(i)w(i+1) \dots w(i+n-1) = w(k)w(k+1) \dots w(k+n-1)$ . Similarly, we may fix indices  $\ell < j \in [N-n, N]$  such that  $w(\ell-n+1) \dots w(\ell-1)w(\ell) = w(j-n+1) \dots w(j-1)w(j)$ . Set  $w' = w(i)w(i+1) \dots w(j-1)w(j)$ .

It suffices to show that  $w'$  is periodic of period less than or equal to  $n$ ; if this is true, then taking  $t = w(1) \dots w(n)$ ,  $u = w(n+1) \dots w(N-n)$ , and  $v = w(N-n+1) \dots w(N)$  completes the proof since  $u$  is a subword of  $w'$ .

Let us now consider the number of  $m$ -letter subwords of  $w'$  for values of  $m \in [1, n]$ . If the number of one-letter subwords of  $w'$  is equal to 1, then  $w'$  is of the form  $ss \dots s$  for some symbol  $s$  and we are done. If not, then the number of one-letter subwords of  $w'$  is greater than 1, whereas the number of  $n$ -letter subwords of  $w'$  is less than or equal to  $n$ . Therefore, there must be an  $m \in [1, n-1]$  for which the number of  $(m+1)$ -letter subwords of  $w'$  is less than or equal to the number of  $m$ -letter subwords of  $w'$ . Fix  $m$  to be this number for the remainder of the proof.

We now claim that for every  $m$ -letter subword  $t$  of  $w'$ , there exists  $a \in A$  so that  $ta$  is a subword of  $w'$  as well. For any choice of  $t$  aside from the  $m$ -letter suffix of  $w'$ , this is obvious. But it is true for the suffix as well, since by construction of  $w'$ , if  $t$  is a suffix of  $w'$  then  $t$  is also the suffix of  $w(\ell-n+1) \dots w(\ell-1)w(\ell)$  which means  $tw(\ell+1)$  is a subword of  $w'$ . A similar argument shows that for every  $m$ -letter subword  $t$  of  $w'$ , there exists  $b \in A$  so that  $bt$  is a subword of  $w'$  as well.

Note that because the number of  $m$ -letter subwords is less than or equal to the number of  $(m + 1)$ -letter subwords of  $w'$ , the  $a$  and  $b$  constructed in the previous paragraph are always unique.

Let  $p = k - i$ , and note that  $w(i)w(i + 1) \dots w(i + m - 1) = w(i + p)w(i + 1 + p) \dots w(i + m - 1 + p)$ . Since there is a unique  $a$  which extends the word  $w(i)w(i + 1) \dots w(i + m - 1)$  as a subword of  $w'$ , we get that  $w(i + 1)w(i + 2) \dots w(i + m) = w(i + 1 + p)w(i + 2 + p) \dots w(i + m + p)$ . Using the same argument and working inductively, we see that  $w(i + r)w(i + r + 1) \dots w(i + r + p) = w(i + r + p)w(i + r + p + 1) \dots w(i + r + p + p)$  for any  $0 \leq r \leq j - i - p$ . In other words,  $w'$  is periodic with period  $p \leq n$ .  $\square$

We remark that since the word  $u$  in the previous lemma is periodic with period less than or equal to  $n$ , this clearly implies that  $u$  is periodic with period  $n!$  (though this may be a meaningless statement if  $|u| \leq n!$ )

*Proof of Theorem 1.2.* Consider a  $\mathbb{Z}^d$  subshift  $X$  and  $n$  so that  $|N_{[1, n]^d}(X)| \leq n$ . Define an equivalence relation on  $L_{[1, n]^d}(X)$  by  $w \sim w'$  iff  $E_X(w) = E_X(w')$ . For each of the  $k \leq n$  equivalence classes, choose a lexicographically maximal element, and denote the collection of these words by  $M$ . Then for every  $w \in L_{[1, n]^d}(X)$ , there exists  $w' \in M$  so that  $E_X(w) = E_X(w')$ . Equivalently, in any  $x \in X$  containing  $w$ ,  $w$  can be replaced by  $w'$  to make a new point  $x' \in X$ .

Now, consider any rectangular prism  $R = \prod_{i=1}^d [1, n_i]$  with  $n_i > \max\{4n, 2n + n!\}$  for all  $i$ , and any finite word  $v \in L_R(X)$ . Iterate the following procedure: if  $v$  contains a subword with shape  $[1, n]^d$  which is not in  $M$ , then replace it by the element of  $M$  in its equivalence class. Since each of these replacements increases the entire word on  $R$  in the lexicographic ordering, the procedure will eventually terminate, yielding a word  $v'$  in which every subword with shape  $[1, n]^d$  is in  $M$ . (These replacements could possibly be done in many different ways or orders; simply choose a particular one and call the result  $v'$ .) In particular,  $v'$  contains less than or equal to  $n$  distinct subwords with shape  $[1, n]^d$ . Since  $v'$  was obtained from  $v$  by a sequence of replacements with identical extender sets,  $E_X(v) = E_X(v')$ .

We wish to bound the number of such possible  $v'$  for a given  $R$ . For any translate of the  $(d - 1)$ -dimensional hypercube  $t + [1, n]^{d-1} \subset \prod_{i=2}^d [1, n_i]$ , consider the subpattern  $v'|_{[1, n_1] \times (t + [1, n]^{d-1})}$ . This can be viewed as an  $n_1$ -letter word in the  $x_1$ -direction, where each “letter” is a cross-section with shape  $t + [1, n]^{d-1}$ . When viewed in this way, each  $n$ -letter subword of  $v'|_{[1, n_1] \times (t + [1, n]^{d-1})}$  is a subpattern of  $v'$  with shape  $[1, n]^d$ , and there are less than or equal to  $n$  such subpatterns. Therefore, by Lemma 3.1,  $v'|_{[n+1, n_1-n] \times (t + [1, n]^{d-1})}$  is periodic with period  $n!e_1$ . Since  $t + [1, n]^{d-1}$  was arbitrary, in fact  $v'|_{[n+1, n_1-n] \times \prod_{i=2}^d [1, n_i]}$  is periodic with period  $n!e_1$  as well. In other words, if  $t$  and  $t + n!e_1$  both have first coordinate between  $n + 1$  and  $n_1 - n$  inclusive, then  $v'(t) = v'(t + n!e_1)$ . A similar proof shows that if  $t$  and  $t + n!e_i$  both have  $i$ th coordinate between  $n + 1$  and  $n_i - n$  inclusive, then  $v'(t) = v'(t + n!e_i)$ .

The above shows that except for sites within  $n$  of the boundary of  $R$ ,  $v'$  is determined by the subpattern occurring within a  $d$ -dimensional hypercube of side length  $n!$ . More specifically, the values of  $v'$  on the sites in  $\prod_{i=1}^d ([1, n] \cup [n_i - n + 1, n_i] \cup [n + 1, n + n!])$  uniquely determine  $v'$ , and there are  $(2n + n!)^d$  such sites. So, regardless of our choice for  $R$ , there are less than or equal to  $|A_X|^{(2n + n!)^d}$  possible  $v'$ . Since  $E_X(v) = E_X(v')$  for every  $v$ , this shows that  $|N_R(X)| \leq |A_X|^{(2n + n!)^d}$  for

every  $R$  with all dimensions at least  $n$ , completing the proof for  $K = 2n + n!$  and  $N = |A_X|^{(2n+n!)^d}$ .  $\square$

We now need a few lemmas for the proof of Theorem 1.3. The first shows that for the purposes of proving  $X$  sofic, we may always without loss of generality pass to a higher power shift.

**Lemma 3.2.** *For any  $d$ , for any  $\mathbb{Z}^d$  subshift  $X$  and rectangular prism  $R \subseteq \mathbb{Z}^d$ ,  $X$  is sofic if and only if the higher power shift  $X^{[R]}$  is sofic.*

*Proof.*  $\implies$ : Suppose that  $X$  is sofic. Then there is a 1-block factor map  $\phi$  and  $\mathbb{Z}^d$  nearest-neighbor SFT  $Y$  so that  $X = \phi(Y)$ . But then it is easy to check that  $X^{[R]} = \phi^{[R]}(Y^{[R]})$ , where  $\phi^{[R]}$  acts on patterns in  $A_Y^R$  via coordinatewise action of  $\phi$ . Since  $Y^{[R]}$  is a  $\mathbb{Z}^d$  SFT and  $\phi^{[R]}$  is a factor map, clearly  $X^{[R]}$  is sofic.

$\impliedby$ : Suppose that  $X^{[R]}$  is sofic, and without loss of generality, write  $R = \prod_{i=1}^d [0, n_i - 1]$ . Then there is a 1-block factor map  $\psi$  and  $\mathbb{Z}^d$  nearest-neighbor SFT  $Z$  so that  $X^{[R]} = \psi(Z)$ . Define a  $\mathbb{Z}^d$  nearest-neighbor SFT  $Z'$  with alphabet  $A_Z \times R$  by the following rules:

- (1) In the  $x_i$ -direction, a letter of the form  $(a, (v_1, \dots, v_d))$  must be followed by a letter of the form  $(b, (v_1, \dots, v_{i-1}, v_i + 1 \pmod{n_i}, v_{i+1}, \dots, v_d))$ .
- (2) In rule (1), if  $v_i \neq n_i - 1$ , then  $b = a$ .
- (3) In rule (1), if  $v_i = n_i - 1$ , then  $b$  must be a legal follower of  $a$  in the  $x_i$ -direction in the nearest-neighbor SFT  $Z$ .

The effect of these rules is that in any point of  $Z'$ ,  $\mathbb{Z}^d$  is partitioned into translates of  $R$ , each translate of  $R$  has a constant “label” from  $A_Z$ , and the “labels” of these translates comprise a legal point of  $Z$ . We now define a 1-block factor map  $\phi'$  on  $Z'$  by the rule  $\phi'(a, v) = (\phi(a))(v)$ , i.e. the letter of  $A_Z$  appearing at location  $v$  in  $\phi(a)$ , which was by definition a pattern in  $A_Z^R$ . This has the effect of, in each point of  $Z'$ , filling every translate of  $R$  with the image under  $\phi$  of the letter of  $A_Z$  which was its label. Since these labels comprise a point of  $Z$  and since  $\phi(Z) = X^{[R]}$ , the reader may check that  $\phi'(Z') = X$ , and so  $X$  is sofic.  $\square$

Our next lemma shows that an upper bound for  $N_R(X)$  over all large finite rectangular prisms  $R$  must also be an upper bound for  $N_R(X)$  even when we allow  $R$  to have some infinite dimensions.

**Lemma 3.3.** *For any  $d$  and any  $\mathbb{Z}^d$  subshift  $X$ , if there exist  $K, N$  so that  $N_R(X) \leq N$  for any rectangular prism  $R$  with dimensions at least  $K$ , then it is also the case that  $N_{R'}(X) \leq N$  for any “infinite rectangular prism” of the form  $R' = \prod_{i=1}^d I_i$ , where each of the  $I_i$  is either an interval of integers with length at least  $K$  or  $\mathbb{Z}$ .*

*Proof.* Consider any  $K, N, X$  satisfying the hypotheses of the theorem, and any “infinite rectangular prism”  $R'$  with all dimensions either finite and greater than  $K$  or infinite. Suppose for a contradiction that there exist  $N+1$  distinct configurations  $w_1, \dots, w_{N+1}$  in  $L_{R'}(X)$  and that their extender sets  $E_X(w_i)$  are distinct. Then, for each pair  $i < j \in [1, N+1]$ , there exists a pattern  $v_{ij} \in L_{R'^c}(X)$  s.t.  $v_{ij}w_i \in X$  and  $v_{ij}w_j \notin X$  or vice versa. By compactness, for each  $v_{ij}$ , there exists  $n_{ij}$  so



that  $v_{ij}w_i|_{[-n_{ij}, n_{ij}]^d \cap R'} \in L(X)$  and  $v_{ij}w_i|_{[-n_{ij}, n_{ij}]^d \cap R'} \notin L(X)$ , or vice versa. This property is clearly preserved by increasing  $n_{ij}$ . Therefore, if we define  $M = \max(K, \{n_{ij}\}_{i < j})$ , then for every  $i < j \in [1, N+1]$ , either  $v_{ij}w_i|_{[-M, M]^d \cap R'} \in L(X)$  and  $v_{ij}w_i|_{[-M, M]^d \cap R'} \notin L(X)$  or vice versa. Put another way,  $E_X(w_i|_{[-M, M]^d \cap R'})$  contains a pattern which equals  $v_{ij}$  on  $R'^c$ , and  $E_X(w_j|_{[-M, M]^d \cap R'})$  contains no such pattern, or vice versa. Either way, this shows that  $E_X(w_i|_{[-M, M]^d \cap R'}) \neq E_X(w_j|_{[-M, M]^d \cap R'})$  and, since  $i, j$  were arbitrary, that all  $N+1$  of the extender sets  $E_X(w_i|_{[-M, M]^d \cap R'})$ ,  $i \in [1, N+1]$ , are distinct. Since  $[-M, M]^d \cap R'$  is a finite rectangular prism with all dimensions at least  $K$ , this contradicts the hypotheses of the theorem. Our original assumption was therefore wrong, and  $N_{R'}(X) \leq N$ .  $\square$

Our final preliminary lemma shows that for  $d = 1$ , boundedness of  $N_{[1, n]}(X)$  is equivalent to soficity of  $X$ .

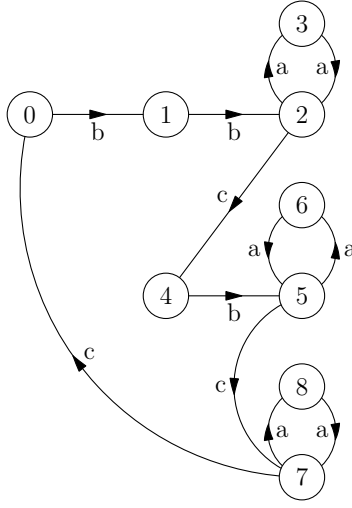
**Lemma 3.4.** *For a  $\mathbb{Z}$  subshift  $X$ ,  $X$  is sofic if and only if  $N_{[1, n]}(X)$  is a bounded sequence.*

*Proof.*  $\implies$ : If  $X$  is sofic, then there is a 1-block map  $\phi$  and nearest-neighbor SFT  $Y$ , with alphabet  $A_Y$ , so that  $X = \phi(Y)$ . Then, for any finite word  $w \in L_{[1, n]}(Y)$ , clearly  $E_X(w) = \bigcup_{y \in \phi^{-1}(w)} \phi(E_Y(y))$ . Since  $Y$  is a nearest-neighbor SFT, this clearly depends only on the set of pairs of first and last letters of  $\phi$ -preimages of  $w$ , and there are fewer than  $2^{|A_Y|^2}$  such sets. Therefore,  $N_{[1, n]}(X) \leq 2^{|A_Y|^2}$  for all  $n$ , and so the sequence  $N_{[1, n]}(X)$  is bounded.

$\impliedby$ : We prove the contrapositive, the proof will be similar to Lemma 3.3. Suppose that  $X$  is not sofic. Then there are infinitely many follower sets  $F(p)$  for infinite pasts  $p \in A_X^{\mathbb{Z}^-}$ . For any  $N$ , choose  $N$  pasts  $p_1, \dots, p_N$  with distinct follower sets. This means that for every  $i < j \in [1, N]$ , there exists a future  $f_{ij} \in A_X^{\mathbb{N}}$  so that either  $p_i f_{ij} \in X$  and  $p_j f_{ij} \notin X$ , or vice versa. By compactness, there exists  $N_{ij}$  so that for any  $n > N_{ij}$ , either  $p_i|_{(-n, 0)} f_{ij} \in X$  and  $p_j|_{(-n, 0)} f_{ij} \notin X$  or vice versa. But then if we take  $M = \max N_{ij}$ , then for every  $i < j \in [1, N]$ , either  $p_i|_{(-M, 0)} f_{ij} \in X$  and  $p_j|_{(-M, 0)} f_{ij} \notin X$  or vice versa, meaning that the  $N$  extender sets  $E_X(p_i|_{(-M, 0)})$ ,  $i \in [1, N]$ , are distinct. Therefore,  $N_{[1, M]}(X) \geq N$ . Since  $N$  was arbitrary,  $N_{[1, n]}(X)$  is not bounded.  $\square$

As an aside, before proving Theorem 1.3 we present the example mentioned in the introduction, of a  $\mathbb{Z}$  sofic shift  $X$  where  $N_{[1, n]}(X)$  is bounded, but does not stabilize. In fact, the number of distinct follower sets of words of length  $n$  also fails to stabilize for this shift, which may be of independent interest.

**Example 3.5.** ([1]) Define  $X$  to be the sofic shift consisting of all labels of biinfinite paths on the labeled graph  $\mathcal{G}$  below. Then for all  $n > 1$ ,  $M_{[1, 2n]}(X) = 14$ ,  $M_{[1, 2n+1]}(X) = 13$ ,  $N_{[1, 2n]}(X) = 46$ , and  $N_{[1, 2n+1]}(X) = 44$ .



*Proof.* The reader may check that  $\mathcal{G}$  is follower-separated (see [5] for a definition), and so for any  $w \in L(X)$ , the follower set  $F_X(w)$  is determined by the set of terminal vertices for paths in  $\mathcal{G}$  with label  $w$ , which we'll denote by  $T(w)$ . We now simply describe the possible sets  $T(w)$  for words of even and odd length, with examples of words realizing each set. We use the notation  $*$  to indicate that any word may replace the  $*$ , and  $n$  to represent any nonnegative integer.

(even length)	(odd length)
$T(w)$	$T(w)$
{0}	{0}
{1}	{1}
{2}	{2}
{3}	{3}
{4}	{4}
{5}	{5}
{6}	{6}
{7}	{7}
{8}	{8}
{1, 5}	{1, 5}
{3, 6}	{2, 5}
{4, 7}	{0, 4, 7}
{0, 4, 7}	{2, 3, 5, 6, 7, 8}
{2, 3, 5, 6, 7, 8}	
$w$	$w$
*cc	*cc
*ccb	*ccb
*cbb	*cbb
*bba	*bba
*bbc	*bbc
*bcb	*bcb
*cba	*cba
*cbc	*cbc
*bca	*bca
$a^n cb$	$a^n cb$
$ba^{2n+1}$	$ba^{2(n+1)}$
$ba^{2n} c$	$a^n c$
$a^n c$	$a^{2n+1}$
$a^{2(n+1)}$	

We leave it to the reader to check that there are no follower sets aside from the ones described here, and so  $M_{[1,2n]}(X) = 14$  and  $M_{[1,2n+1]}(X) = 13$  for all  $n > 1$ . Informally, the reason that words of even length have an additional follower set is that the word  $ba^{2n}c$  has a follower set (given by the set  $\{3, 6\}$  of terminating states) which can not be recreated by odd length; every cycle has even length, knowledge of at least one letter on each side of the cycle is required to create a new follower set, and knowledge of two letters on either side makes the word synchronizing (meaning there is only a single terminating state.)

Since listing 46 and 44 extender sets similarly (for even and odd lengths respectively) would be quite cumbersome, we will not give a complete list of these, but will give a sketch of how they appear. First, note that  $\mathcal{G}$  is also predecessor separated, and so the extender set of a word  $w \in L(X)$  is determined entirely by the set  $\{v \rightarrow v'\}$  of possible pairs of initial and terminal vertices of paths in  $\mathcal{G}$  with label  $w$ , which we denote by  $S(w)$ . Note that partitioning the vertices into  $\{0, 2, 5, 7\}$  and  $\{1, 3, 4, 6, 8\}$  shows that  $\mathcal{G}$  is bipartite. The reader may check that every possible singleton  $\{v \rightarrow v'\}$  for pairs  $v, v'$  in the same vertex class occurs as  $S(w)$  for a word  $w$  of even length, and every possible singleton  $\{v \rightarrow v'\}$  for pairs  $v, v'$  in opposite vertex classes occurs as  $S(w)$  for a word  $w$  of odd length. This contributes  $5^2 + 4^2 = 41$  extender sets to  $N_{[1,2n]}(X)$  and  $2 \cdot 5 \cdot 4 = 40$  extender sets to  $N_{[1,2n+1]}(X)$  for every  $n > 1$ . The remaining sets  $S(w)$ , along with  $w$  presenting them, appear in the table below. Note that though we can informally pair up the first through fourth types in each case, again the word  $ba^{2n-2}c$  creates an extender set with no analogous extender set for a word of odd length.

(even length)

$S(w)$	$w$
$\{2 \rightarrow 2, 3 \rightarrow 3, 5 \rightarrow 5, 6 \rightarrow 6, 7 \rightarrow 7, 8 \rightarrow 8\}$	$a^{2n}$
$\{1 \rightarrow 3, 4 \rightarrow 6\}$	$ba^{2n-1}$
$\{2 \rightarrow 5, 7 \rightarrow 1\}$	$a^{2n-2}cb$
$\{3 \rightarrow 4, 6 \rightarrow 7, 8 \rightarrow 0\}$	$a^{2n-1}c$
$\{1 \rightarrow 4, 4 \rightarrow 7\}$	$ba^{2n-2}c$

(odd length)

$S(w)$	$w$
$\{2 \rightarrow 3, 3 \rightarrow 2, 5 \rightarrow 6, 6 \rightarrow 5, 7 \rightarrow 8, 8 \rightarrow 7\}$	$a^{2n-1}$
$\{1 \rightarrow 2, 4 \rightarrow 5\}$	$ba^{2n}$
$\{3 \rightarrow 5, 8 \rightarrow 1\}$	$a^{2n-1}cb$
$\{2 \rightarrow 4, 5 \rightarrow 7, 7 \rightarrow 0\}$	$a^{2n}c$

□

*Proof of Theorem 1.3.* Our proof proceeds by induction on  $d$ . The base case  $d = 1$  is precisely Lemma 3.4. We now assume that the result holds for  $\mathbb{Z}^{d-1}$  subshifts, and will prove it for  $\mathbb{Z}^d$  subshifts. To that end, assume that  $X$  is a  $\mathbb{Z}^d$  subshift and that there exist  $K, N$  so that for any rectangular prism  $R$  with dimensions at least  $K$ ,  $N_R(X) \leq N$ . By Lemma 3.3, the same is true even if  $R$  has some infinite dimensions.

Note that by Lemma 3.2, we may without loss of generality replace  $X$  by the higher power shift  $X^{[[1,K]^d]}$ . Since  $N_R(X^{[[1,K]^d]}) \leq N$  for all rectangular prisms, with no restrictions on the dimension, we will assume this property for  $X$  in the remainder of the proof.

Define  $X' = \{x|_{\mathbb{Z}^{d-1} \times \{0\}} : x \in X\}$ , the set of restrictions of points of  $X$  to hyperplanes spanned by the first  $d - 1$  cardinal directions. By the assumption above, there are fewer than  $N$  distinct extender sets for  $x \in X'$ , and so we define equivalence classes  $C_i$ ,  $i \in [1, M]$ ,  $M \leq N$ , for the equivalence relation defined by

$x \sim y$  if  $E_X(x) = E_X(y)$ . In a slight abuse of notation, we denote by  $E_X(C_i)$  the common extender set shared by all  $x \in C_i$ .

Now, consider any  $x \in X'$  and  $k \in [1, d-1]$ . By the pigeonhole principle, there exist  $i < j \in [1, M+1]$  so that  $\sigma_{ie_k}x \sim \sigma_{je_k}x$ . But then for  $y \in L_{\mathbb{Z}^{d-1} \times \{0\}^c}(X)$ ,  $y \in E_X(x) \iff \sigma_{ie_k}y \in E_X(\sigma_{ie_k}x) \iff \sigma_{ie_k}y \in E_X(\sigma_{je_k}x) \iff y \in E_X(\sigma_{(j-i)e_k}x)$ . Therefore,  $x \sim \sigma_{(j-i)e_k}x$ , and the same logic shows that  $\sigma_{(j-i)me_k}x \sim x$  for any  $m \in \mathbb{Z}$ . Since  $j-i \leq M \leq N$ , this shows that the  $C_i$  containing  $x$  is invariant under shifts by  $N!e_k$  for  $k \in [1, d-1]$ . Since  $x \in X'$  was arbitrary, this means that every  $C_i$  is invariant under shifts by each  $N!e_k$ . We may then, again by Lemma 3.2, replace  $X$  by its higher power shift  $X^{[[1, N!]^{d-1} \times \{0\}]}$ , which allows us to assume without loss of generality that all of the  $C_i$  are shift-invariant subsets of  $A_X^{\mathbb{Z}^{d-1}}$ . The classes  $C_i$  need not, however, be closed. Their closures are  $\mathbb{Z}^{d-1}$  subshifts though, and we will show that they in fact must be sofic.

---

**Claim 1:**  $\overline{C_i}$  is sofic for each  $i$ .

It suffices to show that for any rectangular prism  $R \subseteq \mathbb{Z}^{d-1}$  and  $w, w' \in L_R(\overline{C_i})$ ,  $E_X(w) = E_X(w') \implies E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$ , since then  $N_{R \times \{0\}}(X) \leq N \implies N_{R \times \{0\}}(\overline{C_i}) \leq N$  for all rectangular prisms  $R$ , which will imply the desired conclusion by our inductive hypothesis.

So, assume that  $E_X(w) = E_X(w')$  for  $w, w' \in L_R(\overline{C_i})$ . Suppose also that  $vw \in \overline{C_i}$  for some  $v \in A_X^{\mathbb{Z}^{d-1} \setminus R}$ . Then there exists  $v_n \in A_X^{\mathbb{Z}^{d-1} \setminus R}$  so that  $v_n \rightarrow v$  and  $v_n w \in C_i$  for all  $n$ . Then for any  $y \in E_X(C_i)$ ,  $yv_n w \in X$ , since all  $v_n w$  share the same class  $C_i$ . Since  $E_X(w) = E_X(w')$ ,  $yv_n w' \in X$  as well. Similarly, for any  $y \notin E_X(C_i)$ ,  $yv_n w \notin X$ , and so  $yv_n w' \notin X$ . But then  $E_X(v_n w') = E_X(C_i)$ , and so  $v_n w' \in C_i$ . By taking limits,  $vw' \in \overline{C_i}$ . We've then shown that  $vw \in \overline{C_i} \implies vw' \in \overline{C_i}$ . The converse is true by the same proof, and so  $E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$ , completing the proof of soficity of  $\overline{C_i}$  as described above.

---

Since all elements of any class  $C_i$  are interchangeable in points of  $X$ , we can define  $V \subseteq [1, M]^{\mathbb{Z}}$  which lists legal sequences of classes (in the  $e_d$ -direction) within points in  $X$ :

$$V := \{(i_n) \in [1, M]^{\mathbb{Z}} : \exists x \in X \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}\}.$$

It is obvious that  $V$  is shift-invariant since  $X$  is shift-invariant. However, it is not immediately clear that  $V$  is closed since the  $C_i$  are not necessarily closed. We will show that  $V$  is closed by proving the following claim.

---

**Claim 2:**  $X = \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}\}.$

In other words, given  $v \in V$ , not only can you make points of  $X$  by substituting in configurations from the classes given by the letters in  $v$ , but you may also substitute configurations from the closures of these classes.

$$X \subseteq \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}\}:$$

First, we note that for any  $x \in X$ , by definition  $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in X'$  for all  $n \in \mathbb{Z}$ , and so each of these is in some class  $C_i$ . Define  $v = (i_n) \in [1, M]^{\mathbb{Z}}$  by saying that the  $x|_{\mathbb{Z} \times \{n\}}$  of  $x$  is in  $C_{i_n}$ . Then by definition of  $V$ ,  $v \in V$ . This clearly shows the desired containment.

$$X \supseteq \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z} \times \{n\}} \in \overline{C_{i_n}}\}:$$

Choose any  $x \in A_X^{\mathbb{Z}^d}$  so that there is  $v = (i_n) \in V$  with the property that  $\forall n \in \mathbb{Z}$ ,  $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}$ . Then, for each  $n \in \mathbb{Z}$ , there exists a sequence  $x^{(k,n)} \in C_{i_n}$  so that  $x^{(k,n)} \xrightarrow[k \rightarrow \infty]{} x|_{\mathbb{Z}^{d-1} \times \{n\}}$  for all  $n$ . Also, since  $v \in V$ , there exists  $x' \in X$  so that  $\forall n \in \mathbb{Z}$ ,  $x'|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$ .

We now define, for every  $k$ , the point  $x^{(k)} \in A_X^{\mathbb{Z}^d}$  by

$$x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = \begin{cases} x^{(k,n)} & \text{if } |n| \leq k \\ x'|_{\mathbb{Z}^{d-1} \times \{n\}} & \text{if } k \leq |n| \end{cases}$$

The central  $2k+1$   $(d-1)$ -dimensional hyperplanes of  $x^{(k)}$  are given by the  $x^{(k,n)}$ , and the remaining  $(d-1)$ -dimensional hyperplanes are unchanged from  $x'$ . We note that  $x^{(k)}$  can be obtained from  $x' \in X$  by making  $2k+1$  consecutive replacements of  $x'|_{\mathbb{Z}^{d-1} \times \{n\}}$  by  $x^{(k,n)}$ . Since these replacements involve configurations in the same class  $C_{i_n}$ , each of these replacements preserves being in  $X$ , and so  $x^{(k)} \in X$  for all  $k$ . Finally, we note that  $x^{(k)} \rightarrow x$ , so  $x \in X$  as well, showing the desired containment.

---

**Claim 3:**  $V$  is a sofic subshift.

We first show that  $V$  is closed and therefore a subshift. Let  $v^{(k)} \in V$  and  $v^{(k)} \rightarrow v = (i_n)$ . By passing to a subsequence, we may assume that for all  $k \geq |n|$ ,  $v_n^{(k)} = i_n$ . By definition of  $V$ , for every  $k \in \mathbb{N}$  there exists  $x^{(k)} \in X$  where  $x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v_n^{(k)}}$  for every  $n$ . For all  $n \leq k$ , since  $C_{v_n^{(k)}} = C_{i_n} = C_{v_n^{(n)}}$  we may replace the pattern  $x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}}$  in  $x^{(k)}$  with  $x^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}}$  to form a legal point in  $X$ . In such a way we obtain a new sequence of points  $y^{(k)} \in X$  where  $y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v_n^{(k)}}$ , but with the additional property that  $y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}}$  for  $k \geq |n|$ . The sequence  $y^{(k)}$  clearly converges to a point  $y \in A_X^{\mathbb{Z}^d}$ , and  $y \in X$  since  $X$  is closed. Since  $y|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$ , we have  $v \in V$ .

We claim that  $N_{[1,m]}(V) \leq N$  for all  $m \in \mathbb{N}$ , which will prove the claim by Lemma 3.4. Suppose for a contradiction that there are  $N+1$  words  $v^{(1)}, \dots, v^{(N+1)} \in L_m(V)$  s.t. the extender sets  $E_V(v^{(i)})$  are all distinct. Then, for every  $i < j \in$

$[1, N + 1]$ , there exists  $w^{(ij)} \in L_{[1,m]^c}(V)$  s.t. either  $v^{(i)}w^{(ij)} \in V$  and  $v^{(j)}w^{(ij)} \notin V$  or vice versa.

For each  $v^{(i)} = v_1^{(1)} \dots v_m^{(1)}$ , define  $S^{(i)} \in A_X^{\mathbb{Z}^{d-1} \times [1,m]}$  by choosing  $S^{(i)}|_{\mathbb{Z}^{d-1} \times \{n\}}$  to be any row in  $C_{v_n^{(i)}}$ . Similarly, for each  $w^{(ij)}$ , define a pattern  $O^{(ij)} \in A_X^{\mathbb{Z}^{d-1} \times [1,m]^c}$  by choosing  $O^{(ij)}|_{\mathbb{Z}^{d-1} \times \{b\}}$  to be any row in  $C_{w_n^{(ij)}}$ . Then, by Claim 2,  $S^{(i)}O^{(ij)} \in X$  and  $S^{(j)}O^{(ij)} \notin X$  or vice versa, meaning that all extender sets  $E_X(S^{(i)})$ ,  $i \in [1, N + 1]$ , are distinct. This is a contradiction to Lemma 3.3, and so our original assumption was wrong,  $N_{[1,m]}(V)$  is a bounded sequence, and  $V$  is sofic.

---

We may now finally construct an SFT cover of  $X$  to show that it is sofic. Since  $V$  is sofic by Claim 3, we may define a 1-block factor  $\psi$  and nearest-neighbor SFT  $W$  so that  $\psi(W) = V$ . For each  $a \in A_V$ , since  $\overline{C_a}$  is sofic by Claim 1, there is a 1-block factor  $\phi_a$  and nearest-neighbor  $\mathbb{Z}^{d-1}$  SFT  $Y_a$  (whose alphabet we denote by  $A_a$ ) so that  $\phi_a(Y_a) = \overline{C_a}$ . Now, define a nearest-neighbor  $\mathbb{Z}^d$  SFT  $Y$  with alphabet  $A_Y := \bigcup_{a \in A_W} (\{a\} \times A_{Y_{\psi(a)}})$  by the following adjacency rules:

- (1) Any pair of letters  $(a, s)$ ,  $(b, t)$  which are adjacent in one of the first  $d - 1$  cardinal directions must share the same first coordinate, i.e.  $a = b$ .
- (2)  $(a, s)$  may legally precede  $(a, t)$  in the  $e_i$ -direction for  $i \in [1, d - 1]$  if and only if  $s$  may legally precede  $t$  in the same direction in  $Y_{\psi(a)}$ .
- (3)  $(a, s)$  may legally precede  $(b, t)$  in the  $e_d$ -direction if and only if  $a$  may legally precede  $b$  in  $W$ . (There is no restriction on the second coordinates  $s, t$ .)

Clearly for any  $y \in Y$ , these rules force the  $(d - 1)$ -dimensional hyperplanes  $y|_{\mathbb{Z}^{d-1} \times \{n\}}$  to have constant first coordinate (say  $a_n$ ), force the second coordinates to form a point in  $Y_{\psi(a_n)}$ , and force the sequence  $(a_n)$  to be in  $W$ . We now define the 1-block factor map  $\phi$  on  $Y$  by  $\phi(a, s) = \phi_{\psi(a)}(s)$ .

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**Claim 4:**  $\phi(Y) = X$ .

$\phi(Y) \subseteq X$ : Take any  $y \in Y$ , and define  $a_n$  to be the first coordinate shared by all letters in  $y|_{\mathbb{Z}^{d-1} \times \{n\}}$ . Then by definition of  $Y$ ,  $(a_n) \in W$ . Also by definition of  $Y$ , the second coordinates of the letters in  $y|_{\mathbb{Z}^{d-1} \times \{n\}}$  form a point of  $Y_{\psi(a_n)}$ , call it  $b^{(n)}$ . Then,  $(\psi(a_n)) \in V$ , and for every  $n \in \mathbb{Z}$ ,  $(\phi(y))|_{\mathbb{Z}^{d-1} \times \{n\}} = \phi_{\psi(a_n)}(b^{(n)})$  is the  $\phi_{\psi(a_n)}$ -image of a point of  $Y_{\psi(a_n)}$ , and so is in  $\overline{C_{a_n}}$ . But then, by Claim 2,  $\phi(y) \in X$ , and since  $y \in Y$  was arbitrary,  $\phi(Y) \subseteq X$ .

$\phi(Y) \supseteq X$ : Choose any  $x \in X$ . For every  $n \in \mathbb{Z}$ ,  $x|_{\mathbb{Z}^{d-1} \times \{n\}}$  is in one of the  $C_i$ , and if we define a sequence  $i_n$  by  $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$ , then  $(i_n) \in V$ . Choose any  $(a_n) \in W$  s.t.  $(\psi(a_n)) = (i_n)$ . For each  $n \in \mathbb{Z}$ , since  $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}} = \overline{C_{\psi(a_n)}}$ , there exists  $b^{(n)} \in Y_{\psi(a_n)}$  s.t.  $\phi_{\psi(a_n)}(b^{(n)}) = x|_{\mathbb{Z}^{d-1} \times \{n\}}$ . Define a point  $y \in A_Y^{\mathbb{Z}^2}$  by setting, for all  $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$ ,  $y(t) = (a_{t_d}, b^{(n)}(t_1, \dots, t_{d-1}))$ . Then  $y \in Y$  since  $(a_n)$  is in  $W$ , and for all  $n \in \mathbb{Z}$ , the second coordinates in each hyperplane

$y|_{\mathbb{Z}^{d-1} \times \{n\}}$  create a point of  $Y_{\psi(a_n)}$ . It is also clear that  $\phi(y) = x$ , and since  $X$  was arbitrary,  $\phi(Y) \supseteq X$ .

Since  $Y$  was an SFT and  $\phi$  a factor map, this shows that  $X$  is sofic, completing the proof of Theorem 1.3.  $\square$

We would like to say a bit more about shifts satisfying the hypotheses of Theorem 1.3, i.e. those with eventually bounded numbers of extender sets, because in fact they satisfy a much stronger (though technical) condition than just being sofic.

**Definition 3.6.** We say that a  $\mathbb{Z}^d$  nearest-neighbor SFT  $X$  is **decouplable** if either  $d = 1$  (in which case  $X$  is automatically called decouplable) or there exist  $i \in [1, d]$ , a  $\mathbb{Z}$  nearest-neighbor SFT  $W$ , and decouplable  $\mathbb{Z}^{d-1}$  nearest-neighbor SFTs  $Y_a$  for each  $a \in A_W$ , with disjoint alphabets, so that

$$X = \{x \in A_X^{\mathbb{Z}^d} : \exists w = (w_n) \in W \text{ s.t. } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{i-1} \times \{n\} \times \mathbb{Z}^{d-i}} \in Y_{w_n}\}.$$

(Here, we have made the obvious identification between  $\mathbb{Z}^{i-1} \times \{n\} \times \mathbb{Z}^{d-i}$  and  $\mathbb{Z}^{d-1}$ .)

In other words,  $X$  is decouplable if one can construct it by starting from a one-dimensional nearest neighbor SFT and then arbitrarily replacing occurrences of each letter in its alphabet by points from a  $\mathbb{Z}^{d-1}$  decouplable nearest-neighbor SFT associated to that letter. This definition is obviously recursive;  $X$  is decouplable if its  $(d-1)$ -dimensional hyperplanes are given by decouplable SFTs, whose  $(d-2)$ -dimensional hyperplanes are given by decouplable SFTs, and so on. This means that though  $X$  is a  $\mathbb{Z}^d$  SFT, its behavior is in some sense one-dimensional.

**Remark 3.7.** In fact the SFT cover in the proof of Theorem 1.3 can always be chosen to be decouplable. By the inductive hypothesis, the covers  $Y_a$  can each be chosen to be decouplable  $\mathbb{Z}^{d-1}$  SFTs, and then the construction of  $X$  from  $W$  and all  $Y_a$  clearly yields a decouplable  $\mathbb{Z}^d$  SFT.

In order to give an application of Theorem 1.3 and to elucidate the idea of decouplable SFTs, we will present a brief example.

**Example 3.8.** Define  $X$  to be the  $\mathbb{Z}^2$  subshift on  $\{0, 1\}$  consisting of all  $x \in \{0, 1\}^{\mathbb{Z}^2}$  with either no 1s, a single 1, or two 1s.

Then it is not hard to see that for any  $S \subseteq \mathbb{Z}^2$  with  $|S| > 1$ ,  $N_S(X) = 3$ . It is easily checked that the three possible extender sets for  $w \in L_S(X)$  are:

- If  $w$  contains no 1s, then  $E_X(w)$  consists of all patterns on  $S^c$  with either no 1s, a single 1, or two 1s.
- If  $w$  contains a single 1, then  $E_X(w)$  consists of all patterns on  $S^c$  with either no 1 or a single 1.
- If  $w$  contains two 1s, then  $E_X(w)$  consists of the single pattern on  $S^c$  with no 1s, namely  $0^{S^c}$ .

We will now describe how the proof of Theorem 1.3 yields an SFT cover for  $X$ . Using the language of the proof of Theorem 1.3,  $X'$  consists of all biinfinite 0-1

sequences with either no 1s, a single 1, or two 1s.  $X'$  is broken into three classes of rows with the same extender sets in  $X$ , which are again classified by number of 1s contained:

- $C_0 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains no 1s}\} = \{0^{\mathbb{Z}}\}$ .
- $C_1 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains exactly one 1}\}$ .
- $C_2 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains exactly two 1s}\}$ .

(We've written the  $C_i$ s with subscripts starting from 0 rather than 1 so that  $V$  can be more easily described; clearly this has no effect on the proof.) Note that each of the  $C_i$  is shift-invariant, but  $C_1$  and  $C_2$  are not closed. However, each closure  $\overline{C_i}$  is sofic. This is easily checked, but we will momentarily explicitly describe SFT covers of the  $\overline{C_i}$  anyway.

We now wish to find  $V$ , the  $\mathbb{Z}$  subshift with alphabet  $\{0, 1, 2\}$  which describes how the rows in various classes can fit together to make points of  $X$ . This is not so hard to see: since points of  $X$  must have at most two 1s, and since the classes  $C_i$  are partitioned by number of 1s,

$$V = \{v \in \{0, 1, 2\}^{\mathbb{Z}} : v \text{ has only finitely many nonzero digits, and } \sum v_n \leq 2\}.$$

Points of  $X$  are then constructed by beginning with a sequence in  $V$ , writing it vertically, and replacing each letter  $v_n$  with an arbitrary element of  $C_{v_n}$ . So, for instance, one could start with  $\dots 0002000 \dots \in V$ , replace all 0s by the single sequence  $\dots 000000 \dots \in C_0$ , and replace 2 by any sequence in  $C_2$ , for instance  $\dots 0001001000 \dots$ . Clearly every point obtained in this way will have at most two 1s and so will be in  $X$ . In addition though, as described in Claim 2, one can also replace each  $v_n$  by an arbitrary element of the closure  $\overline{C_{v_n}}$ . For instance, if we chose to replace the 2 in our earlier sequence by  $\dots 0001000 \dots$ , which is not in  $C_2$  (in fact it's in  $C_1$ ), but is in  $\overline{C_2}$ , we would still arrive at a legal point of  $X$ .

We now note that  $V$  is a sofic shift, with nearest-neighbor SFT cover  $W$  defined as follows:  $A_W = \{A, B, C, D, E, F\}$ , and legal adjacent pairs in  $W$  are  $AA, AB, BC, CC, CD, DE, EE, AF$ , and  $FE$ . So,

$$W = \{A^\infty, C^\infty, E^\infty, A^\infty BC^\infty, C^\infty DE^\infty, A^\infty BC^n DE^\infty, A^\infty FE^\infty\}.$$

The factor  $\psi$  is defined by  $\psi(A) = \psi(C) = \psi(E) = 0$ ,  $\psi(B) = \psi(D) = 1$ , and  $\psi(F) = 2$ , and it is easily checked that  $\psi(W) = V$ .

Following our proof of Theorem 1.3, the next step is to construct SFT covers of each  $\overline{C_i}$ , which is straightforward. Define  $Y_0$  to consist of the single fixed point  $\{a^\infty\}$ , and  $\phi_0$  by  $\phi_0(a) = 0$ . Define  $Y_1$  to have alphabet  $\{a, b, c\}$  with legal adjacent pairs  $aa, ab, bc$ , and  $cc$ ; then  $Y_1 = \{a^\infty, c^\infty, a^\infty bc^\infty\}$ . Define  $\phi_1$  by  $\phi_1(a) = \phi_1(c) = 0$  and  $\phi_1(b) = 1$ . Finally, define  $Y_2$  to have alphabet  $\{a, b, c, d, e\}$  with legal adjacent pairs  $aa, ab, bc, cd, de$ , and  $ee$ ; then  $Y_2 = \{a^\infty, c^\infty, e^\infty, a^\infty bc^\infty, c^\infty de^\infty, a^\infty bc^n de^\infty\}$ . Define  $\phi_2$  by  $\phi_2(a) = \phi_2(c) = \phi_2(e) = 0$  and  $\phi_2(b) = \phi_2(d) = 1$ . The reader may check that  $\phi_i(Y_i) = \overline{C_i}$  for each  $i$ .

We may now construct an SFT cover  $Y$  for  $X$  following our proof. The alphabet  $A_Y := \bigcup_{a \in A_W} (\{a\} \times A_{\psi(a)}) = \{(A, a), (B, a), (B, b), (B, c), (C, a), (D, a), (D, b), (D, c), (E, a), (F, a), (F, b), (F, c), (F, d), (F, e)\}$ . The adjacency rules are that horizontally adjacent letters have the same first (capital) coordinate  $\Pi$  and second (lowercase) coordinates satisfying the adjacency rules given by  $Y_{\psi(\Pi)}$ , and that vertically adjacent letters have first (capital) coordinates satisfying the adjacency rules of  $W$ . So,



for instance,  $(D, a)$  cannot appear immediately to the left of  $(D, c)$ , since  $ac$  is not a legal pair in  $Y_{\psi(D)} = Y_1$ . On the other hand,  $(A, a)$  can appear below  $(F, d)$ , since  $AF$  is a legal pair in  $W$ . The map  $\phi$  is defined, as before by  $\phi(a, b) = \phi_{\psi(a)}(b)$ , meaning that  $\phi(B, b) = \phi(D, b) = \phi(F, b) = \phi(F, d) = 1$ , and all other letters of  $A_Y$  have  $\phi$ -image 0. It's easy to see that  $\phi(Y) = X$ ; the rules defining  $Y$  mean that there are at most two letters with second coordinate  $b$  or  $d$ , and these are the only letters of  $A_Y$  which have  $\phi$ -image 1. □

Finally, we will prove Theorem 1.4, but first need the following definition and theorem from [3].

**Definition 3.9.** ([3]) A  $\mathbb{Z}^d$  subshift  $X$  is **effective** if there exists a forbidden list  $(w_n)$  for  $X$  and a Turing machine which, on input  $n$ , outputs  $w_n$ .

It is easy to see that not every subshift is effective; there are only countably many Turing machines, and so only countably many effective subshifts.

**Theorem 3.10.** ([3]) *Any  $\mathbb{Z}^d$  sofic shift is effective.*

*Proof of Theorem 1.4.* For any Sturmian  $\mathbb{Z}$  subshift  $S$ , we can extend  $S$  to a  $\mathbb{Z}^d$  subshift  $\tilde{S}$  by enforcing constancy along all cardinal directions  $e_i$ ,  $i \in [2, d]$ . There are uncountably many Sturmian subshifts, and so there exists one, call it  $S'$ , s.t.  $\tilde{S}'$  is not effective. (In fact, effectiveness of Sturmian  $S$  and/or the shift  $\tilde{S}$  is equivalent to computability of the rotation number defining  $S$ , but we will not need this fact.) By Theorem 3.10,  $\tilde{S}'$  is not sofic. In addition, by the minimal complexity definition of Sturmian subshifts, for every  $n \in \mathbb{N}$ ,  $|L_{[1,n]^d}(\tilde{S}')| = |L_{[1,n]}(S')| = n + 1$ , and so trivially  $N_{[1,n]}(\tilde{S}') \leq n + 1$ . Since  $\tilde{S}'$  is not sofic, by Theorem 1.1, in fact  $N_{[1,n]}(\tilde{S}') = n + 1$  for all  $n$ . □

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