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Formalizing the fretboard’s phantasmic fingers

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Abstract

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Cover Page Footnote

Lecture given at The 21 Century Guitar Conference 2021

Formalizing the fretboard's phantasmic fingers[1]

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This paper shows how the symmetric group S_4 can be used to analyze the manifold ways fingers connect with fretted instruments. S_4, visualized as the symmetrical manipulations of a cube, consists of all possible permutations of four elements. Therefore, the operations can be used for the fingers of *both* the fretting and picking hands. I highlight two types of subgroups in S_4, dihedral ($\mathbb{D}_8$ and $\mathbb{D}_6$) and cyclic ($\mathbb{Z}_4$ and $\mathbb{Z}_3$), in order to analytically model the experiential differentiation between grouped and isolated conceptions of finger action, respectively. In addition to finger transformations, I define contextual operations that function componentwise on ordered n-tuples containing representatives of both fretboard and finger spaces – thus, attending to the nexus of instrument and performer. I illustrate the applicability of these transformations with musical examples by Villa-Lobos and Jonathan Kreisberg, culminating in an analysis of jazz guitarist Ben Monder's *Windowpanes*. These explorations show how motions through finger space can both support and diverge from motions in fretboard space.

In the domain of music theory, the past decade bore witness to a number of compelling theorizations of the instrument/performer interface that draw primarily upon transformational methodologies (Bennet, 2019; Bungert, 2015; De Souza, 2017, 2018; Koozin, 2011; Momii, 2020; Rockwell, 2007; 2009).[2] Rather than formalizing transformations of more traditional spaces such as pitch, rhythm, or harmony, these studies take instruments themselves and the bodies that play them as their subject matter. This line of thought nicely mirrors the "transformational attitude" that David Lewin characterized as: "'If I am *at* s and wish to get to t, what characteristic gesture ⋯ should I perform in order to arrive there?'" (Lewin, 1987/ 2007, p. 159). The gestures that Lewin refers to are typically intra-musical actions such as *transposing*, *sequencing*, or *modulating*. Although transformational theories of instrumental spaces can coincide with more abstract music theoretical ones, this correspondence is only a portion of the insights that investigating instrumental topographies generates.

This divergence of music-theoretic spaces from instrumental ones is effectively shown in the outro to Opeth's 2008 track, *Burden*, transcribed in Figure 1.[3] Over the course of the forty-eight-second-long section, guitarist Mikael Åkerfeldt repeats the example almost six times before being cut off by dark, ominous laughter. All the while, fellow guitarist Fredrik Åkesson slowly detunes Åkerfeldt's guitar in an ad hoc manner – creating a fluctuating cacophony that slowly morphs in and out of relative tune. Notwithstanding the difficulties resulting from uneven/unusual string tensions, Åkerfeldt faithfully repeats the same physical motions. As the guitar slips further out of tune, the pitches in my transcription slowly lose their value – essentially transforming it into a form of tablature. These physical invariants – rather than pitch-based parameters – are what are taken as objects of analytical inquiry in instrumental transformational analysis. This, of course, does not mean that pitch is not important. Rather, this highlights the fact that pitch and temperament are often tacitly *known* by the instrument.

[1] Lecture given at The 21 Century Guitar Conference 2021.

[2] Koozin (2011) and Bennet (2019) studied the guitar; Bungert (2015) the piano; De Souza (2017) most prominently, the harmonica, guitar, piano, and violin; De Souza (2018) fretted instruments in general; Momii (2020) the shō; and Rockwell (2007, 2009) the banjo.

[3] Opeth is a Swedish progressive metal band.

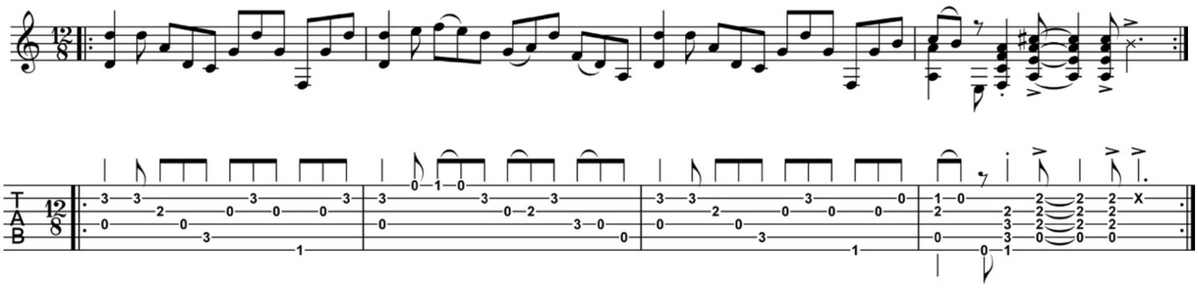

Figure 1 Transcription of the outro section of the song *Burden* (6:41–7:29) from Opeth's album *Watershed*.[4]

Fretted string instruments, in particular, have received a considerable amount of theoretical attention. The most thorough being Jonathan De Souza's (2018) article, aptly entitled *Fretboard Transformations*, which expands upon the work stated in his book, *Music at Hand* (De Souza 2017, pp. 55–56, 83–108), as well as addresses some mathematical formalisms that plagued previous transformational studies of instruments. Although generative, De Souza's transformations act on the instrumental space alone, while the fingers are merely shadows of the fretboard. By this I mean that, for instance, if the fifth fret of the sixth string is represented in a transformational network, the reader is left to assume that some finger must be there pressing it down. This doesn't quite capture the role that economic finger usage plays in performance. To represent this experience theoretically one needs an approach for the fingers themselves.

The aim of this paper is to explicate how the symmetric group S_4 can be used to analyze the manifold ways fingers connect with fretted instruments. S_4, visualized as the symmetrical manipulations of a cube, consists of all possible permutations of four elements. Therefore, the operations can be equally used for the fingers of both the fretting and picking hands. Flattening the cube into dihedral subgroups ($\mathbb{D}_8$ and $\mathbb{D}_6$)[5] provides transformations that metaphorically map the experience of isolating a particular finger cycle and letting it spin. These dihedral subgroups contain the cyclic subgroups ($\mathbb{Z}_4$ and $\mathbb{Z}_3$)[6] that supply transformations for singular motions through a finger ordering. This formalistic redundancy of transformations affords analytical differentiation between isolated and grouped conceptions of finger action. In addition to finger transformations, I define contextual operations that function componentwise on ordered n-tuples containing representatives of both fretboard and finger spaces – thus, attending to the nexus of instrument and performer. I illustrate the applicability of these transformations with musical examples by Villa-Lobos and Jonathan Kreisberg, culminating in an analysis of jazz guitarist Ben Monder's *Windowpanes*. These explorations show how the diversity of transformations afford analytical plasticity in modeling the conceptualizations of the body that underwrite performance.

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[4] All transcriptions are by the author.

[5] A dihedral group, $\mathbb{D}_n$, can be thought of as the symmetrical rotations and flips of a regular polygon with $n/2$ vertices. This paper will focus on $\mathbb{D}_8$ and $\mathbb{D}_6$, which will be depicted as a square and triangle, respectively.

[6] A cyclic group, $\mathbb{Z}_n$ (sometimes written as C_n), is a group generated by a single element, meaning all members of the group can be expressed in terms of a single element. For the purposes of this paper, it can be thought of as the symmetrical rotations of a polygon with n vertices. However, the most common exemplar is clock time. Whether you count 24 hours or twice through 12 hours a day, clock math (where $23 + 1 = 0_{\text{mod}24}$ or $12 + 1 = 1_{\text{mod}12}$) is based on cyclic group structure where one hour is the generator.

Transformational theory: A far too brief primer

Transformation theory is an application of abstract algebra, particularly group theory, to music that was pioneered by music theorist David Lewin (1987/2007).[7] Transformational models typically contain two things: 1) a *set* of *objects*, and 2) a *group* of *transformations* (functions or actions) that send said objects to other objects of the same set.[8] To take a traditional musical example, imagine that pitches are our objects, and our functions/actions are chromatic transpositions (see Fig. 2). Here, we are at the pitch C_4, and transposition T_2 indicates that we need to move up two half steps to arrive at the pitch D_4. Phrased to mirror my above wording: the transformation T_2 sends C_4 (an object in the set) to D_4 (another object of the *same* set). There are many more stipulations and limitations pertaining to group structure and how functions act on objects that I will not go into, here. The main idea to keep in mind, though, is that I will be dealing with a set of *objects* and a group of *actions* acting on those objects.

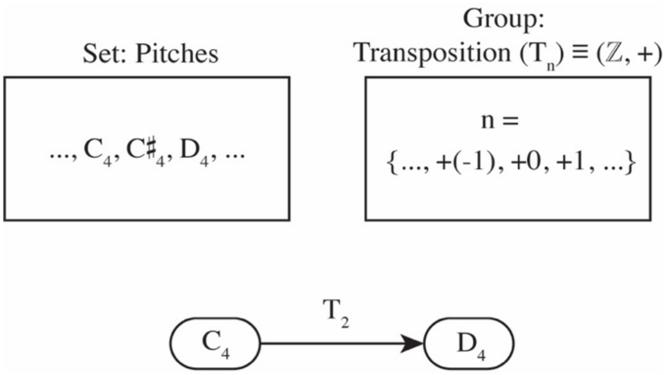


Figure 2 Chromatic transposition in Pitch space.

Formalizing the guitar/performer nexus

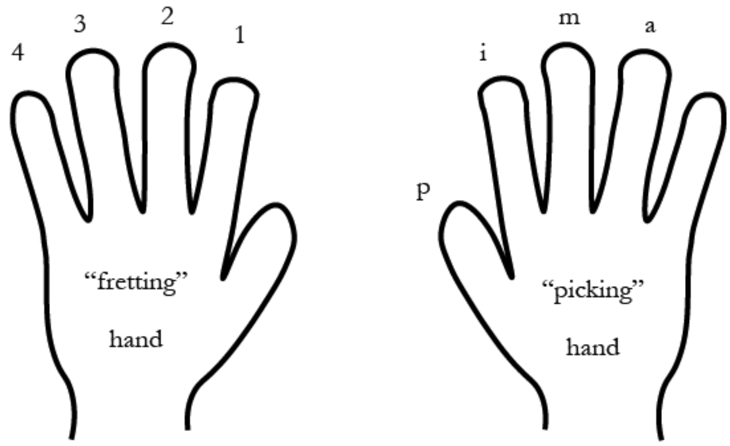
Turning now to the guitar, let's start with the sets of objects: Figure 3 depicts the traditional labels for both hands (Fig. 3a) as well as the fretboard (Fig. 3b). In fingerstyle technique, the picking hand plucks the strings of the guitar with the thumb (p), index (i), middle (m), and ring (a) fingers. These four elements are the sole members of pick space (P). The set of fretting hand fingers (F) numbers the index through pinky as 1 through 4, respectively. Figure 3b shows the neck portion of the guitar. We will return to generalize this portion shortly, but as an initial gloss, the set of strings (S) are labeled 1 to s_{max} in descending order from highest to lowest sounding open string. In the horizontal dimension, the frets are the elements of fret space (R) and are numbered from 0 (open) to r_{max}.[9] Following Joti Rockwell's (2009)

[7] I have done my best to present mathematical information in as reader friendly a way as possible – full unpacking of all the constraints (and the insights they enable) is far beyond the scope of this paper. Naturally, this has entailed me leaving vague – or out entirely – many definitions that are crucial to the infrastructure of this work. For those looking for a more general and complete introduction to the mathematics for musicians, I strongly recommend Satyendra (2004) and Rings (2011, Chapter 1). Rings' book also contains a wonderfully pithy glossary of relevant algebraic terms (pp. 223–230). For a rigorous, purely mathematical, explanation see Dummit and Foote (2004). Carter (2009) provides a more approachable account that abounds in visualizations.

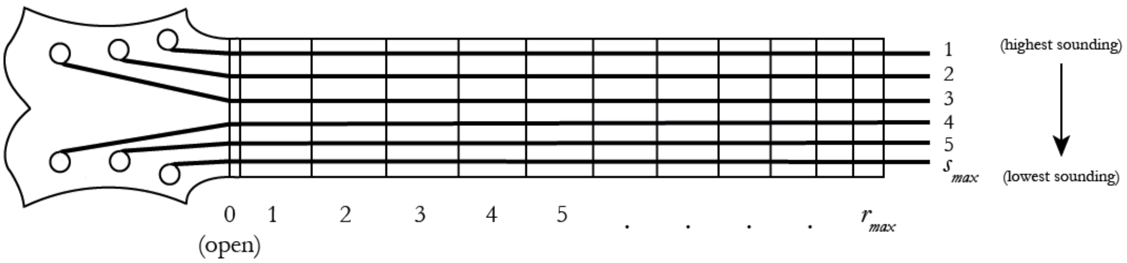
[8] This is accordance with classic, Lewinian transformational theory (1987/2007, p. 3). However, as Julian Hook (2002) has pointed out, "there are no mathematical impediments to" constructing "cross-type transformations" (p. 117). For an in-depth exploration of such transformations, see Hook (2007a).

[9] NB. De Souza (2017) uses f to denote fret (p. 55). However, since the present paper looks to include fretting fingers, I, following Rockwell (2009), use *r* for fret and *f* for fretting finger. Thus, all formulations from De Souza presented here have been relabeled.

“B-set” for the banjo (p. 140), we can define for the guitar a G-set consisting of the following ordered quadruple G = (r, s, f, p), containing a fret, string, fretting finger, and picking finger respectively.[10] As noted earlier, *pitch is not itself included in G*. This omission simplifies the guitar’s many-to-one pitch mapping property, that is, that the exact same pitch can be performed on different strings.[11] The omission of pitch reflects a performative leaning in which sometimes it is more intuitive to think of *moving up two frets* rather than, say, *ascending a major second from C to D*.



a)



b)

Figure 3 Guitar Objects: a) labeling of fingers for both hands; b) labeling of guitar topology

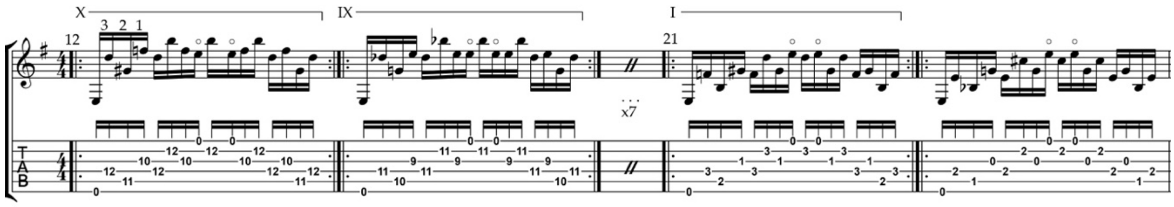
Groups of actions can be constructed to act on various subsets of G. The most intuitive division would be to split transformational actions on the instrument from those on the hands. De Souza’s (2018) fretboard transformations do exactly this by focusing exclusively on string and fret space. In this formulation, every placement on the instrument can be represented with an ordered pair from $R \times S$: $\{(r, s) \mid r \in R, s \in S\}$. In other words, the objects are ordered pairs consisting of a fret and a string; for instance, the ordered pair (3, 4) stands for the third fret on the fourth string. The actions are also ordered pairs distinguished by a plus or minus sign indicating direction of motion; (+3, –1) represents a motion up three frets and down one string. To take a simple example, submitting the object (3, 4) to the action (+3, –1) yields a new

[10] Rockwell’s (2009) B-set does not include fretting fingers (f). However, it does include a different fourth element (t) for time point. Although t does not figure prominently enough to warrant much formalization here, it is discussed as needed.

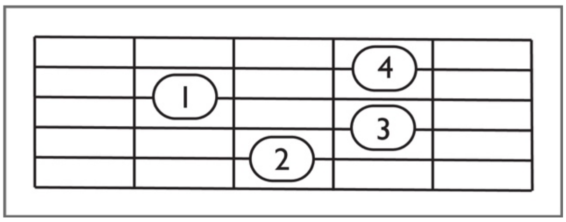
[11] Rockwell (2007) presents a function PITCH that maps the Cartesian product $R \times S$ into Y (pitch space). Given a specific tuning ($Y_{tune} = \{(a_1, a_2, \cdots, a_{smax}) \mid a_s \in Y$, PITCH$(0, s) = a_s\}$), one can calculate the pitch of any fret/string combination as PITCH$(r, s) = a_s + r$. See, also, De Souza (2018, pp. 5–6).

object, (6, 3) – again, paraphrasing Lewin and translating into language common to guitarists: "if I am at the third fret on the fourth string and wish to get to the sixth fret on the third string, I must move up three frets and down one string." The utility of De Souza's approach becomes more apparent when considering chords, which are understood as collections of fret/string pairs.

To illustrate this, Villa-Lobos' first étude for guitar serves a succinct example of how idiomatic gestures on the fretboard can be exploited compositionally. As shown in Figure 4, measure 12 initiates a systematic process in which the same chord shape is slid down the fretboard from tenth to open position, one fret each measure. Ignoring the outside open strings, Figure 5 shows this motion as a transformational network where the objects are collections of (fret, string) pairs that are connected by the transformation (–1, 0), signifying a shift down one fret. Alternately, the distance between the starting and ending shapes can be represented as (–10, 0), capturing the global motion of the progression.



a)



b)

Figure 4 a) Sequence from Villa-Lobos' Étude No. 1 (1929); b) Chord shape throughout the sequence. Score excerpt adapted from *Heitor Villa-Lobos: Collected works for solo guitar* (pp. 36–37) by H. Villa-Lobos, 1990, Editions Max Eschig.

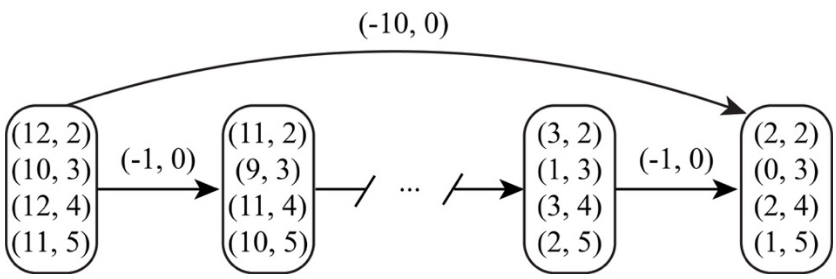


Figure 5 Transformation network of Villa-Lobos' Étude No. 1, mm. 12–22.

The fretboard revisited

However, mathematical formalisms cause problems when modeling bounded spaces – such as instruments, which only have so many strings and frets. Lewin's "Condition (B)" for generalized interval systems (GISes) states, "for every s in S and every i in IVLS, there is a unique t in S which lies the interval i from s" (Lewin, 1987/2007, p. 26).[12] Roughly translating this into the transformational language I have been using, if we have an element *s* and a transformation *i* there must be some other element *t* that is the result of performing action *i* on *s*. Drawing on Lewin's comment that mathematically formal spaces model "theoretical potentialities, rather than musical practicalities" (Lewin, 1987/2007, p. 27), De Souza's (2017) solution to this problem is to define *R* and *S* as theoretically infinite, that is, equivalent to $\mathbb{Z} \times \mathbb{Z}$ (p. 56).[13] Thus defined, a generalized interval system (GIS) can be fashioned for the fretboard.[14] From this infinite space the analyst or performer focuses on the subset that a particular instrument uses. Thus, we can redefine the fretboard portion of G (the Cartesian product of fret and string space) as follows:

- Generalized fretboard space: $R \times S = \{(r, s) \mid r \in \mathbb{Z}, s \in \mathbb{Z}\}$
- Particular fretboard subset: $R' \times S' = \{(r, s) \mid r \in \mathbb{Z}, 0 \leq r \leq r_{max}, s \in \mathbb{Z}, 1 \leq s \leq s_{max}\}$

Figure 6 provides one possible visual representation of this subset relation.

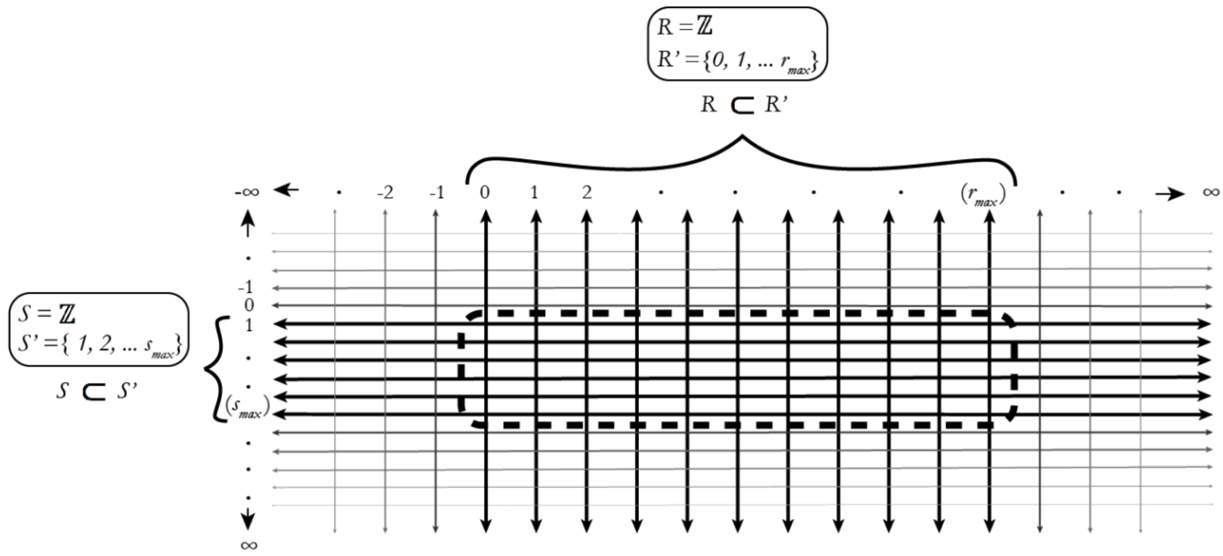


Figure 6 A particular fretboard subset in infinite fretboard space.

[12] "IVLS" is Lewin's term for the group of intervals in a generalized interval system (Lewin, 1987/2007, p. 26).

[13] Robert Wells (2017) has recently commented on this tension between mathematical and musical thought in Lewin's work. Wells uses the terms "math-forward" and "music-forward" to indicate when one way of thinking comes to the fore and shapes the other.

[14] In a sense, GISes are a subcategory of transformational theory that formalize intervals. They are more strictly defined and place the analyst in an objective, Cartesian relationship to musical elements (Lewin, 1987/2007, pp. 158–159). As a subcategory of transformational theory, they can easily be translated into transformational language; however, the reverse is not always true. On the Cartesian tint of GISes, see Rings (2011, pp. 16–17). For an explanation of how to translate between GISes and transformations (and the two analytical perspectives associated with them), see Rings (2011, pp. 27–29) and Satyendra (2004, pp. 102–103).

Although this may at first sound needlessly abstract, this definition captures nicely the notion that certain gestures transfer between different fretted instruments, *despite* their differences. For instance, none of the five fretted instruments that I have lying around my apartment inhabit the same subset of the infinite fretboard space (shown in Table 1). However, I could technically perform the Villa-Lobos étude on any one of these guitars – even the 8-string! Furthermore, as pitch is not being modeled here, with a few alterations of picking I could easily realize the network in Figure 5 on the ukulele – enacting an arrangement, of sorts, of the Villa-Lobos étude. These commonalities also underwrite more mundane acts, such as when one picks up a friend's guitar at an unplanned jam session. In short, yes, this is an abstraction that makes mathematical structures more robust, but it also captures something of the standardizations that subtend many everyday actions of fretted-string instrument players.

Table 1 Sample of different particular fretboard spaces.

Guitar	# of frets	# of strings
Mitchell Ukulele	16	4
Cordoba C10	19	6
PRS CE24	24	6
Carvin DC 7X	24	7
strandberg Boden 8	24 (fanned)	8

Leaving De Souza's fretboard theorizations intact, I now turn to formalizing the other half of the instrument/performer nexus: the fingers. Joti Rockwell (2009) formulates a group of transformations that represent the various actions of the picking hand's fingers in banjo performance (pp. 142-143).[15] The particular group is the symmetric group S_3 of permutations that is isomorphic to the dihedral group $\mathbb{D}_6$; both groups can be visualized as the symmetrical flips and rotations of a triangle.[16] Rockwell's formulation provides the ability to conceptualize picking as either an isolated action (i.e., a singular flip or rotation of the triangle) or as a continuous series (i.e., letting the triangle continuously oscillate due to flips or rotate freely like a top on a frictionless surface). These two conceptualizations provide a compelling correlate to the performative experience of playing a plucked instrument where it might make more sense to mentally fix an order of fingers and let them spin rather than micromanaging each individual motion (Rockwell, 2009, pp. 144–145).

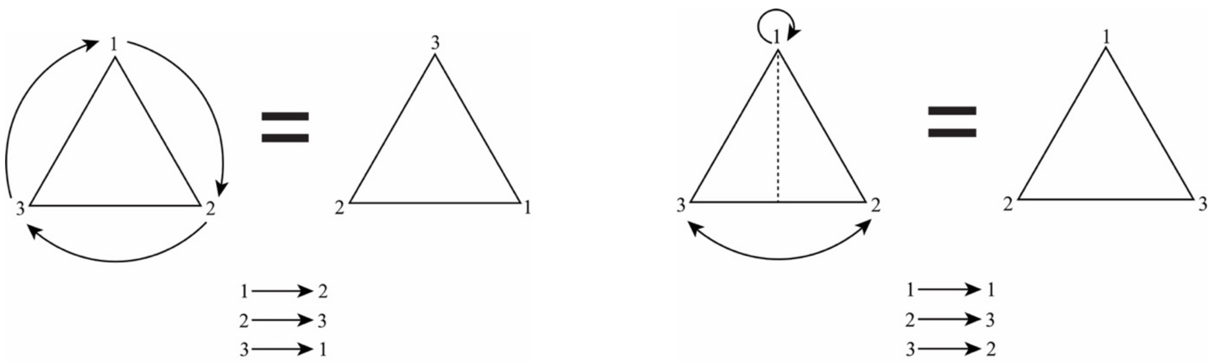
Banjo performance, however, only utilizes three picking fingers (p, i, m), while guitar performance typically uses four.[17] Furthermore, Rockwell's formulations are restricked to the picking hand – leaving, like De Souza, the fretting hand to merely reflect the fretboard. To that end, I propose using S_4 to model the permutations of four elements, here the picking and fretting fingers. Like Rockwell's triangles, S_4 can also

[15] For other musical theoretic utilizations of permutations, see: Harrison (1988); Hook (2007b); Callender et al. (2008); drawing from the previous study, Tymoczko (2011, pp. 36–45); and De Souza (2017, pp. 111–118).

[16] There are two ways to denote the dihedral group or order n: $\mathbb{D}_n$ or $\mathbb{D}_{n/2}$. The former emphasizes the number of transformations in the group while the latter reflects the number of sides of the n-gon that is subject to the rotations and flips of the group.

[17] The fourth finger is the ring finger (*a*) that was included in G above. The four-finger model to follow can also be used to explore the *hybrid picking* technique. Hybrid picking refers to the use of the thumb and first finger to hold a pick (plectrum) while the remaining fingers are utilized in a fingerstyle fashion. This technique allows for quick transitioning between more traditional flat-pick style and fingerstyle technique. To allow the theory to work, simply fuse the thumb and index finger (which hold the pick) and add the pinky as the fourth element. Playing with only a pick can be modeled as a two-element set that is acted on by $\mathbb{Z}_2$ (De Souza, 2017, p. 56; Hook, 2002, p. 62). The same holds for bowing a fretted instrument such as the viola da gamba. Both the gamba and flatpicking technique can fit into a modified G that replaces the four picking fingers with the set $\{+, –\}$.

be represented geometrically, namely as the twenty-four rotations of a cube about its axes of symmetry. Before getting carried away, I want to first unpack how the geometric visualizations operate in this context.



a) b)

Figure 7 Geometric manipulations of a triangle: a) rotation; b) flip.

Figure 7 presents two different manipulations of the triangle. These visualizations are intended to provide a tangible grounding of the permutational transformations acting on the fingers. Here we will only be concerned with two types: rotations and flips. Each vertex of the triangle is labeled with a number that, in our application, represents a finger. A rotation takes each finger and sends it to the next object in the cycle. In Figure 7a, the arrows around the triangle indicated where each individual object is being sent. Put more abstractly, we can think of this single clockwise rotation as the action that sends 1 to 2, 2 to 3, and 3 to 1, as shown below the triangle pairs. With flips we fold the object over a line of symmetry. As seen in Figure 7b, this action sends 2 to 3 and 3 to 2. Note that 1, though lying on the axis, is still involved in this permutation: *it is sent to itself* by the flip. As that finger remains unchanged, it will occasionally be useful to leave it out and proceed with fewer elements. With this in mind, let us return to the cube.

Figure 8 depicts all three types of a cube's axes of symmetry. Although complete, the manipulation of a cube can be a bit unwieldy and often presents more information than is needed for analyzing or performing a given example. To preserve the association with the mentality of the performer, we can separate S_4 (the twenty-four rotations of a cube about its axes of symmetry) into some of its subgroups. Rotating the cube about the axis running through the front and back faces (labeled z in Fig. 8a) produces the rotational subgroup of $\mathbb{D}_8$. A 180° rotation of either the axis shown in Figure 8b or one of the two remaining axes running through opposing faces in 8a provides the flip operation that, in conjunction with the rotations, generates the rest of the $\mathbb{D}_8$ actions. In this light, the cube is meerly a square stretched into a third dimension with the back side rotated 180°. However, this provides only eight of the twenty-four possible arrangements of four elements.

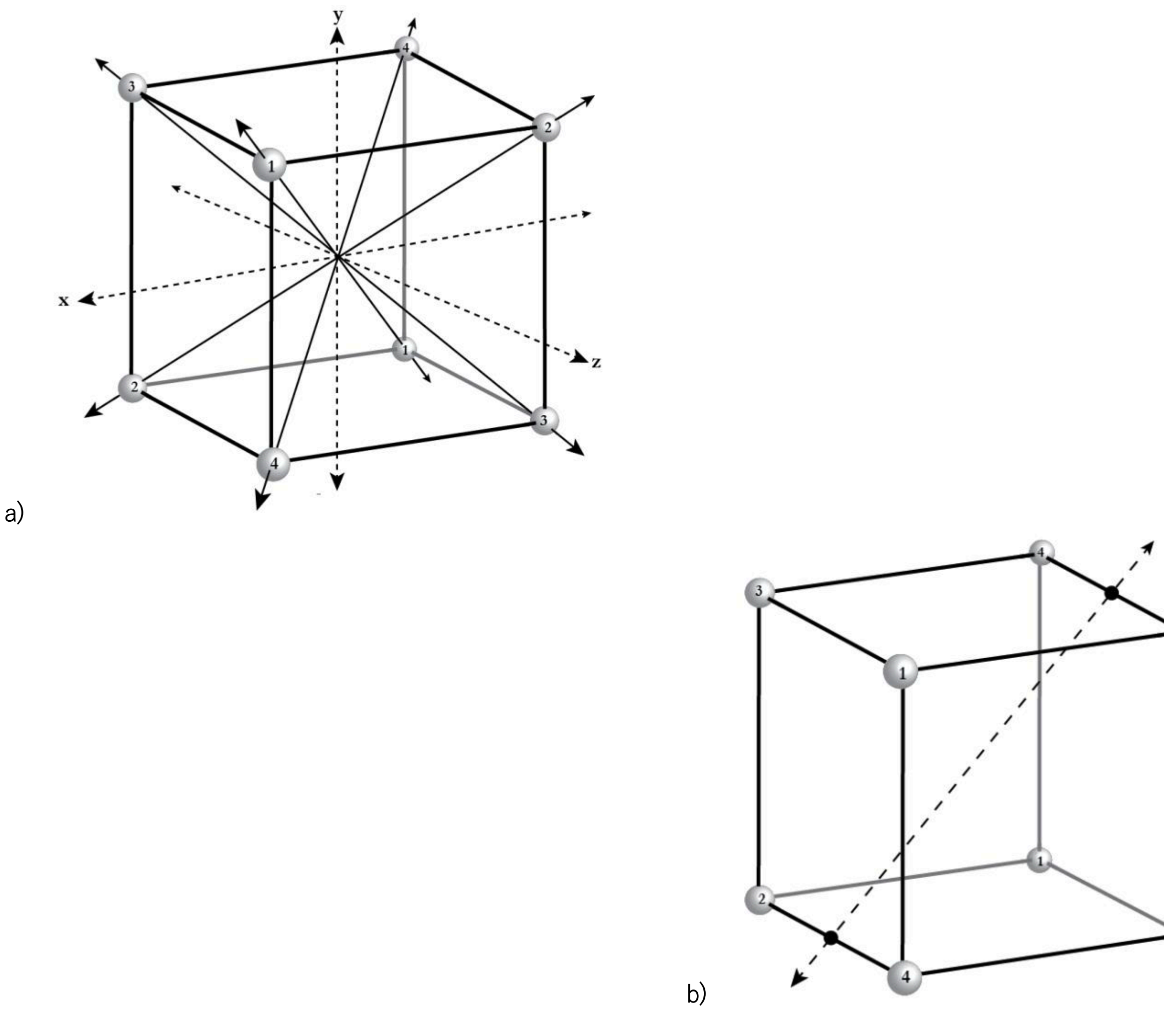


Figure 8 The cube's axes of symmetry: a) solid lines run through opposite vertices, dashed lines run through the center of opposing faces.; b) axis through midpoint of opposing edges (only one of six shown).

The other sixteen arrangements result from flattening the other two pairs of opposing sides of the cube. These other $\mathbb{D}_8$-inflected sets are not equivalent, though; as each of the three flattened-squares has a unique ordering of elements around its four vertices. Being more precise, and remembering that we are dealing with a set of objects and a group of actions, $\mathbb{D}_8$ remains the same group of actions in all of three collections. However, when acting on a set of four objects, it partitions them into three distinct *orbits*.[18] Figure 9 shows the orbits of $\mathbb{D}_8$ as distinct squares. In order to distinguish them, each orbit is labeled with a representative element; note that the ordering of four elements in the parentheses differ in each orbit. This representative can be read by traveling clockwise around each square, starting at the top. Each action is labeled with a greek letter and an integer. The greek letter indicates the orbit of objects on which it operates, while the integer denotes which action of $\mathbb{D}_8$ it enacts. For instance, the index 1 is the clockwise labeling of the vertices (see Table 2 for the other actions).

[18] Essentially, orbits result from running an element through every action in a subgroup and collecting all the results. Since there are eight transformations in the $\mathbb{D}_8$ subgroup, there are three orbits containing eight distinct objects, which, added together, reassemble the twenty-four possible permutations of four elements.

Although the guitarist is trained to use all four fingers, there are often musical examples that only utilize three of the picking hand fingers. Rockwell's triangle model functions perfectly, here, since $\mathbb{D}_6$ is yet another subgroup of S_4. Thinking back to our cube in Figure 8, rotation about any of the axes running through opposing verticies rotates the remaining verticies in the same fashion as the triangle. Figure 10 shows the four orbits generated by $\mathbb{D}_6$, again labeled with picking hand finger elements. Note that these 3-cycles do in fact have a fourth element. As they are members of S_4, they *must* be bijections from a set of four elements to itself. The fourth element in these groupings of 3 is the element through which the axis passes, that is, the axis element is sent to itself by the permutation.[19] There are four orbits of $\mathbb{D}_6$ because there are four elements that can be thusly fixed. As with the $\mathbb{D}_8$ orbits, all of the operations are labeled with a greek letter and an integer. For each of the two types of orbits, the integer represents the actual action, while the greek letter conveys the orbit being acted upon and their dispersal around the given shape.

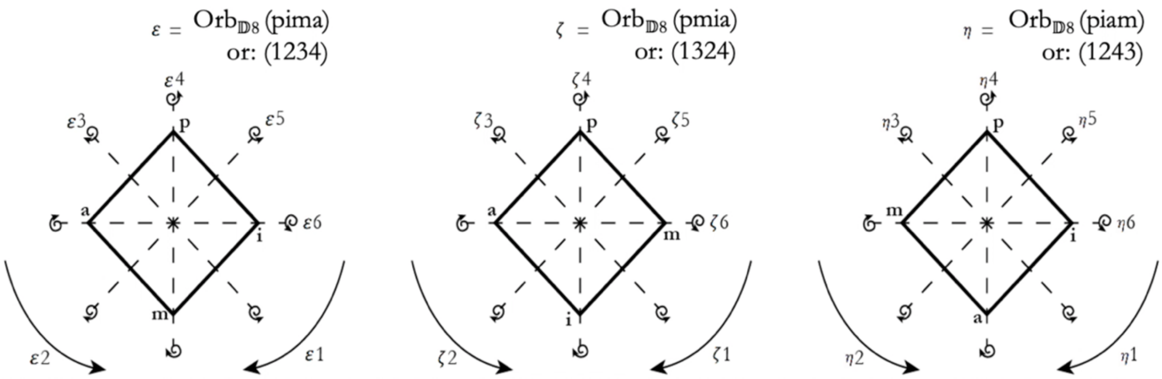


Figure 9 Orbits of $\mathbb{D}_8$.

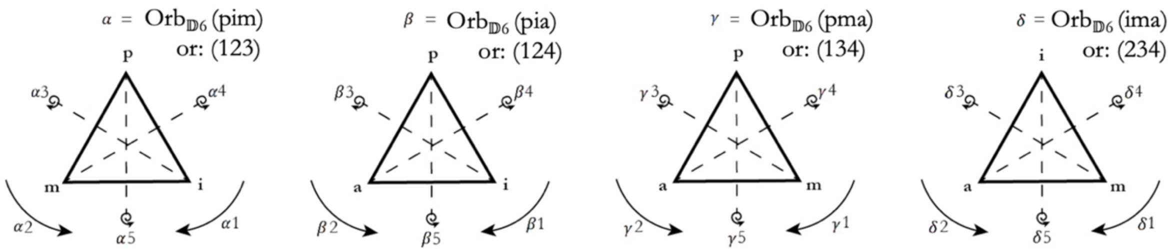


Figure 10 Orbits of $\mathbb{D}_6$.

[19] Concerning these "missing" elements (or 1-cycles), Dummit and Foote (2004) "adopt the convention that 1-cycles will not be written. Thus if some integer, *i*, does not appear in the cycle decomposition of a permutation τ, it is understood that $\tau(i)=i$, that is., that τ fixes *i*" (p. 31).

Table 2 shows a simplification of the subgroups' commonalities. The cyclic decompositions show the rotations in each permutation (the last element of each parenthesis wraps around to the first element). Furthermore, note that all 1-cycles (fixed elements) are omitted for clarity except for the identity transformations (x6 in $\mathbb{D}_6$ and x8 in $\mathbb{D}_8$), in which all elements are sent to themselves. This table fixes one ordering of each shape that places 1 on the top corner and continues the labeling by moving clockwise through the remaining corners. When given a particular greek letter, one simply substitutes the correct orbit shown above each shape in Figures 9 and 10 for the general ordering in Table 2. For example, γ1 is in Orb$\mathbb{D}_6$(pma). To adjust the table's labeling, substitute p in for every 1, m for 2, and a for 3. γ1 now apears as the cyclic permutation (pma).

Table 2 Cycle decomposition of dihedral subgroups

Subgroup type	Permutation	Cyclic Decomposition
$\mathbb{D}_6$ $(\alpha, \beta, \gamma, \delta)$	x1	(123)
	x2	(132)
	x3	(13)
	x4	(12)
	x5	(23)
	x6	(1)(2)(3) (i.e., e)
$\mathbb{D}_8$ $(\varepsilon, \zeta, \eta)$	x1	(1234)
	x2	(1432)
	x3	(14)(23)
	x4	(24)
	x5	(12)(34)
	x6	(13)
	x7	(13)(24)
	x8	(1)(2)(3)(4) (i.e., e)

Lastly, the rotations of each dihedral subgroup themselves form a subgroup isomorphic to the cyclic groups $\mathbb{Z}_3$ or $\mathbb{Z}_4$ for the triangles and squares, respectively. These are equivalent to the rotational portions of the dihedral subgroups, that is, all flips are omitted. Figure 11 shows geometric representations for these subgroups. As with the dihedral groups, these cyclic groups also have multiple orbits, which correspond to their given dihedral group's orbit. The cyclic group $\mathbb{Z}_3$ is latent in Rockwell's formulations. However, by excavating these cyclic groups we can visually represent the split between moment-to-moment movement and continuous cycling. For example, Rockwell's permutations label a single motion from p to i *and* the continuous alteration of these two fingers as α4. I propose using directed integers *modn*, that is, ($\mathbb{Z}_n$, +), for single alterations and preserving the dihedral permutations for continuous alternation.

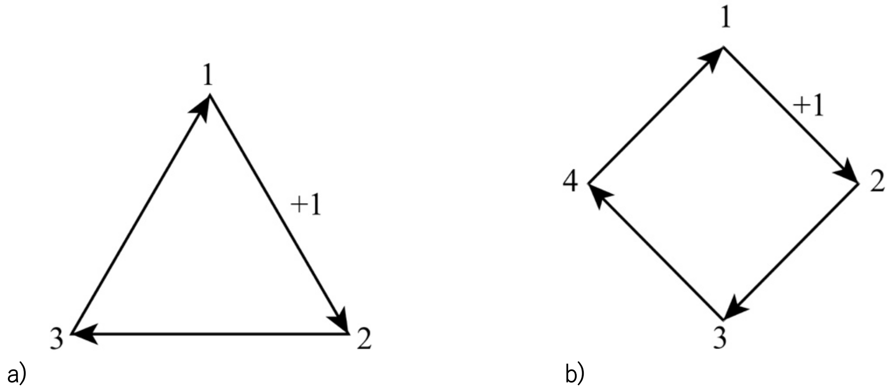


Figure 11 Cyclic subgroups $\mathbb{Z}_3$ (a) and $\mathbb{Z}_4$ (b).

Analytical applications

Turning to some applications, I will first show how the fingers can both reinforce and deviate from fretboard transformations, thus showing their independence, before turning to a brief analysis of Ben Monder's solo guitar work, *Windowpane*. Figure 12a presents an annotated excerpt from Villa-Lobos' fourth prelude for guitar. Note that ε1 drives the picking hand throughout the excerpt (refer to Figs. 9 and 10 and/or Table 2 to decipher Greek-letter labels). The transformational network just above the notated score uses a variant of De Souza's (2017) contextual fretboard transformations (pp. 83–108).[20] S_n (Shift) represents a shift along the fret dimension where *n* is a directed distance traversed. Note that this S_n (without italics) is different from S_n, the symmetric group of order *n*. K is the contextual transformation that moves the middle fretted note up one fret; K's inverse (K^{-1}) sends the middle node back down one fret. The two transformations, S_n and K, commute.

Figure 12b shows the chord shapes used in this excerpt and how they are related via K. Above the bold line of Figure 12a, I listed all possible fingerings that maintain the same fingers on the outer fretted notes of the chord shape through the entire excerpt; note that the ordered pairs now show string and fretting finger, respectively. Surprisingly, to me, the four fingerings that require no exchanging of fingers are *less* comfortable than the one that does require action. Although requiring action, the fingering of (a) maintains the same relative position of the hand throughout: as the chord shape fluctuates via the K operations, the fingers remain positioned over all three frets used between the shapes. The other fingerings require the performer to continually adjust the relative position of the fingers, which is exacerbated by the S_n shifts. Looking at (a)'s transition from the second chord to the third (the first K transformation), the second finger of the hand is not engaged with the fretboard directly – thus, entirely outside the realm of fretboard transformation – but it hovers just behind fingers 3 and 4, right over its (relative) fretted position for the following chord. The same holds for motion from the third to fourth chords but with finger 1. Thus, the transformation ζ6 represents the mapping that fixes fingers 3 and 4 on the fretboard and transfers pressure from 1 or 2 (and vice versa). In this excerpt, the fretboard and fingers work together in creating idiomatic gestures.

[20] For a more general overview of fretboard transformations see De Souza (2018, pp. 25–31).

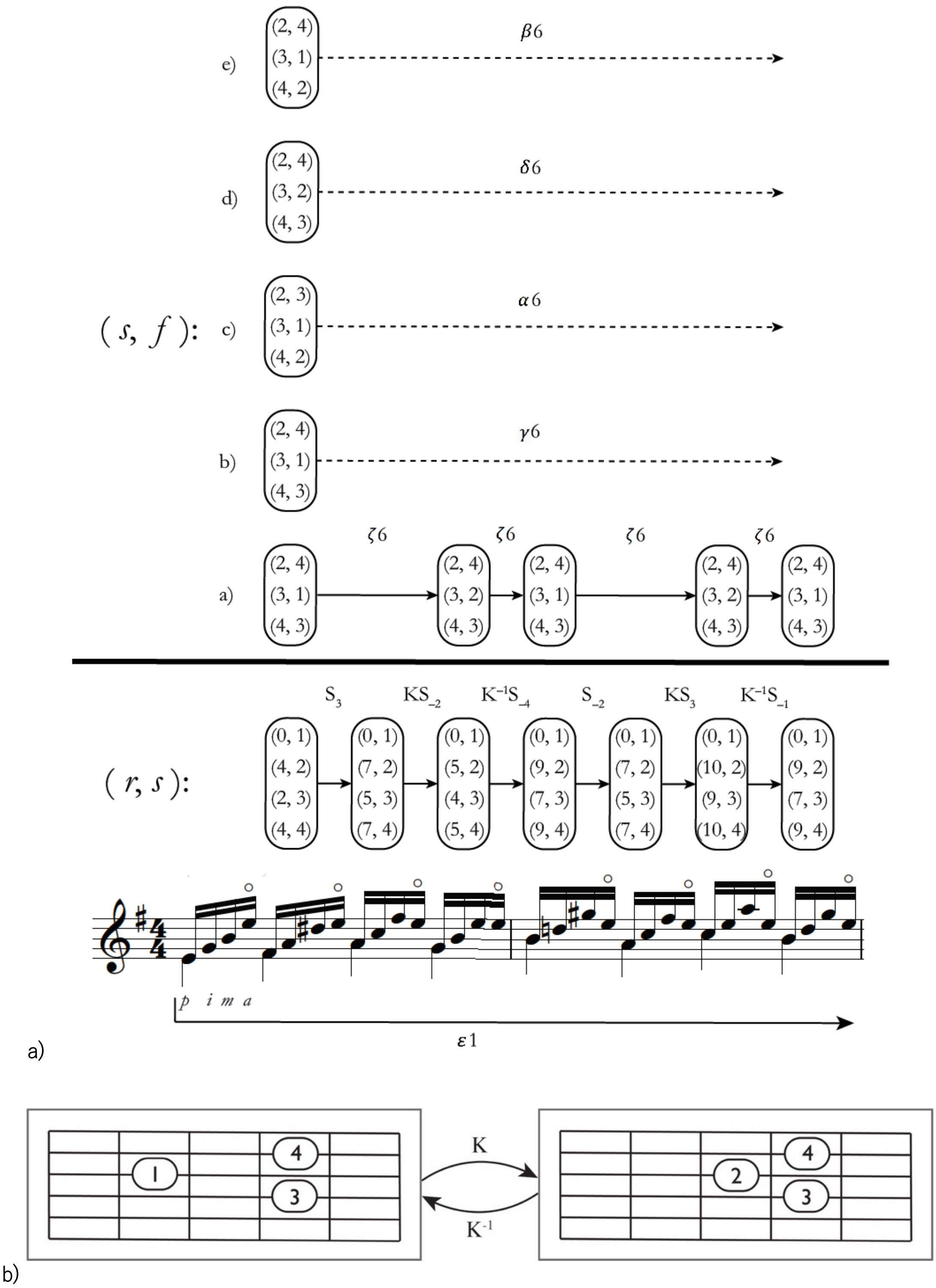


Figure 12 a) Transformational networks (r: fret; s: string; f: fretting finger) and b) chord shapes in an excerpt of Villa-Lobos' Prelude No. 4 in E minor (1940; mm. 11–12). Score excerpt adapted from *Heitor Villa-Lobos: Collected works for solo guitar* (p. 86) by H. Villa-Lobos, 1990, Editions Max Eschig.

Featured on his 2013 album, *One*, Jonathan Kreisberg's solo guitar arrangement of Leonard Cohen's classic *Hallelujah* offers a related though differing example in which the fretboard and fingers *diverge* rather than reinforce one another (Fig. 13a). As in the Villa-Lobos, Kreisberg alternates between two related chord shapes, shown in Figure 13b. The contextual transformation L simultaneously takes the fretted note of the lowest string up one fret and the fretted note of the second-to-lowest string down one fret. ζ6 once again acts on the fingers. In the Villa-Lobos prelude, the fretting finger change was within the same voice, or on the same string. Here, however, the fingers take a different path than the L transformation, as shown in Figure 14. Whereas the fretboard perspective highlights parsimonious motions along the string, the finger perspective not only travels in the opposition direction, but also incorporates a string change. Here the fretboard and fingers are divergent but complimentary takes on the same chord change – each adding to the progression's particular character.

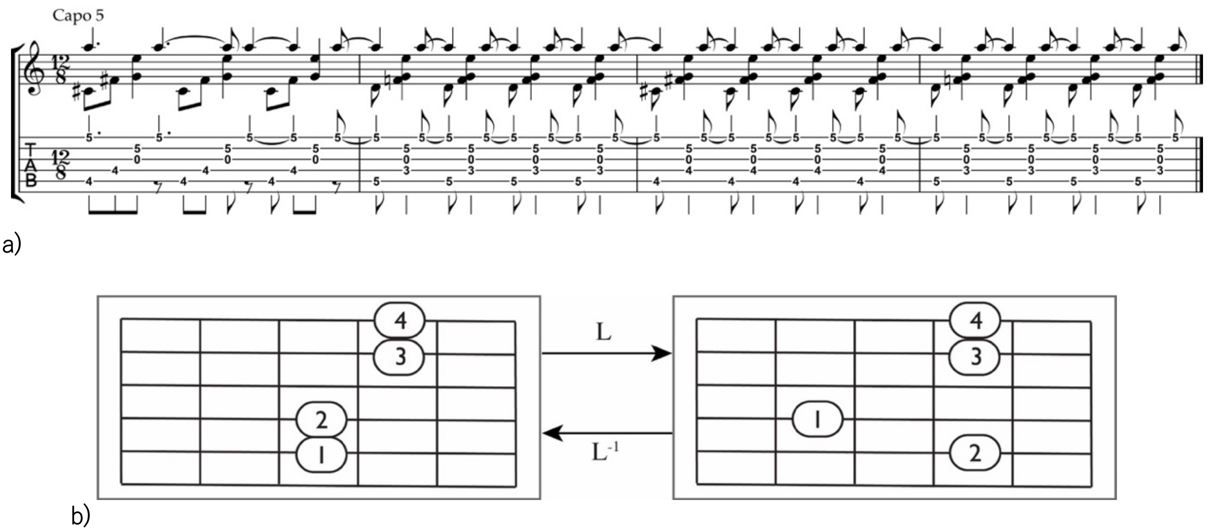


Figure 13 a) Transcription and b) chord shapes (alternated every measure) in *Hallelujah* of Jonathan Kreisberg's album *One* (0:37–0:51).

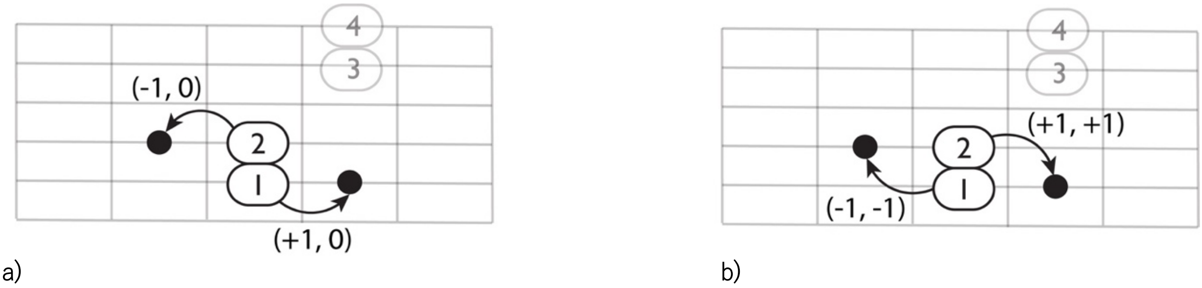


Figure 14 Diverging perspectives in Jonathan Kreisberg's *Hallelujah* (from the album *One*): a) fretboard perspective; b) finger perspective. In parentheses: (fret motion, string motion).

Lastly, Ben Monder's *Windowpane* offers an opportunity to explore how smooth finger motions can mediate between unconventional chord shapes that are less amenable to fretboard transformation. This work consists almost entirely of rapid arpeggiations. As such, picking patterns are integral in articulating the two formal sections that alternate throughout the work; Figure 15 shows these two patterns in a network of ordered pairs (picking finger, string). The A section, Figure 15a, takes a zig-zag path that covers all six strings while alternating thumb articulations with the index and middle fingers. The first four articulations establish a pattern that is shifted down two strings by (e, -2). The last, falling gesture alters the string component of the transformation while maintaining the same finger pattern. The picking pattern of the B section, conversely, moves directly through both finger and string space in opposite directions, which limits the range covered in string space, while exhausting all elements in finger space (Fig. 15b). Note that the angular journey in Figure 15a led me to use directed integers (the $\mathbb{Z}_n$ subgroup) in the transformations. The journey in Figure 15b, however, lends itself to the dihedral interpretation, capturing the continuity of the cycle.

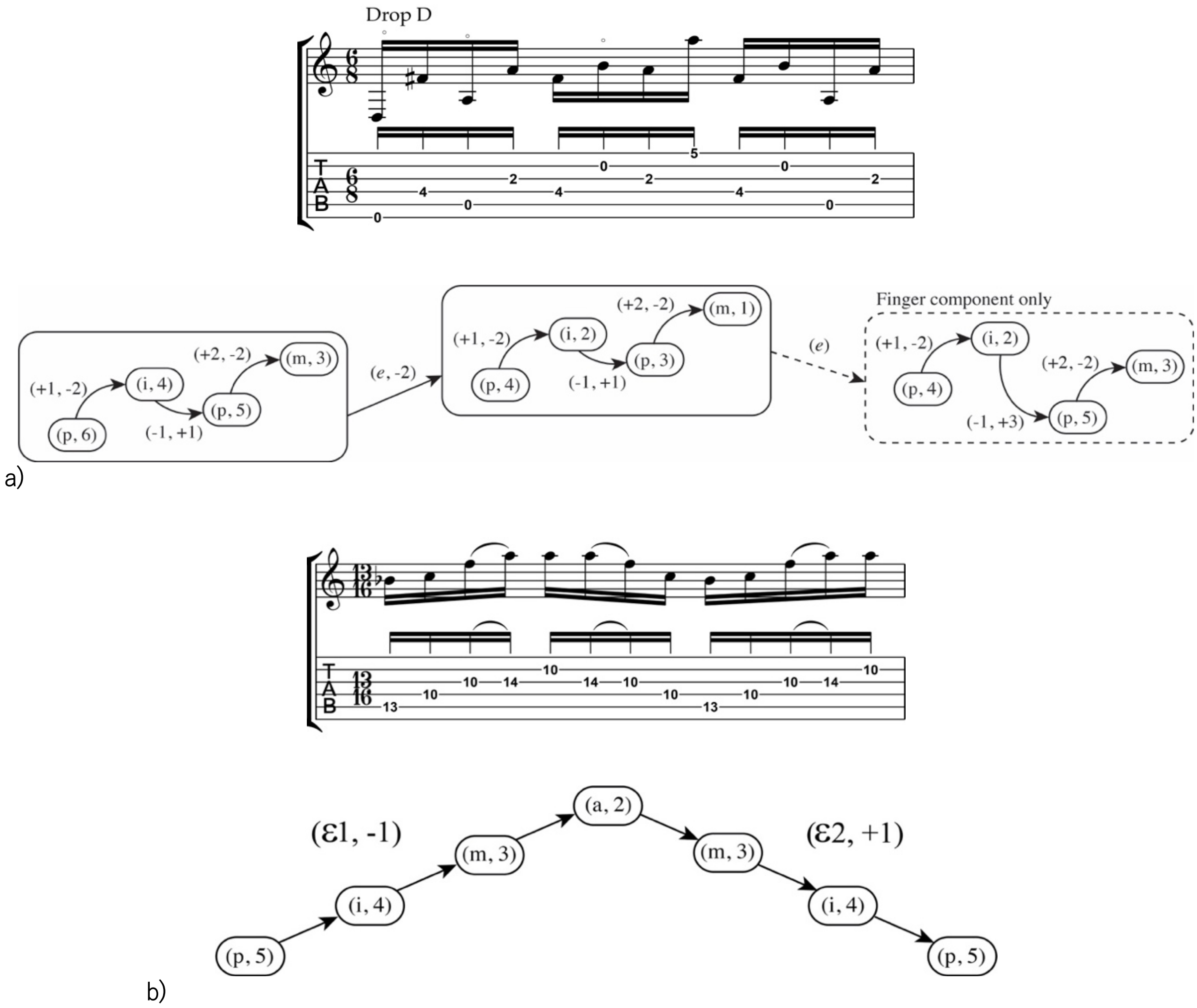
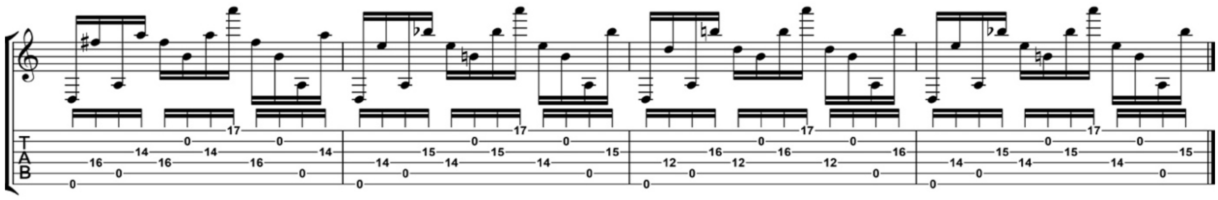
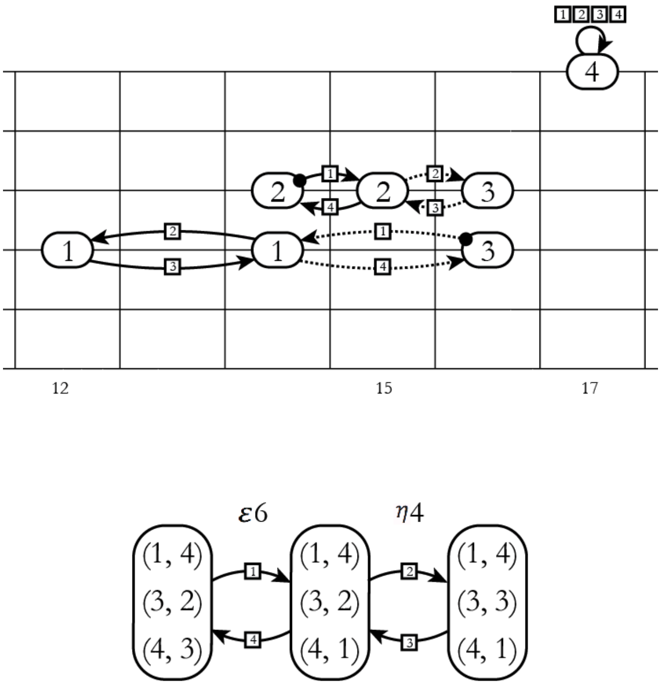


Figure 15 Patterns in score excerpts of Ben Monder's *Windowpane* (2006) and in their transformational network (picking finger, string): a) A section (m. 1); b) B section (m. 417). Score excerpts adapted from *Ben Monder Compositions* (pp. 240 and 254, respectively) by B. Monder, 2008, Mel

Turning now to the fretting hand, Figure 16 presents a pathway traversed on the fretboard that is emblematic of the A section as a whole. Throughout the example, Monder always keeps two fingers on the fretboard. The first and fourth motions map fingers 2 and 4 onto themselves and exchanging fingers 1 and 3 via $\varepsilon 6$. With finger 3 now freed from the fretboard, the second and third motions map fingers 1 and 4 onto themselves, while switching fingers 2 and 3 (via $\eta 4$). Thinking back the commonalities between transformations (Table 2), $\varepsilon 6$ and $\eta 4$ are both permutations that fix two elements and sent the other two onto each other.[21] The resultant actions exhibit economic finger motions that provide a degree of smoothness in performance that persists alongside the smooth motions in pitch space.



a)

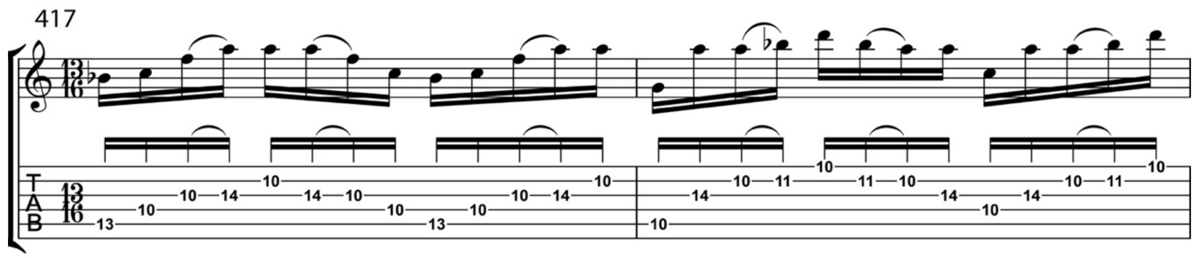


b)

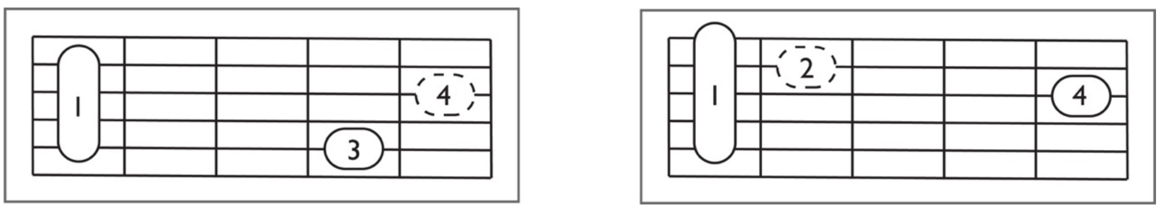
Figure 16 Fretting hand in A section of Ben Monder's *Windowpane*. a) score, m. 45–48; b) fretboard visualization (above) and transformational network (string, picking finger) (below). Score excerpt adapted from *Ben Monder Compositions* (p. 241) by B. Monder, 2008, Mel Bay.

[21] Indeed, the metaphorical connection underwriting the orbit labeling is strained, here. I have maintained the Greek-letter notation for consistency, but note that reverting to the cycle notation of Table 2 is equally valid in such cases.

An exemplar from the B section provides a different approach to the fretting fingers. Instead of focusing on along the string connections, this example explores what *function* the fingers play in a given section. Figure 17 provides a score as well as the two different chord shapes that Monder utilizes in this excerpt. In both chords, each finger serves a specific function: the finger 1 holds the barré (i.e., maps onto itself), some finger is firmly planted, some finger is off the fretboard, and the last finger functions as the initiator of hammer-ons and pull-offs. Monder's first pass through these chords is a direct shift that rotates the three non-barred fingers, not to the string the resultant fingers inhabit, but to the *function* those fingers played in the previous chord shape. This first pass is shown in the top transformational network of Figure 18. After repeating this alternation four times, Monder adds a passing motion on the lowest string, shown in Figure 19. This passing fret requires a change of finger that subdivides the rotation show in the top network of Figure 18 into two stages that preserve a common finger/function node between them, as shown in the lower network. Although this second figuration requires an extra step, it maintains a common thread that helps conceptually mediate the ungrounded rotation of the first pass.



a)



b)

Figure 17 a) Score excerpt of Ben Monder's *Windowpane* and b) its chord shapes. Dashed-line boxes indicate hammer-on/pull-off fingers. Score excerpt adapted from *Ben Monder Compositions* (p. 254) by B. Monder, 2008, Mel Bay.

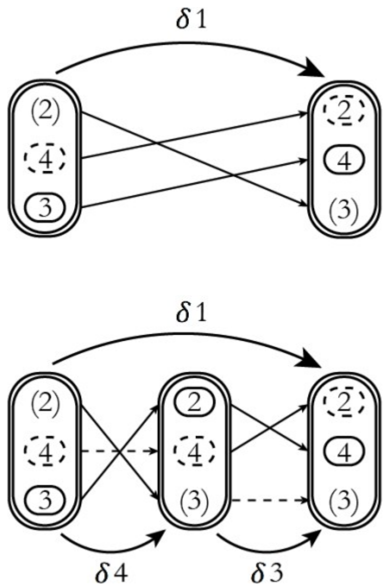
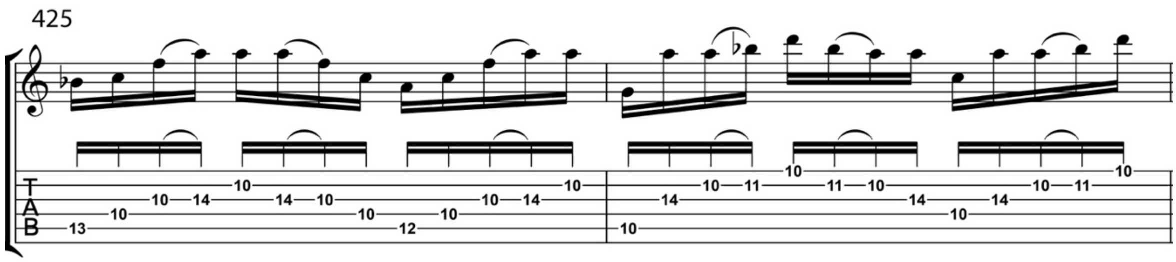
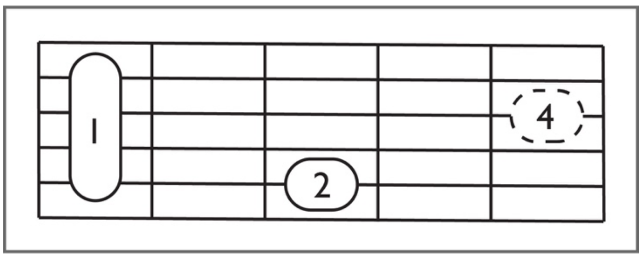


Figure 18 Network of finger functions where parenthesis represent a finger off the fretboard; a solid rectangle represents a finger firmly planted on the fretboard; and dashed rectangle represents a hammer-on/pull-off of the finger.



a)



b)

Figure 19 a) Score excerpt of Ben Monder's *Windowpane* and b) its chord shapes. Dashed-line boxes indicate hammer-on/pull-off fingers. Score excerpt adapted from *Ben Monder Compositions* (p. 254) by B. Monder, 2008, Mel Bay.

Conclusion

The types of motion that De Souza's fretboard transformational model captures best are staples of fretted instrument performance that don't require much, if any, finger gymnastics. However, the wealth of literature for the guitar contains just as many moments of compromise where abstract music theoretic principles or stylistic norms limit such a blatant traversal of the instrument's topology. Monder's *Windowpane*, for instance, highlights moments of idiomacity that are irreducible to the fretboard alone. Although still graspable with fretboard transformations, Monder's famous penchant for unusual chord voicings renders such an account a bit clunky and unintuitive – betraying the sense of continuity afforded by economic finger usage. The fretboard certainly exerts a tremendous influence on composition and performance; the theorizations presented here are intended to complement and deepen, rather than critique, the research already done in this area. However, by also attending to the fingers formally, we can model smooth coordination between body and instrument that transcend explicit capitalization of the fretboard's logic.

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