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A DYNAMICS FOR
DISCRETE QUANTUM GRAVITY

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Abstract

This paper is based on the causal set approach to discrete quantum gravity. We first describe a classical sequential growth process (CSGP) in which the universe grows one element at a time in discrete steps. At each step the process has the form of a causal set (causet) and the "completed" universe is given by a path through a discretely growing chain of causets. We then quantize the CSGP by forming a Hilbert space H on the set of paths. The quantum dynamics is governed by a sequence of positive operators ρ_n on H that satisfy normalization and consistency conditions. The pair $(H, \{\rho_n\})$ is called a quantum sequential growth process (QSGP). We next discuss a concrete realization of a QSGP in terms of a natural quantum action. This gives an amplitude process related to the "sum over histories" approach to quantum mechanics. Finally, we briefly discuss a discrete form of Einstein's field equation and speculate how this may be employed to compare the present framework with classical general relativity theory.

1 Introduction

This paper builds on the causal set approach to discrete quantum gravity [1, 10] and we refer the reader to [7, 12] for more details and motivation. The origins of this approach stem from studies of the causal structure $(M, <)$ of a Lorentzian space-time manifold (M, g). For $a, b \in M$ we write $a < b$ if b is in the causal future of a. If $a \le b$ or $b \le a$ we say that a and b are *comparable* and otherwise a and b are *incomparable*. If there are no closed causal curves in (M, g), then $(M, <)$ is a partially ordered set (poset). It has been shown that $(M, <)$ determines the topology and even the differential structure of

the manifold (M, g) [10, 12] and it is believed that the order structure $(M, <)$ provides a viable candidate for describing a discrete quantum gravity.

To remind us that we are dealing with a causal structure, we call a finite poset a *causet*. A causet is assumed to be unlabeled and isomorphic causets are identified. In the literature, causets are frequently labeled according to the order of "birth" and this causes complications because covariant properties are independent of labeling [7, 10, 12]. In this way, our causets are automatically covariant.

Section 2 describes a classical sequential growth process in which the universe grows one element at a time in discrete steps. At each step, the process has the form of a causet and the "completed" universe is given by a path through a discretely growing chain of causets. The transition probability at each step is given by an expression due to Rideout-Sorkin [8, 13]. Letting Ω be the set of paths, $\mathcal{A}$ be the σ-algebra generated by cylinder sets and ν_c the probability measure determined by the transition probabilities, the classical dynamics is described by a Markov chain in the probability space $(\Omega, \mathcal{A}, \nu_c)$.

In Section 3 we quantize the classical framework by forming the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$. The quantum dynamics is governed by a sequence of positive operators ρ_n on H that satisfy normalization and consistency conditions. We employ ρ_n to construct decoherence functionals and a quantum measure μ on a "quadratic algebra" $\mathcal{S}$ of subsets of Ω. In general, $\mathcal{S}$ is between the collection of cylinder sets and $\mathcal{A}$. We then nominate $(\Omega, \mathcal{S}, \mu)$ as a candidate model for a discrete quantum gravity.

Section 4 discusses a concrete realization of the quantum sequential growth process ρ_n considered in Section 3. This realization is given in terms of a natural quantum action and is called an amplitude process. The amplitude process is related to the "sum over histories" approach to quantum mechanics [7, 12]. Section 5 briefly discusses a discrete form of Einstein's field equation and speculates how this may be employed to compare the present framework with classical general relativity theory.

2   Classical Sequential Growth Processes

Let $\mathcal{P}_n$ be the collection of all causets of cardinality n, $n = 1, 2, \ldots$, and let $\mathcal{P} = \cup \mathcal{P}_n$ be the collection of all causets. An element $a \in x$ for $x \in \mathcal{P}$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x *produces* y if y is obtained from x by adjoining a single maximal element a to x. In this case we write $y = x \uparrow a$ and use the notation $x \to y$. If

$x \to y$, we also say that x is a *producer* of y and y is an *offspring* of x. Of course, x may produce many offspring and a causet may be the offspring of many producers. Moreover, x may produce y in various isomorphic ways. To describe this precisely, we consider labeled causets where a *labeling* of $x \in \mathcal{P}_n$ is a function $\ell \colon x \to \{1, 2, \ldots, n\}$ such that $a, b \in x$ with $a < b$ implies $\ell(a) < \ell(b)$. In Figure 1, the labeled causet x produces the labeled causets u, v, w. In this paper we mainly consider unlabeled causets (which we simply call causets) and identify isomorphic copies of a causet so we identify u, v, w and say the *multiplicity* of $x \to u$ is three and write $m(x \to u) = 3$. We then replace Figure 1 by the simpler Figure 2.
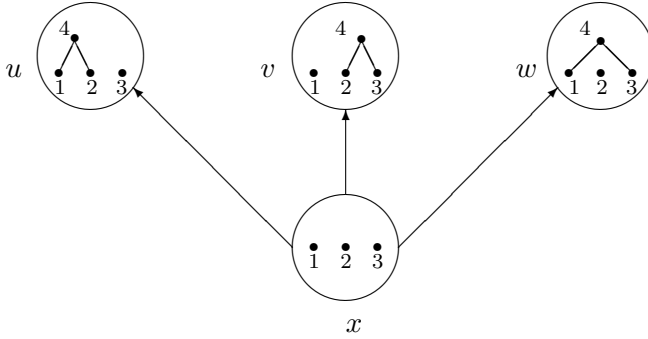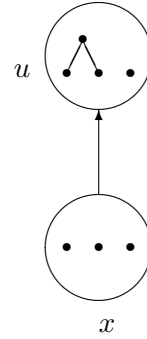
Figure 1   **Figure 2**

The transitive closure of $\to$ makes $\mathcal{P}$ into a poset itself and we call $(\mathcal{P}, \to)$ the *causet growth process*. A *path* in $\mathcal{P}$ is a sequence (string) $\omega_1 \omega_2 \cdots$, where $\omega_i \in \mathcal{P}_i$ and $\omega_i \to \omega_{i+1}$, $i = 1, 2, \ldots$. An *n-path* in $\mathcal{P}$ is a finite string $\omega_1 \omega_2 \cdots \omega_n$ where, again, $\omega_i \in \mathcal{P}$ and $\omega_i \to \omega_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n.

If $a, b \in x$ with $x \in \mathcal{P}$, we say that a is an *ancestor* of b and b is a *successor* of a if $a < b$. We say that a is a *parent* of b and b is a *child* of a if $a < b$ and there is no $c \in x$ with $a < c < b$. A *chain* in x is a set of mutually comparable elements of x and an *antichain* in x is a set of mutually incomparable elements of x. By convention, the empty set is both a chain and an antichain and clearly singleton sets also have this property. In Figure 2, any subset of x is an antichain while u has two chains of cardinality 2.

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Theorem 2.1. *If x is a labeled causet, then the number of labeled offspring of x is the number of distinct antichains in x.*

Proof. Let x have cardinality n and suppose $A = \{a_1, \ldots, a_k\}$ is an antichain in x. Let $b \notin x$ and form the labeled causet $y = x \uparrow b$ where y inherits the labeling of x, $\ell(b) = n + 1$ and the elements of A are the parents of b. Then y is a labeled offspring of x and different antichains give different labeled offspring because they give different parents of b. Conversely, if $y = x \uparrow b$ is a labeled offspring of x and $A = \{a_1, \ldots, a_k\}$ is the set of parents of b, then A is an antichain because $a_i < a_j$ contradicts the fact that a_i is a parent of b. Hence, we have a bijection between the antichains in x and labeled offspring of x. $\qquad\square$

We denote the cardinality of a set A by $|A|$.

Corollary 2.2. *The number r of labeled offspring of a labeled causet x satisfies $|x| + 1 \le r \le 2^{|x|}$ and both bounds are achieved.*

Corollary 2.3. *The number of offspring of a causet x, including multiplicity, is the number of distinct antichains in x.*

Corollary 2.4. *The number r of offspring of a causet x, including multiplicity, satisfies $|x| + 1 \le r \le 2^{|x|}$ and both bounds are achieved.*

For example, the causet x in Figure 2 has eight distinct antichains so x has eight offspring, including multiplicity. The causet u in Figure 2 has ten distinct antichains so u has ten offspring, including multiplicity.

Let $c = (c_0, c_1, \ldots)$ be a sequence of nonnegative numbers (called *coupling constants*) [9, 13]. For $r, s \in \mathbb{N}$ with $r \le s$ define

$$\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s - r}{k - r} c_k = \sum_{k=0}^{s-r} \binom{s - r}{k} c_{r+k}$$

For $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$ with $y = x \uparrow a$ define the transition probability [9, 13]

$$p_c(x \to y) = m(x \to y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}$$

where α is the number of ancestors and π the number of parents of a. By convention we define $p_c(x \to y) = 0$ if $x \not\to y$. It is shown in [13] that $p_c(x \to y)$ is a probability distribution in that it satisfies the Markov-sum rule $\sum p_c(x \to y) = 1$. The distribution $p_c(x \to y)$ is essentially the most general that is consistent with principles of causality and covariance

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[8, 13]. It is hoped that other theoretical principles or experimental data will determine the coupling constants One suggestion is to take $c_k = 1/k!$ [10]. Another is the *percolation dynamics* $c_k = c^k$ for some $c > 0$ [12]. For this later choice we have the simple form

$$p_c(x \to y) = m(x \to y)r^\pi (1 - r)^\beta$$

where β is the number of incomparable elements to a (other than a itself) and $r = c(1 + c)^{-1}$ [4].

We view a causet $x \in \mathcal{P}_n$ as a possible universe at step n while a path may be viewed as a possible (evolved) universe. The set $\mathcal{P}$ together with the set of transition probabilities $p_c(x \to y)$ forms a *classical sequential growth process* (CSGP) which we denote by $(\mathcal{P}, p_c)$ [3, 8, 13]. It is clear that $(\mathcal{P}, p_c)$ is a Markov chain. As with any Markov chain, the probability of an n-path $\omega = \omega_1 \omega_2 \cdots \omega_n$ is

$$p_c^n(\omega) = p_c(\omega_1 \to \omega_2)p_c(\omega_2 \to \omega_3)\cdots p_c(\omega_{n-1} \to \omega_n)$$

Of course, $\omega \mapsto p_c^n(\omega)$ is a probability measure on Ω_n. Figure 3 illustrates the first two steps of a CSGP. It follows from Corollary 2.3 that the number of offspring including multiplicity of x_4, x_5, x_6, x_7 and x_8 are $4, 5, 6, 5, 8$, respectively. In this case, Corollary 2.4 tells us that $4 \leq r \leq 8$.
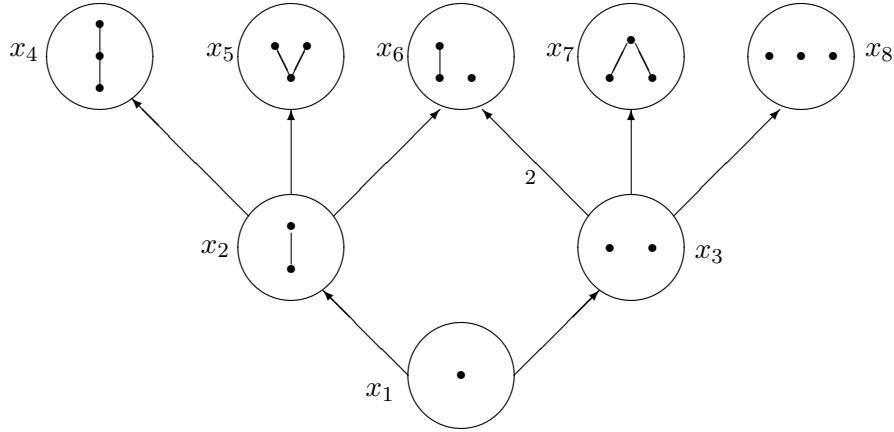
Figure 3

The set of all paths beginning with $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$ is called an *elementary cylinder set* and is denoted by $\mathrm{cyl}(\omega)$. If $A \subseteq \Omega_n$, then the *cylinder set* $\mathrm{cyl}(A)$ is defined by

$$\mathrm{cyl}(A) = \bigcup_{\omega \in A} \mathrm{cyl}(\omega)$$

Using the notation

$$\mathcal{C}(\Omega_n) = \{\mathrm{cyl}(A) \colon A \subseteq \Omega_n\}$$

we see that

$$\mathcal{C}(\Omega_1) \subseteq \mathcal{C}(\Omega_2) \subseteq \cdots$$

is an increasing sequence of subalgebras of the *cylinder algebra* $\mathcal{C}(\Omega) = \cup \mathcal{C}(\Omega_n)$. For $A \subseteq \Omega$ we define the set $A^n \subseteq \Omega_n$ by

$$A^n = \{\omega_1 \omega_2 \cdots \omega_n \in \Omega_n \colon \omega_1 \omega_2 \cdots \omega_n \omega_{n+1} \cdots \in A\}$$

That is, A^n is the set of n-paths that can be continued to a path in A. We think of A^n as the n-step approximation to A.

For $A = \mathrm{cyl}(A_1) \in \mathcal{C}(\Omega_n)$, $A_1 \subseteq \Omega_n$ define $\widehat{p}_c(A) = p_c^n(A)$. Notice that $\widehat{p}_c$ becomes a well-defined probability measure on the algebra $\mathcal{C}(\Omega)$. By the Kolmogorov extension theorem, $\widehat{p}_c$ has a unique extension to a probability measure ν_c on the σ-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space and the restriction $\nu_c \,|\, \mathcal{C}(\Omega_n) = \widehat{p}_c$.

3   Quantum Sequential Growth Processes

We now show how to "quantize" the CSGP $(\mathcal{P}, p_c)$ to obtain a quantum sequential growth process (QSGP). Letting $H = L_2(\Omega, \mathcal{A}, \nu_c)$ be the *path Hilbert space* we see that the *n-path Hilbert spaces* $H_n = L_2(\Omega, \mathcal{C}(\Omega_n), \widehat{p}_c)$ form an increasing sequence $H_1 \subseteq H_2 \subseteq \cdots$ of closed subspaces of H. Let

$$\Omega_n' = \{\omega \in \Omega_n \colon p_c^n(\omega) \neq 0\}$$

be the set of n-paths with nonzero measure. For $\omega \in \Omega_n'$, letting

$$e_\omega^n = \chi_{\mathrm{cyl}(\omega)} / p_c^n(\omega)^{1/2}$$

it is clear that $\{e_\omega^n \colon \omega \in \Omega_n'\}$ forms an orthonormal basis for H_n.

Letting $1 = \chi_\Omega$ we see that 1 is a unit vector in H. More generally, if $A \in \mathcal{C}(\Omega)$ then the characteristic function $\chi_s \in H$ with $\|\chi_s\| = \nu_c(A)^{1/2}$. If T is an operator on H_n, we shall assume that T is also an operator on H by

defining $Tf = 0$ for all $f \in H_n^\perp$. A *probability operator* on H_n is a positive operator ρ on H_n that satisfies the normalization condition $\langle \rho 1, 1 \rangle = 1$. If ρ is a probability operator on H_n, we define the *decoherence functional* $D_\rho \colon \mathcal{C}(\Omega_n) \times \mathcal{C}(\Omega_n) \to \mathbb{C}$ by

$$D_\rho(A, B) = \mathrm{tr}\,(\rho |\chi_B\rangle \langle \chi_A|) = \langle \rho \chi_B, \chi_A \rangle$$

It is easy to check that D_ρ has the usual properties of a decoherence functional. Namely, $D_\rho(\Omega, \Omega) = 1$, $D_\rho(A, B) = \overline{D_\rho(B, A)}$, $A \mapsto D_\rho(A, B)$ is a complex measure on $\mathcal{C}(\Omega_n)$ and if $A_i \in \mathcal{C}(\Omega_n)$, $i = 1, \dots, r$, then the $r \times r$ matrix with components $D_\rho(A_i, A_j)$ is positive semidefinite. We interpret $D_\rho(A, B)$ as a measure of the interference between the events A and B when the system is described by ρ. We also define the *q-measure* $\mu_\rho \colon \mathcal{C}(\Omega_n) \to \mathbb{R}^+$ by $\mu_\rho(A) = D_\rho(A, A)$ and interpret $\mu_\rho(A)$ as the quantum propensity of the event A. In general, μ_ρ is not additive so μ_ρ is not a measure on $\mathcal{C}(\Omega_n)$. However, μ_ρ is *grade-2 additive* [2, 9, 10] in the sense that if $A, B, C \in \mathcal{C}(\Omega_n)$ are mutually disjoint, then

$$\mu_\rho(A \cup B \cup C)$$
$$= \mu_\rho(A \cup B) + \mu_\rho(A \cup C) + \mu_\rho(B \cup C) - \mu_\rho(A) - \mu_\rho(B) - \mu_\rho(C) \quad (3.1)$$

A subset $\mathcal{Q} \subseteq \mathcal{A}$ is a *quadratic algebra* if $\emptyset, \Omega \in \mathcal{Q}$ and if $A, B, C \in \mathcal{Q}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$, then $A \cup B \cup C \in \mathcal{Q}$. A *q-measure* on a quadratic algebra $\mathcal{Q}$ is a map $\mu \colon \mathcal{Q} \to \mathbb{R}^+$ satisfying (3.1) whenever, $A, B, C \in \mathcal{Q}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$. In particular $\mathcal{C}(\Omega_n)$ is a quadratic algebra and $\mu_\rho \colon \mathcal{C}(\Omega_n) \to \mathbb{R}^+$ is a q-measure in this sense.

Let ρ_n be a probability operator on H_n, $n = 1, 2, \dots$, which we view as a probability operator on H. We say that the sequence ρ_n is *consistent* if

$$D_{\rho_{n+1}}(A, B) = D_{\rho_n}(A, B)$$

for every $A, B \in \mathcal{C}(\Omega_n)$. We call a consistent sequence ρ_n a *discrete quantum process* and we call $(H, \{\rho_n\})$ a *quantum sequential growth process* (QSGP).

Let $(H, \{\rho_n\})$ be a QSGP. If $C \in \mathcal{C}(\Omega)$ has the form $C = \mathrm{cyl}(A)$, $A \in \mathcal{C}(\Omega_n)$, we define $\mu(C) = \mu_{\rho_n}(A)$. It is easy to check that μ is well-defined and gives a q-measure on $\mathcal{C}(\Omega)$. In general, μ cannot be extended to a q-measure on $\mathcal{A}$, but it is important to extend μ to other physically relevant sets [2, 11]. We say that a set $A \in \mathcal{A}$ is *suitable* if $\lim \mathrm{tr}\,(\rho_n |\chi_A\rangle \langle \chi_A|)$ exists and is finite and if this is the case, we define $\widetilde{\mu}(A)$ to be the limit. We denote the collection of suitable sets by $\mathcal{S}(\Omega)$ and it is shown in [4] that $\mathcal{S}(\Omega)$ is a quadratic algebra with $\widetilde{\mu}$ a q-measure on $\mathcal{S}(\Omega)$ that extends μ from $\mathcal{C}(\Omega)$.

In general, $\mathcal{S}(\Omega)$ is strictly between $\mathcal{C}(\Omega)$ and $\mathcal{A}$. For example, if $A \in \mathcal{A}$ with $\nu_c(A) = 0$, then $\chi_A = 0$ almost everywhere so $|\chi_A\rangle\langle\chi_A| = 0$ and $\widetilde{\mu}(A) = 0$. To be specific, if $\omega \in \Omega$ then $\{\omega\} \in \mathcal{A}$ but $\{\omega\} \notin \mathcal{C}(\Omega)$. Although there are exceptions, a typical $\omega \in \Omega$ satisfies $\nu_c(\{\omega\}) = 0$ so $\widetilde{\mu}(\{\omega\}) = 0$. It follows from Schwarz's inequality that if $A \in \mathcal{S}(\Omega)$ then $\widetilde{\mu}(A) \leq \nu_c(A) \sup \|\rho_n\|$.

Theorem 3.1. *If $(H, \{\rho_n\})$ is a QSGP and $A \in \mathcal{A}$, then $A \in \mathcal{S}(\Omega)$ if and only if $\lim \mu_{\rho_n} [\mathrm{cyl}(A^n)]$ exists. If this is the case, then $\widetilde{\mu}(A) = \lim \mu_{\rho_n}[\mathrm{cyl}(A^n)]$.*

Proof. Let P_n be the orthogonal projection from H onto H_n. We first show that $P_n\chi_A = \chi_{\mathrm{cyl}(A^n)}$. Now for e_ω^n, $\omega \in \Omega_n'$ with $\omega = \omega_1\omega_2\cdots\omega_n$ we have

$$\left\langle \chi_{\mathrm{cyl}(A^n)}, e_\omega^n \right\rangle = \frac{1}{p_c^n(\omega)^{1/2}} \left\langle \chi_{\mathrm{cyl}(A^n)}, \chi_{\mathrm{cyl}(\omega)} \right\rangle$$

$$= \begin{cases} p_c^n(\omega)^{1/2} & \text{if } \omega_1\omega_2\cdots\omega_n\omega_{n+1}\cdots \in A \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{p_c^n(\omega)^{1/2}} \left\langle \chi_a, \chi_{\mathrm{cyl}(\omega)} \right\rangle$$

$$= \langle \chi_A, e_\omega^n \rangle = \langle P_n\chi_A, e_\omega^n \rangle$$

Since $P_n\chi_A, \chi_{\mathrm{cyl}(A^n)} \in H_n$ is follows that $P_n\chi_A = \chi_{\mathrm{cyl}(A^n))}$. Hence,

$$\mathrm{tr}\left(\rho_n|\chi_A\rangle\langle\chi_A|\right) = \mathrm{tr}\left(\rho_n P_n|\chi_A\rangle\langle\chi_A|\right)$$

$$= \sum_{\omega\in\Omega_n'} \langle \rho_n P_n|\chi_A\rangle\langle\chi_A|e_\omega^n, e_\omega^n\rangle$$

$$= \sum_{\omega\in\Omega_n'} \langle e_\omega^n, \chi_A\rangle\langle\rho_n P_n\chi_{A^i} e_\omega^n\rangle$$

$$= \sum_{\omega\in\Omega_n'} \left\langle e_\omega^n, \chi_{\mathrm{cyl}(A^n)} \right\rangle\left\langle \rho_n\chi_{\mathrm{cyl}(A^n)}, e_\omega^n \right\rangle$$

$$= \mathrm{tr}\left(\rho_n\left|\chi_{\mathrm{cyl}(A^n)}\right\rangle\left\langle\chi_{\mathrm{cyl}(A^n)}\right|\right) = \mu_{\rho_n}(A^n)$$

The result now follows. $\qquad\square$

4 Amplitude Processes

Various ways of constructing discrete quantum processes on $H = L_2(\Omega, \mathcal{A}, \nu_c)$ have been considered [3, 4, 5]. After we introduce general amplitude processes, we present a concrete realization of a discrete quantum process in terms of a natural quantum action.

A *transition amplitude* is a map $a\colon \mathcal{P}_n \times \mathcal{P}_{n+1} \to \mathbb{C}$ such that $a(x, y) = a(x \to y) = 0$ if $p_c(x \to y) = 0$ and

$$\sum \{a(x \to y)\colon y \in \mathcal{P}_{n+1}, x \to y\} = 1 \tag{4.1}$$

for all $x \in \mathcal{P}_n$. The *amplitude process* (AP) corresponding to a is given by the maps $a_n\colon \Omega_n \to \mathbb{C}$ where

$$a_n(\omega_1 \omega_2 \cdots \omega_n) = a(\omega_1 \to \omega_2) a(\omega_2 \to \omega_3) \cdots a(\omega_{n-1} \to \omega_n)$$

We define the *probability vector* $\widehat{a}_n\colon \Omega_n \to \mathbb{C}$ by $\widehat{a}_n(\omega) = 0$ if $p_c^n(\omega) = 0$ and if $\omega \in \Omega_n'$ then $\widehat{a}_n(\omega) = p_c^n(\omega)^{-1} a_n(\omega)$. For a given AP a_n define the positive operators ρ_n on H_n by

$$\langle \rho_n \chi_{\{\omega'\}}, \chi_{\{\omega\}} \rangle = a_n(\omega) \overline{a_n(\omega')}$$

for all $\omega, \omega' \in \Omega_n'$. Then

$$\langle \rho_n e_{\omega'}^n, e_\omega^n \rangle = p_c^n(\omega')^{-1/2} p_c^n(\omega)^{-1/2} a_n(\omega) \overline{a_n(\omega')}$$

It follows that ρ_n is the rank 1 operator given by $\rho_n = |\widehat{a}_n\rangle\langle\widehat{a}_n|$.

Theorem 4.1. *The operators ρ_n, $n = 1, 2, \ldots$, form a discrete quantum process.*

Proof. We have seen that ρ_n is a positive operator on H_n, $n = 1, 2, \ldots$. To show that ρ_n is a probability operator we have

$$\langle e_n 1, 1 \rangle = \left\langle \rho_n \sum \chi_{\{\omega\}}, \sum \chi_{\{\omega\}} \right\rangle = \sum_{\omega, \omega'} \left\langle e_n \chi_{\{\omega\}}, \chi_{\{\omega'\}} \right\rangle$$

$$= \sum_{\omega, \omega'} a_n(\omega) \overline{a_n(\omega)} = \left| \sum_\omega a_n(\omega) \right|^2 \tag{4.2}$$

Applying (4.1) we obtain

$$\sum a_n(\omega) = \sum a(\omega_1 \to \omega_2) a(\omega_2 \to \omega_3) \cdots a(\omega_{n-1} \to \omega_n)$$

$$= \sum a(\omega_1 \to \omega_2) \cdots a(\omega_{n-2} \to \omega_{n-1}) \sum_{\omega_n} a(\omega_{n-1} \to \omega_n)$$

$$= \sum a(\omega_1 \to \omega_2) \cdots a(\omega_{n-2} \to \omega_{n-1})$$

$$\vdots$$

$$= \sum_{\omega_2} a(\omega_1 \to \omega_2) = 1 \tag{4.3}$$

By (4.2) and (4.3) we conclude that $\langle \rho_n 1, 1 \rangle = 1$ so P_n is a probability operator. To show that ρ_n is a consistent sequence, let $\omega, \omega' \in \Omega'_n$ with $\omega = \omega_1 \omega_2 \cdots \omega_n$, $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n$. By (4.1) we have

$$
\begin{aligned}
D_{n+1}\left[\mathrm{cyl}(\omega), \mathrm{cyl}(\omega')\right] &= \langle \rho_{n+1} \chi_{\mathrm{cyl}(\omega')}, \chi_{\mathrm{cyl}(\omega)} \rangle \\
&= \sum \left\{ a_n(\omega) a_n(\omega_n \to x) \overline{a(\omega')} a_n(\omega'_n \to y) \colon \omega_n \to x, \omega'_n \to y \right\} \\
&= a_n(\omega) \overline{a_n(\omega')} \sum \left\{ a(\omega_n \to x \colon \omega_n \to x \right\} \sum \left\{ \overline{a(\omega'_n \to y)} \colon \omega'_n \to y \right\} \\
&= a_n(\omega) \overline{a_n(\omega')} = D_n\left[\mathrm{cyl}(\omega), \mathrm{cyl}(\omega')\right]
\end{aligned}
\tag{4.4}
$$

For $A, B \in \mathcal{C}(\Omega_n)$ by (4.4) we have

$$
\begin{aligned}
D_{n+1}(A, B) &= \sum \left\{ D_{n+1}\left[\mathrm{cyl}(\omega), \mathrm{cyl}(\omega')\right] : \omega \in A, \omega' \in B \right\} \\
&= \sum \left\{ D_n\left[\mathrm{cyl}(\omega), \mathrm{cyl}(\omega')\right] : \omega \in A, \omega' \in B \right\} \\
&= D_n(A, B) \qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Since $\rho_n = |\widehat{a}_n\rangle\langle\widehat{a}_n|$, we see that

$$
\|\rho_n\| = \||\widehat{a}_n\rangle\langle\widehat{a}_n|\| = \|\widehat{a}_n\|^2 = \sum |a_n(\omega)|^2
$$

The decoherence functional corresponding to ρ_n becomes

$$
\begin{aligned}
D_n(A, B) &= \langle \rho_n \chi_B, \chi_A \rangle = \langle |\widehat{a}_n\rangle\langle\widehat{a}_n| \chi_B, \chi_A \rangle \\
&= \langle \widehat{a}_n, \chi_A \rangle \langle \chi_B, \widehat{a}_n \rangle \\
&= \sum \left\{ a_n(\omega) \colon \omega \in A \cap \Omega'_n \right\} \sum \left\{ \overline{a_n(\omega')} \colon \omega' \in B \cap \Omega'_n \right\}
\end{aligned}
$$

The corresponding q-measure is given by

$$
\mu_n(A) = D_n(A, A) = \left| \sum \left\{ a_n(\omega) \colon \omega \in A \cap \Omega'_n \right\} \right|^2
$$

It follows from Theorem 3.1 that if $A \in \mathcal{A}$ then $A \in \mathcal{S}(\Omega)$ if and only if

$$
\lim \mu_{\rho_n}\left[\mathrm{cyl}(A^n)\right] = \lim \left| \sum \left\{ a_n(\omega) \colon \omega \in A_n \cap \Omega'_n \right\} \right|^2
$$

exists and is finite in which case $\widetilde{\mu}(A)$ is this limit.

We now present a specific example of an AP that arises from a natural quantum action. For $x \in \mathcal{P}$, the *height* $h(x)$ of x is the cardinality of a largest chain is x. The *width* $w(x)$ of x is the cardinality of a largest antichain in x. Finally, the *area* $A(x)$ of x is given by $A(x) = h(x)w(x)$. Roughly speaking,

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$h(x)$ corresponds to an internal time in x, $w(x)$ corresponds to the mass or energy of x [5] and $A(x)$ corresponds to an action for x. If $x \to y$, then $h(y) = h(x)$ or $h(x) + 1$ and $w(y) = w(x)$ or $w(x) + 1$. In the case of $h(y) = h(x)+1$ we call y a *height offspring* of x, in the case $w(y) = w(x)+1$ we call y a *width offspring* of x and if both $h(y) = h(x)$, $w(y) = w(x)$ hold we call y a *mild offspring* of x. Let $H(x)$, $W(x)$ and $M(x)$ be the sets of height, width and mild offspring of x, respectively.

Lemma 4.2. *The sets $H(x)$, $W(x)$, $M(x)$ form a partition of the set of offspring x.*

Proof. Since $\{y \colon x \to y\} = H(x) \cup W(x) \cup M(x)$, we only need to show that $H(x)$, $W(x)$, $M(x)$ are mutually disjoint. Clearly, $H(x) \cap M(x) = W(x) \cap M(x) = \emptyset$ so we must show that $H(x) \cap W(x) = \emptyset$. Suppose $y \in H(x)$ where $y = x \uparrow a$. If $y \in W(x)$ then a is incomparable with every element of some largest antichain $\{b_1, \ldots, b_r\}$ in x. Also, $a > a_s > a_{s-1} > \cdots > a_1$ where $\{a_1, \ldots, a_s\}$ is a largest chain in x. It follows that $b_i \neq a_j$ for every i, j. Now $a_s \not> b_i$ for some i because otherwise $a > b_i$ which is a contradiction. Hence, $a_s < b_i$ for some i because otherwise $\{b_1, \ldots, b_r, a_s\}$ would be a larger antichain in x. But then $\{a_1, \ldots, a_s, b_i\}$ is a larger chain in x. But this is a contradiction. $\qquad\square$

If $x \to y$ we have the following three possibilities: $y \in M(x)$ in which case $A(y) - A(x) = 0$, $y \in H(x)$ in which case

$$A(y) - A(x) = [h(x) + 1]\,w(x) - h(x)w(x) = w(x)$$

$y \in W(x)$ in which case

$$A(y) - A(x) = h(x)\,[w(x) + 1] - h(x)w(x) = h(x)$$

We define the transition amplitude $a(x \to y)$ in terms of the "action" change from x to y. We first define the *partition function*

$$z(x) = \sum_y \left\{ e^{2\pi i [A(y) - A(x)]/|x|} \colon p_c^{|x|}(x \to y) \neq 0 \right\}$$

For $x \to y$ define the transition amplitude $a(x \to y)$ to be 0 if $p_c^{|x|}(x \to y) = 0$ and otherwise
$$a(x \to y) = \tfrac{1}{z(x)} e^{2\pi i [A(y) - A(x)]/|x|}$$

As before, we have three possibilities. If $y \in M(x)$ then $a(x \to y) = z(x)^{-1}$, if $y \in H(x)$ then

$$a(x \to y) = \frac{e^{2\pi i w(x)/|x|}}{z(x)}$$

if $y \in W(x)$ then

$$a(x \to y) = \frac{e^{2\pi i h(x)/|x|}}{z(x)}$$

Since the transition amplitudes $a(x \to y)$ satisfy (4.1) it follows from Theorem 4.1 that the corresponding ρ_n form a discrete quantum process. For any $x \in \mathcal{P}$ there are only three possible values for $a(x \to y)$. This is roughly analogous to a 3-dimensional Markov chain. Does this indicate the emergence of 3-dimensional space?

5   Discrete Einstein Equation

Let $Q_n = \bigcup_{i=1}^{n} \mathcal{P}_i$ and let K_n be the Hilbert space $\mathbb{C}^{Q_n}$ with the standard inner product

$$\langle f, g \rangle = \sum_{x \in Q_n} \overline{f(x)} g(x)$$

Let $L_n = K_n \otimes K_n$ which we identify with $\mathbb{C}^{Q_n \times Q_n}$. Suppose $(H, \{\rho_n\})$ is a QSGP with corresponding decoherence matrices

$$D_n(\omega, \omega') = D_n \left[\mathrm{cyl}(\omega), \mathrm{cyl}(\omega') \right] \quad \omega, \omega' \in \Omega_n$$

If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$ and $\omega_i = x$ for some i, then ω *contains* x. For $x, y \in Q_n$ we define

$$D_n(x, y) = \sum \left\{ D_n(\omega, \omega') \colon \omega \text{ contains } x, \ \omega' \text{ contains } y \right\}$$

Due to the consistency of ρ_n, $D_n(x, y)$ is independent of n if $n \geq |x|, |y|$. Also $D_n(x, y)$, $x, y \in Q_n$, are the components of a positive semidefinite matrix. Moreover, if

$$A_x = \left\{ \omega \in \Omega_n \colon \omega \text{ contains } x \right\}$$

then we define the q-measure $\mu_n(x)$ of x by

$$\mu_n(x) = \mu_n \left[\mathrm{cyl}(A_x) \right] = D_n(x, x)$$

We think of Q_m as an analogue of a differentiable manifold and $D_n(x, y)$ as an analogue of a metric tensor. For $\omega, \omega' \in \Omega_n$ we define the *covariant bidifference operator* $\nabla^n_{\omega,\omega'} \colon L_n \to L_n$ [6] by

$$\nabla^n_{\omega,\omega'} f(x, y) = \left[D_n(\omega_{|x|-1}, \omega'_{|y|-1}) f(x, y) - D_n(x, y) f(\omega_{|x|-1}, \omega'_{|y|-1}) \right]$$
$$\cdot \, \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}}$$

In analogy to the curvature operator on a manifold, we define the *discrete curvature operator* $\mathcal{R}^n_{\omega,\omega'} \colon L_n \to L_n$ by

$$\mathcal{R}^n_{\omega,\omega'} = \nabla^n_{\omega,\omega'} - \nabla^n_{\omega',\omega}$$

We also define the *discrete metric operator* $\mathcal{D}^n_{\omega,\omega'}$ on L_n by

$$\mathcal{D}^n_{\omega,\omega'} f(x, y)$$
$$= D_n(x, y) \left[f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x,\omega'_{|x|}} \delta_{y,\omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \right]$$

and the *discrete mass-energy operator* $\mathcal{T}^n_{\omega,\omega'}$ on L_n by

$$\mathcal{T}^n_{\omega,\omega'} f(x, y)$$
$$= \left[D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x,\omega'_{|x|}} \delta_{y,\omega_{|y|}} \right] f(x, y)$$

It is not hard to show that

$$\mathcal{R}^n_{\omega,\omega'} = \mathcal{D}^n_{\omega,\omega'} + \mathcal{T}^n_{\omega,\omega'} \tag{5.1}$$

We call (5.1) the *discrete Einstein equation* [6]

If we can find $D_n(\omega, \omega')$ such that the classical Einstein equation is an approximation to (5.1), then it would give information about $D_n(\omega, \omega')$. Moreover, an important problem in discrete quantum gravity theory is how to test whether general relativity is a close approximation to the theory. Whether Einstein's equation is an approximation to (5.1) would provide such a test.

As with the classical Einstein equation (5.1) is difficult to analyze. We obtain a simplification by considering the *contractive discrete curvature, metric* and *mass-energy operators* $\widehat{\mathcal{R}}^n_{\omega,\omega'}, \widehat{\mathcal{D}}^n_{\omega,\omega'}, \widehat{\mathcal{T}}^n_{\omega,\omega'} \colon L_n \to K_n$, respectively, given by $(\widehat{\mathcal{R}}^n_{\omega,\omega'} f)(x) = \mathcal{R}^n_{\omega,\omega'} f(x, x)$, $(\widehat{\mathcal{D}}^n_{\omega,\omega'} f)(x) = \mathcal{D}^n_{\omega,\omega'} f(x, x)$, $(\widehat{\mathcal{T}}^n_{\omega,\omega'} f)(x) = \mathcal{T}^n_{\omega,\omega'} f(x, x)$. We then have the *contracted discrete Einstein equation*

$$\widehat{\mathcal{R}}^n_{\omega,\omega'} = \widehat{\mathcal{D}}^n_{\omega,\omega'} + \widehat{\mathcal{T}}^n_{\omega,\omega'}$$

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where

$$\widehat{\mathcal{D}}^n_{\omega,\omega'} f(x) = \mu_n(x) \left[f(\omega'_{|x|-1}, \omega_{|x|-1}) - f(\omega_{|x|-1}, \omega'_{|x|-1}) \right]$$
$$\cdot \, \delta_{x,\omega'_{|x|}} \delta_{x,\omega_{|x|}}$$
$$\widehat{\mathcal{T}}^n_{\omega,\omega'} f(x) = 2i \, Im \, D_n(\omega_{|x|-1}, \omega'_{|x|-1}) \delta_{x,\omega_{|x|}} \delta_{x,\omega'_{|x|}} f(x,x)$$

Any $f \in L_n$ can be decomposed into a sum of its symmetric and anti-symmetric parts: $f = f_s + f_a$ where

$$f_s(x,y) = \frac{f(x,y) + f(y,x)}{2}$$
$$f_a(x,y) = \frac{f(x,y) - f(y,x)}{2}$$

and $f_s(x,y) = f_s(y,x)$, $f_a(x,y) = -f_a(y,x)$. We then obtain the simpler forms

$$\widehat{\mathcal{D}}^n_{\omega,\omega'} f(x) = \mu_n(x) f_a(\omega'_{|x|-1}, \omega_{|x|-1}) \delta_{x,\omega'_{|x|}} \delta_{x,\omega_{|x|}}$$
$$\widehat{\mathcal{T}}^n_{\omega,\omega'} f(x) = 2i \, Im \, D_n(\omega_{|x|-1}, \omega'_{|x|-1}) \delta_{x,\omega_{|x|}} \delta_{x,\omega'_{|x|}} f_s(x,x)$$

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