The Causal Poset is Directed but not Lattice Ordered

S. Gudder

Follow this and additional works at: https://digitalcommons.du.edu/math_preprints

Part of the Mathematics Commons

Recommended Citation
https://digitalcommons.du.edu/math_preprints/28

This Article is brought to you for free and open access by the Department of Mathematics at Digital Commons @ DU. It has been accepted for inclusion in Mathematics Preprint Series by an authorized administrator of Digital Commons @ DU. For more information, please contact jennifer.cox@du.edu, dig-commons@du.edu.
THE CAUSAL POSET IS DIRECTED
BUT NOT LATTICE ORDERED

S. Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208, U.S.A.
sgudder@du.edu

Abstract

In the causal set approach to discrete quantum gravity the universe grows one element at a time in discrete steps. At each step the process has the form of a causal set (causet) and the "completed" universe is given by a path through a discretely growing chain of causets. The collection of causets forms a partially ordered set (poset) in a natural way. We first show that this poset is directed. We then give a counterexample which shows it is not lattice ordered.

1 Notation and Definitions

This section sets the notation and definitions that we will need concerning the causal set approach to discrete quantum gravity. For motivation and further details we refer the reader to [1, 2, 3, 4, 5, 6]. In this paper a finite partially ordered set (poset) will be called a causet. For a causet (x, <) we denote the cardinality of x by |x|. All of our causets are unlabeled and we identify isomorphic causets. Let P_n be the collection of all causets with cardinality n, n = 1, 2, . . . and let P = ∪P_n be the collection of all causets. An element a ∈ x for x ∈ P is maximal if there is no b ∈ x with a < b. If x ∈ P_n y ∈ P_{n+1} then x produces y if y is obtained from x by adjoining a single maximal element a to x. If x produces y we write x → y.

The transitive closure of → makes P into a poset itself and we call (P, →) the causal poset. For x, y ∈ P with |x| < |y|, a path from x to y is a finite sequence ω_1ω_2 · · · ω_m where ω_1 = x, ω_m = y and ω_i → ω_{i+1}, i = 1, . . . , m − 1. It is clear that x < y if and only if there is a path from x to y. We view a causet x as a possible universe at time |x|. An infinite path starting at
the one vertex causet $x$, is considered to be a "completed" universe that includes its histories.

If $a, b \in x$ with $x \in \mathcal{P}$ we say that $a$ and $b$ are comparable if one of the following hold: $a = b$, $a < b$, $b < a$. If $a$ and $b$ are not comparable, we say they are incomparable. Let $(P, <)$ be an arbitrary poset and let $A \subseteq P$. We say that $b \in P$ is an upper bound for $A$ if $a \leq b$ for all $a \in A$. The poset $P$ is directed if for any pair $a, b \in P$, the set $\{a, b\}$ has an upper bound. We call $P$ lattice ordered if for any pair $a, b \in P$, the set $\{a, b\}$ has a least upper bound. That is, there exists an upper bound $c$ for $\{a, b\}$ such that $c \leq d$ for any upper bound $d$. In the next section we show that the causal poset $(\mathcal{P}, \rightarrow)$ is directed, but is not lattice ordered. We interpret this as saying that for a given time, the paths of any two universes will cross at some later time.

2 Directed But Not Lattice Ordered

Theorem. $\mathcal{P}$ is directed.

Proof. Let $x, y \in \mathcal{P}$ with $|x| = |y|$. We can assume that $x \cap y = \emptyset$. Construct a path beginning with $x$ and growing one element at a time with the vertices of $y$ until the path arrives at the causet $z = x \cup y$ where the vertices of $y$ are incomparable with all the vertices of $x$ in $z$ (in technical terms, $z$ is the horizontal sum of $x$ and $y$). We then have $x \leq z$. In a similar way, we can construct a path from $y$ to $z$ so that $y \leq z$. In the other case $|x| \neq |y|$ we can assume without loss of generality that $|y| < |x|$. Form a path beginning at $y$ and ending at any caused $y_1$ with $|y_1| = |x|$. Then $y \leq y_1$ and from our previous work there exists a $z \in \mathcal{P}$ with $x, y \leq z$. Since $y \leq y_1$, we have that $x, y \leq z$. 

We now present a counterexample which shows that $\mathcal{P}$ is not lattice ordered. The counterexample is given in Figure 1 which employs the usual Hasse diagram method for displaying causets. In Figure 1, causets $x$ and $y$ both produce the distinct causets $u$ and $v$. Thus $u$ and $v$ are both upper bounds for $x$ and $y$. However, there is no smaller upper bound for $x$ and $y$ so $\{x, y\}$ has no least upper bound. We conjecture that this is the smallest counterexample possible.
References


