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# Models for Discrete Quantum Gravity

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**Abstract**

We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of $q$-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is "quantized" to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

## 1 Introduction

In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) $\rho_n$, $n = 1, 2, \ldots$. Instead of requiring that $\rho_n$ be a state on a Hilbert space $H_n$, we require that $\rho_n$ be a $q$-probability operator on $H_n$. This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring $\mathrm{tr}(\rho_n) = 1$, this condition normalizes the corresponding quantum measure. By superimposing a concrete DQP on a classical sequential growth process we obtain a model for discrete quantum gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of $q$-probability operators is a convex set and find its set of extreme elements. We discuss the property of consistency for a DQP and introduce the so-called quadratic algebra of suitable sets. The suitable sets are those on which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process (CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets are discussed. In Section 4 we show how to "quantize" a CSGP to obtain a quantum sequential growth process (QSGP). Some results concerning the consistency of a DQP are given. Finally, Section 5 provides two methods for constructing examples of QSGP.

# 2 Discrete Quantum Processes

Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and let

$$H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f \colon \Omega \to \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\}$$

be the corresponding Hilbert space. Let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$ be an increasing sequence of sub $\sigma$-algebras of $\mathcal{A}$ that generate $\mathcal{A}$ and let $\nu_n = \nu \mid \mathcal{A}_n$ be the restriction of $\nu$ to $\mathcal{A}_n$, $n = 1, 2, \ldots$. Then $H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)$ forms an increasing sequence of closed subspaces of $H$ called a *filtration* of $H$. A bounded operator $T$ on $H_n$ will also be considered as a bounded operator on $H$ by defining $Tf = 0$ for all $f \in H_n^\perp$. We denote the characteristic function $\chi_\Omega$ of $\Omega$ by 1. Of course, $\|1\| = 1$ and $\langle 1, f \rangle = \int f d\nu$ for every $f \in H$. A *$q$-probability operator* is a bounded positive operator $\rho$ on $H$ that satisfies $\langle \rho 1, 1 \rangle = 1$. Denote the set of $q$-probability operators on $H$ and $H_n$ by $\mathcal{Q}(H)$ and $\mathcal{Q}(H_n)$, respectively. Since $1 \in H_n$, if $\rho \in \mathcal{Q}(H_n)$ by our previous convention, $\rho \in \mathcal{Q}(H)$. Notice that a positive operator $\rho \in \mathcal{Q}(H)$ if and only if $\left\| \rho^{1/2} 1 \right\| = 1$ where $\rho^{1/2}$ is the unique positive square root of $\rho$.

A rank 1 element of $\mathcal{Q}(H)$ is called a *pure $q$-probability operator*. Thus $\rho \in \mathcal{Q}(H)$ is pure if and only if $\rho$ has the form $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in H$ satisfying

$$|\langle 1, \psi \rangle| = \left| \int \psi d\nu \right| = 1$$

We then call $\psi$ a *q-probability vector* and we denote the set of $q$-probability vectors by $\mathcal{V}(H)$ and the set of pure $q$-probability operators by $\mathcal{Q}_p(H)$. Notice that if $\psi \in \mathcal{V}(H)$, then $\|\psi\| \geq 1$ and $\|\psi\| = 1$ if and only if $\psi = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|c| = 1$. Two operators $\rho_1, \rho_2 \in \mathcal{Q}(H)$ are *orthogonal* if $\rho_1 \rho_2 = 0$.

**Theorem 2.1.** (i) *$\mathcal{Q}(A)$ is a convex set and $\mathcal{Q}_p(H)$ is its set of extreme elements.* (ii) *$\rho \in \mathcal{Q}(H)$ is of trace class if and only if there exists a sequence of mutually orthogonal $\rho_i \in \mathcal{Q}_p(H)$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$ such that $\rho = \sum \alpha_i \rho_i$ in the strong operator topology. The $\rho_i$ are unique if and only if the corresponding $\alpha_i$ are distinct.*

*Proof.* (i) If $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$, then $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ is a positive operator and

$$\langle \rho 1, 1\rangle = \langle (\lambda\rho + (1-\lambda)\,\rho_2)1, 1\rangle = \lambda\langle \rho_1 1, 1\rangle + (1-\lambda)\langle \rho_2 1, 1\rangle = 1$$

Hence, $\rho \in \mathcal{Q}(H)$ so $\mathcal{Q}(H)$ is a convex set. Suppose $\rho \in \mathcal{Q}_p(H)$ and $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ where $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$. If $\rho_1 \neq \rho_2$, then $\operatorname{rank}(\rho) \neq 1$ which is a contradiction. Hence, $\rho_1 = \rho_2$ so $\rho$ is an extreme element of $\mathcal{Q}(H)$. Conversely, suppose $\rho \in \mathcal{Q}(H)$ is an extreme element. If the cardinality of the spectrum $|\sigma(\rho)| > 1$, then by the spectral theorem $\rho = \rho_1 + \rho_2$ where $\rho_1, \rho_2 \neq 0$ are positive and $\rho_1 \neq \alpha\rho_2$ for $\alpha \in \mathbb{C}$. If $\rho_1 1, \rho_2 1 \neq 0$, then $\langle \rho_1 1, 1\rangle, \langle \rho_2 1, 1\rangle \neq 0$ and we can write

$$\rho = \langle \rho_1 1, 1\rangle \frac{\rho_1}{\langle \rho_1 1, 1\rangle} + \langle \rho_2 1, 1\rangle \frac{\rho_2}{\langle \rho_2 1, 1\rangle}$$

Now $\langle \rho_1 1, 1\rangle^{-1}\rho_1, \langle \rho_2 1, 1\rangle^{-1}\rho_2 \in \mathcal{Q}(H)$ and

$$\langle \rho_1 1, 1\rangle + \langle \rho_2 1, 1\rangle = \langle \rho 1, 1\rangle = 1$$

which is a contradiction. Hence, $\rho_1 1 = 0$ or $\rho_2 1 = 0$. Without loss of generality suppose that $\rho_2 1 = 0$. We can now write

$$\rho = \tfrac{1}{2}\rho_1 + \tfrac{1}{2}(\rho_1 + 2\rho_2)$$

Now $\rho_1 1 \neq 0$, $(\rho_1 + 2\rho_2)1 \neq 0$ and as before we get a contradiction. We conclude that $|\sigma(\rho)| = 1$. Hence, $\rho = \alpha P$ where $P$ is a projection and $\alpha > 0$. If $\operatorname{rank}(P) > 1$, then $P = P_1 + P_2$ where $P_1$ and $P_2$ are orthogonal nonzero projections so $\rho = \alpha P_1 + \alpha P_2$. Proceeding as before we obtain a contradiction. Hence, $\operatorname{rank}(P) = 1$ so $\rho = \alpha P$ is pure. (ii) This follows from the spectral theorem. $\square$

Let $\{H_n : n = 1, 2, \ldots\}$ be a filtration of $H$ and let $\rho_n \in \mathcal{Q}(H_n)$, $n = 1, 2, \ldots$. The *$n$-decoherence functional* $D_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ defined by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

gives a measure of the interference between $A$ and $B$ when the system is described by $\rho_n$. It is clear that $D_n(\Omega_n, \Omega_n) = 1$, $D_n(A, B) = \overline{D_n(B, A)}$ and $A \mapsto D_n(A, B)$ is a complex measure for all $B \in \mathcal{A}_n$. It is also well-known that if $A_1, \ldots, A_r \in \mathcal{A}_n$ then the matrix with entries $D_n(A_j, A_k)$ is positive semidefinite. We define the map $\mu_n : \mathcal{A}_n \to \mathbb{R}^+$ by

$$\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle$$

Notice that $\mu_n(\Omega_n) = 1$. Although $\mu_n$ is not additive, it does satisfy the *grade*-2 *additivity condition*: if $A, B, C \in \mathcal{A}_n$ are mutually disjoint, then

$$\begin{aligned} \mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) \\ - \mu_n(A) - \mu_n(B) - \mu_n(C) \end{aligned} \tag{2.1}$$

We say that $\rho_{n+1}$ is *consistent* with $\rho_n$ if $D_{n+1}(A, B) = D_n(A, B)$ for all $A, B \in \mathcal{A}_n$. We call the sequence $\rho_n$, $n = 1, 2, \ldots$, *consistent* if $\rho_{n+1}$ is consistent with $\rho_n$ for $n = 1, 2, \ldots$. Of course, if the sequence $\rho_n$, $n = 1, 2, \ldots$, is consistent, then $\mu_{n+1}(A) = \mu_n(A)$ $\forall A \in \mathcal{A}_n$, $n = 1, 2, \ldots$. A *discrete quantum process* (DQP) is a consistent sequence $\rho_n \in \mathcal{Q}(H_n)$ for a filtration $H_n$, $n = 1, 2, \ldots$. A DQP $\rho_n$ is *pure* if $\rho_n \in \mathcal{Q}_p(H_n)$, $n = 1, 2, \ldots$.

If $\rho_n$ is a DQP, then the corresponding maps $\mu_n : \mathcal{A}_n \to \mathbb{R}^+$ have the form

$$\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \left\| \rho_n^{1/2} \chi_A \right\|^2$$

Now $A \to \rho_n^{1/2} \chi_A$ is a vector-valued measure on $\mathcal{A}_n$. We conclude that $\mu_n$ is the squared norm of a vector-valued measure. In particular, if $\rho_n = |\psi_n\rangle\langle\psi_n|$ is a pure DQP, then $\mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2$ so $\mu_n$ is the squared modulus of the complex-valued measure $A \mapsto \langle \psi_n, \chi_A \rangle$.

For a DQP $\rho_n \in \mathcal{Q}(H_n)$, we say that a set $A \in \mathcal{A}$ is *suitable* if $\lim \langle \rho_j \chi_A, \chi_A \rangle$ exists and is finite and in this case we define $\mu(A)$ to be the limit. We denote the set of suitable sets by $\mathcal{S}(\rho_n)$. If $A \in \mathcal{A}_n$ then

$$\lim \langle \rho_j \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle$$

so $A \in \mathcal{S}(\rho_n)$ and $\mu(A) = \mu_n(A)$. This shows that the algebra $\mathcal{A}_0 = \cup \mathcal{A}_n \subseteq \mathcal{S}(\rho_n)$. In particular, $\Omega \in \mathcal{S}(\rho_n)$ and $\mu(\Omega) = 1$. In general, $\mathcal{S}(\rho_n) \neq \mathcal{A}$ and $\mu$

may not have a well-behaved extension from $\mathcal{A}_0$ to all of $\mathcal{A}$ [2, 7]. A subset $\mathcal{B}$ of $\mathcal{A}$ is a *quadratic algebra* if $\emptyset, \Omega \in \mathcal{B}$ and whenever $A, B, C \in \mathcal{B}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{B}$, we have $A \cup B \cup C \in \mathcal{B}$. For a quadratic algebra $\mathcal{B}$, a *q-measure* is a map $\mu_0 \colon \mathcal{B} \to \mathbb{R}^+$ that satisfies the grade-2 additivity condition (2.1). Of course, an algebra of sets is a quadratic algebra and we conclude that $\mu_n \colon \mathcal{A}_n \to \mathbb{R}^+$ is a $q$-measure. It is not hard to show that $\mathcal{S}(\rho_n)$ is a quadratic algebra and $\mu \colon \mathcal{S}(\rho_n) \to \mathbb{R}^+$ is a $q$-measure on $\mathcal{S}(\rho_n)$ [3].

# 3 Classical Sequential Growth Processes

A *partially ordered set* (*poset*) is a set $x$ together with an irreflexive, transitive relation $<$ on $x$. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $\mathcal{P}_n$ be the collection of all posets with cardinality $n$, $n = 1, 2, \ldots$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then $x$ *produces* $y$ if $y$ is obtained from $x$ by adjoining a single new element to $x$ that is maximal in $y$. We also say that $x$ is a *producer* of $y$ and $y$ is an *offspring* of $x$. If $x$ produces $y$ we write $x \to y$. We denote the set of offspring of $x$ by $x \to$ and for $A \subseteq \mathcal{P}_n$ we use the notation

$$A \to\, = \{y \in \mathcal{P}_{n+1} \colon x \to y, x \in A\}$$

The transitive closure of $\to$ makes the set of all finite posets $\mathcal{P} = \cup \mathcal{P}_n$ into a poset.

A *path* in $\mathcal{P}$ is a string (sequence) $x_1, x_2, \ldots$ where $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$, $i = 1, 2, \ldots$. An *$n$-path* in $\mathcal{P}$ is a finite string $x_1 x_2 \cdots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$. We denote the set of paths by $\Omega$ and the set of $n$-paths by $\Omega_n$. The set of paths whose initial $n$-path is $\omega_0 \in \Omega_n$ is denoted by $\omega_0 \Rightarrow$. Thus, if $\omega_0 = x_1 x_2 \cdots x_n$ then

$$\omega_0 \Rightarrow\, = \{\omega \in \Omega \colon \omega = x_1, x_2 \cdots x_n y_{n+1} y_{n+2} \cdots\}$$

If $x$ produces $y$ in $r$ isomorphic ways, we say that the *multiplicity* of $x \to y$ is $r$ and write $m(x \to y) = r$. For example, in Figure 1, $m(x \to y) = 3$. (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)

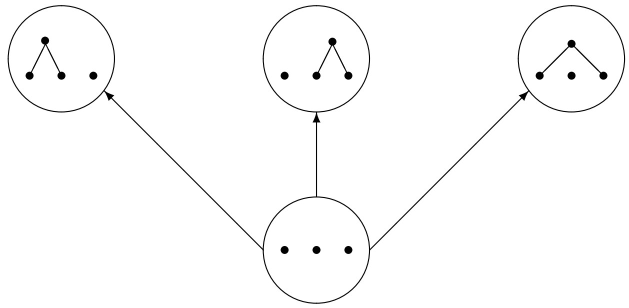


**Figure 1**

If $x \in \mathcal{P}$ and $a, b \in x$ we say that $a$ is an *ancestor* of $b$ and $b$ is a *successor* of $a$ if $a < b$. We say that $a$ is a *parent* of $b$ and $b$ is a *child* of $a$ if $a < b$ and there is no $c \in x$ such that $a < c < b$. Let $c = (c_0, c_1, \ldots)$ be a sequence of nonnegative numbers called *coupling constants* [5, 9]. For $r, s \in \mathbb{N}$ with $r \leq s$, we define

$$\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}$$

For $x \in \mathcal{P}_n$ $y \in \mathcal{P}_{n+1}$ with $x \to y$ we define the *transition probability*

$$p_c(x \to y) = m(x \to y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}$$

where $\alpha$ is the number of ancestors and $\pi$ the number of parents of the adjoined maximal element in $y$ that produces $y$ from $x$. It is shown in [5, 9] that $p_c(x \to y)$ is a probability distribution in that it satisfies the Markov-sum rule

$$\sum \{p_c(x \to y) \colon y \in \mathcal{P}_{n+1}, x \to y\} = 1$$

In discrete quantum gravity, the elements of $\mathcal{P}$ are thought of as causal sets and $a < b$ is interpreted as $b$ being in the causal future of $a$. The distribution $y \mapsto p_c(x \to y)$ is essentially the most general that is consistent with principles of causality and covariance [5, 9]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take $c_k = 1/k!$ [6, 7]. The case $c_k = c^k$ for some $c > 0$ has been previously studied and is called a *percolation dynamics* [5, 6, 8].

We call an element $x \in \mathcal{P}$ a *site* and we sometimes call an $n$-path an $n$-*universe* and a path a *universe* The set $\mathcal{P}$ together with the set of transition probabilities $p_c(x \to y)$ forms a *classical sequential growth process* (CSGP) which we denote by $(\mathcal{P}, p_c)$ [4, 5, 6, 8, 9]. It is clear that $(\mathcal{P}, p_c)$ is a Markov chain and as usual we define the probability of an $n$-path $\omega = y_1 y_2 \cdots y_n$ by

$$p_c^n(\omega) = p_c(y_1 \to y_2) p_c(y_2 \to y_3) \cdots p_c(y_{n-1} \to y_n)$$

Denoting the power set of $\Omega_n$ by $2^{\Omega_n}$, $(\Omega_n, 2^{\Omega_n}, p_c^n)$ becomes a probability space where

$$p_c^n(A) = \sum \{p_c^n(\omega) \colon \omega \in A\}$$

for all $A \in 2^{\Omega_n}$. The probability of a site $x \in \mathcal{P}_n$ is

$$p_c^n(x) = \sum \{p_x^n(\omega) \colon \omega \in \Omega_n, \omega \text{ ends at } x\}$$

Of course, $x \mapsto p_c^n(x)$ is a probability measure on $\mathcal{P}_n$ and we have

$$\sum_{x \in \mathcal{P}_n} p_c^n(x) = 1$$

**Example 1**. Figure 2 illustrates the first two steps of a CSGP where the 2 indicates the multiplicity $m(x_3 \to x_6) = 2$. Table 1 lists the probabilities of the various sites for the general coupling constants $c_k$ and the particular coupling constants $c_k' = 1/k!$ where $d = (c_0 + c_1)(c_0 + 2c_1 + c_2)$.

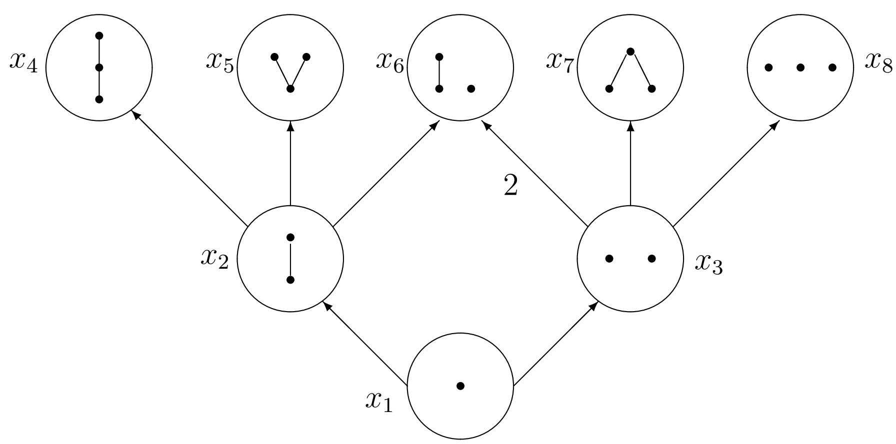


**Figure 2**

| $x_i$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $p_c^{(n)}(x_i)$ | 1 | $\frac{c_1}{c_0+c_1}$ | $\frac{c_0}{c_0+c_1}$ | $\frac{c_1(c_1+c_2)}{d}$ | $\frac{c_1^2}{d}$ | $\frac{3c_0c_1}{d}$ | $\frac{c_0c_2}{d}$ | $\frac{c_0^2}{d}$ |
| $p_{c'}^n(x_i)$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{14}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{14}$ | $\frac{1}{7}$ |

**Table 1**

For $A \subseteq \Omega_n$ we use the notation

$$A \Rightarrow = \cup \{\omega \Rightarrow : \omega \in A\}$$

Thus, $A \Rightarrow$ is the set of paths whose initial $n$-paths are elements of $A$. We call $A \Rightarrow$ a *cylinder set* and define

$$\mathcal{A}_n = \{A \Rightarrow : A \subseteq \Omega_n\}$$

In particular, if $\omega \in \Omega_n$ then the *elementary cylinder set* $\mathrm{cyl}(\omega)$ is given by $\mathrm{cyl}(\omega) = \omega \Rightarrow$. It is easy to check that the $\mathcal{A}_n$ form an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on $\Omega$ and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of $\Omega$. Also for $A \in \mathcal{C}(\Omega)$ of the form $A = A_1 \Rightarrow$, $A_1 \subseteq \Omega_n$, we define $p_c(A) = p_c^n(A_1)$. It is easy to check that $p_c$ is a well-defined probability measure on $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that $p_c$ has a unique extension to a probability measure $\nu_c$ on the $\sigma$-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space, the increasing sequence of subalgebras $\mathcal{A}_n$ generates $\mathcal{A}$ and that the restriction $\nu_c \mid \mathcal{A}_n = p_c^n$. Hence, the subspaces $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ form a filtration of the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$.

# 4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP $(\mathcal{P}, p_c)$ developed in Section 3. We have seen that the *$n$-path Hilbert space* $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ forms a filtration of the *path Hilbert space* $H = L_2(\Omega, \mathcal{A}, \nu_c)$. In the sequel, we assume that $p_c^n(\omega) \neq 0$ for every $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then the set of vectors

$$e_\omega^n = p_c^n(\omega)^{1/2} \chi_{\mathrm{cyl}(\omega)}, \omega \in \Omega_n$$

form an orthonormal basis for $H_n$, $n = 1, 2, \dots$. For $A \in \mathcal{A}_n$, notice that $\chi_A \in H$ with $\|\chi_A\| = p_c^n(A)^{1/2}$.

We call a DQP $\rho_n \in \mathcal{Q}(H_n)$ a *quantum sequential growth process* (QSGP). We call $\rho_n$ the *local operators* and $\mu_n(A) = D_n(A, A)$ the *local q-measures* for the process. If $\rho = \lim \rho_n$ exists in the strong operator topology, then $\rho$ is a $q$-probability operator on $H$ called the *global operator* for the process. If the global operator $\rho$ exists, then $\widehat{\mu}(A) = \langle \rho\chi_A, \chi_A \rangle$ is a (continuous) $q$-measure on $\mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \dots$. Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators [2, 3, 7]. In this case, we still have the $q$-measure $\mu$ on the quadratic algebra $\mathcal{S}(\rho_n) \subseteq \mathcal{A}$ that extends $\mu_n$ $n = 1, 2, \dots$. We frequently identify a set $A \subseteq \Omega_n$ with the corresponding cylinder set $(A \Rightarrow) \in \mathcal{A}_n$. We then have the $q$-measure, also denoted by $\mu_n$, on $2^{\Omega_n}$ defined by $\mu_n(A) = \mu_n(A \Rightarrow)$. Moreover, we define the $q$-measure, again denoted by $\mu_n$, on $\mathcal{P}_n$ by

$$\mu_n(A) = \mu_n\left(\{\omega \in \Omega_n \colon \omega \text{ end in } A\}\right)$$

for all $A \subseteq \mathcal{P}_n$. In particular, for $x \in \mathcal{P}_n$ we have

$$\mu_n\left(\{x\}\right) = \mu_n\left(\{\omega \in \Omega_n \colon \omega \text{ ends with } x\}\right)$$

If $A \in \mathcal{A}_n$ has the form $A_1 \Rightarrow$ for $A_1 \subseteq \Omega_n$ then $A \in \mathcal{A}_{n+1}$ and $A = (A_1 \to) \Rightarrow$ where $A_1 \to \subseteq \Omega_{n+1}$. Let $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and let $D_n(A, B) = \langle \rho_n\chi_B, \chi_A \rangle$, $D_{n+1}(A, B) = \langle \rho_{n+1}\chi_B, \chi_A \rangle$ be the corresponding decoherence functionals. Then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $A, B \subseteq \Omega_n$ we have

$$D_{n+1}\left[(A \to) \Rightarrow, (B \to) \Rightarrow\right] = D_n(A \Rightarrow, B \Rightarrow) \tag{4.1}$$

**Lemma 4.1.** *For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $\omega, \omega' \in \Omega_n$ we have*

$$D_{n+1}\left[(\omega \to) \Rightarrow, (\omega' \to) \Rightarrow\right] = D_n(\omega \Rightarrow, \omega' \Rightarrow) \tag{4.2}$$

*Proof.* Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every $A, B \subseteq \Omega_n$ we have

$$\begin{aligned} D_{n+1}\left[(A \to) \Rightarrow, (B \to) \Rightarrow)\right] &= \sum_{\omega \in A} \sum_{\omega' \in B} D_{n+1} D_{n+1}\left[(\omega \to) \Rightarrow, (\omega' \to) \Rightarrow\right] \\ &= \sum_{\omega \in A} \sum_{\omega' \in B} D_n(\omega \Rightarrow, \omega' \Rightarrow) = D_n(A \Rightarrow, B \Rightarrow) \end{aligned}$$

and the result follows from (4.1). $\square$

For $\omega = x_1x_2\cdots x_n \in \Omega_n$ and $y \in \mathcal{P}_{n+1}$ with $x_n \to y$ we use the notation $\omega y \in \Omega_{n+1}$ where $\omega y = x_1x_2\cdots x_n y$. We also define $p_c(\omega \to y) = p_c(x_n \to y)$ and write $\omega \to y$ whenever $x_n \to y$.

**Theorem 4.2.** *For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have*

$$\langle \rho_n e_{\omega'}^n, e_\omega^n \rangle = \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega' \to x}} \sum_{\substack{y \in \mathcal{P}_{n+1} \\ \omega \to y}} p_c(\omega' \to x)^{1/2} p_c(\omega \to y)^{1/2} \left\langle \rho_{n+1} e_{\omega' x}^{n+1} e_{\omega y}^{n+1} \right\rangle \tag{4.3}$$

*Proof.* By Lemma 4.1, $\rho_{n+1}$ is consistent with $\rho_n$ if and only if (4.2) holds. But

$$\begin{aligned} D_n(\omega \Rightarrow, \omega' \Rightarrow) &= \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle = \left\langle \rho_n \chi_{\mathrm{cyl}(\omega')}, \chi_{\mathrm{cyl}(\omega)} \right\rangle \\ &= p_c^n(\omega')^{1/2} p_c^n(\omega)^{1/2} \langle \rho_n e_{\omega'}^n, e_\omega^n \rangle \end{aligned}$$

Moreover, we have

$$\begin{aligned} D_{n+1}\left[(\omega \to) \Rightarrow, (\omega' \to) \Rightarrow\right] &= \left\langle \rho_{n+1} \chi_{(\omega' \to) \Rightarrow}, \chi_{(\omega \to) \Rightarrow} \right\rangle \\ &= \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega' \to x}} \sum_{\substack{y \in \mathcal{P}_{n+1} \\ \omega \to y}} \left\langle \rho_{n+1} \chi_{\omega' x \Rightarrow}, \chi_{\omega y \Rightarrow} \right\rangle \\ &= \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega' \to x}} \sum_{\substack{y \in \mathcal{P}_{n+1} \\ \omega \to y}} \left\langle \rho_{n+1} \chi_{\mathrm{cyl}(\omega' x)}, \chi_{\mathrm{cyl}(\omega y)} \right\rangle \\ &= \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega' \to x}} \sum_{\substack{y \in \mathcal{P}_{n+1} \\ \omega \to y}} p_c^n(\omega' x)^{1/2} p_c^n(\omega y)^{1/2} \left\langle \rho_{n+1} e_{\omega' x}^{n+1}, e_{\omega y}^{n+1} \right\rangle \\ &= p_c^n(\omega')^{1/2} p_c^n(\omega)^{1/2} \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega' \to x}} \sum_{\substack{y \in \mathcal{P}_{n+1} \\ \omega \to y}} p_c(\omega' \to x) p_c(\omega \to y)^{1/2} \left\langle \rho_n e_{\omega' x}^{n+1}, e_{\omega y}^{n+1} \right\rangle \end{aligned}$$

The result now follows. $\square$

Viewing $H_n$ as $L_2(\Omega_n, 2^{\Omega_n}, p_c^n)$ we can write (4.3) in the simple form

$$\left\langle \rho_n \chi_{\{\omega'\}}, \chi_{\{\omega\}} \right\rangle = \langle \rho_{n+1} \chi_{\omega' \to}, \chi_{\omega \to} \rangle \tag{4.4}$$

**Corollary 4.3.** *A sequence $\rho_n \in \mathcal{Q}(H_n)$ is a QSGP if and only if (4.3) or (4.4) hold for every $\omega, \omega' \in \Omega_n$, $n = 1, 2, \ldots$.*

We now consider pure $q$-probability operators. In the following results we again view $H_n$ as $L_2(\Omega_n, 2^{\Omega_n}, p_c^n)$.

**Corollary 4.4.** *If $\rho_n \in \mathcal{Q}_p(H_n)$, $\rho_{n+1} \in \mathcal{Q}_p(H_{n+1})$ with $p_n = |\psi_n\rangle\langle\psi_n|$, $\rho_{n+1} = |\psi_{n+1}\rangle\langle\psi_{n+1}|$, then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have*

$$\left\langle\psi_n, \chi_{\{\omega\}}\right\rangle\left\langle\chi_{\{\omega'\}}, \psi_n\right\rangle = \langle\psi_{n+1}, \chi_{\omega\to}\rangle\langle\chi_{\omega'\to}, \psi_{n+1}\rangle \tag{4.5}$$

**Corollary 4.5.** *A sequence $|\psi_n\rangle\langle\psi_n| \in \mathcal{Q}_p(H_n)$ is a QSGP if and only if* (4.5) *holds for every $\omega, \omega' \in \Omega_n$.*

We say that $\psi_{n+1} \in \mathcal{V}(H_{n+1})$ is *strongly consistent* with $\psi_n \in \mathcal{V}(H_n)$ if for every $\omega \in \Omega_n$ we have

$$\left\langle\psi_n, \chi_{\{\omega\}}\right\rangle = \langle\psi_{n+1}, \chi_{\omega\to}\rangle \tag{4.6}$$

By (4.5) strong consistency implies the consistency of the corresponding $q$-probability operators.

**Corollary 4.6.** *If $\psi_{n+1} \in \mathcal{V}(H_{n+1})$ is strongly consistent with $\psi_n \in \mathcal{V}(H_n)$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n| \in \mathcal{Q}_p(H_n)$ is a QSGP.*

**Lemma 4.7.** *If $\psi_n \in \mathcal{V}(H_n)$ and $\psi_{n+1} \in H_{n+1}$ satisfies* (4.6) *for every $\omega \in \Omega_n$, then $\psi_{n+1} \in \mathcal{V}(H_{n+1})$.*

*Proof.* Since $\psi_n \in \mathcal{V}(H_n)$ we have by (4.6) that

$$\begin{aligned}|\langle\psi_{n+1}, 1\rangle| &= \left|\sum_{\omega\in\Omega_n}\langle\psi_{n+1}, \chi_{\omega\to}\rangle\right| = \left|\sum_{\omega\in\Omega_n}\left\langle\psi_n, \chi_{\{\omega\}}\right\rangle\right|\\ &= |\langle\psi_n, 1\rangle| = 1\end{aligned}$$

The result now follows. □

**Corollary 4.8.** *If $\|\psi_1\| = 1$ and $\psi_n \in H_n$ satisfies* (4.6) *for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n|$ is a QSGP.*

*Proof.* Since $\|\psi_1\| = 1$, it follows that $\psi_1 \in \mathcal{V}(H_1)$. By Lemma 4.7, $\psi_n \in \mathcal{V}(H_n)$, $n = 1, 2, \ldots$. Since (4.6) holds, the result follows from Corollary 4.6. □

Another way of writing (4.6) is

$$\sum_{\omega\to x} p_c^{n+1}(\omega x)\psi_{n+1}(\omega x) = p_c^n(\omega)\psi_n(x) \tag{4.7}$$

for every $\omega \in \Omega_n$.

# 5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for discrete quantum gravity. The simplest way to construct a QSGP is to form the constant pure DQP $\rho_n = |1\rangle\langle 1|$, $n = 1, 2, \ldots$. To show that $\rho_n$ is indeed consistent, we have for $\omega \in \Omega_n$ that

$$\sum_{\omega \to x} p_c^{n+1}(\omega x) = \sum_{\omega \to x} p_c^n(\omega) p_c(\omega \to x) = p_c^n(\omega) \sum_{\omega \to x} p_c(\omega \to x) = p_c^n(\omega)$$

so consistency follows from (4.7). The corresponding $q$-measures are given by

$$\mu_n(A) = |\langle 1, \chi_A \rangle|^2 = p_c^n(A)^2$$

for every $A \in \mathcal{A}_n$. Hence, $\mu_n$ is the square of the classical measure. Of course, $|1\rangle\langle 1|$ is the global $q$-probability operator for this QSGP and in this case $\mathcal{S}(\rho_n) = \mathcal{A}$. Moreover, we have the global $q$-measure $\mu(A) = \nu_c(A)^2$ for $A \in \mathcal{A}$.

Another simple way to construct a QSGP is to employ Corollary 4.8. In this way we can let $\psi_1 = 1$, $\psi_2$ any vector in $L_2(\Omega_2, 2^{\Omega_2}, p_c^2)$ satisfying

$$\left\langle \psi_2, \chi_{\{x_1 x_2\}} \right\rangle + \left\langle \psi_2 \chi_{\{x_1 x_3\}} \right\rangle = \left\langle \psi_1, \chi_{\{x_1\}} \right\rangle = 1$$

and so on, where $x_1, x_2, x_3$ are given in Figure 2. As a concrete example, let $\psi_1 = 1$,

$$\psi_2 = \tfrac{1}{2} \left[ p_c^2(x_1 x_2)^{-1} \chi_{\{x_1 x_2\}} + p_c^2(x_1 x_3) \chi_{\{x_1 x_3\}} \right]$$

and in general

$$\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p_c^n(\omega)^{-1} \chi_{\{\omega\}}$$

The $q$-measure $\mu_1$ is $\mu_1(\{x_1\}) = 1$ and $\mu_2$ is given by

$$\mu_2(\{x_1 x_2\}) = \left| \left\langle \psi_2, \chi_{\{x_1 x_2\}} \right\rangle \right|^2 = \tfrac{1}{4}$$
$$\mu_2(\{x_1 x_3\}) = \left| \left\langle \psi_2, \chi_{\{x_1 x_3\}} \right\rangle \right|^2 = \tfrac{1}{4}$$
$$\mu_2(\Omega_2) = |\langle \psi_2, 1 \rangle|^2 = 1$$

In general, we have $\mu_n(A) = |A|^2 / |\Omega_n|^2$ for all $A \in \Omega_n$. Thus $\mu_n$ is the square of the uniform distribution. The global operator does not exist because there is no $q$-measure on $\mathcal{A}$ that extends $\mu_n$ for all $n \in \mathbb{N}$. For $A \in \mathcal{A}$ we have

$$\langle \psi_n, \chi_A \rangle = \int \psi_n \chi_A d\nu_c = \frac{|A \cap \{\mathrm{cyl}(\omega) \colon \omega \in \Omega_n\}|}{|\Omega_n|}$$

Letting $\rho_n = |\psi_n\rangle\langle\psi_n|$ we conclude that $A \in \mathcal{S}(\rho_n)$ if and only if

$$\lim_{n\to\infty} \frac{|A \cap \{\mathrm{cyl}(\omega)\colon \omega \in \Omega_n\}|}{|\Omega_n|}$$

exists. For example, if $|A| < \infty$ then for $n$ sufficiently large we have

$$|A \cap \{\mathrm{cyl}(\omega)\colon \omega \in \Omega_n\}| = |A|$$

so $A \in \mathcal{S}(\rho_n)$ and $\mu(A) = 0$. In a similar way if $|A| < \infty$ then for the complement $A'$, if $n$ is sufficiently large we have

$$|A' \cap \{\mathrm{cyl}(\omega)\colon \omega \in \Omega_n\}| = |\Omega_n| - |A|$$

so $A' \in \mathcal{S}(\rho_n)$ with $\mu(A') = 1$.

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let $\alpha_\omega \in \mathbb{C}$, $\omega \in \Omega_n$ satisfy

$$\left|\sum_{\omega\in\Omega_n} \alpha_\omega p_c^n(\omega)^{1/2}\right| = 1 \tag{5.1}$$

and let $\rho_n$ be the operator on $H_n$ satisfying

$$\langle \rho_n e_\omega^n, e_{\omega'}^n\rangle = \alpha_{\omega'}\overline{\alpha_\omega} \tag{5.2}$$

Then $\rho_n$ is a positive operator and by (5.1), (5.2) we have

$$\begin{aligned}
\langle \rho_n 1, 1\rangle &= \left\langle \rho_n \sum_\omega p_c^n(\omega)^{1/2} e_\omega^n, \sum_{\omega'} p_c^n(\omega')^{1/2} e_{\omega'}^n \right\rangle \\
&= \sum_{\omega,\omega'} p_c^n(\omega)^{1/2} p_c^n(\omega')^{1/2} \langle \rho_n e_\omega^n, e_{\omega'}^n\rangle \\
&= \left|\sum_\omega p_c^n(\omega)^{1/2}\alpha_\omega\right|^2 = 1
\end{aligned}$$

Hence, $\rho_n \in \mathcal{Q}(H_n)$. Now

$$\Omega_{n+1} = \{\omega x\colon \omega \in \Omega_n, x \in \mathcal{P}_{n+1}, \omega \to x\}$$

and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy

$$\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} p_c^{n+1}(\omega x)^{1/2} \right| = 1$$

Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying

$$\left\langle \rho_{n+1} e_{\omega x}^{n+1}, e_{\omega' x'}^{n+1} \right\rangle = \beta_{\omega' x'} \overline{\beta_{\omega x}} \tag{5.3}$$

As before, we have that $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** *The operator $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have*

$$\alpha_{\omega'} \overline{\alpha_\omega} = \sum_{\substack{x' \in \mathcal{P}_{n+1} \\ \omega' \to x'}} \beta_{\omega' x'} p_c(\omega' \to x')^{1/2} \sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega \to x}} \overline{\beta_{\omega x}} p_c(\omega \to x)^{1/2} \tag{5.4}$$

A sufficient condition for (5.4) to hold is

$$\sum_{\substack{x \in \mathcal{P}_{n+1} \\ \omega \to x}} \beta_{\omega x} p_c(\omega \to x)^{1/2} = \alpha_\omega \tag{5.5}$$

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** *Let $\rho_n \in \mathcal{Q}(H_n)$ be defined by* (5.2) *and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by* (5.3)*. If* (5.5) *holds, then $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_n$.*

The next result gives the general construction.

**Corollary 5.3.** *Let $\rho_1 = I \in \mathcal{Q}(H_1)$ and define $\rho_n \in \mathcal{Q}(H_n)$ inductively by* (5.3)*. Then $\rho_n$ is a QSGP.*

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