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LIFTING MODULE MAPS BETWEEN DIFFERENT NONCOMMUTATIVE
DOMAIN ALGEBRAS

A DISSERTATION
PRESENTED TO
THE FACULTY OF NATURAL SCIENCES AND MATHEMATICS
UNIVERSITY OF DENVER

IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY
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AUGUST 2010
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Abstract

The classical Carathéodory interpolation problem is the following: let $n \in \mathbb{N}$, $a_0, a_1, \dots, a_N \in \mathbb{C}$, and \mathbb{D} the unit disk. When does there exist an analytic function $F : \mathbb{D} \rightarrow \mathbb{C}$ and complex numbers a_{N+1}, a_{N+2}, \dots such that $F(z) = a_0 + a_1z + a_2z^2 + \dots + a_Nz^N + a_{N+1}z^{N+1} + \dots$ and $\|F\|_\infty < 1$? In 1967, Sarason used operator theory techniques to give an elegant solution to the Carathéodory interpolation problem. In 1968, Sz.-Nagy and Foias extended Sarason's approach into a commutant lifting theorem. Both the theorem and the technique of the proof have become standard tools in control theory. In particular, the commutant lifting theorem approach lends itself to a wide range of generalizations. This thesis concerns one such generalization.

Arias presented generalizations of the original commutant lifting theorem relating to the full Fock space. Popescu then refined the approach by introducing domain algebras. While Arias and Popescu focused on module maps from one noncommutative domain algebra to itself, we present a unitarily equivalent representation which allows us to easily generalize their results to module maps between different noncommutative domain algebras. We use a renorming technique to prove that as long as the "formal identity map" is bounded, we can lift module maps between different noncommutative domains generated by finite positive regular free holomorphic functions. Finally, we apply the lifting theorem to create projective resolutions.

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Chapter 1

Introduction

In [21], Sarason used an operator theoretic approach to solve two different interpolation problems. The first of these problems, the Carathéodory interpolation problem, was introduced by Carathéodory in [6] and extensively studied by Schur in [22] and [23]. The second of these problems, the Nevanlinna-Pick interpolation problem, was posed and solved by Pick in [16]. Unaware of Pick's work, Nevanlinna posed and solved the problem in [13] and expanded his solution in [14]. By taking an operator theoretic approach to these problems, Sarason was able to solve both problems in almost identical fashion. This approach was refined by Sz.-Nagy and C. Foiaş in [24] and [25] into a commutant lifting theorem. The advantage of this approach is that it is very generalizable. This thesis is concerned with one such generalization.

The main results of this thesis are contained in Chapters 5 and 6. In Chapter 4, the notion of a module map is introduced. This notion is tied to that of a weighted Fock space in Chapter 5, where we attempt to categorize when module maps exist. This is where the first major result, Theorem 5.3.1 is

obtained. In Theorem 5.3.1, equivalent conditions to the existence of non-zero module maps are obtained. This result allows us to determine when non-zero module maps exist by asking easily computable questions of the underlying weighted Fock spaces.

In chapter 6, a generalization of the commutant lifting theorem of [25] is suggested. However, the usual machinery of proving such lifting theorems, Parrott's Lemma ([15]), fails. The second major result of this thesis is obtained in the form of the lifting theorem of Theorem 6.1.7, which is proved using a renorming technique. This renorming technique allows us to partially sidestep Parrott's Lemma, and thus avoid the difficulty encountered using the usual methods of commutant lifting theorems.

In [19], Popescu studied noncommutative domains $\mathcal{D}_f(\mathcal{H}) \subset B(\mathcal{H})^n$ generated by positive regular free holomorphic functions f . He proved that each such domain has a universal model (W_1, W_2, \dots, W_n) of weighted orthogonal shifts acting on the full Fock space with n generators. In this thesis, we will look at a unitarily equivalent version of the weighted Fock spaces used by Popescu, which we will denote $\mathcal{F}^2(f)$. We will use a Hilbert module language similar to that of Douglas and Paulsen [8] and Muhly and Solel [12].

As stated in [19], a commutant lifting theorem for $\mathcal{F}^2(f)$ follows directly from [2], which was adapted from a paper of Clancy and McCullough [7]. Assume \mathcal{M} and \mathcal{N} are both $*$ -submodules of $\mathcal{F}^2(f)$ and $X : \mathcal{M} \rightarrow \mathcal{N}$ is a non-zero module map. This commutant lifting theorem states that there exists a module map $\widehat{X} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(f)$ with $\|X\| = \|\widehat{X}\|$ such that the diagram in Figure 1.1 commutes.

$$\begin{array}{ccccc}
\mathcal{F}^2(f) & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\exists \downarrow \widehat{X} & & \downarrow X & & \\
\mathcal{F}^2(f) & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 1.1: The commutant lifting theorem of [19].

Notice here that both \mathcal{N} and \mathcal{M} are $*$ -submodules of the same Hilbert module $\mathcal{F}^2(f)$. In this thesis we ask a more general question. What if \mathcal{M} and \mathcal{N} are instead $*$ -submodules of $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ respectively? This gives the diagram shown in Figure 1.2.

$$\begin{array}{ccccc}
\mathcal{F}^2(f) & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
& & \downarrow X & & \\
\mathcal{F}^2(g) & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 1.2: Setup for a lifting theorem.

In this case, there is certainly no reason why we should be able to lift X to a module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ in general. Indeed, there are examples of spaces $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ such that there does not exist a non-zero module map $\widehat{X} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ of any kind. But if non-zero module maps do exist between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$, under what cases can a lifting theorem be proved?

Assuming that the “formal identity map” $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is a contraction, the commutant lifting theorems of [2] and [7] can be adjusted with some care to prove that there exists an \widehat{X} with $\|X\| = \|\widehat{X}\|$ such that the diagram in Figure 1.3 commutes.

$$\begin{array}{ccccc}
\mathcal{F}^2(f) & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\exists \downarrow \widehat{X} & & \downarrow X & & \\
\mathcal{F}^2(g) & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 1.3: A lifting theorem between different noncommutative domain algebras.

We follow the standard argument of lifting element by element, then iterating using Parrott’s lemma [15]. However, this technique breaks down if the “formal identity map” $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is not a contraction.

In the case where $\mathcal{F}^2(g)$ is the full Fock space and $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is bounded, it immediately follows that ε is a contraction, and thus any module map $X : \mathcal{M} \rightarrow \mathcal{N}$ can be lifted to an $\widehat{X} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ with $\|X\| = \|\widehat{X}\|$. This suggests that $X : \mathcal{M} \rightarrow \mathcal{N}$ can be lifted (with some penalty in the norm) in general, as long as there exists a non-zero module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$.

To prove that this is indeed the case, we apply a renorming technique to $\mathcal{F}^2(f)$ to create the situation where we can apply our lifting theorem. It turns out that if the “formal identity map” $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ has norm $\|\varepsilon\| = C$, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ with $\|\widehat{X}\| \leq C \|X\|$ such that the diagram in Figure 1.3 commutes.

Chapter 2 will present some background material needed to approach the subject. This will include a brief look at both the Carathéodory and Nevanlinna-Pick interpolation problems. In order to present a solution to these problems, both Sarason’s approach and the Sz.-Nagy-Foiaş refinement will be presented. Chapter 3 will introduce the full Fock space. In Chapter 4 we will give a brief overview of the Hilbert module language. Chapter

5 will discuss the weighted Fock spaces given in [2] and [19]. A unitarily equivalent description of these weighted Fock spaces will be presented, which will allow some aspects of weighted Fock spaces to be addressed with greater transparency. Finally, a lifting theorem for a specific class of weighted Fock spaces will be given in Chapter 6. For convenience, we will assume that our functions f and g are finite.

Chapter 2

Motivation

We will now introduce some preliminaries which will be necessary for this thesis. We begin by discussing two representations of the infinite separable Hilbert space as well as the Banach algebra H^∞ . We will then look at the unilateral shift operators and multiplication operators. These operators play a fundamental role in our study of noncommutative domain algebras. Finally, we will introduce some historical interpolation results.

2.1 Preliminaries

2.1.1 Two Representations of a Hilbert Space

We will begin by discussing two representations, ℓ^2 and the Hardy space H^2 , of the infinite separable Hilbert space.

Definition 2.1.1. *The Hilbert space ℓ^2 is the collection of all square-summable sequences (a_0, a_1, a_2, \dots) , where $a_i \in \mathbb{C}$:*

$$\ell^2 := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

The norm of the sequence $(a_n)_{n=0}^{\infty}$ is given by

$$\|(a_n)_{n=0}^{\infty}\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

The space ℓ^2 is an infinite dimensional separable Hilbert space, and thus has a countable orthonormal basis given by the set $\{e_n\}_{n=0}^{\infty}$, where $e_i = (0, 0, \dots, 0, 1, 0, 0, \dots)$ is the sequence with zero in every coordinate except the i th coordinate, which is 1.

Definition 2.1.2. *The Hardy space H^2 is the collection of all analytic functions on the unit disc whose power series representation has square-summable complex coefficients:*

$$H^2 := \left\{ f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

The norm of the analytic function $f = \sum_{n=0}^{\infty} a_n z^n$ is given by

$$\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function in H^2 and let $\theta \in \mathbb{T}$ be given.

Define $f_r(\theta)$ by $f_r(\theta) = f(re^{i\theta})$. Now $\lim_{r \rightarrow 1^-} f_r(\theta)$ exists almost everywhere. Thus, define $F(\theta) = \lim_{r \rightarrow 1^-} f_r(\theta)$. Since $f_r(\theta) = \sum_{n=0}^{\infty} a_n r^n \theta^n$, we get that for $|r| < 1$,

$$\|f_r\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 |r|^{2n} < \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_{H^2}^2$$

Thus, the functions f_r are bounded for each $r < 1$. By Hoffman ([10], pg 32), it follows that $F \in L^2(\mathbb{T})$. From here we get the following alternative definition of the Hardy space H^2 :

Definition 2.1.3. *The Hardy space H^2 can be thought of as the following subset of L^2 :*

$$H^2 := \left\{ f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for } n < 0 \right\}.$$

With the following norm:

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}.$$

Define a map $F : \ell^2 \rightarrow H^2$ by $(a_n)_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} a_n z^n$. It is easy to see that F defines an onto isometry between ℓ^2 and H^2 . Thus, H^2 is an infinite dimensional separable Hilbert space. Furthermore, the orthonormal basis for H^2 is $1, z, z^2, z^3, \dots$. While the Hilbert spaces ℓ^2 and H^2 are isomorphic, it is useful to have both definitions, as each definition will give a different set of insights into the Fock spaces discussed in Chapters 3 and 5.

2.1.2 The Banach Algebra H^∞

Another space which will be of importance is H^∞ , the Banach algebra of holomorphic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with a finite essential supremum. The space H^∞ is given in the following definition:

Definition 2.1.4. *The Banach algebra H^∞ is given by:*

$$\begin{aligned} H^\infty &:= \left\{ f = \sum_{n=0}^{\infty} a_n z^n : \text{ess sup}_{z \in \mathbb{D}} |f(z)| < \infty \right\} \\ &= \left\{ f \in L^\infty(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \right\}. \end{aligned}$$

The norm of the bounded holomorphic function $f = \sum_{n=0}^{\infty} a_n z^n$ is given by

$$\|f\|_\infty = \text{ess sup}_{z \in \mathbb{D}} |f(z)|.$$

While H^∞ is not a Hilbert space, it plays a fundamental role in the study of interpolation problems.

2.1.3 Unilateral Shift Operators

The unilateral shift operators are the foundation upon which the study of domain algebras is based. We will now define the shift operators $S : \ell^2 \rightarrow \ell^2$ and $M_z : H^2 \rightarrow H^2$:

Definition 2.1.5. *The unilateral shift operator $S : \ell^2 \rightarrow \ell^2$ is defined by $Se_i = e_{i+1}$ for $i \geq 0$.*

Definition 2.1.6. *The multiplication operator $M_z : H^2 \rightarrow H^2$ is defined by $M_z f = zf$.*

$$e_0 \xrightarrow{S} e_1 \xrightarrow{S} e_2 \xrightarrow{S} e_3 \xrightarrow{S} \dots$$

Figure 2.1: The unilateral shift operator on ℓ^2 .

$$1 \xrightarrow{M_z} z \xrightarrow{M_z} z^2 \xrightarrow{M_z} z^3 \xrightarrow{M_z} \dots$$

Figure 2.2: The multiplication operator on H^2 .

The unilateral shift operator and multiplication operator above can both be generalized. Let p be an arbitrary polynomial in H^2 , and define $M_p : H^2 \rightarrow H^2$ as multiplication by p . The unilateral shift operator $S : \ell^2 \rightarrow \ell^2$ can be extended similarly. The multiplication operator M_p gives a strong connection between H^2 and H^∞ , as shown in the following lemma:

Lemma 2.1.7. *Let $p \in H^\infty$ be given. Then:*

$$\|M_p\| = \|p\|_\infty$$

Moreover, if $f \in H^2$, then M_f is bounded if and only if $f \in H^\infty$.

Proof. (Sz.-Nagy and Foiaş [24]) First, let $p \in H^\infty$. Then:

$$\|p\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |a_{n-1}|^2 \right)^{\frac{1}{2}} = \|zp\|_2 = \|M_z p\|_2$$

Let $p \in H^\infty$ be given. Then for every $f \in H^2$, we have

$$\begin{aligned} \|M_p f\|_2 &= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})f(e^{i\theta})|^2 d\theta} \\ &\leq \sqrt{\left(\sup_{z \in \mathbb{D}} |p|\right)^2 \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta} \\ &= \|p\|_\infty \|f\|_2 \end{aligned}$$

Thus, it follows that $\|M_p\| \leq \|p\|_\infty$.

Next, let $\epsilon > 0$ be given. Now since p is a bounded holomorphic function, there exists $U \subset \mathbb{T}$ with $\mu(U) > 0$ such that $|p(z)| \geq \|p\|_\infty - \epsilon$ for all $z \in U$. There also exists an open set $G \subset U$ with $\mu(U \setminus G) < \epsilon$. Without loss of generality, we may assume that $G = (a, b) \subset \mathbb{T}$. Now define $g \in C(\mathbb{T})$ by $g(z) = \frac{1}{(b-a)-\epsilon}$ if $z \in (a+\epsilon, b-\epsilon)$ and $g(z) = 0$ if $z \in (a, b)^c$. Define $g(z)$ in the obvious linear fashion when $z \in (a, a+\epsilon)$ and $(b-\epsilon, b)$. It is easy to see that $\|g\|_2 = 1$. Now we can find a polynomial $q \in C(\mathbb{T})$ such that $\|g(z) - q(z)\|_\infty < \sqrt{\frac{\epsilon}{(b-a)}}$. If q is analytic, let $f(z) = q(z)$. Otherwise, $q(z) = \sum_{n=-m}^{\infty} a_n z^n$ where $m > 0$. Define $f(z) = z^m q(z)$. Now $f(z)$ is analytic and thus $f \in H^2$. Then:

$$\begin{aligned}
\|M_p f\|_2 &= \|pf\| = \|pz^m q\| = \|pq\| = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})q(e^{i\theta})|^2 d\theta} \\
&\geq \sqrt{\frac{1}{2\pi} \int_{(a,b)} |p(e^{i\theta})q(e^{i\theta})|^2 d\theta} \\
&\geq \sqrt{\frac{1}{2\pi} \int_{(a,b)} (\|p\|_\infty - \epsilon)^2 |q(e^{i\theta})|^2 d\theta} \\
&= (\|p\|_\infty - \epsilon) \sqrt{\frac{1}{2\pi} \int_{(a,b)} |q(e^{i\theta})|^2 d\theta} \\
&\geq (\|p\|_\infty - \epsilon) \sqrt{\frac{1}{2\pi} \int_{(a,b)} (|g(e^{i\theta})|^2 - |g(e^{i\theta}) - q(e^{i\theta})|^2) d\theta} \\
&= (\|p\|_\infty - \epsilon) \sqrt{\frac{1}{2\pi} \int_{(a,b)} |g(e^{i\theta})|^2 d\theta - \int_{(a,b)} |g(e^{i\theta}) - q(e^{i\theta})|^2 d\theta} \\
&\geq (\|p\|_\infty - \epsilon) \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta - \int_{(a,b)} \frac{\epsilon}{(b-a)} d\theta} \\
&= (\|p\|_\infty - \epsilon) \sqrt{\|g\|_2^2 - \epsilon} = (\|p\|_\infty - \epsilon) \sqrt{1 - \epsilon}
\end{aligned}$$

Since ϵ is arbitrary, it follows that $\|M_p\| \geq \|p\|_\infty$. Therefore, $\|M_p\| = \|p\|_\infty$ as desired. \square

This leads immediately to the following corollary:

Corollary 2.1.8.

$$H^\infty = \{f \in H^2 : \|M_f\| < \infty\}$$

In this sense, the Banach algebra H^∞ can be viewed as an operator algebra.

In particular, we will view H^∞ as the multiplier algebra of H^2 .

2.2 Interpolation Problems

We are now in a position to state some classical problems of interpolation theory.

Problem 2.2.1. (*Carathéodory Interpolation Problem, 1907*)

Let $a_0, a_1, a_2, \dots, a_N$ be complex numbers such that the function $f(z)$ is analytic at 0. When does there exist an analytic function $F : \mathbb{D} \rightarrow \mathbb{C}$ with the following properties:

$$\begin{aligned} F(z) &= a_0 + a_1z + a_2z^2 + \dots + a_Nz^N + \dots \\ \|F\|_\infty &\leq 1? \end{aligned}$$

Problem 2.2.2. (*Nevanlinna-Pick Interpolation Problem, 1916*)

Let N points $\lambda_1, \lambda_2, \dots, \lambda_N$ in the unit disk \mathbb{D} and N complex numbers c_1, c_2, \dots, c_N be given. When does there exist an analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$ with the following properties:

$$\begin{aligned} f(\lambda_i) &= c_i, 1 \leq i \leq N \\ \|f\|_\infty &\leq 1? \end{aligned}$$

Necessary and sufficient conditions for both the Nevanlinna-Pick and the Carathéodory interpolation problems were discovered relatively quickly, but the proofs were cumbersome. Furthermore, the effort to generalize these problems was difficult due to the nature of the proofs. However, Sarason ([20], [21]) developed an operator-theoretic approach which allowed these interpolation problems to be solved with greater ease. This approach was formalized by

Sz.-Nagy and Foiaş ([24], [25]) into the following commutant lifting theorem:

Theorem 2.2.3. (*Sz.-Nagy, Foiaş, 1968*)

Let \mathcal{H} and \mathcal{K} be Hilbert spaces with $\mathcal{H} \subset \mathcal{K}$, $T : \mathcal{H} \rightarrow \mathcal{H}$ and $W : \mathcal{K} \rightarrow \mathcal{K}$ be bounded, and $W^*|_{\mathcal{H}} = T^*$. If $X : \mathcal{H} \rightarrow \mathcal{H}$ is given with $XT = TX$, then there exists $Y : \mathcal{K} \rightarrow \mathcal{K}$ such that:

- $Y^*|_{\mathcal{H}} = X^*$
- $YW = WY$
- $\|Y\| = \|X\|$

ie, the diagram in Figure 2.3 commutes.

$$\begin{array}{ccccc}
 \mathcal{K} & \xrightarrow{P_{\mathcal{H}}} & \mathcal{H} & \longrightarrow & 0 \\
 \exists \downarrow Y & & \downarrow X & & \\
 \mathcal{K} & \xrightarrow{P_{\mathcal{H}}} & \mathcal{H} & \longrightarrow & 0
 \end{array}$$

Figure 2.3: The diagram for Theorem 2.2.3

We immediately arrive at the following corollary:

Corollary 2.2.4. *Let \mathcal{M} be a subspace of H^2 that is $*$ -invariant under S . Let $X : \mathcal{M} \rightarrow \mathcal{M}$ be a contraction that commutes with $T = P_{\mathcal{M}}S|_{\mathcal{M}}$. Then there exists $F : H^2 \rightarrow H^2$ such that $F^*|_{\mathcal{M}} = X^*$, $FS = SF$, and $\|F\| = \|X\|$. Furthermore, there exists $\phi \in H^\infty$ such that $F = M_\phi$.*

$$\begin{array}{ccccc}
H^2 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\downarrow F=M_\phi & & \downarrow X & & \\
H^2 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0
\end{array}$$

Figure 2.4: Diagram for Corollary 2.2.4

Using the above commutant lifting theorem, the Carathéodory interpolation problem can be addressed with considerable ease, as shown in the following theorem:

Theorem 2.2.5. (*Carathéodory Interpolation Problem, 1907*) *Given complex numbers a_0, a_1, \dots, a_N , there exists an analytic function $f(z) = a_0 + a_1z + \dots + a_Nz^N + \dots$ in the unit ball of $H^\infty(\mathbb{C})$ if and only if the Toeplitz matrix*

$$\begin{bmatrix}
a_0 & 0 & 0 & \cdots & 0 \\
a_1 & a_0 & 0 & \cdots & 0 \\
a_2 & a_1 & a_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
a_N & a_{N-1} & a_{N-2} & \cdots & a_0
\end{bmatrix}$$

is a contraction.

Proof. (Sarason, 1968) Choose $\mathcal{M} = \vee\{z^k : 0 \leq k \leq N-1\}$, the closed linear span of $1, z, z^2, \dots, z^{N-1}$. Apply Corollary 2.2.4. \square

The Nevanlinna-Pick interpolation problem can be solved with similar ease by using a different choice of \mathcal{M} , but we will need one definition before we can state and prove the original Nevanlinna-Pick interpolation theorem.

Definition 2.2.6. Let H^2 and $\lambda \in \mathbb{D}$ be given. The Szegő kernel at λ , denoted k_λ , is the element of H^2 with the property that for all $f \in H^2$, $\langle f, k_\lambda \rangle = f(\lambda)$

Note that the Szegő kernel exists since evaluation at any point $\lambda \in \mathbb{D}$ is a continuous linear functional on H^2 . We will now state and prove the original Nevanlinna-Pick interpolation problem:

Theorem 2.2.7. (Nevanlinna-Pick Interpolation Problem, 1916) Given points $\lambda_1, \lambda_2, \dots, \lambda_N$ in the unit disk \mathbb{D} and complex numbers c_1, c_2, \dots, c_N , there is a function $\phi \in H^\infty$ satisfying $\phi(\lambda_i) = c_i$ for $1 \leq i \leq N$ if and only if the Pick matrix

$$\left(\frac{1 - c_i \bar{c}_j}{1 - \lambda_i \bar{\lambda}_j} \right)_{i,j=1}^N \tag{2.2.1}$$

is positive semi-definite.

Proof. Choose $\mathcal{M} = \vee\{k_{\lambda_i} : 1 \leq i \leq N\}$. Apply Corollary 2.2.4. □

The beauty of the approach developed by Sarason and streamlined by Sz.-Nagy and Foiaş is that the proofs of both the Nevanlinna-Pick and the Carathéodory interpolation problems are virtually identical. More importantly, it allowed more general interpolation results to follow. In the remainder of this thesis, we will explore one such generalization.

Chapter 3

The Full Fock Space

In this chapter, we will introduce the full Fock space $\mathcal{F}^2(\mathcal{H})$ of a Hilbert space \mathcal{H} . A formal definition of $\mathcal{F}^2(\mathcal{H})$ will be given in Section 3.1. In Section 3.2, we will define the left and right creation operators. These operators are fundamental in the study of the domain algebras introduced by Popescu in [19] and discussed in Chapter 5. Finally, Section 3.3 introduces the noncommutative disk algebra \mathcal{A}_n , which turns out to be the universal model of all row contractions.

3.1 The Full Fock Space of a Hilbert space \mathcal{H}

Let \mathcal{H} be an n -dimensional Hilbert space with orthonormal basis e_1, e_2, \dots, e_n .

The full Fock space of \mathcal{H} is defined as

$$\mathcal{F}^2(\mathcal{H}) := \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \dots$$

with the following inner product, defined on elementary tensors:

$$\langle \zeta_0 \otimes \zeta_1 \otimes \dots \otimes \zeta_j, \xi_0 \otimes \xi_1 \otimes \dots \otimes \xi_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ \prod_{i=0}^n \langle \zeta_i, \xi_i \rangle & \text{otherwise} \end{cases}$$

If the underlying Hilbert space is clear, we shall denote $\mathcal{F}^2(\mathcal{H})$ by \mathcal{F}^2 and $I_{\mathcal{H}}$ by I .

One representation of the full Fock space is given using the free semigroup on n generators. Let \mathbb{F}_n^+ be the unital free semigroup on n generators, g_1, g_2, \dots, g_n , together with the identity g_0 . Define the length of an element $\alpha \in \mathbb{F}_n^+$ as $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1}g_{i_2} \dots g_{i_k}$, where $1 \leq i_1, i_2, \dots, i_k \leq n$. For each $\alpha \in \mathbb{F}_n^+$, define

$$e_\alpha := \begin{cases} e_{g_{i_1}} \otimes e_{g_{i_2}} \otimes \dots \otimes e_{g_{i_k}} & \text{if } \alpha = g_{i_1}g_{i_2} \dots g_{i_k} \\ 1 & \text{if } \alpha = g_0 \end{cases}$$

It is clear that the set $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis for the full Fock space \mathcal{F}^2 . Thus, we can identify the full Fock space \mathcal{F}^2 with the Hilbert space $\ell^2(\mathbb{F}_n^+)$.

Finally, if (T_1, T_2, \dots, T_n) is an n -tuple of operators $T_i \in B(\mathcal{H})$ and $\alpha = g_{i_1}g_{i_2} \dots g_{i_k} \in \mathbb{F}_n^+$, define T_α to be the product $T_{i_1}T_{i_2} \dots T_{i_k}$.

3.2 Left and Right Creation Operators

The concept of a row contraction is fundamental in both the study of the full Fock space, defined in the previous section, and the domain algebras introduced by Popescu in [19].

Definition 3.2.1. *Let \mathcal{H} be an n dimensional Hilbert space. A row contraction on \mathcal{H} is an n -tuple of operators (T_1, T_2, \dots, T_n) , with $T_i \in B(\mathcal{H})$ for every $1 \leq i \leq n$, such that*

$$\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}$$

It is usually beneficial to think of the n -tuple (T_1, T_2, \dots, T_n) as the operator $T = [T_1 \ T_2 \ \cdots \ T_n]$ acting on \mathcal{H}^n . In this sense, (T_1, T_2, \dots, T_n) is a row contraction if and only if T is a contraction, since $\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}$ holds exactly when $TT^* \leq I_{\mathcal{H}^n}$.

With this definition in mind, we can begin our discussion on the left and right creation operators.

Definition 3.2.2. *For each $i = 1, 2, \dots, n$, the left creation operators $L_i : \ell^2(\mathbb{F}_n^+) \rightarrow \ell^2(\mathbb{F}_n^+)$ are defined by*

$$L_i e_{\alpha} = e_{g_i \alpha}$$

for every $\alpha \in \mathbb{F}_n^+$. Similarly, for each $i = 1, 2, \dots, n$, the right creation operators $R_i : \ell^2(\mathbb{F}_n^+) \rightarrow \ell^2(\mathbb{F}_n^+)$ are defined by

$$R_i e_\alpha = e_{\alpha g_i}$$

for every $\alpha \in \mathbb{F}_n^+$.

The left and right creation operators have some important properties, as shown in the following proposition:

Proposition 3.2.3. *The left (respectively, right) creation operators are isometries with orthogonal ranges. Furthermore, they are row contractions:*

$$\sum_{i=1}^n L_i L_i^* \leq I,$$

$$\sum_{i=1}^n R_i R_i^* \leq I.$$

Proof. Since the left (right) creation operators map the orthonormal basis of $\ell^2(\mathbb{F}_n^+)$ into the orthonormal basis of $\ell^2(\mathbb{F}_n^+)$, it is clear to see that they are isometries. Now the left (right) creation operators have orthogonal ranges since for all $\alpha \in \mathbb{F}_n^+$ and $i \neq j$,

$$\langle L_i \delta_\alpha, L_j \delta_\alpha \rangle = \langle \delta_{g_i \alpha}, \delta_{g_j \alpha} \rangle = 0$$

$$\langle R_i \delta_\alpha, R_j \delta_\alpha \rangle = \langle \delta_{\alpha g_i}, \delta_{\alpha g_j} \rangle = 0$$

Now to show that (L_1, L_2, \dots, L_n) forms a row contraction, let $\delta_\alpha \in \ell^2(\mathbb{F}_n^+)$ be given. If $|\alpha| \geq 1$, then $\alpha = g_j \beta$ for some $j \leq n$ and $\beta \in \mathbb{F}_n^+$ with $|\beta| = |\alpha| - 1$.

Then:

$$\left(\sum_{i=1}^n L_i L_i^* \right) \delta_\alpha = L_j \delta_\beta = \delta_\alpha.$$

However, if $|\alpha| = 0$, then:

$$\left(\sum_{i=1}^n L_i L_i^* \right) \delta_\alpha = 0.$$

Thus,

$$\left(I - \sum_{i=1}^n L_i L_i^* \right) \delta_\alpha \geq 0.$$

Thus, the left creation operators form a row contraction. An identical argument will verify that the right creation operators also form a row contraction. \square

The full Fock space for $n = 2$ together with the left creation operators L_1 and L_2 can be visualized as in figure 3.1.

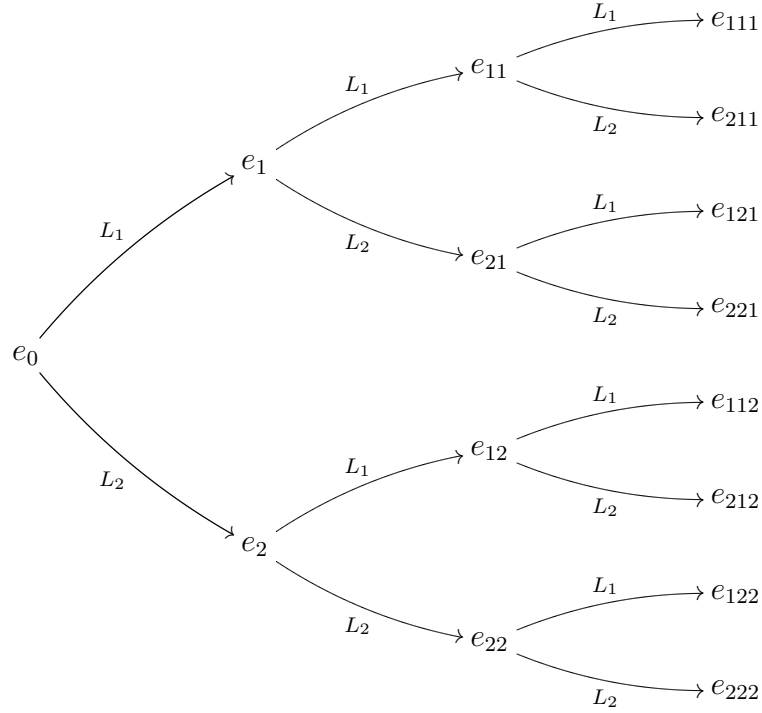


Figure 3.1: The full Fock space for $n = 2$.

3.3 The Noncommutative Disk Algebra \mathcal{A}_n

The study of noncommutative domains is based on the study various creation operators. When looking at the left creation operators, as defined above, we get the following algebra:

Definition 3.3.1. *The noncommutative disk algebra \mathcal{A}_n is the norm closure of the algebra generated by the left creation operators and the identity:*

$$\mathcal{A}_n = \overline{\{L_1, L_2, \dots, L_n, I\}}.$$

The noncommutative disk algebra \mathcal{A}_n is very important in the study of row contractions due to the following property discovered by Popescu:

Theorem 3.3.2. *(G. Popescu, [17]) Let (T_1, T_2, \dots, T_n) be an n -tuple of operators on a Hilbert space \mathcal{H} . Then (T_1, T_2, \dots, T_n) is a row contraction if and only if there exists a unique unital completely contractive morphism $\Phi : \mathcal{A}_n \rightarrow \mathcal{B}(\mathcal{H})$ such that $L_i \mapsto T_i$ for $1 \leq i \leq n$.*

The previous theorem states that the n -tuple of left creation operators (L_1, L_2, \dots, L_n) is the model of all row contractions. In [19], Popescu developed the notion of domain algebras, the natural generalizations of the noncommutative disk algebra \mathcal{A}_n with weighted shift operators. Domain algebras will be discussed in Chapter 5, where the weighted shift operators will be shown to be the models of row contractions in their corresponding domain algebras.

Chapter 4

Hilbert Modules

4.1 Hilbert Modules

Let \mathcal{H} be a Hilbert space, and $V_i : \mathcal{H} \rightarrow \mathcal{H}$ be bounded linear operators for $1 \leq i \leq n$. Then $(\mathcal{H}; V_1, V_2, \dots, V_n)$ is called a Hilbert module over the free algebra generated by n -noncommutative variables. In this thesis, we will primarily consider Hilbert modules of the form $(\mathcal{F}^2(f) \otimes \mathcal{H}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$, where $\mathcal{F}^2(f)$ is a weighted Fock space, which will be defined in chapter 5, \mathcal{H} is an arbitrary Hilbert space, and the L_i 's are the left creation operators $L_i : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(f)$. For the purposes of this chapter, you may consider the weighted Fock space $\mathcal{F}^2(f)$ to be the full Fock space of chapter 3.

4.1.1 Submodules and *-Submodules

If $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is invariant under $L_i \otimes I_{\mathcal{H}}$ for $1 \leq i \leq n$, we say that $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Similarly, if $\mathcal{M} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is invariant under $(L_i \otimes I_{\mathcal{H}})^*$ for $1 \leq i \leq n$ and $V_i = P_{\mathcal{M}}(L_i \otimes$

$I_{\mathcal{H}}|_{\mathcal{M}}$ for $1 \leq i \leq n$, we say that $(\mathcal{M}; V_1, V_2, \dots, V_n)$ is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Every submodule induces a corresponding $*$ -submodule, and every $*$ -submodule induces a corresponding submodule, as shown in the following lemma:

Lemma 4.1.1. *Let $(\mathcal{F}^2(f) \otimes \mathcal{H}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ be a Hilbert module. Then $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$ if and only if $(\mathcal{M}; V_1, V_2, \dots, V_n)$, where $\mathcal{M} = \mathcal{H} \ominus \mathcal{N}$, is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$*

Proof. Let $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ be a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Then for $x \in \mathcal{N}$ and $y \in \mathcal{M}$, $\langle x, L_i^* y \rangle = \langle L_i x, y \rangle$. But since $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$, $L_i x \in \mathcal{N}$, and so $\langle L_i x, y \rangle = 0$. Thus, $\langle x, L_i^* y \rangle = 0$ and so it follows that $(\mathcal{M}; V_1, V_2, \dots, V_n)$ is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$, as desired. The other direction is identical. \square

4.1.2 Semi-invariant Subspaces

Let a submodule $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ and an algebra $A \subset B(\mathcal{F}^2(f) \otimes \mathcal{H})$ be given. Then \mathcal{N} is said to be semi-invariant under A if for every $a, b \in A$ it follows that $P_{\mathcal{N}} a P_{\mathcal{N}} b P_{\mathcal{N}} = P_{\mathcal{N}} a b P_{\mathcal{N}}$. In [20] Sarason proved the following lemma, which we will state and prove in less generality:

Lemma 4.1.2. *(Sarason, [20]) Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be given. Then \mathcal{N} is semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$ if and only if there exist two submodules \mathcal{N}_1 and \mathcal{N}_2 with $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$.*

Proof. (\implies) Assume that \mathcal{N} is semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$. Then define \mathcal{N}_1 and \mathcal{N}_2 as follows:

$$\mathcal{N}_2 := \left\{ p(L)x : p(L) = \sum_{\alpha \in \mathbb{F}_n^+} c_\alpha L_\alpha \otimes I_{\mathcal{H}}, x \in \mathcal{N} \right\} \quad (4.1.1)$$

$$\mathcal{N}_1 := \mathcal{N}_2 \ominus \mathcal{N}, \quad (4.1.2)$$

Clearly, $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$, so we only need to show that both \mathcal{N}_1 and \mathcal{N}_2 are submodules. Now, it is easy to see from 4.1.1 that \mathcal{N}_2 is invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$, and thus, \mathcal{N}_2 is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Thus, we need only show that \mathcal{N}_1 is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. This is equivalent to proving the following for every $p(L) = \sum_{\alpha \in \mathbb{F}_n^+} c_\alpha L_\alpha \otimes I_{\mathcal{H}}$:

$$\begin{aligned} P_{\mathcal{N}_1} p(L) P_{\mathcal{N}_1} &= p(L) P_{\mathcal{N}_1} \\ (P_{\mathcal{N}_2} - P_{\mathcal{N}}) p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) &= p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) \\ P_{\mathcal{N}_2} p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) - P_{\mathcal{N}} p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) &= p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) \\ p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) - P_{\mathcal{N}} p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) &= p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) \quad (4.1.3) \\ P_{\mathcal{N}} p(L) (P_{\mathcal{N}_2} - P_{\mathcal{N}}) &= 0 \end{aligned}$$

$$P_{\mathcal{N}} p(L) P_{\mathcal{N}_2} = P_{\mathcal{N}} p(L) P_{\mathcal{N}} \quad (4.1.4)$$

Note that 4.1.3 follows from the fact that \mathcal{N}_2 is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Next we need to prove that 4.1.4 holds for every $y \in \mathcal{N}_1$. Combining equations 4.1.1 and 4.1.2 gives us that $y = q(L)x$, where $q(L) = \sum_{\alpha \in \mathbb{F}_n^+} c_\alpha L_\alpha \otimes I_{\mathcal{H}}$ and $x \in \mathcal{N}$. We get:

$$\begin{aligned}
P_{\mathcal{N}}p(L)P_{\mathcal{N}_2}q(L)x &= P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)x \\
P_{\mathcal{N}}p(L)P_{\mathcal{N}_2}q(L)P_{\mathcal{N}}x &= P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}}x \\
P_{\mathcal{N}}p(L)q(L)P_{\mathcal{N}}x &= P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}}x \tag{4.1.5}
\end{aligned}$$

Now 4.1.5 is obtained since \mathcal{N}_2 is a submodule. Finally, since \mathcal{N} is semi-invariant, equation 4.1.5 is verified, and thus \mathcal{N}_1 is a submodule, as desired.

(\Leftarrow) Assume that there exist two submodules \mathcal{N}_1 and \mathcal{N}_2 such that $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$. Then:

$$\begin{aligned}
P_{\mathcal{N}}p(L)q(L)P_{\mathcal{N}} - P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}} &= P_{\mathcal{N}}p(L)q(L)P_{\mathcal{N}} - P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}} \\
&= P_{\mathcal{N}}p(L)P_{\mathcal{N}_2}q(L)P_{\mathcal{N}} - P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}} \\
&= P_{\mathcal{N}}p(L)(P_{\mathcal{N}_2} - P_{\mathcal{N}})q(L)P_{\mathcal{N}} \\
&= P_{\mathcal{N}}p(L)P_{\mathcal{N}_1}q(L)P_{\mathcal{N}} \\
&= 0
\end{aligned}$$

The last equality follows from the fact that \mathcal{N}_1 is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$ together with $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$. Thus, $P_{\mathcal{N}}p(L)q(L)P_{\mathcal{N}} = P_{\mathcal{N}}p(L)P_{\mathcal{N}}q(L)P_{\mathcal{N}}$, and \mathcal{N} is semi-invariant, as desired. \square

4.1.3 Subquotients

Now if $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$ and $W_i = P_{\mathcal{N}}(L_i \otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i \leq n$, then $(\mathcal{N}; W_1, W_2, \dots, W_n)$ is called a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$. This leads immediately to the following lemma:

Lemma 4.1.3. *If $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$, then $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Similarly, if $(\mathcal{M}; V_1, V_2, \dots, V_n)$ is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$, then it follows that $(\mathcal{M}; V_1, V_2, \dots, V_n)$ is a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$.*

Proof. Assume $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Define $\mathcal{N}_2 := \mathcal{N}$ and $\mathcal{N}_1 := 0$. Clearly \mathcal{N}_1 and \mathcal{N}_2 are both submodules of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Since $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$, we can apply lemma 4.1.2, giving us that \mathcal{N} is a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$ as desired.

Now assume that $(\mathcal{M}; V_1, V_2, \dots, V_n)$ is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Define $\mathcal{N}_2 := \mathcal{H}$ and $\mathcal{N}_1 := \mathcal{H} \ominus \mathcal{N}$. Now \mathcal{N}_2 is clearly a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$, and by lemma 4.1.1, \mathcal{N}_1 is also a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Thus, since $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$, we can apply lemma 4.1.2, giving us that \mathcal{N} is a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$ as desired. \square

The converse does not hold, as shown in the following example:

Example 4.1.4. *Let $n = 2$, and let the full Fock space $\ell^2(\mathbb{F}_n^+)$ be given. Let $\mathcal{N}_1 = \{\delta_{\alpha} : |\alpha| \geq 2\}$ and $\mathcal{N}_2 = \{\delta_{\alpha} : |\alpha| \geq 1\}$. Then define $\mathcal{N} := \mathcal{N}_2 \ominus \mathcal{N}_1$. It is easy to see that $\mathcal{N} = \{\delta_{\alpha} : |\alpha| = 2\}$. Now $(\mathcal{N}_1; L_1, L_2, \dots, L_n)$ and $(\mathcal{N}_2; L_1, L_2, \dots, L_n)$ are both submodules of $\ell^2(\mathbb{F}_n^+)$, and so lemma 4.1.2 gives us that $(\mathcal{N}; L_1, L_2, \dots, L_n)$ is a subquotient of $\ell^2(\mathbb{F}_n^+)$. However, it is clear that*

for any $i, j \leq n$, $L_i \delta_{g_j} = \delta_{g_i g_j} \notin \mathcal{N}$, so $(\mathcal{N}; L_1, L_2, \dots, L_n)$ is not a submodule of $\ell^2(\mathbb{F}_n^+)$. Similarly, for any $i \leq n$, $L_i^* \delta_{g_0} = \delta_{g_0} \notin \mathcal{N}$, so $(\mathcal{N}; V_1, V_2, \dots, V_n)$ is not a $*$ -submodule of $\ell^2(\mathbb{F}_n^+)$.

4.2 Module Maps

Let $X : \mathcal{H} \rightarrow \mathcal{K}$ be a bounded linear operator between the Hilbert modules $(\mathcal{H}; V_1, V_2, \dots, V_n)$ and $(\mathcal{K}; W_1, W_2, \dots, W_n)$. If $X(V_i h) = W_i(Xh)$ for every $h \in \mathcal{H}$ and $1 \leq i \leq n$, we say that X is a module map. The Hilbert modules $(\mathcal{H}; V_1, V_2, \dots, V_n)$ and $(\mathcal{K}; W_1, W_2, \dots, W_n)$ are isomorphic if there exists an invertible map $X : \mathcal{H} \rightarrow \mathcal{K}$ such that both X and X^{-1} are isometric module maps.

For our purposes, it is important to determine when the orthogonal projection and the inclusion map are module maps. To this end, we give the following propositions:

Proposition 4.2.1. *The orthogonal projection $P_{\mathcal{N}} : \mathcal{F}^2(f) \otimes \mathcal{H} \rightarrow \mathcal{N}$ is a module map if and only if $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$.*

Proof. (\implies) Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$. Assume that the orthogonal projection $P_{\mathcal{N}} : \mathcal{F}^2(f) \otimes \mathcal{H} \rightarrow \mathcal{N}$ is a module map. Let $x \in \mathcal{N}$ be given. Then:

$$L_i^* x = L_i^* P_{\mathcal{N}} x = (P_{\mathcal{N}} L_i)^* x = (V_i P_{\mathcal{N}})^* x = P_{\mathcal{N}} V_i^* x = P_{\mathcal{N}} L_i^* x$$

Thus, \mathcal{N} is a $*$ -submodule, as desired.

(\impliedby) Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be a $*$ -submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Let $x \in$

$\mathcal{F}^2(f) \otimes \mathcal{H}$ be given. Then:

$$(P_{\mathcal{N}}L_i)^*x = L_i^*P_{\mathcal{N}}x = P_{\mathcal{N}}L_i^*P_{\mathcal{N}}x = V_i^*x = P_{\mathcal{N}}V_i^*x = (V_iP_{\mathcal{N}})^*x$$

Thus, $P_{\mathcal{N}}$ is a module map, as desired. \square

Proposition 4.2.2. *The inclusion map $\iota : \mathcal{N} \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}$ is a module map if and only if \mathcal{N} is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$.*

Proof. (\implies) Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$. Assume that the inclusion map $\iota : \mathcal{N} \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}$ is a module map. Let $x \in \mathcal{N}$ be given. Then:

$$L_i x = L_i \iota x = \iota L_i x$$

But since the domain of ι is \mathcal{N} , it follows that $L_i x \in \mathcal{N}$. Thus, \mathcal{N} is a submodule, as desired.

(\impliedby) Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Let $x \in \mathcal{N}$ be given. Then:

$$L_i \iota x = L_i \iota P_{\mathcal{N}} x = L_i x = \iota P_{\mathcal{N}} L_i x = \iota L_i x$$

Thus, ι is a module map, as desired. \square

Chapter 5

Weighted Fock Spaces

In this chapter we will first present the non-commutative domain algebras introduced by Popescu ([19]). Then we will present a unitarily equivalent description of the non-commutative domain algebras which will allow us to more easily lift module maps between these algebras. We will spend some time determining when module maps exist between two different weighted Fock spaces. Finally, we will present some examples which demonstrate different properties of the weighted Fock spaces.

5.1 Domain Algebras

5.1.1 Positive Regular Free Holomorphic Functions

Definition 5.1.1. (Popescu, [19]) *Look at the formal power series over n free variables (X_1, X_2, \dots, X_n) given by $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f X_\alpha$ for scalar a_α^f . Then f is a positive regular free holomorphic function on $B(\mathcal{H})^n$ if the following properties on a_α^f are satisfied:*

$$\begin{aligned}
a_{g_0}^f &= 0 \\
a_{g_i}^f &> 0 \quad 1 \leq i \leq n \\
a_\alpha^f &\geq 0 \quad \alpha \in \mathbb{F}_n^+, |\alpha| > 1 \\
\sup_{k \in \mathbb{N}} \left(\sum_{|\alpha|=k} |a_\alpha^f|^2 \right)^{\frac{1}{2k}} &= M < \infty
\end{aligned}$$

In Chapter 3 we looked at n -tuples of operators of the form (T_1, T_2, \dots, T_n) . Specifically, we asked the following question: when is (T_1, T_2, \dots, T_n) a row contraction? The question was answered in Theorem 3.3.2, where it was discovered that the unilateral shift operators (L_1, L_2, \dots, L_n) acted as the model of all such row contractions. In [19], Popescu asked a more general question. Given a positive regular free holomorphic function f , when did the n -tuple $(T_1, T_2, \dots, T_n) \in B(\mathcal{H})^n$ belong to the noncommutative domain algebra

$$\mathcal{D}_f(\mathcal{H}) = \left\{ (X_1, X_2, \dots, X_n) \in B(\mathcal{H})^n : \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f X_\alpha X_\alpha^* \leq I_{\mathcal{H}} \right\} ?$$

The answer to this question rests with weighted shift operators, which we will describe in the next section.

5.1.2 Weighted Shift Operators

We will now briefly summarize the construction of the weighted shift operators $W_i^f : \ell^2(\mathbb{F}_n^+) \longrightarrow \ell^2(\mathbb{F}_n^+)$ which together will act as the model of n -tuples in $\mathcal{D}_f(\mathcal{H})$.

Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|}$ be a positive regular free holomorphic function. Choose $r \in \mathbb{R}$ such that $rM < \frac{1}{2}$. Then:

$$\begin{aligned}
\left\| \sum_{|\alpha|=k} a_\alpha r^k S_\alpha \right\|^2 &= \left\langle \sum_{|\alpha|=k} a_\alpha r^k S_\alpha, \sum_{|\beta|=k} a_\beta r^k S_\beta \right\rangle \\
&= r^{2k} \left\langle \sum_{|\alpha|=k} a_\alpha S_\alpha, \sum_{|\beta|=k} a_\beta S_\beta \right\rangle \\
&= r^{2k} \sum_{|\alpha|=k} \sum_{|\beta|=k} \langle a_\alpha S_\alpha, a_\beta S_\beta \rangle \\
&= r^{2k} \sum_{|\alpha|=k} \langle a_\alpha S_\alpha, a_\alpha S_\alpha \rangle \\
&= r^{2k} \sum_{|\alpha|=k} |a_\alpha|^2 \\
&\leq (rM)^{2k}
\end{aligned}$$

Therefore, it follows that $\left\| \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha \right\| < 1$, so $I - \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha$ is invertible in \mathcal{A}_n with its inverse given by $\sum_{\alpha \in \mathbb{F}_n^+} b_\alpha r^{|\alpha|} S_\alpha$ for some $b_\alpha \in \mathbb{C}$. Since we know that $\frac{1}{1-x} = \sum_{k \geq 0} x^k$ for any $x \in B(\mathcal{H})^n$ with $\|x\| < 1$, we get that

$$\sum_{\alpha \in \mathbb{F}_n^+} b_\alpha r^{|\alpha|} S_\alpha = \left(I - \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha \right)^{-1} = \sum_{k=0}^{\infty} \left(\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha \right)^k \quad (5.1.1)$$

In addition, we also know the following:

$$\left(I - \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha \right) \left(\sum_{\alpha \in \mathbb{F}_n^+} b_\alpha r^{|\alpha|} S_\alpha \right) = 1 \quad (5.1.2)$$

$$\left(\sum_{\alpha \in \mathbb{F}_n^+} b_\alpha r^{|\alpha|} S_\alpha \right) \left(I - \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} S_\alpha \right) = 1 \quad (5.1.3)$$

From equation 5.1.1 we get several properties of the b_α 's:

$$b_{g_0} = 1 \quad (5.1.4)$$

$$b_\alpha > 0 \text{ for all } \alpha \in \mathbb{F}_n^+ \quad (5.1.5)$$

$$b_{g_i} = a_{g_i} \quad (5.1.6)$$

$$b_\alpha = \sum_{i=1}^{|\alpha|} \sum_{\substack{\gamma_1 \gamma_2 \cdots \gamma_i = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_i| \geq 1}} a_{\gamma_1} a_{\gamma_2} \cdots a_{\gamma_i} > 0 \quad (5.1.7)$$

$$b_{\alpha\beta} \geq b_\alpha b_\beta \text{ for any } \alpha, \beta \in \mathbb{F}_n^+ \quad (5.1.8)$$

From equations 5.1.2 and 5.1.3 we get another important property of the b'_α 's which holds when $|\gamma| \geq 1$:

$$b_\gamma = \sum_{\substack{\beta\alpha=\gamma \\ |\beta|\geq 1}} a_\beta b_\alpha = \sum_{\substack{\alpha\beta=\gamma \\ |\beta|\geq 1}} a_\beta b_\alpha \quad (5.1.9)$$

We can now define the weighted left creation operators $W_i^f : \ell^2(\mathbb{F}_n^+) \longrightarrow \ell^2(\mathbb{F}_n^+)$ for $i = 1, 2, \dots, n$ associated with the noncommutative domain $\mathcal{D}_f(\mathcal{H})$.

In particular, for $\alpha \in \mathbb{F}_n^+$, define

$$W_i^f e_\alpha = \sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} e_{g_i \alpha}.$$

A few easy calculations give us the following:

$$\begin{aligned} \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha W_\alpha W_\alpha^* &\leq I \\ W_\beta e_\alpha &= \sqrt{\frac{b_\alpha}{b_{\beta \alpha}}} e_{\beta \alpha} \\ W_\beta^* e_\alpha &= \begin{cases} \sqrt{\frac{b_\alpha}{b_\beta}} e_\beta & \text{if } \alpha = \beta \gamma; \\ 0 & \text{else} \end{cases} \\ \|W_\alpha\| &= \frac{1}{\sqrt{b_\alpha}} \end{aligned}$$

We can define the weighted right creation operators $\Lambda_i^f : \ell^2(\mathbb{F}_n^+) \rightarrow \ell^2(\mathbb{F}_n^+)$ for $i = 1, 2, \dots, n$ associated with the noncommutative domain $\mathcal{D}_f(\mathcal{H})$ in a similar fashion. In particular, for $\alpha \in \mathbb{F}_n^+$, define

$$\Lambda_i^f e_\alpha = \sqrt{\frac{b_\alpha}{b_{\alpha g_i}}} e_{\alpha g_i}.$$

The weighted right creation operators have similar properties to the weighted left creation operators. Some of these properties are shown in Figure 5.1.

We are now in a position to state the following theorem:

Theorem 5.1.2. (G. Popescu, [19]) Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f X_\alpha$ be a positive regular free holomorphic function. Then:

$$\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f W_\alpha^f W_\alpha^{f*} \leq I$$

Furthermore, if (T_1, T_2, \dots, T_n) is an n -tuple of operators acting on a Hilbert space \mathcal{H} , then $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f T_\alpha T_\alpha^* \leq I$ if and only if there exists a unique unital completely contractive morphism Φ from the algebra generated by the weighted left creation operators $\{W_1^f, W_2^f, \dots, W_n^f\}$ into $B(\mathcal{H})$ such that $W_i^f \mapsto T_i$ for $1 \leq i \leq n$.

A similar theorem holds for the weighted right creation operators. If we consider the Fock space $\ell^2(\mathbb{F}_n^+)$ together with the weighted shift operators $W_1^f, W_2^f, \dots, W_n^f$ defined in the previous theorem, we get the Hilbert module $(\ell^2(\mathbb{F}_n^+); W_1^f, W_2^f, \dots, W_n^f)$.

5.2 Weighted Fock Spaces

It is more useful for us to work with a unitarily equivalent version of the weighted Fock space, which we will call $\mathcal{F}^2(f)$. For a positive regular free holomorphic function f , set $\mathcal{F}^2(f)$ to be the Hilbert space with a complete orthogonal basis $\{\delta_\alpha^f : \alpha \in \mathbb{F}_n^+\}$, where the norm is given by $\|\delta_\alpha^f\| = \frac{1}{\sqrt{b_\alpha^f}}$. For each $i \leq n$, define the left creation operators $L_i : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(f)$ by $L_i \delta_\alpha = \delta_{g_i \alpha}$. Similarly, for each $i \leq n$, define the right creation operators $R_i : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(f)$ by $R_i \delta_\alpha = \delta_{\alpha g_i}$. We obtain this unitarily equivalent version via the unitary operator $U : \ell^2(\mathbb{F}_n^+) \rightarrow \mathcal{F}^2(f)$ defined as $U e_\alpha =$

$\sqrt{b_\alpha}\delta_\alpha$. Furthermore, $U^*\delta_\alpha = \frac{1}{\sqrt{b_\alpha}}e_\alpha$. Verifying that this unitary satisfies the conditions is easy. We will verify the most important relation:

$$U^*L_\beta U e_\alpha = \sqrt{b_\alpha^f} U^* L_\beta \delta_\alpha = \sqrt{b_\alpha^f} U^* \delta_{\beta\alpha}^f = \frac{\sqrt{b_\alpha^f}}{\sqrt{b_{\beta\alpha}^f}} e_{\beta\alpha} = W_\beta^f e_\alpha$$

The correspondence with Popescu's version is shown in figure 5.1, with the left column containing Popescu's representation and the right column containing our representation.

$$\begin{array}{lll}
\ell^2(\mathbb{F}_n^+) & \iff & \mathcal{F}^2(f) \\
e_\alpha & \iff & \delta_\alpha \\
a_\alpha & \iff & a_\alpha \\
\|e_\alpha\| = 1 & \iff & \|\delta_\alpha\| = \sqrt{\frac{1}{b_\alpha}} \\
W_\beta e_\alpha = \sqrt{\frac{b_\alpha}{b_{\beta\alpha}}} e_{\beta\alpha} & \iff & L_\beta \delta_\alpha = \delta_{\beta\alpha} \\
\|W_\beta\| = \sqrt{\frac{1}{b_\beta}} & \iff & \|L_\beta\| = \sqrt{\frac{1}{b_\beta}} \\
\Lambda_\beta e_\alpha = \sqrt{\frac{b_\alpha}{b_{\alpha\tilde{\beta}}}} e_{\alpha\tilde{\beta}} & \iff & R_\beta \delta_\alpha = \delta_{\alpha\tilde{\beta}} \\
\|\Lambda_\beta\| = \sqrt{\frac{1}{b_{\tilde{\beta}}}} & \iff & \|R_\beta\| = \sqrt{\frac{1}{b_{\tilde{\beta}}}}
\end{array}$$

Figure 5.1: Correspondence between domain algebras and weighted Fock spaces .

If we consider the weighted Fock space $\mathcal{F}^2(f)$ together with the weighted shifts $L_1^f, L_2^f, \dots, L_n^f$, we get the Hilbert module $(\mathcal{F}^2(f); L_1^f, L_2^f, \dots, L_n^f)$. If the underlying positive regular free holomorphic function is understood, we will refer to L_α^f and δ_α^f simply as L_α and δ_α , respectively.

It turns out that not only is U a unitary operator, but it is also a module map, as shown in the following lemma:

Lemma 5.2.1. *Let $(\ell^2(\mathbb{F}_n^+); W_1^f, W_2^f, \dots, W_n^f)$ and $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ be given, and let $Ue_\alpha = \sqrt{b_\alpha}\delta_\alpha$ as above. Then $U : \ell^2(\mathbb{F}_n^+) \rightarrow \mathcal{F}^2(f)$ is an isometric module map.*

Proof. To prove that U is a module map, let $e_\alpha \in F_n^+$ be given. Then:

$$\begin{aligned} UW_\beta e_\alpha &= U \frac{\sqrt{b_\alpha}}{\sqrt{b_{\beta\alpha}}} e_{\beta\alpha} = \sqrt{b_{\beta\alpha}} \frac{\sqrt{b_\alpha}}{\sqrt{b_{\beta\alpha}}} \delta_{\beta\alpha} \\ &= \sqrt{b_\alpha} \delta_{\beta\alpha} = \sqrt{b_\alpha} L_\beta \delta_\alpha = L_\beta U e_\alpha \end{aligned}$$

Clearly U is isometric, since $\|\sqrt{b_\alpha}\delta_\alpha\| = \sqrt{b_\alpha}\|\delta_\alpha\| = \frac{\sqrt{b_\alpha}}{\sqrt{b_\alpha}} = 1 = \|e_\alpha\| \quad \square$

Notice that the b_α 's end up being identical between spaces, as they are solely determined by the a_α 's. Let $\alpha = g_{i_1}g_{i_2}\dots g_{i_k}$ be given. Denote by $\tilde{\alpha}$ the reverse of α , $\tilde{\alpha} = g_{i_k}\dots g_{i_2}g_{i_1}$. It is clear to see that if $f(X_1, X_2, \dots, X_n) = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ is a positive regular free holomorphic function on $B(\mathcal{H})^n$, then so is $\tilde{f}(X_1, X_2, \dots, X_n) = \sum_{\alpha \in \mathbb{F}_n^+} a_{\tilde{\alpha}} X_\alpha$. We will now state a result we will need for this work:

Proposition 5.2.2. *(G. Popescu, [19]) Let $(L_1, L_2, \dots, L_n), (R_1, R_2, \dots, R_n)$, and $f = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$ be given. Then:*

(i) $\sum_{|\beta| \geq 1} a_\beta L_\beta L_\beta^* \leq I$ and

(ii) $\sum_{|\beta| \geq 1} a_{\tilde{\beta}} R_\beta R_\beta^* \leq I$.

(iii) $V^* R_i^f V = L_i^{\tilde{f}}$ for $1 \leq i \leq n$, where $V : \mathcal{F}^2(\tilde{f}) \rightarrow \mathcal{F}^2(f)$ is the unitary operator defined by $V\delta_{\tilde{\alpha}} = \delta_\alpha$ for $\alpha \in \mathbb{F}_n^+$.

Proof. We will first verify (i). To do this, we will take advantage of our unitary equivalence and use the following result from Theorem 5.1.2:

$$\sum_{|\beta| \geq 1} a_\beta W_\beta W_\beta^* \leq I$$

Thus, it follows that

$$\begin{aligned} \sum_{|\beta| \geq 1} a_\beta L_\beta L_\beta^* &= \sum_{|\beta| \geq 1} a_\beta U W_\beta U^* U W_\beta^* U^* = \sum_{|\beta| \geq 1} a_\beta U W_\beta W_\beta^* U^* \\ &= U \left(\sum_{|\beta| \geq 1} a_\beta W_\beta W_\beta^* \right) U^* \leq U I U^* = I \end{aligned}$$

as desired. The proof of (ii) is identical. The third is as follows. Let $\alpha \in \mathbb{F}_n^+$:

$$V^* R_i^f V \delta_{\tilde{\alpha}} = V^* R_i^f \delta_\alpha = V^* \delta_{\alpha g_i} = \delta_{\tilde{\alpha} g_i} = \delta_{g_i \tilde{\alpha}} = L_i^f \delta_{\tilde{\alpha}}$$

This proves part (iii). This part of the proof is deceptively easy, and does not explain why the change from $\mathcal{F}^2(\tilde{f})$ to $\mathcal{F}^2(f)$ is necessary. Here is a more cumbersome proof utilizing the unitary equivalence to the domain algebras of [19] and the result in that space that states that $\bar{V}^* \Lambda_i^f \bar{V} = W_i^f$ with $\bar{V} e_{\tilde{\alpha}} = e_\alpha$:

$$\begin{array}{ccc} (\ell^2(\mathbb{F}_n^+); W_1^{\tilde{f}}, W_2^{\tilde{f}}, \dots, W_n^{\tilde{f}}) & \xrightarrow{\bar{V}} & (\ell^2(\mathbb{F}_n^+); W_1^f, W_2^f, \dots, W_n^f) \\ \downarrow \tilde{U} & & \downarrow U \\ \mathcal{F}^2(\tilde{f}) & \xrightarrow{V} & \mathcal{F}^2(f) \end{array}$$

Figure 5.2: Diagram for part (iii) of Proposition 5.2.2

Since the diagram in figure 5.2 commutes, we can do the following computations:

$$\begin{aligned}
\bar{V}^* \Lambda_i^f \bar{V} e_{\tilde{\alpha}} &= W_i^{\tilde{f}} e_{\tilde{\alpha}} \\
\bar{V}^* U^* R_i^f U \bar{V} e_{\tilde{\alpha}} &= \tilde{U}^* L_i^{\tilde{f}} \tilde{U} e_{\tilde{\alpha}} \\
\tilde{U} \bar{V}^* U^* R_i^f U \bar{V} \tilde{U}^* \sqrt{b_{\tilde{\alpha}}^{\tilde{f}}} \delta_{\tilde{\alpha}} &= L_i^{\tilde{f}} \sqrt{b_{\tilde{\alpha}}^{\tilde{f}}} \delta_{\tilde{\alpha}} \\
\tilde{U} \bar{V}^* U^* R_i^f U \bar{V} \tilde{U}^* \delta_{\tilde{\alpha}} &= L_i^{\tilde{f}} \delta_{\tilde{\alpha}}
\end{aligned}$$

The right side is easy to compute, since $L_i^{\tilde{f}} \delta_{\tilde{\alpha}} = \delta_{g_i \tilde{\alpha}}$. Computing the left side:

$$\begin{aligned}
\tilde{U} \bar{V}^* U^* R_i^f U \bar{V} \tilde{U}^* \delta_{\tilde{\alpha}} &= \frac{1}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \tilde{U} \bar{V}^* U^* R_i^f U \bar{V} e_{\tilde{\alpha}} = \frac{1}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \tilde{U} \bar{V}^* U^* R_i^f U e_{\alpha} \\
&= \frac{\sqrt{b_{\alpha}^f}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \tilde{U} \bar{V}^* U^* R_i^f \delta_{\alpha} = \frac{\sqrt{b_{\alpha}^f}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \tilde{U} \bar{V}^* U^* \delta_{\alpha g_i} \\
&= \frac{\sqrt{b_{\alpha}^f}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \frac{1}{\sqrt{b_{\alpha g_i}^f}} \tilde{U} \bar{V}^* e_{\alpha g_i} = \frac{\sqrt{b_{\alpha}^f}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}}} \frac{1}{\sqrt{b_{\alpha g_i}^f}} \tilde{U} e_{\tilde{\alpha} g_i} \\
&= \frac{\sqrt{b_{\alpha}^f} \sqrt{b_{\tilde{\alpha} g_i}^{\tilde{f}}}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}} \sqrt{b_{\alpha g_i}^f}} \delta_{\tilde{\alpha} g_i} = \frac{\sqrt{b_{\alpha}^f} \sqrt{b_{\tilde{\alpha} g_i}^{\tilde{f}}}}{\sqrt{b_{\tilde{\alpha}}^{\tilde{f}}} \sqrt{b_{\alpha g_i}^f}} \delta_{g_i \tilde{\alpha}} \\
&= \delta_{g_i \tilde{\alpha}}
\end{aligned}$$

The last equality comes from the fact that $b_{\tilde{\alpha}}^{\tilde{f}} = b_{\alpha}^f$ for all $\alpha \in \mathbb{F}_n^+$. This is where the change from $\mathcal{F}^2(\tilde{f})$ to $\mathcal{F}^2(f)$ is necessary, since it is not always the case that $b_{\tilde{\alpha}}^{\tilde{f}} = b_{\alpha}^f$. \square

5.3 Module Maps Between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$

Let $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \dots, L_n)$ be two Hilbert modules. Define the “formal identity map” $\varepsilon(f, g) : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ to be the linear map $\varepsilon(\delta_\alpha^f) = \delta_\alpha^g$. When f and g are clear, we will just refer to $\varepsilon(f, g)$ as ε . By this definition, ε , if bounded, acts like the “formal identity map.” However, it is important to note that ε does not have to be an isometry, need not be bounded, and isn’t necessarily well-defined. Now an element $x \in \mathcal{F}^2(f)$ looks like $x = \sum_{\alpha \in \mathbb{F}_n^+} x_\alpha \delta_\alpha$, so we can think of an element in $\mathcal{F}^2(f)$ as the sequence $x = \{x_\alpha : \alpha \in \mathbb{F}_n^+\}$. We will therefore use $\mathcal{F}^2(f) \subset \mathcal{F}^2(g)$ to denote sequence inclusion.

The main result of this section is

Theorem 5.3.1. *Let $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \dots, L_n)$ be two Hilbert modules. Then the following are equivalent:*

- (i) $\mathcal{F}^2(f) \subset \mathcal{F}^2(g)$.
- (ii) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is well defined.
- (iii) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is bounded.
- (iv) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is a module map.
- (v) $\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} < \infty$

Furthermore, if f has finitely many terms, then:

- (vi) *There exists a non-zero module map $X : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$.*

We start with a lemma.

Lemma 5.3.2. Let $f = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$, where f has only finitely many terms. Then there exists $C > 0$ such that for all $\alpha \in \mathbb{F}_n^+$ and $i \leq n$,

$$\frac{b_\alpha}{C} \geq b_{g_i \alpha} \geq b_{g_i} b_\alpha \quad (5.3.1)$$

$$\frac{b_\alpha}{C} \geq b_{\alpha g_i} \geq b_\alpha b_{g_i} \quad (5.3.2)$$

Proof. We will only prove 5.3.1, as 5.3.2 has a similar proof. The right inequality of 5.3.1 was proven by Popescu in [19]. Thus, it remains to prove the left inequality of 5.3.1. Now

$$b_{g_i \alpha} = \sum_{\substack{\beta \gamma = g_i \alpha \\ |\beta| \geq 1}} a_\beta b_\gamma$$

Since f is finite, there exists a $k \in \mathbb{N}$ such that for all α with $|\alpha| > k$, $a_\alpha = 0$. Thus:

$$b_{g_i \alpha} = \sum_{\substack{\beta \gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} a_\beta b_\gamma \leq \max_{\substack{\beta \gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} \{a_\beta\} \sum_{\substack{\beta \gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} b_\gamma$$

Let $A = \max_{1 \leq |\beta| \leq k} \{a_\beta\}$. Then:

$$\begin{aligned} b_{g_i \alpha} &\leq A \sum_{\substack{\sigma \gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} b_\gamma = A \sum_{\substack{\sigma \gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_\sigma b_\gamma}{b_\sigma} \leq A \sum_{\substack{\sigma \gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_{\sigma \gamma}}{b_\sigma} \\ &= A \sum_{\substack{\sigma \gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_\alpha}{b_\sigma} = A b_\alpha \sum_{\substack{\sigma \gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{1}{b_\sigma} \\ &\leq C b_\alpha \end{aligned}$$

where $C = A \sum_{|\sigma| \leq k-1} \frac{1}{b_\sigma}$. Thus, $\frac{b_\alpha}{C} \geq b_{g_i\alpha}$, as desired. \square

It is important to note that this result does not extend to arbitrary f 's, as shown in the following example:

Example 5.3.3. Define $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ by:

$$a_\alpha = \begin{cases} 1 & \text{if } |\alpha| = 1 \\ 2^k & \text{if } \alpha = 12^k 1 \\ 0 & \text{else} \end{cases}$$

Let $\alpha = 2^k 1$ and $i = 1$. Let's calculate $\frac{b_\alpha}{b_{g_i\alpha}}$:

$$\frac{b_\alpha}{b_{g_i\alpha}} = \frac{b_{2^k 1}}{b_{12^k 1}} = \frac{\sum_{\substack{\beta\gamma=2^k 1 \\ |\beta| \geq 1}} a_\beta b_\gamma}{\sum_{\substack{\beta\gamma=12^k 1 \\ |\beta| \geq 1}} a_\beta b_\gamma} = \frac{(a_2)^k a_1}{a_1 (a_2)^k a_1 + a_{12^k 1}} = \frac{1}{1 + 2^k}$$

This converges to 0 as $k \rightarrow \infty$, and therefore, no $C > 0$ exists such that $\frac{b_\alpha}{b_{g_i\alpha}} \geq C$ and therefore no C exists such that $\frac{b_\alpha}{C} \geq b_{g_i\alpha}$.

We are now ready to complete the proof of Theorem 5.3.1.

Proof. The following implications are clear: (iv) \implies (iii) \implies (ii) \implies (i). We will now complete the loop by showing that (i) \implies (iv):

Assume that $x_m \rightarrow x$ in $\mathcal{F}^2(f)$ and that $\varepsilon(x_m) \rightarrow y$ in $\mathcal{F}^2(g)$. Now we know that $x_m = \sum_{\beta \in \mathbb{F}_n^+} c_{m,\beta} \delta_\beta$. A quick calculation reveals that $x = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta$, where $c_\beta = \lim_{m \rightarrow \infty} c_{m,\beta}$ for every $\beta \in \mathbb{F}_n^+$:

$$\left\langle \lim_{m \rightarrow \infty} x_m, \delta_\alpha \right\rangle_f = \lim_{m \rightarrow \infty} \left\langle \sum_{\beta \in \mathbb{F}_n^+} c_{m,\beta} \delta_\beta, \delta_\alpha \right\rangle_f = \lim_{m \rightarrow \infty} c_{m,\alpha} \langle \delta_\alpha, \delta_\alpha \rangle_f$$

On the other hand:

$$\langle x, \delta_\alpha \rangle_f = \left\langle \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta, \delta_\alpha \right\rangle_f = c_\alpha \langle \delta_\alpha, \delta_\alpha \rangle_f$$

Notice that the above calculations will look exactly the same in $\mathcal{F}^2(g)$. Since $\varepsilon(\delta_\alpha^f) = \delta_\alpha^g$, it is clear to see that $\varepsilon(x) = y$. Thus, by the closed graph theorem, ε is bounded. Now it is trivial to show that ε is a module map:

$$\varepsilon L_\beta^f \delta_\alpha^f = \varepsilon \delta_{\beta\alpha}^f = \delta_{\beta\alpha}^g = L_\beta^g \delta_\alpha^g = L_\beta^g \varepsilon \delta_\alpha^f$$

As desired. Thus, (i) \iff (ii) \iff (iii) \iff (iv).

Now we will show that (v) \iff (iii): Notice the following:

$$\begin{aligned} \|\varepsilon\| &= \sup_{\alpha \in \mathbb{F}_n^+} \left\| \varepsilon \sqrt{b_\alpha^f} \delta_\alpha \right\|_g = \sup_{\alpha \in \mathbb{F}_n^+} \left\| \sqrt{b_\alpha^f} \delta_\alpha \right\|_g \\ &= \sup_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^f} \|\delta_\alpha\|_g = \sup_{\alpha \in \mathbb{F}_n^+} \frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}} \end{aligned}$$

Thus, ε is bounded if and only if $\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} < \infty$, as desired. Thus, (i) \iff (ii) \iff (iii) \iff (iv) \iff (v).

Now assume that f has finitely many terms. Since (iv) \implies (vi) always holds, we need only prove (vi) \implies (v): Assume $X : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is a non-zero module map. Then X is completely determined by what it does to δ_0 : $X\delta_0 = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta$. Then in general, we have:

$$X\delta_\alpha = XL_\alpha\delta_0 = L_\alpha X\delta_0 = L_\alpha \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta = \sum_{\beta \in \mathbb{F}_n^+} c_\beta L_\alpha \delta_\beta = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_{\alpha\beta}$$

Let β_0 be such that $\beta_0 \neq 0$, but for all other $\beta \in \mathbb{F}_n^+$ such that $|\beta| < |\beta_0|$ it follows that $\beta = 0$. Then:

$$X\delta_\alpha = c_{\beta_0} \delta_{\alpha\beta_0} + \sum_{\substack{|\beta| \geq |\beta_0| \\ \beta \neq \beta_0}} c_\beta \delta_{\alpha\beta}$$

Now $\delta_{\alpha\beta_0}$ is orthogonal to $\delta_{\alpha\beta}$ for every $\beta \neq \beta_0$. Thus:

$$\|X\delta_\alpha\|_g = \left\| c_{\beta_0} \delta_{\alpha\beta_0} + \sum_{\substack{|\beta| \geq |\beta_0| \\ \beta \neq \beta_0}} c_\beta \delta_{\alpha\beta} \right\|_g \geq \|c_{\beta_0} \delta_{\alpha\beta_0}\|_g = |c_{\beta_0}| \|\delta_{\alpha\beta_0}\|_g = \frac{|c_{\beta_0}|}{\sqrt{b_{\alpha\beta_0}^g}}$$

By applying Lemma 5.3.2 iteratively, we can peel off the β_0 on the bottom of the fraction. This gives us:

$$\|X\delta_\alpha\|_g \geq \frac{|c_{\beta_0}| K}{\sqrt{b_\alpha^g}}$$

Therefore, it follows that $\frac{1}{\sqrt{b_\alpha^g}} \leq C \|X\delta_\alpha\|_g \leq C \|X\| \|\delta_\alpha\|_f = C \|X\| \frac{1}{\sqrt{b_\alpha^f}}$. Thus,

$$\frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}} \leq C \|X\|$$

Since α was arbitrary, it follows that

$$\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} < \infty$$

as desired. □

This leads immediately to the following corollary, which gives a simple method to calculate the norm of the ε map:

Corollary 5.3.4. *Let $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \dots, L_n)$ be two Hilbert modules. Assume that f has finitely many terms. Then if $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is a module map, its norm is given by:*

$$\|\varepsilon\| = \sup_{\alpha \in \mathbb{F}_n^+} \frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}}$$

5.4 Examples

In order to get some intuition about the b_α 's, we will look at some examples. These examples will show some nice combinatorial connections between the a 's and the b 's for certain classes of functions f :

Example 5.4.1. *Let $f = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$, where $a_\alpha \in \{0, 1\}$ for all $\alpha \in \mathbb{F}_n^+$. Then $b_\alpha = |\{\alpha_1 \alpha_2 \cdots \alpha_i = \alpha : a_{\alpha_j} = 1, 1 \leq j \leq i\}|$. In other words, b_α counts the number of ways you can decompose α into subwords $\alpha = \alpha_1 \alpha_2 \cdots \alpha_i$ such that $a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_i} = 1$.*

Proof. Let $f = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$, with $a_\alpha \in \{0, 1\}$ for all $\alpha \in \mathbb{F}_n^+$ be given. The proof will be by strong induction.

Base: Let $\alpha = g_i$. Now since $a_{g_i} > 0$ for all $i \leq n$ and $a_\alpha \in \{0, 1\}$ for all $\alpha \in \mathbb{F}_n^+$, it follows that $a_{g_i} = 1$. Also recall that $b_0 = 1$. Now

$$b_{g_i} = \sum_{\substack{\beta\gamma = g_i \\ |\beta| \geq 1}} a_\beta b_\gamma = a_{g_i} b_0 = 1.$$

But there is clearly only one way to decompose $\alpha = g_i$, so we're done.

Induction: Assume the claim is true for all α with $|\alpha| \leq m$. Let α be given such that $|\alpha| = m + 1$. Then:

$$b_\alpha = \sum_{\substack{\beta\gamma = \alpha \\ |\beta| \geq 1}} a_\beta b_\gamma$$

By the induction hypothesis, b_γ counts the number of ways γ can be decomposed into subwords $\gamma = \gamma_1\gamma_2 \cdots \gamma_i$ such that $a_{\gamma_1} = a_{\gamma_2} = \cdots = a_{\gamma_i} = 1$. Now if $a_\beta = 1$, then any of the decompositions of γ will lead to proper decompositions of $\beta\gamma = \alpha$, and so we should count them. This will lead to $a_\beta b_\gamma = b_\gamma$ new decompositions. However, if $a_\beta = 0$, then none of the decompositions of γ will lead to proper decompositions of $\beta\gamma = \alpha$, and so we should not count them. This will lead to $a_\beta b_\gamma = 0$ new decompositions. Finally, summing over all $\beta\gamma = \alpha$ with $|\beta| \geq 1$ gives us the final result. Since we've counted all possible decompositions and only those possible decompositions, it follows that b_α counts the number of ways you can decompose α into subwords $\alpha = \alpha_1\alpha_2 \cdots \alpha_i$ such that $a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_i} = 1$, as desired. \square

Example 5.4.2. Let $f = \sum_{i=1}^n \left(X_i + \sum_{j=1}^n X_{ij} \right)$. Then $b_\alpha = F_{|\alpha|+1}$, where F_m is the m^{th} Fibonacci number.

Proof. Clearly $b_0 = 1 = F_1$ and $b_{g_i} = 1 = F_2$ for all $i \leq n$. Assume $b_\alpha = F_{|\alpha|+1}$ for all $|\alpha| < m$. Let α be given such that $|\alpha| = m$. It then follows that $b_\alpha = \sum_{\substack{\beta\gamma=\alpha \\ |\beta|\geq 1}} a_\beta b_\gamma = a_\rho b_\sigma + a_\tau b_\phi$, where $\rho\sigma = \alpha = \tau\phi$ with $|\rho| = 1$ and $|\tau| = 2$. Thus, $b_\alpha = a_\rho b_\sigma + a_\tau b_\phi = F_{|\sigma|+1} + F_{|\tau|+1} = F_{|\alpha|} + F_{|\alpha|-1} = F_{|\alpha|+1}$ as desired. \square

While the finite case is quite pleasant, an example illustrates the potential difficulties encountered when transitioning to the infinite case:

Example 5.4.3. Look at f and g such that

$$a_\alpha^f = \begin{cases} 1 & \text{if } |\alpha| = 1 \\ 2^k & \text{if } \alpha = 12^k 1 \\ 0 & \text{else} \end{cases} \quad a_\alpha^g = \begin{cases} 1 & \text{if } |\alpha| = 1 \\ 4^k & \text{if } \alpha = 12^k 11 \\ & \alpha = 12^k 12 \\ 0 & \text{else} \end{cases}$$

Look at $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$. Is this bounded?

$$\begin{aligned} \|\varepsilon\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} \geq \frac{b_{12^k 1}^f}{b_{12^k 1}^g} = \frac{\sum_{\substack{\gamma\beta=12^k 1 \\ |\gamma|\geq 1}} a_\gamma^f b_\beta^f}{\sum_{\substack{\pi\sigma=12^k 1 \\ |\pi|\geq 1}} a_\pi^g b_\sigma^g} \\ &= \frac{a_1^f (a_2^f)^k a_1^f + a_{12^k 1}^f}{a_1^g (a_2^g)^k a_1^g} = \frac{1 + 2^k}{1} \rightarrow \infty \end{aligned}$$

So ε is unbounded. Now look at $Y : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ by $Y\delta_0 = \delta_1$. Is this bounded? Let α be given. We can decompose α as $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ such

that $\alpha_i \in \{1, 2, 12^{k_1}\}$. This may yield several decompositions. However, it is easy to show that there exists a minimal decomposition in the sense that for every other decomposition $\alpha = \alpha'_1 \alpha'_2 \cdots \alpha'_m$, $m \geq n$. Next, note that for every i , if $|\alpha_i| = 1$, then $b_{\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_i \alpha_{i+1} \cdots \alpha_n}^f = b_{\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_n}^f$, and $b_{\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_i \alpha_{i+1} \cdots \alpha_n}^g \geq b_{\alpha_1 \alpha_2 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_n}^g$. Thus, we only need look at the case when $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ with $\alpha_i = 12^{k_i}$. Thus:

$$\begin{aligned} \frac{b_\alpha^f}{b_{\alpha g_1}^g} &= \frac{\sum_{\substack{\gamma \beta = \alpha \\ |\gamma| \geq 1}} a_\gamma^f b_\beta^f}{\sum_{\substack{\pi \sigma = \alpha \\ |\pi| \geq 1}} a_\pi^g b_\sigma^g} \\ &= \frac{1 + (a_{\alpha_1}^f + \cdots + a_{\alpha_n}^f) + \cdots + (a_{\alpha_1}^f a_{\alpha_2}^f \cdots a_{\alpha_n}^f)}{1 + (a_{\alpha_1}^g + \cdots + a_{\alpha_n}^g) + \cdots + (a_{\alpha_1}^g a_{\alpha_2}^g \cdots a_{\alpha_n}^g)} \\ &= \frac{1 + (2^{k_1} + \cdots + 2^{k_n}) + \cdots + (2^{k_1} 2^{k_2} \cdots 2^{k_n})}{1 + (4^{k_1} + \cdots + 4^{k_n}) + \cdots + (4^{k_1} 4^{k_2} \cdots 4^{k_n})} \end{aligned}$$

It isn't difficult to notice that the bottom is always greater than or equal to the top. This is because if the top has n factors, the bottom can have at most $\lceil \frac{n}{2} \rceil$ factors. Thus, it follows that

$$\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} = \frac{b_1^f}{b_1^g} = \frac{1}{1} = 1$$

Thus, Y is bounded. This produces an example where f has infinitely many terms, ε is unbounded, but a nontrivial map is bounded.

Chapter 6

Lifting Module Maps

6.1 Lifting Theorems

6.1.1 L-Bounded and R-Bounded Functions

If f is a positive regular free holomorphic function, it induces $\{b_\alpha : \alpha \in \mathbb{F}_n^+\}$ with the property that $b_\alpha > 0$ for every $\alpha \in \mathbb{F}_n^+$. We will now generalize this concept by looking at functions $f : \mathbb{F}_n^+ \rightarrow (0, \infty)$ with $f(\alpha) = b_\alpha^f$ and a Hilbert space $\mathcal{F}^2(f)$ with orthonormal basis $\|\delta_\alpha\| = \sqrt{\frac{1}{b_\alpha^f}}$. It is easy to see that f induces a set $\{a_\alpha^f : \alpha \in \mathbb{F}_n^+\}$ satisfying:

$$b_\gamma^f = \sum_{\substack{\beta\alpha=\gamma \\ |\beta|\geq 1}} a_\beta^f b_\alpha^f \text{ if } |\gamma| \geq 1$$
$$b_\gamma^f = \sum_{\substack{\alpha\beta=\gamma \\ |\beta|\geq 1}} a_\beta^f b_\alpha^f \text{ if } |\gamma| \geq 1$$

However, unlike positive regular free holomorphic functions, the a_α^f 's need not be non-negative, as seen in the following example:

Example 6.1.1. *Look at the function $f : \mathbb{F}_1^+ \rightarrow (0, \infty)$ by*

$$f(\alpha) = 10 = b_\alpha^f \text{ for all } |\alpha| \geq 3$$

A quick calculation reveals the following:

$$a_1^f = 10$$

$$a_{11}^f = -90$$

$$a_{111}^f = 810$$

$$a_{1111}^f = -7290$$

$$a_{11111}^f = 65610$$

Clearly not all of the a_α 's are non-negative.

In addition to the fact that the a_α^f 's need not be non-negative, the left and right creation operators may no longer be bounded, since the norm of the left and right creation operators will be given by:

$$\begin{aligned}\|L_i\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{\|\delta_{g_i \alpha}\|^2}{\|\delta_\alpha\|^2} = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_{g_i \alpha}^f} \\ \|R_i\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{\|\delta_{\alpha g_i}\|^2}{\|\delta_\alpha\|^2} = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_{\alpha g_i}^f}\end{aligned}$$

This leads immediately to the following definition:

Definition 6.1.2. *A function $f : \mathbb{F}_n^+ \rightarrow (0, \infty)$ is called L-bounded if $\|L_i\| < \infty$ for every $i \leq n$. Similarly, A function $f : \mathbb{F}_n^+ \rightarrow (0, \infty)$ is called R-bounded if $\|R_i\| < \infty$ for every $i \leq n$.*

We will look at the weighted Fock spaces generated by L-bounded and R-bounded functions. As with positive regular free holomorphic functions, the weighted Fock space generated by a L-bounded or R-bounded function f will be denoted $\mathcal{F}^2(f)$. Due to the two different definitions we now have for $\mathcal{F}^2(f)$, we will assume that f is a positive regular free holomorphic function unless explicitly stated.

6.1.2 A Lifting Theorem When $\varepsilon(f, g)$ is Bounded

Using this more general idea of L-bounded and R-bounded maps, we are now in the position to modify the proof of the commutant lifting theorem for module maps from [19], which was based on [2] and [7]:

Theorem 6.1.3. *Let $f : \mathbb{F}_n^+ \rightarrow (0, \infty)$ be an L-bounded map and $g = \sum_{|\alpha| \geq 1} a_\alpha^g X_\alpha$ be a positive regular free holomorphic function with $\|\varepsilon(f, g)\| \leq 1$.*

In addition, let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|X\| = \|\widehat{X}\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.1 commutes.

$$\begin{array}{ccccccc} \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 & & \\ \exists \downarrow \widehat{X} & & \downarrow X & & & & \\ \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0 & & \end{array}$$

Figure 6.1: Commutative diagram for Theorem 6.1.3.

Proof. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ be a $*$ -L-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be a $*$ -submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$. Assume $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map with $\|X\| = 1$. Let $Y = XP_{\mathcal{M}}$. Then $Y : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{N}$ is also a module map with $\|Y\| = \|X\| = 1$. Now look at $\delta_0 \otimes \mathcal{H}_2$. If $\delta_0 \otimes \mathcal{H}_2 \not\subseteq \mathcal{N}$, let $\mathcal{N}_1 = \overline{\delta_0 \otimes \mathcal{H}_2 + \mathcal{N}}$. If $\delta_0 \otimes \mathcal{H}_2 \subset \mathcal{N}$, find $\alpha \in \mathbb{F}_n^+$ such that $\delta_\alpha \otimes \mathcal{H}_2 \not\subseteq \mathcal{N}$ but if $\alpha = \beta\gamma$ with $|\beta| \geq 1$ then $\delta_\gamma \otimes \mathcal{H}_2 \subset \mathcal{N}$. Then let $\mathcal{N}_1 = \overline{\delta_\alpha \otimes \mathcal{H}_2 + \mathcal{N}}$. Note that in either case,

$$(L_\alpha \otimes I_{\mathcal{H}_2})^* \mathcal{N}_1 \subset \mathcal{N} \quad \text{for every } |\alpha| \geq 1 \quad (6.1.1)$$

Let $T_i^g = P_{\mathcal{N}_1}(L_i^g \otimes I_{\mathcal{H}_2})P_{\mathcal{N}_1}$ for $i \leq n$. Then $(\mathcal{N}_1; T_1^g, T_2^g, \dots, T_n^g)$ is a $*$ -L-submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, and \mathcal{N} is a $*$ -L-submodule of \mathcal{N}_1 . Now if $|\alpha| \geq 1$, it follows that:

$$\begin{aligned}
(T_\alpha^g P_{N_1 \ominus N})^* &= (P_{N_1} (L_\alpha^g \otimes I_{H_2}) P_{N_1} P_{N_1 \ominus N})^* \\
&= (P_{N_1} (L_\alpha^g \otimes I_{H_2}) P_{N_1 \ominus N})^* \\
&= P_{N_1 \ominus N} (L_\alpha^g \otimes I_{H_2})^* P_{N_1} \\
&= 0
\end{aligned}$$

Thus,

$$T_\alpha^g P_{N_1 \ominus N} = 0 \tag{6.1.2}$$

Using this identity, a quick computation will reveal that for $|\alpha| \geq 1$

$$T_{\alpha\beta}^g P_N = T_\alpha^g V_\beta^g P_N \tag{6.1.3}$$

To verify, compute $T_{\alpha\beta}^g P_N - T_\alpha^g V_\beta^g P_N$:

$$\begin{aligned}
T_{\alpha\beta}^g P_N - T_\alpha^g V_\beta^g P_N &= T_\alpha^g (T_\beta^g P_N - V_\beta^g P_N) \\
&= T_\alpha^g (P_{N_1} (L_\beta^g \otimes I_{H_2}) P_{N_1} P_N - P_N (L_\beta^g \otimes I_{H_2}) P_N) \\
&= T_\alpha^g (P_{N_1} (L_\beta^g \otimes I_{H_2}) P_N - P_N (L_\beta^g \otimes I_{H_2}) P_N) \\
&= T_\alpha^g (P_{N_1} (L_\beta^g \otimes I_{H_2}) - P_N (L_\beta^g \otimes I_{H_2})) P_N \\
&= T_\alpha^g (P_{N_1} - P_N) (L_\beta^g \otimes I_{H_2}) P_N \\
&= T_\alpha^g P_{N_1 \ominus N} (L_\beta^g \otimes I_{H_2}) P_N \\
&= 0
\end{aligned}$$

Now we need to find a module map $\widehat{Y} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{N}_1$ with $\|\widehat{Y}\| = \|X\|$ and $P_{\mathcal{N}}\widehat{Y} = Y$. For every $\alpha \in \mathbb{F}_n^+$, define $Y_\alpha : \mathcal{H}_1 \longrightarrow \mathcal{N}$ by $Y_\alpha(x) = Y(\delta_\alpha \otimes x)$. Similarly, define $\widehat{Y}_\alpha : \mathcal{H}_1 \longrightarrow \mathcal{N}_1$ by $\widehat{Y}_\alpha(x) = \widehat{Y}(\delta_\alpha \otimes x)$. Now if \widehat{Y} is to be a module map, each \widehat{Y}_α , and thus \widehat{Y} , will be determined by \widehat{Y}_0 :

$$\widehat{Y}_\alpha(x) = \widehat{Y}(\delta_\alpha \otimes x) = \widehat{Y}((L_\alpha^f \otimes I_{\mathcal{H}_1})(\delta_0 \otimes x)) = T_\alpha^g \widehat{Y}(\delta_0 \otimes x) = T_\alpha^g \widehat{Y}_0(x)$$

However, if $|\alpha| \geq 1$, then:

$$\widehat{Y}_\alpha(x) = T_\alpha^g \widehat{Y}_0(x) = T_\alpha^g P_{\mathcal{N}_1 \ominus \mathcal{N}} \widehat{Y}_0(x) + T_\alpha^g P_{\mathcal{N}} \widehat{Y}_0(x) = T_\alpha^g P_{\mathcal{N}} \widehat{Y}_0(x) = T_\alpha^g Y_0(x)$$

We can decompose $\mathcal{F}^2(f) \otimes \mathcal{H}_1$ as $\mathcal{F}^2(f) \otimes \mathcal{H}_1 = [\delta_0 \otimes \mathcal{H}_1] \oplus [\delta_0 \otimes \mathcal{H}_1]^\perp$ and \mathcal{N}_1 as $\mathcal{N}_1 = [\mathcal{N}_1 \ominus \mathcal{N}] \oplus \mathcal{N}$ and write \widehat{Y} as a block matrix with respect to this decomposition:

$$\widehat{Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : [\delta_0 \otimes \mathcal{H}_1] \oplus [\delta_0 \otimes \mathcal{H}_1]^\perp \longrightarrow [\mathcal{N}_1 \ominus \mathcal{N}] \oplus [\mathcal{N}] \quad (6.1.4)$$

Now $Y = P_{\mathcal{N}}\widehat{Y} = \begin{bmatrix} c & d \end{bmatrix}$, so the second row of \widehat{Y} is already determined, and $\left\| \begin{bmatrix} c & d \end{bmatrix} \right\| \leq 1$. Similarly, the second column of \widehat{Y} is already determined, since

$$\begin{bmatrix} b \\ d \end{bmatrix}^* = \left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp} \right)^* : \mathcal{N}_1 \longrightarrow \bigoplus_{|\alpha| \geq 1} \delta_\alpha \otimes \mathcal{H}_1$$

is given by the following equation for $x \in \mathcal{N}_1$:

$$\left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^*(x) = \sum_{|\alpha| \geq 1} b_\alpha^f \delta_\alpha \otimes \widehat{Y}_\alpha^*(x)$$

This can be seen easily. Let $x \in N_1$ and $|\beta| \geq 1$. Then:

$$\begin{aligned} \langle \widehat{Y}^* x, \delta_\beta \otimes h \rangle &= \langle x, \widehat{Y}(\delta_\beta \otimes h) \rangle \\ &= \langle x, \widehat{Y}_\beta h \rangle \\ &= \langle \widehat{Y}_\beta^* x, h \rangle \\ &= b_\beta^f \langle \delta_\beta, \delta_\beta \rangle \langle \widehat{Y}_\beta^*(x), h \rangle \\ &= \sum_{|\alpha| \geq 1} b_\alpha^f \langle \delta_\alpha, \delta_\beta \rangle \langle \widehat{Y}_\alpha^*(x), h \rangle \\ &= \sum_{|\alpha| \geq 1} \langle b_\alpha^f \delta_\alpha \otimes \widehat{Y}_\alpha^*(x), \delta_\beta \otimes h \rangle \\ &= \left\langle \sum_{|\alpha| \geq 1} b_\alpha^f \delta_\alpha \otimes \widehat{Y}_\alpha^*(x), \delta_\beta \otimes h \right\rangle \end{aligned}$$

Next we need $\left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| \leq 1$.

$$\begin{aligned}
\left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\|^2 &= \left\| \left(\widehat{Y}|_{[\delta_0 \otimes H_1]^\perp} \right)^* \right\|^2 = \sup_{\|x\|=1} \left\langle \left(\widehat{Y}|_{[\delta_0 \otimes H_1]^\perp} \right)^* x, \left(\widehat{Y}|_{[\delta_0 \otimes H_1]^\perp} \right)^* x \right\rangle \\
&= \sup_{\|x\|=1} \left\langle \sum_{|\alpha| \geq 1} b_\alpha^f \delta_\alpha \otimes \widehat{Y}_\alpha^*(x), \sum_{|\beta| \geq 1} b_\beta^f \delta_\beta \otimes \widehat{Y}_\beta^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} \sum_{|\beta| \geq 1} \left\langle b_\alpha^f \delta_\alpha \otimes \widehat{Y}_\alpha^*(x), b_\beta^f \delta_\beta \otimes \widehat{Y}_\beta^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} \sum_{|\beta| \geq 1} \left\langle b_\alpha^f \delta_\alpha, b_\beta^f \delta_\beta \right\rangle \left\langle \widehat{Y}_\alpha^*(x), \widehat{Y}_\beta^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} \sum_{|\beta| \geq 1} b_\alpha^f b_\beta^f \langle \delta_\alpha, \delta_\beta \rangle \left\langle \widehat{Y}_\alpha^*(x), \widehat{Y}_\beta^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} (b_\alpha^f)^2 \langle \delta_\alpha, \delta_\alpha \rangle \left\langle \widehat{Y}_\alpha^*(x), \widehat{Y}_\alpha^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} b_\alpha^f \left\langle \widehat{Y}_\alpha^*(x), \widehat{Y}_\alpha^*(x) \right\rangle = \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} b_\alpha^f \left\langle x, \widehat{Y}_\alpha \widehat{Y}_\alpha^*(x) \right\rangle \\
&= \sup_{\|x\|=1} \left\langle x, \left(\sum_{|\alpha| \geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* \right) x \right\rangle
\end{aligned}$$

Thus it suffices to show that $\sum_{|\alpha| \geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* \leq I$. Notice that

$$\left[\sum_{\gamma \in \mathbb{F}_n^+} b_\gamma^g V_\gamma^g Y_0 Y_0^* V_\gamma^{g*} \right] \leq \left[\sum_{\gamma \in \mathbb{F}_n^+} b_\gamma^g V_\gamma^g V_\gamma^{g*} \right] \leq I$$

since Y is a contraction. In addition, since $\varepsilon(f, g)$ is a contraction, $\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} \leq 1$ by Corollary 5.3.4. Therefore, $b_\alpha^f \leq b_\alpha^g$ for every $\alpha \in \mathbb{F}_n^+$. Thus:

$$\begin{aligned}
\sum_{|\alpha| \geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* &= \sum_{|\alpha| \geq 1} b_\alpha^f T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} \leq \sum_{|\alpha| \geq 1} b_\alpha^g T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} \\
&= \sum_{|\alpha| \geq 1} \left[\sum_{\substack{\beta\gamma=\alpha \\ |\beta| \geq 1}} a_\beta^g b_\gamma^g \right] T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} = \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_{\beta\gamma}^g Y_0 Y_0^* T_{\beta\gamma}^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_{\beta\gamma}^g P_N Y_0 (P_N Y_0)^* T_{\beta\gamma}^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_{\beta\gamma}^g P_N Y_0 Y_0^* P_N^* T_{\beta\gamma}^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_{\beta\gamma}^g P_N Y_0 Y_0^* (T_{\beta\gamma}^g P_N)^* \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_\beta^g V_\gamma^g P_N Y_0 Y_0^* (T_\beta^g V_\gamma^g P_N)^* \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_\beta^g V_\gamma^g Y_0 Y_0^* P_N V_\gamma^{g*} T_\beta^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_\beta^g V_\gamma^g Y_0 (P_N Y_0)^* V_\gamma^{g*} T_\beta^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in F_n^+} a_\beta^g b_\gamma^g T_\beta^g V_\gamma^g Y_0 Y_0^* V_\gamma^{g*} T_\beta^{g*} \\
&= \sum_{|\beta| \geq 1} a_\beta^g T_\beta^g \left[\sum_{\gamma \in F_n^+} b_\gamma^g V_\gamma^g Y_0 Y_0^* V_\gamma^{g*} \right] T_\beta^{g*} \\
&\leq \sum_{|\beta| \geq 1} a_\beta^g T_\beta^g T_\beta^{g*} \leq \sum_{|\beta| \geq 1} a_\beta^g (L_\beta^g \otimes I_{H_2}) (L_\beta^g \otimes I_{H_2})^* \leq I
\end{aligned}$$

as desired. The last inequality follows from Proposition 5.2.2. Thus, $\left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| \leq$

1 and $\left\| \begin{bmatrix} c & d \end{bmatrix} \right\| \leq 1$. As in most commutant lifting theorems, we now apply

Parrott's Lemma [15] to find a such that $\|\widehat{Y}\| = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\| = 1$. Thus, $\widehat{Y} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{N}_1$ is a module map with $\|\widehat{Y}\| = \|X\|$ and $P_{\mathcal{N}}\widehat{Y} = Y$, as desired. By iterating this process, it follows that we can find a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|X\| = \|\widehat{X}\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, completing the proof. \square

We can also prove a similar result for the right creation operators R_i , as shown in the following theorem:

Theorem 6.1.4. *Let $f : \mathbb{F}_n^+ \rightarrow (0, \infty)$ be an R -bounded map and $g = \sum_{|\alpha| \geq 1} a_\alpha^g X_\alpha$ be a positive regular free holomorphic function with $\|\varepsilon(f, g)\| \leq 1$. In addition, let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; R_1 \otimes I_{\mathcal{H}_1}, R_2 \otimes I_{\mathcal{H}_1}, \dots, R_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; R_1 \otimes I_{\mathcal{H}_2}, R_2 \otimes I_{\mathcal{H}_2}, \dots, R_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|X\| = \|\widehat{X}\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.2 commutes.*

$$\begin{array}{ccccc} \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\ \exists \downarrow \widehat{X} & & \downarrow X & & \\ \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0 \end{array}$$

Figure 6.2: Commutative diagram for Theorem 6.1.4.

Proof. The proof of Theorem 6.1.4 is very similar to the proof of Theorem 6.1.3. Thus, we will only provide a sketch of the proof of Theorem 6.1.4. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ be a $*$ - R -submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; R_1 \otimes I_{\mathcal{H}_1}, R_2 \otimes$

$I_{\mathcal{H}_1}, \dots, R_n \otimes I_{\mathcal{H}_1}$) and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be a $*$ -submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; R_1 \otimes I_{\mathcal{H}_2}, R_2 \otimes I_{\mathcal{H}_2}, \dots, R_n \otimes I_{\mathcal{H}_2})$. Assume $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map with $\|X\| = 1$. Let $Y = XP_{\mathcal{M}}$. For every $\alpha \in \mathbb{F}_n^+$, define $Y_\alpha : \mathcal{H}_1 \rightarrow \mathcal{N}$ by $Y_\alpha(x) = Y(\delta_\alpha \otimes x)$. Similarly, define $\widehat{Y}_\alpha : \mathcal{H}_1 \rightarrow \mathcal{N}_1$ by $\widehat{Y}_\alpha(x) = \widehat{Y}(\delta_\alpha \otimes x)$.

If $\delta_0 \otimes \mathcal{H}_2 \not\subseteq \mathcal{N}$, let $\mathcal{N}_1 = \overline{\delta_0 \otimes \mathcal{H}_2 + \mathcal{N}}$. If $\delta_0 \otimes \mathcal{H}_2 \subset \mathcal{N}$, find $\alpha \in \mathbb{F}_n^+$ such that $\delta_\alpha \otimes \mathcal{H}_2 \not\subseteq \mathcal{N}$ but if $\alpha = \beta\gamma$ with $|\beta| \geq 1$ then $\delta_\gamma \otimes \mathcal{H}_2 \subset \mathcal{N}$. Then let $\mathcal{N}_1 = \overline{\delta_\alpha \otimes \mathcal{H}_2 + \mathcal{N}}$. It follows that (6.1.1), (6.1.2), and (6.1.3) of Theorem 6.1.3 hold.

Now we need to find a module map $\widehat{Y} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{N}_1$ with $\|\widehat{Y}\| = \|X\|$ and $P_{\mathcal{N}_1}\widehat{Y} = Y$. As before, \widehat{Y}_α , and thus \widehat{Y} , will be determined by \widehat{Y}_0 , but this time we get that $\widehat{Y}_\alpha(x) = T_\alpha^g \widehat{Y}_0(x)$. Furthermore, if $|\alpha| \geq 1$, it follows that $\widehat{Y}_\alpha(x) = T_\alpha^g Y_0(x)$. Let $T_i^g = P_{\mathcal{N}_1}(R_i^g \otimes I_{\mathcal{H}_2})P_{\mathcal{N}_1}$ for $i \leq n$. Then $(\mathcal{N}_1; T_1^g, T_2^g, \dots, T_n^g)$ is a $*$ -R-submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; R_1 \otimes I_{\mathcal{H}_2}, R_2 \otimes I_{\mathcal{H}_2}, \dots, R_n \otimes I_{\mathcal{H}_2})$, and \mathcal{N} is a $*$ -R-submodule of \mathcal{N}_1 . We now decompose \widehat{Y} into the same 2×2 matrix given by 6.1.4 in Theorem 6.1.3.

As in Theorem 6.1.3, $\left\| \begin{bmatrix} c & d \end{bmatrix} \right\| \leq 1$, and to prove that $\left\| \begin{bmatrix} c \\ d \end{bmatrix} \right\| \leq 1$, it suffices to prove that $\sum_{|\alpha| \geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* = \sum_{|\alpha| \geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* \leq I$:

$$\begin{aligned}
\sum_{|\alpha| \geq 1} b_{\alpha}^f \widehat{Y}_{\alpha} \widehat{Y}_{\alpha}^* &= \sum_{|\alpha| \geq 1} b_{\alpha}^f T_{\alpha}^g Y_0 Y_0^* T_{\alpha}^{g*} \leq \sum_{|\alpha| \geq 1} b_{\alpha}^g T_{\alpha}^g Y_0 Y_0^* T_{\alpha}^{g*} \\
&= \sum_{|\alpha| \geq 1} \left[\sum_{\substack{\tilde{\gamma} \tilde{\beta} = \tilde{\alpha} \\ |\tilde{\beta}| \geq 1}} a_{\tilde{\beta}}^g b_{\tilde{\gamma}}^g \right] T_{\alpha}^g Y_0 Y_0^* T_{\alpha}^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in \mathbb{F}_n^+} a_{\tilde{\beta}}^g b_{\tilde{\gamma}}^g T_{\beta\gamma}^g Y_0 Y_0^* T_{\beta\gamma}^{g*} \\
&= \sum_{|\beta| \geq 1} \sum_{\gamma \in \mathbb{F}_n^+} a_{\tilde{\beta}}^g b_{\tilde{\gamma}}^g T_{\beta}^g V_{\gamma}^g Y_0 Y_0^* V_{\gamma}^{g*} T_{\beta}^{g*} \\
&= \sum_{|\beta| \geq 1} a_{\tilde{\beta}}^g T_{\beta}^g \left[\sum_{\gamma \in \mathbb{F}_n^+} b_{\tilde{\gamma}}^g V_{\gamma}^g Y_0 Y_0^* V_{\gamma}^{g*} \right] T_{\beta}^{g*} \\
&\leq \sum_{|\beta| \geq 1} a_{\tilde{\beta}}^g T_{\beta}^g T_{\beta}^{g*} \leq \sum_{|\beta| \geq 1} a_{\tilde{\beta}}^g (R_{\beta}^g \otimes I_{\mathcal{H}_2}) (R_{\beta}^g \otimes I_{\mathcal{H}_2})^* \leq I
\end{aligned}$$

as desired. The last inequality follows from the second part of Proposition 5.2.2. Finally, iterating finishes the proof. \square

6.1.3 A Lifting Theorem When $\mathcal{F}^2(g)$ is the Full Fock Space

We can actually say quite a bit more when $\mathcal{F}^2(g)$ is the full Fock space. In this case, any module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ ends up being a contraction, as shown in the following proposition:

Proposition 6.1.5. *Let $f = \sum_{|\alpha| \geq 1} a_{\alpha}^f X_{\alpha}$, where f has only finitely many terms, and g be the full Fock space. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes$*

$\mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2}$), respectively. If $\varepsilon(f, g)$ is bounded, then $\varepsilon(f, g)$ is a contraction, and therefore X can be lifted to \widehat{X} such that $\|\widehat{X}\| = \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$.

Proof. Popescu proved in [19] that for all $\alpha \in \mathbb{F}_n^+$ $b_{\alpha\alpha} \geq b_{\alpha}b_{\alpha} = b_{\alpha}^2$. Thus, if $b_{\alpha}^f > 1$ for any α , $\lim_{k \rightarrow \infty} b_{\alpha^k}^f \leq \lim_{k \rightarrow \infty} (b_{\alpha}^f)^k = \infty$. Therefore, if $\varepsilon(f, g)$ is bounded, $\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_{\alpha}^f}{b_{\alpha}^g} = \sup_{\alpha \in \mathbb{F}_n^+} b_{\alpha}^f < \infty$ and so $b_{\alpha} \leq 1$ for all $\alpha \in \mathbb{F}_n^+$. It follows that $\|\varepsilon\| = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_{\alpha}^f}{b_{\alpha}^g} \leq 1$ and thus we can apply Theorem 6.1.3 to obtain our \widehat{X} , as desired. \square

6.1.4 *-Submodule Correspondence Between Two Isomorphic Weighted Fock Spaces

Now if \mathcal{M} is a submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $\varepsilon : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is bounded, then $\varepsilon(\mathcal{M})$ is a submodule of $\mathcal{F}^2(g)$. This follows immediately due to the nice property that $L_i \delta_{\alpha} = \delta_{g_i \alpha}$. However, it is important to note that the L_i^* 's do not behave nearly so nicely. We know that $L_i^* \delta_{g_i \alpha} = c \delta_{\alpha}$. Then:

$$\frac{1}{b_{g_i \alpha}} = \langle \delta_{g_i \alpha}, \delta_{g_i \alpha} \rangle = \langle \delta_{g_i \alpha}, L_i \delta_{\alpha} \rangle = \langle L_i^* \delta_{g_i \alpha}, \delta_{\alpha} \rangle = \langle c \delta_{\alpha}, \delta_{\alpha} \rangle = \frac{c}{b_{\alpha}}$$

It follows that $L_i^* \delta_{g_i \alpha} = \frac{b_{\alpha}}{b_{g_i \alpha}} \delta_{\alpha}$. It follows that if $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ is a *-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $\varepsilon(f, g)$ is bounded, $(\varepsilon(\mathcal{M}); V_1^g, V_2^g, \dots, V_n^g)$ need not be a *-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$.

However, when $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$, there is a nice correspondence between $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and those of $(\mathcal{F}^2(g) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$, as shown in the following lemma:

Lemma 6.1.6. *Let f and g be positive regular free holomorphic functions with $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$. Then if $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ is a $*$ -submodule of $\mathcal{F}^2(f)$, there exists a $*$ -submodule $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ of $\mathcal{F}^2(g)$ such that $\mathcal{M} \cong \mathcal{N}$. Furthermore, there exists a module map $\widehat{\varepsilon}(\mathcal{M}, \mathcal{N}) : \mathcal{M} \rightarrow \mathcal{N}$ with $\|\widehat{\varepsilon}\| \leq \|\varepsilon\|$ such that $\widehat{\varepsilon}P_{\mathcal{M}} = P_{\mathcal{N}}\varepsilon$.*

Proof. Let the $*$ -submodule $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ be given. We can decompose $\mathcal{F}^2(f)$ as $\mathcal{F}^2(f) = \mathcal{M} \oplus \mathcal{L}$. This means that $\mathcal{M} = \mathcal{F}^2(f)/\mathcal{L}$, where \mathcal{L} is a submodule of $\mathcal{F}^2(f)$. Now since $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$, we can look at $\mathcal{N} = \mathcal{F}^2(g)/\varepsilon(\mathcal{L})$. Define $V_i^g : \mathcal{N} \rightarrow \mathcal{N}$ by $V_i^g(x + \varepsilon(\mathcal{L})) = L_i x + \varepsilon(\mathcal{L})$. It is easy to see that V_i^g is well defined. By Proposition 5.9 of Douglas and Paulsen [8], this induces a unique $\widehat{\varepsilon}$ with $\|\widehat{\varepsilon}\| \leq \|\varepsilon\|$ such that the diagram in Figure 6.3 commutes.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{L}^c & \longrightarrow & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \twoheadrightarrow & \mathcal{M} \longrightarrow 0 \\
& & \downarrow \varepsilon|_{\mathcal{L}} & & \downarrow \varepsilon & & \downarrow \widehat{\varepsilon} \\
0 & \longrightarrow & \varepsilon(\mathcal{L})^c & \longrightarrow & \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \twoheadrightarrow & \mathcal{N} \longrightarrow 0
\end{array}$$

Figure 6.3: Commutative diagram for Lemma 6.1.6.

It immediately follows that $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ is a $*$ -submodule as desired. \square

6.1.5 A Lifting Theorem When $\varepsilon(f, g)$ is Bounded

Notice that Theorems 6.1.3 and 6.1.4 only apply to cases where ε is a contraction. However, it turns out that as long as $\varepsilon(f, g)$ is bounded, we can lift all module maps with only a small penalty, as shown in the main theorem of this thesis:

Theorem 6.1.7. *Let f and g be positive regular free holomorphic functions with $\|\varepsilon(f, g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be *-submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|\widehat{X}\| \leq C \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.4 commutes.*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{L}_{\mathcal{M}}^c & \longrightarrow & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
 & & \exists \downarrow \widehat{X}|_{\mathcal{L}_{\mathcal{M}}} & & \exists \downarrow \widehat{X} & & \downarrow X & & \\
 0 & \longrightarrow & \mathcal{L}_{\mathcal{N}}^c & \longrightarrow & \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
 \end{array}$$

Figure 6.4: Commutative diagram for Theorem 6.1.7.

Proof. Assume without loss of generality that $\|\varepsilon(f, g)\| = C > 1$. Define an L-bounded function $h : \mathbb{F}_n^+ \rightarrow (0, \infty)$ as follows:

$$\begin{aligned}
 b_0^h &= 1 \\
 b_\alpha^h &= \frac{b_\alpha^f}{C^2} \quad \text{for every } \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1
 \end{aligned}$$

Since f is a positive regular free holomorphic function, it follows immediately that h is indeed L -bounded. This gives us the weighted Fock space $\mathcal{F}^2(h)$. In addition, $\mathcal{F}^2(f) \cong \mathcal{F}^2(h)$. This is because for $|\alpha| \geq 1$:

$$\|\delta_\alpha^h\| = \frac{1}{\sqrt{b_\alpha^h}} = \sqrt{\frac{C^2}{b_\alpha^f}} = C \|\delta_\alpha^f\|$$

Note that since $\|\delta_0^h\| = \frac{1}{\sqrt{b_0^h}} = 1 = \frac{1}{\sqrt{b_0^f}} = \|\delta_0^f\|$, $\mathcal{F}^2(g)$ and $\mathcal{F}^2(h)$ are not isometric.

Now let $\varepsilon(h, f) : \mathcal{F}^2(h) \otimes \mathcal{H}_2 \rightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_2$ be defined in the typical way. It follows that $\|\varepsilon(h, f)\| = 1$ and $\|\varepsilon(h, f)^{-1}\| = C$. Let $\widetilde{\mathcal{M}}$ be the $*$ -submodule isomorphic to \mathcal{M} given by Lemma 6.1.6.

We get the diagram shown in Figure 6.5.

$$\begin{array}{ccc}
 \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} \\
 \searrow \varepsilon(h, f)^{-1} & & \downarrow \widehat{\varepsilon}(h, f)^{-1} \\
 & \mathcal{F}^2(h) \otimes \mathcal{H}_1 & \xrightarrow{P_{\widetilde{\mathcal{M}}}} \widetilde{\mathcal{M}} \\
 \swarrow & \downarrow & \downarrow X \circ \widehat{\varepsilon}(h, f) \\
 & \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} \mathcal{N}
 \end{array}
 \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} X$$

Figure 6.5: Introducing the isomorphic weighted Fock space $\mathcal{F}^2(h) \otimes \mathcal{H}_1$.

If we focus only on the bottom two rows, we obtain the diagram shown in Figure 6.6.

$$\begin{array}{ccc}
\mathcal{F}^2(h) \otimes \mathcal{H}_1 & \xrightarrow{P_{\widetilde{\mathcal{M}}}} & \widetilde{\mathcal{M}} \\
\downarrow & & \downarrow X \circ \widehat{\varepsilon}(h, f) \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N}
\end{array}$$

Figure 6.6: Simplified diagram with weighted Fock space $\mathcal{F}^2(h) \otimes \mathcal{H}_1$.

A brief calculation gives us

$$\begin{aligned}
\|\varepsilon(h, g)\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^h}{b_\alpha^g} = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{C^2 b_\alpha^g} \\
&= \frac{1}{C^2} \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} = \frac{1}{C^2} \|\varepsilon(f, g)\|^2 \\
&= 1
\end{aligned}$$

We are thus in a position to apply Theorem 6.1.3. This gives us $\widehat{Y} : \mathcal{F}^2(h) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|\widehat{Y}\| \leq \|X\|$ and $(X \circ \widehat{\varepsilon}(h, f))P_{\widetilde{\mathcal{M}}} = P_{\mathcal{N}}\widehat{Y}$.

The inequality is verified by a quick calculation:

$$\begin{aligned}
\|\widehat{Y}\| &= \|X \circ \widehat{\varepsilon}(h, f)\| \\
&\leq \|X\| \|\widehat{\varepsilon}(h, f)\| \\
&\leq \|X\| \|\varepsilon(h, f)\| \\
&= \|X\|
\end{aligned}$$

Finally, define $\widehat{X} := \widehat{Y} \circ \varepsilon(h, f)^{-1}$. Then

$$\begin{aligned}
\|\widehat{X}\| &= \|\widehat{Y} \circ \varepsilon(h, f)^{-1}\| \\
&\leq \|\widehat{Y}\| \|\varepsilon(h, f)^{-1}\| \\
&\leq C \|X\|
\end{aligned}$$

All that is left is to verify that $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$:

$$\begin{aligned}
XP_{\mathcal{M}} &= XP_{\mathcal{M}}\varepsilon(h, f)\varepsilon(h, f)^{-1} \\
&= X\widehat{\varepsilon}(h, f)P_{\widetilde{\mathcal{M}}}\varepsilon(h, f)^{-1} \\
&= P_{\mathcal{N}}\widehat{Y}\varepsilon(h, f)^{-1} \\
&= P_{\mathcal{N}}\widehat{X}
\end{aligned}$$

as desired. □

Theorem 6.1.8. *Let f and g be positive regular free holomorphic functions with $\|\varepsilon(f, g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be *-submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; R_1 \otimes I_{\mathcal{H}_1}, R_2 \otimes I_{\mathcal{H}_1}, \dots, R_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; R_1 \otimes I_{\mathcal{H}_2}, R_2 \otimes I_{\mathcal{H}_2}, \dots, R_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|\widehat{X}\| \leq C \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.7 commutes.*

$$\begin{array}{ccccc}
\mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\downarrow \exists \hat{X} & & \downarrow X & & \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 6.7: Commutative diagram for Theorem 6.1.8.

Proof. Identical to the previous proof. □

6.2 Projective Resolutions

Using Theorems 6.1.7 and 6.1.8, we can build projective resolutions. To build such resolutions, we must first recall, with a slight reformulation, the construction of Poisson kernels developed by Popescu in [19] based on [4]:

Lemma 6.2.1. (*G. Popescu, [19]*) *Let f be a positive regular free holomorphic function, and let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be semi-invariant under the maps $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$, and let $W_i = P_{\mathcal{N}}(L_i \otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i \leq n$. Then:*

(i) *The sequence $\Delta_N = \sum_{|\alpha| \leq N} a_{\alpha} W_{\alpha} W_{\alpha}^*$ is non-negative and non-increasing,*

(ii) $\lim_{N \rightarrow \infty} \sum_{|\gamma| > N} \left[\sum_{\substack{\alpha\beta=\gamma \\ |\alpha| \leq N}} b_{\alpha} a_{\beta} \right] W_{\gamma} W_{\gamma}^* = 0,$

(iii) $\Delta = \lim_{N \rightarrow \infty} \Delta_N$ *exists, $0 \leq \Delta \leq I$, and Δ is not equal to zero, and*

(iv) $\sum_{\alpha \in \mathbb{F}_n^+} b_{\alpha} W_{\alpha} \Delta W_{\alpha}^* = I_{\mathcal{N}}.$

Theorem 6.2.2. (*G. Popescu, [19] - Poisson kernels*) *Let f be a positive regular free holomorphic function and let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$, and let $W_i = P_{\mathcal{N}}(L_i \otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i \leq n$. Define*

$D = \Delta^{\frac{1}{2}}$, where Δ is the non-negative operator given by the previous lemma.

Define $K : \mathcal{N} \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{N}$ by

$$K(x) = \sum_{\alpha \in F_n^+} b_\alpha \delta_\alpha \otimes DW_\alpha^* x.$$

Then K is an isometry and K^* is a module map. Moreover, \mathcal{N} is isomorphic to $K(\mathcal{N})$ and $K(\mathcal{N})$ is a $*$ -invariant submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. We call K the Poisson Kernel of \mathcal{N} .

We use the Poisson kernel to construct resolutions.

Proposition 6.2.3. *Let f be a positive regular free holomorphic function and let $\mathcal{M} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ be semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$. Then there exists Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \dots$, partial isometric module maps $\Phi_i : \mathcal{F}^2(f) \otimes \mathcal{H}_{i+1} \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_i$, $i = 1, 2, \dots$, and a partial isometric module map $\Phi_0 : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{M}$ such that the sequence shown in Figure 6.8 is exact.*

$$\dots \xrightarrow{\Phi_3} \mathcal{F}^2(f) \otimes \mathcal{H}_3 \xrightarrow{\Phi_2} \mathcal{F}^2(f) \otimes \mathcal{H}_2 \xrightarrow{\Phi_1} \mathcal{F}^2(f) \otimes \mathcal{H}_1 \xrightarrow{\Phi_0} \mathcal{M} \longrightarrow 0$$

Figure 6.8: Exact sequence of partial isometric module maps.

Proof. Since \mathcal{M} is a semi-invariant subspace of $\mathcal{F}^2(f) \otimes \mathcal{H}$, it has a Poisson kernel $K_0 : \mathcal{M} \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_1$, for some Hilbert space \mathcal{H}_1 . Set $\Phi_0 = K_0^*$. Notice that $\ker \Phi_0$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}_1$, and hence it has a Poisson kernel $K_1 : \ker \Phi_0 \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_2$, for some Hilbert space \mathcal{H}_2 . Define $\Phi_1 : \mathcal{F}^2(f) \otimes \mathcal{H}_2 \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_1$ by $\Phi_1 = \iota \circ K_1^*$, as illustrated in Figure 6.9.

$$\begin{array}{ccccc}
\mathcal{F}^2(f) \otimes \mathcal{H}_2 & \xrightarrow{\Phi_1} & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{\Phi_0} & \mathcal{M} \\
& \searrow^{K_1^*} & \uparrow \iota & & \\
& & \ker \Phi_0 & & \\
0 & \nearrow & & &
\end{array}$$

Figure 6.9: Construction of the exact sequence of partial isometric module maps.

The other maps are constructed similarly: $\ker \Phi_1$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}_2$, and hence it has a Poisson kernel $K_2 : \ker \Phi_1 \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_3$, for some Hilbert space \mathcal{H}_3 , $\Phi_2 = \iota \circ K_2^*$. \square

This proposition together with 6.1.7 and 6.1.8 give the following result:

Theorem 6.2.4. *Let f and g be positive regular free holomorphic functions. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be subquotients of $(\mathcal{F}^2(f) \otimes \mathcal{H}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \dots, L_n \otimes I_{\mathcal{H}})$ and $(\mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}; L_1 \otimes I_{\widehat{\mathcal{H}}}, L_2 \otimes I_{\widehat{\mathcal{H}}}, \dots, L_n \otimes I_{\widehat{\mathcal{H}}})$, respectively. Assume there exist a module map $T : \mathcal{M} \longrightarrow \mathcal{N}$. Then if $\|\epsilon(f, g)\| = C$, there exists module maps $T_1, T_2, T_3 \dots$ with $\|T_i\| \leq C^i \|T\|$, such that the diagram in Figure 6.10 commutes.*

$$\begin{array}{cccccccc}
\dots & \xrightarrow{\Phi_3} & \mathcal{F}^2(f) \otimes \mathcal{H}_3 & \xrightarrow{\Phi_2} & \mathcal{F}^2(f) \otimes \mathcal{H}_2 & \xrightarrow{\Phi_1} & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{\Phi_0} & \mathcal{M} & \longrightarrow & 0 \\
& & \exists \downarrow T_3 & & \exists \downarrow T_2 & & \exists \downarrow T_1 & & \downarrow T & & \\
\dots & \xrightarrow{\Psi_3} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_3 & \xrightarrow{\Psi_2} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_2 & \xrightarrow{\Psi_1} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_1 & \xrightarrow{\Psi_0} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 6.10: Projective Resolutions.

Proof. Clearly we have that T_1 exists via Theorems 6.1.7 and 6.1.8. $\Phi_1 = \iota \circ K_1^*$ where $K_1 : \ker \Phi_0 \rightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_2$ is the Poisson kernel of $\ker \Phi_0$, and $\Psi_1 = \iota \circ \widehat{K}_1^*$ where $\widehat{K}_1 : \ker \Psi_0 \rightarrow \mathcal{F}^2(f) \otimes \widehat{\mathcal{H}}_2$ is the Poisson kernel of $\ker \Psi_0$, as shown in Figure 6.11.

$$\begin{array}{ccccccc}
& & \Phi_1 & & & & \\
& \curvearrowright & & \curvearrowleft & & & \\
\mathcal{F}^2(f) \otimes \mathcal{H}_2 & \xrightarrow{K_1^*} & \ker \Phi_0 & \xrightarrow{\iota} & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{\Phi_0} & \mathcal{M} \longrightarrow 0 \\
& & \downarrow T_1 & & \downarrow T_1 & & \downarrow T \\
\mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_2 & \xrightarrow{\widehat{K}_1^*} & \ker \Psi_0 & \xrightarrow{\iota} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_1 & \xrightarrow{\Psi_0} & \mathcal{N} \longrightarrow 0 \\
& \curvearrowleft & & \curvearrowright & & & \\
& & \Psi_1 & & & &
\end{array}$$

Figure 6.11: Constructing Projective Resolutions.

Since K_1 and \widehat{K}_1 are isometries, by Theorem 6.2.2, the Poisson kernels induce a module map $\widehat{T}_1 : K_1(\ker \Phi_0) \rightarrow \widehat{K}_1(\ker \Psi_0)$ with $\|\widehat{T}_1\| = \|T_1\|$. Therefore, Theorems 6.1.7 and 6.1.8 give us T_2 with $\|T_2\| \leq C \|T_1\|$ such that the diagram in Figure 6.12 commutes.

$$\begin{array}{ccc}
\mathcal{F}^2(f) \otimes \mathcal{H}_2 & \xrightarrow{P} & K_1(\ker \Phi_0) \\
\downarrow \exists T_2 & & \downarrow \widehat{T}_1 \\
\mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_2 & \xrightarrow{P} & \widehat{K}_1(\ker \Psi_0)
\end{array}$$

Figure 6.12: Lifting the module map \widehat{T}_1 .

By iterating this process, we get the diagram shown in Figure 6.13. This diagram has the property that $\|T_i\| \leq C^i \|T\|$, as desired.

□

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\Phi_3} & \mathcal{F}^2(f) \otimes \mathcal{H}_3 & \xrightarrow{\Phi_2} & \mathcal{F}^2(f) \otimes \mathcal{H}_2 & \xrightarrow{\Phi_1} & \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{\Phi_0} & \mathcal{M} & \longrightarrow & 0 \\
& & \downarrow T_3 & & \downarrow T_2 & & \downarrow T_1 & & \downarrow T & & \\
\cdots & \xrightarrow{\Psi_3} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_3 & \xrightarrow{\Psi_2} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_2 & \xrightarrow{\Psi_1} & \mathcal{F}^2(g) \otimes \widehat{\mathcal{H}}_1 & \xrightarrow{\Psi_0} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 6.13: Completed projective resolution.

6.3 Future Problems

There are a variety of different questions which arise from the above work. In this section, we will pose several questions and conjectures which have arisen, but as yet are without solutions or proofs, through the course of our study of module maps over noncommutative domain algebras. We begin with a straight forward question:

Problem 6.3.1. *Assume that f and g are positive regular free holomorphic functions which are not finitely generated and let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. What conditions need to be satisfied in order to lift a module map $X : \mathcal{M} \longrightarrow \mathcal{N}$ to a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that the diagram in Figure 6.14 commutes?*

$$\begin{array}{ccc}
\mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} \longrightarrow 0 \\
\exists \downarrow \widehat{X} & & \downarrow X \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} \longrightarrow 0
\end{array}$$

Figure 6.14: Commutative diagram for Problem 6.3.1.

In particular, we can ask the following questions:

- Do f and g need to satisfy special conditions, perhaps some sort of C_0 -type condition?
- What conditions must be placed on the formal identity map $\epsilon(f, g)$? In particular, does $\epsilon(f, g)$ still need to be bounded? Does it need to be a contraction?
- Do the $*$ -submodules \mathcal{M} and \mathcal{N} need to satisfy any special properties?

The situation presented in Example 5.4.3 illustrates some of the difficulty encountered when attempting to generalize to positive regular free holomorphic functions which are not finitely generated. Even though the formal identity map $\epsilon(f, g)$ is unbounded, there is still a non-trivial module map $Y : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$. Worse still, there exist $*$ -submodules \mathcal{M} and \mathcal{N} and module maps $X : \mathcal{M} \rightarrow \mathcal{N}$ and $Y : \mathcal{M} \rightarrow \mathcal{N}$ such that X can indeed be lifted to a module map $\widehat{X} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$, but Y cannot be lifted to a module map $\widehat{Y} : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$.

However, in Example 5.4.3, the a_α^f 's and the a_α^g 's both grew very quickly. It is possible if they both satisfy a C_0 -type condition, a lifting theorem can be obtained. With this in mind, the following definition is given:

Definition 6.3.2. *A positive regular free holomorphic function f is of type C_0 if the following holds:*

$$\lim_{k \rightarrow \infty} \left(\sum_{|\alpha|=k} |a_\alpha^f|^2 \right)^{\frac{1}{2k}} = 0$$

This leads to the following conjecture:

Conjecture 1. *Let f and g be positive regular free holomorphic functions of type C_0 with $\|\varepsilon(f, g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|\widehat{X}\| \leq C \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.15 commutes.*

$$\begin{array}{ccccc} \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\ \exists \downarrow \widehat{X} & & \downarrow X & & \\ \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0 \end{array}$$

Figure 6.15: Commutative diagram for Conjecture 1.

This conjecture can be proved if Theorem 5.3.1 can be modified to state that (i) through (v) are equivalent to (vi) if f is a positive regular free holomorphic function of type C_0 .

The next question we will pose relates to the main theorem of the paper:

Problem 6.3.3. *Let f and g be positive regular free holomorphic functions (either finitely generated or possibly of type C_0) with $\|\varepsilon(f, g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map, are there conditions which can be placed on \mathcal{M} and \mathcal{N} such that X can be lifted to a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ with $\|\widehat{X}\| \leq \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.16 commutes?*

$$\begin{array}{ccccc}
\mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\exists \downarrow \widehat{X} & & \downarrow X & & \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0
\end{array}$$

Figure 6.16: Commutative diagram for Problem 6.3.3.

This problem can be illustrated with the following example:

Example 6.3.4. Look at \mathbb{F}_2^+ and f and g such that

$$a_{\alpha}^f = \begin{cases} 1 & \text{if } \alpha = 1 \\ 1 & \text{if } \alpha = 2 \\ 0 & \text{else} \end{cases} \quad a_{\alpha}^g = \begin{cases} \frac{1}{2} & \text{if } \alpha = 1 \\ 1 & \text{if } \alpha = 2 \\ 1 & \text{if } |\alpha| = 2 \\ 0 & \text{else} \end{cases}$$

Take $\mathcal{M} = \text{span} \{\delta_{1^k} : k \in \mathbb{N}\} \subset \mathcal{F}^2(f)$ and $\mathcal{N} = \text{span} \{\delta_{1^k} : k \in \mathbb{N}\} \subset \mathcal{F}^2(g)$. Define $X : \mathcal{M} \rightarrow \mathcal{N}$ by $X\delta_{\alpha} = \delta_{\alpha}$. It is easy to verify the following:

- $(\mathcal{M}; L_1, 0)$ is a $*$ -submodule of $\mathcal{F}^2(f)$.
- $(\mathcal{N}; L_1, 0)$ is a $*$ -submodule of $\mathcal{F}^2(g)$.
- $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map.
- $\|X\| = 2$.
- $\|\epsilon(f, g)\| = 2$.

However, when X is lifted, it will be lifted to $\widehat{X} = \epsilon(f, g)$, and $\|X\| = \|\widehat{X}\|$. Thus, we have an example where $\|\epsilon(f, g)\| > 1$, yet we can lift a module map X with no penalty in the norm.

This leads us to the following conjecture:

Conjecture 2. *Let f and g be positive regular free holomorphic functions (either finitely generated or possibly of type C_0) with $\|\varepsilon(f, g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \dots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \dots, V_n^g)$ be $*$ -submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \dots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \dots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \rightarrow \mathcal{N}$ is a module map with $\|\widehat{\varepsilon}\| = C$, then X can be lifted to a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \rightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ with $\|\widehat{X}\| \leq \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the diagram in Figure 6.17 commutes.*

$$\begin{array}{ccccc} \mathcal{F}^2(f) \otimes \mathcal{H}_1 & \xrightarrow{P_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\ \exists \downarrow \widehat{X} & & \downarrow X & & \\ \mathcal{F}^2(g) \otimes \mathcal{H}_2 & \xrightarrow{P_{\mathcal{N}}} & \mathcal{N} & \longrightarrow & 0 \end{array}$$

Figure 6.17: Commutative diagram for Conjecture 2.

To prove this conjecture, more care needs to be taken with respect to the $\widehat{\varepsilon}$ maps of Theorem 6.1.7. In particular, the norms of $X \circ \widehat{\varepsilon}(h, f)$ and $\widehat{Y} = X \circ \widehat{\varepsilon}(h, f)$ need to be more carefully computed.

Next, let f be a positive regular free holomorphic function and define:

$$F^\infty(f) = \{g \in \mathcal{F}^2(f) : \forall h \in \mathcal{F}^2(f), gh \in \mathcal{F}^2(f)\}$$

with the following norm:

$$\|g\|_\infty = \sup \{\|gh\|_2 : h \in \mathcal{F}^2(f), \|h\|_2 \leq 1\}.$$

Similarly, define:

$$F^\infty(f) = \{g \in \mathcal{F}^2(f) : \forall h \in \mathcal{F}^2(f), hg \in \mathcal{F}^2(f)\}$$

with the following norm:

$$\|g\|_\infty = \sup \{\|hg\|_2 : h \in \mathcal{F}^2(f), \|h\|_2 \leq 1\}.$$

In generalizing Theorem 2.2.3 and 2.2.4, Arias ([2]) was able to prove the following proposition:

Proposition 6.3.5. *Let f be a positive regular free holomorphic function, and $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ be a Hilbert module. Then $X : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(f)$ is a module map if and only if there exists $g \in R^\infty(f)$ such that $X = R_g$. Similarly, let g be a positive regular free holomorphic function and $(\mathcal{F}^2(g); R_1, R_2, \dots, R_n)$ be a Hilbert module. Then $X : \mathcal{F}^2(g) \rightarrow \mathcal{F}^2(g)$ is a module map if and only if there exists $f \in F^\infty(g)$ such that $X = L_f$.*

Let f and g be two positive regular free holomorphic functions and let $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \dots, L_n)$ be Hilbert modules. In our more general setting, it still holds that any module map $X : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ will behave like a multiplication operator R_g . However, there is no reason why g has to belong to either $R^\infty(f)$ or $R^\infty(g)$. Therefore, we propose the following conjecture:

Conjecture 3. *Let f and g be positive regular free holomorphic functions and $(\mathcal{F}^2(f); L_1, L_2, \dots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \dots, L_n)$ be Hilbert modules. Then $X : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is a module map if and only if there exists $h \in R^\infty(f)$*

and $k \in R^\infty(g)$ such that $X = \sum_{i=0}^{\infty} R_{k_i} \circ \varepsilon(f, g) \circ R_{h_i}$. Similarly, let f and g be positive regular free holomorphic functions and $(\mathcal{F}^2(f); R_1, R_2, \dots, R_n)$ and $(\mathcal{F}^2(g); R_1, R_2, \dots, R_n)$ be Hilbert modules. Then $X : \mathcal{F}^2(f) \rightarrow \mathcal{F}^2(g)$ is a module map if and only if there exists $h \in F^\infty(f)$ and $k \in F^\infty(g)$ such that $X = \sum_{i=0}^{\infty} L_{k_i} \circ \varepsilon(f, g) \circ L_{h_i}$.

Finally, in [18], Popescu generalized the notion of the noncommutative domain algebra $\mathcal{D}_f(\mathcal{H})$ by introducing a new class of noncommutative domains given by:

$$\mathbf{D}_p^m(\mathcal{H}) := \{X = (X_1, X_2, \dots, X_n) : (1 - p)^k(X, X^*) \geq 0 \text{ for } 1 \leq k \leq m\}.$$

Note that the domain algebras $\mathcal{D}_f(\mathcal{H})$ studied in this thesis arise when $m = 1$. It is not too difficult to see that, for arbitrary m , these new noncommutative domains are each a subset of the class of L-bounded functions. Indeed, these new domains, when viewed in the context of $\mathcal{D}_f(\mathcal{H})$, seem to generalize the notion of a positive regular free holomorphic function to include the possibility that some of the a_α 's are negative. This seems to suggest that some other condition is required of the f 's rather than positivity. Further study of this area is needed. We end this thesis by posing two natural questions relating to L-bounded functions:

- What conditions on the a_α 's are needed to ensure that f is L-bounded?
- Can we extend Theorems 6.1.3, 6.1.4, 6.1.7, and 6.1.8 to the more general case where both f and g can be arbitrary L-bounded functions?

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