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Permutation Patterns, Reduced Decompositions with Few Repetitions and the Bruhat Order

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Abstract

This thesis is concerned with problems involving permutations. The main focus is on connections between permutation patterns and reduced decompositions with few repetitions. Connections between permutation patterns and reduced decompositions were first studied various mathematicians including Stanley, Billey and Tenner. In particular, they studied pattern avoidance conditions on reduced decompositions with no repeated elements. This thesis classifies the pattern avoidance and containment conditions on reduced decompositions with one and two elements repeated. This classification is then used to obtain new enumeration results for pattern classes related to the reduced decompositions and introduces the technique of counting pattern classes via reduced decompositions. In particular, counts on pattern classes involving 1 or 2 copies of the patterns 321 and 3412 are obtained. Pattern conditions are then used to classify and enumerate downsets in the Bruhat order for the symmetric group and the rook monoid which is a generalization of the symmetric group. Finally, motivated by coding theory, the concepts of displacement, additive stretch and multiplicative stretch of permutations are introduced. These concepts are then analyzed with respect to maximality and distribution as a new prospect for improving interleaver design.

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Chapter 1

Introduction

This chapter introduces the three main objects of study in this thesis: permutation patterns, reduced decompositions of permutations and the Bruhat order on permutations.

1.1 Permutation Patterns

When speaking about permutation patterns, we will always consider a permutation in one-line notation, $\pi = \pi_1 \ldots \pi_n$, where the image of i under the permutation π is π_i. For example, 3142 is the permutation which sends 1 to 3, 2 to 1, 3 to 4 and 4 to 2.

Definition 1.1.1. *A permutation $\pi = \pi_1 \ldots \pi_n \in S_n$ is said to* contain *a permutation $\sigma = \sigma_1 \ldots \sigma_m \in S_m$ if there exists $1 \leq i_1 < i_2 < \ldots i_m \leq n$ such that $\forall j, k \in \{1, \ldots, m\}$ $\sigma_j < \sigma_k$ if and only if $\pi_{i_j} < \pi_{i_k}$. If π does not contain σ, then π* avoids *σ.*

In other words, π contains σ if there exists a subsequence of π which is order-isomorphic to σ. The use of permutation patterns in sorting algorithms dates back to the 1970s and was studied by Donald Knuth. One may consult [24] for further information. The study of permutation patterns began in earnest with the publication in 1985 of the seminal paper [31] by Simion and Schmidt who answered many basic enumerative questions concerning permutation patterns and opened many pathways into fruitful research.

The most basic and one of the most important enumeration questions in the theory of permutation patterns is: given a permutation $\pi \in S_m$, how many permutations in S_n avoid π? For arbitrary $m \leq n$, this question is very difficult, but there are results for small m. Before we discuss them, we need to develop some notation which will be used throughout this thesis. We will denote by $Av_n(\pi)$ the set of permutations in S_n that avoid π. Regrettably, there is no standard notation for this set. If $\pi, \sigma \in S_m$, then π and σ are *Wilf-equivalent* if $|Av_n(\pi)| = |Av_n(\sigma)|$ for all $n \in \mathbb{N}$. Wilf-equivalence is an equivalence relation on S_n.

We may now survey some of the results on the above question which can be restated as: given $\pi \in S_m$, what is $|Av_n(\pi)|$? If $m = 3$, it is a quite surprising result that there is precisely one Wilf-equivalence class containing every permutation in S_3. The following result was originally proved in [31].

Theorem 1.1.2 (Simion and Schmidt, 1985)**.** *If $\pi \in S_3$, then $|Av_n(\pi)| = C_n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.*

When $\pi \in S_3$, the result is surprisingly simple and beautiful. When $\pi \in S_4$, many complications arise. The permutations in S_4 have been split into 3 distinct Wilf-equivalence classes. The usual representatives for these classes are: 1234, 1342, and 1324. The enumeration of both $|Av_n(1234)|$ and $|Av_n(1342)|$ was accomplished by Gessel (who found two distinct formulas for the same class) [17] and Bona [5] respectively; however, both the formulas for both classes are long and complicated. The enumeration of $|Av_n(1324)|$ is still an open problem.

Proving two permutations are Wilf-equivalent aids in the enumeration of pattern questions. There are three very useful maps from S_n to S_n which respect Wilf-equivalence. They are the Reverse, Complement and Inverse maps which we now define.

Definition 1.1.3. *Assume* $\pi = \pi_1 \dots \pi_n \in S_n$.

- *(reverse map)* $\pi \mapsto \pi^r$ *where* $\pi^r = \pi_1^r \dots \pi_n^r$ *and* $\pi_i^r = \pi_{n-i+1}$.
- *(complement map)* $\pi \mapsto \pi^c$ *where* $\pi^c = \pi_1^c \dots \pi_n^c$ *and* $\pi_i^c = n - \pi_i + 1$.
- *(inverse map)* $\pi \mapsto \pi^{-1}$ *where* π^{-1} *is the group theoretic inverse of* π.

For example, let $\pi = 3217456 \in S_7$. Then $\pi^r = 6547123$, $\pi^c = 5671432$ and $\pi^{-1} = 3215674$.

These maps are all involutions and give rise to the following theorem.

Theorem 1.1.4. *Let* $\pi \in S_n$ *and* $\sigma \in S_m$. π *avoids* $\sigma \iff \pi^r$ *avoids* $\sigma^r \iff \pi^c$ *avoids* $\sigma^c \iff \pi^{-1}$ *avoids* σ^{-1}.

For a good overview of the combinatorics of pattern avoidance, the interested reader is advised to consult [6], especially chapters 4 and 5.

Since 1985, many questions concerning patterns have been posed and there has been a prodigious amount of research published in the area. Patterns are connected with many other areas of mathematics such as representation theory (Young tableaux), algebraic geometry (Schubert varieties) and theoretical computer science (sorting algorithms). Good references for the above connections are: [30], [15], [6], [2], [25] and [24].

1.2 Reduced Decompositions

1.2.1 Basic Definitions, Notation Conventions and Braid Moves

It is a well-known result that any permutation in S_n can be written as a product of transpositions of the form $(i, i+1)$ for $i \in \{1, \dots n-1\}$. We then define a reduced decomposition as follows.

Definition 1.2.1. *Let $\pi \in S_n$. A* reduced decomposition *of π is a sequence of transpositions $t_1 \dots t_k$ where each of the t_j is a transposition of the form $(i, i+1)$ for some i such that $\pi = t_1 \dots t_k$ and k is the minimum number of such transpositions required to obtain π.*

Notice first that reduced decompositions are not unique. A permutation may have precisely one reduced decomposition or it may have many. The number of such reduced decompositions is related to the hook-length formula for tableaux and more details can be found in Chapter 7 of [4]. We will denote the set of reduced decompositions of π by $\mathcal{R}(\pi)$.

For example, the permutation $\pi = 1423$ only has one reduced decomposition: $(3,4)(2,3)$. The permutation $\pi = 4132$ has reduced decompositions $(3,4)(2,3)(1,2)(3,4)$, $(3,4)(2,3)(3,4)(1,2)$ and $(2,3)(3,4)(2,3)(1,2)$.

Certain aspects of reduced decompositions remain invariant for a permutation π. The set of transpositions that occur in one reduced decomposition for π is the same for any reduced decomposition of π.

Definition 1.2.2. *If $\mathbf{t} = t_1 \dots t_k$ is a reduced decomposition for π, then we define the* length *of π to be k and this will be denoted $l(\pi)$ or $l(\mathbf{t})$.*

Based on the previous examples, we compute $l(1423) = 2$ and $l(4132) = 4$. Also, boldface type will be used to name a specific reduced decomposition as in the previous definition.

It is notationally inconvenient to continue to write out sequences of transpositions, each of the form $(i, i+1)$, in full every time we wish to discuss them. We shall minimize notation by writing only the first element of each transposition and will distinguish permutations from such sequences by enclosing sequences in brackets.

For example, if $\pi = 1423$, we will denote its reduced decomposition by [32] and the reduced decompositions for $\pi = 4132$ by $[3213], [3231]$ and $[2321]$.

Once a reduced decomposition for a permutation is known, all reduced decompositions may be computed by the use of braid moves.

Definition 1.2.3. *The* braid moves *are:*

- *(Short braid move)* $[ij] = [ji]$ *if* $|i - j| > 1$.
- *(Long braid move)* $[i(i+1)i] = [(i+1)i(i+1)]$ $\forall i$.

Definition 1.2.4. *Two reduced decompositions* $\mathbf{s} = [s_1 \dots s_k]$ *and* $\mathbf{t} = [t_1 \dots t_k]$ *are* equivalent *if and only if* $\mathbf{s}$ *can be obtained from* $\mathbf{t}$ *through a sequence of braid moves. (Notational convention: in the instance that* $\mathbf{s}$ *is equivalent to* $\mathbf{t}$*, we will write* $\mathbf{s} = \mathbf{t}$ *and rely on the context to determine whether equality of permutations or equivalence of reduced decompositions is meant.)*

Theorem 1.2.5. *Two reduced decompositions* $\mathbf{s}$ *and* $\mathbf{t}$ *determine the same permutation if and only if one is obtained from the other by a sequence of braid moves.*

If a permutation $\pi \in S_n$ is fixed, then one may consider the reduced decompositions of π as vertices in a graph with two vertices connected if and only if the two reduced decompositions differ by one application of one braid move. Theorem 1.2.5 then says that the graph of reduced decompositions is connected. Figure 1.1 gives the graph of the reduced decompositions for 4231.

In this thesis, we will be concerned not only with reduced decompositions, but with substrings of reduced decompositions. There are two specific types of substrings that will be of interest.

Definition 1.2.6. *Let* $[s_1 \dots s_k]$ *be a reduced decomposition. A* subword *of* $[s_1 \dots s_k]$ *is a subsequence* $[s_{i_1} \dots s_{i_j}]$ *where* $1 \leq i_1 < i_2 < \cdots < i_j \leq k$*. A* factor *of* $[s_1 \dots s_k]$ *is a subword of the form* $[s_i s_{i+1} s_{i+2} \dots s_{i+j}]$.

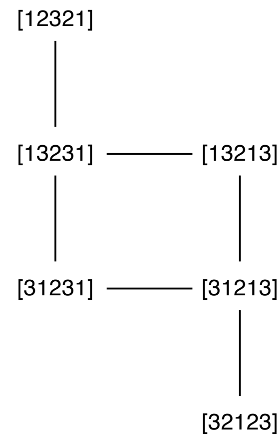

Figure 1.1: Graph of $\mathcal{R}(4231)$

It is important to note that factors must always be reduced decompositions themselves, but subwords of reduced decompositions need not be reduced.

Finally, it is important to note how permutations are composed. We will alternately use two different points of view. The first is the standard cycle multiplication which we will read from right to left. For example, given the permutation $(1,2)(3,4)(1,3)$, then we note that $1 \to 3 \to 4$, $2 \to 1$, etc. The second point of view (used particularly in Chapter 2) is as follows: given a permutation $\pi \in S_n$ and a transposition $t = (u_1, u_2)$, $\pi \cdot t$ is the permutation π with the elements in positions u_1 and u_2 transposed. For example, consider the sequence of transpositions $(1,2)(3,4)(1,3)$. $(1,2)(3,4) = 2143$, therefore $(1,3)$ transposes the elements in positions 1 and 3 giving that $(1,2)(3,4)(1,3) = 4123$.

1.2.2 Constructing Reduced Decompositions

In the previous section we discussed how to construct one reduced decomposition from another, but the question of how to actually build a reduced decomposition given only the permutation was not discussed. The goal of this section is to give one way to build a reduced decomposition given only the permutation. The following algorithm is very thoroughly discussed in [25], but it will be very useful for results to come later, so we introduce it here. In order to build a reduced decomposition, it is necessary to use the graph of a permutation. Visually, one can draw the graph of a permutation just as one would draw the graph of a function in first-year calculus and the visualization of graphs of permutations will be very helpful in both creating reduced decompositions and for providing intuition for results in later chapters.

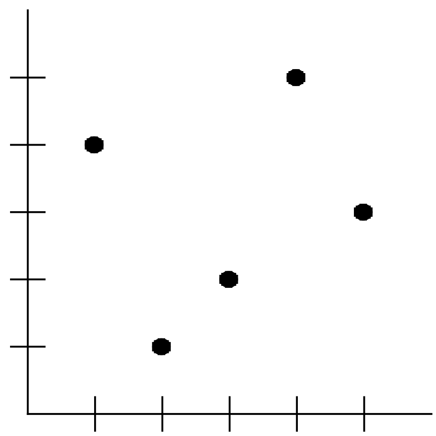

Figure 1.2: Graph of 41253

Figure 1.2 gives the graph of the permutation 41253. To build a reduced decomposition from the graph of a permutation, the algorithm is as follows:

1. For each $i \in \{1, \ldots, n\}$, draw the vertical line from (i, π_i) to (i, n) and then the horizontal line from (i, π_i) to (n, π_i). The unions of these line segments are called *hooks*.

2. For each $i \in \{1, \ldots, n\}$, let $\{i_1, \ldots, i_k\}$ be the set of numbers in $\{1, \ldots, n\}$ such that (i, j_l) does not have a hook through that position, ordered such that $i_1 > i_2 \cdots > i_k$. Label each cell (i, i_l) with the number $i + l - 1$.

3. For each $i \in \{1, \ldots, n\}$, list the numbers in column i from bottom to top and concatenate moving from left to right.

Theorem 1.2.7. *The list of numbers generated by the above algorithm gives a reduced decomposition for π.*

The proof of the above theorem can be found in [25].

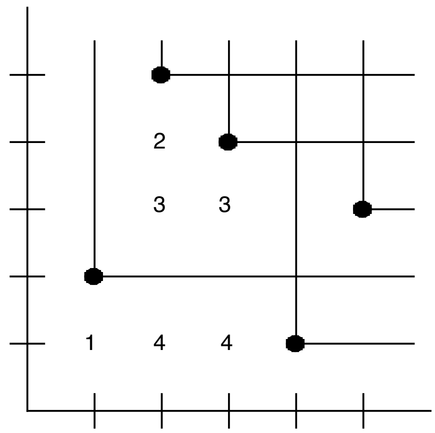

Figure 1.3: Generating a reduced decomposition for 25413

Figure 1.3 gives a graph showing the graph for 25413 after step two of the algorithm. After step three, we have that [143243] is a reduced decomposition for 25413.

1.2.3 Properties

Reduced decompositions may be more generally defined for any Coxeter group, although we will restrict our attention solely to the Coxeter group A_n, and in general enjoy several very nice properties. Although stated for permutations, the properties stated in this section hold for reduced decompositions in any Coxeter group.

Definition 1.2.8. *The set of* reflections *of S_n is the set $T_n = \{\pi(i, i+1)\pi^{-1} : \pi \in S_n, i \in \{1, \ldots, n-1\}\}$.*

Example 1.2.9. $T_3 = \{132, 213, 321\}$.

Note that the permutations represented by the transpositions $(i, i+1)$ are all elements of T_n. In fact, for S_n, T_n is just the set of transpositions, i.e. $T_n = \{(i, j) :$

$i, j \in \{1, \ldots, n\}, i \neq j\}$. Throughout this thesis, we use the notation $[s_1 \ldots \hat{s_i} \ldots s_k]$ to denote $[s_1 \ldots s_{i-1} s_{i+1} \ldots s_k]$. We may now introduce two properties which will be very useful for this study.

Theorem 1.2.10. *[Strong Exchange Property] Let $[s_1 \ldots s_k]$ be a reduced decomposition for $\pi \in S_n$ and let $t \in T_n$. If $l(t\pi) < l(\pi)$, then $t\pi = [s_1 \ldots \hat{s_i} \ldots s_k]$ for some $i \in \{1, \ldots, k\}$.*

Theorem 1.2.11. *[Deletion Property] Let $[s_1 \ldots s_k]$ be a sequence of transpositions each of the form $(i, i+1)$ such that $[s_1 \ldots s_k] = \pi \in S_n$ and $l(\pi) < k$, then there exists $i, j \in \{1, \ldots, k\}$, $i \neq j$, such that $[s_1 \ldots s_k] = [s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_k]$.*

The proofs of these two properties are standard and can be found in any book on Coxeter groups such as [4] or [23].

1.3 Bruhat order for S_n

This section defines the third and final main object of our study: the Bruhat order for permutations. Again, as in the case of reduced decompositions, the Bruhat order is well-defined for any Coxeter group though we will use it mainly, though not exclusively, for permutations. There are many different ways to define Bruhat order for S_n. The definition used here generalizes very naturally to any Coxeter group.

Definition 1.3.1. *Fix $n \in \mathbb{N}$. Let π, $\sigma \in S_n$ and let T_n be the set of reflections as defined in the previous section. Define*

- *$\pi \xrightarrow{t} \sigma$ to mean $\pi^{-1}\sigma = t \in T_n$ and $l(\pi) < l(\sigma)$.*
- *$\pi \rightarrow \sigma$ to mean there exists $t \in T_n$ such that $\pi \xrightarrow{t} \sigma$*
- *$\pi \leq \sigma$ to mean that there exist $\pi_i \in S_n$ such that*
  $\pi = \pi_0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \cdots \rightarrow \pi_{k-1} \rightarrow \pi_k = \sigma$.

The partial order defined above is the Bruhat Order *on* S_n*.*

Note that, in the definition above, unlike the convention used throughout the rest of this work, the π_i are actual permutations and do not represent the image of i under the permutation π.

The definition above can be restated solely in terms of pattern avoidance which gives a much more intuitive understanding of this partial order.

Definition 1.3.2. *An* inversion *is a* 21 *pattern.*

One can now define the Bruhat order on S_n by defining $\pi \leq \sigma$ if σ can be obtained from π by a sequence of moves starting at π, each transposing the elements of a 12 pattern. This is the same as finding a sequence of moves that transform σ into π with each move transposing the elements of an inversion. Throughout this work, we will think of the Bruhat order from this latter perspective.

Example 1.3.3. *To prove* $21354 \leq 52143$ *we produce the following sequence of permutations:* $52143 \rightarrow 25143 \rightarrow 21543 \rightarrow 21453 \rightarrow 21354$*.*

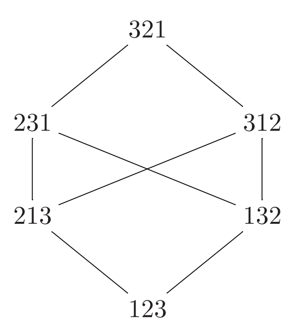

Figure 1.4: Bruhat Order on S_3

Figures 1.4 and 1.5 show the Hasse diagrams for S_3 and S_4 respectively.

There are many connections between permutation patterns, reduced decompositions and the Bruhat order. One of the first results connecting reduced decompositions and the Bruhat order which will be used many times is the Subword property.

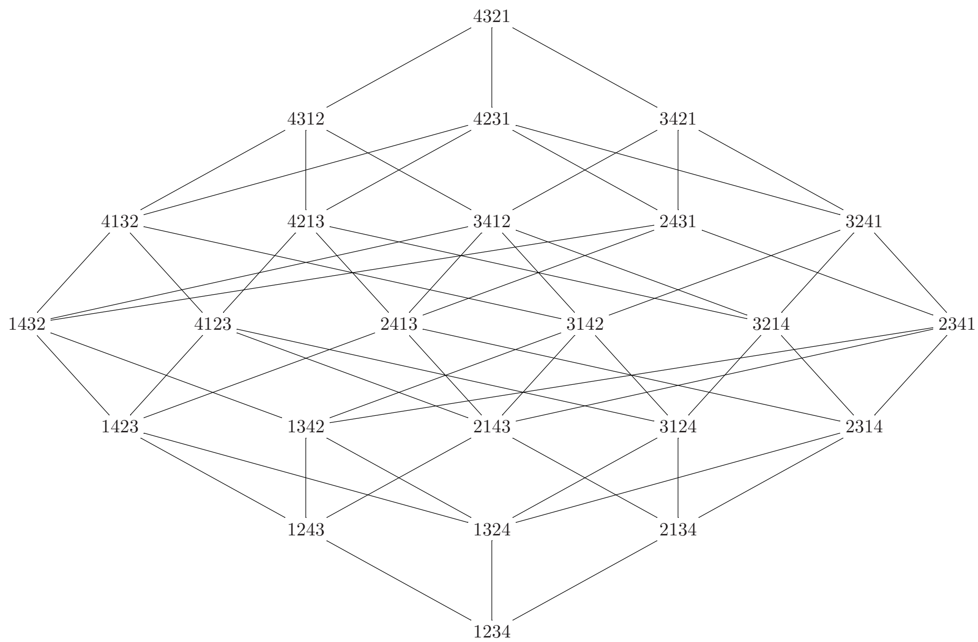

Figure 1.5: Bruhat Order on S_4

Theorem 1.3.4. *[Subword Property] Let $\sigma \in S_n$ have reduced decomposition $[s_1 \ldots s_k]$ and let $\pi \in S_n$. Then $\pi \leq \sigma \Leftrightarrow$ there exists a subword $[s_{i_1} \ldots s_{i_m}]$ of $[s_1 \ldots s_k]$ which is a reduced decomposition for π.*

The Subword property is of special significance when considering downsets of permutations.

Definition 1.3.5. *Given a partial order $(P, \leq)$ and a subset $S \subseteq P$, the* downset *of S, denoted $d(S)$, is the set $\{p \in P : p \leq s$ for some element $s \in S\}$. We write $d(s)$ for $d(\{s\})$.*

If $\pi \in S_n$, $d(\pi)$ will denote the downset of π in the Bruhat order. By the Subword property, $d(\pi)$ consists of all permutations obtainable as subwords of a reduced decomposition of π. The Subword property is independent of the choice of reduced decomposition for π. If $[s_1 \ldots s_k]$ is a reduced decomposition for $\pi \in S_n$, then any subword of $[s_1 \ldots s_k]$ represents a permutation in $d(\pi)$ (even though that subword need not be reduced) and any permutation $\sigma \in d(\pi)$ has a reduced decomposition $[s_{i_1} \ldots s_{i_m}]$ that is a subword of $[s_1 \ldots s_k]$.

Definition 1.3.6. *A poset is* graded *if all of its maximal chains have the same number of elements.*

Theorem 1.3.7. *The Bruhat order on S_n is a graded poset.*

The proof of Theorem 1.3.7 can be found in [6]. One aspect of the proof which will be important for us later is that if $\pi, \sigma \in S_n$, then σ covers π if and only if $l(\sigma) = l(\pi) + 1$.

Chapter 2

Reduced Decompositions with One Repetition

2.1 Reduced Decompositions with No Repetitions

The connection between reduced decompositions and permutation patterns was first studied by Billey, Jockusch and Stanley in [3] and further by Tenner in [35] and [36]. In particular, the pattern avoidance conditions connected with reduced decompositions with no repetitions are very well studied.

Theorem 2.1.1 (Billey, Jockusch, Stanley, [3]). *$\pi \in S_n$ contains 321 if and only if π has a reduced decomposition with $[i(i+1)i]$ as a factor for some i.*

For example, $\pi = 45231$ has many 321 patterns and so there must be a reduced decomposition of π with $[i(i+1)i]$ as a factor. There are many such reduced decompositions for π, one of which is [3214*232*4].

Tenner greatly generalizes this result via the vexillary characterization.

Definition 2.1.2. *A permutation is* vexillary *if and only if it avoids 2134.*

Theorem 2.1.3 (Tenner, [35])**.** *$\pi \in S_n$ is vexillary if and only if for every permutation σ containing π, there exists a reduced decomposition of σ containing a reduced decomposition of π as a factor.*

321 and 3412 are both vexillary permutations and so by Theorem 2.1.3 any permutation which contains 321 or 3412 must contain a factor with the same structure as a reduced decomposition of 321 or 3412. $\mathcal{R}(321) = \{[121], [212]\}$ and $\mathcal{R}(3412) = \{[2132], [2312]\}$.

Definition 2.1.4. *Let $\mathbf{s} = \mathbf{abc}$ be a reduced decomposition which is the concatenation of three distinct reduced decompositions with $\mathbf{a} \in \mathcal{R}(\pi)$, $\mathbf{c} \in \mathcal{R}(\sigma)$ for $\pi, \sigma \in S_n$. Suppose $\mathbf{b}$ contains only elements in $S = \{i, i+1 \ldots i+k\}$. If no element of $\mathcal{R}(\pi)$ has rightmost element in S and no element of $\mathcal{R}(\sigma)$ has leftmost element in S, then the factor $\mathbf{b}$ is* isolated *in $\mathbf{s}$.*

For example, let $\mathbf{s} = [123454678]$ where $\mathbf{a} = [123]$, $\mathbf{b} = [454]$ and $\mathbf{c} = [678]$. $\mathbf{b}$ is an isolated factor in $\mathbf{s}$. If $\mathbf{s} = [421323]$ where $\mathbf{a} = [4]$, $\mathbf{b} = [2132]$ and $\mathbf{c} = [3]$, then $\mathbf{b}$ is not isolated.

Theorem 2.1.5 (Tenner, [35])**.** *If a reduced decomposition of σ contains an isolated factor which is a reduced decomposition of π, then σ contains π. If π is vexillary, the converse is true.*

When considering isolated factors, the following intuition may be helpful. Isolated factors preserve the pattern they represent. Factors which are not isolated may cause the pattern they represent to be destroyed. For example, [521324] gives the permutation 341652. [2132] gives the permutation 3412 and the fact that it is isolated means that the 3412 created by [2132] is preserved. However, the reduced decomposition [32132] gives the permutation 4312 which does not contain 3412.

Definition 2.1.6. *Let $\pi \in S_n$. π is* Boolean *if and only if $d(\pi)$ is a Boolean algebra.*

Motivated by the structure of downsets in the Bruhat order, Tenner proves the following two results using reduced decompositions.

Lemma 2.1.7 (Tenner, [36]). *$\pi \in S_n$ is Boolean if and only if π has a reduced decomposition with no repeated elements.*

Theorem 2.1.8 (Tenner, [36]). *$\pi \in S_n$ is Boolean if and only if π avoids* 321 *and* 3412*.*

Boolean permutations have been enumerated by various mathematicians. One can find the enumeration of these permutations under different points of view in [14], [37] and [36].

Theorem 2.1.9. *Let F_k be the k^{th} Fibonacci number. The number of Boolean permutations in S_n is F_{2n-1}.*

2.2 Structure of Reduced Decompositions with One Repetition

The material in this section forms the paper [9]. It is a natural question to ask what is the relation between reduced decompositions that have exactly one repeated letter and the corresponding permutation avoids. We now prove some preliminary results that will aid us in our analysis of reduced decompositions of permutations with exactly one repeated letter. Note that if π is a permutation and $\tau = (\tau_1, \tau_2)$ is a transposition then $\pi\tau$ will be the permutation π with the two elements in positions τ_1 and τ_2 switched.

Example 2.2.1. *If we have the reduced decomposition* [2132] *in S_4 then the effect this decomposition will have on the identity permutation is as follows:* $1234 \to 1324 \to 3124 \to 3142 \to 3412$*.* [2132] *is a reduced decomposition of* 3412*.*

Lemma 2.2.2. *Let $\pi \in S_n$ have reduced decomposition $[j_1 \dots j_l]$. Let x and y be the two values in positions j_i and $j_i + 1$ in the permutation represented by $[j_1 \dots j_{i-1}]$. Then $x < y$. In particular, if $\pi_a > \pi_b$ for some $a < b$, then there is precisely one j_i that interchanges the values π_a and π_b.*

Proof. If $x > y$ in the permutation represented by $[j_1 \dots j_{i-1}]$, the permutation $[j_1 \dots j_i]$ would have one less inversion than $[j_1 \dots j_{i-1}]$ and thus it would not be reduced which is a contradiction. The rest is clear, since the two values $\pi_a > \pi_b$ have to switch positions at one point. □

Lemma 2.2.3. *Let $\mathbf{j} = [j_1 \dots j_l]$ be a reduced decomposition of $\pi \in S_n$. If the permutation with reduced decomposition $[j_1 \dots j_m]$ $(m < l)$ has p occurrences of 321, then π has at least p occurrences of 321.*

Proof. It is enough to demonstrate that if $[j_1 \dots j_m]$ is a reduced decomposition with p 321 patterns, then $[j_1 \dots j_{m+1}]$ has at least p 321 patterns. j_{m+1} switches positions j_{m+1} and $j_{m+1} + 1$, so assume x is in position j_{m+1} and y is in position $j_{m+1} + 1$ before the application of the transposition (j_m, j_{m+1}). Now $x < y$ by Lemma 2.2.2. Any 321 pattern in $[j_1 \dots j_m]$ will occur in $[j_1 \dots j_{m+1}]$ because interchanging two values in consecutive positions will not affect any existing 321 patterns. □

Lemma 2.2.4. *Let $\mathbf{j} = [j_1 \dots j_q]$ be a reduced decomposition of $\pi \in S_n$ and let $m < q$. If $[j_1 \dots j_m]$ has a 3412 pattern, then π has the same 3412 pattern or the positions of the 3412 pattern are interchanged to create at least two 321 patterns.*

Proof. Consider the permutation $[j_1 \dots j_m]$ with a 3412 pattern. If none of the elements in the 3412 pattern are interchanged by $[j_{m+1} \dots j_q]$, then π will have a 3412 pattern. If elements of the 3412 pattern are interchanged, then either the "34" or the "12" must be interchanged first. If j_k, $k > m$, interchanges the "34", then $[j_1 \dots j_m \dots j_k]$ has a 4312 pattern, which has two 321 patterns. If j_k, $k > m$, interchanges the "12", then $[j_1 \dots j_m \dots j_k]$ has a 3421 pattern which also has two 321 patterns. By Lemma 2.2.3, this means π has at least two 321 patterns. $\square$

Lemma 2.2.5. *If there exists a reduced decomposition of* π *which contains factors* $[i(i+1)i]$ *and* $[j(j+1)j]$*, for* $i \neq j$*, then* π *contains at least two* 321 *patterns.*

Proof. Without loss of generality, assume $i(i+1)i$ occurs before $j(j+1)j$ in the reduced decomposition, so the reduced decomposition looks like $[\dots i(i+1)i \dots j(j+1)j \dots]$. Consider the permutation σ formed by the elements to the left of $i(i+1)i$ and assume $\sigma_i = a$, $\sigma_{i+1} = b$ and $\sigma_{i+2} = c$. Lemma 2.2.2 implies that $a < b < c$. After applying $i(i+1)i$, cba will be a 321 pattern in σ. Now consider the permutation σ' formed by all elements in the reduced decomposition occurring to the left of $j(j+1)j$. By Lemma 2.2.3, σ' has at least one 321 pattern. Assume $\sigma'_j = d$, $\sigma'_{j+1} = e$ and $\sigma'_{j+2} = f$. As before, we have $d < e < f$. Apply $j(j+1)j$ to σ' to get σ'' and note that Lemma 2.2.4 implies cba still forms a 321 pattern in σ'' (if it did not then two of a, b and c would have to interchange positions at least twice). If $\{a, b, c\} \cap \{d, e, f\} = \emptyset$ then fed forms another 321 pattern in σ'' completely disjoint from the 321 pattern cba. If $\{a, b, c\} \cap \{d, e, f\} \neq \emptyset$, then $|\{a, b, c\} \cap \{d, e, f\}| = 1$ since $d < e < f$ and so switching two nonshared numbers will not affect the 321 pattern cba and we will have two 321 patterns in σ''. By Lemma 2.2.3, we have at least two 321 patterns in π. $\square$

2.2.1 Trajectories

Definition 2.2.6. *Given a reduced decomposition* $[j_1 \dots j_l]$ *of* $\pi \in S_n$ *and an element* $x \in \{1 \dots n\}$*, define the* trajectory *of* x *as the sequence of positions at which* x *is found by applying the permutations* $()$*,* $[j_1]$*,* $[j_1 j_2]$*,* $\dots$ $[j_1 \dots j_l]$ *to* x*.*

Consider the permutation 43251 with reduced decomposition [3213234]. Here is the trajectory of 3.

position of 3	permutation
3	()
4	[3]
4	[32]
4	[321]
3	[3213]
2	[32132]
2	[321323]
2	[3213234]

Therefore the trajectory of 3 is the sequence $< 3, 4, 4, 4, 3, 2, 2, 2 >$.

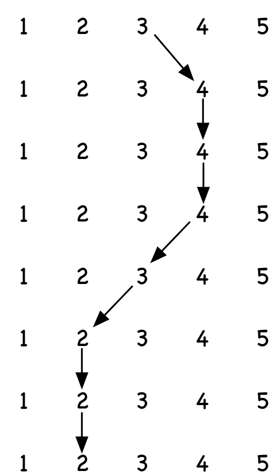

Figure 2.1: Trajectory of 3 in [3213234]

Figure 2.2 shows all of the trajectories of 43251 which are just a "flattening" of the standard braid diagram as shown in Figure 2.1.

Note, given the trajectories for all points $x \in \{1 \ldots n\}$ for a permutation π, they can be projected onto the x-axis allowing us to count the multiplicities of arrows coming in to position i from the right and leaving position i moving to the right. This multiplicity is denoted m_i. Also note that arrows come in pairs, i.e., for each arrow that moves a point to the right, there always exists an arrow in the same

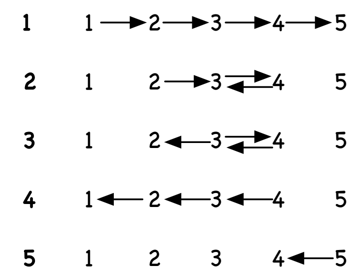

Figure 2.2: Trajectories of 43251

position moving to the left.

The multiplicities in the example permutation are: $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 2$.

An immediate observation is:

Lemma 2.2.7. *i is repeated k times in a reduced decomposition if and only if $m_i = 2k$.*

Definition 2.2.8. *An element i* turns (or reverses direction) *in a trajectory if there exists an element j such that between elements j and $j+1$ the arrow in the trajectory of i going from left to right and the arrow going from right to left both appear.*

2.2.2 One Repetition of the Form $i(i+1)i$

We will now consider the case where we have a permutation $\pi \in S_n$ with a reduced decomposition with a factor of the form $i(i+1)i$ for some $i \in \{1, \ldots, n-1\}$ and no other repetitions. We will show that such permutations are precisely those that contain exactly one copy of 321 and avoid 3412.

Consider a permutation which contains exactly one 321 pattern and avoids 3412. The vexillary characterization of Tenner (Theorem 2.1.3), implies that there exists a reduced decomposition of π containing $i(i+1)i$ as a factor. Call this reduced decomposition **j**. In any theorem or lemma that uses a permutation that contains exactly one 321 and avoids 3412, we will always assume **j** is the reduced decomposition of the appropriate form guaranteed by Theorem 2.1.3.

Lemma 2.2.9. *If $\pi \in S_n$ contains exactly one* 321 *pattern and avoids* 3412*, then in the reduced decomposition* $\mathbf{j}$ *there is no* $j \in \{1, \ldots, n\} \setminus \{i, i+1\}$ *appearing twice on the left or twice on the right of the factor* $[i(i+1)i]$.

Proof. Assume the condition of the lemma does not hold. Without loss of generality, assume the repeated element j appears twice on the left of $[i(i+1)i]$ (the argument will be analogous for the other case), so $\mathbf{j}$ has the form $[\ldots j \ldots j \ldots i(i+1)i \ldots]$. Let us pick the repeated element j such that the length of the factor $[j \ldots j]$ is minimal. It is not hard to see that there are only four possible factors that can occur between the two occurrences of j. We have either $[j(j+1)(j-1)j]$, $[j(j-1)(j+1)j]$, $[j(j-1)j]$ or $[j(j+1)j]$. If we have either $[j(j+1)(j-1)j]$ or $[j(j-1)(j+1)j]$, then consider the permutation formed by the first elements of $\mathbf{j}$ up through the second j. Lemma 2.2.4 gives that π contains either 3412 or multiple 321 patterns, both of which are contradictions. If we have a factor of the form $[j(j+1)j]$ or $[j(j-1)j]$, then Lemma 2.2.5 implies π has multiple 321 patterns, a contradiction. $\square$

Lemma 2.2.10. *If $\pi \in S_n$ contains exactly one* 321 *pattern and avoids* 3412*, then in the reduced decomposition* $\mathbf{j}$*, neither* i *nor* $i+1$ *appears outside of the factor* $[i(i+1)i]$.

Proof. Again, we assume the repeated element occurs to the left of the factor $i(i+1)i$ in $\mathbf{j}$. $\mathbf{j}$ will have the form $[\ldots x \ldots i(i+1)i \ldots]$ where $x = i$ or $x = i+1$. There may be many occurrences of i or $i+1$ to the left of $[i(i+1)i]$, so let us pick the one such that the length of the factor $[x \ldots i(i+1)i]$ is minimal and apply braid moves to minimize the factor $[x \ldots i(i+1)i]$ as much as possible. Note that there has to be at least one element in between x and i, since neither $[ii(i+1)i]$ nor $[(i+1)i(i+1)i]$ is reduced. If there is one element in between x and i, the possibilities are $[i(i-1)i(i+1)i]$ or $[(i+1)(i+2)i(i+1)i]$. Applying a braid move to the first case gives $[(i-1)i(i-1)(i+1)i]$ which means π has a reduced decomposition with $[i(i-1)(i+1)i]$ as a factor. It was noted in Lemma 2.2.9 that such a factor leads to a contradiction. The second case has the factor $[(i+1)(i+2)i(i+1)] = [(i+1)i(i+2)(i+1)]$, which also gives a contradiction by Lemma 2.2.9.

Thus the number of elements between x and i is greater than one. First, consider the case where $x = i$. We can assume there are no extra occurrences of $i+1$ in between x and i because if there were, we could apply the braid move $[i(i+1)i] = [(i+1)i(i+1)]$ and obtain a factor $[(i+1)\dots(i+1)i(i+1)]$ with a smaller number of elements in between the $(i+1)$'s. So, the factor of the reduced decomposition we are interested in is of the form $[it_1 \dots t_m i(i+1)i]$ where $i \neq t_j \neq i+1$ for all t_j. $t_1 = i-1$ and since there are no repetitions among the t_j, we must have $t_m = i+2$. Continuing in this fashion we have $t_2 = i-2$, $t_{m-1} = i+3$, etc. At some point we must stop. If m is even, we have $[it_1 \dots t_m i(i+1)i] = [i(i-1)(i-2)\dots(i-k)(i-k-1)(i+k)\dots(i+2)i(i+1)i]$. If m is odd, we have $[it_1 \dots t_m i(i+1)i = i(i-1)(i-2)\dots(i-k)(i+k)\dots(i+2)i(i+1)i]$. In either case, apply braid moves to obtain the factor $[i(i-1)i(i+1)i]$, which is impossible by the first paragraph. The case where $x = i+1$ is similar (apply the braid move $[i(i+1)i] = [(i+1)i(i+1)]$ and consider the factor $[(i+1)t_1 \dots t_m(i+1)i(i+1)]$). $\square$

Lemma 2.2.11. *If $\pi \in S_n$ contains exactly one copy of* 321 *and avoids* 3412*, then no factor of the form $[ji(i+1)ij]$ can occur in* $\mathbf{j}$.

Proof. Note, j cannot commute with $[i(i+1)i]$ by braid moves, so $j = i-1$ or $j = i+2$; therefore, we only need to check these two possibilities for j. Note, one can apply braid moves to the factor, $[(i+2)i(i+1)i(i+2)]$ to obtain the factor $[i(i+1)(i+2)(i+1)i]$ which is the same ordering of elements as $[(i-1)i(i+1)i(i-1)]$. It therefore suffices to check that no factor of the form $[(i-1)i(i+1)i(i-1)]$ can occur in $\mathbf{j}$. Consider the permutation π' formed by all the elements before the factor $[(i-1)i(i+1)i(i-1)]$ in $\mathbf{j}$ and consider applying the factor $[(i-1)i(i+1)i(i-1)]$ to

π'. If $\pi'_{i-1} = a$, $\pi'_i = b$, $\pi'_{i+1} = c$, and $\pi'_{i+2} = d$, then applying $[(i-1)i(i+1)i(i-1)]$ to π' gives π'' where $\pi''_{i-1} = d$, $\pi''_i = b$, $\pi''_{i+1} = c$ and $\pi''_{i+2} = a$, as described in the following table. Recall that Lemma 2.2.2 forces order relations between elements every time a transposition in a reduced decomposition is applied to a permutation π.

i-1	i	i+1	i+2	action	$<$
a	b	c	d	initial positions	—
b	a	c	d	apply $i-1$	a $<$ b
b	c	a	d	apply i	a $<$ c
b	c	d	a	apply $i+1$	a $<$ d
b	d	c	a	apply i	c $<$ d
d	b	c	a	apply $i-1$	b $<$ d

Thus the ordering of a, b, c, and d is determined except for the relation between b and c. If $b < c$, then $a < b < c < d$, and we have a 4231 pattern which gives π at least two 321 patterns. If $c < b$ then $a < c < b < d$, and we have a 4321 pattern which gives π at least six 321 patterns. Both are contradictions with π having exactly one 321 pattern. □

Lemma 2.2.12. *Assume π contains exactly one* 321 *pattern and avoids* 3412*. Then, there can be no element j which occurs both to the left and right of the factor* $[i(i+1)i]$ *in* $\mathbf{j}$.

Proof. Note, by Lemma 2.2.10, $j \neq i$ and $j \neq i+1$. We may also assume at this point, by Lemmas 2.2.9 and 2.2.10, that there are no repeated elements between the two occurrences of j other than those in the factor $[i(i+1)i]$. Since braid moves cannot move $i(i+1)i$ to the left of j, we must have an occurrence of $i-1$ or $i+2$ to the left of $[i(i+1)i]$. Similarly, we must have an occurrence of $i-1$ or $i+2$ to the right of $[i(i+1)i]$. If $i-1$ occurs both to the left and right, then $j = i-1$. Similarly if $i+2$

occurs on both sides. The only other possibility is that $i-1$ occurs on one side and $i+2$ occurs on the other. Assume $i-1$ occurs on the left and $i+2$ occurs on the right. The factor then looks like $[j \ldots (i-1) \ldots i(i+1)i \ldots (i+2) \ldots j]$. Note that neither $i-1$ nor $i+2$ can be moved outside of the j's by braid moves, therefore our reduced decomposition looks like $[j \ldots (i-2) \ldots (i-1) \ldots i(i+1)i \ldots (i+2) \ldots (i+3) \ldots j]$. We can continue this reasoning and force on the left $j = i - k$ for some k and on the right $j = i + k'$ for some k', a contradiction.

We will now consider all factors of the form $[(i-1)t_1 \ldots t_m i(i+1)iu_1 \ldots u_p(i-1)]$. For convenience let us assume that all elements that can be moved to the left of $i(i+1)i$ by braid moves have been so moved. We claim there can be no u_j's at all in this factor. To see this, consider u_1. Since u_1 cannot be moved to the left of $i(i+1)i$, $u_1 = i-1$ or $u_1 = i+2$. If the former, then we are done, so assume $u_1 = i+2$. This means $u_k = i+k+1$ for all $2 \leq k \leq p$ and all of these u_k can be moved to the right of $(i-1)$ by braid moves. Our factor is now of the form $[(i-1)t_1 \ldots t_m i(i+1)i(i-1)]$. t_1 must be $i-2$, $t_2 = i-3$, $\ldots$, $t_m = i-m-1$. Any factor of this form can be reduced to a factor of the form $[(i-1)(i-2)i(i+1)i(i-1)]$.

One may make a similar argument for $j = i+2$ and so we must only verify that $[(i-1)(i-2)i(i+1)i(i-1)]$ and $[(i+2)(i+3)i(i+1)i(i+2)]$ cannot occur as a factor in a reduced decomposition of π. This is easily verified by an argument similar to that of Lemma 2.2.11 and we shall leave it to the reader. $\square$

Lemmas 2.2.9, 2.2.10 and 2.2.12 combine to yield:

Proposition 2.2.13. *If a permutation $\pi \in S_n$ contains exactly one* 321 *pattern and avoids* 3412*, then there exists a reduced decomposition of π containing $[i(i+1)i]$ as a factor with no other repetitions.*

We now consider what patterns are contained and avoided by a permutation $\pi \in S_n$ with reduced decomposition $\mathbf{j} = [j_1 \ldots j_l]$ where $\mathbf{j}$ has a factor of the form

$[i(i+1)i]$ and no other repetitions. In such a $\mathbf{j}$, Theorem 2.1.3 implies the existence of at least one 321 pattern. Moreover, since there are no other repetitions, there cannot be a 3412 pattern, as we cannot transform $\mathbf{j}$ into a reduced decomposition with factor $[j(j+1)(j-1)j]$. We now show there cannot be more than one 321 pattern in π.

Lemma 2.2.14. *If cba forms a 321 pattern in $\pi = \pi_1 \dots \pi_n$, then b must have reversed direction in its trajectory and so there exists an i such that $m_i > 2$.*

Proof. cba is a 321 pattern so $a < b < c$. At some point a and b must switch positions, so this means that b must travel to the left in its trajectory. Similarly, at some point b and c must switch positions so this means that b must travel to the right in its trajectory. This means that b reversed directions at some position k and $m_i \geq 4$ follows by Lemma 2.2.7. □

Lemma 2.2.15. *Assume π has a reduced decomposition with only one repetition. There cannot be cba and fed with $b \neq e$ as 321 patterns in π.*

Proof. If there were such 321 patterns, then b and e must turn at some point in their trajectories by Lemma 2.2.14, so either there exists $m_i > 2$ and $m_j > 2$ with $i \neq j$ (this corresponds to b and e turning at different positions) or there exists $m_i > 4$ (which corresponds to b and e turning at the same position). In either case we have a contradiction, for Lemma 2.2.7 gives either two elements repeated or an element repeated at least three times. A word should be said about why $m_i \neq 4$ in the second case. If $m_i = 4$, then b and e switch positions at position i twice, a contradiction by Lemma 2.2.2. □

Lemma 2.2.16. *Assume π has a reduced decomposition with only one repetition. Then 321 appears at most once in π.*

Proof. Assume there are at least two 321 patterns, cba and fed, in π. If $b \neq e$, then we're done by Lemma 2.2.15. Therefore our two 321 patterns are cba and fbd

and either $c \neq f$ or $a \neq d$. Assume $c \neq f$ and $c > f$ (one can make an analogous argument if $a \neq d$). If b reverses directions twice in its trajectory then we are done, so assume b reverses directions only once. We have $a < b < f < c$. a and b must switch positions so b must go left at least once in its trajectory. b must also switch positions with c and f so b must go right at least twice. Since $m_i > 2$ for exactly one i, we can only have that b goes left once, turns and goes right for at least two positions. b going to the left at the beginning means that a and b switched at that position. a itself must also switch positions with c, so a must go to the right at least once more, a contradiction with only one multiplicity greater than 2. □

Thus:

Proposition 2.2.17. *If there exists a reduced decomposition of $\pi \in S_n$ containing $i(i+1)i$ as a factor and no other repetitions, then π contains exactly one 321 pattern and avoids 3412.*

Propositions 2.2.13 and 2.2.17 yield the following result:

Theorem 2.2.18. *$\pi \in S_n$ contains exactly one 321 pattern and avoids 3412 if and only if there exists a reduced decomposition of π containing $i(i+1)i$ as a factor and no other repetitions.*

2.2.3 One Repetition of the Form $i(i+1)(i-1)i$

The goal of this section is to show that those permutations which have reduced decompositions with factors of the form $[i(i+1)(i-1)i]$ and no other repetitions are precisely those that avoid 321 and contain exactly one copy of 3412.

Similarly to the previous section, we will associate with permutation $\pi \in S_n$ which contains 3412 and avoids 321, the reduced decomposition $\mathbf{j}$, guaranteed by the vexillary characterization (Theorem 2.1.3), which has a factor of the form $i(i+$

$1)(i-1)i$. Again, we always assume $\mathbf{j}$ to be this reduced decomposition in the lemmas and theorems that follow.

Lemma 2.2.19. *If $\pi \in S_n$ avoids 321 and contains exactly one 3412, then no element in the set $\{i, i+1, i-1\}$ occurs outside of the factor $[i(i+1)(i-1)i]$.*

Proof. Assume not. Let us choose the element in the set $\{i, i+1, i-1\}$ which occurs closest to the factor $[i(i+1)(i-1)i]$ and assume it occurs to the left of the factor (analogous argument for occurring to the right). Call this element a and let us consider each possibility for a.

1. $a = i$. This implies we have a factor of the form $it_1 \dots t_m i(i+1)(i-1)i$ in $\mathbf{j}$ with each $t_j \neq i+1, i-1$, but t_1 can only be $i+1$ or $i-1$, a contradiction.

2. $a = i+1$. This implies we have a factor of the form $[(i+1)t_1 \dots t_m i(i+1)(i-1)i]$. If $m = 0$, then we have $[(i+1)i(i+1)(i-1)i]$, but this gives a factor of the form $[(i+1)i(i+1)]$, which means π has a 321 pattern by Theorem 2.1.5. We cannot have $m > 1$ since that would mean $t_1 = i+2$, $t_m = i-2$, $t_2 = i+3$, ... and we would be able to move everything outside of the factor by braid moves with the exception of t_1. It therefore remains to consider the possibility of a factor of the form $(i+1)(i+2)i(i+1)(i-1)i$. Let π' be formed from the reduced decomposition consisting of all elements of $\mathbf{j}$ before the factor $[(i+1)(i+2)i(i+1)(i-1)i]$. Assume $\pi'_{i-1} = a$, $\pi'_i = b$, $\pi'_{i+1} = c$, $\pi'_{i+2} = d$ and $\pi'_{i+3} = e$. The application of $[(i+1)(i+2)i(i+1)(i-1)i]$ on π' will give π'' where $\pi''_{i-1} = d$, $\pi''_i = e$, $\pi''_{i+1} = a$, $\pi''_{i+2} = b$ and $\pi''_{i+3} = c$. The reduced decomposition forces the relations $c < d$, $c < e$, $b < d$, $b < e$, $a < d$, and $a < e$, so the relations between a and b, a and c, b and c, and d and e have not been determined. Consider what the relation could be between d and e. If $e < d$, then there must be a 321 pattern in $deabc$ which means π will have at least one 321 pattern. If $d < e$, then unless $a < b < c$, there will also be

a 321 pattern in *deabc*. Now, $d < e$ and $a < b < c$ means that *deabc* forms a 45123 pattern. 45123 contains two 3412 patterns and so either they remain in π, which is a contradiction, or Lemma 2.2.4 gives multiple 321 patterns which is also a contradiction.

3. $a = i - 1$. This implies a factor of the form $[(i-1)t_1 \dots t_m i(i+1)(i-1)i]$ in the reduced decomposition. If $m = 0$ we have $[(i-1)i(i+1)(i-1)i] = [(i-1)i(i-1)(i+1)i]$. This has $[(i-1)i(i-1)]$ as a factor so π would have a 321 pattern, a contradiction. If $m = 1$, then $[(i-1)(i-2)i(i+1)(i-1)i]$ is a possibility. One can verify using a method similar to the case $a = i+1$ that this is impossible, and similarly that $m > 1$ is also impossible.

□

Lemma 2.2.20. *If* π *avoids* 321 *and contains exactly one* 3412*, then no element* $j \in \{1, \dots, n\}$ *can appear twice on one side of the factor* $[i(i+1)(i-1)i]$ *in* $\mathbf{j}$.

Proof. Assume there exists a repetition that appears twice to the left of the factor $[i(i+1)(i-1)i]$. There is an analogous argument for the right. So $\mathbf{j} = [j_1 \dots j \dots j \dots i(i+1)(i-1)i \dots j_k]$. This means there exists an additional factor of the form $[j(j+1)j]$ or $[j(j+1)(j-1)j]$ where $\{j, j+1, j-1\} \cap \{i, i+1, i-1\} = \emptyset$, by Lemma 2.2.19. There cannot be a factor of the form $[j(j+1)j]$ since the occurrence of such a factor means the occurrence of a 321 pattern. So the only possibility is a factor of the form $j(j+1)(j-1)j$. The occurrence of a factor $[j(j+1)(j-1)j]$ in a reduced decomposition of a permutation that avoids 321 means that the elements in positions $j-1$, j, $j+1$ and $j+2$ after applying the factor must form a 3412 pattern (and hence formed a 1234 pattern before the application of the factor). If they do not, there will be a 321 pattern. This means that after applying the $[j(j+1)(j-1)j]$ factor there will be a 3412 pattern and after applying the $[i(i+1)(i-1)i]$ factor there will be another 3412 pattern. Hence there will be two 3412 patterns in the permutation represented by $[j_1 \dots j \dots j \dots i(i+1)(i-1)i]$. Lemma 2.2.4 implies that if the two 3412 patterns do not occur in π as well, there will be multiple 321 patterns in π. Either option gives a contradiction. □

Lemma 2.2.21. *If π avoids 321 and contains exactly one 3412, then no element $j \in \{1, \dots, n\}$ can appear both on the left and right of the factor $i(i+1)(i-1)i$.*

Proof. We use the same reasoning as in Lemma 2.2.12. $\mathbf{j}$ has the form $[\dots j \dots i(i+1)(i-1)i \dots j \dots]$. To keep $[i(i+1)(i-1)i]$ in between the j's, we must have that $j = i-2$ or $j = i+2$. One may then reduce the form of the factor $[j \dots i(i+1)(i-1)i \dots j]$ to four possibilities: $[(i-2)i(i+1)(i-1)i(i-2)]$, $[(i-2)(i-3)i(i+1)(i-1)i(i-2)]$, $[(i+2)i(i+1)(i-1)i(i+2)]$ and $[(i+2)(i+3)i(i+1)(i-1)i(i-2)]$. We eliminate $[(i-2)i(i+1)(i-1)i(i-2)] = [i(i+1)(i-2)(i-1)(i-2)i]$ and $[(i+2)i(i+1)(i-1)i(i+2)] = [i(i+2)(i+1)(i+2)(i-1)i]$ because they contain a factor of the form $[j(j-1)j]$. The other two may be eliminated by computing what happens to the elements switched by those elements, and observing that they either cause the occurrence of 321 patterns or two 3412 patterns. □

Lemmas 2.2.19, 2.2.20, and 2.2.21 prove the following proposition.

Proposition 2.2.22. *If $\pi \in S_n$ avoids 321 and contains exactly one 3412, then there exists a reduced decomposition of π with $[i(i+1)(i-1)i]$ as a factor with no other repetitions.*

Consider what permutations are avoided and contained by a permutation $\pi \in S_n$ with reduced decomposition $\mathbf{j} = [j_1 \dots j_l]$ where $\mathbf{j}$ has a factor of the form $[i(i+1)(i-1)i]$ and no other repetitions. In such a $\mathbf{j}$, Theorem 2.1.3 implies the existence of at least one 3412 pattern, as we cannot transform $\mathbf{j}$ into a reduced decomposition with factor $[j(j+1)j]$. We now show there cannot be more than one 3412 pattern in π.

Let us consider what happens to the elements in positions $i-1$, i, $i+1$, $i+2$ at the time the factor $i(i+1)(i-1)i$ is applied on a permutation avoiding 321. Assume the elements in these positions are a, b, c and d.

The trajectories at these four positions are:

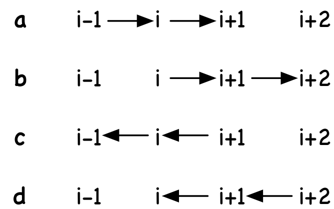

Figure 2.3: Trajectories for [i(i+1)(i-1)i]

Note, because there can be no 321 patterns, the *cdba* pattern formed by the $i(i+1)(i-1)i$ factor must be a 3412 pattern in π.

Lemma 2.2.23. *If π has a reduced decomposition with $[i(i+1)(i-1)i]$ as a factor and no other repetitions, then all trajectories are straight lines (no elements turn).*

Proof. Assume an element turns. Let x turn at position $j+1$. x moved into position $j+1$ through the action of j in the reduced decomposition. In order for x to turn again, we must have another occurrence of j, therefore $i = j$; however, no element turns during the factor $[i(i+1)(i-1)i]$. □

Lemma 2.2.24. *Assume only one repetition of the form $[i(i+1)(i-1)i]$ in a reduced decomposition of $\pi \in S_n$. Then, if cdab forms a 3412 pattern in π, $c = b + 1$.*

Proof. Consider the trajectories and recall the trajectories of such a permutation must be straight lines by Lemma 2.2.23. Assume b moves to the right at least once before it switches positions with c. Now c and a must also switch positions, therefore either c must move to the right again which gives a position of multiplicity 4, or a must move to the left, which also gives a position of multiplicity 4. Either way, the current pattern is *cabd*. a cannot move any farther to the left because doing so would create two positions of multiplicity 4. Therefore a and d cannot switch.

Contradiction. There is a similar argument if c has to move left at least once before switching with b. Therefore, b and c must have been next to each other originally and the result follows. □

It is worth noting at this point that since the factor $[i(i+1)(i-1)i]$ creates a 3412 pattern, then $b = i$ and $c = i+1$ and also that a must remain in position $i+1$ and d in position i.

Lemma 2.2.25. *Assume there is only one repetition of the form* $[i(i+1)(i-1)i]$ *in a reduced decomposition of* $\pi \in S_n$*. If* $cdab$ *and* $ghef$ *form* 3412 *patterns in* π*, then* $c = g$ *and* $b = f$*.*

Proof. Assume not. Note that this means $c \neq g$ and $b \neq f$. Also, $b \neq g$, because $b = g$ would mean π has a $cdaghef$ pattern. cgf would then be a 321 pattern since $g = f+1$ and $c = b+1 = g+1 = f+2$. Similarly, $c \neq f$. Then $\{b, c\} \cap \{f, g\} = \emptyset$. We have by Lemma 2.2.24, that $c = b+1$ and $g = f+1$. b and c must switch positions at position b and f and g must switch positions at position f. a and d must swap positions and this cannot occur without at least one of them going through position b, so $m_b = 4$. Similarly h and e must swap positions and this cannot occur without one of them going through position f, so $m_f = 4$. $b \neq f$, so we have two positions of multiplicity 4. □

Lemma 2.2.26. *Assume only one repetition of the form* $[i(i+1)(i-1)i]$ *in a reduced decomposition of* $\pi \in S_n$*. There cannot exist two* 3412 *patterns in* π *of the form* $cdab$ *and* $cfeb$*.*

Proof. Assume the factor $[i(i+1)(i-1)i]$ forms the $cdab$ pattern in π. We know that $m_b = 4$ by looking at the trajectory formed by the factor $[i(i+1)(i-1)i]$. Suppose there exist two extra elements e and f such that $cfeb$ forms a 3412 pattern (i.e., $e < b < c < f$). e and f must cross. One is on the left of position i and the other is on the right of position i. They cannot cross without one of them going through position i. This means $m_i = 6$ which is a contradiction. □

Lemmas 2.2.25 and 2.2.26 imply the following proposition:

Proposition 2.2.27. *If $\pi \in S_n$ has a reduced decomposition with $[i(i+1)(i-1)i]$ as a factor and no other repetitions, then π contains exactly one 3412 pattern and avoids 321.*

Propositions 2.2.22 and 2.2.27 yield the following theorem:

Theorem 2.2.28. *$\pi \in S_n$ avoids 321 and contains exactly one 3412 pattern if and only if there exists a reduced decomposition of π containing $[i(i+1)(i-1)i]$ as a factor and no other repetitions.*

We combine Theorems 2.2.18 and 2.2.28 for easy reference.

Theorem 2.2.29. *$\pi \in S_n$ has a reduced decomposition with exactly one element repeated if and only if either $\pi \in Av_n(3412)$ and contains exactly one 321 pattern, or $\pi \in Av_n(321)$ and contains exactly one 3412 pattern. More specifically,*

- *$\pi \in Av_n(3412)$ and contains exactly one 321 pattern if and only if π has a reduced decomposition with $[i(i+1)i]$ as a factor for some $i \in \{1, \dots, n-2\}$ and no other repetitions.*

- *$\pi \in Av_n(321)$ and contains exactly one 3412 pattern if and only if π has a reduced decomposition with $[i(i-1)(i+1)i]$ as a factor for some $i \in \{2, \dots, n-2\}$ and no other repetitions.*

2.3 Counting permutations in $Av_n(3412)$ that contain exactly one 321 or in $Av_{n+1}(321)$ that contain exactly one 321

Using the classification of reduced decompositions with one repetition, it is possible to count the number of permutations in $Av_n(321)$ that contain exactly one 3412 pattern and to count the number of permutations in $Av_n(3412)$ that contain exactly one 321 pattern. In fact, there actually exists a bijection from the set of permutations in $Av_n(3412)$ that contain exactly one 321 pattern to the set of permutations in $Av_{n+1}(321)$ that contain exactly one 3412 pattern. Once the permutations in these two sets are counted, we consider the case of involutions. The material in this section forms the paper [10].

2.3.1 Bijection

As our goal is to count the number of permutations in $Av_n(3412)$ that contain exactly one 321 pattern and the number of permutations in $Av_n(321)$ that contain exactly one 3412 pattern, we first provide a bijection between the two classes. Let $\mathcal{A}_n = \{\pi \in Av_n(3412) \mid \pi \text{ contains exactly one 321 pattern}\}$ and let $\mathcal{B}_n = \{\pi \in Av_n(321) \mid \pi \text{ contains exactly one 3412 pattern}\}$. We will show $|\mathcal{A}_n| = |\mathcal{B}_{n+1}|$.

Before we create the bijection, some propositions concerning the properties of reduced decompositions with one repetition are required.

Proposition 2.3.1. *Let $\pi \in S_n$ have a reduced decomposition $\mathbf{s} = [s_1 \dots s_k]$ with $[i(i+1)i]$ as a factor and no other repetitions and let $\sigma \in S_n$ have a reduced decomposition $\mathbf{t} = [t_1 \dots t_l]$ with $[j(j+1)j]$ as a factor and no other repetitions. If $i \neq j$, then $\pi \neq \sigma$.*

Proof. Suppose $[i-1]$ does not occur in $[s_1 \dots s_k]$ or $[i-1]$ occurs to the left of the factor $[i(i+1)i]$. Then the image of i under $[i(i+1)i \dots s_k]$ is $i+2$. Since $[i(i+1)i]$ contains only the repetition in $[s_1 \dots s_k]$, we must have $\pi_i \geq i+2$. Thus, $\pi_{i+2}, \dots, \pi_n$ must contain some element of $\{1, 2, \dots i\}$, so $\pi_{i+1} = i+1$ is the middle element of the 321 pattern. Now suppose $[i-1]$ occurs to the right of $[i(i+1)i]$ in $[s_1 \dots s_k]$. If l is the greatest integer such that $i-1, i-2, \dots i-l$ appear in that order to the right of $[i(i+1)i]$, then $\pi_{i-l} \geq i-2$. As in the previous case, this implies $\pi_{i+1} = i+1$ is the middle element of the 321 pattern. By a similar argument $j+1$ is the middle element of the 321 pattern in σ. Since $i \neq j$, we also have $\pi \neq \sigma$. □

Proposition 2.3.2. *Let $\pi \in S_n$ have a reduced decomposition $\mathbf{s} = [s_1 \dots s_k]$ with $[i(i+1)(i-1)i]$ as a factor with no other repetitions, and let $\sigma \in S_n$ have a reduced decomposition $\mathbf{t} = [t_1 \dots t_l]$ with $[j(j+1)(j-1)j]$ as a factor and no other repetitions. If $i \neq j$, then $\pi \neq \sigma$.*

Proof. The occurrence of $[i(i+1)(i-1)i]$ in a reduced decomposition with no other repetitions means the following must occur by methods similar to the proof of Proposition 2.3.1:

- there exists an element $a \leq i-1$ such that $\pi_a = i+1$. Note $a = i-1$ if $[i-2]$ does not exist to the right of the factor.
- $\pi_i \geq i+2$.
- $\pi_{i+1} \leq i-1$.
- there exists an element $b \geq i+2$ such that $\pi_b = i$. Note $b = i+2$ if $[i+2]$ does not exist to the right of the factor.

Hence, π_i and π_{i+1} are the middle elements of the 3412 pattern of π. We can similarly conclude that σ_j and σ_{j+1} are the middle elements of the 3412 pattern of σ. Since $i \neq j$, $\pi \neq \sigma$. □

Proposition 2.3.3. *Let $\pi \in \mathcal{A}_n$ and let $\mathbf{s} = [s_1 \dots s_k]$ and $\mathbf{t} = [t_1 \dots t_k]$ be reduced decompositions for π. Let i be the element such that $[i(i+1)i]$ appears in a reduced decomposition of π. $\mathbf{s}$ can be transformed into $\mathbf{t}$ using only short braid moves and the long braid move $[i(i+1)i] = [(i+1)i(i+1)]$ for the specific element i.*

Proof. Starting with a reduced decomposition with $[i(i+1)i]$ as a factor and no other repetitions. $[i-1]$ cannot commute with $[i]$, so there can be no occurrence of $[i(i-1)i]$ in any equivalent reduced decomposition for $\mathbf{s}$. Similarly, $[i(i+1)i] = [(i+1)i(i+1)]$ and $[i+2]$ cannot commute with $[i+1]$ so there can be no occurrence of $[(i+1)(i+2)(i+1)]$ in any equivalent reduced decomposition for $\mathbf{s}$. □

Proposition 2.3.4. *The 321-avoiding permutations are the permutations for which all of the reduced decompositions may be derived from one to the other without using braid relations - in other words, by allowing only commutation of simple transpositions with nonconsecutive indices.*

Proof. This is Proposition 2.2.15 in [25]. □

Proposition 2.3.5. *Let $\pi \in \mathcal{B}_n$ and let $\mathbf{s} = [s_1 \dots s_k]$ and $\mathbf{t} = [t_1 \dots t_k]$ be reduced decompositions for π. $\mathbf{s}$ can be transformed into $\mathbf{t}$ using only short braid moves.*

Proof. This is a specific instance of Proposition 2.3.4. □

Proposition 2.3.5 implies that each element s_j occurs the same number of times in every reduced decomposition of π as short braid moves do not change the number of times an element occurs in $\mathbf{s}$.

The bijection from $\mathcal{A}_n$ to $\mathcal{B}_n$ can now be constructed. By Theorem 2.2.29, $\pi \in \mathcal{A}_n$ implies that π has a reduced decomposition with $[i(i+1)i]$ as a factor for some i and no other repetitions. Hence, by an application of a long braid move, π has a reduced decomposition with $[(i+1)i(i+1)]$ as a factor and no other repetitions.

Let $\mathcal{R}_k^i$ be the set of all reduced decompositions of length k with $[(i+1)i(i+1)]$ as a factor and no other repetitions. Note these are all such reduced decompositions, not just those for a particular permutation π. Let $\mathcal{S}_k^i$ be the set of all reduced decompositions of length k with $[i(i-1)(i+1)i]$ as a factor and no other repetitions. Define a map $g_i^k : \mathcal{R}_k^i \to \mathcal{S}_{k+1}^{i+1}$ by the following method. Let $\mathbf{s} = [s_1 \dots s_k] \in \mathcal{R}_k^i$. Replace each s_j in $\mathbf{s}$ with s_j' where

$$s_j' = \begin{cases} s_j & \text{if } s_j \leq i+1 \\ s_j + 1 & \text{if } s_j > i+1 \end{cases}.$$

Lastly, insert the element $[i+2]$ into the factor $[(i+1)i(i+1)]$ giving $[(i+1)i(i+2)(i+1)]$. Call this new expression $\mathbf{s}'$. Define $g_i^k(\mathbf{s}) := \mathbf{s}'$.

Now consider the lexicographic ordering on $\mathcal{R}_k^i$. It is clear that if $\mathbf{s} < \mathbf{t}$ in lexicographic ordering, then $g_i^k(\mathbf{s}) < g_i^k(\mathbf{t})$ in the lexicographic ordering on $\mathcal{S}_{k+1}^{i+1}$. This gives the following lemma.

Lemma 2.3.6. *Assume* $\mathbf{s}$ *is the lexicographically smallest reduced decomposition for* $\pi \in S_n$ *with* $[(i+1)i(i+1)]$ *as a factor and that* $l(\pi) = k$. *The reduced decomposition* $g_i^k(\mathbf{s})$ *is the lexicographically smallest reduced decomposition among all reduced decompositions of* π' *with* $[(i+1)i(i+2)(i+1)]$ *as a factor.*

One may produce $(g_i^k)^{-1}$ by taking an element $\mathbf{s} = [s_1 \dots s_k]$ with $[(i+1)i(i+2)(i+1)]$ as a factor and no other repetitions, removing the $[i+2]$ element from the $[(i+1)i(i+2)(i+1)]$ factor and then for all remaining $s_j > i+2$, replace s_j with $s_j - 1$. It is therefore clear that g_i^k is a bijection. The bijection $f : \mathcal{A}_n \to \mathcal{B}_{n+1}$ may now be induced from g_i^k. Let $\pi \in \mathcal{A}_n$ such that $l(\pi) = k$. Let $\mathbf{s}$ be the lexicographically smallest reduced decomposition for π with $[i(i+1)i]$ as a factor. Define $f(\pi) := g_k^i(\mathbf{s})$.

Example 2.3.7. *Consider the permutation* $\pi = 243165 \in S_6$. *The set of reduced de-*

compositions with a factor of the form $[(i{+}1)i(i{+}1)]$ *of* π *is* $\{[15323], [51323], [13235]\}$. *The least element under lexicographic ordering is* $\mathbf{s} = [13235]$. *Transforming* $\mathbf{s}$ *into* $\mathbf{s}'$ *gives* $\mathbf{s}' = [132436]$. 2451376 *is the permutation represented by* $[132436]$. *Therefore* $f(243165) = 2451376$.

Restricting to the lexicographically smallest reduced decomposition for $\pi \in S_n$ and using the fact that g_k^i is a bijection gives the desired bijection.

Lemma 2.3.8. $f : \mathcal{A}_n \to \mathcal{B}_{n+1}$ *as defined above is a bijection.*

Theorem 2.3.9. *The number of permutations in* $Av_n(3412)$ *that contain exactly one* 321 *pattern is equal to the number of permutations in* $Av_{n+1}(321)$ *that contain exactly one* 3412 *pattern.*

2.3.2 Counting $\pi \in Av_n(3412)$ that contain exactly one 321

By Theorem 2.3.9, in order to count the number of permutations in $Av_n(3412)$ that contain exactly one 321 and the number of permutations in $Av_n(321)$ that contain exactly one 3412, it suffices to count the former. The strategy to count such permutations is to count equivalence classes of reduced decompositions. Two reduced decompositions $\mathbf{s}$ and $\mathbf{t}$ are considered equivalent if and only if $\mathbf{s}$ and $\mathbf{t}$ represent the same permutation. For a fixed n and $i \in \{1 \ldots n-2\}$, we will count the number of equivalence classes having $[i(i+1)i]$ as a factor and then sum over all i.

To accomplish the count, we will construct sets $E_j^i(n)$ of reduced decompositions such that the elements of $E_j^i(n)$ are representatives of distinct equivalence classes of reduced decompositions with the property that every permutation in $\mathcal{A}_n$ which has a reduced decomposition $[s_1 \ldots s_k]$ in which $[i(i+1)i]$ is a factor and $s_m \leq j$ for all $1 \leq m \leq k$ has a representative in $E_j^i(n)$.

Table 2.1: $E_j^i(4)$ (i - rows; j - cols)

	2	3
1	$\{[121]\}$	$\{[121],[3121],[1213]\}$
2	$\emptyset$	$\{[232],[1232],[2321]\}$

Table 2.2: $E_j^i(5)$ (i - rows; j - cols)

	2	3	4
1	$\{[121]\}$	$\{[121],[3121],[1213]\}$	$\{[121],[3121],[1213],[4121],$ $[43121],[31214],[41213],[12134]\}$
2	$\emptyset$	$\{[232],[1232],[2321]\}$	$\{[232],[1232],[2321],[4232],$ $[2324],[41232],[12324],[42321],[23214]\}$
3	$\emptyset$	$\emptyset$	$\{[343],[2343],[3432],[1343],$ $[12343],[23431],[13432],[34321]\}$

Note $E_j^i(n)$ is empty when $j < i+1$. Tables 2.1 and 2.2 give the sets $E_j^i(4)$ and $E_j^i(5)$ in terms of their reduced decompositions. Note, only one reduced decomposition from each equivalence class is listed.

It is clear $|\mathcal{A}_n| = \sum_{i=1}^{n-2} |E_{n-1}^i(n)|$. To compute the cardinalities of the sets $E_{n-1}^i(n)$, we will show how to construct each set. This procedure is broken into two parts: i.) how to construct the set $E_{i+1}^i(n)$ and ii.) how to construct $E_{j+1}^i(n)$ given $E_j^i(n)$. Assuming we know how to construct $E_{i+1}^i(n)$, we will first show how to construct $E_{j+1}^i(n)$ and then we will go back and show how to construct $E_{i+1}^i(n)$.

Constructing the set $E_j^i(n)$ from $E_{j-1}^i(n)$

The smallest j for which $E_j^i(n)$ is nonempty is $i+1$, so let us assume we have $E_{i+1}^i(n)$ and show how to construct $E_{i+2}^i(n)$. For all reduced decompositions $\mathbf{s} \in E_{i+1}^i(n)$, construct a set X of reduced decompositions as follows:

1. Add $\mathbf{s}$ to X.
2. Concatenate (i+2) to $\mathbf{s}$ on both sides, giving $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(i+2)]$, and add them to X.

$\mathbf{s}$ only contains elements in $\{1, \ldots, i+1\}$ so both $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(i+2)]$ are reduced.

Lemma 2.3.10. *The set X is $E^i_{i+2}(n)$.*

Proof. First, it must be shown why all reduced decompositions in X represent distinct permutations. No reduced decomposition from step 1 can be equivalent to any reduced decomposition from step 2, as the set of elements in a reduced decomposition is invariant among all equivalent reduced decompositions. The elements of the set created in step 1 are all distinct as they are assumed to be distinct from being in the set $E^i_{i+1}(n)$. Assume $\mathbf{s}$ and $\mathbf{t}$ are distinct elements of $E^i_{i+1}(n)$ and that $[(i+2)\mathbf{s}] = [(i+2)\mathbf{t}]$. Multiplying both sides by $(i+2)$ gives $\mathbf{s} = \mathbf{t}$ which is a contradiction. Now assume $[(i+2)\mathbf{s}] = [\mathbf{t}(i+2)]$. This implies $(i+2)$ must commute with every element of $\mathbf{s}$. This can happen only by short braid moves or by the use of long braid moves of the form $[(i+2)(i+1)(i+2)] = [(i+1)(i+2)(i+1)]$ or $[(i+2)(i+3)(i+2)] = [(i+3)(i+2)(i+3)]$. Such long braid moves are impossible by Proposition 2.3.3. This implies $(i+2)$ commutes with every element. $(i+1)$ is one of the elements in $\mathbf{s}$ and $\mathbf{t}$, so $(i+2)$ does not commute with every element which is a contradiction.

Second, it must be shown why every permutation is represented by an element in X. Let π be a permutation with a reduced decomposition $\mathbf{s}$ with $[i(i+1)i]$ as a factor and no other repetitions such that the elements of $\mathbf{s}$ are a subset of $\{1, \ldots, i+2\}$. If the element $(i+2)$ does not appear in $\mathbf{s}$, then $\mathbf{s} \in E^i_{i+1}(n)$ by assumption and so must be in $E^i_{i+2}(n)$. If the element $(i+2)$ does appear in $\mathbf{s}$, then $\mathbf{s} = [s_1 \dots (i+2) \dots s_k]$. The element $(i+1)$ either occurs to the left or to the right of $(i+2)$. If $(i+1)$ occurs to the left of $(i+2)$, then applying short braid moves produces $[s_1 \dots s_k(i+2)]$ which is equivalent to $\mathbf{s}$. $[s_1 \dots s_k]$ is a reduced decomposition on the elements $\{1, \dots, i+1\}$ and so is a member of $E^i_{i+1}(n)$ and so by step 2 of the construction $[s_1 \dots s_k(i+2)]$ is an element of E^i_{i+2}. The argument is similar if $(i+1)$ occurs to the right of $(i+2)$. □

Once the set $E^i_{i+2}(n)$ is built, we can generalize the procedure for building inductively $E^i_{j+1}(n)$ from $E^i_j(n)$ and $E^i_{j-1}(n)$ as follows for all $\mathbf{s} \in E^i_j(n)$.

1. Add $\mathbf{s}$ to $E^i_{j+1}(n)$.
2. If $\mathbf{s} \in E^i_j(n) \cap E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ to $E^i_{j+1}(n)$.
3. If $\mathbf{s} \in E^i_j(n) \setminus E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ and $[\mathbf{s}(j+1)]$ to $E^i_{j+1}(n)$.

Example 2.3.11. *Here is an example of this procedure to produce $E^1_5(6)$ from $E^1_4(6) = \{[121], [3121], [1213], [4121], [43121], [31214], [41213], [12134]\}$ and $E^1_3(6) = \{[121], [3121], [1213]\}$.*
Step 1 adds $\{[121], [3121], [1213], [4121], [43121], [31214], [41213], [12134]\}$.
Step 2 adds $\{[5121], [53121], [51213]\}$.
Step 3 adds $\{[54121], [41215], [543121], [431215], [531214], [312145], [541213], [412135], [512134], [121345]\}$.

Lemma 2.3.12. *The procedure outlined above correctly produces the set $E^i_{j+1}(n)$ for all $i+1 \le j < n-1$.*

Proof. The proof is by induction on j. The base case is Lemma 2.3.10. Every element created by the above procedure is reduced. It must be shown that each reduced decomposition represents a distinct permutation and every permutation is represented by a reduced decomposition.

All the reduced decompositions from step 1 are distinct by the induction hypothesis and are also distinct from the reduced decompositions from step 2 and 3 since none of the expressions from step 1 have j as an element. The distinction of the elements from steps 2 and 3 follows in a fashion similar to Lemma 2.3.10.

To show that every permutation is represented, consider the reduced decomposition $\mathbf{s}$ of a permutation that has not been created by the above procedure. If the element $j+1$ does not appear in $\mathbf{s}$, then $\mathbf{s} \in E_j^i(n)$ and step 1 puts $\mathbf{s}$ in $E_{j+1}^i(n)$. If the element $j+1$ does appear in $\mathbf{s}$, then it only appears once. Also, the element $j+2$ does not appear in s, so a series of short braid moves can be applied to $\mathbf{s}$ to make $j+1$ either the rightmost element or the leftmost element. Once $j+1$ is the leftmost or rightmost element, remove the element $j+1$ and get a reduced decomposition $\mathbf{t}$ having only the elements $\{1, \ldots, j\}$ appearing. $\mathbf{t}$ appears in $E_j^i(n)$ and so applying step 2 or step 3 will create $\mathbf{s}$. □

As stated previously, there is precisely one reduced decomposition in $E_{n-1}^i(n)$ for each permutation in $\mathcal{A}_n$ and no others. Define $a_i(k) := |E_{i+k}^i(n)|$ for $k \geq 0$ and note $a_i(0) = 0$. By our construction of these sets $a_i(k)$ satisfies the following recurrence: $a_i(k) = a_i(k-1) + a_i(k-2) + 2(a_i(k-1) - a_i(k-2)) = 3a_i(k-1) - a_i(k-2)$. It is a well-known result that when a recurrence of the form $b(k) = 3b(k-1) - b(k-2)$ with $b(0) = 0$ and $b(1) = 1$, then $b(k) = F_{2k}$ where F_j is the j^{th} Fibonacci number. Note that $a_i(1) = |E_{i+1}^i(n)|$. Note, in particular, that for each i, the value of k that makes $i+k = n-1$ is $k = n-i-1$. Therefore $|E_{n-1}^i(n)| = a_i(n-i-1) = |E_{i+1}^i(n)| \cdot F_{2(n-i-1)}$.

Constructing the sets $E_{i+1}^i(n)$

Note that $E_2^1(n) = \{[121]\}$ for all n. The sets $E_{i+1}^i(n)$ are constructed inductively from $E_i^{i-1}(n)$ by the following procedure for all $\mathbf{s}$ in $E_i^{i-1}(n)$:

1. If $\mathbf{s} = [s_1 \ldots s_k]$, then add $[(s_1+1)(s_2+1)\ldots(s_k+1)]$ to $E_{i+1}^i(n)$.

2. If $\mathbf{t}$ was added in step 1 and only contains elements in $\{3, \ldots, i+1\}$, then add the reduced decomposition $[1\mathbf{t}]$ to $E^i_{i+1}(n)$.

3. If $\mathbf{t}$ was added in step 1 and contains the element 2, then add the reduced decompositions $[1\mathbf{t}]$ and $[\mathbf{t}1]$ to $E^i_{i+1}(n)$.

Example 2.3.13. *Here is an example of the procedure creating* $E^4_5(6)$ *from* $E^3_4(6) = \{[343], [2343], [3432], [1343], [12343], [23431], [13432], [34321]\}$.

Step 1 adds $\{[454], [3454], [4543], [2454], [23454], [34542], [24543], [45432]\}$.
Step 2 adds $\{[1454], [13454], [14543]\}$.
Step 3 adds $\{[12454], [24541], [123454], [234541], [134542], [345421], [124543], [245431], [145432], [454321]\}$.

By an induction similar to Lemma 2.3.12, we have the following lemma.

Lemma 2.3.14. *The procedure above correctly produces the sets* $E^i_{i+1}(n)$.

Define $b(i) := |E^i_{i+1}(n)|$. The construction of the sets implies $b(i)$ satisfies the recurrence: $b(i) = b(i-1)+b(i-2)+2(b(i-1)-b(i-2)) = 3b(i-1)-b(i-2)$ where $b(0) = 0$ and $b(1) = 1$. Similar to the previous construction, we have $b(i) = F_{2i}$ where F_i is the i^{th} Fibonacci number.

Completing the Count

Combining all the lemmas above gives

$$|\mathcal{A}_n| = \sum_{i=1}^{n-2} |E^i_{n-1}(n)| = \sum_{i=1}^{n-2} |E^i_{i+1}(n)| F_{2(n-i-1)} = \sum_{i=1}^{n-2} F_{2i} F_{2(n-i-1)}$$

Theorem 2.3.15. *The number of permutations in* $Av_n(3412)$ *that contain exactly one* 321 *is* $\sum_{i=1}^{n-2} F_{2i} F_{2(n-i-1)}$.

Table 2.3: $|A_n|$ for small n

n	3	4	5	6	7	8	9	10	11	12
$\lvert\mathcal{A}_n\rvert$	1	6	25	90	300	954	2939	8850	26195	76500

Table 2.3 shows these numbers for the first few n.

Using the Online Encyclopedia of Integer Sequences, [33], one sees for small values of n that this sequence matches sequence A001871 which count the number of ordered trees of height at most 4 where only the right-most branch at the root achieves this height. Using Maple, one can verify that the sum $\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)}$ satisfies the recurrence

$$a(n) = \frac{2a(n-1) + (n+1)F_{2n+4}}{3}$$

for A001871, giving the following theorem.

Theorem 2.3.16. $|\mathcal{A}_n|$ *is counted by sequence A001871 and so*

$$a(n) = \sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)} = \frac{2(2n-5)F_{2n-6} + (7n-16)F_{2n-5}}{5}$$

satisfies the recurrence

$$a(n) = \frac{2a(n-1) + (n+1)F_{2n+4}}{3}$$

and has generating function

$$\frac{x^3}{(1-3x+x^2)^2}$$

2.3.3 Involutions

In [13], Egge enumerated the number of 3412-avoiding involutions that contained exactly one decreasing sequence of length k using lattice paths and Chebyshev polynomials. Here we reproduce that result for $k = 3$ using reduced decompositions. To enumerate the involutions requires characterizing the reduced decompositions of the involutions in $\mathcal{A}_n$.

Lemma 2.3.17. *Let* $[i(i+1)is_1 \dots s_k]$ *be a reduced decomposition such that* $s_a \neq s_b$ *when* $a \neq b$ *and for all* a, $s_a \neq i$ *and* $s_a \neq i+1$. *Assume* $[i(i+1)is_1 \dots s_k i]$ *is not reduced. Then* $[i(i+1)s_1 \dots s_k]$ *is a reduced decomposition for* $[i(i+1)is_1 \dots s_k i]$ *and* $[is_j] = [s_j i]$ *for all* $1 \leq j \leq k$.

Proof. To show $[is_j] = [s_j i]$ for all $1 \leq j \leq k$, it suffices to show that $s_j \neq [i-1]$ for all j. Assume for purposes of a contradiction that there exists j (and this j must be unique by assumption $s_a \neq s_b$ for all $a \neq b$) such that $s_j = [i-1]$. Therefore $[i(i+1)is_1 \dots s_k] = [i(i+1)is_i \dots s_{j-1}(i-1)s_{j+1} \dots s_k]$. Let π be the permutation whose reduced decomposition is $[i(i+1)is_1 \dots s_k i]$ where $s_j = [i-1]$. Consider π_i. By the right-most transposition i is mapped to $i+1$. By the assumptions, none of the s_a for $1 \leq a \leq k$ move $i+1$. The factor $[i(i+1)i]$ leaves $i+1$ fixed, so $\pi_i = i+1$. It is a well-known result (see [4] for more details) that if $\mathbf{s}$ is a reduced decomposition and s is a transposition not necessarily appearing in $\mathbf{s}$ then $l([\mathbf{s}s]) = l([\mathbf{s}]) \pm 1$. $l([i(i+1)is_1 \dots s_k i]) = l([i(i+1)is_1 \dots s_k]) - 1$ because by assumption $[i(i+1)is_1 \dots s_k i]$ is not reduced. By the Exchange property, theorem 1.2.10, $[i(i+1)is_1 \dots s_k i]$ is equivalent to one of the following reduced decompositions: $[i(i+1)s_1 \dots s_k]$, $[(i+1)is_1 \dots s_k]$, or $[i(i+1)is_1 \dots \hat{s_a} \dots s_k]$ for some $1 \leq a \leq k$.

Consider the cases:

1. $[i(i+1)s_1 \dots s_k]$. i is mapped to $i-1$ by the occurrence of $[i-1]$ in position j and so i must be mapped to $i-a$ where $a \geq 1$. Therefore $[i(i+1)s_1 \dots s_k] \neq$

$[i(i+1)is_1 \dots s_k i]$ since $\pi_i = i+1$.

2. $[(i+1)is_1 \dots s_k]$. i is mapped to $i-1$ by the transposition $[i-1]$ and so again must be mapped to $i-a$ for some $a \geq 1$. Therefore, $[(i+1)is_1 \dots s_k] \neq [i(i+1)is_1 \dots s_k i]$.

3. $[i(i+1)is_1 \dots \hat{s_b} \dots s_k]$ for some $1 \leq b \leq k$. If $b \neq j$, then i is mapped to $i-a$ for some $a \geq 1$. If $b = j$, then the factor $[i(i+1)i]$ sends i to $i+2$. So, $[i(i+1)is_1 \dots \hat{s_b} \dots s_k] \neq [i(i+1)is_1 \dots s_k i]$.

Since all three possibilities lead to a contradiction we must have $s_j \neq [i-1]$ for all j and so $[is_j] = [s_j i]$ for all j. Therefore $[i(i+1)is_1 \dots s_k i] = [i(i+1)iis_1 \dots s_k] = [i(i+1)s_1 \dots s_k]$. $[i(i+1)s_1 \dots s_k]$ must be reduced because of the length. □

Theorem 2.3.18. *Suppose $\pi \in \mathcal{A}_n$. π is an involution if and only if π has a reduced decomposition $[s_1 \dots s_k]$ for which the following hold.*

1. *$[s_1 \dots s_k]$ has a factor of the form $[i(i+1)i]$ for some i and no other repetitions.*

2. *If $|s_j - s_m| = 1$ then $\{s_j, s_m\} = \{i, i+1\}$.*

Proof. ($\Leftarrow$) By Theorem 2.2.29, such a reduced decomposition for π implies $\pi \in \mathcal{A}_n$. By the structure of the reduced decomposition and since $[i(i+1)i]^2 = [\emptyset]$, π is easily seen to be an involution.

($\Rightarrow$) Assume π is an involution. $\pi \in \mathcal{A}_n$ implies that π has a reduced decomposition $\mathbf{s} = [s_1 \dots s_k]$ with $[i(i+1)i]$ as a factor with no other repetitions. Assume there exist s_j, s_m such that $|s_j - s_m| = 1$ and $\{s_j, s_m\} \neq \{i, i+1\}$. Without loss of generality, we may assume $s_m = s_j + 1$.

If $[s_m]$ occurs to the left of $[s_j]$, then consider the image of s_j under π.

If $[s_j - 1]$ occurs as an element of $\mathbf{s}$ to the right of $[s_j]$, then s_j must be mapped to $s_j - k$ for some $k \geq 1$. This implies $[s_j - (k-1)]$, $[s_j - (k-2)]$, $\ldots$, $[s_j - 2]$ occur in that order to the left of $[s_j - 1]$. Now, if $[s_j - k]$ occurs to the right of $[s_j - (k-1)]$, then $s_j - k$ must be mapped to $s_j - (k + l)$ for some $l \geq 1$ and therefore π is not an involution. If $[s_j - k]$ does not occur to the right of $[s_j - (k-1)]$, then $s_j - k$ is mapped to $s_j - (k-1)$ and again π is not an involution.

If $[s_j - 1]$ does not occur as an element of $\mathbf{s}$ to the right of $[s_j]$, then s_j must be mapped to $s_j + k$ for some $k \geq 2$. This implies $[s_j + k - 1]$, $[s_j + k - 2]$, $\ldots$, $[s_j + 2]$ occur to the left of $[s_j + 1]$. If $[s_j + k]$ occurs to the right of $[s_j + k - 1]$, then $s_j + k$ must be mapped to $s_j + k + l$ for some $l \geq 1$ and hence π is not an involution. If not, $s_j + k$ must be mapped to $s_j + k - 1$ and again π is not an involution.

The argument for the case $[s_m]$ occurring to the right of $[s_j]$ is similar to the above argument. □

In order to count the number of involutions, we define sets similar to those constructed in section 3.

Let $I_j^i(n)$ be the sets of reduced decompositions as in the construction of $E_j^i(n)$ in the previous section with the additional property that the reduced decompositions represent involutions.

Table 2.4 gives the sets $I_j^i(7)$.

Constructing the sets $I_j^i(n)$

The sets $I_{j+1}^i(n)$ are constructed inductively from $\mathbf{s} \in I_j^i(n)$ and $\mathbf{s} \in I_{j-1}^i(n)$ as follows:

1. If $\mathbf{s} \in I_j^i(n)$, then add $\mathbf{s}$ to $I_{j+1}^i(n)$

Table 2.4: $I^i_j(7)$ (i - rows; j - cols)

	2	3	4	5	6
1	$\{[121]\}$	$\{[121]\}$	$\{[121],[4121]\}$	$\{[121],[4121],[5121]\}$	$\{[121],[4121],[5121],[6121],[64121]\}$
2	$\emptyset$	$\{[232]\}$	$\{[232]\}$	$\{[232],[5232]\}$	$\{[232],[5232],[6232]\}$
3	$\emptyset$	$\emptyset$	$\{[343],[1343]\}$	$\{[343],[1343]\}$	$\{[343],[1343],[6343],[61343]\}$
4	$\emptyset$	$\emptyset$	$\emptyset$	$\{[454],[1454],[2454]\}$	$\{[454],[1454],[2454]\}$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{[565],[1565],[2565],[3565],[13565]\}$

2. If $\mathbf{s} \in I^i_j(n) \cap I^i_{j-1}(n)$, then add $[j\mathbf{s}]$ to $I^i_{j+1}(n)$

Similarly to the case of general permutations, $I^1_2(n) = \{[121]\}$ for all n. The set $I^i_{i+1}(n)$ is constructed from $I^{i-1}_i(n)$ as follows:

1. If $\mathbf{s} = [s_1 \dots s_k]$ then add $[(s_1+1)(s_2+1)\dots(s_k+1)]$ to $I^i_{i+1}(n)$.
2. If $\mathbf{t}$ was added in step 1 and the element 2 does not occur in $\mathbf{t}$ then add $[1\mathbf{t}]$ to $I^i_{i+1}(n)$.

Lemma 2.3.19. *The above procedures correctly produce the sets $I^i_j(n)$ for all i and j.*

Proof. Similar to the proofs of Lemmas 2.3.10 and 2.3.12. □

Counting the involutions in $\mathcal{A}_n$

The number of inversions in $\mathcal{A}_n$ is given by $\sum_{i=1}^{n-2} |I^i_{n-1}(n)|$. Counting similarly to the case for permutations, define $c_i(k) := |I^i_{i+k}(n)|$ for $k \geq 0$. Note, $c_i(0) = 0$. The $c_i(k)$ satisfy the recurrence $c_i(k) = c_i(k-1) + c_i(k-2)$. Such a recurrence generates the Fibonacci numbers when the initial conditions are 0 and 1. Therefore, $|I^i_{n-1}(n)| = |I^i_{i+1}|F_{n-i-1}$.

Now define $d(i) := |I^i_{i+1}(n)|$. By the second procedure $d(i) = d(i-1) + d(i-2)$ and so $d(i) = F_i$. Therefore, the number of involutions in $\mathcal{A}_n$ is:

$$\sum_{i=1}^{n-2} |I^i_{n-1}(n)| = \sum_{i=1}^{n-2} |I^i_{i+1}(n)| \cdot F_{n-i-1} = \sum_{i=1}^{n-2} F_i F_{n-i-1}$$

.

The closed form of the above sum corresponds to Egge's result from [13] cited below.

Table 2.5: Number of Involutions in $\mathcal{A}_n$ for small n.

n	3	4	5	6	7	8	9	10	11	12
Involutions in $\mathcal{A}_n$	1	2	5	10	20	38	71	130	235	420

Theorem 2.3.20. *(Egge) The number of involutions in* $Av_n(3412)$ *that contain exactly one* 321 *pattern is*

$$\frac{2(n-1)F_n - nF_{n-1}}{5}$$

.

In the Online Encyclopedia of Integer Sequences this is sequence A001629 and is a very well studied sequence. Table 2.5 gives these numbers for the first few n.

Chapter 3

Reduced Decompositions with Two Repetitions

Theorem 2.2.29 characterizes reduced decompositions with precisely one element repeated and no other repetitions in terms of pattern conditions. In this chapter, we study the reduced decompositions with precisely two elements each repeated once, how pattern conditions characterize such reduced decompositions and obtain new counting results for pattern classes through that characterization.

3.1 Entangled Factors and Patterns

3.1.1 Definitions and Preliminary Results

In attempting to generalize the results of the previous chapter, one may ask whether to consider reduced decompositions with one element occurring exactly three times and no other repetitions or to consider reduced decompositions with two elements occurring exactly two times each and no other repetitions. Considering the former case is actually the same as considering a special subset of the latter case.

Theorem 3.1.1. *If $\pi \in S_n$ has a reduced decomposition $\mathbf{s} = [s_1 \dots s_k]$ with exactly* *one element $[i]$ occurring three times and no other repetitions then π has a reduced decomposition with exactly two elements occurring exactly twice.*

Proof. Consider a reduced decomposition $\mathbf{s}$ of π with one element occurring precisely three times. $\mathbf{s}$ has the form $[s_1 \ldots i \ldots i \ldots i \ldots s_k]$ and assume without loss of generality that the number of elements in the factor $\mathbf{f} = [i \ldots i \ldots i]$ is as small as possible (otherwise apply braid moves). Consider the element j_1 which is directly to the right of the leftmost occurrence of $[i]$. This element must not commute with $[i]$ for if it did, this would contradict minimality. Therefore, $j = i+1$ or $j = i-1$. If $j = i+1$, then the factor is $[i(i+1) \ldots i \ldots i]$. Now consider the element $[k]$ which is directly to the left of the rightmost $[i]$. $[k]$ cannot commute with $[i]$ and so must be $[i+1]$ or $[i-1]$. Since $[i]$ is the only element occurring more than once in the reduced decomposition, $k = i-1$ and so $\mathbf{f} = [i(i+1) \ldots i \ldots (i-1)i]$. Now consider the element to the right of $[i+1]$ in $\mathbf{f}$. Such an element cannot commute with both $[i]$ and $[i+1]$ and so must be either $[i+2]$ or $[i-1]$. Therefore $\mathbf{f} = [i(i+1)(i+2) \ldots i \ldots (i-1)i]$. Similarly the element to the left of $[i-1]$ in $\mathbf{f}$ must be $[i-2]$. Continuing this argument we conclude that $\mathbf{f} = [i(i+1)(i+2) \ldots (i+l)i(i-m) \ldots (i-2)(i-1)i]$ for some natural numbers $l, m \geq 1$. If $l > 1$ or $m > 1$, apply short braid moves to $\mathbf{f}$ to get a contradiction with minimality, so $\mathbf{f} = [i(i+1)i(i-1)i]$. Applying a long braid move gives $\mathbf{f}$ is equivalent to $[(i+1)i(i+1)(i-1)i]$ a factor with two elements occurring exactly twice. A similar argument works for the case $j = i-1$. $\square$

By Theorem 3.1.1, the study of reduced decompositions having exactly two repeated elements encompasses the study of reduced decompositions having exactly one element repeated three times. The opposite inclusion is not true. For example, consider the reduced decomposition [12321]. The set of reduced decompositions equivalent to [12321] is shown in graph form in Figure 1.1. None of the equivalent reduced decompositions have one element repeated three times and no other repetitions.

In chapter 2, understanding what was meant by "repetition" was fairly straightforward. Now, with Theorem 3.1.1, a "repetition" is a little more ambiguous. We now formalize what is meant by having a reduced decomposition have a certain number of "repetitions." We preface this material by noting that while the following material could easily have been presented in Chapter 2, it is not useful until now and so the presentation was delayed until Chapter 3.

Definition 3.1.2. *If* $\mathbf{s} = [s_1 \dots s_k]$ *is a reduced decomposition, then the* number of repetitions *is defined to be* k *minus the number of distinct symbols in* $\mathbf{s}$.

For example, given the reduced decomposition [3213543], the number of repetitions is $l([3213543]) - |\{1, 2, 3, 4, 5\}| = 7 - 5 = 2$. From now on, a repeated element will be meant in terms of the previous definition.

Proposition 3.1.3. *The number of repetitions of* $\mathbf{s} = [s_1 \dots s_k]$ *is an invariant among all equivalent reduced decompositions.*

Proof. Let d be the number of distinct symbols of $\mathbf{s}$, then the number of repetitions of $\mathbf{s}$ is $k - d$. It is enough to show that if we apply either a short braid move or a long braid move to $\mathbf{s}$ then the number of repetitions remains the same. The length is invariant under equivalent reduced decompositions, so we only need to show d does not change under the application of braid moves. Applying a short braid move only commutes two elements, so d does not change. Applying a long braid move changes $[i(i+1)i]$ into $[i(i+1)i]$ or vice versa. The number of distinct elements again does not change. □

Definition 3.1.4. *If* $\mathbf{s} = [s_1 \dots s_k]$ *is a reduced decomposition of* $\pi \in S_n$ *then a* repetition factor *of* $\mathbf{s}$ *is a factor* $[s_u \dots s_v]$ *of* $\mathbf{s}$ *with* $1 \leq u < v \leq k$ *such that all elements that occur more than once in* $\mathbf{s}$ *occur in the factor* $[s_u \dots s_v]$.

Here are some examples of repetition factors:

[232] is a repetition factor of [12324]. [2342] is a repetition factor of [12342]. [423452] is a repetition factor of [4234521].

Definition 3.1.5. *Let* $\mathbf{s} = [s_1 \dots s_k]$ *be a reduced decomposition. A repetition factor* $[s_u \dots s_v]$ *is* minimal *if* $v - u$ *is minimal among equivalent reduced decompositions for* $\mathbf{s}$*.*

Minimal repetition factors will play the same role as $[i(i+1)i]$ and $[i(i+1)(i-1)i]$ did in Chapter 2. In the above example, [232] is a minimal repetition factor for [12324], but [2342] is not minimal for [12342].

The classification of the structure of minimal repetition factors with two repetitions is one of the main objectives of this chapter.

3.1.2 Entangled Factors

Definition 3.1.6. *A minimal repetition factor* $[s_u \dots s_v]$ *with two repetitions is* entangled *if it is not equivalent to a factor of the form* $[i \dots i \dots j \dots j]$*. A minimal repetition factor with two repetitions is* unentangled *if it is equivalent to a factor of the form* $[i \dots i \dots j \dots j]$*.*

The intuition here is that entangled factors are those factors for which the two repeated elements occur in alternating order in every equivalent factor. For example, [12132] is entangled, but [214524] is not, as it is equivalent to the factor [212454].

Entangled factors are, in some sense, the special minimal repetition factors with two repetitions. In order to classify them, we will consider only the structural properties of the reduced decompositions. By that, we mean not only up to equivalence as reduced decompositions, but also in terms of the orderings of the elements. For example, $[i(i+1)i]$ is structurally the same as $[(i+2)(i+3)(i+2)]$. The classification of entangled factors begins here.

Entangled factors of length 5

The easiest entangled factors to classify are (not surprisingly) the factors of length 5. We shall classify the entangled factors by cases. First, consider the case where the two repeated elements are distance one apart.

In what follows, A, B and C are subfactors of the factor. Consider an entangled factor of length 5 of the form $[iA(i+1)BiC(i+1)]$ where A, B and C represent either the empty string or one element. Call this factor $\mathbf{f}$. Since the factor is of length 5, two of A, B and C must be empty. Note, we may assume B is empty because any element placed there may be commuted with either $[i]$ or $[i+1]$ into positions A or C. A is either empty or contains an element which cannot commute with $[i]$.

- Case 1: A is empty. In this case, C is not empty and so must contain an element which does not commute with $[i+1]$. The only option for this is $C = [i+2]$. Therefore, this case yields the factor
  $[i(i+1)i(i+2)(i+1)]$.

- Case 2: A contains an element which does not commute with $[i]$. The only such available element is $[i-1]$ and so the factor is
  $[i(i-1)(i+1)i(i+1)]$.

Structurally these two cases yield the same reduced decomposition as $[i(i+1)i(i+2)(i+1)] = [(i+1)i(i+1)(i+2)(i+1)] = [(i+1)i(i+2)(i+1)(i+2)]$ which is the same structurally as $[i(i-1)(i+1)i(i+1)]$.

By symmetry, we have that the only entangled factor of the form $[(i+1)AiB(i+1)Ci]$ is $[(i+1)i(i+1)(i-1)i]$.

Now, consider an entangled factor of the form $[(i+1)AiBiC(i+1)]$. If B is empty then the factor would not be reduced, so we must have A and C empty and an element in position B. That element cannot commute with $[i]$, so the only entangled factor of that form is $[(i+1)i(i-1)i(i+1)]$. Similarly, the only factor of the form $[iA(i+1)B(i+1)Ci]$ is $[i(i+1)(i+2)(i+1)i]$. Because $[i(i+1)(i+2)(i+1)i] = [i(i+2)(i+1)(i+2)i] = [(i+2)i(i+1)i(i+2)] = [(i+2)(i+1)i(i+1)(i+2)]$, there is really only one new entangled factor here.

At this point, the different entangled factors of length 5 are (up to equivalence by braid moves): $\{[i(i-1)(i+1)i(i+1)], [(i+1)i(i+1)(i-1)i], [(i+1)i(i-1)i(i+1)]\}$. Of course, there may be entangled factors where the distance between the two repeated elements is greater than one. We consider the same cases for distance two.

If $[iA(i+2)BiC(i+2)]$ is an entangled factor, then going through similar cases as before, we have that only $[i(i+2)(i+1)i(i+2)]$, $[(i+2)i(i+1)(i+2)i]$, and $[i(i+2)(i+1)(i+2)i]$ are possibilities. Each one has the same structure as one previously found, so new entangled factors are to be found.

Lastly, we consider the reduced decompositions of the form $[iAjBiCj]$ and $[iAjBjCi]$ for elements $[i]$ and $[j]$ such that $|i-j| > 2$.

- Case 1: $[iAjBiCj]$. B cannot be empty without the factor being unentangled, so B must be an element which cannot commute with $[i]$ and $[j]$. Since $|i-j| > 2$, no such element exists.

- Case 2: $[iAjBjCi]$. B cannot be empty without the $[j]$ elements cancelling contradicting the reduced property, so B is either $[j+1]$ or $[j-1]$; however, since $|i-j| > 2$, we have $[ij(j+1)ji] = [iij(j+1)j]$ or $[ij(j-1)ji] = [iij(j-1)j]$. Either one is a contradiction.

Theorem 3.1.7. *The entangled factors of length 5 (up to equivalence) with two repetitions are:* $[i(i-1)(i+1)i(i+1)]$, $[(i+1)i(i+1)(i-1)i]$, *and* $[(i+1)i(i-1)i(i+1)]$.

Proof. By the above construction, there cannot be any more factors of length 5. We only need to check that these three factors are not equivalent. To check this, consider the image of $i+1$ under each factor. The first factor sends $i+1$ to i. The second sends $i+1$ to $i-1$. The third sends $i+1$ to $i+1$. Since each factor sends $i+1$ to a different element, the three factors must be distinct. □

Entangled Factors of Length 6

We first consider the cases of having an entangled factor where the repeated elements are $[i]$ and $[i+1]$. The cases are similar to the length 5 cases.

- Case 1: $[iA(i+1)BiC(i+1)]$. If B contains one element, then that element is either $> i+1$ in which case it commutes with i or it is $< i$ in which case it commutes with $i+1$. If B contains two elements, $B = b_1b_2$, then if $b_1 > i+1$ and $b_2 < i$, $[b_1b_2] = [b_2b_1]$ and both would commute with i or $i+1$. If b_1 and b_2 are both greater than $i+1$ or both less than i, then they will commute with one of i or $i+1$. Therefore B can be assumed empty. If A contains two elements and C is empty then we must have $[i(i-1)(i-2)(i+1)i(i+1)]$. This factor is not minimal. A similar factor occurs if A is empty and C contains two elements. We must have A and C containing one element each. This gives the factor $[i(i-1)(i+1)i(i+2)(i+1)]$ and by symmetry the factor $[(i+1)(i+2)i(i+1)(i-1)i]$.

- Case 2: $[iA(i+1)B(i+1)Ci]$. B cannot be empty in this case, but can be assumed to contain only one element as any element other than $[i+2]$ will commute with $[i+1]$. Assuming A is nonempty, we have the factor $[i(i-1)(i+1)(i+2)(i+1)i]$. Note, by braid moves, we have $[i(i-1)(i+1)(i+2)(i+1)i] = [i(i+1)(i+2)(i+1)(i-1)i]$ which is the result if A is empty and C is nonempty.

- Case 3: $[(i+1)AiBiC(i+1)]$. B cannot be empty in this case, but can again be assumed to contain only one element which is $[i-1]$. By an argument similar to case 2, the only entangled factor possible is $[(i+1)(i+2)i(i-1)i(i+1)]$.

Therefore, we have four distinct entangled factors: $[i(i-1)(i+1)i(i+2)(i+1)]$, $[(i+1)(i+2)i(i+1)(i-1)i]$, $[i(i-1)(i+1)(i+2)(i+1)i]$, $[(i+1)(i+2)i(i-1)i(i+1)]$.

If the two repeated factors are $[i]$ and $[i+2]$, a similar case analysis gives the following four entangled factors : $[i(i-1)(i+2)(i+1)(i+2)i]$, $[(i+2)i(i-1)(i+1)i(i+2)]$, $[i(i+2)(i+1)(i+3)(i+2)i]$, and $[(i+2)(i+3)i(i+1)i(i+2)]$. The first two are equivalent via braid moves to $[i(i-1)(i+1)(i+2)(i+1)i]$ and the last two have the same order structure as factors equivalent to $[(i+1)(i+2)i(i-1)i(i+1)]$.

If an entangled factor of length 6 has two repeated elements $[i]$ and $[j]$ such that $|i-j| > 2$, then we have the usual three cases.

- Case 1: $[iAjBiCj]$. In this case, we may assume without loss of generality, that $i < j$. If $i > j$, then reverse the order of elements. If B is nonempty and contains two elements then B must be $(j-1)(j-2)$ and hence the factor is $[ij(j-1)(j-2)ij]$ which is equivalent to a factor of the form $[iAjBjCi]$ and will be discussed in case 2. If B contains one element then $[iAjBiCj]$ is equivalent to either a factor of the form $[iAjBjCi]$ or $[jAiBiCj]$ and will be discussed in either case 2 or 3. If B is empty, then $[iAjiCj] = [iAijCj]$ which is not entangled.

- Case 2: $[iAjBjCi]$ where $i < j$. B cannot be empty in this case without a contradiction with being reduced. If B contains two elements, the two elements must be $[j-1]$ and $[j+1]$ which gives a factor equivalent to $[ij(j-1)(j+1)ji]$. Since $|i-j| > 2$, $[i]$ commutes with $[j-1]$, so this factor is equivalent to $[iij(j-1)(j+1)j]$ which is not reduced. If B contains one element, then either A or C is empty. The element in B cannot commute with $[j]$, so the factor must be $[iAj(j \pm 1)ji]$ or $[ij(j \pm 1)jCi]$. In either case, the factor is unentangled.

- Case 3: $[iAjBjCi]$ where $i > j$. Similar to case 2 yielding no new entangled factors.

There are therefore no entangled factors of length 6 which have $[i]$ and $[j]$ repeated with $|i - j| > 2$.

Theorem 3.1.8. *The entangled factors of length 6 (up to equivalence) with two repetitions are:* $[i(i-1)(i+1)i(i+2)(i+1)]$, $[(i+1)(i+2)i(i+1)(i-1)i]$, $[i(i-1)(i+1)(i+2)(i+1)i]$, *and* $[(i+1)(i+2)i(i-1)i(i+1)]$.

Proof. As in the 5 case, the previous discussion means that there are no more than these 4 factors. To prove they are all distinct, consider where $i+1$ is mapped under each factor. The first maps $i+1$ to $i+3$. The second maps $i+1$ to $i-1$. The third maps $i+1$ to $i+1$. The fourth maps $i+1$ to i. Since all four factors map $i+1$ to distinct elements, the factors must all be distinct. □

Entangled Factors of Length 7

The strategy for length 7 is precisely the same as the strategy for lengths 5 and 6. Consider the cases for an entangled factor of length 7 where the two repeated elements are $[i]$ and $[i+1]$.

- Case 1: $[iA(i+1)BiC(i+1)]$. B can be assumed to be empty.
  - If A is nonempty, then the leftmost element of A is $[i-1]$ and the factor must be of the form $[i(i-1)A(i+1)iC(i+1)]$. (We abuse notation slightly here by writing A for the factor between $[i]$ and $[i+1]$ and for the factor between $[i-1]$ and $[i+1]$.) If C is empty, then the factor must be of the form $[i(i-1)(i-2)(i-3)(i+1)i(i+1)]$ which is not minimal and hence we obtain a contradiction. If C is not empty, then the factor must have the form $[i(i-1)A(i+1)iC(i+2)(i+1)]$. This means the factor must be either

$[i(i-1)(i-2)(i+1)i(i+2)(i+1)]$ or $[i(i-1)(i+1)i(i+3)(i+2)(i+1)]$. Both of these factors, however, are not minimal.

  - If A is empty, then the factor must be of the form $[i(i+1)i(i+3)(i+2)(i+1)]$ which is not minimal.

- Case 2: $[iA(i+1)B(i+1)Ci]$. B cannot be empty in this case and can be assumed to have only one element: $[i+2]$. (Any other element can be moved into factors A or C via braid moves). So the factor must be $[iA(i+1)(i+2)(i+1)Ci]$.

  - If A is nonempty, then the factor must be of the form $[i(i-1)A(i+1)(i+2)(i+1)Ci]$. If C is empty, then the factor must be $[i(i-1)(i-2)(i+1)(i+2)(i+1)i]$ which is not minimal. If C is not empty, then there must exist an element $[j]$ directly to the left of the rightmost $[i]$ such that $[j]$ does not commute with $[i]$. This implies $j \in \{i+1, i-1\}$ which is a contradiction.
  - If A is empty, then the factor must be of the form $[i(i+1)(i+2)(i+1)(i-2)(i-1)i]$ which is not minimal and a contradiction.

- Case 3: $[(i+1)AiBiC(i+1)]$. Similar to case 2 yielding no new entangled factors.

Therefore, there are no entangled factors of length 7 with two repeated elements of distance one. Consider the possibilities of an entangled factor of length 7 with two repeated elements of distance two.

- Case 1: $[iA(i+2)BiC(i+2)]$. If B is empty, then the factor is unentangled. Without loss of generality, B can be assumed to be $[i+1]$ making the factor $[iA(i+2)(i+1)iC(i+2)]$.

- If A is nonempty, then the factor must look like $[i(i-1)A(i+2)(i+1)iC(i+2)]$. If C is empty, then the factor is $[i(i-1)(i-2)(i+2)(i+1)i(i+2)]$ which is not minimal. If C is not empty, then the factor is $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$.
- If A is empty, then the factor must be $[i(i-1)(i+2)(i+1)i(i+4)(i+3)(i+2)]$ which is not minimal.

From Case 1, we have the entangled factor $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$. In the next cases, we will refer to this factor as $\mathbf{f}$.

- Case 2: $[iA(i+2)B(i+2)Ci]$. If B is empty, then the factor is not reduced and in this case, there are three choices for B: $[i+1]$ or $[i+3]$ or both.
  - If B is $[i+1]$, the factor is $[iA(i+2)(i+1)(i+2)Ci]$. If A is nonempty, then C must be empty and the factor must be $[i(i-1)(i-2)(i-3)(i+2)(i+1)(i+2)i]$ which is not entangled (or minimal). If A is empty, then a similar phenomenon occurs before the rightmost $[i]$.
  - If B is $[i+3]$, the factor is $[iA(i+2)(i+3)(i+2)Ci]$. If A is nonempty, then the factor must be either $[i(i-1)A(i+2)(i+3)(i+2)Ci]$ or $[i(i+1)A(i+2)(i+3)(i+2)Ci]$. In the former case, the minimality of the factor forces A or C to be $[i+1]$ giving either $[i(i-1)(i+1)(i+2)(i+3)(i+2)i]$ or $[i(i-1)(i+2)(i+3)(i+2)(i+1)i]$. Both of which are equivalent to $\mathbf{f}$. By symmetry, the latter case produces factors $[i(i+1)(i-1)(i+2)(i+3)(i+2)i]$ or $[i(i+1)(i+2)(i+3)(i+2)(i-1)i]$ both of which are also equivalent to $\mathbf{f}$.
  - If B is $[(i+3)(i+1)]$, the factor is $[iA(i+2)(i+3)(i+1)(i+2)Ci]$. The only possibility is $[i(i-1)(i+2)(i+3)(i+1)(i+2)i]$ which is equivalent to $\mathbf{f}$.

We have now found there is precisely one (up to equivalence) entangled factor of length 7 with repeated elements of distance two, namely $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$.

As the last case, let us consider the possible entangled factors with $[i]$ and $[j]$ repeated and $|i-j| > 2$.

- Case 1: $[iAjBiCj]$. Assume without loss of generality that $i < j$ (otherwise reverse the factor and use symmetry). If B is empty, then the factor is not entangled. If B is not empty, then $j = i+3$ or $i+4$ and the factor is either $[i(i+4)(i+3)(i+2)(i+1)i(i+4)]$, $[i(i-1)(i+3)(i+2)(i+1)i(i+3)]$ or $[i(i+3)(i+2)(i+1)i(i+4)(i+3)]$. None of these factors are entangled.

- Case 2: $[iAjBjCi]$ with $i < j$. If B is empty, then the factor is not reduced, so B is either $[j+1]$, $[j-1]$, or $[(j+1)(j-1)]$.

  – If B is $[j+1]$, then the factor is $[iAj(j+1)jCi]$. If either A or C is empty, the factor is not entangled. Therefore, the factor is either $[i(i+1)j(j+1)j(i-1)i]$ or $[i(i-1)j(j+1)j(i+1)i]$.

  – If B is $[j-1]$, a similar argument applies.

  – If B is $[(j+1)(j-1)]$, then the factor is $[iAj(j+1)(j-1)jCi]$ which means either A or C is empty as there is only one element left to place. It then follows that the factor is equivalent to $[iAij(j+1)(j-1)j]$ or $[j(j+1)(j-1)jiCi]$ which are not entangled.

- Case 3: $[iAjBjCi]$ with $i > j$. This case is similar to case 2 and yield no new reduced decompositions.

Theorem 3.1.9. *The only entangled factor of length 7 (up to equivalence) with two repetitions is* $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$.

Entangled Factors of Length > 7

The end is near in the classification of entangled factors with two repetitions. The goal in this section is to show that all entangled factors have now been found and there are no others. To show this, we consider the possibilities for an entangled factor of length greater than 7.

Consider a factor of length greater than 7 with $[i]$ and $[i+1]$ each occurring twice in the factor.

- Case 1: $[iA(i+1)BiC(i+1)]$. B can be assumed empty without loss of generality.
  - If A is empty, then the element directly to the left of the rightmost $[i+1]$ must be $[i+2]$. So, the factor is $[i(i+1)iC(i+2)(i+1)]$. The element directly to the left of $[i+2]$ cannot commute with both $[i+2]$ and $[i+1]$ and so must be $[i+3]$. Continuing in this fashion means the factor must be $[i(i+1)i(i+k)(i+k-1)\ldots(i+2)(i+1)]$ where $k \geq 5$ as the factor must be of length greater than 7. This factor is not minimal as it is equivalent to $[(i+k)(i+k-1)\ldots(i+3)i(i+1)i(i+2)(i+1)]$.
  - A similar situation occurs if C is empty.
  - If both A and C are nonempty, the element directly to the right of the leftmost $[i]$ must be $[i-1]$ and the element directly to the left of the rightmost $[i+1]$ must be $[i+2]$. Therefore, the factor must be $[i(i-1)\ldots(i-l)(i+1)i(i+k)\ldots(i+2)(i+1)]$ where $l \geq 1$, $k \geq 1$ and $l+k > 3$. This factor is equivalent to $[(i+k)\ldots(i+3)i(i-1)(i+1)i(i+2)(i+1)(i-2)\ldots(i-l)]$ and hence is not minimal.
- Case 2: $[iA(i+1)B(i+1)Ci]$. B cannot be empty without a contradiction with being reduced and B must be $[i+2]$ which makes the factor $[iA(i+1)(i+2)(i+1)Ci]$.

- If A is empty, the factor is $[i(i+1)(i+2)(i+1)Ci]$. This means the factor must be $[i(i+1)(i+2)(i+1)(i-l)(i-l+1)\dots(i-1)i]$ where $l \geq 3$. This factor is equivalent to $[(i-l)\dots(i-2)i(i+1)(i+2)(i+1)(i-1)i]$ which is not minimal.
- If C is empty, the argument is similar and the factor is not minimal.
- If both A and C are nonempty, then the element directly to the right of the leftmost $[i]$ must be $[i-1]$ and the element directly to the left of the rightmost $[i]$ must be $[i-1]$ as well. This is a contradiction with $[i]$ and $[i+1]$ being the only two repeated elements.

- Case 3: $[(i+1)AiBiC(i+1)]$. This case is similar to Case 2 and yield no entangled factors.

There are no entangled factors of length > 7 with $[i]$ and $[i+1]$ the only repeated elements. We now consider the possibilities for entangled factors of length > 7 with $[i]$ and $[i+2]$ the only repeated elements.

- Case 1: $[iA(i+2)BiC(i+2)]$. If B is empty, then the factor is not entangled, so B must be $[i+1]$.

  - If A is empty, then using arguments similar to the $[i]-[i+1]$ case, the factor must be $[i(i+2)(i+1)i(i+k)(i+k-1)\dots(i+3)(i+2)]$ where $k \geq 5$. This factor is equivalent to $[(i+k)\dots(i+4)i(i+2)(i+1)i(i+3)(i+2)]$ which is not minimal.
  - If C is empty, a similar argument yields again no entangled factors.
  - If A and C are not empty, the factor must be $[i(i-1)\dots(i-l)(i+2)(i+1)i(i+k)\dots(i+3)(i+2)]$ where $l \geq 1$, $k \geq 1$ and $l+k \geq 3$. This factor is equivalent to $[(i+k)\dots(i+4)i(i-1)(i+2)(i+1)i(i-2)\dots(i-l)]$ which is not minimal.

- Case 2: $[iA(i{+}2)B(i{+}2)Ci]$. B cannot be empty, so there are three possibilities for B: $[i+1]$, $[i+3]$ or $[(i+1)(i+3)]$.
  - If B is $[i+1]$, then the factor is $[iA(i+2)(i+1)(i+2)Ci]$. If A is empty, then the factor must be $[i(i+2)(i+1)i(i-l)(i-l+1)\dots(i-1)i]$ which is not minimal. Similarly if C is empty. If both A and C are nonempty, then $[i-1]$ must occur as both the first element of A and the last element of C which is a contradiction with $[i]$ and $[i+2]$ as the only repeated elements.
  - If B is $[i+3]$, then the argument is similar to above.
  - If B is $[(i+1)(i+3)]$, then the factor is $[iA(i+2)(i+1)(i+3)(i+2)Ci]$ and the argument follows similarly to the above.
- Case 3: $[(i+2)AiBiC(i+2)]$. Similar to Case 2.

There are no entangled factors of length > 7 with repeated elements of distance one or two. We now consider the three cases where $[i]$ and $[j]$ are repeated and $|i-j| > 2$.

- Case 1: $[iAjBiCj]$. Without loss of generality, we can assume $i < j$ (otherwise, reverse the string and use symmetry). If B is empty, then the factor is unentangled. B must be $[(j-1)(j-2)\dots(i+1)]$ and so the factor is $[iAj(j-1)(j-2)\dots(i+1)iCj]$.
  - If A is empty, then the factor is $[ij(j-1)\dots(i+1)iCj]$. C must be of the form $[(j+k)\dots(j+1)]$ and so the factor is not entangled. Similarly, the factor is not entangled if C is empty.
  - If both A and C are not empty, the factor is $[i(i-1)\dots(i-l)j(j-1)(j-2)\dots(i+1)i(j+k)(j+k-1)\dots(j+1)j]$. This factor is not minimal as it is equivalent to $[(j+k)\dots(j+2)i(i-1)j(j-1)\dots(i+1)i(j+1)j(i-2)\dots(i-l)]$.

- Case 2: $[iAjBjCi]$ where $i < j$. B cannot be empty and we have the usual three cases: B is $[j+1]$, $[j-1]$ or $[(j+1)(j-1)]$.
  - If B is $[j+1]$, then the factor is $[iAj(j+1)jCi]$. If either A or C is empty, then the factor is not entangled. If both A and C are nonempty, then there are two possibilities. Either $A = [(i+1)\dots(i+k)]$ where $i+k < j$ and $C = [(i-l)(i-l+1)\dots(i-1)]$ or $A = [(i-1)\dots(i-l)]$ and $C = [(i+k)\dots(i+1)]$ where $i+k < j$. In either case, the resulting factor is not entangled.
  - The other two cases are similar to the above.
- Case 3: $[iAjBjCi]$ where $i > j$. This case is similar to case 2.

The above discussion yields the following theorem.

Theorem 3.1.10. *There are no entangled factors of length > 7 with two repetitions.*

Combining Theorems 3.1.7, 3.1.8, 3.1.9, and 3.1.10 yields the following classification of entangled factors.

Theorem 3.1.11. *[Classification of Entangled Factors] The entangled factors (up to equivalence) with two repetitions are:*

$$[i(i-1)(i+1)i(i+1)], [(i+1)i(i+1)(i-1)i], [(i+1)i(i-1)i(i+1)],$$

$$[i(i-1)(i+1)i(i+2)(i+1)], [(i+1)(i+2)i(i+1)(i-1)i], [i(i-1)(i+1)(i+2)(i+1)i],$$

$$[(i+1)(i+2)i(i-1)i(i+1)] \text{ and } [i(i-1)(i+2)(i+1)i(i+3)(i+2)].$$

3.1.3 Connecting Entangled Factors and Patterns

The ultimate goal of classifying the entangled factors is to determine what having two repetitions mean in terms of pattern avoidance and containment. As a starting

Table 3.1: Permutations obtained from entangled factors

General Factor	Numeric Factor	Permutation
$[i(i-1)(i+1)i(i+1)]$	[21323]	3421
$[(i+1)i(i+1)(i-1)i]$	[32312]	4312
$[(i+1)i(i-1)i(i+1)]$	[32123]	4231
$[i(i-1)(i+1)i(i+2)(i+1)]$	[213243]	34512
$[(i+1)(i+2)i(i+1)(i-1)i]$	[342312]	45123
$[i(i-1)(i+1)(i+2)(i+1)i]$	[213432]	35142
$[(i+1)(i+2)i(i-1)i(i+1)]$	[342123]	42513
$[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$	[2143254]	351624

point, Table 3.1 calculates the permutations from each of the entangled factors if the smallest element in each factor is 1. It is important to note that of all permutations listed in Table 3.1, the only one that is not vexillary is 351624.

In order to connect these reduced decompositions more firmly to pattern criteria, we need a preliminary proposition and lemma.

Proposition 3.1.12. *Assume* $\mathbf{s} = [s_1 \dots s_k]$ *is a reduced decomposition such that the only elements that occur in* $\mathbf{s}$ *are* $\{[i], [i-1], \dots, [i-l]\}$ *and* $\{[j], [j+1], \dots, [j+m]\}$ *for* $i < j$ *and* $l, m \geq 0$*. Further if the elements in* $\{[i], [i-1], \dots, [i-l]\}$ *occur in that order in* $\mathbf{s}$ *and the elements in* $\{[j], [j+1], \dots, [j+m]\}$ *occur in that order in* $\mathbf{s}$*, then* $\mathbf{s}$ *is equivalent either to* $[i(i-1)\dots(i-l)j(j+1)\dots(j+m)]$ *or to* $[j(j+1)\dots(j+m)i(i-1)\dots(i-l)]$.

Proof. By induction on k. If $k = 1$, then the statement is vacuously true. Let $\mathbf{s} = [s_1 \dots s_k s_{k+1}]$ be a reduced decomposition of length $k+1$ satisfying the properties of the proposition. s_{k+1} is either $[j+m]$ or $[i-1]$. Assume first that $s_{k+1} = [j+m]$. There are two possibilities for $[s_1 \dots s_k]$ by the induction hypothesis.

- Case 1: $[s_1 \dots s_k] = [i(i-1)\dots(i-l)j(j+1)\dots(j+m-1)]$. This case gives that $[s_1 \dots s_k s_{k+1}] = [i(i-1)\dots(i-l)j(j+1)\dots(j+m-1)(j+m)]$ which gives the proposition.

- Case 2: $[s_1 \dots s_k] = [j(j+1)\dots(j+m-1)i(i-1)\dots(i-l)]$. If $m = 0$, then we have that $[s_1 \dots s_k s_{k+1}] = [i(i-1)\dots(i-l)j]$ and the proposition is done. If $m > 0$, then $[s_1 \dots s_k s_{k+1}] = [j(j+1)\dots(j+m-1)i(i-1)\dots(i-l)(j+m)]$. Let a be such that $0 \leq a \leq l$. Note that $j-i > 0$ by assumption. $|j+m-(i-a)| = |(j-i)+(m+a)| \geq |1+(m+a)| \geq 2$. Therefore, $[(j+m)(i-a)] = [(i-a)(j+m)]$ for all $0 \leq a \leq m$ and hence $[j(j+1)\dots(j+m-1)i(i-1)\dots(i-l)(j+m)] = [j(j+1)\dots(j+m)i(i-1)\dots(i-l)]$.

The case where $s_{k+1} = [i-l]$ is similar. □

Lemma 3.1.13. *Let* $\mathbf{s} = [s_1 \dots s_u \dots s_v \dots s_k]$ *where* $\mathbf{f} = [s_u \dots s_v]$ *is an entangled repetition factor. Assume* i *is the smallest number to appear in* $\mathbf{f}$ *and* j *is the largest number to appear in* $\mathbf{f}$. *Then the factor* $[s_{v+1} \dots s_k]$ *can be assumed to be* $[(i-1)(i-2)\dots(i-l)(j+1)(j+2)\dots(j+m)]$ *where* $l, m \geq 0$. *(If* $l = 0$, $[(i-1)(i-2)\dots(i-l)] = [\emptyset]$ *is meant. Similarly for* $m = 0$.*)*

Proof. Assume first that all elements s_w that can be moved to the left of $\mathbf{f}$ via braid moves are so moved. Note that in any entangled factor, all elements between i and j occur, so the only elements that can occur to the right of $\mathbf{f}$ are either less than i or greater than j. s_{v+1} must be either $[i-1]$ or $[j+1]$ for if s_{v+1} is any other element it will commute with all elements between $[i]$ and $[j]$. s_{v+2} either cannot commute with elements between $[i]$ and $[j]$ or it cannot commute with s_{v+1} and so must be one of $[i-1]$, $[i-2]$, $[j+1]$, or $[j+2]$. Continuing this line of reasoning, the factor $[s_{v+1} \dots s_k]$ must exactly a factor of the form described in Proposition 3.1.12. By Proposition 3.1.12 and noting that $[(i-1)(i-2)\dots(i-l)(j+1)\dots(j+m)] = [(j+1)\dots(j+m)(i-1)\dots(i-l)]$ for $i < j$, the lemma is proven. □

Entangled Factors of Length 5 and Patterns

Lemma 3.1.14. *If* $\pi \in S_n$ *contains exactly two* 321 *patterns of the form* 3421 *and avoids* 3412, *then* π *has a reduced decomposition with* $[i(i-1)(i+1)i(i+1)]$ *as a* *factor and no other repetitions.*

Proof. 3421 is vexillary, so Theorem 2.1.3 applies and so there exists a reduced decomposition of π with a factor of the form $[i(i-1)(i+1)i(i+1)]$. To show there are no other repetitions requires checking exactly the same type of details as in the proof of Proposition 2.2.13 and so are omitted here. □

Lemma 3.1.15. *If* $\pi \in S_n$ *has a reduced decomposition* $\mathbf{s}$ *with* $[i(i-1)(i+1)i(i+1)]$ *as a factor and no other repetitions, then* π *contains exactly two* 321 *patterns of the form* 3421.

Proof. Since there are no other repetitions $[i(i-1)(i+1)i(i+1)]$ is an isolated factor, Theorem 2.1.5 gives that π contains 3421. The only fact that needs to be verified is that there are no other 321 patterns in π. Assume xyz is another 321 pattern in π. By Lemma 2.2.23, y must turn during its trajectory. Assume elements $abcd$ are in positions $i-1$, i, $i+1$, and $i+2$ directly before the factor $[i(i-1)(i+1)i(i+1)]$ is applied. The trajectories of these four elements is shown in Figure 3.1 and note the only element that turns is b.

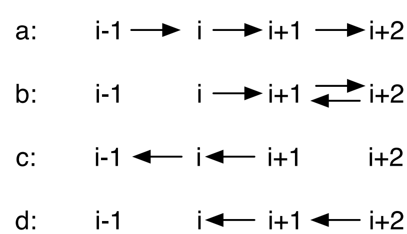

Figure 3.1: Trajectory of 3421

Recall via Lemma 2.2.7 that occurrences of pairs of arrows is equivalent to repetitions in reduced decompositions. From the trajectories, b must be i and c must be $i+1$ otherwise there would be more arrows in those positions and hence more repetitions. Since $b = i$ is the only element that turns, y must be $b = i$ in the additional 321 pattern. This implies there must be an element, not equal to a, c, or

d, that is less than i which is mapped to a position greater than $i+1$ or an element greater than $i+1$, not equal to a, c, or d, which is mapped to the left of position i which is a contradiction with the number of repetitions. □

Lemma 3.1.16. *If $\pi \in S_n$ has a reduced decomposition* $\mathbf{s}$ *with* $[i(i-1)(i+1)i(i+1)]$ *as a factor and no other repetitions, then π avoids* 3412*.*

Proof. By Lemma 3.1.15, we know π contains precisely two 321 patterns which form a 3421 pattern. The proof will be by induction on the length k of the reduced decomposition $\mathbf{s}$. If $k = 5$, then the permutation is the identity everywhere except on positions $i-1$, i, $i+1$ and $i+2$ which has a 3421 pattern. Such a permutation avoids 3412. Assume that all reduced decompositions with $[i(i-1)(i+1)i(i+1)]$ as a factor and no other repetitions of length k avoid 3412. Consider two cases.

- Case 1: $\mathbf{s} = [s_1 \dots i(i-1)(i+1)i(i+1)]$. Consider the reduced decomposition $\mathbf{s}$ without the last element: $[\dots i(i-1)(i+1)i]$. By Theorem 2.2.29, the permutation π represented by this permutation contains precisely one 3412 and avoids 321. Let $abcd$ be the elements in positions $i-1$ through $i+2$ respectively before the application of the factor $[i(i-1)(i+1)i]$. After the application of the factor, the permutation has $cdab$ in those positions and from Lemma 2.2.2, it follows that $a < c$, $b < c$, $a < d$, and $b < d$. Since $[i+2]$ follows in the reduced decomposition, we also have $a < b$. If $d < c$, then $cdab$ is 4312 before the application of the final $[i+1]$. This means that the permutation represented by $\mathbf{s}$ has a 4321 pattern which is a contradiction with Lemma 3.1.15. Therefore, $c < d$ and $cdab$ is the 3412 pattern guaranteed by Theorem 2.2.29. Therefore, the application of the final $[i+1]$ transforms 3412 into 3421.

  The question that remains is whether or not $[i+1]$ causes another 3412 pattern to be formed. Assume there exists a 3412 pattern. Then $[s_1 \dots i(i-1)(i+1)i]$

must have a 3142 pattern as every transposition in a reduced decomposition must transpose a pair of consecutive increasing elements into a pair of consecutive decreasing elements. In the 3412 pattern, the only pair of consecutive decreasing elements is 41.

Since $[i+1]$ changes ab into ba, ab must be the "41" of the 3142 pattern, which implies there exists an element x to the right of a such that abx is the "142" of the 3142 pattern. In the permutation represented by $\mathbf{s}$, $cdba$ is a 3421 pattern and bax is a 312 pattern. This implies $a < b < c < d$ and $a < x < b$ which means cbx and dbx are 321 patterns which is a contradiction with Lemma 3.1.15.

- Case 2: $\mathbf{s} = [s_1 \dots i(i-1)(i+1)i(i+1)\dots s_k]$ (i.e. the factor does not end with $[i+1]$). By Lemma 3.1.13, we may assume without loss of generality that the factor $[(i+1)\dots s_k]$ is $[(i+1)(i+2)(i+3)\dots(i+m)]$ or $[(i+1)(i-2)(i-3)\dots(i-l)$ for some $m \geq 2$ or $l \geq 2$. The two cases are handled similarly, so the $[i+m]$ case will be shown here. Assume $cdba$ is the result of applying the factor $[i(i-1)(i+1)i(i+1)]$. By the induction hypothesis $[s_1 \dots i(i-1)(i+1)i(i+1)(i+2)(i+3)\dots(i+m-1)]$ contains 3421 and avoids 3412. If $[s_1 \dots i(i-1)(i+1)i(i+1)\dots(i+m-1)(i+m)]$ contains 3412, then the permutation represented by $[i(i-1)(i+1)i(i+1)(i+2)(i+3)\dots(i+m-1)]$ contains 3142 as noted in the previous case. Note the application of $[(i+1)(i+2)\dots(i+m-1)]$ will move a into position $i+m$ in the permutation. Let π be the permutation represented by $[s_1 \dots i(i-1)(i+1)i(i+1)(i+2)(i+3)\dots(i+m-1)]$. $\pi = \dots cdb \dots ax \dots$. Note that ax is the "41" of the 3142 pattern in π. Since there are no repeated elements other than those occurring in the entangled factor, $\{\pi_{i+m}, \pi_{i+m+1}, \dots, \pi_n\} = \{i+m, i+m+1, \dots, n\}$. We also have that $a \leq i$ since there are no other repeated elements. This is most easily seen from the trajectory picture in Figure 3.1 as there would be more repeats in the reduced decomposition if not). Let y be the '2' element in the 3142 pattern in π. This means that $y > i + m$. $\pi(i + m) = \ldots cdb \ldots xa \ldots y \ldots$. Consider b, c and d. We know $b = i$ and $c = i + 1$ by the previous case. Since $m \geq 2$, then we must have $d = i + 2$ as having $d > i + 2$ requires more arrows in the trajectory and hence more repetitions. The '3' element of the 3412 pattern of $\pi(i + m)$ must be greater than i+m since $y > i + m$, but no element to the left of position $i + m$ is greater than $i + m$ which is a contradiction.

Since both cases lead to a contradiction there is no 3412 pattern in $\mathbf{s}$. □

Lemmas 3.1.14, 3.1.15, and 3.1.16 give the following theorem.

Theorem 3.1.17. *$\pi \in S_n$ contains exactly two* 321 *patterns of the form* 3421 *and avoids* 3412 *if and only if there exists a reduced decomposition for π with $[i(i-1)(i+1)i(i+1)]$ as a factor for some i and no other repetitions.*

By similar methods as above, the following two theorems complete the pattern classification of the entangled factors of length 5.

Theorem 3.1.18. *$\pi \in S_n$ contains exactly two* 321 *patterns of the form* 4312 *and avoids* 3412 *if and only if there exists a reduced decomposition for π with $[(i+1)i(i+1)(i-1)i]$ as a factor for some i and no other repetitions.*

Theorem 3.1.19. *$\pi \in S_n$ contains exactly two* 321 *patterns of the form* 4231 *and avoids* 3412 *if and only if there exists a reduced decomposition for π with $[(i+1)i(i-1)i(i+1)]$ as a factor for some i and no other repetitions.*

It is interesting here to note an additional connection to the literature.

Definition 3.1.20. *A permutation π is* freely-braided *if and only if it avoids the four patterns* 4321, 3421, 4231 *and* 4312.

Such permutations have been studied in [18] and [19]. In addition, the generating function for these permutations was found in [26].

Entangled Factors of Length 6 and Patterns

34512, 45123, 35142 and 42513 are all vexillary and so Theorems 2.1.3 and 2.1.5 apply to reduced decompositions of permutations involving these patterns. The following theorems follow in exactly the same way as the previous three theorems. The details are omitted.

Theorem 3.1.21. *$\pi \in S_n$ contains exactly three 3412 patterns of the form 34512 and avoids 321 if and only if there exists a reduced decomposition for π with $[i(i-1)(i+1)i(i+2)(i+1)]$ as a factor for some i and no other repetitions.*

Theorem 3.1.22. *$\pi \in S_n$ contains exactly three 3412 patterns of the form 45123 and avoids 321 if and only if there exists a reduced decomposition for π with $[(i+1)(i+2)i(i+1)(i-1)i]$ as a factor for some i and no other repetitions.*

Theorem 3.1.23. *$\pi \in S_n$ contains exactly one 3412 and exactly one 321 pattern of the form 35142 if and only if there exists a reduced decomposition for π with $[i(i-1)(i+1)(i+2)(i+1)i]$ as a factor for some i and no other repetitions.*

Theorem 3.1.24. *$\pi \in S_n$ contains exactly one 3412 and exactly one 321 pattern of the form 42513 if and only if there exists a reduced decomposition for π with $[(i+1)(i+2)i(i-1)i(i+1)]$ as a factor for some i and no other repetitions.*

The Entangled Factor of Length 7 and Patterns

In this section, the goal is to connect the entangled factor $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$ with the pattern 351624 which contains exactly two 3412 patterns and avoids 321. 351624 is not vexillary, so Theorem 2.1.3 does not apply. One direction of the classification is proven similarly to previous classification and so the proof is omitted here.

Lemma 3.1.25. *If there exists a reduced decomposition for* π *with* $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$ *with no other repetitions, then* π *avoids* 321 *and contains precisely two* 3412 *patterns of the form* 351624*.*

For the other direction, it suffices to prove that if π avoids 321 and contains precisely two 3412 patterns of the form 351624 then π must have a reduced decomposition with a factor of the form $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$. The fact that such a reduced decomposition has no other repetitions follows from an analysis similar to what has been shown in chapter 2.

In order to produce such a reduced decomposition, we first consider the properties of a permutation that avoids 321 and contains exactly two 3412 patterns of the form 351624.

Lemma 3.1.26. *Suppose* π *avoids* 321 *and contains exactly two* 3412 *patterns of the form* 351624*, then the* 5162 *pattern occurs in consecutive elements in* π*.*

Proof. Let π be a permutation satisfying the above conditions.
Let $\pi = \pi_1 \ldots \pi_c \ldots \pi_e \ldots \pi_a \ldots \pi_f \ldots \pi_b \ldots \pi_d \ldots \pi_n$ where $\pi_c\pi_e\pi_a\pi_f\pi_b\pi_d$ forms the 351624 pattern (i.e. $\pi_a < \pi_b < \pi_c < \pi_d < \pi_e < \pi_f$). We wish to show that $\pi_e\pi_a\pi_f\pi_b$ form a consecutive sequence in π.

- Assume there exists a number g in between π_e and π_a in π. If $g < \pi_a$, then $\pi_c\pi_e g\pi_a$ forms another 3412 pattern which is a contradiction. If $\pi_a < g < \pi_e$, then $\pi_e g\pi_a$ forms a 321 pattern which is a contradiction. If $\pi_e < g < \pi_f$, then $\pi_e g\pi_a\pi_b$ forms a 3412 pattern. If $g > \pi_f$, then $g\pi_f\pi_b$ forms a 321. Since all possibilities for g lead to a contradiction, we have that there is no number in between π_e and π_a in π.

- Assume there exists a number g in between π_a and π_f in π. If $g < \pi_a$, then $\pi_c\pi_e g\pi_b$ is a 3412 pattern. If $\pi_a < g < \pi_c$, then $\pi_c\pi_e\pi_a g$ forms a 3412 pattern. If $\pi_c < g < \pi_e$ then $\pi_e g\pi_b$ is a 321 pattern. If $\pi_e < g < \pi_f$, then $g\pi_f\pi_b\pi_d$ is a 3412 pattern. If $g > \pi_f$, then $g\pi_f\pi_b$ is a 321 pattern. Since all possibilities for g lead to a contradiction, we have that there is no number in between π_a and π_f in π.

- Assume there exists a number g in between π_f and π_b in π. If $g < \pi_a$, then $\pi_c\pi_a g$ is a 321 pattern. If $\pi_a < g < \pi_b$, then $\pi_e\pi_f g\pi_b$ is a 3412 pattern. If $\pi_b < g < \pi_f$, then $\pi_f g\pi_b$ is a 321 pattern. If $g > \pi_f$, then $\pi_f g\pi_b\pi_d$ is a 3412 pattern. Since all possibilities for g lead to a contradiction, we have that there is no number in between π_f and π_b in π.

This proves the lemma and shows $\pi = \pi_1 \dots \pi_c \dots \pi_e\pi_a\pi_f\pi_b \dots \pi_d \dots \pi_n$. □

Using the terminology of the previous lemma, we now must consider the elements in π to the left of π_c, in between π_c and π_e, in between π_b and π_d, and to the right of π_d.

Let π be as in the previous lemma and structure π as follows: let $\pi = W\pi_c X\pi_e\pi_a\pi_f\pi_b Y\pi_d Z$ where $W = w_1 \dots w_{k_1}$, $X = x_1 \dots x_{k_2}$, $Y = y_1 \dots y_{k_3}$ and $Z = z_1 \dots z_{k_4}$. We must determine the allowable orderings of elements in W, X, Y and Z if they exist.

First, consider X. If $k_2 \neq 0$, then $x_i < \pi_a$ for all $1 \leq i \leq k_2$. If that were not the case, then if $\pi_a < x_i < \pi_c$, $\pi_c x_i\pi_a$ would be a 321 pattern and if $x_i > \pi_c$ then $\pi_c x_i\pi_a\pi_b$ would be a 3412 pattern. Also, if $i < j$, then $x_i < x_j$ for all $1 \leq i, j \leq k_2$. If this were not the case, then $\pi_c\pi_i\pi_j$ would be a 321 pattern.

Similarly, in considering Y, we must have $y_i > \pi_f$ for all $1 \leq i \leq k_3$ and $y_i < y_j$ implying $i < j$ for all $1 \leq i, j \leq k_3$.

Now, consider W. If $k_2 = 0$, $w_i < \pi_b$ for all $1 \leq i \leq k_1$. If that were not the case, then if $w_i > \pi_c$, $w_i\pi_c\pi_a$ would be a 321 pattern and if $\pi_b < w_i < \pi_c$, then $w_i\pi_c\pi_a\pi_b$ would be a 3412 pattern. If $k_2 \neq 0$, then $w_i < \pi_a$ for all $1 \leq i \leq k_1$. If not, then $\pi_a < w_i < \pi_b$ and $w_i\pi_c x_j \pi_a$ would be a 3412 pattern.

Similarly, in considering Z, we must have $z_i > \pi_e$ for all $1 \leq i \leq k_4$ if $k_3 = 0$ and $z_i > \pi_f$ for all $1 \leq i \leq k_4$ if $k_3 \neq 0$.

It follows that π_b, π_c, π_d, and π_e are consecutive numbers (i.e. if $\pi_b = j$, then $\pi_c = j+1$, $\pi_d = j+2$, and $\pi_e = j+3$).

The main tool we have for constructing reduced decompositions from permutations is to compute the graph of the permutation and apply the algorithm described in Chapter 1. To construct the reduced decomposition with the appropriate factor that we require, we will graph the permutation in two cases: when $k_2 = 0$ and $k_2 \neq 0$. Figure 3.2 gives a partial graph of π and it is useful to refer to the figure when following the argument. The graph is partial because only the elements less than or equal to b have been graphed.

When $k_2 = 0$, π_c and π_e are consecutive elements in π. Note that all numbers that are less than π_b occur in π to the left of position c with the exception of π_a. Therefore, when we graph the permutation π there will be precisely two blank spaces which will be filled with c and $c+1$ respectively. In position $e = c+1$, the only numbers with blanks will be π_d, π_b and π_a. These blanks will be filled with $c+1$, $c+2$, and $c+3$ respectively. In position $a = c+2$, there are no blanks at all. In position $f = c+4$, there will be at least two blanks: those in positions π_d, and π_b. There may be more in positions greater than π_e. These blanks will be filled with $c+3$, $c+4$, $\ldots$, $c+k$ for some $k \geq 4$. Figure 3.2 shows the case where $k = 4$.

Therefore, we may read a factor of the reduced decomposition from the positions c, e, a, and f as: $[(c+1)c(c+3)(c+2)(c+1)(c+k)(c+k-1)\ldots(c+4)(c+3)]$. Using braid moves, we have that this reduced decomposition is equivalent to $[(c+k)(c+k-1)\ldots(c+5)(c+1)c(c+3)(c+2)(c+1)(c+4)(c+3)]$. This means that

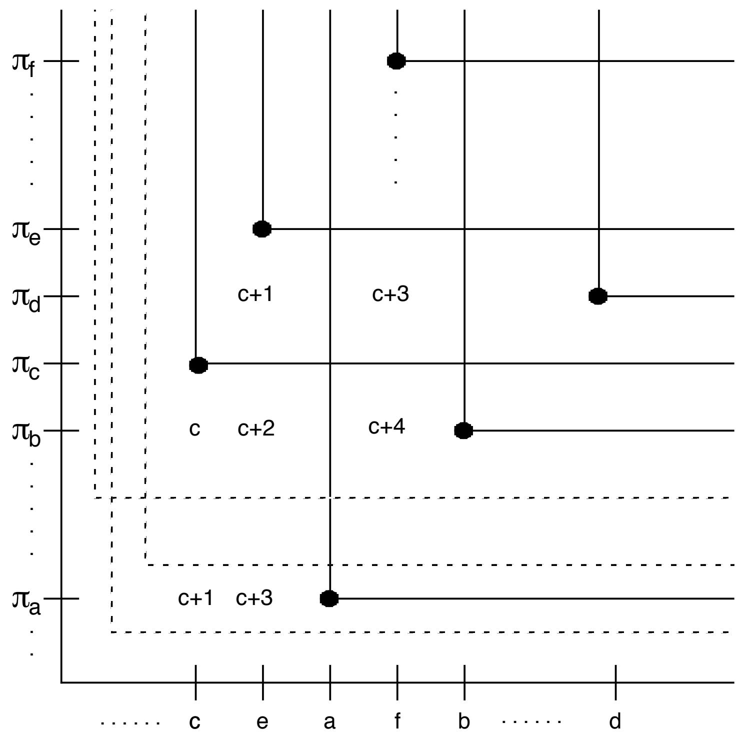

Figure 3.2: Partial graph of π. $k_2 = 0$

π has a reduced decomposition with $[(c+1)c(c+3)(c+2)(c+1)(c+4)(c+3)]$ as a factor. Taking $i = c+1$ gives precisely the required factor.

When $k_2 \neq 0$, this means there is an increasing sequence of numbers occurring in between π_c and π_e that are all less than π_a. Figure 3.3 gives a picture of this situation.

All numbers x such that $\pi_a < \pi_b$ must occur in π to the left of position c. All numbers x that are less than π_a must occur to the left of position e. Assume $e = c+k$ for some integer $k \geq 2$. Note $k \neq 1$ since $k_2 \neq 0$.

Now consider the graph of the permutation π. In position c, there will be blank spaces at π_b, π_a and at least one element less than π_a. There will be precisely as many blank spaces below π_a as there are elements in between c and e. Since $e = c+k$, there must be $k-1$ such blanks. Therefore, the algorithm will fill those blanks with

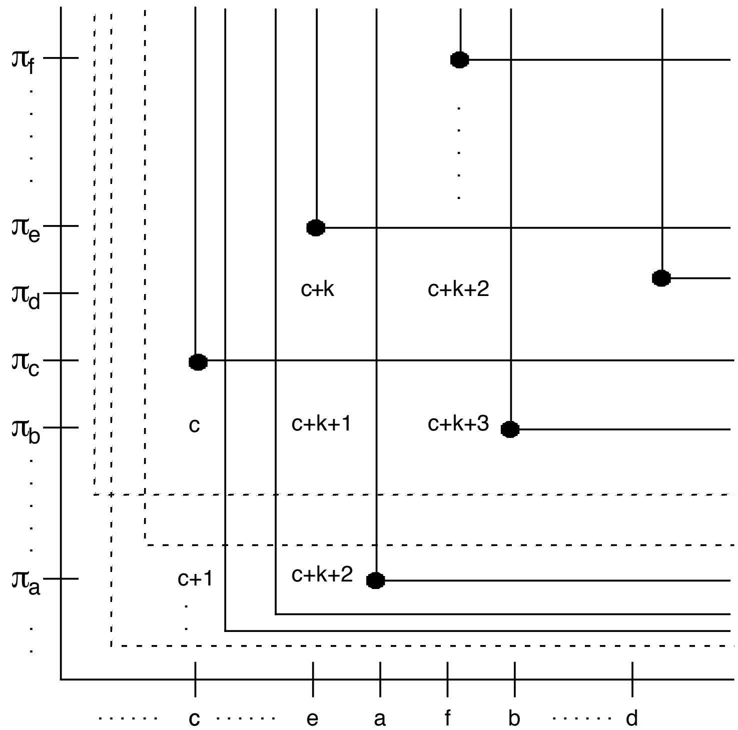

Figure 3.3: Partial graph of π. $k_2 \neq 0$

c, $c+1$, $c+2$, $\ldots$, $c+k$. If x is such that $c < x < e$, there will be no blank spaces at position x in the graph as the x's form an increasing sequence. At position e, there will be precisely three blank spaces at elements π_d, π_b and π_a. These will be filled with $c+k$, $c+k+1$ and $c+k+2$ respectively. In position $a = c+k+1$, there will be no blanks. In position $f = c+k+2$, there will be two blank spaces: π_d and π_b which the algorithm will fill with $c+k+2$ and $c+k+3$ respectively. Reading the graph from positions c to b gives the factor $[(c+k)(c+k-1)\ldots(c+1)c(c+k+2)(c+k+1)(c+k)(c+k+3)(c+k+2)]$ which is equivalent to the factor $[(c+k)(c+k-1)(c+k+2)(c+k+1)(c+k)(c+k+3)(c+k+2)(c+k-2)\ldots(c+1)c]$ which has $[(c+k)(c+k-1)(c+k+2)(c+k+1)(c+k)(c+k+3)(c+k+2)]$ as a subfactor. Taking $c+k = i$ gives the desired factor.

The above discussion proves the following lemma:

Lemma 3.1.27. *If* π *avoids* 321 *and contains exactly two* 3412 *patterns of the form* 351624*, then* π *has a reduced decomposition with* $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$ *as a factor.*

To show there are no other repetitions requires a similar analysis to previous theorems and so the detail here is omitted.

Lemmas 3.1.25, 3.1.27 and the additional analysis to show no other repetitions give the following theorem:

Theorem 3.1.28. $\pi \in S_n$ *avoids* 321 *and contains exactly two* 3412 *patterns of the form 351624 if and only if* π *has a reduced decomposition with* $[i(i-1)(i+2)(i+1)i(i+3)(i+2)]$ *as a factor and no other repetitions.*

This completes the classification of entangled factors.

3.2 Unentangled Factors and Patterns

We now turn our attention to the study of unentangled factors and what such factors tell us about patterns.

Ultimately, the goal is to classify not only the reduced decompositions with two repetitions, but also to classify the permutations in S_n that contain exactly two copies of 321 and avoid 3412, contain exactly one 321 and exactly one 3412, and contain exactly two copies of 3412 and avoid 321.

In order for a permutation to contain exactly two copies of 321 and avoid 3412, we may have that the two copies share two of the three elements forming the 321 patterns, that the two copies share one element, or that the two copies are disjoint. For a permutation of length n to contain exactly two copies of 321 sharing two elements, there must be precisely four elements that make up the two 321 patterns. Similarly, if a permutation contains exactly two copies of 321 sharing one element, there must be precisely five elements that make up the two 321 patterns and again

Table 3.2: Permutations containing exactly two 321 patterns and avoiding 3412

Number of Elements Shared	Permutations
2	3421, 4312, 4231
1	32541, 52143
0	321654, 326154, 421653

Table 3.3: Permutations containing exactly one 321 and exactly one 3412

Number of Elements Shared	Permutations
2	35142, 42513
1	325614, 341652, 361254, 521634
0	3216745, 3261745, 3412765, 3417265, 3512764, 4216735

sharing no elements means there must be precisely six elements that make up the two 321 patterns. One can therefore exhaust all possible means of containing precisely two 321 patterns and avoiding 3412 by considering permutations of length 4, 5 and 6. Table 3.2 gives all of the possible permutations of these lengths containing exactly two 321 patterns and avoiding 3412. Any permutation of greater length containing two 321 patterns and avoiding 3412 must necessarily contain one of these patterns. Similar data is shown in Tables 3.3 and 3.4.

3.2.1 Structure of Unentangled Factors

An unentangled factor is a minimal factor of the form $[i \ldots i \ldots j \ldots j]$ with no other repetitions than i and j. Any minimal factor of the form $[i \ldots i]$ with no other repetitions is equivalent to a factor of the form $[i(i+1)i]$ or $[i(i+1)(i-1)i]$. Therefore, an unentangled factor is equivalent to a factor of the form $[i(i+1)i \ldots j(j+1)j]$, $[i(i+1)i \ldots j(j+1)(j-1)j]$, $[i(i+1)(i-1)i \ldots j(j+1)j]$, or $[i(i+1)i \ldots j(j+1)(j-1)j]$.

Table 3.4: Permutations containing exactly two 3412 patterns and avoiding 321

Number of Elements Shared	Permutations
2	351624
1	3416725, 3612745
0	34127856, 34172856, 35127846

Definition 3.2.1. *We will call factors equivalent to one of :* $[i(i+1)ij(j+1)j]$, $[i(i+1)(i-1)ij(j+1)j]$, $[i(i+1)ij(j+1)(j-1)j]$, *and* $[i(i+1)(i-1)ij(j+1)(j-1)j]$ elementary unentangled factors.

Lemma 3.2.2. *Let* $[i*is_1\dots s_kj*j]$ *be a non-elementary unentangled factor where* $[i*i]$ *is equal to* $[i(i+1)i]$ *or* $[i(i+1)(i-1)i]$ *(similar for* j*) and assume* $i<j$*. Let* $S=\{i+2,i+3,\dots j-1\}$ *if* $[j*j]=[j(j+1)j]$ *and* $\{i+2,i+3,\dots j-2\}$ *otherwise. If there exists* m*,* $1\leq m\leq k$*, such that* $s_m\notin S$*, then* $[i*is_1\dots s_kj*j]$ *is not minimal.*

Proof. If there exists m, $1\leq m\leq k$, such that $s_m>j+1$, choose s_m such that $s_m\geq s_t$ for all $1\leq t\leq k$. In such a case, the element s_m-1 may exist either to the left or to the right of s_m. On the other side, all s_t will be strictly less than s_m-1 as s_m was chosen to be the largest element of the factor. Hence $[s_m]$ commutes with all $[s_t]$ on that side of s_m and so $[i*is_1\dots s_kj*j]=[s_mi*is_1\dots\hat{s_m}\dots s_kj*j]$ or $[i*is_1\dots\hat{s_m}\dots s_kj*js_m]$ and we have the proposition.

If there exists m, $1\leq m\leq k$, such that s_m is less than the smallest element in $[i*i]$ (either i or $i-1$), then choose s_m such that $s_m\leq s_t$ for all $1\leq t\leq k$. In such a case, the element s_m+1 may exist to one side of s_m but to the other side all elements will be strictly greater than s_m+1 and so s_m will commute with all of these elements and so again we have the proposition. □

We have a similar lemma if $i>j$.

Lemma 3.2.3. *Let* $[i*is_1\dots s_kj*j]$ *be a non-elementary unentangled factor where* $[i*i]$ *is equal to* $[i(i+1)i]$ *or* $[i(i+1)(i-1)i]$ *(similar for* j*) and assume* $i>j$*. Let* $S=\{j+2,j+3,\dots i-1\}$ *if* $[j*j]=[j(j+1)j]$ *and* $\{j+2,j+3,\dots i-2\}$ *otherwise. If there exists* m*,* $1\leq m\leq k$*, such that* $s_m\notin S$*, then* $[i*is_1\dots s_kj*j]$ *is not minimal.*

Proof. Similar to Lemma 3.2.2. □

Lemma 3.2.4. *Let $[i*is_1 \dots s_k j*j]$ be a non-elementary unentangled factor where $[i*i]$ is equal to $[i(i+1)i]$ or $[i(i+1)(i-1)i]$ (similar for j) and assume $i<j$. If $\{i+2, i+3, \dots, j-1\} \not\subseteq \{s_1 \dots s_k\}$, then $[i*is_1 \dots s_k j*j]$ is not minimal.*

Proof. By assumption, there exists $i+m$ such that $i+1 \leq i+m \leq j-1$ such that $i+m \notin \{s_1 \dots s_k\}$. Choose m such that $i+m$ is the smallest member of $S \setminus \{s_1 \dots s_k\}$.

If $m \geq 3$, consider $s_t = i+m-1$. Since $i+m \notin \{s_1 \dots s_k\}$, either the factor $[i*is_1 \dots s_{t-1}]$ or the factor $[s_{t+1} \dots s_k j*j]$ where $i+m-2$ does not appear. In the factor missing $i+m-2$, all the elements are either less than $i+m-2$ or greater than $i+m$. Hence $[s_t]$ commutes with all elements in that factor and so $[i*is_1 \dots s_k j*j] = [s_t i*is_1 \dots \hat{s}_t \dots s_k j*j]$ or $[i*is_1 \dots \hat{s}_t \dots s_k j*js_t]$ and so the factor is not minimal.

If $m=2$, then consider s_1. $s_1 \geq i+3$ and so s_1 commutes with i, $i+1$ and $i-1$. Therefore $[i*is_1 \dots s_k j*j] = [s_1 i*is_2 \dots s_k j*j]$ and the factor is not minimal. □

We have a similar lemma again if $i>j$.

Lemma 3.2.5. *Let $[i*is_1 \dots s_k j*j]$ be a non-elementary unentangled factor where $[i*i]$ is equal to $[i(i+1)i]$ or $[i(i+1)(i-1)i]$ (similar for j) and assume $i>j$. Let $S = \{j+2, j+3, \dots i-1\}$ if $[j*j] = [j(j+1)j]$ and $\{j+2, j+3, \dots i-2\}$ otherwise. If $S \not\subseteq \{s_1 \dots s_k\}$, then $[i*is_1 \dots s_k j*j]$ is not minimal.*

At this point, an unentangled factor $[i*is_1 \dots s_k j*j]$ is either elementary (k =0) or $\{s_1, \dots, s_k\} = \{i+2, i+3, \dots, j-1\}$ if $i<j$ or $\{s_1, \dots, s_k\} = \{j+2, j+3, \dots, i-1\}$ if $i>j$.

Lemma 3.2.6. *If an unentangled factor $[i*is_1 \dots s_k j*j]$ is not equivalent to an elementary entangled factor, then it must be equivalent to one of the following:*

1. *$[i(i+1)i(i+2)\dots(i+k)(i+k+1)(i+k)]$ for $k \geq 3$.*

2. $[i(i+1)i(i-1)\dots(i-k)(i-k-1)(i-k)]$ *for* $k \geq 2$.

3. $[i(i+1)(i-1)i(i+2)\dots(i+k)(i+k+1)(i+k)]$ *for* $k \geq 3$.

4. $[i(i+1)(i-1)i(i-2)\dots(i-k)(i-k-1)(i-k)]$ *for* $k \geq 2$.

5. $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k-1)(i+k+1)(i+k)]$ *for* $k \geq 4$.

6. $[i(i+1)i(i-1)\dots(i-k+2)(i-k)(i-k-1)(i-k+1)(i-k)]$ *for* $k \geq 3$.

7. $[i(i+1)(i-1)i(i+2)\dots(i+k-2)(i+k)(i+k-1)(i+k+1)(i+k)]$ *for* $k \geq 4$.

8. $[i(i+1)(i-1)i(i-2)\dots(i-k+2)(i-k)(i-k-1)(i-k+1)(i-k)]$ *for* $k \geq 3$.

Proof. Consider an unentangled factor $[i * is_1 \dots s_m j * j]$ where $m \neq 0$ and assume momentarily that $i < j$ and $[j * j] = [j(j+1)j]$. From the previous lemmas, we must have $\{s_1 \dots s_k\} = \{i+2, i+3, \dots j-1\}$ in this case. In order for this factor to be minimal, s_1 cannot commute with the factor $[i*i]$. This can only occur if $s_1 = i+2$. Similarly $s_k = j-1$. Therefore, $[i * is_1 \dots s_k j * j] = [i * i(i+2)s_2 \dots s_{k-1}(j-1)j(j+1)j]$. If s_2 is greater than $i+3$, then s_2 commutes with every element to its left and so the factor would not be minimal. Therefore $s_3 = i+3$. Similarly $s_{k-1} = j-2$. Continuing the argument in this fashion gives $[i * is_1 \dots s_k j * j] = [i * i(i+2)(i+3)\dots(j-1)j(j+1)j]$. This gives items 1 and 3 in the lemma. The $k \geq 3$ follows from $m \neq 0$. The other six items in the lemma follow in a similar fashion and the ardent reader shall not be bored by the extreme amount of mundane detail required to check them. □

3.2.2 Patterns Sharing One Element

Two 321 Patterns Sharing Exactly One Element

There are precisely two permutations of length 5 that contain exactly two 321 patterns and avoid 3412 such that the two 321 patterns share precisely one element.

They are 32541 and 52143. In this section, we will go through the detail of analyzing 32541. The process for 52143 is precisely the same.

Lemma 3.2.7. *Suppose* π *contains precisely two* 321 *patterns of the form* 32541 *and avoids* 3412 *then the* 54 *pattern occurs in consecutive elements in* π*.*

Proof. Let π be a permutation satisfying the above conditions.

Let $\pi = \pi_1 \dots \pi_c \dots \pi_b \dots \pi_e \dots \pi_d \dots \pi_a \dots \pi_n$ where $\pi_c \pi_b \pi_e \pi_d \pi_a$ forms the 32541 pattern (i.e. $\pi_a < \pi_b < \pi_c < \pi_d < \pi_e$). We wish to show that there is no element g in between π_e and π_d.

Assume there exists a number g in between π_e and π_d in π. If $g < \pi_b$, then $\pi_c \pi_b g$ is 321. If $\pi_b < \pi_c$, then $\pi_c g \pi_a$ is 321. If $\pi_c < g < \pi_e$, then $\pi_e g \pi_a$ is 321. Finally, if $g > \pi_e$, then $g \pi_d \pi_a$ is 321. Since all possibilities for g lead to a contradiction, the lemma is proved. □

Let π be as in the previous lemma and structure π as follows: let $\pi = V \pi_c W \pi_b X \pi_e \pi_d Y \pi_a Z$ where $V = v_1 \dots v_{k_1}$, $W = w_1 \dots w_{k_2}$, $X = x_1 \dots x_{k_3}$, $Y = y_1 \dots y_{k_4}$ and $Z = z_1 \dots z_{k_5}$. We must determine the allowable orderings of elements in V, W, X, Y, and Z if they exist.

First, consider $X = x_1 \dots x_{k_3}$. If $k_3 \neq 0$, then $\pi_c < x_i < \pi_d$ for all $1 \leq i \leq k_3$. If that were not the case, then if $x_i < \pi_b$ for some i, $\pi_c \pi_b x_i$ would be a 321 pattern. If $\pi_b < x_i < \pi_c$, then $\pi_c x_i \pi_a$ would be a 321 pattern. If $\pi_d < x_i < \pi_e$, then $x_i \pi_d \pi_a$ would be a 321 pattern and if $x_i > \pi_e$, then $x_i \pi_e \pi_d$ would be a 321 pattern. Also, $x_i < x_j$ (for $1 \leq i, j \leq k_3$), then $i < j$. If not, $x_i x_j \pi_a$ would form a 321 pattern.

Similarly, for all $1 \leq i \leq k_j$ (for the appropriate j), we have $v_i < \pi_b$, $w_i < \pi_a$, $z_i > \pi_d$ and in particular $\pi_c < y_i < \pi_d$. This forces $\pi_c = \pi_b + 1$.

The properties of these permutations are now sufficient to construct the appropriate factor of the reduced decomposition using the graph. Figure 3.4 gives this graph.

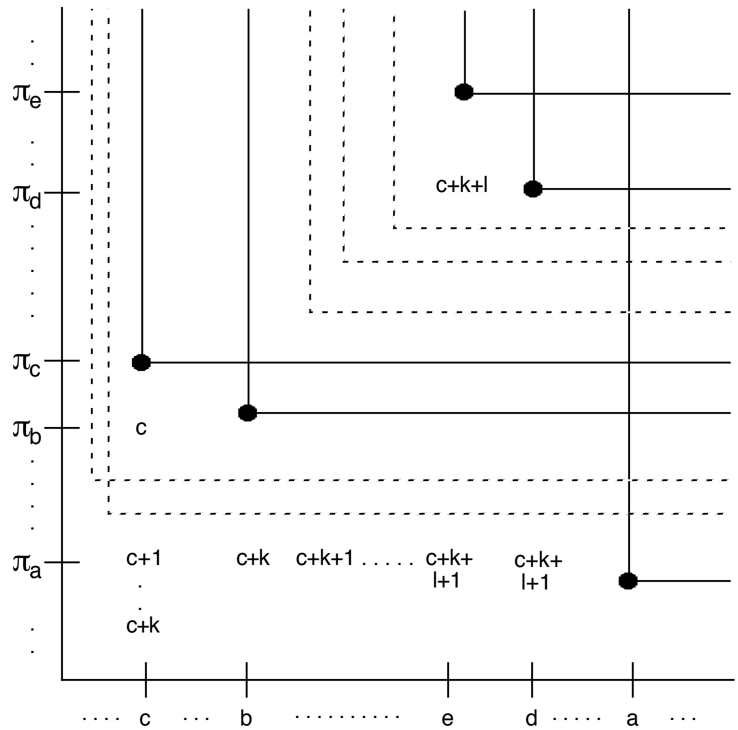

Figure 3.4: Partial graph associated with 32541

Note that all elements that are less than π_b are positioned to the left of b and that all elements x that satisfy $\pi_a < x < \pi_b$ occur to the left of position c. In the graph, position c will have blanks at π_a, π_b and at $k-1$ elements all of which are between 1 and π_a. These positions will be filled with numbers c, $c+1$, $\ldots c+k$. If there are $k-1$ blanks below π_a, then $b = c+k$. π_a is the only blank at position b. Between positions b and e we all of the elements x that satisfy $\pi_c < x < \pi_d$ in increasing order. Therefore the only blank in the positions between b and e is π_a. If there $e = b + l$, then $e = c + k + l$. At positions $b + i$ for $1 \leq i < l$, the blank at position i will be filled with $b+i = c+k+1$. At position $e = b+l = c+k+l$, there will be at least two blanks: one at position π_d and one at position π_a. There may be other blanks at any x satisfying $\pi_d < x < \pi_e$. (Note, the graph shows only the case where there are exactly two blanks.) Finally, at position d, there will be again precisely one blank: at position π_a which will be filled with $d = e+1 = c+k+l+1$. Reading the reduced decomposition off of the graph from positions c to d gives the factor $[(c+k)(c+k-1)\ldots(c+1)c(c+k)(c+k+1)\ldots(c+k+l-1)(c+k+l+m)(c+k+l+m-1)\ldots(c+k+l+1)(c+k+l)(c+k+l+1)]$ where $k, l, m \geq 1$. Using only short braid moves this factor is equivalent to $[(c+k+l+m)(c+k+l+m-1)\ldots(c+k+l+2)(c+k)(c+k-1)(c+k)(c+k+1)\ldots(c+k+l-1)(c+k+l+1)(c+k+l)(c+k+l+1)(c+k-2)\ldots(c+1)c]$ which has a factor $[(c+k)(c+k-1)(c+k)(c+k+1)\ldots(c+k+l-1)(c+k+l+1)(c+k+l)(c+k+l+1)]$. Using long braid moves on the first and last three elements, this factor is equivalent to $[(c+k-1)(c+k)(c+k-1)(c+k+1)\ldots(c+k-l-1)(c+k+l)(c+k+l+1)(c+k+l)]$ which is a factor of the form $[i(i+1)i(i+2)\ldots(i+k)(i+k+1)(i+k)]$. We have therefore proved the following lemma.

Lemma 3.2.8. *If π avoids 3412 and contains exactly two 321 patterns of the form 32541 then π has a reduced decomposition with $[i(i+1)i(i+2)\ldots(i+k)(i+k+1)(i+k)]$.*

Using techniques described in Chapter 2 and Lemma 3.2.8, we have the following theorem.

Theorem 3.2.9. *π has a reduced decomposition with $[i(i+1)i(i+2)\ldots(i+k)(i+k+1)(i+k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π avoids 3412 and contains exactly two 321 patterns of the form 32541.*

The other pattern, 52143, corresponds to the factor $[(i+k)(i+k+1)(i+k)(i+k-1)\ldots(i+2)i(i+1)i]$.

Theorem 3.2.10. *π has a reduced decomposition with $[(i+k)(i+k+1)(i+k)(i+k-1)\ldots(i+2)i(i+1)i]$ for $k \geq 2$ as a factor and no other repetitions if and only if π avoids 3412 and contains exactly two 321 patterns of the form 52143.*

One 321 and One 3412 Pattern Sharing Exactly One Element

There are four permutations of length 6 that contain exactly one 321 pattern and exactly one 3412. They are: 325614, 341652, 361254, and 521634. The following results follow in the same fashion as the previous subsection and so the proofs are omitted.

Theorem 3.2.11. *π has a reduced decomposition with $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely one* 321 *pattern and one* 3412 *pattern of the form* 325614*.*

Theorem 3.2.12. *π has a reduced decomposition with $[i(i-1)(i+1)i(i+2)\dots(i+k)(i+k+1)(i+k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely one* 321 *pattern and one* 3412 *pattern of the form* 341652*.*

Theorem 3.2.13. *π has a reduced decomposition with $[i(i+1)i(i-1)\dots(i-k+2)(i-k)(i-k+1)(i-k-1)(i-k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely one* 321 *pattern and one* 3412 *pattern of the form* 361254*.*

Theorem 3.2.14. *π has a reduced decomposition with $[i(i-1)(i+1)i(i-2)\dots(i-k+2)(i-k)(i-k+1)(i-k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely one* 321 *pattern and one* 3412 *pattern of the form* 521634*.*

Two 3412 Patterns Sharing Exactly One Element

There are two permutations of length 7 that contain exactly two 3412 patterns and avoid 321. They are: 3416725 and 3612745.

As the reader has probably guessed the two factors which classify these two patterns are the last two factors in Lemma 3.2.6.

Theorem 3.2.15. *π has a reduced decomposition with $[i(i+1)(i-1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely two 3412 patterns of the form 3416725 and no other repetitions.*

Theorem 3.2.16. *π has a reduced decomposition with $[i(i+1)(i-1)i(i-2)\dots(i-k+2)(i-k)(i-k-1)(i-k+1)(i-k)]$ for $k \geq 2$ as a factor and no other repetitions if and only if π contains precisely two 3412 patterns of the form 3612745 and no other repetitions.*

3.2.3 Patterns Sharing No Elements

Two 321 Patterns Sharing No Elements

There are exactly three permutations that described the behavior of a permutation having precisely two 321 patterns and avoiding 3412 with no elements shared between them. They are: 321654, 326154, and 421653. We now begin our consideration of elementary factors of the form $[i(i+1)ij(j+1)j]$ with $|i-j| > 2$. The case $|i-j| = 2$ is covered by our discussion in the previous section.

Lemma 3.2.17. *If $\pi \in S_n$ has a reduced decomposition with $[i(i+1)ij(j+1)j]$ such that $|i-j| > 2$. as a factor and no other repetitions, then π has exactly two 321 patterns that share no elements and avoids 3412.*

Proof. Without loss of generality, we may assume $i < j$. If not, since $|i-j| > 2$, we may use short braid moves to commute $[i(i+1)i]$ with $[j(j+1)j]$. The $[i(i+1)i]$ factor creates a 321 pattern from the elements that are in positions i, $i+1$, and $i+2$ when that factor is applied. Similarly the $[j(j+1)j]$ factor creates a 321 pattern from whichever elements are in positions j, $j+1$ and $j+2$ when that factor is applied. Since $|i-j| > 2$ and the two factors occur one immediately after the other, by Lemma 2.2.3, π must contain at least two 321 patterns. To show that π only has

2 321 patterns and avoids 3412 requires an argument that is analogous to previous ones. □

For each one of 321654, 326154 and 421653, one may calculate the properties of permutations containing those patterns and no other 321 or 3412 patterns and then calculate the reduced decompositions via the graphs. From there, one may work through the detail to show that each one yields a reduced decomposition with $[i(i+1)ij(j+1)j]$ as a factor such that $|i-j| > 2$. This yields the following theorem:

Theorem 3.2.18. *$\pi \in S_n$ has a reduced decomposition with $[i(i+1)ij(j+1)j]$ as a factor with $|i-j| > 2$ and no other repetitions if and only if π avoids 3412 and contains precisely two 321 patterns that share no elements.*

Other Patterns Sharing No Elements

We will omit the proofs in this section and the next section and just quote the result.

Theorem 3.2.19. *$\pi \in S_n$ has a reduced decomposition with $[i(i+1)ij(j+1)(j-1)j]$ such that $|i-j| > 4$ as a factor and no other repetitions if and only if π contains exactly one 3412 and exactly one 321 pattern that share no elements.*

Theorem 3.2.20. *$\pi \in S_n$ has a reduced decomposition with $[i(i+1)(i-1)ij(j+1)(j-1)j]$ such that $|i-j| > 4$ as a factor and no other repetitions if and only if π contains exactly two 3412 patterns sharing no elements and avoids 321.*

We have now classified all unentangled factors and this completes the classification of reduced decompositions with two repetitions.

3.3 Counting Pattern Classes with Reduced Decompositions

The goal of this section is to count the number of permutations in $Av_n(3412)$ that contain exactly two 321 patterns, the number of permutations in S_n that contain exactly one 321 and exactly one 3412 pattern, and the number of permutations in $Av_n(321)$ that contain exactly two 3412 patterns using the reduced decomposition classifications thus far developed.

3.3.1 Fibonacci Convolution Theorem

Theorem 2.3.15 is an instance of a more general phenomenon which we call here the Fibonacci Convolution Theorem. First, we will state it and then we shall give the proof.

Theorem 3.3.1. *[Fibonacci Convolution Theorem (FCT)] Let* $\mathbf{t}$ *be a reduced decomposition containing only the elements* $\{i, \ldots, i+k\}$ *for some* $k \geq 1$*. Further assume* $\mathbf{t}$ *contains at least one occurrence each of* i *and* $i+k$*. The number of distinct reduced decompositions on* $\{1, \ldots, n-1\}$ *having* $\mathbf{t}$ *as a factor with no other repetitions is* $\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}$ *where* F_m *is the* m^{th} *Fibonacci number.*

Note that there are no restrictions on the number of occurrences of elements in $\{i, \ldots, i+k\}$.

The proof of this theorem is very similar to the proof of Theorem 2.3.15, but as this theorem will be so crucial to all of the counting done in this section we produce the details of the proof that differ from the proof of Theorem 2.3.15.

In the discussion that follows fix a reduced decomposition $\mathbf{t}$ and $k \geq 1$ satisfying the hypotheses of the FCT.

As in the proof of Theorem 2.3.15, we construct sets $E_j^i(n)$ of reduced decompositions on $\{1, \ldots, n-1\}$ such that the elements of $E_j^i(n)$ are representatives of distinct equivalence classes of reduced decompositions with the property that every reduced decomposition $[s_1 \dots s_m]$ which has $\mathbf{t}$ as a factor and no other repetitions and $s_h \leq j$ for all $1 \leq h \leq m$ has a representative in $E^i_j(n)$.

The number of reduced decompositions on $\{1, \dots, n-1\}$ with $\mathbf{t}$ as a factor and no other repetitions will be $\sum_{h=1}^{n-k-1} |E^h_{n-1}(n)|$. If $j < i + k$, then $E^i_j(n)$ is empty. We show how to count the cardinality of $E^i_{n-1}(n)$, we show to construct all of the sets $E^i_j(n)$ for $i + k \leq j \leq n - 1$. We first show how to construct $E^i_j(n)$ given the set $E^i_{j-1}(n)$ and then we will show how to construct $E^i_{i+k}(n)$.

Constructing the set $E^i_j(n)$ from $E^i_{j-1}(n)$

The smallest j for which $E^i_j(n)$ is nonempty is $j = k+1$, so first we assume we have the set $E^i_{i+k}(n)$ and show how to construct $E^i_{i+k+1}(n)$. For all reduced decompositions $\mathbf{s} \in E^i_{i+k}(n)$, construct a set X of reduced decompositions as follows:

1. Add $\mathbf{s}$ to X.

2. Concatenate $(i+k+1)$ to $\mathbf{s}$ on both sides, giving $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(\mathbf{i}+\mathbf{2})]$, and add them to X.

$\mathbf{s}$ only contains elements in $\{1, \dots, i+k\}$ so $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(i+2)]$ are reduced for all $\mathbf{s} \in E^i_{i+k}(n)$.

Lemma 3.3.2. *The set X is $E^i_{i+k+1}(n)$.*

Proof. First, it must be shown why all reduced decompositions in X represent distinct permutations. The argument is similar to the proof of Lemma 2.3.10 and we will not repeat all of the details here, but we will show why $[(i+k+1)\mathbf{s_1}] = [\mathbf{s_2}(i+k+1)]$ leads to a contradiction for $\mathbf{s_1}$, $\mathbf{s_2} \in E^i_{i+k}(n)$ (note $\mathbf{s_1}$ and $\mathbf{s_2}$ may be equal, but that is not assumed). As all elements in $\mathbf{s_1}$ and $\mathbf{s_2}$ are elements in $\{1, \dots i+k\}$, we must have that the permutation represented by $[(i+k+1)\mathbf{s_1}]$ sends

$i+k+1$ to $i+k-l$ for some $l \geq 0$. The permutation represented by $[\mathbf{s_2}(i+k+1)]$ sends $i+k+1$ to $i+k+2$. Therefore $[(i+k+1)\mathbf{s_1}] \neq [\mathbf{s_2}(i+k+1)]$.

To show why every reduced decomposition is represented by an element in X, the argument is exactly the same as in the proof of Lemma 2.3.10 with $i+1$ and $i+2$ replaced with $i+k$ and $i+k+1$ respectively. □

Once the set $E^i_{i+k+1}(n)$ is built, the procedure for building $E^i_{j+1}(n)$ from $E^i_j(n)$ and $E^i_{j-1}(n)$ is as follows for all $\mathbf{s} \in E^i_j(n)$.

1. Add $\mathbf{s}$ to $E^i_{j+1}(n)$.

2. If $\mathbf{s} \in E^i_j(n) \cap E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ to $E^i_{j+1}(n)$.

3. If $\mathbf{s} \in E^i_j(n) \setminus E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ and $[\mathbf{s}(j+1)]$ to $E^i_{j+1}(n)$.

Lemma 3.3.3. *The procedure outlined above correctly produces the set $E^i_{j+1}(n)$ for all $i+k+1 \leq j < n-1$.*

Proof. Same as Lemma 2.3.12 □

Define $a_i(m) := |E^i_{i+k+m-1}(n)|$ for $m \geq 0$. The reason for such a unique definition is that since E^i_{i+k} is the first nonempty set, we want $m = 0$ to give $a_i(0) = 0$ in order to be consistent with Chapter 2. Again $a_i(m) = 3a_i(m-1) - a_i(m-2)$ with $a_i(0) = 0$ and $a_i(1) = |E^i_{i+k}(n)|$. This recurrence gives $|E^i_{i+k+m-1}(n)| = a_i(m) = |E^i_{i+k}| * F_{2m}|$. The value of m which makes $i+k+m-1 = n-1$ is $m = n-i-k$. Therefore $|E^i_{n-1}(n)| = a_i(n-i-k) = |E^i_{i+k}| \cdot F_{2(n-i-k)}$.

Constructing the sets $E^i_{i+k}(n)$

Note that E^1_{k+1} contains only the reduced decomposition $\mathbf{t}$ where $i = 1$. The sets $E^i_{i+k}(n)$ are constructed inductively from $E^{i-1}_{i+k-1}(n)$ by the following procedure for all $\mathbf{s} \in E^{i-1}_{i+k-1}$.

1. If $\mathbf{s} = [s_1 \dots s_k]$, then add $[(s_1+1)(s_2+1)\dots(s_k+1)]$ to $E^i_{i+k}(n)$.

2. If $\mathbf{u}$ was added in step 1 and only contains elements in $\{3, \dots, i+k\}$, then add the reduced decomposition $[1\mathbf{u}]$ to $E^i_{i+1}(n)$.

3. If $\mathbf{u}$ was added in step 1 and contains the element 2, then add the reduced decompositions $[1\mathbf{u}]$ and $[\mathbf{u}1]$ to $E^i_{i+1}(n)$.

Lemma 3.3.4. *The procedure above correctly produces the sets* $E^i_{i+k}(n)$ *and* $|E^i_{i+k}(n)| = F_{2i}$ *where* F_m *is the* m^{th} *Fibonacci number.*

Proof. Both claims follow similarly from the proof of Theorem 2.3.15.

□

This proves the FCT, Theorem 3.3.1.

3.3.2 Counting Reduced Decompositions with Entangled Factors

Now that we have the FCT, counting the reduced decompositions with entangled factors is very easy. The following results are direct consequences of the FCT.

Theorem 3.3.5. *The following quantities are equal:*

1. $|\{\pi \in Av_n(3412) : \pi$ *contains exactly two* 321 *patterns of the form* $3421\}|$.

2. $|\{\pi \in Av_n(3412) : \pi$ *contains exactly two* 321 *patterns of the form* $4312\}|$.

3. $|\{\pi \in Av_n(3412) : \pi$ *contains exactly two* 321 *patterns of the form* $4231\}|$.

4. $\sum_{i=1}^{n-3} F_{2i}F_{2(n-i-2)}$ *where* F_m *is the* m^{th} *Fibonacci number.*

Theorem 3.3.6. *The following quantities are equal:*

1. $|\{\pi \in Av_n(321)$: π *contains exactly three* 3412 *patterns of the form* $34512\}|$.

2. $|\{\pi \in Av_n(321)$: π *contains exactly three* 3412 *patterns of the form* $45123\}|$.

3. $|\{\pi \in S_n$: π *contains exactly one* 3412 *and exactly one* 321 *pattern of the form* 35142$\}|$.

4. $|\{\pi \in S_n$: π *contains exactly one* 3412 *and exactly one* 321 *pattern of the form* 42513$\}|$.

5. $\sum_{i=1}^{n-4} F_{2i}F_{2(n-i-3)}$ *where* F_m *is the* m^{th} *Fibonacci number.*

Theorem 3.3.7. *The following quantities are equal:*

1. $|\{\pi \in Av_n(321)$: π *contains exactly two* 3412 *patterns of the form* 351624$\}|$.

2. $\sum_{i=1}^{n-5} F_{2i}F_{2(n-i-4)}$ *where* F_m *is the* m^{th} *Fibonacci number.*

3.3.3 Counting Reduced Decompositions with Unentangled Factors

Patterns that share precisely one element

Due to the pattern conditions proved earlier, if a reduced decomposition **s** contains one of the factors listed in Lemma 3.2.6 and another reduced decomposition **t** contains a different factor from the list in Lemma 3.2.6, then the two reduced decompositions must be different. We now show that if **s** and **t** are reduced decompositions both containing the same type of factor for some factor in Lemma 3.2.6 with no other repetitions, but are of different lengths then the two reduced decompositions represent distinct permutations.

Lemma 3.3.8. *Let* **s** *and* **t** *be two reduced decompositions both with a factor of the same type* j *for some* $1 \leq j \leq 8$ *in the list in Lemma 3.2.6 with no other repetitions. Then if the length of the factor in* **s** *is not the same as the length of the factor in* **t**, *then the two permutations represented by* **s** *and* **t** *are distinct.*

Proof. If the lengths of **s** and **t** are different, then the two permutations must be distinct as length is an invariant.

Consider the case where the factor is of type $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$ for $k \geq 2$. There are three subcases to consider.

- Case 1: The factor in $\mathbf{s}$ is $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$ for some $k \geq 2$ and the factor in $\mathbf{t}$ is $[i(i+1)i(i+2)\dots(i+k+l-2)(i+k+l)(i+k+l+1)(i+k+l-1)(i+k+l)]$ for some $l \geq 1$. In this case, consider the mapping of element $i+k$. In the first factor $i+k$ is mapped to $i+k+m$ for some $m \geq 2$. In the second, $i+k$ is mapped to $i+k+1$. Therefore, the two permutations must be distinct.

- Case 2: The factor in $\mathbf{s}$ is $[(i-k)(i-k+1)(i-k)(i-k+2)\dots(i-2)i(i+1)(i-1)i]$ for some $k \geq 2$ and the factor in $\mathbf{t}$ is $[(i-k-l)(i-k-l+1)(i-k-l)(i-k-l+2)\dots(i-2)i(i+1)(i-1)i]$ for some $l \geq 1$. Consider the mapping of element $i-k$. The first factor maps $i-k$ to $i-k+2$. The second maps $i-k$ to $i-k+1$. Therefore, the two permutations must be distinct.

- Case 3: The factor in $\mathbf{s}$ is $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$ and the factor in $\mathbf{t}$ is $[(i-l)(i-l+1)(i-l)(i-l+2)\dots i(i+1)(i+2)\dots(i+k-1)(i+k)(i+k+1)\dots(i+k+m)(i+k+m+1)(i+k+m)]$ for some $l \geq 1$. Consider the mapping of element i. In the first factor i is mapped to $i+2$. In the second, i is mapped to $i+1$. Therefore, the two permutations must be distinct.

The theorem is proved for factors of type $[i(i+1)i(i+2)\dots(i+k-2)(i+k)(i+k+1)(i+k-1)(i+k)]$. The proof is similar for the other factors. □

We shall now count the number of permutations in S_n that avoid 3412 and contain exactly two 321 patterns of the form 32541. By Theorem 3.2.9, we must count the number of reduced decompositions having a factor of the form $[i(i+1)i(i+2)\dots(i+k)(i+k+1)(i+k)]$ for some $k \geq 2$ and no other repetitions.

Fix $n \in \mathbb{N}$. By the FCT, the number of reduced decompositions having a factor of the form $[i(i+1)i(i+2)(i+3)(i+2)]$ (the case $k=2$) is $\sum_{j=1}^{n-4} F_{2j}F_{2(n-j-3)}$. The number of reduced decompositions having a factor of the form $[i(i+1)i(i+2)(i+3)(i+4)(i+3)]$ is $\sum_{j=1}^{n-5} F_{2j}F_{2(n-j-4)}$. In general, the number of reduced decompositions having a factor of the form $[i(i+1)i(i+2)\ldots(i+k)(i+k+1)(i+k)]$ is $\sum_{j=1}^{n-k-2} F_{2j}F_{2(n-j-k-1)}$. All of the reduced decompositions are distinct by Lemma 3.3.8 and the pattern classifications. Therefore, to count all of these reduced decompositions we just sum from $k=2$ to $n-3$ giving the following theorem.

Theorem 3.3.9. *The number of permutations in S_n that avoid 3412 and contain exactly two 321 patterns of the form 32541 is*

$$\sum_{k=3}^{n-2}\left(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}\right)$$

.

Continuing in this fashion gives us counts for all of the non elementary unentangled factors. Here is the complete count.

Theorem 3.3.10. *The following quantities are equal:*

1. $|\{\pi \in Av_n(3412) : \pi$ *contains exactly two* 321 *patterns of the form* $32541\}|$.
2. $|\{\pi \in Av_n(3412) : \pi$ *contains exactly two* 321 *patterns of the form* $52143\}|$.
3. $\sum_{k=3}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)})$.

Theorem 3.3.11. *The following quantities are equal:*

1. $|\{\pi \in S_n : \pi$ *contains exactly one* 321 *pattern pattern and exactly one* 3412 *pattern of the form* $325614\}|$.
2. $|\{\pi \in S_n : \pi$ *contains exactly one* 321 *pattern pattern and exactly one* 3412 *pattern of the form* $341652\}|$.

Table 3.5: $\sum_{k=1}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)})$ for small n.

n	3	4	5	6	7	8	9	10	11	12
	1	7	32	122	422	1376	4315	13165	39360	115860

3. $|\{\pi \in S_n : \pi$ *contains exactly one* 321 *pattern pattern and exactly one* 3412 *pattern of the form* $361254\}|$.

4. $|\{\pi \in S_n : \pi$ *contains exactly one* 321 *pattern pattern and exactly one* 3412 *pattern of the form* $521634\}|$.

5. $\sum_{k=4}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)})$.

Theorem 3.3.12. *The following quantities are equal:*

1. $|\{\pi \in Av_n(321)$*:* π *contains exactly two* 3412 *patterns of the form* $3416725\}|$.

2. $|\{\pi \in Av_n(321)$*:* π *contains exactly two* 3412 *patterns of the form* $3612745\}|$.

3. $\sum_{k=5}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)})$.

Note, that all of the sums in the theorems above represent the same sequence of numbers. Table 3.5 shows these numbers for $\sum_{k=1}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)})$.

Patterns that share no elements

We now count the number of reduced decompositions having a $[i*ij*j]$ factor and no other repetitions where $[i*ij*j] = [j*ji*i]$, $[i*i] = [i(i+1)i]$ or $[i(i-1)(i+1)i]$ and $[j*j] = [j(j+1)j]$ or $[j(j-1)(j+1)j]$. Recall that such reduced decompositions correspond either to counting the number of permutations in S_n that contain two disjoint 321 patterns and avoid 3412, that contain one 321 and one 3412 pattern that are disjoint, or that contain two disjoint 3412 patterns and avoid 321.

Since $[i*ij*j] = [j*ji*i]$, we may assume without loss of generality that $i<j$. Let $min[i*i]$ be the smallest element in the factor $[i*i]$. Our strategy for

counting these reduced decompositions will be to first count the number of reduced decompositions having $[i * ij * j]$ as a factor with no other repetitions such that each element s of the reduced decomposition satisfies $min[i * i] < s < j + 1$. Once these reduced decompositions are counted, we may use the FCT to finish the count.

Let $\mathbf{s} = [S_1 i * ij * jS_2]$ where $S_1 = s^1_1 \ldots s^1_{k_1}$ and $S_2 = s^2_1 \ldots s^2_{k_2}$. Without loss of generality, we may assume that S_2 is minimal in the sense that any element that can be commuted with $[i * ij * j]$ and be made an element of S_1 has been so commuted.

Consider the possibilities for S_2 under this assumption. The first possibility is that $S_2 = \emptyset$. It is also possible that $S_2 = [(min[j * j] - 1)(min[j * j] - 2) \ldots (min[j * j] - k]$ or $S_2 = [(i + 2)(i + 3) \ldots (i + l)]$ for $1 \leq k \leq i - min[j * j] + 2$ and $2 \leq l \leq min[j * j] - i - 1$.

Proposition 3.3.13. *If $S_2 = [s^2_1 \ldots s^2_k]$ is not of one of the forms previously listed, then $S_2 = [(i + 2)(i + 3) \ldots (i + k)(min[j * j] - 1) \ldots (min[j * j] - l)]$ or $S_2 = [(min[j * j] - 1) \ldots (min[j * j] - l)(i + 2)(i + 3) \ldots (i + k)]$.*

Proof. By induction on k. If $k = 1$, then $s^2_1 = i + 2$ or $s^2_1 = min[j * j] - 1$ otherwise, $[i * ij * js^2_1] = [s^2 i * ij * j]$. Assume the proposition is true for k and we shall now show it for $k + 1$. If $S_2 = [s_1 \ldots s_k s_{k+1}]$, then since no s_m $(1 \leq m \leq k + 1)$ can commute with $[i * ij * j]$, the factor $[s_1 \ldots s_k]$ either is $[(i + 2)(i + 3) \ldots (i + k + 1)]$, $[(min[j*j]-1) \ldots (min[j*j]-k)]$, or satisfies the induction hypothesis. This implies $[s_1 \ldots s_k]$ is either equal to $[(i + 2) \ldots (i + m)(min[j * j] - 1) \ldots (min[j * j] - l)]$ or $[(min[j*j]-1) \ldots (min[j*j]-l)(i+2) \ldots (i+m)]$. In the former case $[s_1 \ldots s_k s_{k+1}] = [(i+2) \ldots (i+m)(min[j*j]-1) \ldots (min[j*j]-l)s_{k+1}]$. We must have $i+m < s_{k+1} < min[j*j]-l$. If $s_{k+1} \neq i+m+1$ and $s_{k+1} \neq min[j*j]-l-1$, then $[s_{k+1}s_m] = [s_m s_{k+1}]$ for all $1 \leq m \leq k$ and hence $[i * ij * js_{k+1}] = [s_{k+1} i * ij * j]$ which is a contradiction. Therefore $s_{k+1} = i+m+1$ or $[min[j*j]-l-1]$. If $s_{k+1} = i+m+1 \neq min[j*j]-l-1$, then $[s_1 \ldots s_{k+1}] = [(i+2) \ldots (i+m)(min[j*j]-1) \ldots (min[j*j]-l)(i+m+1)] = [(i + 2) \ldots (i + m)(i + m + 1)(min[j * j] - 1) \ldots (min[j * j] - l)]$ which gives the proposition. Otherwise, $[s_1 \dots s_{k+1}] = [(i+2)\dots(i+m)(i+m+1)(min[j*j]-1)\dots(min[j*j]-l)(min[j*j]-l-1)]$ which also gives the proposition. The same argument applies if $[s_1 \dots s_k] = [(min[j*j]-1)\dots(min[j*j]-l)(i+2)\dots(i+m)]$. □

To recap, S_2 is either empty, an increasing or decreasing sequence of elements, or an increasing sequence followed by a decreasing sequence or vice versa.

If $S_2 = \emptyset$, then there are $min[j*j]-i-2$ possible elements that can occur in S_1. In particular they are the elements $\{i+2, i+3, \dots min[j*j]-1\}$. The number of reduced decompositions on these elements with no repetitions is the same as the number of reduced decompositions on $\{1, \dots, min[j*j]-i-2\}$ which is the same as the number of permutations in $S_{min[j*j]-i-1}$. By Theorem 2.1.9, there are $F_{2(min[j*j]-i-2)+1}$ such permutations.

If S_2 is not empty, then there are many possibilities. If S_2 is either an increasing sequence or a decreasing sequence (but not a combination of the two), then by Theorem 2.1.9, there are $F_{2(min[j*j]-i-2-m)+1}$ such reduced decompositions for any increasing or decreasing sequence of size m as there are $min[j*j]-i-2-m$ distinct elements possible for S_1. Therefore, there are $2\sum_{m=1}^{min[j*j]-i-2} F_{2(min[j*j]-i-2-m)+1}$ such reduced decompositions.

Now consider the case where S_2 is a reduced decomposition of the form $[(i+2)(i+3)\dots(i+k)(min[j*j]-1)\dots(min[j*j]-l)]$ or $[(min[j*j]-1)\dots(min[j*j]-l)(i+2)(i+3)\dots(i+k)]$ for $k \geq 2$ and $l \geq 1$. If $i+k+1 = min[j*j]-l$, we have the following proposition.

Proposition 3.3.14. *Any sequence of the form* $[(i+2)(i+3)\dots(i+k)(min[j*j]-1)\dots(min[j*j]-l)]$ *where* $i+k+1 = min[j*j]-l$ *is equivalent to a sequence of the form* $[(min[j*j]-1)\dots(min[j*j]-l')(i+2)(i+3)\dots(i+k')]$

Proof. By braid moves, we have $[(i+2)(i+3)\dots(i+k)(min[j*j]-1)\dots(min[j*j]-l)] = [(min[j*j]-1)(min[j*j]-2)\dots(min[j*j]-l+1)(i+2)(i+3)\dots(i+$

$k)(i+k+1)]$. $\square$

By Proposition 3.3.14, we only need to count the number of sequences that are increasing first and then decreasing. Such sequences are described by the position of $min[j*j]-1$ in the sequence. That number can be one of s_2^2, s_3^2, $\ldots$, $s_{min[j*j]-i-3}^2$. Therefore, there are $min[j*j]-i-3-2+1 = min[j*j]-i-4$ such sequences.

Lastly, if $i+k+1 \neq min[j*j]-l$, then $[(i+2)\ldots(i+k)(min[j*j]-1)\ldots(min[j*j]-l)] = [(min[j*j]-1)\ldots(min[j*j]-l)(i+2)(i+3)\ldots(i+k)]$. The number of reduced decompositions with $S_2 = [(i+2)\ldots(i+k)(min[j*j]-1)\ldots(min[j*j]-l)]$ depends solely on k, l and the number of ways to arrange $\{i+k+1,\ldots,min[j*j]-l-1\}$ in S_1. The smallest size of $\{i+k+1,\ldots,min[j*j]-l-1\}$ is 1. There are $min[j*j]-i-2$ possibilities for it; however, $i+2$ and $min[j*j]-1$ are not allowed as both must appear in S_2 for there to be both an increasing and a decreasing sequence in S_2. Therefore, there are $min[j*j]-i-4$ possibilities for $\{i+k+1,\ldots,min[j*j]-l-1\}$. The largest $\{i+k+1,\ldots,min[j*j]-l-1\}$ can be is $min[j*j]-i-4$. In general, if there are m elements in $\{i+k+1,\ldots,min[j*j]-l-1\}$, then there are m elements possible in S_1 and there are $(min[j*j]-i-2)-m-1$ different ways of selecting a set of the form $\{i+k+1,\ldots,min[j*j]-l-1\}$ and F_{2m+1} reduced decompositions possible by Theorem 2.1.9. Therefore there are $\sum_{m=1}^{min[j*j]-i-4}(min[j*j]-i-m-3)F_{2m+1}$ reduced decompositions with S_1 nonempty and S_2 composed of both an increasing and decreasing sequence.

Combining all of the above counts gives an upper bound on the total number of reduced decompositions on $\{min[i*i],\ldots,j+1\}$ with $[i*ij*j]$ as a factor and no other repetitions. We must show this is the actual count by showing that each class of reduced decompositions above is distinct.

In Propositions 3.3.15, 3.3.16, 3.3.17, 3.3.18, 3.3.19, 3.3.20, 3.3.21, 3.3.22 and 3.3.23 fix a factor $[i*ij*j]$ such that $[i*ij*j] = [j*ji*i]$ where $[i*i] \in \{[i(i+1)i], [i(i+1)(i-1)i]\}$ and similarly for $[j*j]$. We may assume without loss of generality that $i < j$.

Proposition 3.3.15. *If* $\mathbf{s} = [S_1 i * ij * j]$ *and* $\mathbf{t} = [S_1' i * ij * j(i+2)\dots(i+k)]$ *for some* $2 \leq k \leq min[j*j] - 1$ *and such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. $\mathbf{s}$ maps $i+2$ to i and then to $i-m$ for some $m \geq 0$. $\mathbf{t}$ maps $i+2$ to $i+3$ and then if $k = 2$ possibly to $i+l$ for some $l \geq 2$. Therefore, the permutations represented by $\mathbf{s}$ and $\mathbf{t}$ must be distinct. □

Proposition 3.3.16. *If* $\mathbf{s} = [S_1 i * ij * j]$ *and* $\mathbf{t} = [S_1' i * ij * j(min[j*j]-1)\dots(min[j*j]-k)]$ *for some* $1 \leq k \leq min[j*j] - i - 2$ *and such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. $\mathbf{s}$ maps $min[j*j]-1$ to $min[j*j]$. $\mathbf{t}$ maps $min[j*j]-1$ to either $j+1$ or $j+2$ depending on $[j*j]$. $min[j*j]$ is either j or $j-1$, so $\mathbf{s}$ and $\mathbf{t}$ must be distinct. □

Proposition 3.3.17. *If* $\mathbf{s} = [S_1 i * ij * j]$ *and* $\mathbf{t} = [S_1' i * ij * j(i+2)(i+3)\dots(i+k)(min[j*j]-1)\dots(min[j*j]-l)]$ *for some* k*,* l *such that* $i+k < min[j*j]-l$ *and such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. $\mathbf{s}$ maps $i+2$ to i and then to $i-m$ for some $m \geq 0$. $\mathbf{t}$ maps $i+2$ to $i+3$. □

Proposition 3.3.18. *If* $\mathbf{s} = [S_1 i * ij * j(i+2)\dots(i+k)]$ *and* $\mathbf{t} = [S_1' i * ij * j(i+2)\dots(i+l)]$ *for* $2 \leq k < l \leq min[j*j] - 1$ *and such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. $\mathbf{s}$ maps $i+k+1$ to i. $\mathbf{t}$ maps $i+k+1$ to $i+k+m$ for some $m \geq 2$. □

Proposition 3.3.19. *If* $\mathbf{s} = [S_1 i * ij * j(min[j*j]-1)\dots(min[j*j]-k)]$ *and* $\mathbf{t} = [S_1' i * ij * j(min[j*j]-1)\dots(min[j*j]-l)]]$ *for* $1 < k < l \leq min[j*j] - 1$ *and* *such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. $\mathbf{s}$ maps $min[j*j]-k$ to $j+2$. $\mathbf{t}$ maps $min[j*j]-k$ to $min[j*j]-k-m$ for some $m \geq 1$. □

Proposition 3.3.20. *If* $\mathbf{s} = [S_1 i*ij*j(i+2)\dots(i+k)]$ *and* $\mathbf{t} = [S_1' i*ij*j(min[j*j]-1)\dots(min[j*j]-l)]$ *for some* $2 \leq k \leq min[j*j]-1$ *and* $1 \leq l \leq min[j*j]-l-2$ *such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. If $i+k+1 = min[j*j]-l$, then $\mathbf{s}$ maps $i+k+1$ to i and $\mathbf{t}$ maps $i+k+1$ to $j+2$. If $i+k+1 < min[j*j]-l$, then $\mathbf{s}$ maps $i+k+1$ to i and $\mathbf{t}$ maps $i+k+1$ to $i+k+m$ for some $m \geq 2$. □

Proposition 3.3.21. *If* $\mathbf{s} = [S_1 i*ij*j(i+2)\dots(i+k)]$ *and* $\mathbf{t} = [S_1' i*ij*j(i+2)(i+3)\dots(i+k')(min[j*j]-1)\dots(min[j*j]-l)]$ *for some* $k, k' \geq 2$ *and* $l \geq 1$ *such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. If $k < k'$, then $\mathbf{s}$ maps $i+k+1$ to i and $\mathbf{t}$ maps $i+k+1$ to $i+k+m$ for $m \geq 2$. If $k' < k$, then $\mathbf{s}$ maps $i+k'+1$ to $i+k'+m$ for $m \geq 2$ and $\mathbf{t}$ maps $i+k'+1$ to i. If $k = k'$, then $\mathbf{s}$ maps $min[j*j]$ to $j+1$ or $j+2$ and $\mathbf{t}$ maps $min[j*j]$ to $min[j*j]-m$ for some $m \geq 1$. □

Proposition 3.3.22. *If* $\mathbf{s} = [S_1 i*ij*j(min[j*j]-1)\dots(min[j*j]-l)]$ *and* $\mathbf{t} = [S_1' i*ij*j(i+2)(i+3)\dots(i+k)(min[j*j]-1)\dots(min[j*j]-l')]$ *for some* $k \geq 2$ *and* $l, l' \geq 1$ *such that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$*, then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. If $l < l'$, then $\mathbf{s}$ maps $min[j*j]-l$ to $j+2$ and $\mathbf{t}$ maps $min[j*j]-l$ to $min[j*j]-l-m$ for $m \geq 1$. If $l' < l$, then $\mathbf{s}$ maps $min[j*j]-l'$ to $min[j*j]-l'-m$

for $m \geq 1$ and $\mathbf{t}$ maps $min[j*j]-l'$ to $j+2$. If $l'=l$, then $\mathbf{s}$ maps $i+2$ to i and $\mathbf{t}$ maps $i+2$ to $i+m$. □

Proposition 3.3.23. *Assume* $\mathbf{s} = [S_1 i{*}ij{*}j(i+2)\ldots(i+k)(min[j{*}j]-1)\ldots(min[j{*}j]-l)]$ *and* $\mathbf{t} = [S_1' i*ij*j(i+2)(i+3)\ldots(i+k')(min[j*j]-1)\ldots(min[j*j]-l')]$ *for some* $k, k' \geq 2$, $l, l' \geq 1$. *Assume also that* $k \neq k'$ *or* $l \neq l'$ *and that the elements of* $\mathbf{s}$ *are the same as the elements of* $\mathbf{t}$, *then the permutations represented by* $\mathbf{s}$ *and* $\mathbf{t}$ *are distinct.*

Proof. If $l < l'$, then $\mathbf{s}$ maps $min[j*j]-l$ to $j+2$ and $\mathbf{t}$ maps $min[j*j]-l$ to $min[j*j]-l-m$ for $m \geq 1$. If $l' > l$, then reverse the roles of $\mathbf{s}$ and $\mathbf{t}$ in the previous sentence. If $l = l'$ and $k < k'$, then $\mathbf{s}$ maps $i+k+1$ to i and $\mathbf{t}$ maps $i+k+1$ to $i+k+m$ for $m \geq 2$. If $l = l'$ and $k < k'$, then reverse the roles of $\mathbf{s}$ and $\mathbf{t}$ in the previous sentence. □

By the propositions above, we have that all of the classes of reduced decompositions that were counted are distinct. Since these are the only possibilities, we conclude that the number of reduced decompositions with $[i*ij*j]$ as a factor with no other repetitions and all elements s_k in the reduced decomposition satisfying $min[i*i] < s < j+1$ is $F_{2(min[j*j]-i-2)+1} + 2\sum_{m=1}^{min[j*j]-i-2} F_{2(min[j*j]-i-2-m)+1} + (min[j*j]-i-4) + \sum_{m=1}^{min[j*j]-i-4}(min[j*j]-i-m-3)F_{2m+1}$.

Let $a = min[j*j]-i-2$, then this sum is

$$F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}.$$

We have now proved the following theorem.

Theorem 3.3.24. *Let* X *be the set of reduced decompositions* $\mathbf{s}$ *with the following properties for a fixed* i *and* j:

Table 3.6: $F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}$ for small a

a	1	2	3	4	5	6	7	8	9	10	11	12
	3	11	32	87	231	608	1595	4179	10944	28655	75023	196416

1. $\mathbf{s}$ *has* $[i*ij*j]$ *as a factor with* $[i*i] \in \{[i(i+1)i], [i(i-1)(i+1)i]\}$ *and* $[j*j]$ *defined similarly.*

2. $[i*ij*j] = [j*ji*i]$.

3. $\mathbf{s}$ *has no repeated elements other than* i *and* j.

4. $\mathbf{s}$ *only contains elements* s *that satisfy* $min[i*i] < s < j+1$.

Then $|X| = F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}$ *where* $a = min[j*j] - i - 2$.

Table 3.6 gives these numbers for small a.

In order to use Theorem 3.3.24 we need one more lemma.

Lemma 3.3.25. *Assume* $\mathbf{s} = [s_1 \dots s_k]$ *is a reduced decomposition satisfying items 1-3 of Theorem 3.3.24. Then* $\mathbf{s}$ *is equivalent to a reduced decomposition* $\mathbf{t}$ *having a factor satisfying items 1-4 of Theorem 3.3.24.*

Before we prove this lemma, let us look at an example. $\mathbf{s} = [51343787296]$ is a reduced decomposition satisfying items 1-3 of Theorem 3.3.24. $\mathbf{t}$ is then [1*534378762*9] where the italicized factor satisfies items 1-4. This lemma will allow to us to count all reduced decompositions having $[i*ij*j]$ as a factor such that $[i*ij*j] = [j*ji*i]$ by counting the number of reduced decompositions with a factor satisfying items 1-4 and then using the FCT.

Proof of Lemma 3.3.25. By induction on the cardinality of the set $X = \{s_m : s_m \in \mathbf{s}, s_m < min[i*i] \text{ or } s_m > j+1\}$. If the cardinality is 1, then choose that element $s_u \in X$. $\mathbf{s} = [s_1 \ldots s_u \ldots s_k]$. By definition of X, s_u is either greater than all other elements of $\mathbf{s}$ or smaller than all other elements of $\mathbf{s}$. s_u commutes with all elements in $\mathbf{s}$ with exception of $s_u - 1$ if s_u is the biggest element or $s_u + 1$ if s_u is the smallest element. Note, neither of these is guaranteed to exist, but it is possible. Therefore, s_u must commute with all elements in at least one of the factors $[s_1 \ldots s_{u-1}]$ or $[s_{u+1} \ldots s_k]$. Therefore either $\mathbf{s} = [s_u s_1 \ldots \hat{s_u} \ldots s_k s_u]$ or $\mathbf{s} = [s_1 \ldots \hat{s_u} \ldots s_k]$ and the base case is shown.

Now assume the lemma is true if $|X| = n$. If $|X| = n+1$, let s_u be the smallest element in $\mathbf{s}$ that is less than $min[i*i]$. If such an element does not exist, let s_u be the largest element in $\mathbf{s}$ that is greater than $j+1$. Now apply the argument for the base case to get either $\mathbf{s} = [s_u s_1 \ldots \hat{s_u} \ldots s_k]$ or $\mathbf{s} = [s_1 \ldots \hat{s_u} \ldots s_k s_u]$ and now the induction hypothesis applies to $[s_1 \ldots \hat{s_u} \ldots s_k]$. □

Theorem 3.3.26. *Let* $f(a) = F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}$. *The number of reduced decompositions having* $[i*ij*j]$ *as a factor with* $[i*ij*j] = [j*ji*i]$ *and no other repetitions is*

$$\sum_{k=k_1}^{n-2} (f(k-k_1+1) \cdot \sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$$

where k_1 *is the number of distinct elements in* $[i*ij*j]$.

Proof of Theorem 3.3.26. Let $\mathbf{s}$ be a reduced decomposition with a factor $[i*ij*j]$ satisfying items 1-4 of Theorem 3.3.24. By the FCT, the number of reduced decompositions with elements in $\{1,\ldots,n-1\}$ with $\mathbf{s}$ as a factor is $\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}$ where $k+1$ is the number of elements in the set $\{min[i*i], min[i*i]+1, \ldots, j, j+1\}$. Now, the number of different reduced decompositions satisfying items 1-4 of Theorem 3.3.24 is (by that same theorem), $f(min[j*j]-i-2)$. $min[j*j]-i-2$

counts the number of elements in the set $\{i+2, i+3, \ldots, min[j*j]-1\}$. If k_1 is the number of distinct elements that occur in $[i*ij*j]$, then $k+1-k_1$ enumerates the same set. Therefore, $f(k-k_1+1) \cdot \sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}$ counts all of the reduced decompositions with $[i*ij*j]$ as a factor and no other repetitions subject to $|\{min[i*i], min[i*i]+1, \ldots j, j+1\}| = k+1$. Therefore, to finish the count, we need only sum over all possible values of k. In order to keep the $[i*ij*j] = [j*ji*i]$ property, the smallest $|\{min[i*i], min[i*i]+1, \ldots j, j+1\}|$ can be is k_1+1, so the smallest k can be is k_1. The largest $|\{min[i*i], min[i*i]+1, \ldots j, j+1\}|$ can be is $n-1$, so the largest k can be is $n-2$ and we have the formula. □

Theorem 3.3.27. *Let* $f(a) = F_{2a+1} + 2\sum_{m=1}^{a} F_{2(a-m)+1} + (a-2) + \sum_{m=1}^{a-2}(a-m-1)F_{2m+1}$. *The formulas for the number of reduced decompositions on* $\{1, \ldots, n-1\}$ *with* $[i*ij*j]$ *as a factor such that* $[i*ij*j] = [j*ji*i]$ *with no other repetitions for each specific* $[i*ij*j]$ *are:*

1. $[i(i+1)ij(j+1)j]$.

$$\sum_{k=4}^{n-2} f(k-3)\Big(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\Big)$$

2. $[i(i-1)(i+1)ij(j+1)j]$.

$$\sum_{k=5}^{n-2} f(k-4)\Big(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\Big)$$

3. $[i(i+1)ij(j-1)(j+1)j]$.

$$\sum_{k=5}^{n-2} f(k-4)\Big(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)}\Big)$$

Table 3.7: $\sum_{k=4}^{n-2} f(k-3)(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$ for small n

n	6	7	8	9	10	11	12	13	14
	3	29	173	824	3443	13211	47759	165246	552894

Table 3.8: Reduced Decompositions with $[i(i+1)ij(j+1)j]$ for $n=6,7$

6	[121454], [3121454], [1214543]
7	[121454], [3121454], [1214543], [6121454], [1214546], [63121454], [31214546], [61214543], [12145436], [121565], [3121565], [4121565], [3412565], [43121565], [1215653], [1215654], [12156534], [12156543], [31215654], [41215653], [232565], [4232565], [2325654], [1232565] , [2325651], [14232565], [42325651], [12325654], [23256541]

4. $[i(i+1)(i-1)ij(j+1)(j-1)j]$.

$$\sum_{k=6}^{n-2} f(k-5)(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$$

Proof. These formulas are a direct application of Theorem 3.3.26. □

Since all of these formulas give the same sequence, Table 3.7 gives the first few values of n for formula 1 in Theorem 3.3.27

Table 3.8 gives the reduced decompositions for $[i(i+1)ij(j+1)j]$ factors for $n=6,7$.

3.3.4 Final Counts for Pattern Classes

We may now finally count the pattern classes that were our objective at the beginning of this chapter. In all of the theorems that follow $f(a)$ is the same as defined in the previous section.

Theorem 3.3.28. *The number of permutations in* $Av_n(3412)$ *that contain exactly*

Table 3.9: $|\pi \in Av_n(3412)$ that contain exactly two 321 patterns $|$

n	4	5	6	7	8	9	10	11	12
	3	20	92	363	1317	4530	15012	48391	152674

Table 3.10: $|\pi \in S_n$ that contain exactly one 321 and exactly one 3412 pattern $|$

n	5	6	7	8	9	10	11	12
	2	16	84	366	1434	5244	18268	61382

two 321 *patterns is*

$$3\sum_{k=1}^{n-3} F_{2k}F_{2(n-k-2)} + 2\sum_{k=3}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}) + \sum_{k=4}^{n-2} f(k-3)(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$$

(See Table 3.9.)

Proof. Theorems 3.3.5, 3.3.10, and 3.3.27. □

Theorem 3.3.29. *The number of permutations in S_n that contain exactly one* 321 *pattern and exactly one* 3412 *pattern is*

$$2\sum_{k=1}^{n-4} F_{2k}F_{2(n-k-3)} + 4\sum_{k=4}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}) + 2\sum_{k=5}^{n-2} f(k-4)(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$$

(See Table 3.10.)

Proof. Theorems 3.3.6, 3.3.11, and 3.3.27. □

Theorem 3.3.30. *The number of permutations in $Av_n(321)$ that contain exactly two* 3412 *patterns is*

$$\sum_{k=1}^{n-5} F_{2k}F_{2(n-k-4)} + 2\sum_{k=5}^{n-2}(\sum_{j=1}^{n-k-1} F_{2j}F_{2(n-j-k)}) + \sum_{k=6}^{n-2} f(k-5)(\sum_{m=1}^{n-k-1} F_{2m}F_{2(n-m-k)})$$

Table 3.11: $|\pi \in Av_n(321)$ that contain exactly one 3412 pattern $|$

n	6	7	8	9	10	11	12
	1	8	42	183	717	2622	9134

(See Table 3.11.)

Proof. Theorems 3.3.7, 3.3.12, and 3.3.27. □

Chapter 4

Bruhat Order

4.1 Survey of Known Results on Intervals and Downsets

We now turn to the study of the Bruhat order on S_n. The study of reduced decompositions leads directly to Tenner's results on Boolean permutations, Lemma 2.1.7 and Theorem 2.1.8. In [36], Tenner makes some mention of the one repetition case and discusses briefly the downset structure for such reduced decompositions (including the number of elements in such downsets). In this section, we aim to survey the current known results about downsets and intervals in the Bruhat order for S_n and to add new counting results for the size of downsets in terms of the pattern conditions.

4.1.1 Intervals

As heavily studied as the Bruhat order has been over the past decade, is perhaps surprising that not more is known about the structure of intervals in the Bruhat order. There have been many topological results which are not of interest to us here, but the structure of intervals in S_n remains for the most part open.

For a more in-depth survey of these results, the reader is advised to consult [4].

Much of the work on intervals is due to Hultman in his Ph.D. dissertation, [22]. His paper [21] forms a chapter of that dissertation.

There is only one type of interval of length 2 in the Bruhat order. It is shown in Figure 4.1.

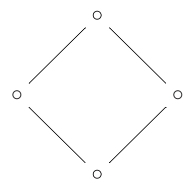

Figure 4.1: Interval of Length 2

Definition 4.1.1. *A* k-crown *is a poset that is order isomorphic to that shown in Figure 4.2.*

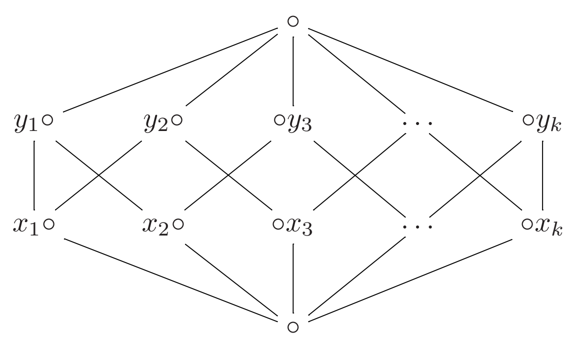

Figure 4.2: k-crown

Theorem 4.1.2. *In S_n, the only intervals of length 3 in the Bruhat order are k crowns for* $2 \leq k \leq 4$.

There are precisely seven intervals of length 4 in S_n. They can be viewed topologically as CW-complexes on the sphere. Figure 4.3 gives these seven complexes. The faces form the first level, the edges form the second level and the vertices form the third level.

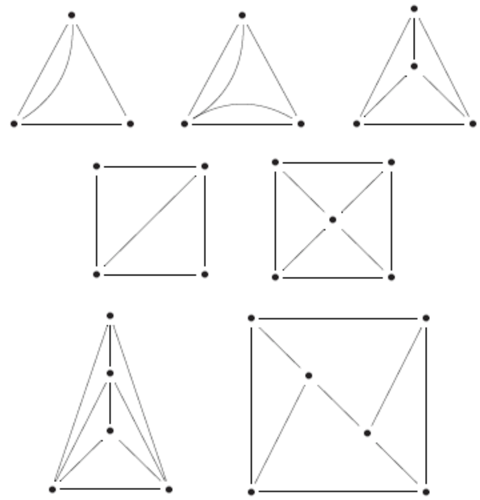

Figure 4.3: Intervals of Length 4

In [36], Tenner notes that the number of elements in the $d(\pi)$ for $\pi \in Av_n(3412)$ that contains exactly one 321 pattern is $3*2^{l(\pi)-2}$. Similarly, the number of elements in $d(\pi)$ for $\pi \in Av_n(321)$ that contain exactly one 3412 pattern is $7 * 2^{l(\pi)-3}$.

4.2 Catalogue of Downsets

We now prove a theorem which will make describing downsets for permutations described in Chapter 3 (and more besides) very easy.

Theorem 4.2.1. *Assume* $\mathbf{s} = [s_1 \dots s_k]$ *is a reduced decomposition with minimal repetition factor* $[s_u \dots s_v]$, $1 \leq u < v \leq k$. *Let* $l = v - u + 1$. *Let* d *be the number of elements in the downset of* $[s_u \dots s_v]$. *The number of elements in* $d(\mathbf{s})$, *hereafter denoted* d, *is* $2^{k-l} \cdot d$.

Proof. By induction on $k - l$. If $k - l = 0$, then $[s_1 \dots s_k] = [s_u \dots s_v]$ and hence $|d(\mathbf{s})| = 2^0 * d = d$. Now, assume the result for all $\mathbf{s}$ with $k-l = n > 0$. Since $k-l > 0$, it must be true that $\mathbf{s} = [s_1 \dots s_u \dots s_v \dots s_k]$ where either $1 \neq u$, $k \neq v$ or both. Assume first that $1 \neq u$. Then by the induction hypothesis, $|d([s_2 \dots s_u \dots s_v \dots s_k])| =$

$2^{k-l-1} \cdot d$. Recall by Theorem 1.3.4 that all elements of $d([s_1 \dots s_u \dots s_v \dots s_k])$ are subwords of $[s_1 \dots s_u \dots s_v \dots s_k]$. All subwords of $[s_1 \dots s_u \dots s_v \dots s_k]$ that do not include s_1 must be included in $d([s_2 \dots s_u \dots s_v \dots s_k])$.

All subwords of $\mathbf{s}$ that include s_1 are of the form $[s_1 w]$ for w a subword of $[s_2 \dots s_k]$. Consider the set $S = \{[s_1 w] : w \in d([s_2 \dots s_k])\}$. We claim that S is the set of all distinct subwords of $[s_1 \dots s_k]$. If $[w_1], [w_2] \in S$, then $[s_1 w_1] = [s_1 w_2]$ implies $[w_1] = [w_2]$ and hence w_1 and w_2 were not distinct. Therefore, all elements of S represent distinct subwords. Now consider $[s_1 w]$ where $[w] \notin d([s_2 \dots s_k])$. Since $[w] \notin d(s_2 \dots s_k)]$, there exists $[w'] \in d([s_2 \dots s_k])$ such that $[w] = [w']$. Therefore $[s_1 w] = [s_1 w']$ and $[s_1 w]$ has a representative in S. Now we have that $|S| = |d([s_2 \dots s_k])| = 2^{k-l-1} \cdot d$ by the induction hypothesis. Therefore, $|d([s_1 \dots s_k])| = |S| + |d([s_2 \dots s_k])| = 2(2^{k-l-1} \cdot d) = 2^{k-l} \cdot d$. □

The structure of downsets now depends entirely on the size of the reduced decomposition and the minimal repetition factor. We may now catalogue the downsets for the permutation classes described in Chapter 3.

Throughout this section l denotes the length of the permutation π.

4.2.1 Entangled Factors

[i(i-1)(i+1)i(i+1)] and [(i+1)i(i+1)(i-1)i]

The entangled factors $[i(i-1)(i+1)i(i+1)]$ and $[(i+1)i(i+1)(i-1)i]$ are symmetric, therefore we have:

Theorem 4.2.2. *If $\pi \in Av_n(3412)$ and π contains exactly two 321 patterns of the form 3421 or 4312, then $|d(\pi)| = 9 \cdot 2^{l-4}$. (See Figure 4.4)*

Proof. $|d([i(i-1)(i+1)i(i+1)]| = |d([(i+1)i(i+1)(i-1)i])| = 18$, therefore $|d(\pi)| = 2^n \cdot 18$ where $n = l(\pi) - 5$. □

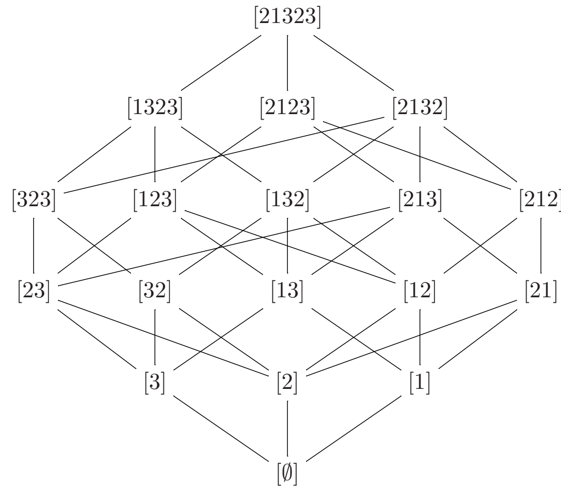

Figure 4.4: Downset of 3421

[(i+1)i(i-1)i(i+1)]

There are 20 elements in the downset of $[(i+1)i(i-1)i(i+1)]$. Therefore,

Theorem 4.2.3. *If* $\pi \in Av_(3412)$ *and* π *contains exactly two* 321 *patterns of the form* 4231*, then* $|d(\pi)| = 5 \cdot 2^{l-3}$*. (See Figure 4.5)*

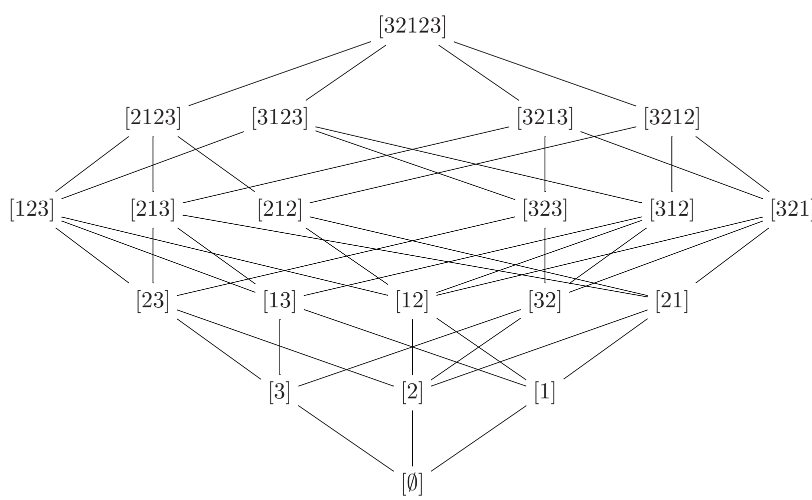

Figure 4.5: Downset of 4231

[i(i-1)(i+1)i(i+2)(i+1)] and [(i+1)(i+2)i(i+1)(i-1)i]

$[i(i-1)(i+1)i(i+2)(i+1)]$ and $[(i+1)(i+2)i(i+1)(i-1)i]$ are symmetric, therefore, since $|d[i(i-1)(i+1)i(i+2)(i+1)]| = |d([(i+1)(i+2)i(i+1)(i-1)i]| = 45$, we have:

Theorem 4.2.4. *If $\pi \in Av_n(321)$ contains exactly three 3412 patterns of the form 34512 or 45123, then $|d(\pi)| = 45 \cdot 2^{l-6}$. (See Figure 4.6)*

[i(i-1)(i+1)(i+2)(i+1)i] and [(i+1)(i+2)i(i-1)i(i+1)]

$[i(i-1)(i+1)(i+2)(i+1)i]$ and $[(i+1)(i+2)i(i-1)i(i+1)]$ are symmetric, therefore, since $|d([i(i-1)(i+1)(i+2)(i+1)i])| = d([(i+1)(i+2)i(i-1)i(i+1)])| = 44$, we have:

Theorem 4.2.5. *If $\pi \in S_n$ such that π contains exactly one 3412 and exactly one 321 pattern of the form 35142 or 42513, then $|d(\pi)| = 11 \cdot 2^{l-4}$. (See Figure 4.7)*

[i(i-1)(i+2)(i+1)i(i+3)(i+2)]

There are 102 elements in $d([i(i-1)(i+2)(i+1)i(i+3)(i+2)])$, therefore, we have:

Theorem 4.2.6. *If $\pi \in Av_n(321)$ such that π contains exactly two 3412 patterns of the form 351624 then $|d(\pi)| = 51 \cdot 2^{l-6}$.*

4.2.2 Nonentangled Factors

Theorem 4.2.7. *If **s** is a reduced decomposition with $[i * i(i+2)(i+3)\dots(min[j * j]-1)j * j]$ or $[i * i(min[i * i]-1)\dots(j+3)(j+2)j * j]$ as a factor and no other repetitions, then $|d(\mathbf{s})| = d \cdot 2^n$, where $d = |d([i * ij * j])|$.*

Proof. Mimic the proof of Theorem 4.2.1.

□

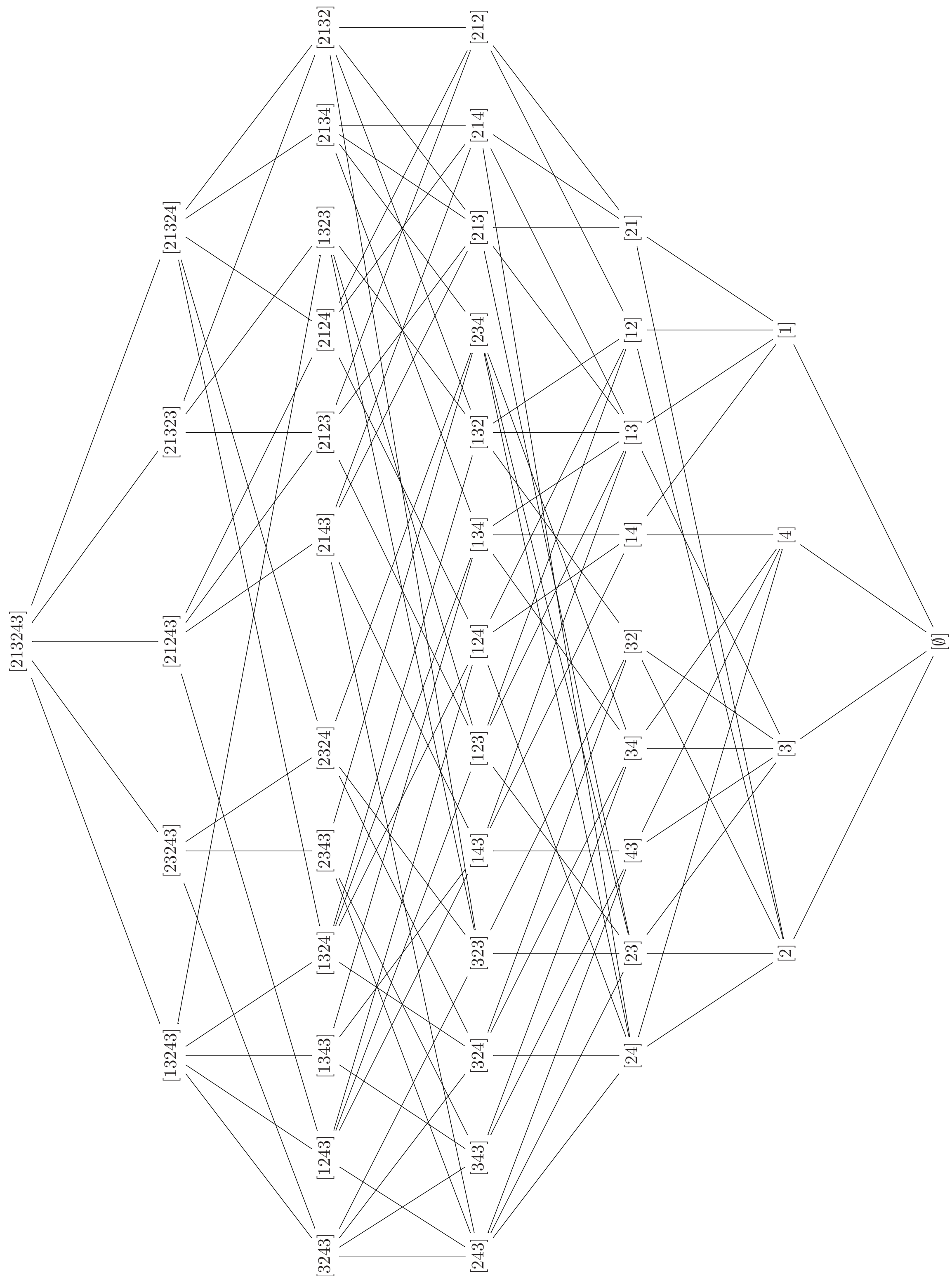

Figure 4.6: Downset of 34512

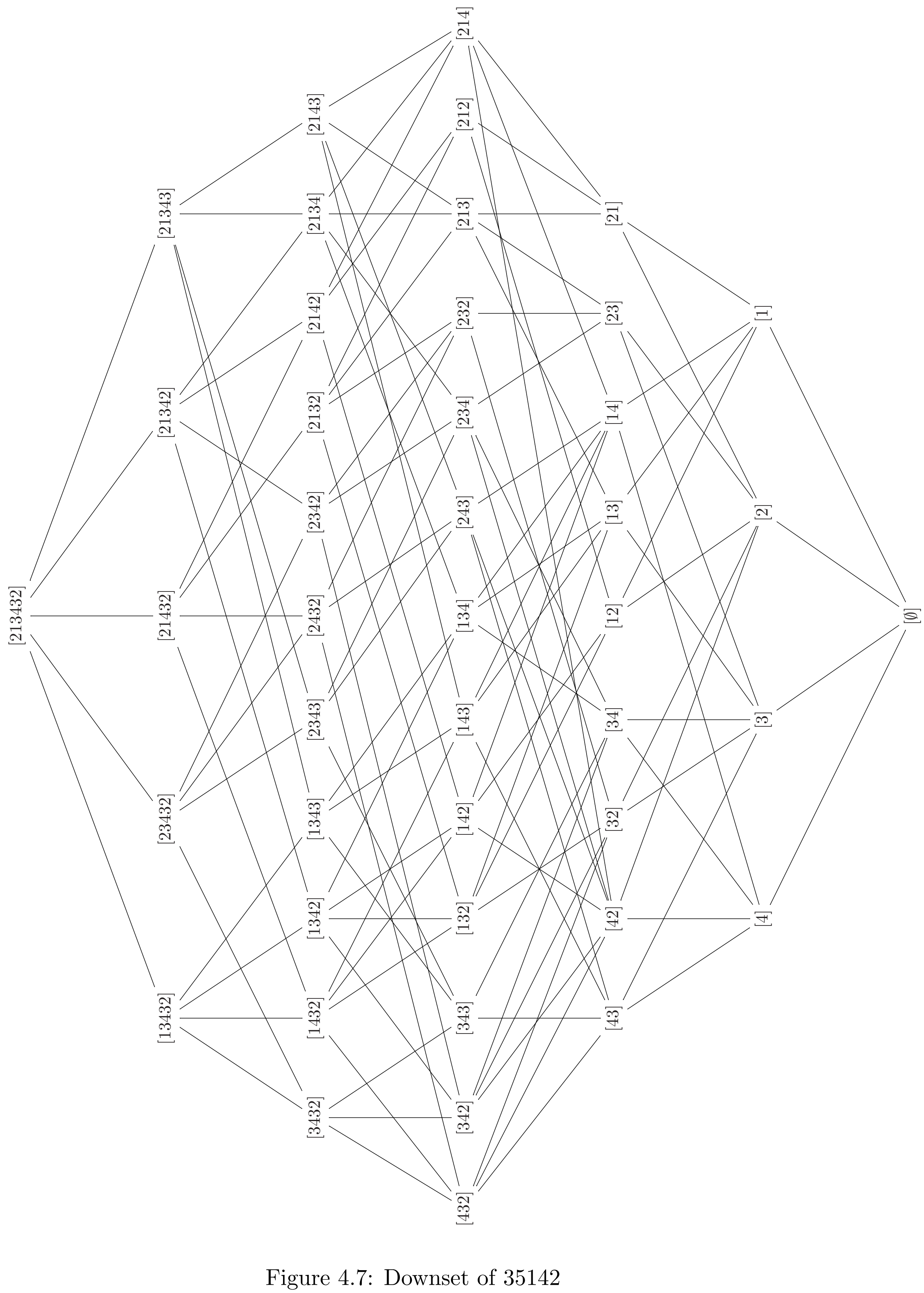

Figure 4.7: Downset of 35142

There are 36 elements in $d([i(i+1)ij(j+1)j])$. Therefore,

Theorem 4.2.8. *If $\pi \in Av_n(3412)$ such that π contains exactly two 321 patterns that share either one or zero elements , then $|d(\pi)| = 9 \cdot 2^{l-4}$. (See Figure 4.8)*

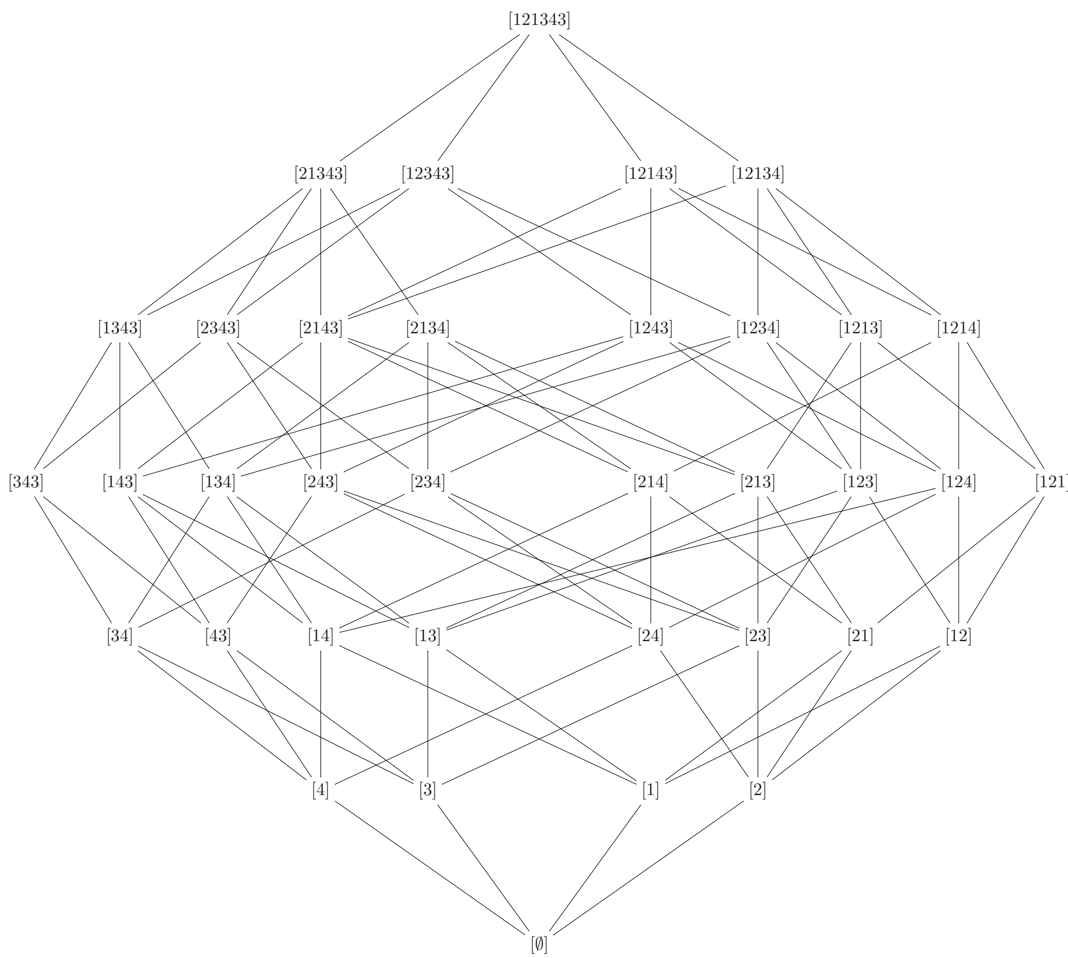

Figure 4.8: Downset of 32541

$|d([i(i+1)ij(j+1)(j-1)j]| = |d([j(j+1)ji(i-1)(i+1)i]| = 84$, therefore,

Theorem 4.2.9. *If $\pi \in S_n$ contains exactly one 321 and exactly one 3412 pattern such that the two patterns share either one or zero elements, then $|d(\pi)| = 21 \cdot 2^{l-5}$.*

$|d([i(i+1)(i-1)ij(j+1)(j-1)j])| = 196$, therefore,

Theorem 4.2.10. *If $\pi \in Av_n(321)$ such that π contains exactly two 3412 patterns that share either one or zero elements, then $|d(\pi)| = 49 \cdot 2^{l-6}$.*

4.3 Bruhat Order for the Rook Monoid

Besides the symmetric group, there are many other algebraic structures where the Bruhat order is well-defined. For example, the Bruhat order can be defined for any Coxeter group. In this section, we will define the Bruhat order for a more general algebraic structure, called the Rook Monoid, which is a generalization of the symmetric group, and attempt to classify downsets combinatorially in terms of patterns.

4.3.1 Definitions and Examples

Definition 4.3.1. *Let $n \in \mathbb{N}$. The* rook monoid*, R_n, is the set of all $\{0,1\}$ $n \times n$ matrices such that each row and column contains at most one 1.*

$S_n \subseteq R_n$, but R_n has a much more complicated structure. $\pi \in R_n$ can be represented like a permutation as a string $\pi = \pi_1 \dots \pi_n$ with the properties that for all $1 \leq i \leq n$, $\pi_i \in \{0, 1, \dots, n\}$ and if $\pi_i = \pi_j > 0$ then $i = j$. For example $R_2 = \{00, 01, 02, 10, 20, 12, 21\}$.

In general, the size of R_n is $\sum_{k=0}^{n}(\binom{n}{k}^2 \cdot (n-k)!)$.

Patterns can be defined in R_n as in S_n, where 0 is smaller than all other elements and may appear more than once.

Definition 4.3.2. *$\pi = \pi_1 \dots \pi_n \in R_n$ contains a* nonzero pattern *$\sigma = \sigma_1 \dots \sigma_m \in R_m$ if there exists $1 \leq i_1 < \dots < i_m \leq m$ such that $\pi_{i_j} < \pi_{i_k}$ if and only if $\sigma_j < \sigma_k$ and $\pi_{i_j} > 0$ for all $1 \leq j \leq m$.*

Definition 4.3.3. *Let $\pi = \pi_1 \dots \pi_n \in R_n$. Define $coinv(\pi) := \{(i,j) : 1 \leq i < j \leq n$ such that $\pi_i \pi_j$ is a nonzero 12 pattern $\}$.*

(Note, coinv is used to mean “coinversion”, but we shall not use such terminology here.)

For example, if $\pi = 31024 \in R_5$, then 324 is a nonzero 213 pattern and $coinv(31024) = \{(1,5),(2,4),(2,5),(4,5)\}$.

We shall now define the Bruhat order on rook monoids following the style of [7] and [28].

Definition 4.3.4. *Let π, $\sigma \in R_n$. The Bruhat order on R_n is the smallest partial order on R_n generated by declaring $\pi \leq \sigma$ if either $\exists i$ such that $1 \leq i \leq n$ such that $\sigma_i > \pi_i$ and $\sigma_j = \pi_j$ for all $j \neq i$ or $\exists i, j$ such that $1 \leq i < j \leq n$, such that $\sigma_i = \pi_j$, $\sigma_j = \pi_i$ and $\pi_i\pi_j$ forms a 12 pattern in π.*

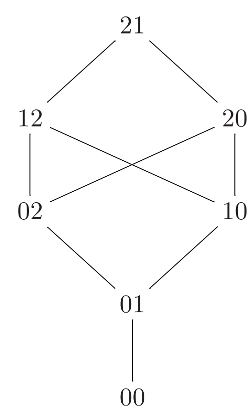

Figure 4.9: Bruhat Order on R_2

Figure 4.9 shows the Bruhat order on R_2.

Just as there is a notion of length for elements in S_n (in terms of reduced decompositions) which is respected by the Bruhat order, there is also such a notion for elements in R_n. We will follow the terminology from [7] for this definition and the reader is advised to consult the same for more details.

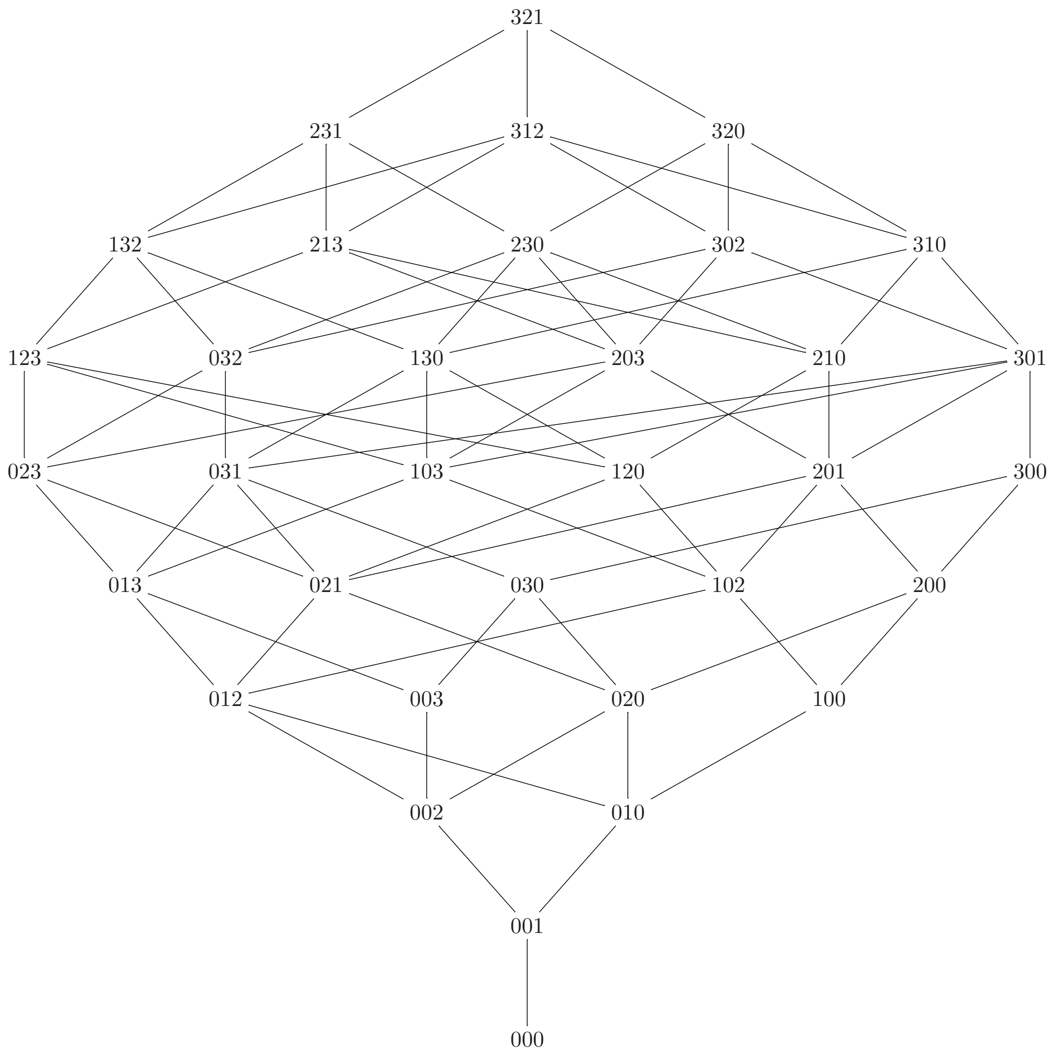

Figure 4.10: Bruhat Order on R_3

Definition 4.3.5. *Let* $\pi = \pi_1 \dots \pi_n \in R_n$*, then the* length, $l(\pi)$ *is defined to be*

$$l(x) = (\sum_{i=1}^{n} \pi_i^*) - coinv(\pi)$$

where

$$\pi_i^* = \begin{cases} \pi_i + n - i & \text{if } \pi_i \neq 0 \\ 0 & \text{else} \end{cases}$$

For example, $l(31024) = (3+5-1)+(1+5-2)+0+(2+5-4)+(4+5-5)-3 = 15$. Note, as well that the notion of length in R_n is not a generalization of the notion of length in S_n, but the Bruhat order on R_n is a generalization of that on S_n.

R_n is also a graded poset based on the length just defined.

4.3.2 Downsets in the rook monoid

In [36], Tenner shows that the only downsets, $d(\pi)$, in the Bruhat order which are lattices are those which are Boolean algebras. Hence $d(\pi)$ is a lattice for $\pi \in S_n$ if and only if π avoids 321 and 3412. This is obviously not true in the case when $\pi \in R_n$. In this section, we aim to classify which $\pi \in R_n$ yield lattice downsets and the structure of those downsets.

If $d(\pi)$, $\pi \in R_n$, contains a 2-crown, then it will not be a lattice. In the Bruhat order of R_n, there is a 2-crown very close to the bottom which will make lattice downsets very rare in this order.

Before we locate this 2-crown, let us make a notational convention: instead of writing $0 \dots 0$ for a sequence of 0's in an element of $\pi \in R_n$, we will write 0^m to denote a sequence of m zeros.

Proposition 4.3.6. *The set* $\{0^{n-1}2, 0^{n-2}10, 0^{n-2}12, 0^{n-2}20\}$ *is a 2-crown in the Bruhat order of* R_n.

Proof. $0^{n-1}2 \leq 0^{n-2}20$, $0^{n-2}10 \leq 0^{n-2}20$, $0^{n-1}2 \leq 0^{n-2}12$ and $0^{n-2}12 \leq 0^{n-2}20$.

To verify that these are indeed covering relations, we compute the lengths of all four of these elements: $l(0^{n-1}2) = l(0^{n-2}10) = 2$ and $l(0^{n-1}12) = l(0^{n-2}20) = 3$. $\square$

Note, that in order to prove that $d(\pi)$ is not a lattice, it suffices to prove that $0^{n-2}20 \leq \pi$ and $0^{n-2}12 \leq \pi$.

Lemma 4.3.7. *Let $\pi \in R_n$. If $d(\pi)$ is a lattice, then π avoids nonzero 21 patterns.*

Proof. Assume π contains a nonzero 21 pattern. Let $\pi = \pi_1 \ldots \pi_i \ldots \pi_j \ldots \pi_n$ where $i < j$ and $\pi_i\pi_j$ is the nonzero 21 pattern. This implies the following sequence of inequalities $\pi \geq 0^{i-1}\pi_i 0^{i-j-1}\pi_j 0^{n-j+1} \geq 0^{n-2}\pi_i\pi_j$. Because $\pi_i\pi_j$ is a nonzero 21, $\pi_i \geq 2$ and $\pi_j \geq 1$. We then have that $0^{n-2}\pi_i\pi_j \geq 0^{n-2}\pi_i 0 \geq 0^{n-2}20$ and $0^{n-2}\pi_i\pi_j \geq 0^{n-2}\pi_j\pi_i \geq 0^{n-2}12$. Therefore, by Proposition 4.3.6, $d(\pi)$ contains a 2-crown and hence is not a lattice. $\square$

Lemma 4.3.8. *Let $\pi \in R_n$. If $d(\pi)$ is a lattice and π contains a nonzero 12 pattern, then $\pi = 0^{i-1}10^{n-i-1}m$ for some $2 \leq m \leq n$.*

Proof. Let $\pi = \pi_1 \ldots \pi_i \ldots \pi_j \ldots \pi_n$ where $\pi_i\pi_j$ is the nonzero 12 pattern. We have the following inequalities: $\pi \geq 0^{i-1}\pi_i 0^{i-j-1}\pi_j 0^{n-j+1} \geq 0^{n-2}\pi_i\pi_j \geq 0^{n-2}12$. If $\pi_i > 1$, then we have $\pi \geq 0^{i-1}\pi_i 0^{n-i+1} \geq 0^{n-2}\pi_i 0 \geq 0^{n-2}20$. If $j < n$, then $\pi \geq 0^{j-1}\pi_j 0^{n-j+1} \geq 0^{n-2}\pi_j 0 \geq 0^{n-2}20$. In either case, $d(\pi)$ contains a 2-crown by Proposition 4.3.6 and hence is not a lattice. $\square$

Lattices are clearly very rare in the rook monoid. Let us consider what elements of the rook monoid actually have lattices as their downsets.

Lemma 4.3.9. *Let $n \in \mathbb{N}$, $1 \leq i \leq n$ and $0 \leq m \leq n$. $d(0^{i-1}m0^{n-1})$ is a lattice.*

Proof. It suffices to show $d(m0^{n-1})$ is a lattice. The elements of $d(m0^{n-1})$ are: $\{0^{i-1}m'0^{n-i} : 1 \leq i \leq n, m' \leq m\}$. We will show that every element has a meet and a join. Let $x, y \in d(m0^{n-1})$. Without loss of generality, we may assume

$x = 0^{i-1}m_1 0^{n-i}$ and $y = 0^{j-1}m_2 0^{n-j}$ for $i \leq j$. Then, $x \vee y = 0^{i-1}\max\{m_1, m_2\}0^{n-i}$ and $x \wedge y = 0^{j-1}\min\{m_1, m_2\}0^{n-j}$. □

Lemma 4.3.10. *Let $n \in \mathbb{N}$, $1 \leq i \leq n-1$ and $2 \leq m \leq n$. $d(0^{i-1}10^{n-i-1}m)$ is a lattice.*

Proof. It suffices to show that $d(10^{n-2}m)$ is a lattice. The elements of the downset of $10^{n-2}m$ are: $\{0^{i-1}10^{n-i-1}m' : 1 \leq i \leq n-1, 2 \leq m' \leq n-1\} \cup \{0^{n-1}m' : 0 \leq m' \leq m\} \cup \{0^{i-1}10^{n-i} : 1 \leq i \leq n-1\}$. Let $x, y \in d(10^{n-2}m)$. We shall exhibit $x \vee y$ and $x \wedge y$ for all x and y. If both x and y come from $\{0^{n-1}m' : 0 \leq m' \leq m\} \cup \{0^{i-1}10^{n-i} : 1 \leq i \leq n-1\}$, then $x \vee y$ and $x \wedge y$ are as in Lemma 4.3.9. Now assume $x = 0^{i-1}10^{n-i-1}m_1$.

1. If $y = 0^{j-1}10^{n-j-1}m_2$ for $i \leq j$, then $x \vee y = 0^{i-1}10^{n-i-1}\max\{m_1, m_2\}$ and $x \wedge y = 0^{j-1}10^{n-j-1}\min\{m_1, m_2\}$.

2. If $y = 0^{n-1}m_2$, then $x \vee y = 0^{i-1}10^{n-1}\max\{m_1, m_2\}$ and $x \wedge y = 0^{n-2}\min\{m_1, m_2\}$.

3. If $y = 0^{j-1}10^{n-j}$, then $x \vee y = 0^{min\{i,j\}-1}10^{n-min\{i,j\}-1}m_1$ and $x \wedge y = 0^{\max\{i,j\}-1}10^{n-\max\{i,j\}}$.

□

By Lemmas 4.3.7, 4.3.8, 4.3.9 and 4.3.10, we have the following theorem.

Theorem 4.3.11. *Let $\pi \in R_n$. Then $d(\pi)$ is a lattice if and only if $\pi = 0^{i-1}10^{n-i-1}m$ for some $1 \leq i \leq n-1$ and $m \geq 1$ or $\pi = 0^{i-1}m0^{n-i}$ for some $1 \leq i \leq n$ and $m \geq 0$.*

Figures 4.11 and 4.12 show examples of these types of lattices.

Theorem 4.3.11 describes which elements of R_n will yield lattice downsets, but does not describe what kind of lattices they are. In the S_n case, all lattice downsets are Boolean algebras. This is not the case for R_n.

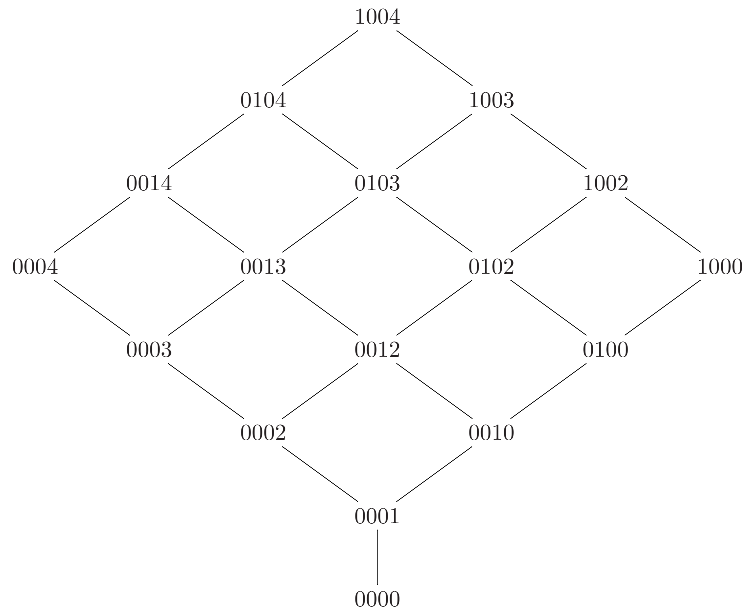

Figure 4.11: Downset of 1004 in R_4

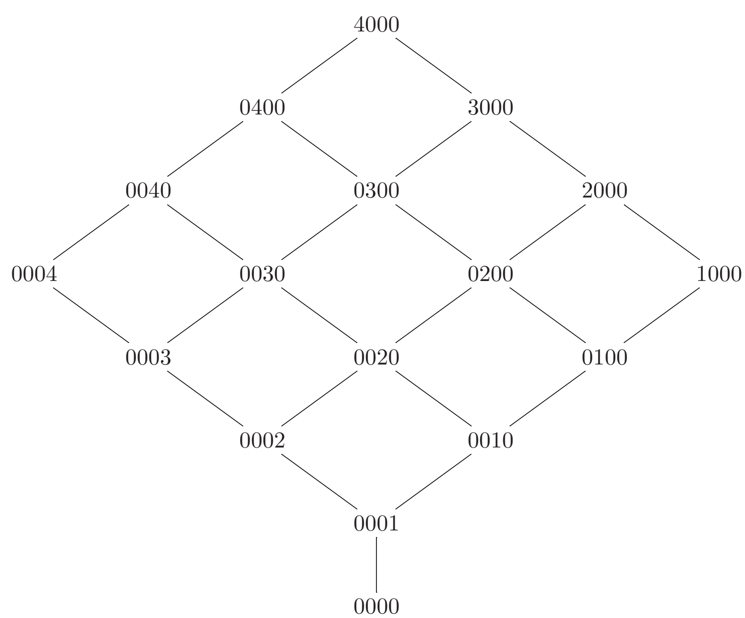

Figure 4.12: Downset of 4000 in R_4

Lemma 4.3.12. *Let $n \in \mathbb{N}$ and $m \in \{0, \ldots, n\}$. The posets $d(10^{n-2}m)$ and $d(m0^{n-1})$ in R_n are isomorphic.*

Proof. Define $\varphi : d(m0^{n-1}) \to d(10^{n-2}m)$ by

$$\varphi(0^{i-1}m'0^{n-i}) = \begin{cases} 0^{i-1}m'0^{n-i} & \text{if } m' = 1 \text{ or } i = n \\ 0^{i-1}10^{n-i-1}m' & \text{else} \end{cases}$$

To show that φ is a homomorphism, let $0^{i-1}m_1 0^{n-i} \le 0^{j-1}m_2 0^{n-j}$. This implies that $m_1 \le m_2$ and $i \ge j$.

- If $\varphi(0^{i-1}m_1 0^{n-i}) = 0^{i-1}m_1 0^{n-i}$ and $\varphi(0^{j-1}m_2 0^{n-j}) = 0^{j-1}m_2 0^{n-j}$, then we are done.
- If $\varphi(0^{i-1}m_1 0^{n-i}) = 0^{i-1}m_1 0^{n-i}$ and $\varphi(0^{j-1}m_2 0^{n-j}) = 0^{j-1}10^{n-j-1}m_2$, then either $m_1 = 1$ or $i = n$. If $m_1 = 1$, then $0^{i-1}10^{n-i} \le 0^{j-1}10^{n-j-1}m_2$ and we are done. If $i = n$, then $0^{n-1}m_1 \le 0^{n-1}m_2 \le 0^{j-1}10^{n-j-1}m_2$ and again we are done.
- If $\varphi(0^{i-1}m_1 0^{n-i}) = 0^{i-1}10^{n-i-1}m_1$ and $\varphi(0^{j-1}m_2 0^{n-j}) = 0^{j-1}m_2 0^{n-j}$, then $m_1 > 1$ and either $m_2 = 1$ or $j = n$. If $m_2 = 1$, then $0^{i-1}m_1 0^{n-i} \le 10^{n-j}$ which is impossible since $m_1 > 1$. If $j = n$, then $0^{i-1}m_1 0^{n-i} \le 0^{n-1}m_2$ which is also a contradiction. This case is therefore impossible.
- If $\varphi(0^{i-1}m_1 0^{n-i}) = 0^{i-1}10^{n-i-1}m_1$ and $\varphi(0^{j-1}m_2 0^{n-j}) = 0^{j-1}10^{n-j-1}m_2$, then since $i \ge j$ and $m_1 \le m_2$ we have $0^{i-1}10^{n-i-1}m_1 \le 0^{i-1}10^{n-i-1}m_2 \le 0^{j-1}10^{n-j-1}m_2$.

We have that φ is a poset homomorphism.

The fact that φ is a bijection is routine and is omitted. □

To show the structure of these downsets, it is now sufficient by Lemma 4.3.12 to concentrate on $d(m0^{n-1})$. Any element in the downset of $m0^{n-1}$ has at most two elements covering it and itself covers at most two elements. To see this, consider $0^{i-1}m'0^{n-i}$ for $0 \le i \le n$ and $m' \le m$. $l(0^{i-1}m'0^{n-i}) = m' + n - i$. The only two possible elements, if they exist, that can cover $0^{i-1}m'0^{n-i}$ are $0^{i-2}m'0^{n-i+1}$ and $0^{i-1}(m'+1)0^{n-i}$. The inequalities certainly hold. To show that the covering relation holds it suffices to compute the lengths of both elements since the Bruhat order is graded. $l(0^{i-2}m'0^{n-i+1}) = m' + n - (i-1) = m' + n - i + 1 = l(0^{i-1}m'0^{n-i}) + 1 = m' + 1 + n - i = l(0^{i-1}(m'+1)0^{n-i})$. If any element is greater than $0^{i-1}m'0^{n-i}$, then it must be of the form $0^{j-1}m''0^{n-j}$ where $j \le i$ or $m'' \ge m'$. The length of such an element must be $m'' + n - j \ge m' + n - i$. $m'' + n - j = m' + n - i + 1$ if and only if $j = i$ and $m'' = m' + 1$ or $j = i + 1$ and $m'' = m'$. A similar argument shows that the only two elements $0^{i-1}m'0^{n-i}$ can cover are $0^{i}m'0^{n-i-1}$ and $0^{i-1}(m'-1)0^{n-i}$. See Figure 4.13.

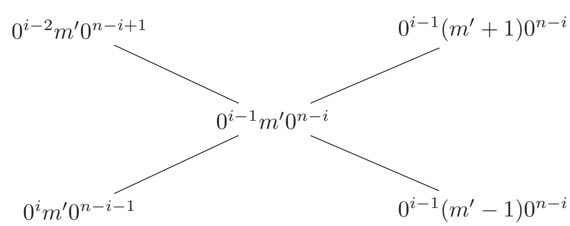

Figure 4.13: Covering relations for $0^{i-1}m'0^{n-i}$.

We therefore conclude,

Theorem 4.3.13. *Let $\pi \in R_n$. If $d(\pi)$ is a lattice, then the interval $[0^{n-1}1, \pi]$ is a direct product of chains.*

Chapter 5

How Permutations Displace Points and Stretch Intervals

This chapter constitutes the paper [11] of the same name which is joint work with my advisor Prof. Petr Vojtěchovský. Please note some of the concepts in this chapter are independent of those in previous chapters and so the notation from the previous chapters does not carry through to the present chapter.

5.1 Motivation and introduction

Allow us to begin with a motivation from the area of turbo coding [20, 32]: Starting with the very first example [1], every turbo code employs a permutation, called the *interleaver*. Although the interleaver has several functions within the coding process, its main objective is to scramble the input bits so that input sequences with a few nonzero bits do not produce output sequences with many nonzero bits, upon being encoded with a convolutional code. The interleaver is typically of length at least one thousand.

While it is easy to simulate the transmission channel and measure the performance of a turbo code with a particular interleaver statistically, it appears to be difficult to characterize those permutations that will perform well as interleavers without actually testing them. Indeed, early publications on turbo coding recommend to select the interleaver at random—an advice still followed in practice.

Nevertheless, it has now become clear that it is sometimes possible to match or outperform random interleavers with deterministic or semi-random interleavers by carefully analyzing the channel and the decoding algorithm, among other parameters.

As an illustration, we mention three properties of permutations that have been suggested in the literature as desirable for the purposes of turbo coding. Let n be an integer, S_n the set of permutations on $\{1, \dots, n\}$, and $\pi \in S_n$. Then:

(a) π should have no fixed points and, more generally, the *delay* $i - \pi(i)$ should be far from zero for every i [16, 29],

(b) the quantity $\min\{|i-j|+|\pi(i)-\pi(j)|;\ 1 \le i < j \le n\}$ should be large [12, 29],

(c) the *dispersion* $|\{(i-j, \pi(i)-\pi(j));\ 1 \le i < j \le n\}| \cdot (n(n-1)/2)^{-1}$ should be large [34, 20].

Viewed in this way, interleaver design is very much a combinatorial problem.

In this paper, we define and discuss two properties of permutations similar to (a)–(c), namely *displacement* and *stretch*. Most of our arguments are combinatorial in nature and no knowledge of coding is needed. While the results obtained here can be considered complete from the mathematical point of view (in their narrow scope), the investigation of the impact of the results on turbo coding is in preliminary stages, is carried out by a different group of researchers, and is mentioned only once below.

Here are the two properties and a summary of results:

5.1.1 Displacement

For $\pi \in S_n$, let

$$\mathrm{d}(\pi) = \sum_{i=1}^{n} \frac{|i - \pi(i)|}{n}. \tag{5.1.1}$$

The value $\mathrm{d}(\pi)$ has been defined in [16, Thm. 2], where it is called descriptively *the average of the absolute values of the delays*. We prefer to call it the *displacement* of π, and $\mathrm{d}(\pi)/n$ the *normalized displacement* of π.

We prove that the normalized displacement of a permutation ranges between 0 and 1/2, and we find all permutations with extreme displacement. Among all permutations in S_n, the average normalized displacement approaches 1/3 as n approaches ∞. Moreover, the distribution of displacements is such that a long, randomly chosen permutation will very likely have normalized displacement close to 1/3.

Hence, by selecting the interleaver at random, the class of permutations with large or small displacement is rarely (never!) put to the test. Preliminary results of Ramya Chandramohan [8] indicate that an S-random interleaver (see [12]) with larger than average displacement performs slightly better than an S-random interleaver.

It is easy to construct permutations with normalized displacement arbitrarily close to a given $0 \leq d \leq 1/2$. The problem is more difficult when the permutation is supposed to have additional properties.

5.1.2 Stretching

The two quantities defined in (b), (c) are telling us something about how the permutation π stretches intervals. To measure the average stretch of an arbitrary collection $\mathcal{A}$ of subsets of $N = \{1, \ldots, n\}$, we propose the following two definitions:

For $A \subseteq N$, let $\mathrm{diam}(A) = \max\{i;\ i \in A\} - \min\{i;\ i \in A\}$. When $\mathcal{A} \subseteq 2^N$ and

$\pi \in S_n$, let

$$\mathrm{s}^+_{\mathcal{A}}(\pi) = |\mathcal{A}|^{-1} \cdot \left(\sum_{A \in \mathcal{A}} \frac{\mathrm{diam}(\pi(A))}{\mathrm{diam}(A)} \right), \tag{5.1.2}$$

and

$$\mathrm{s}^*_{\mathcal{A}}(\pi) = \left(\prod_{A \in \mathcal{A}} \frac{\mathrm{diam}(\pi(A))}{\mathrm{diam}(A)} \right)^{1/|\mathcal{A}|}. \tag{5.1.3}$$

We call both formulas the *stretch of* π *with respect to* $\mathcal{A}$. Formula (5.1.3), which gives equal weight to relative stretching and shrinking, is merely the multiplicative version of (5.1.2).

Since the formulas (5.1.2), (5.1.3) emphasize average stretch instead of extreme stretch, they become trivial when $\mathcal{A} = 2^N$, $\mathcal{A} = \{\{i,j\};\ i < j \in N\}$, etc. However, they are not meaningless. For instance, when $n = 3$ and $\mathcal{A} = \{\{1,2\},\{2,3\}\}$, we have $\mathrm{s}^+_{\mathcal{A}}((1,3,2)) = 3/2 > 1 = \mathrm{s}^+_{\mathcal{A}}(\mathrm{id})$ and $\mathrm{s}^*_{\mathcal{A}}((1,3,2)) = \sqrt{2} > 1 = \mathrm{s}^*_{\mathcal{A}}(\mathrm{id})$, as one would expect.

It appears to be hopelessly complicated to analyze s^+ and s^* for an arbitrary collection $\mathcal{A}$. We therefore focus on stretching with respect to $\mathcal{B} = \{\{i, i+1\};\ 1 \le i < n\}$.

Roughly speaking, the additive formula (5.1.2) with $\mathcal{A} = \mathcal{B}$ is maximized by any permutation that starts in the middle of the interval N and keeps oscillating between the two halves of N. The multiplicative formula (5.1.3) with $\mathcal{A} = \mathcal{B}$ leads to a much more intricate solution. The maximum of s^* is

$$\begin{aligned} &(m^m m^{m-1})^{1/(n-1)}, && \text{when } n = 2m, \text{ and} \\ &(m^m (m+1)(m+2)^{m-1})^{1/(n-1)}, && \text{when } n = 2m+1. \end{aligned}$$

(See Acknowledgement.) Furthermore, the maximum is attained by two permutations when n is even, and by four permutations when $n > 1$ is odd.

5.2 Displacement

5.2.1 Average displacement

We are first going to determine the average value of $\mathrm{d}(\pi)$ over all permutations $\pi \in S_n$. The formula (5.2.1) can be obtained by combining Theorems 2 and 4 of [16] but our proof is shorter and more straightforward.

Theorem 5.2.1. *Let $n \geq 1$ be an integer. Then*

$$\frac{1}{n!} \sum_{\pi \in S_n} \mathrm{d}(\pi) = \frac{n^2 - 1}{3n}. \tag{5.2.1}$$

Proof. Pick $m \in N$. Since the number of permutations $\pi \in S_n$ mapping m onto some m' is equal to $(n-1)!$, we have

$$\frac{1}{n!} \sum_{\pi \in S_n} |m - \pi(m)| = \frac{((m-1) + \cdots + 1) + (1 + \cdots + (n-m))}{n}.$$

Thus

$$\begin{aligned}
\frac{1}{n!} \sum_{\pi \in S_n} \mathrm{d}(\pi) &= \frac{1}{n!} \sum_{\pi \in S_n} \frac{1}{n} \sum_{m=1}^{n} |m - \pi(m)| = \frac{1}{n} \sum_{m=1}^{n} \frac{1}{n!} \sum_{\pi \in S_n} |m - \pi(m)| \\
&= \frac{1}{n} \sum_{m=1}^{n} \frac{(m-1)m + (n-m)(n-m+1)}{2n} \\
&= \frac{1}{n} \sum_{m=1}^{n} \frac{(n-m)^2 + (m-1)^2 + n - 1}{2n}.
\end{aligned}$$

We now note that

$$\sum_{m=1}^{n} (n-m)^2 = \frac{(n-1)n(2n-1)}{6} = \sum_{m=1}^{n} (m-1)^2,$$

and the result follows. ☐

The average displacement over all permutations from S_n is therefore about $n/3$.

Asymptotically:

Corollary 1. *We have*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n!} \sum_{\pi \in S_n} \mathrm{d}(\pi) = \frac{1}{3}.$$

5.2.2 Extreme displacement

The minimal displacement $\mathrm{d}(\pi) = 0$ is attained by exactly one permutation—the identity permutation. The dual question concerning maximal displacement is more interesting.

Let us call a permutation $\pi \in S_n$ *crossing* if for every i, j in N the two closed intervals $[i, \pi(i)]$, $[j, \pi(j)]$ intersect (possibly at a single point). Otherwise, π is said to be *noncrossing*.

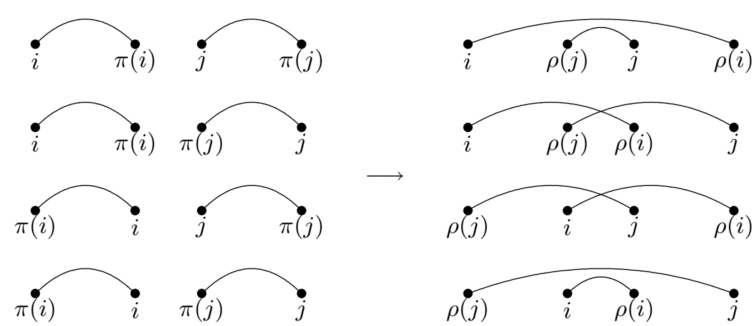

Figure 5.1: Increasing displacement of noncrossing permutations

Lemma 5.2.2. *Let $\pi \in S_n$ be a noncrossing permutation. Then there is $\rho \in S_n$ with $\mathrm{d}(\rho) > \mathrm{d}(\pi)$.*

Proof. Since π is noncrossing, there are $i < j$ in N such that the intervals $[i, \pi(i)]$, $[j, \pi(j)]$ are disjoint. Let $\rho = \pi \circ (i, j)$, where the transposition (i, j) is applied first. Then

$$|i - \rho(i)| + |j - \rho(j)| = |i - \pi(i)| + |j - \pi(j)| + 2(\min\{j, \pi(j)\} - \max\{i, \pi(i)\}),$$

which is perhaps best apparent from Figure 5.1. Since $i < j$ and π is noncrossing, the term $\min\{j, \pi(j)\} - \max\{i, \pi(i)\}$ is positive, proving that $\mathrm{d}(\rho) > \mathrm{d}(\pi)$. □

Now when we have seen that only crossing permutations can attain maximal displacement, we characterize them.

Lemma 5.2.3. *Let $\pi \in S_n$. If $n = 2m$ then π is crossing if and only if it maps $\{1, \ldots, m\}$ onto $\{m + 1, \ldots, n\}$. If $n = 2m + 1$ then π is crossing if and only if it maps $\{1, \ldots, m\}$ to $\{m + 1, \ldots, n\}$ and $\{m + 2, \ldots, n\}$ to $\{1, \ldots, m + 1\}$.*

Proof. Suppose first that $n = 2m$. Assume that π is crossing. If there is $i \in \{1, \ldots, m\}$ with $\pi(i) \in \{1, \ldots, m\}$ then, by the pigeon-hole principle, there must also be $j \in \{m+1, \ldots, n\}$ with $\pi(j) \in \{m+1, \ldots, n\}$. But then the points i, j and their images $\pi(i)$, $\pi(j)$ witness that π is noncrossing, a contradiction. Conversely, every permutation π mapping $\{1, \ldots, m\}$ onto $\{m+1, \ldots, n\}$ must also map $\{m+1, \ldots, n\}$ onto $\{1, \ldots, m\}$, and hence is a crossing permutation.

Now suppose that $n = 2m + 1$. Assume that π is crossing and that $\pi(m + 1) \geq m + 1$. Then the image of $\{1, \ldots, m\}$ must be contained in $\{m + 1, \ldots, n\}$, which forces π to map $\{m + 2, \ldots, n\}$ onto $\{1, \ldots, m\}$. Similarly when π is crossing and $\pi(m + 1) \leq m + 1$. Conversely, assume that π maps $\{1, \ldots, m\}$ to $\{m + 1, \ldots, n\}$ and $\{m + 2, \ldots, n\}$ to $\{1, \ldots, m + 1\}$. Looking at two points at a time, it is easy to see that π is crossing. □

Note that the odd case of Lemma 5.2.3 imposes no restriction on the image of the midpoint $m + 1$. Nevertheless, once $m + 1$ is mapped somewhere, condition (ii) of Lemma 5.2.3 forces π to behave in a certain way. For instance, when $\pi(m + 1) > m + 1$, it follows that $\pi^{-1}(m + 1) < m + 1$. We will need this fact in the next theorem.

Theorem 5.2.4. *Given $n \geq 1$, let $d_n = \max\{\mathrm{d}(\pi);\ \pi \in S_n\}$, and $D_n = \{\pi \in$*

$S_n;\ \mathrm{d}(\pi) = d_n\}$. *Then* $\pi \in D_n$ *if and only if* π *is crossing. Moreover,* $d_n = n/2$ *when* n *is even, and* $d_n = (n-1)(n+1)(2n)^{-1}$ *when* n *is odd.*

Proof. Suppose that $n = 2m$, and let $\pi \in S_n$ be a crossing permutation. By Lemma 5.2.3, π maps $\{1, \ldots, m\}$ onto $\{m+1, \ldots, n\}$ and vice versa. Therefore

$$\begin{aligned} n\mathrm{d}(\pi) &= \sum_{i=1}^{m} |i - \pi(i)| + \sum_{i=m+1}^{n} |i - \pi(i)| \\ &= \sum_{i=1}^{m} (\pi(i) - i) + \sum_{i=m+1}^{n} (i - \pi(i)) \\ &= 2\left(\sum_{i=m+1}^{n} i - \sum_{i=1}^{m} i\right) = 2\left(\frac{n(n+1)}{2} - 2 \cdot \frac{m(m+1)}{2}\right) = \frac{n^2}{2}. \end{aligned}$$

This short calculation proves that, as far as π is crossing, the value of $\mathrm{d}(\pi)$ is independent of π and is equal to $n/2$. The set D_n then coincides with crossing permutations by Lemma 5.2.2, and $d_n = n/2$ follows.

Suppose that $n = 2m+1$, and let $\pi \in S_n$ be a crossing permutation. If $\pi(m+1) \neq m+1$, we construct a crossing permutation ρ with $\rho(m+1) = m+1$ satisfying $\mathrm{d}(\rho) = \mathrm{d}(\pi)$ as follows: Without loss of generality, suppose $c = \pi(m+1) > m+1$. Then $a = \pi^{-1}(m+1) < m+1$, as we have remarked before this theorem. Let $\rho(a) = c$, $\rho(c) = a$, $\rho(m+1) = m+1$ and $\rho(k) = \pi(k)$ for $k \notin \{a, m+1, c\}$. By the construction, $\mathrm{d}(\pi) = \mathrm{d}(\rho)$.

We can therefore assume that the crossing permutation π fixes $m+1$. Then, by Lemma 5.2.3,

$$\begin{aligned} n\mathrm{d}(\pi) &= \sum_{i=1}^{m} (\pi(i) - i) + \sum_{i=m+2}^{n} (i - \pi(i)) \\ &= 2\left(\sum_{i=m+2}^{n} i - \sum_{i=1}^{m} i\right) = 2m(m+1) = \frac{(n-1)(n+1)}{2}. \end{aligned}$$

As in the even case, we see that the value of $\mathrm{d}(\pi)$ does not depend on π, that D_n

consists exactly of all crossing permutations, and that $d_n = (n-1)(n+1)(2n)^{-1}$. □

5.2.3 Distribution of displacements

The reader may wish to select a permutation π of length $n = 1000$ at random and calculate its displacement $\mathrm{d}(\pi)$. We predict that $330 < \mathrm{d}(\pi) < 336$. We could be wrong, of course, as there are permutations with displacement ranging from 0 to $n/2$. Using the characterization of permutations with maximal displacement (Lemma 5.2.3), we count exactly $(m!)^2$ such permutations in the even case $n = 2m$. The ratio $(2m)!/((m!)^2)$ approaches 0 exponentially fast, so such permutations are rare. This is an instance of a much more general notion known to measure theorists as *concentration of measure phenomena*. Let us talk about it briefly, imitating [27, Ch. 6].

Let (X, ρ, μ) be a metric space equipped with a Borel probability measure μ. For a subset A of X and $\varepsilon > 0$ define $A_\varepsilon = \{x \in X;\ \rho(x, A) \leq \varepsilon\}$, where $\rho(x, A)$ is the distance of x from the set A. The *concentration function* $\alpha(X,\ ) : \mathbb{R}^+ \to \mathbb{R}_0^+$ is defined by

$$\alpha(X, \varepsilon) = 1 - \inf\{\mu(A_\varepsilon);\ A \subseteq X, A \text{ is Borel}, \mu(A) \geq 1/2\}.$$

In words, $\alpha(X, \varepsilon)$ measures how much space remains in X when one half of X is inflated by ε.

Let $\mathcal{X} = \{(X_n, \rho_n, \mu_n);\ n = 1, 2, \dots\}$ be a family of metric probability spaces. Then $\mathcal{X}$ is called a *normal Levy family* with constants c_1, c_2 if for every $\varepsilon > 0$ and for every n we have $\alpha(X_n, \varepsilon) \leq c_1 e^{-c_2 \varepsilon^2 n}$.

Let ρ_n be the (normalized Hamming) metric on S_n defined by

$$\rho_n(\pi, \sigma) = \frac{1}{n} |\{i;\ \pi(i) \neq \sigma(i)\}|,$$

and let μ_n be the (normalized counting) measure on S_n defined by

$$\mu_n(\pi) = \frac{1}{n!}.$$

Then $\{(S_n, \rho_n, \mu_n)\}$ is a normal Levy family with constants $c_1 = 2$, $c_2 = 1/64$, according to [27, Sec. 6.4].

Although the defining condition for normal Levy families only restricts the interplay of the measure and the metric in (X_n, ρ_n, μ_n), one can say a lot about the behavior of reasonable functions $f_n : X_n \to \mathbb{R}$. We will assume here that f_n is Lipschitz with constant 1 (i.e., $|f_n(x) - f_n(y)| \leq \rho_n(x, y)$ for every x, $y \in X_n$), but a more general requirement would do (cf. [27]).

So, assume that $f : (X, \rho, \mu) \to \mathbb{R}$ is Lipschitz with constant 1. Denote by M_f the median value of f on X, and let $A = \{x \in X;\ f(x) \leq M_f\}$, $B = \{x \in X;\ f(x) \geq M_f\}$. Then, by definition, $\mu(A) \geq 1/2$, $\mu(B) \geq 1/2$, and $\mu(\{x \in X;\ |f(x) - M_f| \leq \varepsilon\}| \geq \mu(A_\varepsilon \cap B_\varepsilon) \geq 1 - 2\alpha(X, \varepsilon)$. When $X = X_n$ is a member of a normal Levy family, we thus obtain

$$\mu(\{x \in X_n;\ |f(x) - M_f| \leq \varepsilon\}) \geq 1 - 2c_1 e^{-c_2 \varepsilon^2 n}.$$

When $X_n = S_n$ is equipped with the above metric and measure, we get

$$\mu(\{x \in X_n;\ |f(x) - M_f| \leq \varepsilon\}) \geq 1 - 4e^{-\varepsilon^2 n/64}.$$

This inequality explains why the values of f on S_n are packed near the median. Moreover, with such a spike in the distribution, *the median will be close to the average value of f.*

We are about to clinch the argument with the following observation:

Proposition 5.2.5. *Let (S_n, ρ_n, μ_n) be as above. Then all functions $f_n : S_n \to \mathbb{R}$*

defined by $f_n(\pi) = \mathrm{d}(\pi)/n$ *are Lipschitz with constant* 1.

Proof. Let π, σ be two permutations in S_n. Then

$$\begin{aligned}\frac{1}{n}|\mathrm{d}(\pi) - \mathrm{d}(\sigma)| &= \frac{1}{n^2}\left|\sum_{i=1}^{n}|i-\pi(i)| - \sum_{i=1}^{n}|i-\sigma(i)|\right| \\ &\leq \frac{1}{n^2}\left|\sum_{i=1}^{n}|i-\pi(i)-i+\sigma(i)|\right| = \frac{1}{n^2}\sum_{i=1}^{n}|\pi(i)-\sigma(i)| \\ &\leq \frac{1}{n^2}\cdot n \cdot |\{i; \pi(i) \neq \sigma(i)\}| = \rho_n(\pi,\sigma),\end{aligned}$$

and we are through. □

5.2.4 Prescribed displacement

Since S_n is finite, the values of $\mathrm{d}(\pi)/n$ for a fixed n cannot cover the interval $[0, 1/2]$. However, we can get arbitrarily close to any value in $[0, 1/2]$ if we allow n to be sufficiently large; as we are going to show.

The idea is to leave π identical on a certain proportion of N and displace the remaining points as much as possible.

Proposition 5.2.6. *Let d be such that $0 \leq d \leq 1/2$. Then there is a sequence of permutations $\pi_n \in S_n$ such that* $\lim_{n\to\infty} \mathrm{d}(\pi_n)/n = d$.

Proof. Let $\delta = \sqrt{2d}$, and let $u_n = \lceil \delta n/2 \rceil$. Define $\pi_n \in S_n$ as follows:

$$\pi(i) = \begin{cases} i + u_n, & 1 \leq i \leq u_n, \\ i - u_n, & u_n + 1 \leq i \leq 2u_n, \\ i, & i > 2u_n. \end{cases}$$

Then $\mathrm{d}(\pi_n)/n = 2u_n^2/n^2 = 2\lceil \delta n/2 \rceil^2/n^2$. Since both $2(\delta n/2)^2/n^2$ and $2(\delta n/2 + 1)^2/n^2$ tend to $\delta^2/2 = d$ when n approaches infinity, we are done by the Squeeze theorem. □

5.3 Stretching with additive formula

In this section, we answer the following question: *For which permutations* $\pi \in S_n$ *is* $\mathrm{s}^+_{\mathcal{B}}(\pi)$ *maximal, where* $\mathcal{B} = \{\{i, i+1\};\ 1 \le i < n\}$? Note that with this choice of $\mathcal{B}$ we have

$$\mathrm{s}^+_{\mathcal{B}}(\pi) = \frac{|\pi(1)-\pi(2)| + |\pi(2)-\pi(3)| + \cdots + |\pi(n-1)-\pi(n)|}{n-1}.$$

For two subsets A, B of N, we say that $\pi \in S_n$ *oscillates* between A and B if for every $1 \le i < n$ we have either $\pi(i) \in A$, $\pi(i+1) \in B$, or $\pi(i) \in B$, $\pi(i+1) \in A$.

Theorem 5.3.1. *The maximum value of* $\mathrm{s}^+_{\mathcal{B}}(\pi)$ *over all* $\pi \in S_n$ *is*

$$\begin{aligned} &(2m^2-1)/(2m-1) && \text{when } n = 2m, \text{ and} \\ &(2m^2+2m-1)/(2m) && \text{when } n = 2m+1. \end{aligned}$$

When $n = 2m$*, the maximum is attained by precisely those permutations* π *that oscillate between* $\{1, \dots, m\}$, $\{m+1, \dots, n\}$ *and satisfy* $(\pi(1), \pi(n)) \in \{(m, m+1), (m+1, m)\}$.

When $n = 2m+1$*, the maximum is attained precisely by those permutations* π *that oscillate between* $\{1, \dots, m\}$, $\{m+1, \dots, n\}$ *and satisfy* $(\pi(1), \pi(n)) \in \{(m+1, m+2), (m+2, m+1)\}$*, and by those that oscillate between* $\{1, \dots, m+1\}$, $\{m+2, \dots, n\}$ *and satisfy* $(\pi(1), \pi(n)) \in \{(m, m+1), (m+1, m)\}$.

Proof. Let $n = 2m$. Consider the sum $|\pi(1)-\pi(2)| + \cdots + |\pi(n-1)-\pi(n)|$. It consists of $2n-2$ integers from N, $n-1$ with positive and $n-1$ with negative signs. Now, if we are to maximize the sum of $2n-2$ integers out of $1, 1, \dots, n, n$ with $n-1$ integers having negative sign, we must choose

$$-1-1-\cdots-(m-1)-(m-1)-m+(m+1)+(m+2)+(m+2)+\cdots+n+n, \quad (5.3.1)$$

which equals $2m^2 - 1$.

Is there a permutation π such that $|\pi(1)-\pi(2)|+\cdots+|\pi(n-1)-\pi(n)| = 2m^2-1$? The fact that m, $m+1$ appear just once in (5.3.1) means that $\pi(1) = m$ and $\pi(n) = m+1$, or vice versa. Moreover, the distribution of signs implies that π must oscillate between $\{1,\dots,m\}$ and $\{m+1,\dots,n\}$. Any such permutation will do.

When $n = 2m+1$, we proceed similarly. The two maximal sums analogous to (5.3.1) are

$$-1-1-\cdots-(m-1)-(m-1)-m-(m+1)+(m+2)+(m+2)+\cdots+n+n,$$

and

$$-1-1-\cdots-m-m+(m+1)+(m+2)+(m+3)+(m+3)+\cdots+n+n,$$

since deleting both occurrences of $m+1$ would not correspond to any permutation.

□

5.4 Stretching with multiplicative formula

We answer the following question: *For which permutations $\pi \in S_n$ is $\mathrm{s}^*_{\mathcal{B}}(\pi)$ maximal, where $\mathcal{B} = \{\{i, i+1\};\ 1 \le i < n\}$?* Note that with this choice of $\mathcal{B}$ we have

$$\mathrm{s}^*_{\mathcal{B}}(\pi) = \left(\prod_{i=1}^{n-1} |\pi(i) - \pi(i+1)|\right)^{1/(n-1)}.$$

5.4.1 Maximizing products of n integers with a given sum

We obviously have:

Lemma 5.4.1. *Let $x \le y$ be positive integers. Then $(x-1)(y+1) < xy$.*

For positive integers $n \leq s$, let

$$D_{n,s} = \{(x_1, \ldots, x_n);\ x_i \in \mathbb{Z},\, x_i > 0,\, x_1 + \cdots + x_n = s\},$$

and

$$M_{n,s} = \max\{x_1 \cdots x_n;\ (x_1, \ldots, x_n) \in D_{n,s}\}.$$

The following result is certainly well known. We offer a short proof:

Theorem 5.4.2. *Let $n \leq s$ be positive integers, $a = s/n$. Then*

$$M_{n,s} = \lfloor a \rfloor^m \cdot \lceil a \rceil^{n-m},$$

where $m = n\lceil a \rceil - s$. Moreover, $M_{n,s} < M_{n,s+1}$.

Proof. Let $\overrightarrow{x} = (x_1, \ldots, x_n)$ be the unique point in D such that $x_1 \leq \cdots \leq x_n$ and $x_n - x_1 \leq 1$. It is easy to see that $x_1 = \cdots = x_m = \lfloor a \rfloor$, $x_{m+1} = \cdots = x_n = \lceil a \rceil$, where $m = n\lceil a \rceil - s$.

Let $\overrightarrow{y} = (y_1, \ldots, y_n) \in D$ be such that $y_i \leq y_{i+1}$ and $\overrightarrow{y} \neq \overrightarrow{x}$. Let $d_i = y_i - x_i$ and note that $d_1 < 0$, $d_n > 0$, $d_1 + \cdots + d_n = 0$. Assume for a while that $d_i > 0$ and $d_j < 0$ for some $i < j$. Then $x_i < y_i \leq y_j < x_j$ shows that x_i, x_j differ by more than 1, which is impossible. Hence there is k such that $d_i \leq 0$ for every $i \leq k$, and $d_i \geq 0$ for every $i > k$.

The integers d_i count how many times do we have to add or subtract 1 to obtain y_i from x_i. Since $d_1 + \cdots + d_n = 0$, we can reach $\overrightarrow{y}$ from $\overrightarrow{x}$ by repeatedly decreasing one coordinate by 1 and increasing other coordinate by 1 at the same time. Moreover, we have just shown that we can do this in such a way that only the first k coordinates will possibly decrease, and only the remaining $n - k$ coordinates will possibly increase. Since $x_k \leq x_{k+1}$, Lemma 5.4.1 implies that the product will diminish with every step.

It remains to show that $M_{n,s} < M_{n,s+1}$. When $(x_1, \dots, x_n) \in D_{n,s}$ then $(x_1+1, x_2, \dots, x_n) \in D_{n,s+1}$, and, clearly, $x_1 \cdots x_n < (x_1+1)x_2 \cdots x_n$. □

5.4.2 The even case

Let $n = 2m$. Theorem 5.3.1 shows that $(n-1)\mathrm{s}^+_{\mathcal{B}}(\pi) \le 2m^2 - 1$, and that the equality holds if and only if π oscillates between $\{1, \dots, m\}$, $\{m+1, \dots, n\}$ and $(\pi(1), \pi(n)) \in \{(m, m+1), (m+1, m)\}$. By Theorem 5.4.2, the product of $2m-1$ positive integers with sum $2m^2 - 1$ is maximized by $m \cdot m + (m-1)(m+1)$.

Lemma 5.4.3. *Let $n = 2m$. Let $\pi \in S_n$ be a permutation oscillating between $\{1, \dots, m\}$, $\{m+1, \dots, n\}$ such that $\pi(1) = m$, $\pi(n) = m+1$ and such that $|\pi(i) - \pi(i+1)| \in \{m, m+1\}$ for every $1 \le i < n$. Then π is uniquely determined, namely: $\pi(2i) = n - i + 1$, $\pi(2i-1) = m - i + 1$.*

Proof. We must have $\pi(2) = 2m$. Then $\pi(3) = m-1$ since $\pi(1) = m$, etc. □

Dually:

Lemma 5.4.4. *Let $n = 2m$. Let $\pi \in S_n$ be a permutation oscillating between $\{1, \dots, m\}$, $\{m+1, \dots, n\}$ such that $\pi(1) = m+1$, $\pi(n) = m$ and such that $|\pi(i) - \pi(i+1)| \in \{m, m+1\}$ for every $1 \le i < n$. Then π is uniquely determined, namely: $\pi(2i) = i$, $\pi(2i-1) = m + i$.*

Theorem 5.4.5. *Let $n = 2m$. Then the maximum of $\mathrm{s}^*_{\mathcal{B}}$ over all permutations of S_n is $(m^m (m+1)^{m-1})^{1/(2m-1)}$, and it is attained precisely by the two permutations of Lemmas* 5.4.3*,* 5.4.4*.*

Proof. Let $\pi \in S_n$. Let $x_i = |\pi(i) - \pi(i+1)|$, $s = x_1 + \cdots + x_{2m-1}$. Then $s \le 2m^2 - 1$ by Theorem 5.3.1. If $s < 2m^2 - 1$ then $\mathrm{s}^*_{\mathcal{B}}(\pi)^{n-1} \le M_{n,2m^2-2} < M_{n,2m^2-1}$ by Theorem 5.4.2. If $s = 2m^2 - 1$, we have $\mathrm{s}^*_{\mathcal{B}}(\pi)^{n-1} \le M_{n,s} = m^m \cdot (m+1)^{m-1}$, and the equality holds only for the two permutations of Lemma 5.4.3, 5.4.4. □

5.4.3 Local improvements

When $n = 2m + 1$, we are going to see that the maximum of $(\mathrm{s}^*_{\mathcal{B}})^{n-1}$ is $M = m^m(m+1)(m+2)^{m-1}$, which is far less than $M_{2m,2m^2+2m-1}$ (cf. Theorems 5.3.1 and 5.4.2). In fact, it can happen that $M < M_{2m,s}$ even if $s < 2m^2+2m-1$. A more detailed understanding of permutations π with maximal $\mathrm{s}^*_{\mathcal{B}}(\pi)$ is therefore needed.

There is a one-to-one correspondence between the permutations of S_n and the n-cycles of S_n with designated beginning. To see this, identity $\pi \in S_n$ with the n-cycle ρ defined by $\rho(\pi(i)) = \pi(i+1)$ if $i < n$, $\rho(\pi(n)) = \pi(1)$, and designate $\pi(1)$ as the beginning of ρ. Therefore, finding the maximum of $\mathrm{s}^*_{\mathcal{B}}$ on S_n is equivalent to finding the maximum of s^* over all n-cycles ρ in S_n, where

$$\mathrm{s}^*(\rho) = \max \left\{ \prod_{i \neq j} |i - \rho(i)|;\ 1 \leq j \leq n \right\}.$$

In this subsection we show that a number of conditions on ρ must hold, should $\mathrm{s}^*(\rho)$ be maximal.

The following terminology will allow us to communicate more efficiently. We say that two jumps $a \mapsto \rho(a)$, $b \mapsto \rho(b)$ of a cycle ρ *have distinct endpoints* if $|\{a, \rho(a), b, \rho(b)\}| = 4$. The two jumps are *disjoint* if the intervals $[a, \rho(a)]$, $[b, \rho(b)]$ do not intersect. The jump $a \mapsto \rho(a)$ *skips over* the jump $b \mapsto \rho(b)$ if $[b, \rho(b)] \subseteq [a, \rho(a)]$. (Note that a jump skips over itself.) The jump $a \mapsto \rho(a)$ *bridges* $b \mapsto \rho(b)$ if it skips over it and the two jumps have distinct endpoints. Two jumps *intersect nontrivially* if they are not disjoint, one does not skip over the other, and they have distinct endpoints. A jump $a \mapsto \rho(a)$ is *short* if $|a-\rho(a)| \leq |b-\rho(b)|$ for all b. All other jumps are called *long*. Finally, the jumps have the *same direction* if $(a-\rho(a))(b-\rho(b)) > 0$, otherwise they have *opposite direction*.

Given a cycle ρ and two jumps $i \mapsto \rho(i)$, $j \mapsto \rho(j)$ with distinct endpoints, let $\rho_{i,j}$ denote the cycle depicted in Figure 5.2.

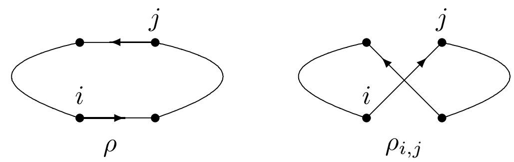

Figure 5.2: The cycles ρ and $\rho_{i,j}$.

Lemma 5.4.6. *Let $\rho \in S_n$ be an n-cycle. Let $i \mapsto \rho(i)$, $j \mapsto \rho(j)$ be jumps with distinct endpoints such that $i \mapsto \rho(i)$ is a short jump and $|i - j| > |j - \rho(j)|$. Then* $\mathrm{s}^*(\rho_{i,j}) > \mathrm{s}^*(\rho)$.

Proof. Since $i \mapsto \rho(i)$ is short, $\mathrm{s}^*(\rho) = \prod_{k \neq i} |k - \rho(k)|$. Now, $\mathrm{s}^*(\rho_{i,j}) \geq |i - j| \prod_{k \neq i,\, k \neq j} |k - \rho(k)| > \prod_{k \neq i} |k - \rho(k)|$. □

Lemma 5.4.7. *Let $\rho \in S_n$ be an n-cycle such that one of the following conditions holds:*

(i) there are disjoint jumps in the same direction,

(ii) a short jump nontrivially intersects a jump in opposite direction,

(iii) a short jump is disjoint from a jump in opposite direction,

(iv) there are disjoint jumps in opposite direction (generalizing (iii)*),*

(v) a jump bridges a long jump in opposite direction.

Then there is an n-cycle $\sigma \in S_n$ such that $\mathrm{s}^*(\sigma) > \mathrm{s}^*(\rho)$.

Proof. In case (i), write $a < \rho(a) < b < \rho(b)$ without loss of generality, and let $\sigma = \rho_{a,b}$. Note that the two old jumps $a \mapsto \rho(a)$, $b \mapsto \rho(b)$ have been replaced by two longer jumps $a \mapsto b$, $\rho(a) \mapsto \rho(b)$, respectively.

In case (ii), let $a \mapsto \rho(a)$ be a short jump, and let b be such that $a < \rho(b) < \rho(a) < b$. Let $\sigma = \rho_{a,b}$ and note that the new jump $a \mapsto b$ is longer that the old jump $b \mapsto \rho(b)$. We are done by Lemma 5.4.6.

In case (iii), let $a \mapsto \rho(a)$ be a short jump and $a < \rho(a) < \rho(b) < b$. Let $\sigma = \rho_{a,b}$. The new jump $a \mapsto b$ is then longer than the old jump $b \mapsto \rho(b)$, and we are again done by Lemma 5.4.6.

In case (iv), we can assume that none of the two jumps $a \mapsto \rho(a)$, $b \mapsto \rho(b)$ in question is short, else (iii) applies. Let $c \mapsto \rho(c)$ be a short jump. We can assume that $c \mapsto \rho(c)$ is not disjoint from $a \mapsto \rho(a)$ nor $b \mapsto \rho(b)$, otherwise either (i) or (iii) applies. Without loss of generality, assume $\max\{a, \rho(a)\} < \max\{b, \rho(b)\}$. Since the two jumps are in opposite directions, $c \mapsto \rho(c)$ cannot intersect both jumps trivially. Again without loss of generality, assume $c \mapsto \rho(c)$ intersects $a \mapsto \rho(a)$ nontrivially. If $a \mapsto \rho(a)$, $c \mapsto \rho(c)$ are in opposite direction, then (ii) applies. So suppose that they are in the same direction. Then $c \mapsto \rho(c)$ and $b \mapsto \rho(b)$ are in opposite direction, and we can assume that they intersect trivially, else (ii) applies. But that is impossible.

In case (v), let $\rho(b) < a < \rho(a) < b$ and $\sigma = \rho_{a,b}$. Let x, y, z be the lengths $a-\rho(b)$, $\rho(a)-a$ and $b-\rho(a)$, respectively. Then we have lost the factor $(x+y+z)y = xy+y^2+yz$ and gained the factor $(x+y)(y+z) = xy+xz+y^2+yz$ while comparing $\mathrm{s}^*(\rho)$ to $\mathrm{s}^*(\sigma)$. Hence $\mathrm{s}^*(\sigma) > \mathrm{s}^*(\rho)$. □

5.4.4 Short jumps

We say that a jump $a \mapsto \rho(a)$ is *right* if $a < \rho(a)$, else it is *left*.

Proposition 5.4.8. *Let $\rho \in S_n$ be an n-cycle with maximal $\mathrm{s}^*(\rho)$. Assume that ρ has a short jump $c \mapsto c+t$, $t > 0$. Then one of the following scenarios holds:*

(i) $t = 1$, all jumps skip over $c \mapsto c+1$, $n = 2m$, $c = m$, there are m left and m right jumps in ρ,

(ii) $t = 1$, the only jump not skipping $c \mapsto c+1$ is the right jump following it, $n = 2m+1$, $c = m$, there are $m+1$ right and m left jumps in ρ,

(iii) $t = 1$*, the only jump not skipping* $c \mapsto c+1$ *is the right jump preceding it,* $n = 2m+1$*,* $c = m+1$*, there are* $m+1$ *right and* m *left jumps in* ρ*,*

(iv) $t = 2$*, precisely two jumps do not skip over* $c \mapsto c+2$ *and these jumps are right,* $n = 2m+1$*,* $c = m$*, there are* $m+1$ *right and* m *left jumps in* ρ*.*

Proof. If there is d such that $c < d < c+t$, consider a such that $d = \rho(a)$. By Lemma 5.4.7(ii), $a < c$. Similarly, $\rho(c) < \rho(d)$. The three jumps $c \mapsto \rho(c)$, $a \mapsto \rho(a)$, $\rho(a) \mapsto \rho(\rho(a)) = \rho(d)$ are thus all right.

If $\rho(c) - c > 2$, there are $c < d < e < \rho(c)$. As above, there are jumps $a \mapsto d \mapsto \rho(d)$, $b \mapsto e \mapsto \rho(e)$, all right. But then Lemma 5.4.7(i) applies to $a \mapsto \rho(a)$ and $e \mapsto \rho(e)$, a contradiction. Hence $t = \rho(c) - c \leq 2$.

Assume $\rho(c) - c = 2$ and let $a \mapsto \rho(a) = c+1 \mapsto \rho(c+1)$ be the two right jumps found above. Let $b \mapsto \rho(b)$ be a right jump different from $a \mapsto c+1$, $c+1 \mapsto \rho(c+1)$, $c \mapsto c+2$. Then $b < c$, else $a \mapsto c+1$, $b \mapsto \rho(b)$ are disjoint and Lemma 5.4.7(i) applies. If $\rho(b) \leq c$, the jump $b \mapsto \rho(b)$ is disjoint from $\rho(a) \mapsto \rho(\rho(a))$, a contradiction with Lemma 5.4.7(i). If $\rho(b) > c$, we must have $\rho(b) > \rho(c)$, and so $b \mapsto \rho(b)$ skips over $c \mapsto \rho(c)$. Now let $b \mapsto \rho(b)$ be any left jump. If $b < c+2$ then, in fact, $b < c$, thus $b \mapsto \rho(b)$ and $c \mapsto c+2$ are disjoint, a contradiction by Lemma 5.4.7(iii). Thus $b \geq c+2$. If $\rho(b) > c+1$ then $b \mapsto \rho(b)$, $a \mapsto \rho(a)$ are disjoint and Lemma 5.4.7(iv) applies. If $\rho(b) \leq c+1$, we must have $\rho(b) \leq c$, and $b \mapsto \rho(b)$ skips over $c \mapsto \rho(c)$. The rest of (iv) is easy.

The case $\rho(c) - c = 1$ can be analyzed similarly, with help of Lemma 5.4.7. □

In view of Theorem 5.4.5, we are only interested in scenarios (ii), (iii) and (iv) of Proposition 5.4.8.

5.4.5 Long jumps

The following Lemma follows immediately from Lemma 5.4.7(iv), (v):

Lemma 5.4.9. *Let ρ be an n-cycle with maximal* $\mathrm{s}^*(\rho)$*. Let $a \mapsto \rho(a)$, $b \mapsto \rho(b)$ be two long jumps of opposite directions. Then at least one of the endpoints of $b \mapsto \rho(b)$ is in the interval $[a, \rho(a)]$.*

Proposition 5.4.10. *Let $\rho \in S_n$ be an n-cycle with maximal* $\mathrm{s}^*(\rho)$ *and with a short cycle $c \mapsto c + t$, $t > 0$, where $n = 2m + 1$. Then every long jump of ρ is of length m, $m + 1$ or $m + 2$.*

Proof. Let $k \mapsto k + t$, $0 < t < m$, be a long right jump of ρ. By Proposition 5.4.8, $m + 1$ is the unique point at which 2 right jumps are consecutive, and, moreover, $m + 1 \in [k, k + t]$. By the same Proposition, there are m left jumps, no two consecutive. By Lemma 5.4.9, each of these left jumps has an endpoint in $[k, k + t]$. Then there are not enough points in $[k, k + t]$ for m nonconsecutive left jumps to start or end at.

Let $k \mapsto k - t$, $0 < t < m$, be a left jump of ρ. By Proposition 5.4.8 and Lemma 5.4.9, there are m long right jumps and each of them has an endpoint in $[k - t, k]$. In scenario (ii) of Proposition 5.4.8, $m \in [k - t, k]$, no long right jump starts or ends at $m + 1$, and no two long right jumps are consecutive. In scenario (iii), $m + 2 \in [k - t, k]$, no long right jump starts or ends at m, and no two long right jumps are consecutive. In scenario (iv), m, $m + 2 \in [k - t, k]$, no long right jump starts or ends at m, $m+2$, and precisely two long right jumps are consecutive. In any case, there are not enough points in $[k - t, k]$ to accommodate all long right jumps.

Consider a jump $a \mapsto \rho(a)$ of length at least $m + 3$. Then there are at most $2m + 1 - (m + 2) = m - 1$ points outside of $(a, \rho(a))$. Assume that $a < \rho(a)$. Then one of the m left jumps, no two of which are consecutive, must have both endpoints in $(a, \rho(a))$. Assume that $a > \rho(a)$. Note that no point outside of $(a, \rho(a))$ can be both the starting and the terminating point of a right jump (this is obvious for a, $\rho(a)$, and it is true for the remaining points by Lemma 5.4.7(iv)). Hence one of the

$m+1$ long right jumps must have both endpoints in $(a, \rho(a))$. In any case, we have reached a contradiction by Lemma 5.4.7(v). $\square$

Lemma 5.4.11. *Let ρ be as in scenario* (ii) *of Proposition* 5.4.8*. Then every long jump is of length m, $m+1$, or $m+2$, ρ is uniquely determined, and* $\mathrm{s}^*(\rho) = m^m \cdot (m+1) \cdot (m+2)^{m-1}$*. When m is odd, we have*

$$\rho(i) = \begin{cases} i+1, & i = m, \\ i-(m+1), & i = m+2, \\ i+m, & i \text{ even}, i < m+2, \\ i+(m+2), & i \text{ odd}, i < m, \\ i-m, & i \text{ even}, i > m+1, \\ i-(m+2), & i \text{ odd}, i > m+2. \end{cases}$$

When m is even, we have

$$\rho(i) = \begin{cases} i+1, & i = m, \\ i+(m+1), & i = 1, \\ i+m, & i \text{ odd}, 1 < i < m+2, \\ i+(m+2), & i \text{ even}, i < m, \\ i-m, & i \text{ even}, i > m+1, \\ i-(m+2), & i \text{ odd}, i > m+2. \end{cases}$$

Proof. We work out two examples, one for $m = 3$ and one for $m = 4$. It will then become clear that the cycle ρ is unique, that its structure is determined by the parity of m, and that the formulae in the statement of the Lemma are correct. We will build the cycle from the shortest jump $m \mapsto m+1$ by alternatively extending it by one jump forward and one jump backwards.

Let $m = 3$. By our assumption, $\rho(3) = 4$. We now determine $\rho(4)$ (building the cycle forward) and $\rho^{-1}(3)$ (building the cycle backwards). Since $\rho(4) > 4$ by the assumption, we must have $\rho(4) = 7$ (else the jump is too short). Then $\rho^{-1}(3) = 6$, since $\rho^{-1}(3) = 7$ would result in a short cycle, and all other values yield a jump that is too short. We next determine $\rho(7)$ and $\rho^{-1}(6)$. We must have $\rho(7) = 2$, since $\rho(7) = 1$ would be too long. Then $\rho^{-1}(6) = 1$ follows, avoiding a short cycle. Now we obviously have $\rho(2) = 5 = \rho^{-1}(1)$.

Let $m = 4$. By our assumption, $\rho(4) = 5$. Proceeding as in the case $m = 3$, we have $\rho(5) = 9$, $\rho^{-1}(4) = 8$, $\rho(9) = 3$, $\rho^{-1}(8) = 2$, $\rho(3) = 7$, $\rho^{-1}(2) = 6$, $\rho(7) = 1$, and $\rho^{-1}(6) = 1$. □

Similarly:

Lemma 5.4.12. *Let ρ be as in scenario* (iii) *of Proposition* 5.4.8*. Then every long jump is of length m, $m+1$, or $m+2$, ρ is uniquely determined, and* $\mathrm{s}^*(\rho) = m^m \cdot (m+1) \cdot (m+2)^{m-1}$*. The formulae for ρ are similar to those of Lemma* 5.4.11*.*

Lemma 5.4.13. *Let ρ be as in scenario* (iv) *of Proposition* 5.4.8*. Then there are at least $m-1$ jumps of length m in ρ.*

Proof. We use Proposition 5.4.10 without reference throughout this proof.

For $i \in \{1, \ldots, m-1\}$, let $L(i)$ denote the length of the left jump ending at i, and $R(i)$ the length of the right jump starting at i. Note that we cannot have $L(i) = R(i)$, else a 2-cycle arises. We claim that in at most one case among 1, $\ldots$, $m-1$ both $L(i)$, $R(i)$ are bigger than m, hence proving the lemma (since $m+1 \mapsto 2m+1$ is also of length m).

For a contradiction, let $i < j$ be the two smallest integers in $\{1, \ldots, m-1\}$ such that $L(i)$, $R(i)$, $L(j)$, $R(j) > m$. Assume that $L(i) = m+1$, $R(i) = m+2$. (The case $L(i) = m+2$, $R(i) = m+1$ is similar.) Let $k = j - i$.

Assume $k = 1$. Since $R(i+1) \neq m+1$, we have $L(i+1) = m+1$, $R(i+1) = m+2$. Since $R(i+2) \neq m$ and $R(i+2) \neq m+1$, we have $R(i+2) = m+2$. Since $L(i+2) \neq m$, we have $L(i+2) = m+1$. Continuing in this fashion, we arrive at $R(m-1) = m+2$, contradicting $m+1 \mapsto 2m+1$.

Assume $k = 2$. Since $L(i+1) \neq m$, we have $R(i+1) = m$. If $L(i+1) = m+1$, we have a 4-cycle. Hence $L(i+1) = m+2$. Since $j = i+2$, $L(i+2) \neq m$. Also, $L(i+2) \neq m+1$. Thus $L(i+2) = m+2$. But then the jump starting at $m+i+2$ is not of length m, $m+1$, or $m+2$, a contradiction.

Assume $k = 3$. Then $R(i+1) = m$, and thus $L(i+1) = m+2$ else we have a 4-cycle. Then $L(i+2) = m$, and thus $R(i+2) = m+2$ else we have a 6-cycle. As $R(i+3) \neq m$ and $R(i+3) \neq m+1$, we have $R(i+3) = m+2$. But then no jump can possibly end at $m+i+3$, a contradiction.

This pattern continues for larger k. □

5.4.6 The odd case

Theorem 5.4.14. *Let $n = 2m+1 > 1$. Then the maximum of $\mathrm{s}^*_{\mathcal{B}}$ over all permutations of S_n is $(m^m \cdot (m+1) \cdot (m+2)^{m-1})^{1/n-1}$, and it is attained precisely by the two permutations of Lemmas* 5.4.11 *and* 5.4.12*, and by their mirror images.*

Proof. Let ρ be a permutation obtained in scenario (iv) of Proposition 5.4.8. Its m left jumps start in positions $m+2, \ldots, 2m+1$, and its m long right jumps start in positions $1, \ldots, m-1, m+1$. It is then easy to see that the sum of the lengths of the $2m$ long jumps of ρ is $2m^2 + 2m - 2$. By Proposition 5.4.10, each long jump is of length m, $m+1$ or $m+2$, and by Lemma 5.4.13 there are at least $m-1$ jumps of length m. If $x_1, \ldots, x_{2m}$ are positive integers such that $m \leq x_i \leq m+2$, $x_1 + \cdots + x_{2m} = 2m^2 + 2m - 2$ and such that at least $m-1$ of them are equal to m, then Theorem 5.4.2 implies that the product $x_1 \cdots x_{2m}$ cannot exceed $m^{m-1}(m+1)^4(m+2)^{m-3}$. However, $m^{m-1}(m+1)^4(m+2)^{m-3}$ is less than $m^m(m+1)(m+2)^{m-1}$ if and only if $(m+1)^3$ is less than $m(m+2)^2$, which is true for every positive m. We are done by Lemmas 5.4.11, 5.4.12 and by their mirrored versions. □

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