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Quantum Metrics on Approximately Finite-Dimensional Algebras

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Abstract

Our dissertation focuses on bringing approximately finite-dimensional (AF) algebras into the realm of noncommutative metric geometry. We construct quantum metric structures on unital AF algebras equipped with a faithful tracial state, and prove that for such metrics, AF algebras are limits of their defining inductive sequences of finite dimensional C*-algebras for the quantum Gromov-Hausdorff propinquity. We then study the geometry, for the quantum propinquity, of three natural classes of AF algebras equipped with our quantum metrics: the UHF algebras, the Effros-Shen AF algebras associated with continued fraction expansions of irrationals, and the Cantor space, on which our construction recovers traditional ultrametrics. We also exhibit several compact classes of AF algebras for the quantum propinquity and show continuity of our family of Lip-norms on a fixed AF algebra. Next, given a C*-algebra, the ideal space may be equipped with natural topologies. Motivated by this, we impart criteria for when convergence of ideals of an AF algebra can provide convergence of quotients in quantum propinquity, while introducing a metric on the ideal space of a C*-algebra. We then apply these findings to a certain class of ideals of the Boca-Mundici AF algebra by providing a continuous map from this class of ideals equipped with various topologies including the Jacobson and Fell topologies to the space of quotients with the quantum propinquity topology.

Keywords
AF algebras, Gromov-Hausdorff convergence, Noncommutative metric geometry, Quantum metric spaces, Approximately finite-dimensional algebras

Quantum metrics on approximately finite-dimensional algebras

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Contents

Chapter 1

Introduction

This dissertation details the advancement in the study of Noncommutative Metric Geometry provided by bringing approximately finite-dimensional (AF) C*-algebras of O. Bratteli [11] into this novel area. The reason to consider these algebras in this context can be reduced to the question of continiuty of a particular map, which we now describe. Given a C*-algebra, $\mathfrak{A}$, its space of primitive ideals, denoted by $\mathrm{Prim}(\mathfrak{A})$, can be equipped with various natural topologies including the Jacobson (or hull-kernel) topology and Fell topology, which was introduced by J. M. G. Fell in [26, 27] (see Definition (2.1.52) and Definition (2.1.58), respectively). Since ideals of C*-algebras are C*-algebras and so are their quotients (see Theorem (2.1.44)), one may define a function from the set of primitive ideals to the class of C*-algebras by:

$$\mathcal{Q}_{\mathfrak{A}} : I \in \mathrm{Prim}(\mathfrak{A}) \longmapsto \mathfrak{A}/I,$$

where $\mathfrak{A}/I$ is the quotient C*-algebra. The existence of the Jacobson and Fell topologies sparks curiosity about continuity of the map $\mathcal{Q}_{\mathfrak{A}}$ as the continuity of $\mathcal{Q}_{\mathfrak{A}}$ would informally provide that the act of taking the quotient is continuous. This would prove to be a powerful application of the Jacobson and Fell topologies.

However, the question of continuity of the map $\mathcal{Q}_{\mathfrak{A}}$ has yet to be investigated since one must first have a topology on the range of this map. This is where Noncommutative Metric Geometry can be of great use. Indeed, if one can endow the quotients with some additional structure, then Noncommutative Metric Geomtery can provide a topology on the range of $\mathcal{Q}_{\mathfrak{A}}$. This topology is induced by a metric — the quantum Gromov-Hausdorff propinquity of F. Latrémolière [46] — which provides a topology on certain classes of C*-algebras called quantum compact metric spaces introduced by M. A. Rieffel in [59]. Together, these ideas have instigated many advancements in the study of C*-algebras, including advancing the study of finite-dimensional approximations of infinite-dimensional C*-algebras [62, 40, 69]. Thus, if we find a way to endow the quotients associated to a given C*-algebra with certain quantum metric structure, then continuity of $\mathcal{Q}_{\mathfrak{A}}$ can be discussed with respect to the quantum Gromov-Hausdorff propinquity topology. This is our reason for the use of Noncommutative Metric Geometry in the study of the map $\mathcal{Q}_{\mathfrak{A}}$. In our work, will consider a particular, natural, class of C*-algebras that form quantum metric spaces and are of particular interest: AF algebras.

A main reason to first consider an AF algebra $\mathfrak{A}$ to study the continuity of $\mathcal{Q}_{\mathfrak{A}}$ is due to a prediction and suggestion of O. Bratteli. In Bratteli's pioneering work on AF algebras [11], he stated, "As AF-algebras are relatively simple to handle without being trivial, they are especially well suited to test conjectures and to provide examples in the theory of C*-algebras, and I think their principal interest lies herein" [11, page 195]. This statement has held true many times. For instance, the Elliott classification of C*-algebras program began with study of AF algebras in [23]. Following Bratteli's advice, we study a particular AF algebra called the Boca-Mundici algebra $\mathfrak{F}$ introduced in [54, 10], which seemed to hold promise. Indeed, unique to F. Boca's work on this AF algebra in [10] is his proof showing that the Jacobson topology on a certain subset of primitive ideals of $\mathfrak{F}$ is induced by the usual topology of the irrationals in $[0,1] \subset \mathbb{R}$ [10, Corollary 12], and the quotients by these primitive ideals form the class of Effros-Shen AF algebras [22]. This connection to such a classic topology suggested that further inspection may lead to a positive answer to the continuity of $\mathcal{Q}_{\mathfrak{F}}$. In fact, the final main result of this dissertation, which is Theorem (5.2.21), establishes that AF algebras were an appropriate first consideration by showing the continuity of $\mathcal{Q}_{\mathfrak{F}}$ on a nontrivial class of primitive ideals of $\mathfrak{F}$ equipped with either the Jacobson or Fell topologies. Therefore, the question of the continuity of $\mathcal{Q}_{\mathfrak{A}}$ naturally synthesizes the study of the Noncommutative Metric Geometry and AF algebras. Of course, there are many more reasons to study AF algebras in the context of Noncommutative Metric Geometry, which we now introduce in more detail.

As stated by F. Latrémolière in [39], Noncommutative Metric Geometry is the study of noncommutative generalizations of the algebra of Lipshitz functions on a metric space. A particular noncommuative generalization comes in the form of a quasi-Leibniz quantum compact metric space [46, 45], which was introduced by Latrémolière inspired by the work of Connes [16, 17] and Rieffel [59]. In short, a quasi-Leibniz quantum compact metric space is a unital C*-algebra with a quasi-Leibniz Lip-norm, which serves as noncommutative generalization of the Lipschitz seminorm. Next, Latrémolière constructed the quantum Gromov-Hausdorff propinquity on the class of these spaces as a generalization of the Gromov-Hausdorff distance on compact metric spaces [46]. With this metric, one may address in a new light the notion of approximations of C*-algebras, continuous families of C*-algebras, and identifying compact classes of C*-algebras [45]. One of the most compelling results thus far involving the quantum Gromov-Hausdorff propinquity was that it provided a way to approximate the Quantum Tori, a non-AF algebra, by finite-dimensional C*-algebras. The result is due to Latrémolière [40], in which he provides explicit approximations and also shows that the Irrational Rotation

Algebras form a continuous family in the quantum Gromov-Hausdorff propinquity topology with respect to their irrational parameter space. Other examples of quasi-Leibniz quantum compact metric spaces include Hyperbolic Group C*-algebras [56] and Curved Noncommutative Tori [42]. Therefore, our research focuses on studying new examples of classes of quasi-Leibniz quantum compact metric spaces provided by AF algebras and their topology for the quantum Gromov-Hausdorff propinquity to better understand this new and fascinating topology.

Another motivating factor in studying the Noncommutative Metric Geometry of AF algebras is that AF algebras laid the foundation for the Elliott classification of C*-algebras program that began in [23] and is still an active and deep area of research today [25]. Therefore, as quantum compact metric spaces are built from C*-algebras, the task of bringing AF algebras into the realm of Noncommutative Metric Geometry seemed both natural and imperative. First, AF algebras are constructed from finite-dimensional approximations as the C*-inductive limit of a sequence of finite-dimensional C*-algebras. But, the question remained of whether the sequence of finite-dimensional C*-algebras approximate the inductive limit with resepct to the quantum Gromov-Hausdorff propinquity. Thus, in collaboration with Latrémolière [3], we were able to show that unital AF algebras with faithful tracial states have quasi-Leibniz Lip norms. These quantum metrics allowed us to show that these AF algebras had finite-dimensional approximations with respect to the quantum Gromov-Hausdorff propinquity provided by any defining inductive sequence of finite-dimensional C*-algebras. In particular, our construction recovers the usual ultrametrics on the Cantor set, seen as the Gelfand spectrum of a commutative AF algebra. We then proved that for our quantum ultrametrics, UHF algebras and Effros-Shen AF algebras form continuous families indexed by the Baire space for the quantum Gromov-Hausdorff propinquity, and we exhibit various compact subclasses of these clases of AF algebras.

This dissertation contains 5 chapters including this chapter. Chapter 2 provides the background needed for this dissertation an brief background on AF algebras, topologies on ideal spaces, and Noncommutative Metric Geometry and contains no original results. However, we do note that Chapter 2 does contain proofs of certain classical results that may be difficult to find in the literature. This chapter begins with definition of C*-algebras and ends with the quantum Gromov-Hausdorff propinquity. We provide plenty of definitions and results for which our original results rely in Chapters 3,4, and 5 to make for a more self-contained dissertation, which Chapters 3, 4, and 5 contain the original results of the author found in [1, 2, 3], in which the author collaborated with F. Latrémolière for [3].

Chapter 3 is the first chapter of original results containing some of the author's collaboration with F. Latrémolière [3] and the author's work in [1]. In this chapter, we give various constructions of quantum metric structure for AF algebras coming from tracial states or quotient norms and study the finite dimensional approximations of AF algebras with respect to the quantum Gromov-Hausdorff propinquity. We also validate our construction by considering it in the classical case of continuous functions on the Cantor set.

With the tools provided by Chapter 3, we present our first convergence results of AF algebras in the quantum Gromov-Hausdorff propinquity in Chapter 4. In particular, we show that the Uniformly Hyperfinite (UHF) Algebras of Glimm [28] and the Effros-Shen Algebras [22] form continuous images of the Baire space in collaboration with F. Latrémolière in [3]. These results also allowed us to discover nontrivial compact classes of AF algebras. We conclude this chapter with a generalization of the convergence of AF algebras in quantum propinquity provided by the author in [2].

Finally, Chapter 5 provides an AF algebra, $\mathfrak{F}$, for which the map $\mathcal{Q}_{\mathfrak{F}}$ introduced at the start of this chapter is continuous on a certain set of ideals with respect to either the Jacobson or Fell topologies. This is done by providing general criteria of convergence of quotients of AF algebras in the quantum Gromov-Hausdorff propinquity with respect to convergence of ideals in the Fell topology. In doing so, we also develop a metric on the ideal space of an AF algebra that metrizes the Fell topology. We find that, when this metric is applied to the ideals on the Boca-Mundici AF algebra, we discover a new metric for irrational numbers that behaves much more like the standard metric on the irrationals than the classical Baire metric. In particular, this new metric is totally bounded, whereas the Baire metric is not (see Remark (5.2.13)). The results of this chapter are from the author's work in [2]. Enjoy!

Chapter 2

Background

Our original results in this dissertation, which are the contents of Chapters 3,4 and 5, pertain to the Fell topology on the ideal space of C*-algebras constructed by J. M. G. Fell [26, 27] from the Jacobson topology on primitive ideals, approximately finite-dimensional (AF) algebras of O. Bratteli [11], quantum compact metric spaces of M. A. Rieffel [59], and the quantum Gromov-Hausdorff propinquity of F. Latrémolière [46]. Therefore, to provide a reference and motivation for our work, this chapter serves as a small introduction into each of these topics in which they are covered in Sections (2.1.1,2.1.2, 2.2, 2.3), respectively.

We make a note on the structure of this chapter. The story of AF algebras and quantum compact metric spaces begins with C*-algebras, which is the first section of this chapter. Not only do we discuss ideals of C*-algebras and inductive limits of C*-algebras in Section (2.1), but also Gelfand duality, which we will see provides a powerful analogue for quantum compact metric spaces in Section (2.2) and the quantum Gromov-Hausdorff propinquity in Section (2.3). In Section (2.2), we present the category quantum compact metric spaces which are built from C*-algebras. Next in Section (2.3), we summarize the construction of the quantum Gromov-Hausdorff propinquity, which provides a distance on certain classes of quantum compact metric spaces. This chapter contains mostly definitions and statements of theorems with references to where their proofs may be found. However, proofs some results are often difficult to find in the literature or stated in a context that strays far enough from our terminology. We only provide proofs in these cases.

2.1 C*-algebras

C*-algebras appear in quantum mechanics, representation theory, and dynamical systems. Recent developments have shown that C*-algebras provide a pathway for extensions of ideas from geometry which have proven helpful in the study of singular spaces, mathematical physics, and symbolic dynamics, among others [17]. The category of unital commutative C*-algebras with unital *-homomorphisms is dual to the category of compact Hausdorff spaces with continuous maps, which is one of the main goals of this section and is treated in Theorem (2.1.30) and Theorem (2.1.34). This duality provides a particular persepective on how to study C*-algebras, sometimes called Noncommutative Topology and is a beginning to the understanding of Noncommutative Metric Geometry presented in Sections (2.2,2.3).

A basic reference on functional analysis is *A Course in Functional Analysis* by John B. Conway [18], and we shall assume that the reader is familiar with its content. We will however give a very brief summary of basic C*-algebra theory to help us fix our notations/terminology and provide some proofs to results that may be difficult to find in the literature for the reader's convenience.

Definition 2.1.1. *An* associative algebra over the complex numbers $\mathbb{C}$ *is a vector space* $\mathfrak{A}$ *over* $\mathbb{C}$ *with an associative multiplication, denoted by concatenation, such that:*

$$a(b+c) = ab + ac \text{ and } (b+c)a = ba + ca \text{ for all } a,b,c \in \mathfrak{A}$$

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \text{ for all } a, b \in \mathfrak{A}, \lambda \in \mathbb{C}.$$

In other words, the associative multiplication is a bilinear map from $\mathfrak{A} \times \mathfrak{A}$ *to* $\mathfrak{A}$.

We say that $\mathfrak{A}$ *is* unital *if there exists a multiplicative identity, denoted by* $1_{\mathfrak{A}}$. *That is:*

$$1_{\mathfrak{A}} a = a = a 1_{\mathfrak{A}} \text{ for all } a \in \mathfrak{A}.$$

Convention 2.1.2. *All algebras are associative algebras over the complex number* $\mathbb{C}$ *unless otherwise specified.*

Notation 2.1.3. *When* E *is a normed vector space, then its norm will be denoted by* $\|\cdot\|_E$ *by default and its zero will be denoted by* 0_E. *If* $E = \mathbb{C}$, *then we denote the zero by just* 0.

Definition 2.1.4. *A* normed algebra *is an algebra* $\mathfrak{A}$ *with a norm* $\|\cdot\|_{\mathfrak{A}}$ *such that:*

$$\|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}} \text{ for all } a, b \in \mathfrak{A}.$$

$\mathfrak{A}$ *is a* Banach Algebra *when* $\mathfrak{A}$ *is complete with respect to the norm* $\|\cdot\|_{\mathfrak{A}}$.

Remark 2.1.5. *Note that for a complete normed algebra* $\mathfrak{A}$, *the condition:*

$$\text{there exists } K > 0 \text{ such that } \|ab\| \leqslant K \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}} \text{ for all } a, b \in \mathfrak{A},$$

is equivalent to joint continuity of the multiplication of the algebra. It is simply standard in the Banach Algebra definition to assume that $K = 1$, *which causes no loss of generality. The proof of this equivalence is outlined in [18, Exercise VII.1.1].*

Definition 2.1.6. *A* C*-algebra, $\mathfrak{A}$, *is a Banach algebra such that there exists an anti-multiplicative conjugate linear involution* $^* : \mathfrak{A} \longrightarrow \mathfrak{A}$, *called the* adjoint. *That is,* $*$ *satisfies:*

1. (conjugate linear): $(\lambda(a+b))^* = \overline{\lambda}(a^* + b^*)$ *for all* $\lambda \in \mathbb{C}, a, b \in \mathfrak{A}$;

2. (involution): $(a^*)^* = a$ *for all* $a \in \mathfrak{A}$;

3. (anti-multiplicative): $(ab)^* = b^* a^*$ *for all* $a, b \in \mathfrak{A}$.

Furthermore, the norm, multiplication, and adjoint together satisfy:

$$\|aa^*\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2 \text{ for all } a \in \mathfrak{A} \tag{2.1.1}$$

called the identity the C*-identity*.*

We say that $\mathfrak{B} \subseteq \mathfrak{A}$ *is a* C*-subalgebra of $\mathfrak{A}$ *if* $\mathfrak{B}$ *is a norm closed subalgebra that is also self-adjoint, i.e.* $a \in \mathfrak{B} \iff a^* \in \mathfrak{B}$*.*

We say that $\mathfrak{A}$ *is* commutative *if the multiplication of the underlying algebra is commutative.*

We will present some examples of C*-algebras in Example (2.1.13). But, first, we introduce some more definitions related to C*-algebras, so that we may present these examples in more detail.

Next, we define some fundamental elements in a C*-algebra. These definitions are motivated by and are consistent with the same definitions of these elements in the space of bounded operators on a Hilbert space $\mathcal{H}$ denoted by $\mathfrak{B}(\mathcal{H})$ in Example (2.1.13.3). For example, using the definition of unitary on the following definition, a unitary element in a C*-algebra corresponds to a unitary operator on some Hilbert space. This will be immediate by one of the main results of this section, which is Theorem (2.1.41) and follows from the well-known Gelfand-Naimark-Segal construction — Theorem (2.1.40).

Definition 2.1.7. *Let* $\mathfrak{A}$ *be a C*-algebra. An element* $a \in \mathfrak{A}$ *is* self-adjoint *if* $a = a^*$*, and we denote the set of self-adjoint elements by* $\mathfrak{sa}(\mathfrak{A}) = \{a \in \mathfrak{A} : a = a^*\}$*.*

An element $a \in \mathfrak{A}$ *is a* projection *if it is self-adjoint and* $a^2 = a$*.*

If $\mathfrak{A}$ *is unital, then an element* $a \in \mathfrak{A}$ *is* unitary *if* $aa^* = 1_{\mathfrak{A}} = a^* a$*.*

The set of self-adjoint elements will play a key role in our work (for example, see the definition of a *quantum compact metric space* in Definition (2.2.5)). Projections and unitaries, among others, provide invariants for the classification of C*-algebras. Before we introduce some examples of C*-algebras, we first discuss the morphisms and isomorphisms in the category of C*-algebras. First, we define an isometry.

Definition 2.1.8. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \longrightarrow Y$ is an* isometry *if for all $a, b \in X$:*

$$\mathsf{d}_Y(f(a), f(b)) = \mathsf{d}_X(a, b).$$

If $(E, \|\cdot\|_E)$ is a normed vector space, then we call the metric $\mathsf{d}_E(\cdot,\cdot) = \|\cdot - \cdot\|_E$, the metric induced by $\|\cdot\|_E$. *Let $(F, \|\cdot\|_F)$ be a normed vector spaces. We say that $\pi : E \longrightarrow F$ is an* isometry *if it is an isometry with respect to the metrics induced by $\|\cdot\|_E, \|\cdot\|_F$ which is equivalent to $\|\pi(e)\|_F = \|e\|_E$ for all $e \in E$, when π is linear or conjugate linear.*

An immediate consequence of the C*-identity is that the adjoint is an isometry, which is the following lemma.

Lemma 2.1.9 ([18, Proposition VIII.1.7])**.** *If $\mathfrak{A}$ is a C*-algebra, then the adjoint $^* : a \in \mathfrak{A} \longmapsto a^* \in \mathfrak{A}$ is an isometry.*

Now, we are in a position to define the morphisms between C*-algebras.

Definition 2.1.10. *Let $\mathfrak{A}, \mathfrak{D}$ be a C*-algebras. A function $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$ is a* *-homomorphism *if it is a linear map that is also:*

1. (multiplicative): $\pi(ab) = \pi(a)\pi(b)$ *for all $a, b \in \mathfrak{A}$, and*
2. (*-preserving): $\pi(a^*) = \pi(a)^*$ *for all $a \in \mathfrak{A}$.*

π is a *-monomorphism *if it is an injective *-homomorphism.*

π is a *-epimorphism *if π is a surjective *-homomorphism.*

π is a *-isomorphism *if π is a bijective *-homomorphism.*

$\mathfrak{A}$ is *-isomorphic *to $\mathfrak{D}$ if there exists a *-isomorphism $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$, and we then write $\mathfrak{A} \cong \mathfrak{D}$.*

*If both $\mathfrak{A}, \mathfrak{D}$ are unital, then we call a *-homomorphism $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$* unital *if $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{D}}$.*

*We call a *-homomorphism* non-zero *if it is not the zero *-homomorphism, i.e. there exists $a \in \mathfrak{A}$ such that $\pi(a) \neq 0_{\mathfrak{D}}$.*

We will present examples of *-homomorphisms once we present examples of C*-algebras in Example (2.1.13). Note that in Definition (2.1.10), we see that only algebraic properties are required for the morphisms. The next result shows that there are important analytical properties (such as continuity, contractibility, and isometry) associated to these morphisms without further assumptions. Thus, only algebraic requirements are indeed needed in Definition (2.1.10).

Proposition 2.1.11 ([19, Theorem I.5.5])**.** *Let $\mathfrak{A}, \mathfrak{D}$ be C*-algebras.*

*If $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$ is a *-homomorphism, then π is continuous and contractive. That is, its operator norm:*

$$\|\pi\|_{\mathfrak{B}(\mathfrak{A},\mathfrak{D})} = \sup\{\|\pi(a)\|_{\mathfrak{D}} : \|a\|_{\mathfrak{A}} = 1\} \leqslant 1,$$

or equivalently, for all $a \in \mathfrak{A}$, we have $\|\pi(a)\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}}$.

*If $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$ is a *-homomorphism, then π is an isometry if and only if π is a *-monomorphism. In particular, *-isomorphisms are isometries.*

Convention 2.1.12. *The set of natural numbers, $\mathbb{N}$, contains 0.*

Example 2.1.13. *We present some classical examples of C*-algebras.*

1. *The complex numbers $\mathbb{C}$ is a C*-algebra with the standard algebraic operations, complex conjugation as the adjoint, the modulus as the norm, and 1 is*

the multiplicative identity. We will denote $1_{\mathbb{C}}$ simply by 1. The self-adjoint elements $\mathfrak{sa}(\mathbb{C}) = \mathbb{R}$.

2. *[18, Example VIII.1.4] If X is a compact Hausdorff space, then the space:*

$$C(X) = \{f : X \longrightarrow \mathbb{C} \mid f \text{ is continuous}\}$$

equipped with point-wise operations induced by $\mathbb{C}$ for the algebra and point-wise complex conjugation for the adjoint is a unital commutative C-algebra with supremum norm, which is defined by $\|f\|_{C(X)} = \sup\{|f(x)| : x \in X\}$ for all $f \in C(X)$. We will see by the Gelfand-Naimark Theorem (2.1.30) with its characterization of unital commutative C*-algebras, that this is a natural algebraic structure and norm on the set $C(X)$ for $C(X)$ to be a C*-algebra. Note that the constant 1 function is the multiplicative identity $1_{C(X)}$. The self-adjoint elements $\mathfrak{sa}(C(X))$ are the continuous real-valued functions on X. Also, if $X = \{x\}$ is a single point, then $C(X) \cong \mathbb{C}$.*

[18, Example VIII.1.6] If X is a locally compact Hausdorff space, then $C(X)$ might contain unbounded functions ($X = \mathbb{R}$ with its usual topology and $f(x) = x$ for all $x \in \mathbb{R}$, for example). In fact, for every non-compact locally compact metric space X , there exists an unbounded real-valued continuous function f_u on X — the proof of this fact is outlined in [71, Exercise 17J.3] and is an application of [71, Tietze's Extension Theorem 15.8]—, and thus the quantity $\sup\{|f_u(x)| : x \in X\} = \infty$ would fail to define a norm on $C(X)$. Hence, for a locally compact Hausdorff space X, we instead consider $C_0(X)$ which is the space of complex-valued continuous functions vanishing at infinity equipped with the same algebraic structure as $C(X)$ defined by:

$$C_0(X) = \{f \in C(X) : \forall \varepsilon > 0, \{x \in X : |f(x)| \geqslant \varepsilon\} \text{ is compact}\},$$

which has finite supremum norm for all $f \in C_0(X)$ by [18, Proposition III.1.7] and is still a C^*-algebra under the same operations by [18, Example VIII.1.4]. Next, when X is a non-compact locally compact Hausdorff space, the C^*-algebra $C_0(X)$ is non-unital. Indeed, first note that if $C_0(X)$ had a unit, then it would have to be the constant 1 function on X, which follows from the given algebra of point-wise operations and the fact that for every $x \in X$ there exists a function $f_x \in C(X)$ such that $f_x(x) \neq 0$ — this is because locally compact Hausdorff spaces are Tychonoff [71, Theorem 19.3]. Now, if the constant 1 function $1_{C(X)} \in C_0(X)$, then for $\varepsilon = 1$, the set $\left\{x \in X : 1 = \left|1_{C(X)}(x)\right| \geqslant 1\right\} = X$ is compact by definition of $C_0(X)$. Note that when X is compact $C(X) = C_0(X)$, and in summary, for a locally compact Hausdorff space X, the following assertions are equivalent:

(i) X is compact;

(ii) $C_0(X)$ is unital.

3. *[18, Example 1.2] Given a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The space:*

$$\mathfrak{B}(\mathcal{H}) = \{T : \mathcal{H} \longrightarrow \mathcal{H} \mid T \text{ is linear and continuous}\}$$

*is a unital C^*-algebra with composition as multiplication and point-wise addition. The norm is given by the operator norm $\|T\|_{\mathfrak{B}(\mathcal{H})} = \sup\{\|Tx\|_{\mathcal{H}} : \|x\|_{\mathcal{H}} = 1\}$. If $T \in \mathfrak{B}(\mathcal{H})$, then the adjoint T^* is given by the unique bounded linear operator such that $\langle Tx, y\rangle_{\mathcal{H}} = \langle x, T^*y\rangle_{\mathcal{H}}$, $\forall x, y \in \mathcal{H}$ [18, Theorem II.2.2, Proposition II.2.6, and Proposition II.2.7]. The identity operator is the multiplicative identity. If $\dim(\mathcal{H}) = n < \infty$, then $\mathfrak{B}(\mathcal{H})$ is *-isomorphic to $\mathfrak{M}(n)$, the algebra of $n \times n$-matrices or the full matrix algebra of dimension n^2 in which the adjoint is the conjugate-transpose of a matrix.*

4. *Building from the previous example, let $d \in \mathbb{N} \setminus \{0\}$ and $\{n(1), \ldots, n(d)\} \subset \mathbb{N} \setminus \{0\}$. The vector space sum $\oplus_{j=1}^{d} \mathfrak{M}(n(j))$ of full matrix algebras forms a C*-algebra with coordinate-wise addition, multiplication, and adjoint along with the max norm defined for all $a = (a_1, \ldots, a_d) \in \oplus_{j=1}^{d} \mathfrak{M}(n(j))$ by:*

$$\|a\|_{\oplus_{j=1}^{d} \mathfrak{M}(n(j))} = \max \left\{ \|a_j\|_{\mathfrak{M}(n(j))} : j \in \{1, \ldots, d\} \right\}$$

These C-algebras classify all finite-dimensional C*-algebras. Indeed, if $\mathfrak{A}$ is a finite-dimensional C*-algebra, then there exists $d \in \mathbb{N} \setminus \{0\}$ and $n(1), \ldots, n(d) \in \mathbb{N} \setminus \{0\}$ such that $\mathfrak{A}$ is *-isomorphic to $\oplus_{j=1}^{d} \mathfrak{M}(n(j))$ by [19, Theorem III.1.1].*

Now, that we have examples of C*-algebras, we make note of some *- homomorphisms between C*-algebras. In Theorem (2.1.34), we will see that unital *-homomorpisms between two unital commutative C*-algebras $C(X)$ and $C(Y)$ are completely determined by continous maps between Y and X. In Theorem (2.1.67), we will see how one may construct unital *-homomorphisms from inductive limits of C*-algebras to a given C*-algebra. Next, we present a classification of all *-homomorphisms between finite-dimensional C*-algebras, which will be crucial in our discussion of AF algebras in Section (2.1.2). The following Theorem-Definition provides the first examples of these maps.

Theorem-Definition 2.1.14. *Let $j, k \in \mathbb{N} \setminus \{0\}$. Consider the C*-algebras $\mathfrak{M}(j)$ and $\mathfrak{M}(k)$ of Example (2.1.13.3). If we define a map $\alpha : \mathfrak{M}(j) \longrightarrow \mathfrak{M}(k)$ by the following rule:*

for any $b \in \mathfrak{M}(j)$, the element $\alpha(b)$ is given by a matrix in $\mathfrak{M}(k)$, which has non-overlapping copies of b (allowing for no copies of b) placed on the diagonal and 0 elsewhere and this placement is independent of b and fixed for all $b \in \mathfrak{M}(j)$,

*then α is a *-homomorphism.*

This rule can be displayed as:

$$\alpha : b \in \mathfrak{M}(j) \longmapsto \begin{pmatrix} d_{\alpha,1}(b) & & \\ & \ddots & \\ & & d_{\alpha,k_\alpha}(b) \end{pmatrix} \in \mathfrak{M}(k),$$

with a suitable choice of $k_\alpha \in \mathbb{N} \setminus \{0\}$*, and for* $p \in \{1, \ldots, k_\alpha\}$*, we have that* $d_{\alpha,p}(b)$ *is either* b *or a zero matrix of an appropriate dimension placed on the diagonal of* $\mathfrak{M}(k)$*, in which the diagonal of* $d_{\alpha,p}(b)$ *lines up with the diagonal of* $\mathfrak{M}(k)$*. For each* $p \in \{1, \ldots, k_\alpha\}$*, the values of* $d_{\alpha,p}$ *depend only on* α*. The blank parts of* $\alpha(b)$ *denote zeros.*

*We call *-homomorphisms between full matrix algebras of this form* canonical*, and the number of copies of a matrix that* α *places on the diagonal is called the* multiplicity *of* α*.*

Proof. Since the map α produces block diagonal matrices, it is a basic linear algebra exercise to show that α is a *-homomorphism. □

Consider $\mathfrak{M}(2)$ and $\mathfrak{M}(4)$. An example of a unital canonical *-monomorphism $\alpha : \mathfrak{M}(2) \longrightarrow \mathfrak{M}(4)$ is the following. Let $b = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \in \mathfrak{M}(2)$.

Define $\alpha(b) = \begin{pmatrix} b_{1,1} & b_{1,2} & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ 0 & 0 & b_{1,1} & b_{1,2} \\ 0 & 0 & b_{2,1} & b_{2,2} \end{pmatrix} \in \mathfrak{M}(4)$. When context is clear, we will write $\alpha(b) = \begin{pmatrix} b & \\ & b \end{pmatrix}$ instead. This canonical *-monomorphism has multiplicity 2. Note that $b \in \mathfrak{M}(2) \longmapsto \begin{pmatrix} b & \\ & 0_{\mathfrak{M}(2)} \end{pmatrix} \in \mathfrak{M}(4)$ is also a canonical *-monomorphism, where $0_{\mathfrak{M}(2)}$ is the zero 2×2-matrix, as well as $b \in \mathfrak{M}(2) \longmapsto \begin{pmatrix} 0_{\mathfrak{M}(2)} & \\ & b \end{pmatrix} \in$ $\mathfrak{M}(4)$, and both of these have multiplicity 1 and are non-unital. The canonical *-homomorphisms $b \in \mathfrak{M}(2) \longmapsto 0_{\mathfrak{M}(4)} \in \mathfrak{M}(4)$ and $b \in \mathfrak{M}(4) \longmapsto 0_{\mathfrak{M}(2)} \in \mathfrak{M}(2)$ are examples of non-injective canonical *-homomorphisms that have multiplicity 0.

An example of a non-canonical *-monomorphism is to fix a unitary $U \in \mathfrak{M}(4), U \notin \mathbb{C}1_{\mathfrak{M}(4)}$ and define $\beta(b) = U\alpha(b)U^*$ for all $b \in \mathfrak{M}(4)$, and it is routine to check that β is a *-monomorphism. In fact, the canonical *-monomorphisms along with their conjugation by unitaries comprise all *-monomorphisms between full matrix algebras. This is Theorem (2.1.18), which is presented in the more general case of finite-dimensional C*-algebras and thus contains the case of full matrix algebras.

Next, we extend the notion of a canonical *-monomorphism between full matrix algebras to the finite-dimensional case. However, before we generalize to the finite-dimensional case, we make a note on why the only *-homomorphism from a full matrix algebra onto a full matrix algebra of smaller dimension is the zero map.

Remark 2.1.15. *Let $j, k \in \mathbb{N} \setminus \{0\}, j > k$. Consider the C*-algebras $\mathfrak{M}(j), \mathfrak{M}(k)$ of Example (2.1.13.3). Assume that $\alpha : \mathfrak{M}(j) \longrightarrow \mathfrak{M}(k)$ is a homomorphism. Since α is linear and the dimension of $\mathfrak{M}(k)$ is less than the dimension of $\mathfrak{M}(j)$, the map α cannot be injective, and thus, the set $\ker \alpha = \left\{b \in \mathfrak{M}(j) : \alpha(b) = 0_{\mathfrak{M}(k)}\right\} \supsetneq \left\{0_{\mathfrak{M}(j)}\right\}$. Also, the set $\ker \alpha$ is two-sided ideal of $\mathfrak{M}(j)$ since α is a homomorphism. However, it is a basic ring theoretic fact that the only two-sided ideals of $\mathfrak{M}(j)$ are $\left\{0_{\mathfrak{M}(j)}\right\}$ and $\mathfrak{M}(j)$. Hence, the set $\ker \alpha = \mathfrak{M}(j)$ and α is the zero map.*

A similar argument also shows that unital homomorphisms between $\mathfrak{M}(j), \mathfrak{M}(k)$ are injective, and in fact, any non-zero homomorphism between $\mathfrak{M}(j), \mathfrak{M}(k)$ is injective.

Definition 2.1.16. *Using notation from Example (2.1.13.4), let $\mathfrak{A} = \oplus_{j=1}^{d} \mathfrak{M}(n(j))$ and $\mathfrak{B} = \oplus_{k=1}^{e} \mathfrak{M}(m(k))$ be two finite dimensional C*-algebras. For each $j \in$*

$\{1,\ldots,d\}, k \in \{1,\ldots,e\}$, *let* $\delta_j : \mathfrak{A} \longrightarrow \mathfrak{M}(n(j))$ *and* $\varepsilon_k : \mathfrak{B} \longrightarrow \mathfrak{M}(m(k))$ *denote the projection mapping onto the* j*-th summand of* $\mathfrak{A}$ *and* k*-th summand of* $\mathfrak{B}$*, respectively.*

For each $j \in \{1,\ldots,d\}$ *let* $1_j = (a_p)_{p=1}^d \in \mathfrak{A}$ *such that* $a_p = 0_{\mathfrak{M}(n(p))}$ *for* $p \in \{1,\ldots,d\} \setminus \{j\}$ *and* $a_j = 1_{\mathfrak{M}(n(j))}$*, and note that* $1_j\mathfrak{A} = \{a = (a_p)_{p=1}^d \in \mathfrak{A} : a_p = 0 \text{ if } p \in \{1,\ldots,d\} \setminus \{j\}, a_j \in \mathfrak{M}(n(j))\} \cong \mathfrak{M}(n(j))$.

*We call a *-homomorphism* $\alpha : \mathfrak{A} \longrightarrow \mathfrak{B}$ canonical *if the following hold:*

1. *for each* $j \in \{1,\ldots,d\}$*, there is a *-homomorphism* $\alpha_j : \mathfrak{M}(n(j)) \longrightarrow \mathfrak{B}$ *such that the restriction of* α *to* $1_j\mathfrak{A}$ *is* $\alpha_j \circ \delta_j$*, and thus* $\alpha(a) = \sum_{j=1}^d \alpha_j \circ \delta_j(a)$ *for all* $a \in \mathfrak{A}$*, and*

2. *for each* $j \in \{1,\ldots,d\}, k \in \{1,\ldots,e\}$ *there exists a canonical *-homomorphism* $\alpha_{k,j} : \mathfrak{M}(n(j)) \longrightarrow \mathfrak{M}(m(k))$ *of Theorem-Definition (2.1.14) such that* $\alpha_{k,j} = \varepsilon_k \circ \alpha_j$.

For each $j \in \{1,\ldots,d\}, k \in \{1,\ldots,e\}$*, let* $(A)_{k,j}$ *denote the multiplicity of* $\alpha_{k,j}$*. We call the* $e \times d$*-matrix* $A = ((A)_{k,j})_{k\in\{1,\ldots,e\},j\in\{1,\ldots,d\}}$ *the* matrix of partial multiplicities *of* α.

Notation 2.1.17. *Throughout this dissertation, we shall employ the notation* $x \oplus y \in X \oplus Y$ *to mean that* $x \in X$ *and* $y \in Y$ *for any two vector spaces* X *and* Y *whenever no confusion may arise, as a slight yet convenient abuse of notation.*

The map:

$$\alpha : a \oplus b \in \mathfrak{M}(2) \oplus \mathfrak{M}(3) \longmapsto \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} \oplus a \in \mathfrak{M}(7) \oplus \mathfrak{M}(2),$$

is an example of a canonical *-monomorphism from $\mathfrak{M}(2) \oplus \mathfrak{M}(3)$ to $\mathfrak{M}(7) \oplus \mathfrak{M}(2)$.

In this case, we have $\alpha_1 : a \in \mathfrak{M}(2) \longmapsto \begin{pmatrix} a & & \\ & a & \\ & & 0_{\mathfrak{M}(3)} \end{pmatrix} \oplus a \in \mathfrak{M}(7) \oplus \mathfrak{M}(2)$ and $\alpha_2 : b \in \mathfrak{M}(3) \longmapsto \begin{pmatrix} 0_{\mathfrak{M}(2)} & & \\ & 0_{\mathfrak{M}(2)} & \\ & & b \end{pmatrix} \oplus 0_{\mathfrak{M}(2)} \in \mathfrak{M}(7) \oplus \mathfrak{M}(2)$. Now, using Definition (2.1.16), the map:

$\alpha_{1,1} = \varepsilon_1 \circ \alpha_1 : a \in \mathfrak{M}(2) \longmapsto \begin{pmatrix} a & & \\ & a & \\ & & 0_{\mathfrak{M}(2)} \end{pmatrix} \in \mathfrak{M}(7)$ has multiplicity 2.

Similarly, we have the map $\alpha_{2,1} = \varepsilon_2 \circ \alpha_1$ has multiplicity 1, the map $\alpha_{1,2} = \varepsilon_1 \circ \alpha_2$ has multiplicity 1, and the map $\alpha_{2,2} = \varepsilon_2 \circ \alpha_2$ has multiplicity 0. Hence, the partial multiplicity matrix of α is $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. Next, we present the classification of *-homomorphisms between finite dimensional C*-algebras by partial multiplicity matrices.

Theorem 2.1.18 ([19, Lemma III.2.1 and Corollary III.2.2])**.** *Using notation from Example (2.1.13.4), let $\mathfrak{A} = \oplus_{j=1}^{d} \mathfrak{M}(n(j)), \mathfrak{B} = \oplus_{k=1}^{e} \mathfrak{M}(m(k))$ be two finite dimensional C*-algebras.*

*If $\beta : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a *-homomorphism, then there exists a canonical *- homomorphism $\alpha : \mathfrak{A} \longrightarrow \mathfrak{B}$ of Definition (2.1.16) and a unitary $U \in \mathfrak{B}$ such that $\beta(a) = U\alpha(a)U^*$ for all $a \in \mathfrak{A}$. Furthermore, the entries of the partial multiplicity matrix $A = ((A)_{k,j})_{k\in\{1,\ldots,e\}, j\in\{1,\ldots,d\}}$ of α satisfy:*

$$\sum_{j=1}^{d} (A)_{k,j} n(j) \leqslant m(k) \text{ for all } k \in \{1, \ldots, e\}, \tag{2.1.2}$$

and each column of A has a non-zero entry if β is further assumed to be injective.

*Moreover, if β is a unital *-homomorphism, then α is a unital *-homomorphism and $\sum_{j=1}^{d}(A)_{k,j}n(j) = m(k)$ for all $k \in \{1, \ldots, e\}$, and in particular, the matrix A satisfies the product:*

$$A \begin{pmatrix} n(1) \\ n(2) \\ \vdots \\ n(d) \end{pmatrix} = \begin{pmatrix} m(1) \\ m(2) \\ \vdots \\ m(e) \end{pmatrix}, \tag{2.1.3}$$

and each column of A has a non-zero entry if β is further assumed to be injective.

*Conversely, if there exists an $e \times d$-matrix A with entries in $\mathbb{N}$ that satisfies Equation (2.1.3) and each column of A has a non-zero entry, then there exists a unital canonical *-monomorphism $\alpha : \mathfrak{A} \longrightarrow \mathfrak{B}$ with partial multiplicity matrix A.*

Proof. Everything up to the converse except injectivity is provided by [19, Lemma III.2.1 and Corollary III.2.2]. For injectivity implying non-zero columns, we proceed by contraposition. Thus, assume that $\beta : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a unital *-homomorphism and that column $p \in \{1, \ldots, d\}$ of the associated partial multiplicity matrix A has all zero entries. Then, using notation from Definition (2.1.16) and following the proof of [19, Lemma III.2.1], the set $\ker \beta$ would contain $\mathfrak{M}(n(p)) \cong 1_p\mathfrak{A} \supsetneq \{0_{\mathfrak{A}}\}$ since in the notation of [19, Lemma III.2.1], the map $\mathrm{id}_{n(p)}^{(A_{k,p})} = \mathrm{id}_{n(p)}^{(0)}$ would be the zero map for all $k \in \{1, \ldots, e\}$. Thus, β would not be injective. The same holds whether or not β is unital by [19, Corollary III.2.2].

For the converse, assume there exists an $e \times d$-matrix A with entries in $\mathbb{N}$ that satisfies Equation (2.1.3) and each column has a non-zero entry. Let $k \in \{1, \ldots, e\}$. Since $\sum_{j=1}^{d}(A)_{k,j}n(j) = m(k)$, for each $j \in \{1, \ldots, d\}$, we may choose canonical *-homomorphisms $\alpha_{k,j} : \mathfrak{M}(n(j)) \longrightarrow \mathfrak{M}(m(k))$ of Definition (2.1.14) with multiplicity $(A)_{k,j}$ in such a way that their images populate distinct blocks of the diagonal of $\mathfrak{M}(m(k))$ and populate the entire diagonal. Therefore:

$$\sum_{j=1}^{d} \alpha_{k,j}\left(1_{\mathfrak{M}(n(j))}\right) = 1_{\mathfrak{M}(m(k))}. \tag{2.1.4}$$

Next, for each $j \in \{1, \ldots, d\}$, define a map $\alpha_j : \mathfrak{M}(n(j)) \longrightarrow \mathfrak{B}$ by the direct sum of maps $\alpha_j(a) = \alpha_{1,j}(a) \oplus \cdots \oplus \alpha_{e,j}(a)$ for all $a \in \mathfrak{M}(n(j))$. By construction, we have that α_j is a *-homomorphism and the map $\alpha_{k,j} = \varepsilon_k \circ \alpha_j$ for each $k \in \{1, \ldots, e\}$. Furthermore, since each column of A has a non-zero entry, there exists $k \in \{1, \ldots, e\}$ such that the multiplicty of $\alpha_{k,j}$ is non-zero. Hence, the map α_j is a *-monomorphism.

Lastly, define $\alpha : \mathfrak{A} \longrightarrow \mathfrak{B}$ by $\alpha(a) = \sum_{j=1}^{d} \alpha_j \circ \delta_j(a)$ for all $a \in \mathfrak{A}$, which is a *-linear map (it is linear and *-preserving) by construction whose restriction to $1_j\mathfrak{A}$ of Definition (2.1.16) is $\alpha_j \circ \delta_j$ for all $j \in \{1, \ldots d\}$. Fix $j, p \in \{1, \ldots d\}$. For the injectivity of α, consider $a = (a_1, \ldots, a_d) \in \mathfrak{A}$. By construction, α places at least one copy of a_j for each $j \in \{1, \ldots, d\}$ on distinct blocks of the diagonals of the full matrix algebras that comprise $\mathfrak{B}$ and zeros elsewhere. Thus, since the norm of a block diagonal matrix is the maximum norm of the norm of each of its blocks, we have that α is an isometry by definiton of the norm on $\mathfrak{B}$ from Example (2.1.13.4). Hence, α is an injective *-linear map.

We check that α is multiplicative. Fix $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathfrak{A}$:

$$\begin{aligned}
(\alpha_j \circ \delta_j(a))\,(\alpha_p \circ \delta_p(b)) &= \alpha_j(a_j)\alpha_p(b_p) \\
&= (\alpha_{1,j}(a_j) \oplus \cdots \oplus \alpha_{e,j}(a_j))\,(\alpha_{1,p}(b_p) \oplus \cdots \oplus \alpha_{e,p}(b_p)) \\
&= (\alpha_{1,j}(a_j)\alpha_{1,p}(b_p)) \oplus \cdots \oplus (\alpha_{e,j}(a_j)\alpha_{e,p}(b_p))\,, \text{ and}
\end{aligned}$$

$$(\alpha_j \circ \delta_j(a))\,(\alpha_p \circ \delta_p(b)) = \begin{cases} \alpha_j \circ \delta_j(ab) & : j = p \\ 0_{\mathfrak{B}} & : j \neq p \end{cases}$$

by construction of $\alpha_{k,j}$ and multiplication of block diagonal matrices. It follows that

α is a multiplicative map and therefore a *-monomorphism. For unital, by Equation (2.1.4), we conclude that:

$$\begin{aligned}
\alpha(1_{\mathfrak{A}}) &= \sum_{j=1}^{d} \alpha_j \circ \delta_j(1_{\mathfrak{A}}) \\
&= \sum_{j=1}^{d} \alpha_j \left(1_{\mathfrak{M}(n(j))}\right) \\
&= \sum_{j=1}^{d} \left(\alpha_{1,j}\left(1_{\mathfrak{M}(n(j))}\right) \oplus \cdots \oplus \alpha_{e,j}\left(1_{\mathfrak{M}(n(j))}\right)\right) \\
&= \left(\sum_{j=1}^{d} \alpha_{1,j}\left(1_{\mathfrak{M}(n(j))}\right)\right) \oplus \cdots \oplus \left(\sum_{j=1}^{d} \alpha_{e,j}\left(1_{\mathfrak{M}(n(j))}\right)\right) \\
&= 1_{\mathfrak{M}(m(1))} \oplus \cdots \oplus 1_{\mathfrak{M}(m(e))} = 1_{\mathfrak{B}},
\end{aligned}$$

which completes the proof. □

Remark 2.1.19. *The converse in Theorem (2.1.18) can also be phrased in the case that we are given a given a matrix A satisfying Inequality (2.1.2). Except in this case, the canonical *-homomorphism constructed in the proof need not be unital and need not be injective.*

Representations of C*-algebras are *-homomorphisms to $\mathfrak{B}(\mathcal{H})$. These representations are highly connected to *states* as we will see in Theorem (2.1.40). Thus, we now discuss states and some of their properties. But, in order to define states, first, we introduce positive elements and maps.

Definition 2.1.20. *Let $\mathfrak{A}$ be a C*-algebra. An element $a \in \mathfrak{A}$ is* positive *if there exits $b \in \mathfrak{A}$ such that $a = b^*b$.*

Let $\mathfrak{A}, \mathfrak{B}$ be C-algebras. A function $E : \mathfrak{A} \longrightarrow \mathfrak{B}$ is* positive *if for all positive $a \in \mathfrak{A}$, we have that $E(a)$ is positive in $\mathfrak{B}$.*

All *-homomorphisms are positive maps, which follows directly by definition. We will encounter other examples of positive maps when we introduce conditional expecations in Section (3.1).

By definiton, all positive elements are self-adjoint. Thus, one would hope that positive maps would also preserve self-adjoint elements. This is the case when the positive map is also assumed to be linear.

Lemma 2.1.21. *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. If $E : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a linear positive function, then E is self-adjoint. That is, for all $a \in \mathfrak{sa}(\mathfrak{A})$, we have that $E(a) \in \mathfrak{sa}(\mathfrak{B})$.*

Proof. Assume that $a \in \mathfrak{sa}(\mathfrak{A})$. Then, by [19, Corollary I.4.2], there exist positive elements $a_+, a_- \in \mathfrak{A}$ such that $a = a_+ - a_-$. Since E is positive there exist $b, c \in \mathfrak{B}$ such that $E(a_+) = b^*b, E(a_-) = c^*c$. Since E is linear, we have:

$$\begin{aligned} E(a)^* &= (E(a_+) - E(a_-))^* \\ &= E(a_+)^* - E(a_-)^* \\ &= (b^*b)^* - (c^*c)^* \\ &= b^*(b^*)^* - c^*(c^*)^* \\ &= b^*b - c^*c \\ &= E(a_+) - E(a_-) = E(a), \end{aligned}$$

which completes the proof. □

Definition 2.1.22. *Let $\mathfrak{A}$ be a C^*-algebra. Define the* state space *of $\mathfrak{A}$ by*

$$\mathscr{S}(\mathfrak{A}) = \{\varphi \in \mathfrak{A}' : \varphi \text{ is positive and } \|\varphi\|_{\mathfrak{A}'} = 1\},$$

where $\mathfrak{A}'$ is the dual space, or the space of $\mathbb{C}$-valued bounded linear functions on $\mathfrak{A}$.

Let's present some basic results about the state space. A great advantage of Proposition (2.1.23) is that in the unital case, we have an easy way of checking when a map is a state without having to check positivity. We will see an application of this in the proof of Proposition (2.1.28).

Proposition 2.1.23. *If $\mathfrak{A}$ is a C*-algebra and $\varphi \in \mathfrak{A}'$, then φ is positive if and only if $\varphi(a^*a) \geqslant 0$ for all $a \in \mathfrak{A}$.*

If $\mathfrak{A}$ is a unital C-algebra, then the state space $\mathscr{S}(\mathfrak{A}) = \{\varphi \in \mathfrak{A}' : \|\varphi\|_{\mathfrak{A}'} = 1 = \varphi(1_{\mathfrak{A}})\}$.*

Proof. The first satement follows by definition and the fact that the positive elements of the C*-algebra $\mathbb{C}$ are the non-negative real numbers.

For the second statement, combine [19, Lemma I.9.5] and [19, Lemma I.9.9], and note that in the unital case, the approximate idenitity in [19, Lemma I.9.9] may be replaced with the unit $1_{\mathfrak{A}}$. □

Proposition 2.1.24. *If $\mathfrak{A}$ is a unital C*-algebra, then $\mathscr{S}(\mathfrak{A})$ is a convex set in $\mathfrak{A}'$ that is compact with respect to the weak* topology on $\mathfrak{A}'$.*

Proof. Convexity is a routine argument following from Proposition (2.1.23). Now, note that if $\varphi \in \mathfrak{A}'$ and $1 \geqslant \|\varphi\|_{\mathfrak{A}'} = \sup\{|\varphi(a)| : \|a\|_{\mathfrak{A}} = 1\}$ with $\varphi(1_{\mathfrak{A}}) = 1$, then $1 = \|\varphi\|_{\mathfrak{A}'}$ since $\|1_{\mathfrak{A}}\|_{\mathfrak{A}} = 1$. Next, for $a \in \mathfrak{A}$ and, let $\hat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \mathfrak{A}'$ denote the evaluation map. By definition of the weak* topology, the function $\hat{a}$ is continuous on $\mathfrak{A}'$. By Proposition (2.1.23), we have that:

$$\begin{aligned}
\mathscr{S}(\mathfrak{A}) &= \left\{\varphi \in \mathfrak{A}' : \|\varphi\|_{\mathfrak{A}'} \leqslant 1 = \widehat{1_{\mathfrak{A}}}(\varphi)\right\} \\
&= \left\{\varphi \in \mathfrak{A}' : \|\varphi\|_{\mathfrak{A}'} \leqslant 1, \varphi \in \widehat{1_{\mathfrak{A}}}^{-1}(\{1\})\right\} \\
&= \left\{\varphi \in \mathfrak{A}' : \|\varphi\|_{\mathfrak{A}'} \leqslant 1\right\} \cap \widehat{1_{\mathfrak{A}}}^{-1}(\{1\}).
\end{aligned}$$

The set $\widehat{1_{\mathfrak{A}}}^{-1}(\{1\})$ is closed in the weak* topology since $\{1\}$ is closed in $\mathbb{C}$ and $\widehat{1_{\mathfrak{A}}}$

is continuous by the weak* topology, and the set $\{\varphi \in \mathfrak{A}' : \|\varphi\|_{\mathfrak{A}'} \leqslant 1\}$ is compact in the weak* topology by [18, Banach-Alaoglu Theorem V.3.1]. Hence, the $\mathscr{S}(\mathfrak{A})$ is compact in the weak* topology. $\square$

This next results shows that the state space captures the norm of self-adjoints.

Proposition 2.1.25. *Let $\mathfrak{A}$ be a C*-algebra. If $a \in \mathfrak{sa}(\mathfrak{A})$, then:*

$$\|a\|_{\mathfrak{A}} = \sup\{|\varphi(a)| : \varphi \in \mathscr{S}(\mathfrak{A})\}.$$

Proof. Let $a \in \mathfrak{sa}(\mathfrak{A})$ and $\varphi \in \mathscr{S}(\mathfrak{A})$. Since φ is a state, we have that $|\varphi(a)| \leqslant \|a\|_{\mathfrak{A}}$. Hence, $\sup\{|\varphi(a)| : \varphi \in \mathscr{S}(\mathfrak{A})\} \leqslant \|a\|_{\mathfrak{A}}$. Next, by [19, Lemma I.9.10], there exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that $|\mu(a)| = \|a\|_{\mathfrak{A}}$, which completes the proof. $\square$

A fundamental result about commutative C*-algebras is the Gelfand duality, which we present now. It states that the category of unital commutative C*-algebras with unital *-homomorphisms is dual to the category of compact Hausdorff topological spaces with continuous maps via an equivalence of categories provided by a contravariant Functor. A standard reference on category theory is [52]. We won't provide a complete proof of this equivalence of categories as this would require many more defintions, but we will provide the main tools that motivate and prove this equivalence, which are Theorem (2.1.30) and Thereom (2.1.34). The first main result is the Gelfand-Naimark Theorem (2.1.30). First, we require some definitions. The name of the space in the following definition will be explained expliticty in Section (2.1.1) in Theorem (2.1.47). However, Remark (2.1.27) already alludes to this space's nomenclature.

Definition 2.1.26. *Let $\mathfrak{A}$ be a unital commutative C*-algebra. The* Maximal Ideal Space *is the set:*

$$M_{\mathfrak{A}} = \{\varphi : \mathfrak{A} \longrightarrow \mathbb{C} \mid \varphi \textit{ is non-zero, linear, and multiplicative}\}.$$

Remark 2.1.27. *For any unital commutative C^*-algebra $\mathfrak{A}$, the set $M_{\mathfrak{A}}$ is non-empty. Indeed, the set $\{0_{\mathfrak{A}}\}$ is a closed two-sided ideal of $\mathfrak{A}$ and a standard Kuratowski-Zorn's Lemma [71, Section 1.18] argument then shows that $\mathfrak{A}$ contains a maximal two-sided ideal M of $\mathfrak{A}$ such that $\{0_{\mathfrak{A}}\} \subseteq M \subsetneq \mathfrak{A}$. Then, the proof of [19, Theorem I.2.5] shows that M is closed and can be used to construct a linear multiplicative function $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ such that* $\ker \varphi = M$*, which implies that φ is non-zero, and thus $\varphi \in M_{\mathfrak{A}}$.*

Proposition 2.1.28 ([19, Theorem I.2.5 and Theorem I.2.6])**.** *If $\mathfrak{A}$ is unital commutative C^*-algebra, then $\emptyset \neq M_{\mathfrak{A}} \subseteq \mathscr{S}(\mathfrak{A})$, and when equipped with the weak* topology, the space $M_{\mathfrak{A}}$ is a compact Hausdorff space.*

Proof. By [19, Theorem I.2.5 and Theorem I.2.6], we only need to show that $M_{\mathfrak{A}} \subseteq \mathscr{S}(\mathfrak{A})$, and non-empty is provided by Remark (2.1.27). Now, since φ is multiplicative, we have that $\varphi(1_{\mathfrak{A}}) = \varphi(1_{\mathfrak{A}}1_{\mathfrak{A}}) = \varphi(1_{\mathfrak{A}})^2$, so $\varphi(1_{\mathfrak{A}}) \in \{0,1\}$ since φ is valued in $\mathbb{C}$. Assume by way of contradiction that $\varphi(1_{\mathfrak{A}}) = 0$. Then, we have that $\varphi(a) = \varphi(a1_{\mathfrak{A}}) = \varphi(a)\varphi(1_{\mathfrak{A}}) = 0$ for all $a \in \mathfrak{A}$, which contradicts the assumption that φ is non-zero. Hence, we conclude $1 = \varphi(1_{\mathfrak{A}})$. Lastly, by [19, Theorem I.2.5], we have that $\|\varphi\|_{\mathfrak{A}'} = 1$. Therefore, by Proposition (2.1.23), we are done. $\square$

By definition, given a unital C*-algebra $\mathfrak{A}$, Proposition (2.1.28) shows that the elements of $M_{\mathfrak{A}}$ are unital and *-preserving by Lemma (2.1.21) since they are states. Hence, we may define $M_{\mathfrak{A}}$ as the set of unital *-homomorphisms from $\mathfrak{A}$ to $\mathbb{C}$.

Definition 2.1.29. *Let $\mathfrak{A}$ be a unital commutative C^*-algebra. For $a \in \mathfrak{A}$, define $\widehat{a} : \varphi \in M_{\mathfrak{A}} \longmapsto \varphi(a) \in \mathbb{C}$. The* Gelfand Transform *of $\mathfrak{A}$ is the function:*

$$\Gamma_{\mathfrak{A}} : a \in \mathfrak{A} \longmapsto \widehat{a} \in C(M_{\mathfrak{A}}),$$

which is well-defined by definition of the weak-topology.*

The following theorem is the celebrated Gelfand-Naimark theorem.

Theorem 2.1.30 ([19, Theorem I.3.1])**.** *If $\mathfrak{A}$ is a unital commutative C*-algebra, then the Gelfand Transform $\Gamma_{\mathfrak{A}}$ of $\mathfrak{A}$ is a unital *-isomorphism onto $C(M_{\mathfrak{A}})$.*

Remark 2.1.31. *Also covered in [19, Theorem I.3.1] is the case when $\mathfrak{A}$ is a non-unital commutative C*-algebra. In this case, we have that $M_{\mathfrak{A}}$ is locally compact Hausdorff [19, Corollary I.2.6], and we would replace $C(M_{\mathfrak{A}})$ with $C_0(M_{\mathfrak{A}})$ as we have seen in Example (2.1.13.2). However, since this requires more work and our concern is only for the unital case as we only work with quantum <u>compact</u> metric space, we do not include these details in this dissertation.*

Thus, for every unital commutative C*-algebra, there exists a compact Hausdorff space associated to it and vice versa by Example (2.1.13.1). Hence, we are on our way to building a Functor from the category of compact Hausdorff spaces onto the category of unital commutative C*-algebras. But, a Functor must also send morphisms to morphisms. This is Theorem (2.1.34), which also shows that a homeomorphism between two compact Hausdorff spaces extends to a unital *-isomorphism of the associated unital commutative C*-algebras and vice versa, which implies that the study of compact Hausdorff topological spaces is the same as the study of unital commutative C*-algebras. First, we present a proposition and a basic lemma.

Proposition 2.1.32 ([18, Theorem VII.8.7])**.** *If X is a compact Hausdorff space, then the map defined by:*

$$\Delta_X : x \in X \longmapsto \delta_x \in M_{C(X)},$$

where $\delta_x(f) = f(x)$ for all $f \in C(X)$ is the Dirac point mass of x, is well-defined and a homeomorphism onto $M_{C(X)}$.

Lemma 2.1.33. *Let X be a compact Hausdorff space. If $p \in C(X)$ is a projection, then $p = 1_{C(X)}$ if and only if $p(x) \neq 0$ for all $x \in \mathfrak{A}$.*

Proof. The forward implication of the equivalence is clear. For the backward implication, assume that $p \neq 1_{C(X)}$. Hence, there exists $x \in X$ such that $p(x) \neq 1$. Now, since p is a projection, we have that $p^2 = p \implies p\left(1_{C(X)} - p\right) = 0 \implies p(x)(1 - p(x)) = 0$. Since p is valued in $\mathbb{C}$ and $1 - p(x) \neq 0$, then $p(x) = 0$, which completes the proof by contraposition. □

Theorem 2.1.34. *If X, Y are two compact Hausdorff spaces, then X is homeomorphic to Y if and only if there is a unital *-isomorphism from $C(Y)$ onto $C(X)$.*

In particular, the following hold:

1. *if $f : X \longrightarrow Y$ is continuous, then the map:*

$$\pi_f : b \in C(Y) \longmapsto b \circ f \in C(X)$$

   *is a unital *-homomorphism, which is a unital *-isomorphism when f is a homeomorphism;*

2. *if $\pi : C(Y) \longrightarrow C(X)$ is a unital *-homomorphism, then the map:*

$$f_\pi : x \in X \longmapsto \Delta_Y^{-1}(\Delta_X(x) \circ \pi) \in Y$$

   *is continuous, which is a homeomorphism when π is a unital *-isomorphism, and in fact, if $\pi : C(Y) \longrightarrow C(Y)$ is a *-homomoprhism, then the map f_π is well-defined if and only if π is unital;*

3. *if $f : X \longrightarrow Y$ is continuous, then using the definitions in parts 1. and 2.:*

$$f_{\pi_f} = f;$$

4. *if $\pi : C(Y) \longrightarrow C(X)$ is a unital *-homomorphism, then using the definitions*

in parts 1. and 2.:

$$\pi_{f_\pi} = \pi.$$

Proof. We start with *1.* It is routine to check that π_f is well-defined and a unital *-homomorphism. Assume that f is surjective. Then, if $b \in C(Y)$, then:

$$\|\pi_f(b)\|_{C(X)} = \sup\{|b \circ f(x)| : x \in X\} = \sup\{|b(y)| : y \in Y\} = \|b\|_{C(Y)}.$$

Hence, π_f is an isometry and therefore injective. Next, assume that f is a homeomorphism, then $f^{-1} : Y \longrightarrow X$ is well-defined and continuous. Let $a \in C(X)$. Thus, consider $a \circ f^{-1} \in C(X)$ and $\pi_f(a \circ f^{-1}) = a \circ f^{-1} \circ f = a$. Therefore, π_f is surjective and we are done.

Next, we prove *2.* Assume that $\pi : C(Y) \longrightarrow C(X)$ is a *-homomorphism. We begin by showing that f_π is well-defined if and only if π is unital. First, assume that π is unital. Fix $x \in X$. Since π is a *-homomorphism, we have that $\delta_x \circ \pi$ is a linear multiplicative continuous $\mathbb{C}$-valued function. Since π is unital, we gather that:

$$0 \neq 1 = 1_{C(X)}(x) = \pi\left(1_{C(Y)}\right)(x) = \delta_x\left(\pi\left(1_{C(Y)}\right)\right) = \delta_x \circ \pi\left(1_{C(Y)}\right).$$

Hence, the function $\delta_x \circ \pi$ is a non-zero linear multiplicative $\mathbb{C}$-valued function and thus $\delta_x \circ \pi \in M_{C(Y)}$. Thus, since Δ_Y is a surjection onto $M_{C(Y)}$ by Proposition (2.1.32), we have that $\Delta_Y^{-1}(\delta_x \circ \pi) \in Y$, and so $f_\pi(x) \in M_{C(X)}$. Since $x \in X$ was chosen arbitrarily, the map f_π is well-defined.

Second, assume that f_π is well-defined. Then, for all $x \in X$, we have that:

$$f_\pi(x) = \Delta_Y^{-1}(\delta_x \circ \pi) \in Y \iff \delta_x \circ \pi \in M_{C(Y)} \tag{2.1.5}$$

since Δ_Y is a bijection by Proposition (2.1.32). Assume by way of contradiction that π is non-unital. Since π is a *-homomorphism and $1_{C(Y)}$ is a projection, we have that $1_{C(X)} \neq \pi\left(1_{C(Y)}\right) = p$ is a projection. Now, by Lemma (2.1.33), there exists $z \in X$ such that $p(z) = 0$. But, then $\delta_z \circ \pi\left(1_{C(Y)}\right) = p(z) = 0$. In particular, the function $\delta_z \circ \pi$ is a non-unital linear multiplicative continuous $\mathbb{C}$-valued function, which is a continuous linear and multiplicative since π is a *-homomorphism, such that $\delta_z \circ \pi \in M_{C(Y)}$ by Expression (2.1.5). But, this is a contradiction to the facts that $M_{C(Y)} \subset \mathscr{S}(C(Y))$ by Proposition (2.1.28) and that states are unital by Proposition (2.1.23). Hence, the *-homomorphism π is unital.

For the remainder of the proof of part *2.*, we assume that π is a unital *-homomorphism, so that f_π is well-defined. For continuity of f_π, let $(x_\lambda)_{\lambda\in\Delta} \subset X$ be a net that converges to $x \in X$. Now, if we fix $b \in C(Y)$, then $\delta_{x_\lambda} \circ \pi(b) = \pi(b)(x_\lambda)$. Hence, since π is well-defined and thus $\pi(b)$ is continuous, the net $(\pi(b)(x_\lambda))_{\lambda\in\Delta} \subset \mathbb{C}$ converges to $\pi(b)(x) = \delta_x \circ \pi(b) \in \mathbb{C}$. Since $b \in C(Y)$ was arbitrary, the net $(\delta_{x_\lambda} \circ \pi)_{\lambda\in\Delta} \subset M_{C(Y)}$ converges to $\delta_x \circ \pi \in M_{C(Y)}$ in the weak* topology. However, since Δ_Y is a homeomorphism, we have that the net $\left(\Delta_Y^{-1}(\delta_{x_\lambda} \circ \pi)\right)_{\lambda\in\Delta} \subset Y$ converges to $\Delta_Y^{-1}(\delta_x \circ \pi) \in Y$. Therefore, the map f_π is continuous. Next, assume that π is a unital *-isomorphism. Hence, the map $\pi^{-1} : C(X) \longrightarrow C(Y)$ is a unital *-isomorphism. Let $y \in Y$. Then, we have that $\Delta_X^{-1}(\delta_y \circ \pi^{-1}) \in X$ by the same argument as above. However, we have:

$$
\begin{aligned}
f_\pi\left(\Delta_X^{-1}(\delta_y \circ \pi^{-1})\right) &= \Delta_Y^{-1}(\Delta_X(\Delta_X^{-1}(\delta_y \circ \pi^{-1})) \circ \pi) \\
&= \Delta_Y^{-1}(\delta_y \circ \pi^{-1} \circ \pi) \\
&= \Delta_Y^{-1}(\delta_y) = y.
\end{aligned}
$$

Hence, the map f_π is surjective. Next, we gather that since π is a surjective and Δ_X, Δ_Y are bijections by Proposition (2.1.32):

$$\begin{aligned}
f_\pi(x) = f_\pi(x') &\implies \Delta_Y^{-1}(\Delta_X(x) \circ \pi) = \Delta_Y^{-1}(\Delta_X(x') \circ \pi) \\
&\implies \Delta_X(x) \circ \pi = \Delta_X(x') \circ \pi \\
&\implies \delta_x \circ \pi = \delta_{x'} \circ \pi \\
&\implies \pi(a)(x) = \pi(a)(x') \text{ for all } a \in C(Y) \\
&\implies b(x) = b(x') \text{ for all } b \in C(X) \\
&\implies \delta_x(b) = \delta_{x'}(b) \text{ for all } b \in C(X) \\
&\implies \delta_x = \delta_{x'} \\
&\implies \Delta_X(x) = \Delta_X(x') \\
&\implies x = x',
\end{aligned}$$

where the last implication uses the fact that Δ_X is injective from Proposition (2.1.32). Thus, the map f_π is a continuous bijection between compact Hausdorff spaces and is therefore a homeomorphism.

Next, we prove *3.* Let $f : X \longrightarrow Y$ be continuous. By part *1.*, the map $\pi_f : C(Y) \longrightarrow C(X)$ is a unital *-homomorphism. Thus, by part *2.*, the map $f_{\pi_f} : X \longrightarrow Y$ is well-defined and continuous. Let $x \in X$, we then have $f_{\pi_f}(x) = \Delta_Y^{-1}(\delta_x \circ \pi_f)$. Therefore:

$$\begin{aligned}
\pi_f(b)(x) = b(f(x)) \text{ if } b \in C(Y) &\iff \delta_x(\pi_f(b)) = \delta_{f(x)}(b) \text{ if } b \in C(Y) \\
&\iff \delta_x \circ \pi_f(b) = \delta_{f(x)}(b) \text{ if } b \in C(Y) \\
&\iff \delta_x \circ \pi_f = \delta_{f(x)} \\
&\iff \delta_x \circ \pi_f = \Delta_Y(f(x)) \\
&\iff \Delta_Y^{-1}(\delta_x \circ \pi_f) = f(x)
\end{aligned}$$

$$\iff f_{\pi_f}(x) = f(x),$$

which implies that $f_{\pi_f} = f$ since $x \in X$ was arbitrary.

Lastly, we prove *4.* Let $\pi : C(Y) \longrightarrow C(X)$ be a unital *-homomorphism. By part *2.*, the map $f_\pi : X \longrightarrow Y$ is well-defined and continuous. Thus, by part *1.*, the map $\pi_{f_\pi} : C(Y) \longrightarrow C(X)$ is a unital *-homomorphism. Let $b \in C(Y)$, we then have $\pi_{f_\pi}(b) = b \circ f_\pi$. Therefore:

$$\begin{aligned}
\delta_x \circ \pi(b) = \delta_x \circ \pi(b) \text{ if } x \in X &\iff \Delta_Y(\Delta_Y^{-1}(\delta_x \circ \pi))(b) = \delta_x \circ \pi(b) \text{ if } x \in X \\
&\iff \delta_{\Delta_Y^{-1}(\delta_x \circ \pi)}(b) = \delta_x \circ \pi(b) \text{ if } x \in X \\
&\iff b\left(\Delta_Y^{-1}(\delta_x \circ \pi)\right) = \delta_x \circ \pi(b) \text{ if } x \in X \\
&\iff b\left(\Delta_Y^{-1}(\delta_x \circ \pi)\right) = \pi(b)(x) \text{ if } x \in X \\
&\iff b\left(f_\pi(x)\right) = \pi(b)(x) \text{ if } x \in X \\
&\iff b \circ f_\pi = \pi(b) \\
&\iff \pi_{f_\pi}(b) = \pi(b),
\end{aligned}$$

which implies that $\pi_{f_\pi} = \pi$ since $b \in C(Y)$ was arbitrary.

To complete the proof, note that *1.* and *2.* imply that X is homeomorphic to Y if and only if there exists a unital *-isomomorphism from $C(Y)$ onto $C(X)$. $\square$

Remark 2.1.35. *Much like Remark (2.1.31), one would hope that Theorem (2.1.34) could also be translated to the non-unital case. This can be done, but requires some subtleties. We will not go into full detail since this dissertation does not concern non-unital C*-algebras as we focus our attention on quantum compact metric spaces, but we will give some ideas here. First, the equivalence of categories would be for the category of locally compact Hausdorff space with proper continuous maps (in the case of locally compact Hausdorff spaces, a continuous map is proper if it extends* *to a continuous map between the Alexandroff (one-point) compactifications in the obvious way) and the category of non-unital commutative C*-algebras with non-zero *-homomorphisms (notice, of course, that the unital requirement is no longer there).*

A reason to use proper continuous maps is because of the following. If X is locally compact Hausdorff, then let $X_\infty = X \cup \{\infty_X\}$ denote its Alexandroff compactification, which is compact Hausdorff. Now, the C-algebra $C_0(X)$ is canonically *-isomorphic to the maximal two-sided ideal $\{a \in C(X_\infty) : a(\infty_X) = 0\}$ of $C(X_\infty)$. Next, any *-homomorphism $\pi : C_0(Y) \longrightarrow C_0(X)$, where Y is some other locally compact Hausdorff space, can be uniquely extended to a unital *-homomorphism $\widetilde{\pi} : C(Y_\infty) \longrightarrow C(X_\infty)$ — this can be done with any *-homomorphism between any two C*-algebras extending to their unitizations (see [18, Proposition VIII.1.9] and [19, Proposition I.1.3] for the definition of unitization and a proof of this fact), and note that the unitization of $C_0(X)$ is canonically *-isomorphic to $C(X_\infty)$. Assume also that π is non-zero, i.e. it is not the zero homomorphism. Now, using Theorem (2.1.34.2), construct the continuous map $f_{\widetilde{\pi}} : X_\infty \longrightarrow Y_\infty$. From basic calculations and the fact that locally compact Hausdorff spaces are Tychonoff [71, Theorem 19.3], one could deduce that $f_{\widetilde{\pi}}(X) \subseteq Y$ and $f_{\widetilde{\pi}}(\infty_X) = \infty_Y$. Thus, we can see why we would restrict our attention to proper continuous maps.*

*One reason we consider only non-zero *-homomorphisms and another reason to consider proper continuous maps is the following. Let $f : X \longrightarrow Y$ be a proper continuous map between locally compact Hausdorff spaces. Using notation from the above paragraph, let $\widetilde{f} : X_\infty \longrightarrow Y_\infty$ be a continuous map that extends f such that $\widetilde{f}(\infty_X) = \infty_Y$. By Theorem (2.1.34.1), let $\widetilde{\pi}_{\widetilde{f}} : b \in C(Y_\infty) \longmapsto b \circ \widetilde{f} \in C(X_\infty)$ be the unital *-homomorphism associated to $\widetilde{f}$. Now, since f is proper, we have that $\widetilde{\pi}_{\widetilde{f}}$ restricted to $\{b \in C(Y_\infty) : b(\infty_Y) = 0\}$ — which is canonically *-isomorphic to $C_0(Y)$ — induces a *-homomorphism $\pi : C_0(Y) \longrightarrow C_0(X)$. Now, since there exists at least one $y \in Y$ such that $f(x) = y$ for some $x \in X$ and since locally* *compact Hausdorff spaces are Tychonoff [71, Theorem 19.3], we have that there exists a $b \in C_0(Y)$ such that $\pi(b)$ is not the zero element in $C_0(X)$.*

*Much more work needs to be done to provide an equivalence of categories in this case, but in the very least, we can see why we would restrict our attention to proper continuous maps and non-zero *-homomorphisms instead of allowing for all continuous maps and all *-homomorphisms.*

There is only little more to be done to provide an equivalence between the categories of compact Hausdorff spaces and unital commutative C*-algebras via a contravariant functor since Theorem (2.1.30) and Theorem (2.1.34) are the main ingredients to provide this duality. A natural idea is thus to study C*-algebras as noncommutative generalizations of topological spaces.

Next, we move on to providing the main representation theorem for C*-algebras. We note that this fact relies heavily on the Gelfand-Naimark Theorem (2.1.30) and the continuous functional calculus that it provides. For a description of the continuous functional calculus, see [19, Corollary I.3.2].

We now begin by detailing the Gelfand-Naimark-Segal (GNS) Construction, which will be Theorem (2.1.40). A powerful consequence of this construction is that every C*-algebra is *-isomorphic to an operator-norm closed *-subalgebra of bounded operators on some Hilbert space [19, Theorem I.9.12]. However, the GNS construction is also useful for constructing quantum metrics (see Theorem (3.1.3)).

We will make note of certain cases of the GNS construction for the different Hilbert spaces and representations it may produce, so we introduce the following definitions.

Definition 2.1.36. *Let $\mathfrak{A}$ be a C*-algebra.*

A state $\mu \in \mathscr{S}(\mathfrak{A})$ is faithful *if for $a \in \mathfrak{A}$, $\mu(a^*a) = 0 \iff a = 0_{\mathfrak{A}}$.*

A state $\mu \in \mathscr{S}(\mathfrak{A})$ is tracial *if for all $a, b \in \mathfrak{A}$, we have $\mu(ab) = \mu(ba)$.*

A state $\mu \in \mathscr{S}(\mathfrak{A})$ is pure *if it is an extreme point in $\mathscr{S}(\mathfrak{A})$. Denote the set of pure states of $\mathfrak{A}$ by $\mathscr{P}(\mathfrak{A})$.*

Example 2.1.37. *We provide some examples of states.*

1. *Let $\mathfrak{M}(n)$ be the C*-algebra of $n \times n$-matrices. The map $\mathsf{tr}_n : a \in \mathfrak{M}(n) \longmapsto \frac{1}{n}\mathrm{Tr}(a) \in \mathbb{C}$, where Tr is the trace of a matrix, is a faithful and tracial state, and this is the unique faithful tracial state of $\mathfrak{M}(n)$ by [19, Example IV.5.4].*

2. *Let X be a compact Hausdorff space and consider the C*-algebra, $C(X)$. By Proposition (2.1.32) and [36, Proposition 4.4.1], all pure states of $C(X)$ are of the form δ_x for some $x \in X$.*

   We also note that by the Riesz Representation Theorem [18, Appendix C.18], the state space $\mathscr{S}(C(X))$ can be identified with Borel probability measures on X, denoted by $M(X)$, via:

$$\mu \in M(X) \longmapsto \varphi_\mu \in \mathscr{S}(C(X)),$$

   where $\varphi_\mu(f) = \int_X f \, d\mu$ for all $f \in C(X)$, and the pure states correspond to points (point masses).

The GNS construction allows one to build the representation theory for C*-algebras from states. Thus, we introduce the following definition.

Definition 2.1.38. *Let $\mathfrak{A}$ be a C*-algebra. A *-representation π of $\mathfrak{A}$ is a *-homomorphism $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

π is cyclic *if there exists a vector $h \in \mathcal{H}$ such that the set $\{\pi(a)h : a \in \mathfrak{A}\}$ is norm dense in $\mathcal{H}$. The vector h is called the* cyclic vector*.*

π is irreducible *if for any closed subspace $M \subset \mathcal{H}$ such that $\pi(\mathfrak{A})M \subseteq M$, then we have $M = \{0_{\mathcal{H}}\}$ or $\mathcal{H}$.*

π is faithful *if for $a \in \mathfrak{A}$, $\pi(a^*a) = 0_{\mathfrak{B}(\mathcal{H})} \iff a = 0_{\mathfrak{A}}$.*

Proposition 2.1.39. *Let $\mathfrak{A}$ be a C*-algebra. A *-representation π is faithful if and only if π is isometric.*

Proof. We prove that faithful implies injective. Let $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ be a faithful *-representation for some Hilbert space $\mathcal{H}$. Let $a \in \mathfrak{A}$ such that $a \in \ker \pi$, then by the C*-identity on $\mathfrak{B}(\mathcal{H})$:

$$0 = \|\pi(a)\|^2_{\mathfrak{B}(\mathcal{H})} = \|\pi(a)^*\pi(a)\|_{\mathfrak{B}(\mathcal{H})} = \|\pi(a^*a)\|_{\mathfrak{B}(\mathcal{H})}.$$

Thus $\pi(a^*a) = 0_{\mathfrak{B}(\mathcal{H})} \implies a = 0_{\mathfrak{A}}$ by faithfulness. Therefore, π has trivial kernel and is injective by linearity. Hence, since π is a *-monomorphism, we have that π is an isometry by Proposition (2.1.11). The other implication follows similarly. □

The following Theorem (2.1.40) is the GNS-construction in the unital case. The non-unital case is also covered in [19, Theorem I.9.6] and only differs in part *5.*, which requires the notion of an approximate identity. We provide 2 references for this construction since their proofs complement each other well, and thus allows us to provide a more complete picture of the construction. We note that most of the parts of the statement of the following theorem are gathered from the proofs of [19, Theorem I.9.6] and [18, Gelfand-Naimark-Segal Construction VIII.5.14].

Theorem 2.1.40 ([19, Theorem I.9.6] and [18, Gelfand-Naimark-Segal Construction VIII.5.14])**.** *Let $\mathfrak{A}$ be a unital C*-algebra. If $\mu \in \mathscr{S}(\mathfrak{A})$, then there is a cyclic *-representation $\pi_\mu : \mathfrak{A} \longrightarrow \mathfrak{B}(L^2(\mathfrak{A}, \mu))$for some Hilbert space $L^2(\mathfrak{A}, \mu)$ called the* GNS representation of μ *and a unit cyclic vector $x_\mu \in L^2(\mathfrak{A}, \mu)$ such that:*

$$\mu(a) = \langle \pi_\mu(a)x_\mu, x_\mu \rangle_{L^2(\mathfrak{A},\mu)} \text{ for all } a \in \mathfrak{A}.$$

Moreover:

1. *The set $N_\mu = \{a \in \mathfrak{A} : \mu(a^*a) = 0\}$ is a norm closed left ideal of $\mathfrak{A}$. If μ is*

faithful, then $N_\mu = \{0_{\mathfrak{A}}\}$*. If* μ *is tracial, then* N_μ *is a norm closed two-sided ideal of* $\mathfrak{A}$*.*

2. *Let* $q_\mu : a \in \mathfrak{A} \longrightarrow a + N_\mu \in \mathfrak{A}/N_\mu$ *denote the quotient map. For* $a, b \in \mathfrak{A}$*, we have that* $\langle q_\mu(a), q_\mu(b)\rangle_{L^2(\mathfrak{A},\mu)} = \mu(b^*a)$ *defines a positive definite inner product on* $\mathfrak{A}/N_\mu$*. The space* $L^2(\mathfrak{A},\mu)$ *denotes the Hilbert space obtained by completing* $\mathfrak{A}/N_\mu$ *in the norm induced by this inner product defined by* $\|\cdot\|_{L^2(\mathfrak{A},\mu)} = \sqrt{\langle q_\mu(\cdot), q_\mu(\cdot)\rangle_{L^2(\mathfrak{A},\mu)}}$*.*

3. *For* $a \in \mathfrak{A}$*, define a map* $\pi_{\mu_0}(a) : \mathfrak{A}/N_\mu \longrightarrow \mathfrak{A}/N_\mu$ *by* $\pi_{\mu_0}(a)\left(q_\mu(b)\right) = q_\mu(ab)$ *for all* $b \in \mathfrak{A}$*, which is: well-defined since* N_μ *is a left ideal of* $\mathfrak{A}$*, bounded linear, and satisfies:*

$$\|\pi_{\mu_0}(a)\|_{\mathfrak{B}(\mathfrak{A}/N_\mu)} = \sup\left\{\|\pi_{\mu_0}(a)\left(q_\mu(b)\right)\|_{L^2(\mathfrak{A},\mu)} : \|q_\mu(b)\|_{L^2(\mathfrak{A},\mu)} = 1\right\} \leqslant \|a\|_{\mathfrak{A}}$$

   for all $a \in \mathfrak{A}$*. Thus, for each* $a \in \mathfrak{A}$*, the map* $\pi_{\mu_0}(a)$ *extends to a bounded linear map on* $L^2(\mathfrak{A},\mu)$ *denoted by* $\pi_\mu(a) \in \mathfrak{B}\left(L^2(\mathfrak{A},\mu)\right)$*, and furthermore, the map* $\pi_\mu : a \in \mathfrak{A} \longmapsto \pi_\mu(a) \in \mathfrak{B}\left(L^2(\mathfrak{A},\mu)\right)$ *is a *-representation of* $\mathfrak{A}$ *associated to the Hilbert space* $L^2(\mathfrak{A},\mu)$*.*

4. *If* μ *is faithful, then* q_μ *is injective and* π_μ *is faithful. The state* μ *is pure if and only if* π_μ *is irreducible by [19, Theorem I.9.8].*

5. *The vector* $x_\mu = q_\mu(1_{\mathfrak{A}}) \in L^2(\mathfrak{A},\mu)$ *is a unit cyclic vector for* π_μ *such that:*

$$\mu(a) = \langle \pi_\mu(a)x_\mu, x_\mu\rangle_{L^2(\mathfrak{A},\mu)} \text{ for all } a \in \mathfrak{A}.$$

Proof. By [18, Gelfand-Naimark-Segal Construction VIII.5.14] and [19, Theorem I.9.6] and their proofs, we only check the faithfulness condition of part *4.* Assume that μ is faithful. The injectivity of q_μ is immedaite from part *1.* since $N_\mu = \{0_{\mathfrak{A}}\}$.

Next, let $a \in \mathfrak{A}$ such that $\pi_\mu(a^*a) = 0_{\mathfrak{B}(L^2(\mathfrak{A},\mu))}$. We thus have that:

$$\mu(a^*a) = \langle \pi_\mu(a^*a)x_\mu, x_\mu \rangle_{L^2(\mathfrak{A},\mu)} = \left\langle 0_{\mathfrak{B}(L^2(\mathfrak{A},\mu))} x_\mu, x_\mu \right\rangle_{L^2(\mathfrak{A},\mu)} = 0,$$

which implies that $a = 0_{\mathfrak{A}}$ since μ is faithful and completes the proof. $\square$

With this construction available, we may now state the main representation theorem of C*-algebras.

Theorem 2.1.41 ([19, Theorem I.9.12])**.** *If $\mathfrak{A}$ is a C*-algebra, then there exists a Hilbert space $\mathcal{H}_{\mathfrak{A}}$ and a *-monomorphism $\pi_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H}_{\mathfrak{A}})$. In particular, the abstract C*-algebra $\mathfrak{A}$ is *-isomorphic to a concrete C*-algebra of operators.*

2.1.1 Ideal space of C*-algebras

Ideals are a crucial aspect of the theory of C*-algebras, and in particular, the representation theory. As ideals are natural structural objects in rings, closed ideals are core structures for C*-algebras.

Definition 2.1.42. *Let $\mathfrak{A}$ be a C*-algebra. An ideal $I \subseteq \mathfrak{A}$ of a C*-algebra is a two-sided ideal of the algebra $\mathfrak{A}$ that is also norm closed. We denote the set of ideals of $\mathfrak{A}$ by* $\mathrm{Ideal}(\mathfrak{A})$*, in which we include the trivial ideals $\{0_{\mathfrak{A}}\}, \mathfrak{A}$.*

We say $\mathfrak{A}$ is simple *if* $\mathrm{Ideal}(\mathfrak{A}) = \{\{0_{\mathfrak{A}}\}, \mathfrak{A}\}$.

The following lemma isolates a convenient fact about tracial states on simple C*-algebras, which follows as a consequence of the GNS construction and thus we present this now.

Lemma 2.1.43. *Let $\mathfrak{A}$ be a simple C*-algebra. If μ is a tracial state on $\mathfrak{A}$, then μ is faithful.*

Proof. Let μ be a tracial state on $\mathfrak{A}$. By Theorem (2.1.40.1), the set $N_\mu \in \mathrm{Ideal}(\mathfrak{A})$. Assume by way of contradiction that $N_\mu = \mathfrak{A}$, then $\mu(a) = 0$ for all positive $a \in \mathfrak{A}$.

However, if $x \in \mathfrak{A}$, then it is a linear combination of positive elements of $\mathfrak{A}$. Indeed, the element $x = y + iz$, where $y = \frac{x+x^*}{2}, z = \frac{x-x^*}{2i} \in \mathfrak{sa}(\mathfrak{A})$, and by [19, Corollary I.4.2], there exist positive $y_+, y_-, z_+, z_- \in \mathfrak{A}$ such that $y = y_+ - y_-, z = z_+ - z_-$. Thus, since μ is linear, we have that $\mu(x) = 0$. Since $x \in \mathfrak{A}$ was arbitrary, the map μ is the zero map, which is a contradiction to the assumption that μ is a state. Hence, the set $N_\mu = \{0_\mathfrak{A}\}$, and thus $\mu(a^*a) = 0 \iff a = 0_\mathfrak{A}$, which completes the proof. $\square$

A far-reaching application of the C*-identity is that ideals of C*-algebras are C*-algebras themselves as well as their associated quotients. This is the next theorem due to Segal.

Theorem 2.1.44 ([19, Lemma I.5.1 and Theorem I.5.4])**.** *Let $\mathfrak{A}$ be a C*-algebra. Every ideal of a C*-algebra is a C*-subalgebra of $\mathfrak{A}$, and therefore a C*-algebra itself.*

Moreover, if $I \in \mathrm{Ideal}(\mathfrak{A})$, then the quotient $\mathfrak{A}/I$ is a C-algebra.*

Convention 2.1.45. *Given a C*-algebra, $\mathfrak{A}$, and $I \in \mathrm{Ideal}(\mathfrak{A})$, an element of the quotient C*-algebra $\mathfrak{A}/I$ will be denoted by $a + I$ for some $a \in \mathfrak{A}$. Furthermore, the quotient norm will be denoted $\|a + I\|_{\mathfrak{A}/I} = \inf \{\|a + b\|_\mathfrak{A} : b \in I\}$.*

Next, let's point out some interesting types of ideals.

Definition 2.1.46. *Let $\mathfrak{A}$ be a C*-algebra. An ideal $I \in \mathrm{Ideal}(\mathfrak{A})$ is a* maximal ideal *if for all ideals $J \in \mathrm{Ideal}(\mathfrak{A})$ such that $I \subseteq J \subseteq \mathfrak{A}$, then either $I = J$ or $J = \mathfrak{A}$. Denote the set of maximal ideals as $\mathrm{mIdeal}(\mathfrak{A})$.*

An subset $I \subseteq \mathfrak{A}$ is a primitive ideal *if there exists a non-zero irreducible *-representation π such that the kernel $\ker \pi = I$. We note that this immediately implies that $I \in \mathrm{Ideal}(\mathfrak{A})$. Denote the set of primitive ideals by $\mathrm{Prim}(\mathfrak{A})$.*

The following theorem explains the use of terminology for the maximal ideal space, $M_\mathfrak{A}$, from Definition (2.1.26).

Theorem 2.1.47 ([19, Theorem I.2.5]). *If $\mathfrak{A}$ is a unital commutative C*-algebra. The map:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \mathrm{mIdeal}(\mathfrak{A})$$

is a well-defined bijection.

For now, we provide a basic example of ideals, and we will go into further examples once we introduce inductive limits in Section (2.1.2) and when we discuss ideals of AF algebras in Section (5.1).

Example 2.1.48. *Let X be a compact Hausdorff space. If $U \subseteq X$ is a closed set, then the set $I_U = \{f \in C(X) : f(x) = 0$ for all $x \in U\} \in \mathrm{Ideal}(C(X))$.*

In fact, by [18, Theorem 8.7] and [55, Theorem 5.4.4], we have that:

$$\left\{I_{\{x\}} \in \mathrm{Ideal}(C(X)) : x \in X\right\} = \mathrm{mIdeal}(C(X)) = \mathrm{Prim}(C(X)).$$

Furthermore, by Proposition (2.1.50), we have:

$$\{I_U \in \mathrm{Ideal}(C(X)) : U \subseteq X \textit{ is closed}\} = \mathrm{Ideal}(C(X)).$$

We continue by proving the last statement in the above example. First, a lemma.

Lemma 2.1.49. *Let (X, τ) be a compact Hausdorff space with topology τ. If $U \subseteq X$, then $I_U = \{f \in C(X) : \forall u \in U, f(u) = 0\} \in \mathrm{Ideal}(C(X))$ and $I_{\overline{U}^\tau} = I_U$, where $\overline{U}^\tau$ denotes the closure of U with respect to τ.*

Proof. Fix $x \in X$. Consider $I_{\{x\}}$. It is routine to check that $I_{\{x\}} \in \mathrm{Ideal}(C(X))$. Now, let $U \subseteq X$. It is clear that I_U is a two-sided ideal. But, note that $I_U = \cap_{x \in U} I_{\{x\}}$ is the intersection of closed sets and is therefore closed. Hence, the set $I_U \in \mathrm{Ideal}(C(X))$.

The ideal $I_{\overline{U}^\tau} \subseteq I_U$ since $U \subseteq \overline{U}^\tau$. Let $f \in I_U$. Let $v \in \overline{U}^\tau$, then there exists a net $(u_\lambda)_{\lambda \in \Lambda} \subseteq U$ converging to v. Hence, we have $f(u_\lambda) = 0$ for all $\lambda \in \Lambda$, and since f

is continuous, we conclude $f(v) = 0$. Therefore, the function $f \in \bigcap_{x \in \overline{U}^\tau} I_{\{x\}} = I_{\overline{U}^\tau}$. Thus, the ideal $I_{\overline{U}^\tau} = I_U$. □

Proposition 2.1.50. *Let (X, τ) be a compact Hausdorff space with topology τ. If $I \in \mathrm{Ideal}(C(X))$, then $F_I = \{x \in X : \forall f \in I, f(x) = 0\}$ is closed and $I = I_{F_I}$ of Lemma (2.1.49). Moreover, the map $F \longmapsto I_F$ establishes a one-to-one correspondence between closed subsets of (X, τ) and $\mathrm{Ideal}(C(X))$.*

Proof. Let $F \subseteq X$ be closed. By Lemma (2.1.49), we have that $I_F \in \mathrm{Ideal}(C(X))$, and so, the map $F \longmapsto I_F$ is well-defined.

For surjectivity, assume $I \in \mathrm{Ideal}(C(X))$. If $I = \{0\}$ or $C(X)$, then $F_I = X$ or $\emptyset$, respectively. Also, if I were maximal, then by [18, Theorem VII.8.7], the ideal $I = I_{\{x\}}$ for some $x \in X$. Next, assume that $I \in \mathrm{Ideal}(C(X))$ and not maximal with $\{0\} \subsetneq I \subsetneq C(X)$. We note that:

$$F_I = \{x \in X : \forall f \in I, f(x) = 0\} = \cap_{f \in I} f^{-1}(\{0\}), \tag{2.1.6}$$

which shows that F_I is closed since each $f \in I$ is continuous. As in the statement of the proposition, define $I_{F_I} = \{f \in C(X) : \forall x \in F_I, f(x) = 0\} \in \mathrm{Ideal}(C(X))$ by Lemma (2.1.49).

First, we show that $I \subseteq I_{F_I}$. Let $f \in I$. Let $x \in F_I$, then by definition of F_I, we have $f(x) = 0$. Since $x \in F_I$ was arbitrary, the function $f \in I_{F_I}$ by definition and by Lemma (2.1.49) since F_I is closed.

For the reverse containment, we show that $C(X) \setminus I \subseteq C(X) \setminus I_{F_I}$. Assume $f \in C(X) \setminus I$. Now, by Theorem (2.1.44), the space $C(X)/I$ is a unital commutative C*-algebra since $I \neq C(X)$. By Theorem (2.1.30), let $\Gamma_{C(X)/I} : C(X)/I \to C\left(M_{C(X)/I}\right)$ denote the Gelfand transform of $C(X)/I$, which is a *-isomorphism and $M_{C(X)/I}$ is the space of nonzero multiplicative linear functionals on $C(X)/I$ associated to maximal ideals of $C(X)/I$ as kernels by Theorem (2.1.47). Since $f \notin I$, we have $f+I \neq 0+I \in C(X)/I$. Thus, by injectivity $\Gamma_{C(X)/I}(f+I) \neq 0 \implies \widehat{f+I} \neq 0$. So, there exists $\varphi_m \in M_{C(X)/I}$, where $\ker \varphi_m = m$ is a maximal ideal of $C(X)/I$, such that:

$$0 \neq \widehat{f+I}(\varphi_m) = \varphi_m(f+I). \tag{2.1.7}$$

In particular, we have $f + I \notin \ker \varphi_m = m$.

Next, let $q_I : g \in C(X) \longrightarrow (g+I) \in C(X)/I$ denote the quotient map. For all $g \in C(X)$, define: $\varphi_{m'}(g) = \varphi_m \circ q_I(g)$. Since $\varphi_m \in (C(X)/I)'$ — the dual of $C(X)/I$ —, the map $\varphi_{m'}$ is the unique linear functional $\varphi_{m'} \in C(X)'$ such that $\ker \varphi_{m'} \supseteq I$ by [18, Theorem V.2.2]. Let $m' = \ker \varphi_{m'}$. Note that since $m' \supseteq I$, the space m'/I is well-defined. Therefore:

$$\begin{aligned} m' &= \{g \in C(X) : \varphi_{m'}(g) = 0\} \\ &= \{g \in C(X) : \varphi_m \circ q_I(g) = 0\} \\ &= \{g \in C(X) : (g+I) \in \ker \varphi_m\} \\ &= \{g \in C(X) : (g+I) \in m\} \end{aligned}$$

and it follows that $m'/I = m$. Now, note that since q_I is unital and multiplicative and so is φ_m by Proposition (2.1.28) and Proposition (2.1.23), we have that $\varphi_{m'}$ is a non-zero multiplicative linear functional and thus $\varphi_{m'} \in M_{C(X)}$. Finally, by [19, Theorem I.2.5], the ideal m' is maximal in $C(X)$ such that $m'/I = m$.

Therefore, by [18, Theorem VII.8.7], there exists $y \in X$ such that $m' = I_{\{y\}} = \{g \in C(X) : g(y) = 0\}$. But, the containment $I \subseteq m' = I_{\{y\}}$ implies that $g(y) = 0$ for all $g \in I$. Thus, we gather that $y \in F_I$ by definition of F_I in Expression (2.1.6). Now, Expression (2.1.7) implies that $f+I \notin m$, but then, the function $f \notin m' = I_{\{y\}}$ since $m'/I = m$. Hence, we have $f(y) \neq 0$, yet $y \in F_I$. Therefore, since F_I is closed, we have $f \notin I_{F_I}$ by Lemma (2.1.49), and thus, the ideal $I_{F_I} \subseteq I$, which completes the argument for $I = I_{F_I}$.

Lastly, we have already established that the map $F \mapsto I_F$ is well-defined and onto. What remains is injectivity. Assume $F \neq E$ are closed subsets of X, then choose $e \in E$ such that $e \notin F$. By [71, Urysohn's Lemma 15.6], there exists $f \in C(X)$ such that $f(v) = 0$ for all $v \in F$, but $f(e) \neq 0$. Since F is closed, we have $f \in I_F$ by Lemma (2.1.49). But, also, we have that $f \notin I_E$. Thus, the ideal $I_E \neq I_F$. □

We present a connection between pure states and irreducible *-representations.

Theorem 2.1.51 ([19, Theorem I.9.8])**.** *Let π be a *-representation of a C*-algebra $\mathfrak{A}$ on some Hilbert space $\mathcal{H}$ with a cyclic unit vector $x \in \mathcal{H}$. Then, the for the state μ defined by $\mu(a) = \langle \pi(a)x, x\rangle_{\mathcal{H}}$ for all $a \in \mathfrak{A}$, the following assertions are equivalent:*

1. *μ is a pure state;*
2. *π is irreducible;*
3. *the set $\ker \pi \in \mathrm{Prim}(\mathfrak{A})$.*

Informally speaking, Theorem (2.1.47) and Theorem (2.1.51) suggest that certain classes of ideals may be equipped with natural topologies since the maximal ideal space and the pure states come naturally equipped with the induced weak* topology. We will now introduce a topology on the set $\mathrm{Prim}(\mathfrak{A})$, called the Jacobson topology, which will have a close connection to the weak* topology on pure states via Theorem (2.1.51) and the GNS-construction (Theorem (2.1.40)) provided by Theorem (2.1.54).

Theorem-Definition 2.1.52. *Let $\mathfrak{A}$ be a C*-algebra. Let $\mathcal{I} \subseteq \mathrm{Prim}(\mathfrak{A})$. Define:*

$$\overline{\mathcal{I}}^{\textit{Jacobson}} = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq \cap_{I \in \mathcal{I}} I\}.$$

By [20, Lemma 3.1.1], the operation $\overline{\mathcal{I}}^{\textit{Jacobson}}$ on subsets $\mathcal{I}$ of $\mathrm{Prim}(\mathfrak{A})$ defines a Kuratowski closure operation [71, Theorem 3.7], and therefore induces a unique *topology on* $\mathrm{Prim}(\mathfrak{A})$*, in which the operation* $\overline{\mathcal{I}}^{\boldsymbol{Jacobson}}$ *on subsets* $\mathcal{I}$ *of* $\mathrm{Prim}(\mathfrak{A})$ *is the closure in this topology. We call this topology on* $\mathrm{Prim}(\mathfrak{A})$*, the* Jacobson topology*, denoted by* ***Jacobson****.*

Moreover, if F *is a closed set in the Jacobson topology, then there exists* $I_F \in \mathrm{Ideal}(\mathfrak{A})$ *such that* $F = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ *by [55, Theorem 5.4.7].*

Next, we state some topological properties of $\mathrm{Prim}(\mathfrak{A})$ with the Jacobson topology. We note that a good reference for topology is *General Topology* by Stephen Willard [71].

Theorem 2.1.53. *If* $\mathfrak{A}$ *is a C*-algebra, then* $\mathrm{Prim}(\mathfrak{A})$ *equipped with the Jacobson topology is a locally compact* T_0 *space.*

Moreover, if $\mathfrak{A}$ *is unital, then* $(\mathrm{Prim}(\mathfrak{A}), \boldsymbol{Jacobson})$ *is compact.*

Proof. This is the combination of [20, Proposition 3.1.3, Proposition 3.1.8, and Corollary 3.3.8]. The definition of quasi-compact given in [20] is that every open cover has a finite subcover. Thus, it is the definition of compact. The term quasi-compact is simply the term sometimes used for compact when the space is not necessarily Hausdorff. □

An immediate flaw of this space is that it is not Hausdorff in general. For a non-trivial example of this, see [10, Remark 8.ii], where the Jacobson topology on the Boca-Mundici AF C*-algebra is not even T_1 let alone Hausdorff as there are singletons that are not closed. One of the main remedies to this is the Fell topology, which not only is Hausdorff, but also defines a compact topology on the entire ideal space. It is built using the Jacobson topology. We will introduce this Fell topology shortly in Definition (2.1.58) once we complete our discussion of the Jacobson topology.

In the next Theorem (2.1.54), we continue by noting a powerful connection with the Jacobson topology and that of the weak* topology on pure states.

Theorem 2.1.54 ([57, Theorem 4.3.3]). *Let $\mathfrak{A}$ be a C^*-algebra. The map:*

$$\mu \in \mathscr{P}(\mathfrak{A}) \longmapsto \ker \pi_\mu \in \mathrm{Prim}(\mathfrak{A}),$$

where π_μ is the GNS-representation (Theorem (2.1.40)) of μ, is open and continuous from $\mathscr{P}(\mathfrak{A})$ equipped with the weak-topoogy onto* $\mathrm{Prim}(\mathfrak{A})$ *with the Jacobson topology.*

Proof. By [57, Theorem 4.3.3], we only note that the map is well-defined by Theorem (2.1.51). □

The next Theorem (2.1.55) displays a satsifying consequence of Theorem (2.1.54) in the unital commutative case, which is that the Jacobson topology recovers the weak* topology on the maximal ideal space, and therefore the Jacobson topology is compact Hausdorff in this case.

Theorem 2.1.55. *If $\mathfrak{A}$ is a unital commutative C^*-algebra, then the map:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \mathrm{Prim}(\mathfrak{A}).$$

is a homeomorphism from $M_{\mathfrak{A}}$ equipped with the weak topology onto* $\mathrm{Prim}(\mathfrak{A})$ *equipped with the Jacobson topology, and therefore* $\mathrm{Prim}(\mathfrak{A})$ *equipped with the Jacobson topology is a compact Hausdorff space.*

Proof. By [55, Theorem 5.4.4], the set $\mathrm{Prim}(\mathfrak{A})$ is the set of maximal ideals. However, for all $\varphi \in M_{\mathfrak{A}}$, the ideal $\ker \varphi$ is maximal by Theorem (2.1.47). Hence, the map $\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \mathrm{Prim}(\mathfrak{A})$ is a bijection by Theorem (2.1.47). Furthermore, by [55, Theorem 5.1.6], the set of pure states on $\mathfrak{A}$ is equal to $M_{\mathfrak{A}}$. Therefore, by Theorem (2.1.54), the map $\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \mathrm{Prim}(\mathfrak{A})$ is a homeomorphism onto $\mathrm{Prim}(\mathfrak{A})$ since it is a continuous and open bijection. Since $M_{\mathfrak{A}}$ is compact Hausdorff, $\mathrm{Prim}(\mathfrak{A})$ with its Jacobson topology is a compact Hausdorff space. □

Next, we introduce the Fell topology, which will use the Jacobson topology to produce a compact Hausdorff space on the set of all ideals of a C*-algebra. In fact, the Fell topology is a compact topology on the closed sets of any topological space and we present the construction in this generality and then apply it to the Jacobson topology. Later, when we introduce the Hausdorff metric topology on the set of closed sets of a compact metric space (Definition (2.3.4)), we will see that the Fell topology agrees with the Hausdorff topology in this case (Proposition (2.3.5.3)). This displays that the Fell topology is a generalization of the Hausdorff topology to non-metric spaces. Just as Fell did in [27], we define:

Definition 2.1.56 ([27])**.** *Let X be a topological space (no separation axioms assumed). Let $\mathcal{C}l(X) = \{F \subseteq X : F \text{ is closed}\}$. Fix a compact set $C \subseteq X$ and a finite family $\mathcal{F}$ of nonempty open subsets of X. Define:*

$$U(C, \mathcal{F}) = \{Y \in \mathcal{C}l(X) : Y \cap C = \emptyset \text{ and } Y \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}\}.$$

We denote the collection of the sets as:

$$\mathcal{B}_{fell}(X) = \left\{ U(C, \mathcal{F}) \subseteq \mathcal{C}l(X) : \begin{array}{c} C \subseteq X \text{ is compact and} \\ \mathcal{F} \text{ is a finite family of nonempty} \\ \text{open subsets of } X \end{array} \right\}.$$

Theorem-Definition 2.1.57 ([27, Lemma 1 and Theorem 1])**.** *If X is a topological space, then the set $\mathcal{B}_{fell}(X)$ forms a basis for a topology on the closed sets $\mathcal{C}l(X)$, called the* Fell topology*, and $\mathcal{C}l(X)$ is compact in this topology.*

Moreover, if X is locally compact, then $\mathcal{C}l(X)$ equipped with the Fell topology is a compact Hausdorff space.

Proof. We only verify that $\mathcal{B}_{fell}(X)$ forms a basis for a topology on $\mathcal{C}l(X)$ since the other results are detailed in the proofs of [27, Lemma 1 and Theorem 1]. For this, we first show that $\mathcal{C}l(X) = \cup_{B \in \mathcal{B}_{fell}(X)} B$. Let $\mathcal{F}$ contain no sets. Then, for any compact set $C \subseteq X$, we have that $\emptyset \in U(C, \mathcal{F})$ and thus $\emptyset \in \cup_{B \in \mathcal{B}_{fell}(X)} B$. Next, let $\emptyset \neq Y \in \mathcal{C}l(X)$, then as $\emptyset$ is compact, the set $U(\emptyset, \{X\})$ contains Y and thus $Y \in \cup_{B \in \mathcal{B}_{fell}(X)} B$. Hence, the set $\mathcal{C}l(X) = \cup_{B \in \mathcal{B}_{fell}(X)} B$.

Now, assume that $U(C_1, \mathcal{F}_1), U(C_2, \mathcal{F}_2) \in \mathcal{B}_{fell}(X)$. Let $C = C_1 \cup C_2$, which is compact and let $\mathcal{F} = \{A \subseteq X : A \in \mathcal{F}_1 \text{ or } A \in \mathcal{F}_2\}$, which is a finite family of nonempty open subsets of X. Let $Y \in U(C, \mathcal{F})$, then $Y \cap (C_1 \cup C_2) = \emptyset \implies (Y \cap C_1) \cup (Y \cap C_2) = \emptyset \implies (Y \cap C_1) = \emptyset$ and $(Y \cap C_2) = \emptyset$. Next, let $A \in \mathcal{F}_1$, then $A \in \mathcal{F}$. However, we have $Y \cap A \neq \emptyset$. Thus, the set $Y \in U(C_1, \mathcal{F}_1)$. Also, it follows that $Y \in U(C_2, \mathcal{F}_2)$, so that $Y \in U(C_1, \mathcal{F}_1) \cap U(C_2, \mathcal{F}_2)$, which completes the argument that $\mathcal{B}_{fell}(X)$ is a basis. $\square$

Now, we apply this construction to primitive ideals with the Jacobson topology and utilize a bijection between closed sets in the Jacobson topology and ideals to provide a topology on all ideals.

Definition 2.1.58 ([26]). *Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ be the set of closed subsets of $(\mathrm{Prim}(\mathfrak{A}), \boldsymbol{Jacobson})$ with the Fell topology, denoted $\tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))}$, which is compact Hausdorff by Theorem-Definition (2.1.57) and Theorem (2.1.53). Let $fell : \mathrm{Ideal}(\mathfrak{A}) \to \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ denote the map:*

$$fell(I) = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq I\},$$

which is a bijection by [55, Theorem 5.4.7]. The Fell topology *on* $\mathrm{Ideal}(\mathfrak{A})$*, denoted* **Fell***, is the initial topology on* $\mathrm{Ideal}(\mathfrak{A})$ *induced by $fell$, which is the weakest topology for which $fell$ is continuous. Equivalently:*

$$\mathbf{Fell} = \left\{ U \subseteq \text{Ideal}(\mathfrak{A}) : U = fell^{-1}(V), V \in \tau_{\mathcal{C}l(\text{Prim}(\mathfrak{A}))} \right\},$$

and $(\text{Ideal}(\mathfrak{A}), \mathbf{Fell})$ *is therefore compact Hausdorff since* $fell$ *is a bijection and* $\left(\mathcal{C}l(\text{Prim}(\mathfrak{A})), \tau_{\mathcal{C}l(\text{Prim}(\mathfrak{A}))}\right)$ *is compact Hausdorff.*

As is, the Fell topology on ideals is a complicated construction. However, Fell provided enough framework in his paper [26] to easliy deduce a useful and simple characterization of net convergence in the Fell topology on ideals. This is the following Lemma (2.1.59), which is stated in [6, Section 2], where the Fell topology, **Fell**, is denoted by τ_s. We provide a proof.

Lemma 2.1.59. *Let* $\mathfrak{A}$ *be a* C^**-algebra. Let* $(I_\mu)_{\mu \in \Delta} \subseteq \text{Ideal}(\mathfrak{A})$ *be a net and* $I \in \text{Ideal}(\mathfrak{A})$*. The net* $(I_\mu)_{\mu \in \Delta}$ *converges to* I *with respect to the Fell topology if and only if for all* $a \in \mathfrak{A}$*, the net* $\left(\|a + I_\mu\|_{\mathfrak{A}/I_\mu} \right)_{\mu \in \Delta} \subseteq \mathbb{R}$ *converges to* $\|a + I\|_{\mathfrak{A}/I} \in \mathbb{R}$ *with respect to the usual topology on* $\mathbb{R}$*.*

Proof. By [26, Theorem 2.2], let $Y \in \mathcal{C}l(\text{Prim}(\mathfrak{A}))$, define:

$$M_Y : a \in \mathfrak{A} \longmapsto \sup \left\{ \|a + I\|_{\mathfrak{A}/I} : I \in Y \right\} \in \mathbb{R},$$

since in Fell's notation, given an ideal S, we have $S_a = a + S$ according to his definition of *transform* in [26, Section 2.1] in the context of the primitive ideal space $\check{\mathfrak{A}} = \text{Prim}(\mathfrak{A})$. But, by the first line of the proof of [26, Theorem 2.2], we note that $\cap_{I \in Y} I \in \text{Ideal}(\mathfrak{A})$ and:

$$M_Y(a) = \|a + \cap_{I \in Y} I\|_{\mathfrak{A}/(\cap_{I \in Y} I)}, \tag{2.1.8}$$

for all $a \in \mathfrak{A}$.

Let $P \in \text{Ideal}(\mathfrak{A})$, then $fell(P) = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq P\} \in \mathcal{C}l(\text{Prim}(\mathfrak{A}))$ by Definition (2.1.58). Note that $\cap_{H \in fell(P)} H = P$ by [55, Theorem 5.4.3]. Thus, by

Expression (2.1.8):

$$M_{fell(P)}(a) = \|a + P\|_{\mathfrak{A}/P}. \tag{2.1.9}$$

Now, assume that $(I_\mu)_{\mu\in\Delta} \subset \text{Ideal}(\mathfrak{A})$ converges to $I \in \text{Ideal}(\mathfrak{A})$ with respect to the Fell topology. Since $fell$ is continuous, the net $(fell\,(I_\mu))_{\mu\in\Delta} \subseteq \mathcal{C}l(\text{Prim}(\mathfrak{A}))$ converges to $fell(I) \in \mathcal{C}l(\text{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{C}l(\text{Prim}(\mathfrak{A}))$. By [26, Theorem 2.2], the net of functions $\left(M_{fell(I_\mu)}\right)_{\mu\in\Delta}$ converges pointwise to $M_{fell(I)}$, which completes the forward implication by Equation (2.1.9).

For the reverse implication, assume that the net $\left(\|a + I_\mu\|_{\mathfrak{A}/I_\mu}\right)_{\mu\in\Delta} \subseteq \mathbb{R}$ converges to $\|a+I\|_{\mathfrak{A}/I} \in \mathbb{R}$ with respect to the usual topology on $\mathbb{R}$ for all $a \in \mathfrak{A}$ and for some net $(I_\mu)_{\mu\in\Delta} \subseteq \text{Ideal}(\mathfrak{A})$ and $I \in \text{Ideal}(\mathfrak{A})$. But, then by Equation (2.1.9) and assumption, the net $\left(M_{fell(I_\mu)}\right)_{\mu\in\Delta}$ converges pointwise to $M_{fell(I)}$. By [26, Theorem 2.2], the net $(fell\,(I_\mu))_{\mu\in\Delta} \subseteq \mathcal{C}l(\text{Prim}(\mathfrak{A}))$ converges to $fell(I) \in \mathcal{C}l(\text{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{C}l(\text{Prim}(\mathfrak{A}))$. However, as $fell$ is a continuous bijection between the compact Hausdorff spaces $(\text{Ideal}(\mathfrak{A}), \mathbf{Fell})$ and $\left(\mathcal{C}l(\text{Prim}(\mathfrak{A})), \tau_{\mathcal{C}l(\text{Prim}(\mathfrak{A}))}\right)$, the map $fell$ is a homeomorphism. Thus, we conclude that $(I_\mu)_{\mu\in\Delta}$ converges to I with respect to the Fell topology. □

2.1.2 Inductive Limits of C*-algebras and AF algebras

Inductive limits of C*-algebras provide a powerful tool in constructing C*-algebras using morphisms and other C*-algebras. A primary application of this is seen in the Elliott classification program. In this program, inductive limits have been used to classify C*-algebras since the programs inception in [23] up to now as seen in [25], in which a specific inductive limit called the the Jiang-Su algebra (defined in [35]) is utililized to provide deep classification results. For our purposes in Noncommutative Metric Geometry, inductive limits provide many possibilities of continuous families for the Gromov-Hausdorff Propinquity.

We follow [55, Chapter 6.1] for the definition of an inductive limit of an inductive sequence of C*-algebras and provide some added details for clarity. First, we introduce the notion of an enveloping C*-algebra, which requires the notion of a *-algebra and C*-seminorm and resembles Definition (2.1.6) of a C*-algebra except that we do not require completeness and a norm.

Definition 2.1.60. *Let $\mathfrak{A}$ be an algebra. If $\mathfrak{A}$ is equipped with an anti-multiplicative conjugate linear involution $^* : \mathfrak{A} \longrightarrow \mathfrak{A}$, called an adjoint, then we call $\mathfrak{A}$ a* *-algebra.

A C*-seminorm *on a *-algebra $\mathfrak{A}$ is a seminorm $p : \mathfrak{A} \longrightarrow [0, \infty)$ such that $p(ab) \leqslant p(a)p(b)$ and $p(aa^*) = p(a)^2$ for all $a, b \in \mathfrak{A}$. The map p is called a* C*-norm *if p is also a norm.*

Now, for the definition of an enveloping C*-algebra associated to a *-algebra and a C*-seminorm.

Theorem-Definition 2.1.61. *If $\widetilde{\mathfrak{A}}$ is a *-algebra with a C*-seminorm p, then:*

1. *$\ker p$ is a two-sided self-adjoint ideal of $\widetilde{\mathfrak{A}}$,*

2. *$\widetilde{\mathfrak{A}}/\ker p$ is a *-algebra with the induced quotient operations from $\widetilde{\mathfrak{A}}$,*

3. *the map $\|\cdot\|_{\mathfrak{A}} : a + \ker p \in \widetilde{\mathfrak{A}}/\ker p \longmapsto p(a) \in [0, \infty)$ is a C*-norm on $\widetilde{\mathfrak{A}}/\ker p$, and*

4. *the Banach space completion of $\widetilde{\mathfrak{A}}/\ker p$ with respect to $\|\cdot\|_{\mathfrak{A}}$ denoted by $\mathfrak{A}$ is a C*-algebra with the norm $\|\cdot\|_{\mathfrak{A}}$, in which the algebraic operations and adjoint of $\widetilde{\mathfrak{A}}/\ker p$ are extended uniquely to $\mathfrak{A}$. If $\widetilde{\mathfrak{A}}$ is unital and p is non-zero, then $\mathfrak{A}$ is unital.*

We call $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ the enveloping C*-algebra *of $\widetilde{\mathfrak{A}}$ with respect to the C*-seminorm p.*

Proof. *1.* and *2* are routine to verify and so is *4.* once we establish *3.*

For *3.*, by [18, Proposition V.2.1], we have that $\widetilde{p}(a+\ker p) = \inf\{p(a+x) : x \in \ker p\}$ is a seminorm on $\widetilde{\mathfrak{A}}/\ker p$. Now, fix $a \in \widetilde{\mathfrak{A}}, x \in \ker p$, then:

$$p(a) = p(a+x-x) \leqslant p(x) + p(a+x) = p(a+x) \leqslant p(a) + p(x) = p(a),$$

which implies that $\|\cdot\|_{\mathfrak{A}}$ is a seminorm on $\widetilde{\mathfrak{A}}/\ker p$, which is a C*-seminorm since $\ker p$ is a two-sided self-adjoint ideal and p is a C*-seminorm. By construction, we have $\|\cdot\|_{\mathfrak{A}}$ is a C*-norm on $\widetilde{\mathfrak{A}}/\ker p$. $\square$

Now, we introduce the notion of an inductive sequence of C*-algebras.

Definition 2.1.62. *Let $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ be a sequence of C*-algebras such that for each $n \in \mathbb{N}$ there exists a *-monomorphism $\alpha_n : \mathfrak{A}_n \longrightarrow \mathfrak{A}_{n+1}$. We call the sequence $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$, an* inductive sequence of C*-algebras. *We say $\mathcal{I}$ is* unital *if $\mathfrak{A}_n$ is unital and α_n is unital for all $n \in \mathbb{N}$.*

Proposition 2.1.63. *If $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ is an inductive sequence of C*-algebras, then:*

1. *if we equip the product $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ with coordinate-wise operations, then $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ is a *-algebra and :*

   $$\widetilde{\mathfrak{A}}_{\mathcal{I}} = \left\{ a = (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n : \exists K_a \in \mathbb{N}, \alpha_k(a_k) = a_{k+1}, \forall k \geqslant K_a \right\}$$

   *is a *-subalgebra of $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$, and if $\mathcal{I}$ is unital, then $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ is unital with unit $(1_{\mathfrak{A}_n})_{n\in\mathbb{N}}$ and $\widetilde{\mathfrak{A}}_{\mathcal{I}}$ is unital,*

2. *for all $a = (a_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$, the sequence $(\|a_n\|_{\mathfrak{A}_n})_{n\in\mathbb{N}} \subset \mathbb{R}$ is eventually constant and the map:*

   $$p_{\mathcal{I}} : a = (a_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}} \longmapsto \lim_{n\to\infty} \|a_n\|_{\mathfrak{A}_n} \in [0,\infty) \subset \mathbb{R}$$

is a C-seminorm on* $\widetilde{\mathfrak{A}}_{\mathcal{I}}$*, and*

3. *the kernel of* $p_{\mathcal{I}}$ *is:*

$$\ker p_{\mathcal{I}} = \left\{ a = (a_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}} : \exists K_a \in \mathbb{N}, a_k = 0, \forall k \geqslant K_a \right\}.$$

Proof. For *1.*, the fact that α_n is a *-homomorphism for each $n \in \mathbb{N}$ implies that $\widetilde{\mathfrak{A}}_{\mathcal{I}}$ is a *-subalgebra of $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$. If $\mathfrak{A}_n$ and α_n are unital for all $n \in \mathbb{N}$, then clearly $(1_{\mathfrak{A}_n})_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$. *3.* follows quickly from *2.*

For *2.*, let $a = (a_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$, then since α_n is a *-monomorphism for each $n \in \mathbb{N}$, we have $\|a_{K_a}\|_{\mathfrak{A}_{K_a}} = \|\alpha_{K_a}(a_{K_a})\|_{\mathfrak{A}_{K_a+1}} = \|a_{K_a+1}\|_{\mathfrak{A}_{K_a+1}}$, and an induction argument shows that $\|a_{K_a}\|_{\mathfrak{A}_{K_a}} = \|a_k\|_{\mathfrak{A}_k}$ for all $k \geqslant K_a$. Hence, for each $a \in \widetilde{\mathfrak{A}}$, the sequence $(\|a_n\|_{\mathfrak{A}_n})_{n\in\mathbb{N}} \subset \mathbb{R}$ is eventually constant and therefore converges. Since for each $n \in \mathbb{N}$, the norms $\|\cdot\|_{\mathfrak{A}_n}$ are C*-norms, we have that $p_{\mathcal{I}}$ is a C*-seminorm in $\widetilde{\mathfrak{A}}_{\mathcal{I}}$. □

Finally, we define an inductive limit associated to an inductive sequence of C*-algebras.

Definition 2.1.64. *Let* $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$*, be an inductive sequence of C*-algebras.*

From Proposition (2.1.63), let the inductive limit *of the inductive sequence of C*-algebras* $\mathcal{I}$ *be the enveloping C*-algebra (Theorem-Definition (2.1.61)) of* $\widetilde{\mathfrak{A}}_{\mathcal{I}}$ *with respect to the C*-seminorm* $p_{\mathcal{I}}$*. We denote the inductive limit by* $\mathfrak{A} = \varinjlim \mathcal{I}$ *and its norm by* $\|\cdot\|_{\mathfrak{A}}$*.*

Note that if $\mathcal{I}$ *is unital, then it is immediate from Proposition (2.1.63) that* $\mathfrak{A}$ *is unital.*

Informally speaking, the idea of an inductive limit is to build up a C*-algebra from a sequence of C*-algebras. This is motivated by the fact that we are choosing a sequence of C*-algebras such that each space embeds into the next space by way of *-monomorphisms. Thus, one would hope that there exists a copy of each C*-algebra of the sequence inside the inductive limit and that these copies build up to $\mathfrak{A}$. This is the purpose of the following Notation (2.1.65) and Proposition (2.1.66)

Notation 2.1.65. *Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence of C*-algebras with inductive limit $\mathfrak{A} = \varinjlim \mathcal{I}$. Let $m, n \in \mathbb{N}$ and set $\alpha_{m\to n}$ be the identity map on $\mathfrak{A}_m$ if $m = n$, and otherwise, set $\alpha_{m\to n}$ to be defined by:*

$$\alpha_{m\to n} : a_m \in \mathfrak{A}_m \longmapsto \alpha_{n-1} \circ \cdots \circ \alpha_{m+1} \circ \alpha_m(a_m) \in \mathfrak{A}_n.$$

For each $n \in \mathbb{N} \setminus \{0\}$, define:

$$\alpha^n : a_n \in \mathfrak{A}_n \longmapsto (b_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$$

by $b_k = 0$ for all $k \in \{0, \ldots, n-1\}$, and $b_k = \alpha_{n\to k}(a_n)$ for all $k \geqslant n$, which is well-defined by construction. If $n = 0$, then let $\alpha^0(a_0) = (b_n)_{n\in\mathbb{N}}$ such that $b_k = \alpha_{0\to k}(a_0)$ for all $k \geqslant 0$.

Finally, let $q_{\mathcal{I}} : \widetilde{\mathfrak{A}}_{\mathcal{I}} \longrightarrow \widetilde{\mathfrak{A}}_{\mathcal{I}}/\ker p_{\mathcal{I}} \subseteq \mathfrak{A}$ be the quotient map. For each $n \in \mathbb{N}$, define:

$$\underrightarrow{\alpha}^n = q_{\mathcal{I}} \circ \alpha^n : \mathfrak{A}_n \longrightarrow \widetilde{\mathfrak{A}}_{\mathcal{I}}/\ker p_{\mathcal{I}} \subseteq \mathfrak{A},$$

and call the maps $\underrightarrow{\alpha}^n$, the canonical *-homomorphisms *of $\mathfrak{A}_n$ into $\mathfrak{A}$.*

Next, we show that the maps introduced in the above Notation (2.1.65) provide a way to capture the C*-algebras of the inductive sequence inside the inductive limit.

Proposition 2.1.66. *If $\mathfrak{A} = \varinjlim \mathcal{I}$ is the inductive limit of an inductive sequence of C*-algebras $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$, then using Notation (2.1.65):*

1. *if $m \in \mathbb{N}, a_m \in \mathfrak{A}_m$ and $N \in \mathbb{N}, N > m$, then $\underrightarrow{\alpha}^m(a_m) = \underrightarrow{\alpha}^N(\alpha_{m\to N}(a_m))$, and*

in particular, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{A}_m \arrow[r, "\alpha_m"] \arrow[dr, "\underrightarrow{\alpha^{m}}"'] & \mathfrak{A}_{m+1} \arrow[d, "\underrightarrow{\alpha^{m+1}}"] \\ & \mathfrak{A} \end{tikzcd},$$

2. *for each $n \in \mathbb{N}$, we have that $\underrightarrow{\alpha^{n}} : \mathfrak{A}_n \longrightarrow \mathfrak{A}$ is a *-monomorphism and thus $\underrightarrow{\alpha^{n}}(\mathfrak{A}_n)$ is a C*-subalgebra of $\mathfrak{A}$ such that $\underrightarrow{\alpha^{n}}(\mathfrak{A}_n) \cong \mathfrak{A}_n$, and if $\mathcal{I}$ is unital, then for each $n \in \mathbb{N}$, we have that $\underrightarrow{\alpha^{n}}$ is unital and $\underrightarrow{\alpha^{n}}(\mathfrak{A}_n)$ has the same unit as $\mathfrak{A}$,*

3. *for each $n \in \mathbb{N}$, we have $\underrightarrow{\alpha^{n}}(\mathfrak{A}_n) \subseteq \underrightarrow{\alpha^{n+1}}(\mathfrak{A}_{n+1})$, and*

4. *the *-subalgebra $\cup_{n\in\mathbb{N}}\underrightarrow{\alpha^{n}}(\mathfrak{A}_n)$, which is unital when $\mathcal{I}$ is unital, is dense in $\mathfrak{A}$.*

Proof. We begin with *1.* Let $m \in \mathbb{N}, a_m \in \mathfrak{A}_m$ and $N \in \mathbb{N}, N > m$. By definition, The element $\underrightarrow{\alpha^{m}}(a_m) = \alpha^m(a_m) + \ker p_{\mathcal{I}}$. Now, by construction, we have that $\alpha^m(a_m) - \alpha^N(\alpha_{m\to N}(a_m)) = b = (b_n)_{n\in\mathbb{N}}$, where $b_k = 0$ for $k \in \{0, \ldots, m-1\}$, $b_k = \alpha_{m\to k}(a_m)$ for $k \in \{m, m+1, \ldots, N-1\}$, and $b_k = 0$ for $k \geqslant N$. However, by Proposition (2.1.63.3), we have that $b \in \ker p_{\mathcal{I}}$. Therefore, we conclude that:

$$\begin{aligned} \underrightarrow{\alpha^{m}}(a_m) &= \alpha^m(a_m) + \ker p_{\mathcal{I}} \\ &= \alpha^N(\alpha_{m\to N}(a_m)) + b + \ker p_{\mathcal{I}} \\ &= \alpha^N(\alpha_{m\to N}(a_m)) + \ker p_{\mathcal{I}} \\ &= \underrightarrow{\alpha^{N}}(\alpha_{m\to N}(a_m)). \end{aligned} \tag{2.1.10}$$

For conclusion *2.*, fix $n \in \mathbb{N}$. It is immediate that $\underrightarrow{\alpha^{n}}$ is a *-homomorphism. Next, we check injectivity. Let $a_n, b_n \in \mathfrak{A}_n$ and assume that $\underrightarrow{\alpha^{n}}(a_n) = \underrightarrow{\alpha^{n}}(b_n)$. By definition, we have that $\alpha^n(a_n) - \alpha^n(b_n) \in \ker p_{\mathcal{I}}$. This implies that there exists $K \in \mathbb{N}, K > n$ such that $\alpha_{n\to k}(a_n) - \alpha_{n\to k}(b_n) = 0$ for all $k \geqslant K$. Hence, we have $\alpha_{n\to K}(a_n - b_n) = 0 \iff \|\alpha_{n\to K}(a_n - b_n)\|_{\mathfrak{A}_K} = 0 \iff \|a_n - b_n\|_{\mathfrak{A}_n} = 0 \iff$

$a_n = b_n$ since α_k is an isometry for all $k \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, the map $\underrightarrow{\alpha^n}$ is an isometry on a complete space $\mathfrak{A}_n$. Thus, the image $\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ is complete in the complete space $\mathfrak{A}$ and thereofore closed. In conclusion, the image $\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ is a C*-subalgebra of $\mathfrak{A}$ such that $\underrightarrow{\alpha^n}(\mathfrak{A}_n) \cong \mathfrak{A}_n$. If $\mathcal{I}$ is unital, by construction, the unit $1_{\mathfrak{A}} = 1_{\widetilde{\mathfrak{A}}_{\mathcal{I}}} + \ker p_{\mathcal{I}}$, where $1_{\widetilde{\mathfrak{A}}_{\mathcal{I}}} = (1_{\mathfrak{A}_k})_{k\in\mathbb{N}}$. Now, by definition, the image $\alpha^n(1_{\mathfrak{A}_n}) = (b_k)_{k\in\mathbb{N}}$ such that $b_k = 0$ for all $k \in \{0, \ldots, n-1\}$ and $b_k = 1_{\mathfrak{A}_k}$ for all $k \geqslant n$ since each α_k is a unital map. But, then, we have $1_{\widetilde{\mathfrak{A}}_{\mathcal{I}}} - \alpha^n(1_{\mathfrak{A}_n}) \in \ker p_{\mathcal{I}}$. Therefore, in the quotient we have that $\underrightarrow{\alpha^n}(1_{\mathfrak{A}_n}) = 1_{\mathfrak{A}}$ by the same argument in Expression (2.1.10).

For conclusion *3.*, fix $n \in \mathbb{N}$. Let $b \in \underrightarrow{\alpha^n}(\mathfrak{A}_n)$. Thus, there exists $a_n \in \mathfrak{A}_n$ such that $\underrightarrow{\alpha^n}(a_n) = b$. By part *1.*, we have that $b = \underrightarrow{\alpha^n}(a_n) = \underrightarrow{\alpha^{n+1}}(\alpha_n(a_n)) \in \underrightarrow{\alpha^{n+1}}(\mathfrak{A}_{n+1})$.

For conclusion *4.*, we first note that by definition $\cup_{n\in\mathbb{N}}\underrightarrow{\alpha^n}(\mathfrak{A}_n) \subseteq \widetilde{\mathfrak{A}}_{\mathcal{I}}/\ker p_{\mathcal{I}}$. Next, let $a + \ker p_{\mathcal{I}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}/\ker p_{\mathcal{I}}$. Now, the assumption that $a = (a_k)_{k\in\mathbb{N}} \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$ implies that there exists $K_a \in \mathbb{N}$ such that $\alpha_k(a_k) = a_{k+1}$ for all $k \geqslant K_a$. Next, consider the element $\alpha^{K_a}(a_{K_a}) \in \widetilde{\mathfrak{A}}_{\mathcal{I}}$. By construction and a similar argument to part *1.*, we have that $a + \ker p_{\mathcal{I}} = \alpha^{K_a}(a_{K_a}) + \ker p_{\mathcal{I}} = \underrightarrow{\alpha^{K_a}}(a_{K_a}) \in \cup_{n\in\mathbb{N}}\underrightarrow{\alpha^n}(\mathfrak{A}_n)$. Therefore, we have the sets $\cup_{n\in\mathbb{N}}\underrightarrow{\alpha^n}(\mathfrak{A}_n) = \widetilde{\mathfrak{A}}_{\mathcal{I}}/\ker p_{\mathcal{I}}$, and by definition of the Banach space completion $\mathfrak{A}$, we have that $\cup_{n\in\mathbb{N}}\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ is a dense *-subalgebra of $\mathfrak{A}$, which is unital when $\mathcal{I}$ is unital. □

The following result provides an easy recipe to provide *-homomorphisms and *-monomorphisms from an inductive limit to a C*-algebra.

Theorem 2.1.67. *Let $\mathfrak{A} = \varinjlim \mathcal{I}$ be the inductive limit of an inductive sequence of C*-algebras $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$. If $\mathfrak{B}$ is a unital C*-algebra and there is a *-homomorphism $\psi^n : \mathfrak{A}_n \longrightarrow \mathfrak{B}$ for each $n \in \mathbb{N}$ such that the diagram:*

$$\begin{tikzcd} \mathfrak{A}_n \arrow[r, "\alpha_n"] \arrow[dr, "\psi^n"'] & \mathfrak{A}_{n+1} \arrow[d, "\psi^{n+1}"] \\ & \mathfrak{B} \end{tikzcd}$$

commutes for all $n \in \mathbb{N}$*, then there exists a unique *-homomorphism* $\psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ *:*

$$\begin{tikzcd} \mathfrak{A}_n \arrow[r, "\underrightarrow{\alpha}^n"] \arrow[dr, "\psi^n"'] & \mathfrak{A} \arrow[d, "\psi"] \\ & \mathfrak{B} \end{tikzcd}$$

such that this diagram commutes for each $n \in \mathbb{N}$*.*

Furthermore, if the map $\psi^n : \mathfrak{A}_n \longrightarrow \mathfrak{B}$ *is a unital *-monomorphism for each* $n \in \mathbb{N}$*, then the map* $\psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ *is a unital *-monomorphism.*

Proof. We only prove the last sentence of the theorem since the rest is proven in [55, Theorem 6.1.2]. For unital, fix $n \in \mathbb{N}$, then by Proposition (2.1.66), we have that $\underrightarrow{\alpha}^n(1_{\mathfrak{A}_n}) = 1_{\mathfrak{A}}$. But, by the second commuting diagram in the statement of this theorem, we have that $\psi^n(1_{\mathfrak{A}_n}) = \psi \circ \underrightarrow{\alpha}^n(1_{\mathfrak{A}_n}) = \psi(1_{\mathfrak{A}})$. Since ψ^n is assumed to be unital, we have that ψ is unital.

Next, let $a \in \cup_{n\in\mathbb{N}} \underrightarrow{\alpha}^n(\mathfrak{A}_n)$. Thus, there exists $k \in \mathbb{N}, a_k \in \mathfrak{A}_k$ such that $a = \underrightarrow{\alpha}^k(a_k)$. Hence, by the second commuting diagram in the statement of this theorem, we have:

$$\|\psi(a)\|_{\mathfrak{B}} = \left\|\psi \circ \underrightarrow{\alpha}^k(a_k)\right\|_{\mathfrak{B}} = \left\|\psi^k(a_k)\right\|_{\mathfrak{B}} = \|a_k\|_{\mathfrak{A}_k} = \left\|\underrightarrow{\alpha}^k(a_k)\right\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}$$

since ψ^k is a *-monomorphism by assumption and $\underrightarrow{\alpha}^k$ is a *-monomorphism by Proposition (2.1.66). In particular, ψ is a linear isometry on the dense subspace $\cup_{n\in\mathbb{N}} \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ of $\mathfrak{A}$ that is contractive on $\mathfrak{A}$ by Proposition (2.1.11) as it is a *-homomorphism on $\mathfrak{A}$.

Hence, let $\varepsilon > 0$ and $a \in \mathfrak{A}$, there exists $a' \in \cup_{n\in\mathbb{N}} \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ such that $\|a - a'\|_{\mathfrak{A}} < \varepsilon/2$ by density. We gather that:

$$\begin{aligned}
|\|\psi(a)\|_{\mathfrak{B}} - \|a\|_{\mathfrak{A}}| &\leqslant \left|\|\psi(a)\|_{\mathfrak{B}} - \|\psi(a')\|_{\mathfrak{B}}\right| + \left|\|\psi(a')\|_{\mathfrak{B}} - \|a'\|_{\mathfrak{A}}\right| + \left|\|a'\|_{\mathfrak{A}} - \|a\|_{\mathfrak{A}}\right| \\
&\leqslant \|\psi(a) - \psi(a')\|_{\mathfrak{B}} + 0 + \|a' - a\|_{\mathfrak{A}} \\
&< \|\psi(a - a')\|_{\mathfrak{B}} + \varepsilon/2 \\
&\leqslant \|a - a'\|_{\mathfrak{A}} + \varepsilon/2 \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $a \in \mathfrak{A}$ was arbitrary, we have that ψ is an isometry on $\mathfrak{A}$. Therefore, the map ψ is a *-monomorphism on $\mathfrak{A}$. $\square$

By Proposition (2.1.66) and Theorem (2.1.67), we may now present a more concrete realization of inductive limits of inductive seqeunces of C*-algebras, which will allow for a smooth transition to AF algebras. We note that both settings of inductive limits introduced in the next proposition have useful applications and will both be used throughout this dissertation.

Proposition 2.1.68. *If $\mathfrak{A} = \varinjlim \mathcal{I}$ is the inductive limit for an inductive sequence of C*-algebras $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$, then there exists a non-decreasing sequence of C*-subalgebras $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$ such that $\mathfrak{A}_n \cong \mathfrak{B}_n$ for each $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}\mathfrak{B}_n$ is dense in $\mathfrak{A}$. And, if $\mathcal{I}$ were unital, then the algebras $\mathfrak{B}_n$ for all $n \in \mathbb{N}$ can be chosen to be unital with the same unit.*

Conversely, if $\mathfrak{A}$ is a C-algebra such that there exists a non-decreasing sequence of C*-subalgebras $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$ such that $\cup_{n\in\mathbb{N}}\mathfrak{B}_n$ is dense in $\mathfrak{A}$, then if we let $\iota_n : \mathfrak{B}_n \longrightarrow \mathfrak{A}$ denote the inclusion mappings for each $n \in \mathbb{N}$, then the inductive limit $\mathfrak{B} = \varinjlim \mathcal{I}$, where $\mathcal{I} = (\mathfrak{B}_n, \iota_n)_{n\in\mathbb{N}}$, is *-isomorphic to $\mathfrak{A}$, in which $\underrightarrow{\iota}^n(\mathfrak{B}_n) \cong \mathfrak{B}_n$ for each $n \in \mathbb{N}$. If $\mathfrak{A}$ were unital with $\mathfrak{B}_n$ unital for all $n \in \mathbb{N}$, then $\mathcal{I}$ is unital.*

Proof. The first paragraph of this proposition is provided by Proposition (2.1.66), in which we can take the spaces $\mathfrak{B}_n$ to be $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ for each $n \in \mathbb{N}$.

For the second paragraph of this proposition, for each $n \in \mathbb{N}$, it is clear that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{B}_n \arrow[r, "\iota_n"] \arrow[dr, "\iota_n"'] & \mathfrak{B}_{n+1} \arrow[d, "\iota_{n+1}"] \\ & \mathfrak{A} \end{tikzcd}$$

Therefore, by Theorem (2.1.67), there exists a unique unital *-monomorphism $\psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ such that the following diagram commutes for each $n \in \mathbb{N}$:

$$\begin{tikzcd} \mathfrak{B}_n \arrow[r, "\underrightarrow{\iota}^n"] \arrow[dr, "\iota_n"'] & \mathfrak{B} \arrow[d, "\psi"] \\ & \mathfrak{A} \end{tikzcd}$$

For surjectivity, note that by this commuting diagram, we have:

$$\psi(\mathfrak{B}) \supseteq \psi\left(\cup_{n\in\mathbb{N}} \underrightarrow{\iota}^n(\mathfrak{B}_n)\right) = \cup_{n\in\mathbb{N}} \mathfrak{B}_n.$$

Since $\mathfrak{B}$ is complete and ψ is a linear isometry, we have that ψ surjects onto $\mathfrak{A}$ by the density of $\cup_{n\in\mathbb{N}}\mathfrak{B}_n$ in $\mathfrak{A}$. $\square$

This characterization of inductive limits allows us to present the fact that ideals of inductive limits are determined by the inductive sequence, and thus provides a basic way to determine when two ideals are the same.

Proposition 2.1.69 ([19, Lemma III.4.1])**.** *Let $\mathfrak{A}$ be a C^*-algebra such that there exists a non-decreasing sequence of C^*-subalgebras $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$ such that $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ is dense in $\mathfrak{A}$. If $I \in \mathrm{Ideal}(\mathfrak{A})$, then:*

$$I = \overline{\cup_{n\in\mathbb{N}} (I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{I \cap (\cup_{n\in\mathbb{N}}\mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$$

In particular, if $I, J \in \mathrm{Ideal}(\mathfrak{A})$ *and* $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ *for all* $n \in \mathbb{N}$*, then* $I = J$*.*

As a corollary, we present that inductive limits of simple C*-algebras are simple.

Corollary 2.1.70. *Let* $\mathfrak{A}$ *be a C*-algebra such that there exists a non-decreasing sequence of C*-subalgebras* $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *of* $\mathfrak{A}$ *such that* $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ *is dense in* $\mathfrak{A}$*.*

If $\mathfrak{A}_n$ *is simple for all* $n \in \mathbb{N}$*, then* $\mathfrak{A}$ *is simple.*

Proof. If the C*-algebra is $\mathfrak{A} = \{0_{\mathfrak{A}}\}$, then the proof is trivial. Assume that $\{0_{\mathfrak{A}}\} \subsetneq \mathfrak{A}$ and assume without loss of generality that for all $n \in \mathbb{N}$, the C*-subalgebra $\{0_{\mathfrak{A}}\} \subsetneq \mathfrak{A}_n$. Now, assume that $I \in \mathrm{Ideal}(\mathfrak{A})$. It is routine to check that $I \cap \mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$.

In the first case, assume that for all $n \in \mathbb{N}$ we have that $I \cap \mathfrak{A}_n = \{0_{\mathfrak{A}}\}$. Therefore, by Proposition (2.1.69), we have that $I = \{0_{\mathfrak{A}}\}$.

On the other hand, assume there exists $M \in \mathbb{N}$ such that $\{0_{\mathfrak{A}}\} \subsetneq I \cap \mathfrak{A}_M$. Since $\mathfrak{A}_M$ is simple and $I \cap \mathfrak{A}_M \in \mathrm{Ideal}(\mathfrak{A}_M)$, we have that $I \cap \mathfrak{A}_M = \mathfrak{A}_M$. Now, assume that $k \geqslant M$, then $\{0_{\mathfrak{A}}\} \subsetneq \mathfrak{A}_M = I \cap \mathfrak{A}_M \subseteq I \cap \mathfrak{A}_k$ by $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ non-decreasing, which implies that $I \cap \mathfrak{A}_k = \mathfrak{A}_k$ since $\mathfrak{A}_k$ is simple and $I \cap \mathfrak{A}_k \in \mathrm{Ideal}(\mathfrak{A}_k)$. Next, assume that $k \leqslant M$, then $I \cap \mathfrak{A}_k = I \cap (\mathfrak{A}_k \cap \mathfrak{A}_M) = (I \cap \mathfrak{A}_M) \cap \mathfrak{A}_k = \mathfrak{A}_M \cap \mathfrak{A}_k = \mathfrak{A}_k$ by $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ non-decreasing. Thus, for all $n \in \mathbb{N}$, we have that $I \cap \mathfrak{A}_n = \mathfrak{A}_n$ and:

$$I = \overline{\cup_{n\in\mathbb{N}} (I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} = \mathfrak{A}$$

by Proposition (2.1.69), which completes the proof. $\square$

Remark 2.1.71. *The hypothesis of Proposition (2.1.69) is not necessary. We shall see in the proof of Theorem (4.2.1) that the Effros-Shen algebras are simple, but by* *their construction in Example (2.1.81), we see that they are an inductive limit of non-simple C*-algebras.*

With these tools available, we now introduce the notion of an approximately finite-dimensional C*-algebra or AF algebra, which form a special class of inductive limits (see Theorem (2.1.75)). The theory of AF algebras started with Uniformly Hyperfinite Algebras or UHF algebras (see Example (2.1.79)) and were first systematically studied and classified by J. Glimm [28] for their strong ties to physics via the Canonical Anticommutation Relation algebra or CAR algebra, which was shown to be UHF by O. Bratteli [11, Section 5]. Also, in [11], O. Bratteli introduced the notion of an AF algebra, which comprised a much larger class of C*-algebras that included all the UHF algebras as well as the C*-algebra of $\mathbb{C}$-valued continuous functions on the Cantor Set (see Example (2.1.76)) and the Gauge Invariant CAR algebra or GICAR algebra [11, Section 5]. Also, with the introduction of the Bratteli diagram associated to an AF algebra [11] and Definition (2.1.83), O. Bratteli paved the way for the classification of AF algebras since all AF algebras associated to a single Bratteli diagram are *-isomorphic, which is Theorem (2.1.88). However, one may associate two distinct Bratteli diagrams to single AF algebras (see Remark (2.1.89)), and thus the Bratteli diagram does not provide a complete invariant. But, motivated by Bratteli's work and using K-theory, G. Elliott was able to provide a complete invariant for AF algebras [23].

Let's consider the phrase "approximately finite-dimensional". Given the norm of a C*-algebra $\mathfrak{A}$, it makes sense that this phrase should mean: given any $\varepsilon > 0, a \in \mathfrak{A}$, there exists a finite-dimensional C*-subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ and $b \in \mathfrak{B}$ such that $\|a - b\|_{\mathfrak{A}} < \varepsilon$. This will essentially be equivalent to the definition of an AF algebra (see Theorem (2.1.74)), but we begin with the following definition.

Definition 2.1.72 ([11]). *A C*-algebra* $\mathfrak{A}$ *is an* approximately finite-dimensional (AF) algebra *if there exists a sequence of finite-dimensional C*-subalgebras of* $\mathfrak{A}$*,*

$(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *such that:*

1. *the sequence* $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *is non-decreasing. That is, for each* $n \in \mathbb{N}$*, the C*-subalgebra* $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$*, and*
2. *the C*-algebra* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$*.*

First, we present a basic fact about tracial states on unital AF algebras, which is a careful application of the [18, Hahn-Banach Theorem III.6.4] along with the fact that every finite-dimensional C*-algebra has tracial states by [19, Example IV.5.4] and the characterization of states in the unital case, which is Proposition (2.1.23).

Lemma 2.1.73 ([50, Proposition 3.4.11])**.** *Let* $\mathfrak{A}$ *be a unital C*-algebra. If* $\mathfrak{A}$ *is an AF algebra, then there exists a tracial state on* $\mathfrak{A}$*.*

Next, we present that Definition (2.1.72) truly captures the spirit of the phrase "approximately finite-dimensional." For the following theorem, note that every AF algebra is separable. Indeed, every finite-dimensional C*-algebra is separable, and by Definition (2.1.72.2), we have that AF algebras are separable.

Theorem 2.1.74 ([11, Theorem 2.2])**.** *Let* $\mathfrak{A}$ *be a separable C*-algebra.*

$\mathfrak{A}$ *is an AF algebra if and only if for every finite set* $a_1, \ldots, a_n \in \mathfrak{A}, n \in \mathbb{N}$ *and* $\varepsilon > 0$ *there exists a finite-dimensional C*-subalgebra* $\mathfrak{B} \subseteq \mathfrak{A}$ *and* $b_1, \ldots, b_n \in \mathfrak{B}$ *such that* $\|a_j - b_j\|_{\mathfrak{A}} < \varepsilon$ *for each* $j \in \{1, \ldots, n\}$*.*

Furthermore, if $\mathfrak{A}$ *is unital and the converse of the above statement holds, then the sequence of non-decreasing finite dimensional C*-subalgebras* $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *of* $\mathfrak{A}$ *for which* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *can be chosen so that* $\mathfrak{A}_n$ *is unital for all* $n \in \mathbb{N}$*.*

Now, we cast the definition of AF algebras in the inductive limit setting, thus showing that AF algebras are a subclass of inductive limits up to *-isomorphism, in which the inductive sequence is required to only contain finite-dimensional C*-algebras.

Theorem 2.1.75. *A C*-algebra $\mathfrak{A}$ is AF if and only if it is *-isomorphic to an inductive limit $\mathfrak{B} = \varinjlim \mathcal{I}$, where $\mathcal{I} = (\mathfrak{B}_n, \beta_n)_{n\in\mathbb{N}}$ and $\mathfrak{B}_n$ is finite-dimensional for all $n \in \mathbb{N}$.*

Proof. This follows immediately from Proposition (2.1.68). $\square$

Example 2.1.76 (Continuous functions on the Cantor set)**.** *Let $\mathbb{Z}_2 = \{0, 1\}$ with the discrete topology. The Cantor set is given by:*

$$\mathcal{C} = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$$

with the product topology. To continue with this example, we introduce the following notation, which will be used later in Section (3.1.1).

Notation 2.1.77. *For all $n \in \mathbb{N}$, we denote the evaluation map $(z_m)_{m\in\mathbb{N}} \in \mathcal{C} \mapsto z_n$ by η_n. Note that $\eta_n \in C(\mathcal{C})$ is a projection and $u_n = 2\eta_n - 1_{C(\mathcal{C})}$ is a self-adjoint unitary in $C(\mathcal{C})$. That is for each $n \in \mathbb{N}$, we have $\eta_n^2 = \eta_n, \eta_n = \eta_n^*$ and $u_n u_n^* = 1_{C(\mathcal{C})} = u_n^* u_n, u_n = u_n^*$, which implies that $u_n^2 = 1_{C(\mathcal{C})}$.*

We set $\mathfrak{A}_0 = \mathbb{C}1_{C(\mathcal{C})}$ and, for all $n \in \mathbb{N} \setminus \{0\}$, we set:

$$\mathfrak{A}_n = \mathrm{C}^*\left(\{1_{C(\mathcal{C})}, u_0, \ldots, u_{n-1}\}\right),$$

*where $C^*(A)$ is the *-algebra generated by the set A, which includes finite products of elements in the linear span of elements in $A \cup \{a^* : a \in A\}$, and then closed in norm.*

By definition, for each $n \in \mathbb{N}$, the C-subalgebra $\mathfrak{A}_n$ of $C(\mathcal{C})$ is finite dimensional with the same unit as $C(\mathcal{C})$ and $\dim \mathfrak{A}_n = 2^n$. Moreover, $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}$. Last, it is easy to check that $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ is a unital *-subalgebra of $C(\mathcal{C})$ which separates points; as $\mathcal{C}$ is compact, the [71, Stone-Weierstrass Theorem 44.5] implies that $C(\mathcal{C}) = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{C(\mathcal{C})}}$.*

Remark 2.1.78. *It is no coincidence that the* C^**-algebra* $C(\mathcal{C})$ *of* $\mathbb{C}$*-valued continuous functions on the Cantor space* $\mathcal{C}$ *is AF. In fact, for any totally disconnected compact metric space* X*, the* C^**-algebra* $C(X)$ *is AF and this characterizes unital commutative AF algebras [12, Proposition 3.1] along with Theorem (2.1.55). Thus, a basic example of a non-AF algebra is* $C([0,1])$*.*

Example 2.1.79 ([28],Uniformly Hyperfinite Algebras or UHF algebras)**.** *A unital* C^**-algebra* $\mathfrak{A}$ *is UHF if there exists a sequence of unital simple finite-dimensional* C^**-subalgebras of* $\mathfrak{A}$*,* $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *such that* $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ *for each* $n \in \mathbb{N}$ *and* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$*.*

The simplicity requirement is equivalent to requiring that for each $n \in \mathbb{N}$*, there exists* $k_n \in \mathbb{N}\setminus\{0\}$ *such that* $\mathfrak{A}_n \cong \mathfrak{M}(k_n)$*, the* C^**-algebra of* $k_n \times k_n$*-*$\mathbb{C}$*-valued matrices. Indeed, it is a standard ring theoretic exercise to show that* $\mathfrak{M}(d)$ *is simple for all* $d \in \mathbb{N}\setminus\{0\}$*. This fact along with Example (2.1.13.4), which is the characterization of finite-dimensional* C^**-algebras, establishes this equivalence.*

Combining this with Theorem (2.1.75), a C^**-algebra* $\mathfrak{A}$ *is UHF if and only if it is *-isomorphic to an inductive limit* $\mathfrak{B} = \varinjlim \mathcal{I}$*, where* $\mathcal{I} = (\mathfrak{M}(k_n), \beta_n)_{n\in\mathbb{N}}$ *such that* β_n *is a unital *-monomorphism and* $k_n \in \mathbb{N}\setminus\{0\}$ *for all* $n \in \mathbb{N}$*. Note that by Theorem (2.1.18) and the requirement that each* β_n *must be unital, we have that* k_n *divides* k_{n+1} *for all* $n \in \mathbb{N}$*.*

Lastly, we note that the Canonical Anticommutation Relation Algebra or CAR algebra is UHF by [11, Section 5]. In fact, for the CAR algebra $\mathfrak{CAR}$ *there exists an increasing sequence of unital* C^**-subalgebras* $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ *such that* $\mathfrak{A}_n \cong \mathfrak{M}(2^n)$ *for each* $n \in \mathbb{N}$ *and* $\mathfrak{CAR} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{CAR}}}$*.*

We list some basic facts about UHF algebras.

Lemma 2.1.80. *Let* $\mathfrak{A}$ *be a* C^**-algebra. If* $\mathfrak{A}$ *is UHF, then* $\mathfrak{A}$ *is simple and has a unique faithful tracial state.*

Proof. By Corollary (2.1.70), UHF algebras are simple. By definition of UHF in Example (2.1.79), the C*-algebra $\mathfrak{A}$ is unital. Therefore, there exists some tracial state μ on $\mathfrak{A}$ by Lemma (2.1.73) and this tracial state is faithful by simplicity of $\mathfrak{A}$ and Lemma (2.1.43).

Now, let $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of unital simple finite-dimensional C*-subalgebras of $\mathfrak{A}$ such that $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ is dense in $\mathfrak{A}$. Assume that there is some other faithful tracial state ν on $\mathfrak{A}$. By Example (2.1.37), the restriction of μ and ν to $\mathfrak{A}_n$ agree for all $n \in \mathbb{N}$ since each $\mathfrak{A}_n$ is simple finite-dimensional. Thus, the states μ and ν agree on the dense subspace $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$. By continuity, the states agree on $\mathfrak{A}$. □

The next example is motivated by the classification of the irrational rotation algebras, $\mathfrak{A}_\theta$ [19, Chapter VI] for any $\theta \in (0,1) \setminus \mathbb{Q}$, i.e. the universal C*-algebra generated by two unitaries U and V subject to $UV = \exp(2i\pi\theta)VU$. These algebras form noncommutative deformations of the torus since $\mathfrak{A}_0 \cong C\left(\mathbb{T}^2\right)$ and thus have many fascinating applications in Noncommutative Geometry. However, it was of utmost importance to classify these algebras up to their irrational parameters. In [58], Pimsner and Voiculescu succeeded in this venture and showed that for $\theta, \theta' \in (0,1) \setminus \mathbb{Q}$, the C*-algebras $\mathfrak{A}_\theta$ and $\mathfrak{A}_{\theta'}$ are *-isomorphic if and only if $\theta = \theta'$. To accomplish this, Pimsner and Voiculescu constructed, for any $\theta \in (0,1) \setminus \mathbb{Q}$, a unital *-monomorphism from the irrational rotation C*-algebra $\mathfrak{A}_\theta$ into $\mathfrak{AF}_\theta$— the Effros-Shen AF algebra [22]. This was a crucial step in their classification of irrational rotation algebras and started a long and fascinating line of investigation about AF embeddings of various C*-algebras, which is still active today [24]. In the next example, we utilize certain basic number theoretic facts about continued fractions.

Example 2.1.81 (Effros-Shen AF algebra)**.** *We begin by recalling the construction of the AF algebras $\mathfrak{AF}_\theta$ constructed in [22] for any irrational θ in $(0,1)$. For any $\theta \in (0,1)\setminus\mathbb{Q}$, let $(r_j)_{j\in\mathbb{N}}$ be the unique sequence in $\mathbb{N}$ such that the limit of continued* *fractions formed by finite initial sequences of* $(r_j)_{j\in\mathbb{N}}$ *converge to* θ*. This is displayed as:*

$$\theta = \lim_{n\to\infty} r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\ddots + \cfrac{1}{r_n}}}}. \tag{2.1.11}$$

The sequence $(r_j)_{j\in\mathbb{N}}$ *is called the continued fraction expansion of* θ*, and we will simply denote it by writing* $\theta = [r_0, r_1, r_2, \ldots] = [r_j]_{j\in\mathbb{N}}$*. We note that* $r_0 = 0$ *(since* $\theta \in (0,1)$*) and* $r_n \in \mathbb{N} \setminus \{0\}$ *for* $n \geqslant 1$*. We then obtain a sequence* $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ *with* $p_n^\theta \in \mathbb{N}$ *and* $q_n^\theta \in \mathbb{N} \setminus \{0\}$ *by setting:*

$$\begin{cases} \begin{pmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{pmatrix} = \begin{pmatrix} r_0 r_1 + 1 & r_1 \\ r_0 & 1 \end{pmatrix} \\ \begin{pmatrix} p_{n+1}^\theta & q_{n+1}^\theta \\ p_n^\theta & q_n^\theta \end{pmatrix} = \begin{pmatrix} r_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n^\theta & q_n^\theta \\ p_{n-1}^\theta & q_{n-1}^\theta \end{pmatrix} & \text{for all } n \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{2.1.12}$$

We then note that $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ *converges to* θ*. For a basic number theory referenece see [32].*

Expression (2.1.12) contains the crux for the construction of the Effros-Shen AF algebras. To continue with this example, we introduce the following notation, which will be used later in Section (4.2).

Notation 2.1.82. *Let* $\theta \in (0,1) \setminus \mathbb{Q}$ *and* $\theta = [r_j]_{j\in\mathbb{N}}$ *be the continued fraction expansion of* θ*. Let* $(p_n^\theta)_{n\in\mathbb{N}}$ *and* $(q_n^\theta)_{n\in\mathbb{N}}$ *be defined by Expression (2.1.12). We set* $\mathfrak{AF}_{\theta,0} = \mathbb{C}$ *and, for all* $n \in \mathbb{N} \setminus \{0\}$*, we set:*

$$\mathfrak{AF}_{\theta,n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta),$$

and:

$$\alpha_{\theta,n} : a \oplus b \in \mathfrak{AF}_{\theta,n} \longmapsto \begin{pmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & b \end{pmatrix} \oplus a \in \mathfrak{AF}_{\theta,n+1},$$

*where a appears r_{n+1} times on the diagonal of the right hand side matrix above, which is a unital *-monomorphism by Theorem (2.1.18). We also set α_0 to be the unique unital *-monomorphism from $\mathbb{C}$ to $\mathfrak{AF}_{\theta,1}$, which is unique by Theorem (2.1.18).*

We thus define the Effros-Shen C-algebra $\mathfrak{AF}_\theta$, after [22]:*

$$\mathfrak{AF}_\theta = \varinjlim \mathcal{I}_\theta,$$

where $\mathcal{I}_\theta = \left(\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n}\right)_{n\in\mathbb{N}}$. And, the C-algebra $\mathfrak{AF}_\theta$ is AF by Theorem (2.1.75).*

Another key example of an AF algebra is the Boca-Mundici AF algebra $\mathfrak{F}$ [10, 54], which is crucial to our work in [2] and is presented in Section (5.2.1). We do not present this example here since in Section (5.2.1), we present results which are related to the structure of $\mathfrak{F}$ itself and not only quantum metric structure.

Next, we present the notion of a Bratteli diagram associated to an AF algebra introduced by Bratteli in [11, Section 1.8]. A major result of Bratteli in [11] was that if two AF algebras have the same Bratteli diagram, then they are *-isomorphic, which we present as Theorem (2.1.88). The motivation for the Bratteli diagram comes from the characterization of finite-dimensional C*-algebras in Example (2.1.13.4) and the characterization of *-homomorphisms between finite-dimensional C*-algebras in Theorem (2.1.18). Just as Bratteli did in [11] , we present a Bratteli diagram abstractly as a graph without any knowledge of an AF algebra.

Definition 2.1.83 ([11]). *We define a directed graph with labelled vertices where multiple edges between two vertices is allowed. We denote this graph by* $\mathcal{D} = \left(V^{\mathcal{D}}, E^{\mathcal{D}}\right)$*, where* $V^{\mathcal{D}}$ *will be the vertex set and* $E^{\mathcal{D}}$ *will be the edge set, which consists of ordered pairs from* $V^{\mathcal{D}}$*, in which the ordering denotes the direction.*

For each $n \in \mathbb{N}$*, let* $v_n^{\mathcal{D}} \in \mathbb{N}$*. For each* $n \in \mathbb{N}$*, we let:*

$$V_n^{\mathcal{D}} = \left\{(n,k) \in \mathbb{N} \times \mathbb{N} : k \in \left\{0, \ldots, v_n^{\mathcal{D}}\right\}\right\},$$

and define $V^{\mathcal{D}} = \cup_{n\in\mathbb{N}} V_n^{\mathcal{D}}$ *and call the elements of* $V^{\mathcal{D}}$ *the vertices of* $\mathcal{D}$*. We label of the vertices* $(n,k) \in V^{\mathcal{D}}$ *by* $[n,k]_{\mathcal{D}} \in \mathbb{N} \setminus \{0\}$*.*

Next, the set $E^{\mathcal{D}} \subset V^{\mathcal{D}} \times V^{\mathcal{D}}$ *defines edges of* $\mathcal{D}$ *if it statisfies:*

(i) For all $n \in \mathbb{N}$*, if* $m \in \mathbb{N} \setminus \{n+1\}$*, then* $((n,k),(m,q)) \notin E^{\mathcal{D}}$ *for all* $k \in \left\{0, \ldots, v_n^{\mathcal{D}}\right\}, q \in \left\{0, \ldots, v_m^{\mathcal{D}}\right\}$*.*

(ii) If $(n,k) \in V^{\mathcal{D}}$*, then there exists* $q \in \left\{0, \ldots, v_{n+1}^{\mathcal{D}}\right\}$ *such that* $((n,k),(n+1,q)) \in E^{\mathcal{D}}$*.*

(iii) If $n \in \mathbb{N} \setminus \{0\}$ *and* $(n,k) \in V^{\mathcal{D}}$ *, then there exists* $q \in \left\{0, \ldots, v_{n-1}^{\mathcal{D}}\right\}$ *such that* $((n-1,q),(n,k)) \in E^{\mathcal{D}}$*.*

If $\mathcal{D}$ *satisfies the all of the above properties, then we call* $\mathcal{D}$ *a* Bratteli diagram*, and we denote the set of all Bratteli diagrams by* $\mathscr{BD}$*.*

We also introduce the following notation. For each $n \in \mathbb{N}$*, let:*

$$E_n^{\mathcal{D}} = (V_n^{\mathcal{D}} \times V_{n+1}^{\mathcal{D}}) \cap E^{\mathcal{D}},$$

which by axiom (i), we have that $E^{\mathcal{D}} = \cup_{n\in\mathbb{N}} E_n^{\mathcal{D}}$*. Also, for* $((n,k),(n+1,q)) \in E_n^{\mathcal{D}}$*, we denote* $[(n,k),(n+1,q)]_{\mathcal{D}} \in \mathbb{N}\setminus\{0\}$ *as the number of edges from* (n,k) *to* $(n+1,q)$*. Let* $(n,k) \in V^{\mathcal{D}}$*, define:*

$$R^{\mathcal{D}}_{(n,k)} = \left\{(n+1,q) \in V^{\mathcal{D}}_{n+1} : ((n,k),(n+1,q)) \in E^{\mathcal{D}}\right\},$$

which is non-empty by axiom (ii). For $n \in \mathbb{N}$, we refer to $V^{\mathcal{D}}_n, E^{\mathcal{D}}_n$, and $\left(V^{\mathcal{D}}_n, E^{\mathcal{D}}_n\right)$ as the vertices at level n, edges at level n, and diagram at level n, respectively.

Remark 2.1.84. *It is easy to see that this definition coincides with Bratteli's of [11, Section 1.8] in that we simply trade his arrow notation with that of edges and number of edges. That is, given a Bratteli diagram $\mathcal{D}$, the correspondence is given by: $(n,k) \searrow^p (n+1,q)$ if and only if $((n,k),(n+1,q)) \in E^{\mathcal{D}}$ and $[(n,k),(n+1,q)]_{\mathcal{D}} = p$.*

One of the first of many useful properties of Bratteli diagram is that given a Bratteli diagram there exists a unique AF algebra up to *-isomorphism associated to the diagram [11, Section 1.8], [19, Proposition III.2.7]. How we associate a Bratteli diagram to an AF algebra is described in the following Definition (2.1.85) following [11, Section 1.8].

Definition 2.1.85 ([11])**.** *Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence of finite dimensional C*-algebras with inductive limit $\mathfrak{A}$ of Definition (2.1.64). Thus, $\mathfrak{A}$ is an AF algebra by Theorem (2.1.75). Let $\mathcal{D}_b(\mathfrak{A})$ be a diagram associated to $\mathfrak{A}$ constructed as follows.*

Fix $n \in \mathbb{N}$. Since $\mathfrak{A}_n$ is finite dimensional, Example (2.1.13.4) implies that $\mathfrak{A}_n \cong \oplus_{k=0}^{a_n} \mathfrak{M}(n(k))$ such that $a_n \in \mathbb{N}$ and $n(k) \in \mathbb{N} \setminus \{0\}$ for $k \in \{0, \ldots, a_n\}$. Define:

$$v^{\mathcal{D}_b(\mathfrak{A})}_n = a_n,\ V^{\mathcal{D}_b(\mathfrak{A})}_n = \left\{(n,k) \in \mathbb{N}^2 : k \in \left\{0, \ldots, v^{\mathcal{D}_b(\mathfrak{A})}_n\right\}\right\},$$

and label $[n,k]_{\mathcal{D}_b(\mathfrak{A})} = \sqrt{\dim(\mathfrak{M}(n(k)))}$ for $k \in \left\{0, \ldots, v^{\mathcal{D}_b(\mathfrak{A})}_n\right\}$.

*Let A_n be the $a_{n+1}+1 \times a_n + 1$-partial multiplicity matrix associated to the *-monomorphism $\alpha_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ from Theorem (2.1.18) with entries $(A_n)_{i,j} \in$*

$\mathbb{N}, i \in \{1, \ldots, a_{n+1}+1\}, j \in \{1, \ldots, a_n+1\}$ *given by Definition (2.1.16). Define:*

$$E_n^{\mathcal{D}_b(\mathfrak{A})} = \left\{((n,k),(n+1,q)) \in \mathbb{N}^2 \times \mathbb{N}^2 : (A_n)_{q+1,k+1} \neq 0\right\},$$

and if $((n,k),(n+1,q)) \in E_n^{\mathcal{D}_b(\mathfrak{A})}$*, then let the number of edges be* $[(n,k),(n+1,q)]_{\mathcal{D}_b(\mathfrak{A})} = (A_n)_{q+1,k+1}$.

Let $V^{\mathcal{D}_b(\mathfrak{A})} = \cup_{n\in\mathbb{N}} V_n^{\mathcal{D}_b(\mathfrak{A})}$, $E^{\mathcal{D}_b(\mathfrak{A})} = \cup_{n\in\mathbb{N}} E_n^{\mathcal{D}_b(\mathfrak{A})}$*, and* $\mathcal{D}_b(\mathfrak{A}) = (V^{\mathcal{D}_b(\mathfrak{A})}, E^{\mathcal{D}_b(\mathfrak{A})})$. *By [11, Section 1.8] and Theorem (2.1.18), we conclude* $\mathcal{D}_b(\mathfrak{A}) \in \mathscr{BD}$ *is a Bratteli diagram as in Definition (2.1.83), which completes the construction.*

If $\mathfrak{A}$ *is an AF algebra of the form* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *where* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence of finite dimensional C*-subalgebras of* $\mathfrak{A}$ *of Definition (2.1.72), then the vertices of the diagram* $\mathcal{D}_b(\mathfrak{A})$ *are constructed just as the inductive limit case, and the edges are formed by the partial multiplicity matrix built from the partial multiplicities of the inclusion mappings* $\iota_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ *for all* $n \in \mathbb{N}$ *with respect to the decomposition of* $\mathfrak{A}_n$ *into factors given by* $\mathfrak{A}_n \cong \oplus_{k=0}^{a_n}\mathfrak{M}(n(k))$ *for each* $n \in \mathbb{N}$.

Remark 2.1.86. *We note that the converse of the Definition (2.1.85) is true in the sense that given a Bratteli diagram, one may construct an AF algebra associated to it. The process is described in [11, Section 1.8], and in particular, the vertices and their labels provide the finite-dimensional C*-algebras and one may construct partial multiplicity matrices from the edge set, which then provide *-monomorphisms by Theorem (2.1.18) and Remark (2.1.19) to build an inductive limit.*

As an example, which will be used in Section (5.2.1), we display the Bratteli diagram for the Effros-Shen AF algebras of Notation (2.1.82).

Example 2.1.87. *Fix* $\theta \in (0,1) \setminus \mathbb{Q}$ *with continued fraction expansion* $\theta = [a_j]_{j\in\mathbb{N}}$ *using Expression (2.1.11) with rational approximations* $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ *given by Expression (2.1.12). Let* $\mathfrak{AF}_\theta$ *be the Effros-Shen AF algebra from Notation (2.1.82). Thus,* $v_0^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 0$ *and* $V_0^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = \{(0,0)\}$ *with* $[0,0]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 1$. *For* $n \in \mathbb{N}\setminus\{0\}$*, we have*

$v_n^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 1$ *and* $V_n^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = \{(n,0),(n,1)\}$ *with* $[n,0]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = q_n^\theta, [n,1]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = q_{n-1}^\theta$*. Utilizing Definition (2.1.16), the partial multiplicity matrix for* $n = 0$ *is:*

$$A_0 = \begin{pmatrix} a_1 \\ 1 \end{pmatrix},$$

and let $n \in \mathbb{N} \setminus \{0\}$*, then the partial multiplicity matrix is:*

$$A_n = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix},$$

by Notation (2.1.82). Thus, we now have the edges to complete the construction. We now provide the diagram as a graph, where the label in the edges denotes number of edges and the top row contains the vertices $(n,1)$ *with their labels with* n *increasing from left to right with the bottom row having vertices* $(n,0)$ *with their labels with* n *increasing from left to right. Assume* $n \geqslant 4$:

$$\begin{tikzcd}
& q_0^\theta \arrow[drrr, "1"] & & q_1^\theta \arrow[drrr, "1"] & & q_2^\theta \cdots & & {} \arrow[drrr, "1"] & & q_{n-1}^\theta \arrow[drrr, "1"] & & q_n^\theta \cdots \\
1 \arrow[ur, "1"] \arrow[rr, "a_1"] & & q_1^\theta \arrow[ur, "1"] \arrow[rr, "a_2"] & & q_2^\theta \arrow[ur, "1"] \arrow[rr, "a_3"] & & q_3^\theta \cdots & & {} \arrow[ur, "1"] \arrow[rr, "a_n"] & & q_n^\theta \arrow[ur, "1"] \arrow[rr, "a_{n+1}"] & & q_{n+1}^\theta \cdots
\end{tikzcd}$$

Finally, to conclude this section, we present a main result of Bratteli in [11] that states: two AF algebras with the same Bratteli diagram are *-isomorphic. This was a major step towards the classification of AF algebras in [23]. For the following, we provide a reference from [19], which is more in-line with our notation and stated explcitly, but we note that the original proof can be found in [11, Section 1.8].

Theorem 2.1.88 ([19, Proposition III.2.7])**.** *From Definition (2.1.72), let* $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}, \mathfrak{B} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$ *be two AF algebras, where* $(\mathfrak{A}_n)_{n \in \mathbb{N}}, (\mathfrak{B}_n)_{n \in \mathbb{N}}$ *are non-decreasing sequences of finite-dimensional C*-algebras of* $\mathfrak{A}, \mathfrak{B}$*, respectively.*

*Using Definition (2.1.85), if $\mathcal{D}_b(\mathfrak{A})$ and $\mathcal{D}_b(\mathfrak{B})$ are the associated Bratteli diagrams and $\mathcal{D}_b(\mathfrak{A}) = \mathcal{D}_b(\mathfrak{B})$, then $\mathfrak{A}$ is *-isomorphic to $\mathfrak{B}$. Moreover, for any *-isomorphism $\pi_0 : \mathfrak{A}_0 \longrightarrow \mathfrak{B}_0$, there exists a *-isomorphism $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that π restricted to $\mathfrak{A}_0$ is π_0.*

Remark 2.1.89. *Unfortunately, a single AF algebra can have multiple Bratteli diagrams associated to it. Indeed, if $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ is an infinite-dimensional AF algebra, then if we consider any non-trivial subsequence of $(\mathfrak{A}_n)_{n\in\mathbb{N}}$, then the closure of the union of the subsequence will still be $\mathfrak{A}$, but its associated Bratteli diagram of Defintion (2.1.85) will be different than the Bratteli diagram of the initial sequence by the simple fact that the vertices will not agree. Yet, the possible differences between two Bratteli diagrams associated to a single AF algebra can be characterized by an equivalence relation, which is discussed in [8, Section 23.3, pages 178-180], and one may classify AF algebras up to their Bratteli diagrams up to this equivalence relation.*

Ideals of AF algebras are completely characterized as certain subdiagrams of any Bratteli diagram associated to a given AF algebra. However, for ease of exposition, we reserve our discussion of ideals of AF algebras until Section (5.1) since many results there are immediate from the definitions.

2.2 Quantum compact metric spaces

One main motivation for the study of quantum compact metric spaces — introduced by M. A. Rieffel in [59] — is to explain some finite-dimensional approximations of quantum spaces from Mathematical Physics [62]. A major advancement in this endeavor has been the introduction of noncommutative analogues to the Gromov-Hausdorff distance (see Section (2.3)), which was instigated by M. A. Rieffel in [61]. Later, F. Latrémolière provided his novel quantum Gromov-Hausdorff propinquity in [46] to strengthen Rieffel's distance by providing finite-dimensional approximations in the form of C*-algebras and not just self-adjoint subspaces. We build our quantum spaces on C*-algebras since the study of C*-algebras already provides a noncommutative study of topology via Gelfand duality and commutative C*-algebras (see Theorem (2.1.30) and Theorem (2.1.34)). Thus, in order to introduce the notion of a quantum compact metric space we first look to unital commutative C*-algebras and how they may capture metric geometry.

Therefore, we restrict our attention from compact Hausdorff spaces to compact metric spaces. Fix a compact metric space (X, d_X) with metric d_X. We look to find a structure associated to the unital commutative C*-algebra $C(X)$ that captures the metric space X much like how the maximal ideal space assocaited to $C(X)$ captures the topology of X via Proposition (2.1.32). Towards this, consider the Lipschitz seminorm on $C(X)$ associated to d_X defined for all $f \in C(X)$ by:

$$\mathsf{L}_{\mathsf{d}_X}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}_X(x,y)} : x, y \in X, x \neq y \right\}, \tag{2.2.1}$$

which may take value $+\infty$. With this seminorm, we may define a metric on the state space of $C(X)$ called the Monge-Kantorovich metric, defined, for all two states $\varphi, \psi \in \mathscr{S}(C(X))$, by:

$$\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}(\varphi, \psi) = \sup\left\{ |\varphi(a) - \psi(a)| : a \in \mathfrak{sa}\left(C(X)\right), \mathsf{L}_{\mathsf{d}_X}(a) \leqslant 1 \right\}.$$

The next proposition displays how this structure on the state space captures the metric space (X, d_X) isometrically in the state space, and therefore considerably strengthening the result of Proposition (2.1.32). Thus, this provides an appropriate model for how to define a quantum metric space.

Proposition 2.2.1. *If (X, d_X) is a compact metric space, then:*

1. *the set $\{f \in C(X) : \mathsf{L}_{\mathsf{d}_X}(f) < \infty\}$ is a dense unital *-subalgebra of $C(X)$ and*

the kernel $\mathsf{L}_{\mathsf{d}_X}^{-1}(\{0\}) = \mathbb{C}1_{C(X)}$,

2. *the Monge-Kantorovich metric* $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}$ *metrizes the weak* topology of* $\mathscr{S}(C(X))$ *and the map:*

$$\Delta_X : x \in (X, \mathsf{d}_X) \longmapsto \delta_x \in \left(\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}\right)$$

is an isometry onto its image, which is the maximal ideal space of $C(X)$ *denoted* $M_{C(X)}$ *of Definition (2.1.26), where* $\delta_x(f) = f(x)$ *is the Dirac point mass of* x,

3. *the seminorm* $\mathsf{L}_{\mathsf{d}_X}$ *is lower semi-continuous with respect to* $\|\cdot\|_{C(X)}$, *and*

4. *for all* $f, g \in C(X)$, *we have:*

$$\mathsf{L}_{\mathsf{d}_X}(fg) \leqslant \mathsf{L}_{\mathsf{d}_X}(f)\|g\|_{C(X)} + \|f\|_{C(X)}\mathsf{L}_{\mathsf{d}_X}(g).$$

Proof. A proof of this will be provided in the proof of Theorem (2.2.10). □

Therefore, we propose to define a quantum compact metric space using a metric on the state space of C*-algebras. To this end, we begin with a few well-known technical observations. We do note that there is an established notion of a quantum metric space in the non-unital case [38, 39] developed by F. Latrémolière. However, this is outside the scope of this dissertation.

Convention 2.2.2. *Let* $\mathfrak{A}$ *be a C*-algebra. If we assume that* $\mathfrak{B}$ *is a subspace of* $\mathfrak{A}$, *then we assume that* $\mathfrak{B}$ *is a subspace over* $\mathbb{C}$. *If we assume that* $\mathfrak{B}$ *is a subspace of the self-adjoints* $\mathfrak{sa}(\mathfrak{A})$, *then we assume that* $\mathfrak{B}$ *is a subspace over* $\mathbb{R}$.

Lemma 2.2.3. *If* $\mathfrak{A}$ *is unital C*-algebra and* $\mathfrak{B}$ *is some dense subspace of* $\mathfrak{sa}(\mathfrak{A})$, *then* $\mathfrak{B}$ *separates the points of* $\mathscr{S}(\mathfrak{A})$. *That is, if for* $\mu, \nu \in \mathscr{S}(\mathfrak{A})$ *we have that* $\mu(a) = \nu(a)$ *for all* $a \in \mathfrak{B}$, *then* $\mu = \nu$.

Proof. Let $\mu, \nu \in \mathscr{S}(\mathfrak{A})$ such that $\mu(a) = \nu(a)$ for all $a \in \mathfrak{B}$. By density of $\mathfrak{B} \subseteq \mathfrak{sa}(\mathfrak{A})$ and continuity of μ, ν, we have that μ and ν agree on $\mathfrak{sa}(\mathfrak{A})$.

Next, assume that $b \in \mathfrak{A}$. Then, $b = \frac{b+b^*}{2} + i\frac{b-b^*}{2i}$, where $\frac{b+b^*}{2}, \frac{b-b^*}{2i} \in \mathfrak{sa}(\mathfrak{A})$. Hence, by linearity:

$$\begin{aligned}\mu(b) &= \mu\left(\frac{b+b^*}{2} + i\frac{b-b^*}{2i}\right)\\ &= \mu\left(\frac{b+b^*}{2}\right) + i\mu\left(\frac{b-b^*}{2i}\right)\\ &= \nu\left(\frac{b+b^*}{2}\right) + i\nu\left(\frac{b-b^*}{2i}\right) = \nu(b),\end{aligned}$$

which completes the proof. □

Proposition 2.2.4. *If* $(\mathfrak{A}, \mathsf{L})$ *is an ordered pair where* $\mathfrak{A}$ *is unital* C^**-algebra and* L *is a seminorm defined on* $\mathfrak{sa}(\mathfrak{A})$ *such that its domain* $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ *is a dense subspace of* $\mathfrak{sa}(\mathfrak{A})$*, then the map:*

$$(\varphi, \psi) \in \mathscr{S}(\mathfrak{A}) \times \mathscr{S}(\mathfrak{A}) \longmapsto \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) \in [0, \infty]$$

defined, for all two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$*, by:*

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$

is an extended metric on $\mathscr{S}(\mathfrak{A})$*, where extended means that the metric may take value* $+\infty$.

Proof. Symmetry and triangle inequality are routine to check. What remains is the axiom of coincidence.

Fix $\mu, \nu \in \mathscr{S}(\mathfrak{A})$. Assume that $\mathsf{mk}_{\mathsf{L}}(\mu, \nu) = 0$, then $\mu(a) = \nu(a)$ for all $a \in \mathrm{dom}(\mathsf{L})$ such that $\mathsf{L}(a) \leqslant 1$. Let $b \in \mathrm{dom}(\mathsf{L})$, then $\mathsf{L}\left(\frac{b}{\max\{1, \mathsf{L}(b)\}}\right) \leqslant 1$ and:

$$\mu(b) = \max\{1, \mathsf{L}(b)\}\mu\left(\frac{b}{\max\{1, \mathsf{L}(b)\}}\right) = \max\{1, \mathsf{L}(b)\}\nu\left(\frac{b}{\max\{1, \mathsf{L}(b)\}}\right) = \nu(b).$$

Therefore, by Lemma (2.2.3), we are done. $\square$

Thus, we are in a position to make the following definition and the main definition of this section introduced by Rieffel in [59] and cast in the setting of C*-algberas by Latrémolière in [46, 45].

Definition 2.2.5 ([59, 46, 45])**.** *A* quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ *is an ordered pair where* $\mathfrak{A}$ *is unital C*-algebra and* L *is a seminorm defined on* $\mathfrak{sa}(\mathfrak{A})$ *such that its domain* $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ *is a dense unital subspace of* $\mathfrak{sa}(\mathfrak{A})$ *such that:*

1. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$*,*

2. *the* Monge-Kantorovich metric *defined, for all two states* $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$*, by:*

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$

is a metric on $\mathscr{S}(\mathfrak{A})$ *that metrizes the weak* topology of* $\mathscr{S}(\mathfrak{A})$*,*

3. *the seminorm* L *is lower semi-continuous on* $\mathfrak{sa}(\mathfrak{A})$ *with respect to* $\|\cdot\|_{\mathfrak{A}}$*.*

The seminorm L *of a quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$ *is called a* Lip-norm.

In Rieffel's pioneering work on quantum compact metric spaces [59], certain equilavent conditions were given for the requirement that the Monge-Kantorovich metric metrizes the weak* topology of the state space. These conditions provide a useful tool for verifying this difficult property. Further equivalences were given in [56]. The following theorem summarizes all known characterizations of Lip-norms. We include some proofs which at times vary from the original ones.

Theorem 2.2.6 ([59, 60, 56]). *Let $(\mathfrak{A}, \mathsf{L})$ be an ordered pair where $\mathfrak{A}$ is unital C^*-algebra and L is a lower semi-continuous seminorm defined on $\mathfrak{sa}(\mathfrak{A})$ such that its domain $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ is a dense unital subspace of $\mathfrak{sa}(\mathfrak{A})$ and $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$. The following are equivalent:*

1. *$(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space;*

2. *the metric mk_{L} is bounded and there exists $r \in \mathbb{R}, r > 0$ such that the set:*

$$\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \|a\|_{\mathfrak{A}} \leqslant r\}$$

   is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$;

3. *the set:*

$$\{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$

   is totally bounded in $\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}$ for $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$;

4. *there exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that the set:*

$$\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$$

   is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$;

5. *for all $\mu \in \mathscr{S}(\mathfrak{A})$ the set:*

$$\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$$

   is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$;

Proof. First, we note that by Lemma (2.2.3), we have that $\mathrm{dom}(\mathsf{L})$ separates the points of $\mathscr{S}(\mathfrak{A})$. Second, since $\mathrm{dom}(\mathsf{L}) \subseteq \mathfrak{sa}(\mathfrak{A})$, for all $a \in \mathrm{dom}(\mathsf{L})$, we have that:

$$\|a\|_{\mathfrak{A}} = \|\hat{a}\|_{C(\mathscr{S}(\mathfrak{A}))} = \sup\{|\hat{a}(\varphi)| : \varphi \in \mathscr{S}(\mathfrak{A})\}, \tag{2.2.2}$$

where $\hat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \mathscr{S}(\mathfrak{A})$, and the equality is given by Proposition (2.1.25). Hence, [59, Condition 1.5] is satisfied, and so, the equivalences of *1.*, *2.*, and *3.* are the combination of [59, Theorem 1.8] and [59, Theorem 1.9].

The equivalence between *1.* and *4.* is given in [56, Proposition 1.3], but the direction *1.* $\implies$ *4.* is only given by a hint at the end of the proof. We will avoid the hint suggested in the proof and prove this direction via different approach that utilizes the Monge-Kantorovich metric itself. We begin with the following claim.

Claim 2.2.7. *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space and* $\mu \in \mathscr{S}(\mathfrak{A})$*, then the set* $\{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$ *is bounded in* $\mathfrak{A}$ *for* $\|\cdot\|_{\mathfrak{A}}$*.*

Proof of claim. Assume that $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space and $\mu \in \mathscr{S}(\mathfrak{A})$. Assume by way of contradiction that $E = \{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$ is not bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$. Thus, for each $n \in \mathbb{N}$, there exists $a_n \in E$ such that $\|a_n\|_{\mathfrak{A}} \geqslant n$. Since $\mathrm{dom}\,(\mathsf{L}) \subseteq \mathfrak{sa}\,(\mathfrak{A})$, for each $n \in \mathbb{N}$, there exists a $\nu_n \in \mathscr{S}(\mathfrak{A})$ such that $|\nu_n(a_n)| = \|a_n\|_{\mathfrak{A}}$ by [19, Lemma I.9.10]. Therefore, for each $n \in \mathbb{N}$:

$$\begin{aligned}
\mathsf{mk}_{\mathsf{L}}(\mu, \nu_n) &= \sup\{|\mu(a) - \nu_n(a)| : a \in \mathrm{dom}\,(\mathsf{L}), \mathsf{L}(a) \leqslant 1\} \\
&\geqslant |\mu(a_n) - \nu_n(a_n)| \\
&= |0 - \nu_n(a_n)| \\
&= \|a_n\|_{\mathfrak{A}} \geqslant n.
\end{aligned}$$

In particular, the metric mk_{L} is unbounded, which is a contradiction to the fact that it metrizes a compact topology (see Proposition (2.1.24)). $\square$

Now, assuming *1.*, we will prove *4.* By the claim, there exists an $r \in \mathbb{R}$ with $r > 0$ such that the set:

$$\{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\} \subseteq \{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \|a\|_{\mathfrak{A}} \leqslant r\}.$$

Set $E = \{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$ and set $F = \{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \|a\|_{\mathfrak{A}} \leqslant r\}$.

We show that F is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$. Consider the map $\hat{}: a \in \mathfrak{sa}\,(\mathfrak{A}) \longmapsto \hat{a} \in C(\mathscr{S}(\mathfrak{A}))$. By Equation (2.2.2), this map is a linear isometry. Therefore, the set $\widehat{F}$ is bounded in $C(\mathscr{S}(\mathfrak{A}))$ for $\|\cdot\|_{C(\mathscr{S}(\mathfrak{A}))}$. Also, for $a \in \mathrm{dom}\,(\mathsf{L}), \mathsf{L}(a) \leqslant 1$ and $\mu, \nu \in \mathscr{S}(\mathfrak{A})$, we have that:

$$|\hat{a}(\mu) - \hat{a}(\nu)| = |\mu(a) - \nu(a)| \leqslant \mathsf{mk}_{\mathsf{L}}(\mu, \nu).$$

Therefore, $\widehat{F}$ is equicontinuous in $C(\mathscr{S}(\mathfrak{A}))$. Thus, by [18, Arzela-Ascoli Theorem VI.3.8], the set $\widehat{F}$ is totally bounded in $C(\mathscr{S}(\mathfrak{A}))$ for $\|\cdot\|_{C(\mathscr{S}(\mathfrak{A}))}$. By Equation (2.2.2), this implies that F is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$ and so the same is true for E by containment.

For *4.* $\implies$ *1.*, we will use the already established equivalence between *1.* and *3.* Assume that the set $E = \{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$ is totally bounded in $\mathfrak{A}$ for $\|\cdot\|_{\mathfrak{A}}$. Let $q : \mathfrak{sa}\,(\mathfrak{A}) \longrightarrow \mathfrak{sa}\,(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}$ denote the quotient map, which is uniformly continuous with respect to the norms $\|\cdot\|_{\mathfrak{A}}$ and $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$ since it is bounded and linear. Thus, the image $q(E) = \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}\,(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\}$ is totally bounded in $\mathfrak{sa}\,(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}$ for $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$.

Clearly, the set $q(E) \subseteq \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}\,(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : a \in \mathrm{dom}\,(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$. Let $a + \mathbb{R}1_{\mathfrak{A}} \in \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}\,(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : a \in \mathrm{dom}\,(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$. Next, we have $\mu(a - \mu(a)1_{\mathfrak{A}}) = \mu(a) - \mu(\mu(a)1_{\mathfrak{A}}) = \mu(a) - \mu(a)\mu(1_{\mathfrak{A}}) = \mu(a) - \mu(a) \cdot 1 = 0$. Also, the seminorm $\mathsf{L}(a - \mu(a)1_{\mathfrak{A}}) = \mathsf{L}(a) \leqslant 1$ since L vanishes on $\mathbb{R}1_{\mathfrak{A}}$ and $\mu(a) \in \mathbb{R}$ by Lemma (2.1.21) since $a \in \mathfrak{sa}\,(\mathfrak{A})$ and μ is a state. Hence, the element $a - \mu(a)1_{\mathfrak{A}} \in E$, and therefore $a+\mathbb{R}1_{\mathfrak{A}} = a-\mu(a)1_{\mathfrak{A}}+\mathbb{R}1_{\mathfrak{A}} \in q(E)$. Therefore, the set $q(E) = \{a+\mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$, which completes the proof since $q(E)$ is totally bounded in $\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}$ for $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$.

The equivalence between *1.* and *5.* follows similarly as *1.* and *4.* since the arguments used relied on an arbitrary state. □

With these equivalences at hand, we note that the structure provided by a Lip-norm in Definition (2.2.5) is enough to provide separability of a C*-algebra. This is the following result. One can consider this as a noncommutative analogue to the result that every compact metric space is separable.

Proposition 2.2.8 ([43, Proposition 2.11])**.** *Let $\mathfrak{A}$ be a unital C*-algebra. If there exists seminorm L defined on $\mathfrak{sa}(\mathfrak{A})$ such that its domain $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ is a dense unital subspace of $\mathfrak{sa}(\mathfrak{A})$ and $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, then $\mathfrak{A}$ is separable.*

Now, we introduce the notion of a quasi-Leibniz quantum metric space, which generalizes the relation between the multiplication and the Lipschitz seminorm on $C(X)$ (see Proposition (2.2.1.4)). The purpose to introduce this Leibniz property is far from aesthetic and crucial to proving that the quantum Gromov-Hausdorff propinquity (see Section (2.3)), is a metric up to the appropriate notion of isomorphism (see Theorem-Definition (2.3.16.5)).

Definition 2.2.9 ([46, 45])**.** *A (C, D)-*quasi-Leibniz quantum compact metric space *$(\mathfrak{A}, \mathsf{L})$, for some $C \geqslant 1$ and $D \geqslant 0$, is a quantum compact metric space such that:*

1. *the domain $\mathrm{dom}(\mathsf{L})$ of L is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{A})$, where the Jordan product is $a \circ b = \frac{ab+ba}{2}$ and the Lie product is $\{a,b\} = \frac{ab-ba}{2i}$ for all $a, b \in \mathfrak{sa}(\mathfrak{A})$, and*

2. the seminorm L *is a* (C,D)-quasi-Leibniz seminorm*, i.e. for all* $a,b \in \mathrm{dom}(\mathsf{L})$*:*

$$\max\left\{\mathsf{L}(a\circ b), \mathsf{L}(\{a,b\})\right\} \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b).$$

We call a $(1,0)$*-quasi-Leibniz quantum compact metric space a* Leibniz quantum compact metric space*. If we do not need to specify values for* $C \geqslant 1, D \geqslant 0$*, then we call these spaces* quasi-Leibniz quantum compact metric space*.*

Of course, Proposition (2.2.8) is still true if quantum compact metric spaces are replaced with quasi-Leibniz quantum compact metric spaces.

Our first example of a quasi-Leibniz quantum compact metric space will be the commutative case presented at the start of this section. Also, we note that when X is a metric space, the Monge-Kantorovich metric considerably strengthens the result of Proposition (2.1.32) by providing a surjective isometry instead of only a homeomorphism.

Theorem 2.2.10. *If* (X, d_X) *is a compact metric space, then* $(C(X), \mathsf{L}_{\mathsf{d}_X})$ *is a Leibniz quantum compact metric space, where* $\mathsf{L}_{\mathsf{d}_X}$ *is the Lipschitz seminorm associated to* d_X *defined in Equation (2.2.1) restricted to* $\mathfrak{sa}(C(X))$ *, such that the map:*

$$\Delta_X : x \in (X, \mathsf{d}_X) \longmapsto \delta_x \in \left(\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}\right)$$

is an isometry onto its image, which is the maximal ideal space of $C(X)$ *denoted* $M_{C(X)}$ *of Definition (2.1.32), where* $\delta_x(f) = f(x)$ *is the Dirac point mass of* x*.*

Proof. First, we check lower semi-continuity of $\mathsf{L}_{\mathsf{d}_X}$. Fix $x, y \in X$. It is routine to verify that the map $\mathsf{L}_{x,y} : f \in C(X) \longmapsto \frac{|f(x)-f(y)|}{\mathsf{d}_X(x,y)} \in \mathbb{R}$ is continuous. But, we have that $\mathsf{L}_{\mathsf{d}_X}(f) = \sup\{\mathsf{L}_{x,y}(f) : x, y \in X\}$. Hence, since a supremum of real-valued lower semi-continuous functions is lower semi-continuous, we have that $\mathsf{L}_{\mathsf{d}_X}$ is lower semi-continuous. Next, we show that $\mathsf{L}_{\mathsf{d}_X}$ is Leibniz. Let $f, g \in C(X)$. Fix $x, y \in X$, we have:

$$\begin{aligned} |fg(x) - fg(y)| &= |fg(x) - f(x)g(y) + f(x)g(y) - fg(y)| \\ &\leqslant |f(x)(g(x) - g(y))| + |(f(x) - f(y))g(y)| \\ &\leqslant \|f\|_{C(X)}|g(x) - g(y)| + |f(x) - f(y)|\|g\|_{C(X)}, \end{aligned}$$

and it follows that $\mathsf{L}_{\mathsf{d}_X}$ is Leibniz.

It is routine to show that $\{a \in \mathfrak{sa}(C(X)) : \mathsf{L}_{\mathsf{d}_X}(a) = 0\} = \mathbb{R}1_{C(X)}$. Next, we prove density of $\mathrm{dom}(\mathsf{L}_{\mathsf{d}_X})$ in $\mathfrak{sa}(C(X))$. Since $\mathsf{L}_{\mathsf{d}_X}$ is a Leibniz seminorm, we have that $\mathrm{dom}(\mathsf{L}_{\mathsf{d}_X})$ is a unital subalgebra of $\mathfrak{sa}(C(X))$. Now, fix $a, b \in X, a \neq b$ and consider the function on X defined by $a_{\mathsf{d}}(x) = \mathsf{d}_X(a, x)$ for all $x \in X$. Clearly, the function $a_{\mathsf{d}} \in \mathfrak{sa}(C(X))$. Also, we have for $x, y \in X$ that $|a_{\mathsf{d}}(x) - a_{\mathsf{d}}(y)| = |\mathsf{d}_X(a, x) - \mathsf{d}_X(a, y)| \leqslant \mathsf{d}_X(x, y)$. Hence, the function:

$$a_{\mathsf{d}} \in \mathrm{dom}(\mathsf{L}_{\mathsf{d}_X}) \text{ and } \mathsf{L}_{\mathsf{d}_X}(a_{\mathsf{d}}) \leqslant 1. \tag{2.2.3}$$

Finally, $a_{\mathsf{d}}(b) > 0 = a_{\mathsf{d}}(a)$, which implies that $\mathrm{dom}(\mathsf{L}_{\mathsf{d}_X})$ separates the points of X. Therefore, by [71, Stone-Weierstrass Theorem 44.5], we conclude that $\mathrm{dom}(\mathsf{L}_{\mathsf{d}_X})$ is dense in $\mathfrak{sa}(C(X))$. We note that since $\mathrm{dom}(\mathsf{L}_{\mathsf{d}_X})$ is a subalgebra over $\mathbb{R}$ of $\mathfrak{sa}(C(X))$ as $\mathfrak{sa}(C(X))$ is commutative, it is also a Jordan-Lie subalgebra of the self-adjoints $\mathfrak{sa}(C(X))$. Fix $s \in \mathbb{R}, s > 0$, by [18, Arzela-Ascoli Theorem VI.3.8], the set:

$$\left\{f \in \mathrm{dom}(\mathsf{L}_{\mathsf{d}_X}) : \mathsf{L}_{\mathsf{d}_X}(f) \leqslant 1, \|f\|_{C(X)} \leqslant s\right\}$$

is totally bounded in $C(X)$ for $\|\cdot\|_{C(X)}$. Next, we show that $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}$ is bounded. Note that compact metric spaces are bounded (have finite diameter).

Claim 2.2.11. *If $r \in (0, \infty) \subset \mathbb{R}$ is an upper bound for the diameter of the compact metric space (X, d_X), then the $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}$ is bounded by $2r$. Thus $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}$ is bounded.*

Proof of claim. Let $r \in (0, \infty)$ be an upper bound for the diameter of (X, d_X). Assume that $f \in \operatorname{dom}(\mathsf{L}_{\mathsf{d}_X}), \mathsf{L}_{\mathsf{d}_X}(f) \leqslant 1$ and $x, y \in X$. We have:

$$|f(x) - f(y)| \leqslant \mathsf{d}_X(x, y) \leqslant \sup\{\mathsf{d}_X(a, b) : a, b \in X\} \leqslant r,$$

and thus, $\sup\{|f(x) - f(y)| : x, y \in X\} \leqslant r$. Now, fix $y_0 \in X$, by the above inequality, we have:

$$\begin{aligned} \left\|f - f(y_0)1_{C(X)}\right\|_{C(X)} &= \sup\{|f(x) - f(y_0)1_{C(X)}(x)| : x \in X\} \\ &= \sup\{|f(x) - f(y_0)| : x \in X\} \leqslant r, \end{aligned}$$

where $1_{C(X)}$ is the constant 1 function on X.

In summary, we have for all $f \in \operatorname{dom}(\mathsf{L}_{\mathsf{d}_X}), \mathsf{L}_{\mathsf{d}_X}(f) \leqslant 1$ there exists $k_f \in \mathbb{R}$ such that $\left\|f - k_f 1_{C(X)}\right\|_{C(X)} \leqslant r$. Now, let $\mu, \nu \in \mathscr{S}(C(X))$ and $f \in \operatorname{dom}(\mathsf{L}_{\mathsf{d}_X}), \mathsf{L}_{\mathsf{d}_X}(f) \leqslant 1$. We conclude that:

$$\begin{aligned} |\mu(f) - \nu(f)| &= |\mu(f) - k_f + k_f - \nu(f)| \\ &= \left|\mu(f) - k_f\mu\left(1_{C(X)}\right) + k_f\nu\left(1_{C(X)}\right) - \nu(f)\right| \\ &= \left|\mu(f) - \mu\left(k_f 1_{C(X)}\right) + \nu\left(k_f 1_{C(X)}\right) - \nu(f)\right| \\ &= \left|\mu\left(f - k_f 1_{C(X)}\right) - \nu\left(f - k_f 1_{C(X)}\right)\right| \\ &= \left|(\mu - \nu)\left(f - k_f 1_{C(X)}\right)\right| \\ &\leqslant \|\mu - \nu\|_{C(X)'} \left\|f - k_f 1_{C(X)}\right\|_{C(X)} \leqslant \left(\|\mu\|_{C(X)'} + \|\nu\|_{C(X)'}\right) r \leqslant 2r. \end{aligned}$$

Hence, we have $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}(\mu, \nu) \leqslant 2r$, and since $\mu, \nu \in \mathscr{S}(C(X))$ were arbitrary, the metric $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}$ is bounded by $2r$. □

Therefore, the pair $(C(X), \mathsf{L}_{\mathsf{d}_X})$ is a Leibniz quantum compact metric space by Theorem (2.2.6).

We finish the proof by verifying the isometry in the statement of the theorem. Fix $x, y \in X, x \neq y$. Let $f \in \mathrm{dom}\,(\mathsf{L}_{\mathsf{d}_X}), \mathsf{L}_{\mathsf{d}_X}(f) \leqslant 1$. We have:

$$|\delta_x(f) - \delta_y(f)| = |f(x) - f(y)| \leqslant \mathsf{d}_X(x, y).$$

Therefore, we gather that $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}(\delta_x, \delta_y) \leqslant \mathsf{d}_X(x, y)$.

Next, consider the function $y_{\mathsf{d}}(a) = \mathsf{d}_X(a, y)$. We have that:

$$|\delta_x(y_{\mathsf{d}}) - \delta_y(y_{\mathsf{d}})| = |\mathsf{d}_X(x, y) - \mathsf{d}_X(y, y)| = \mathsf{d}_X(x, y),$$

and by Expression (2.2.3), we conclude that $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}(\delta_x, \delta_y) = \mathsf{d}_X(x, y)$, which completes the proof by Proposition (2.1.32). □

There are many more examples of quasi-Leibniz quantum compact metric spaces. We will not cover them in detail since they lie outside the scope of this dissertation and would require many more definitions. However, we will still make mention of some examples with references. Some but not all examples of C*-algebras that may be equipped with quasi-Leibniz Lip-norms include: noncommutative tori [59], curved noncommutative tori [42], various classes of group C*-algebras including Hyperbolic and Nilpotent groups [63, 56, 15], and noncommutative solenoids [49]. And, of course, one main goal of this dissertation is to present AF algebras as quasi-Leibniz quantum compact metric spaces.

2.3 Gromov-Hausdorff Propinquity

Developed by F. Latrémolière, the Gromov-Hausdorff propinquity [46, 43, 41, 45, 44, 48], a family of noncommutative analogues of the Gromov-Hausdorff distance, provides a new framework to study the geometry of classes of C*-algebras, opening new avenues of research in noncommutative geometry. Various notions of finite dimensional approximations of C*-algebras are found in C*-algebra theory, from nuclearity to quasi-diagonality, passing through exactness, to name a few of the more common notions. They are also a core focus and major source of examples for our research in noncommutative metric geometry. Examples of finite dimensional approximations in the sense of the propinquity include the approximations of quantum tori by fuzzy tori due to F. Latrémolière in [37, 40] and the full matrix approximations C*-algebras of continuous functions on coadjoint orbits of semisimple Lie groups due to M. A. Rieffel in [62, 66, 69]. Moreover, the existence of finite dimensional approximations for quantum compact metric spaces, in the sense of the dual propinquity, were studied in [45], as part of the discovery by F. Latrémolière of a noncommutative analogue of the Gromov compactness theorem [30], we present as Theorem (2.3.23).

Our primary interest in developing a theory of quantum metric spaces is the introduction of various hypertopologies on classes of such spaces, thus allowing us to study the geometry of classes of C*-algebras and perform analysis on these classes. A classical model for our hypertopologies is given by the Gromov-Hausdorff distance. While several noncommutative analogues of the Gromov-Hausdorff distance have been proposed — most importantly Rieffel's original construction of the quantum Gromov-Hausdorff distance [61] — we shall work with a particular metric introduced by F. Latrémolière. This metric, known as the quantum propinquity, is designed to be best suited to quasi-Leibniz quantum compact metric spaces, and in particular, is zero between two such spaces if and only if they are quantum isometric (see Theorem-Definition (2.3.16.5)) (unlike Rieffel's distance). We now provide the definition of the quantum propinquity along with the tools needed to compute upper bounds on this metric.

Definition 2.3.1 ([46, Definition 3.1])**.** *The* 1-level set $\mathscr{S}_1(\mathfrak{D}|\omega)$ *of an element* ω *of a unital C*-algebra* $\mathfrak{D}$ *is* $\{\varphi \in \mathscr{S}(\mathfrak{D}) : \varphi((1_{\mathfrak{D}} - \omega^*\omega)) = \varphi((1_{\mathfrak{D}} - \omega\omega^*)) = 0\}$.

Next, we define the notion of a *Latrémolière bridge*, which is not only crucial in the definition of the quantum propinquity but also the convergence results of Latrémolière in [40] and Rieffel in [69]. In particular, the pivot of Definition (2.3.2) and its use in the height of Definition (2.3.7) are of utmost importance in the convergence results of [40, 69].

Definition 2.3.2 ([46, Definition 3.6])**.** *A* bridge *from $\mathfrak{A}$ to $\mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are unital C*-algebras, is a quadruple $(\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ where:*

1. *$\mathfrak{D}$ is a unital C*-algebra,*
2. *the element ω, called the* pivot *of the bridge, satisfies $\omega \in \mathfrak{D}$ and $\mathscr{S}_1(\mathfrak{D}|\omega) \neq \emptyset$,*
3. *$\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ are unital *-monomorphisms.*

Remark 2.3.3. *There always exists a bridge between any two arbitrary unital C*-algebras [46, 45]. Indeed, let $\mathfrak{A}, \mathfrak{B}$ be two unital C*-algebras and let $\mathfrak{D} = \mathfrak{A} \otimes \mathfrak{B}$ be any C*-algebra formed over the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$, which always exists (see [55, Chapter 6.3]). Now, the maps:*

$$\pi_{\mathfrak{A}} : a \in \mathfrak{A} \longmapsto a \otimes 1_{\mathfrak{B}} \in \mathfrak{D} \text{ and } \pi_{\mathfrak{B}} : b \in \mathfrak{B} \longmapsto 1_{\mathfrak{A}} \otimes b \in \mathfrak{D}$$

*are unital *-monomorphisms. For the pivot, consider $\omega = 1_{\mathfrak{A}} \otimes 1_{\mathfrak{B}} = 1_{\mathfrak{D}}$, which provides that $\mathscr{S}_1(\mathfrak{D}|\omega) = \mathscr{S}(\mathfrak{D}) \neq \emptyset$. Thus, the quadruple $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ is a bridge from $\mathfrak{A}$ to $\mathfrak{B}$.*

A bridge allows us to define a numerical quantity which estimates, for this given bridge, how far our quasi-Leibniz quantum compact metric spaces are. This quantity, called the length of the bridge, is constructed using two other quantities we define shortly. However, to define these, we utilize the *Hausdorff distance* to provide a suitable tool to calulate distance between closed sets of metric space. We define this metric now.

Definition 2.3.4 ([33]). *Let (X, d) be a (pseudo)metric space, where pseudo means that $\mathsf{d}(x, y) = 0$ need not imply $x = y$. For $b \in X$ and $A \subseteq X$, let* $\operatorname{dist}(b, A) = \inf\{\mathsf{d}(b, a) : a \in A\}$.

Let $\mathcal{C}l(X)$ denote the closed sets of X. Define a map:

$$\mathsf{Haus}_{\mathsf{d}} : \mathcal{C}l(X) \times \mathcal{C}l(X) \longrightarrow [0, \infty] \subset \mathbb{R}$$

by the quanitity $\mathsf{Haus}_{\mathsf{d}}(A, B) = \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\}$ *for any $A, B \in \mathcal{C}l(X)$.*

Next, we display some useful properties of $\mathsf{Haus}_{\mathsf{d}}$, which will allow us to define the length of a bridge, while also making note of a nice connection with the Fell topology defined in Theorem-Definition (2.1.57) for the sake of completion and to show that the Fell topology is a valid generalization of the Hausdorff distance to general topological spaces.

Proposition 2.3.5. *If (X, d) be a pseudo metric space, then:*

1. *([14, Proposition 7.3.3]) $\mathsf{Haus}_{\mathsf{d}}$ is an extended pseudo metric on $\mathcal{C}l(X)$. If d is a metric, then $\mathsf{Haus}_{\mathsf{d}}$ is an extended metric on $\mathcal{C}l(X)$.*

2. *([14, Proposition 7.3.7 and Blaschke Theorem 7.3.8]) If (X, d) is a complete metric space, then $\mathsf{Haus}_{\mathsf{d}}$ is a complete extended metric on $\mathcal{C}l(X)$. If (X, d) is a compact metric space, then $\mathsf{Haus}_{\mathsf{d}}$ is a metric on $\mathcal{C}l(X)$ and $(\mathcal{C}l(X), \mathsf{Haus}_{\mathsf{d}})$ is a compact metric space.*

3. *([4, Theorem 3.93]) If (X, d) is a compact metric space, then the Fell topology of Theorem-Definition (2.1.57) on $\mathcal{C}l(X)$ coincides with the topology on $\mathcal{C}l(X)$ induced by $\mathsf{Haus}_{\mathsf{d}}$.*

4. *If we let $K(X)$ denote the set of compact subsets of X, then $\mathsf{Haus}_{\mathsf{d}}$ is a pseudo metric on $K(X)$, which is a metric on $K(X)$ when d is a metric.*

Proof. The fact that we can drop the adjective "extended" in part *4.* follows from the triangle inequality and the fact that compact sets have finite diameter. $\square$

The height of a bridge assesses the error we make by replacing the state spaces of the Leibniz quantum compact metric spaces with the image of the 1-level set of the pivot of the bridge, using the ambient Monge-Kantorovich metric.

Notation 2.3.6. *Let $\mathfrak{A}, \mathfrak{D}$ be C^*-algebras and $\pi : \mathfrak{A} \longrightarrow \mathfrak{D}$ be a *-monomorphism. Let $\pi^* : \mathfrak{D}' \longrightarrow \mathfrak{A}'$ denote the* dual map*, where $\mathfrak{A}'$ is the space of complex-valued bounded linear functions on $\mathfrak{A}$, and same for $\mathfrak{D}'$. The dual map is defined by $\pi^*(\mu) = \mu \circ \pi$ for all $\mu \in \mathfrak{D}'$.*

Definition 2.3.7 ([46, Definition 3.16])**.** *Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ be two quasi-Leibniz quantum compact metric spaces. The* height $\varsigma(\gamma|\mathsf{L}_\mathfrak{A}, \mathsf{L}_\mathfrak{B})$ *of a bridge $\gamma = (\mathfrak{D}, \pi_\mathfrak{A}, \pi_\mathfrak{B}, \omega)$ from $\mathfrak{A}$ to $\mathfrak{B}$, and with respect to $\mathsf{L}_\mathfrak{A}$ and $\mathsf{L}_\mathfrak{B}$, is given by:*

$$\max \left\{ \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_\mathfrak{A}}}(\mathscr{S}(\mathfrak{A}), \pi_\mathfrak{A}^*(\mathscr{S}_1(\mathfrak{D}|\omega))), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_\mathfrak{B}}}(\mathscr{S}(\mathfrak{B}), \pi_\mathfrak{B}^*(\mathscr{S}_1(\mathfrak{D}|\omega))) \right\},$$

where $\pi_\mathfrak{A}^*$ and $\pi_\mathfrak{B}^*$ are the dual maps of $\pi_\mathfrak{A}$ and $\pi_\mathfrak{B}$, respectively.

Remark 2.3.8. *For any two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ and any bridge $\gamma = (\mathfrak{D}, \pi_\mathfrak{A}, \pi_\mathfrak{B}, \omega)$ from $\mathfrak{A}$ to $\mathfrak{B}$, the height $\varsigma(\gamma|\mathsf{L}_\mathfrak{A}, \mathsf{L}_\mathfrak{B})$ is finite. This is immediate from Proposition (2.3.5.2) since by the definition of a quantum compact metric space (Definition (2.2.5)), the state space with the Monge-Kantorovich metric space is a compact metric space as it metrizes the weak* topology and the state space is compact by Proposition (2.1.24).*

The quantum propinquity was originally devised in the framework on Leibniz quantum compact metric spaces (i.e. for the case $C = 1$ and $D = 0$), and as seen in [45], can be extended to many different classes of quasi-Leibniz compact quantum metric spaces. Thus, although the notion of quasi-Leibniz does ot appear until [45], the citations we provide for the following definitions come from [46], which is a more complete reference of the quantum propinquity.

The second quantity measures how far apart the images of the balls for the Lip-norms are in $\mathfrak{A} \oplus \mathfrak{B}$; to do so, we use a seminorm on $\mathfrak{A} \oplus \mathfrak{B}$ built using the bridge.

Definition 2.3.9 ([46, Definition 3.10]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital C*-algebras. The* bridge seminorm $\mathsf{bn}_{\gamma}(\cdot)$ *of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from $\mathfrak{A}$ to $\mathfrak{B}$ is the seminorm defined on $\mathfrak{A} \oplus \mathfrak{B}$ by:*

$$\mathsf{bn}_{\gamma}(a, b) = \|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}$$

for all $(a, b) \in \mathfrak{A} \oplus \mathfrak{B}$.

We implicitly identify $\mathfrak{A}$ with $\mathfrak{A} \oplus \{0_{\mathfrak{B}}\}$ and $\mathfrak{B}$ with $\{0_{\mathfrak{A}}\} \oplus \mathfrak{B}$ in $\mathfrak{A} \oplus \mathfrak{B}$ in the next definition, for any two spaces $\mathfrak{A}$ and $\mathfrak{B}$.

Definition 2.3.10 ([46, Definition 3.14]). *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. The* reach $\varrho(\gamma|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$ *of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from $\mathfrak{A}$ to $\mathfrak{B}$, and with respect to $\mathsf{L}_{\mathfrak{A}}$ and $\mathsf{L}_{\mathfrak{B}}$, is given by:*

$$\mathsf{Haus}_{\mathsf{bn}_{\gamma}(\cdot)}\left(\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}, \{b \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}\right).$$

Remark 2.3.11. *For any two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and any bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from $\mathfrak{A}$ to $\mathfrak{B}$, the reach $\varrho(\gamma|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$ is finite. This is not immediate since although $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}$ and $\{b \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}$ are closed by lower semi-continuity of $\mathsf{L}_{\mathfrak{A}}$ and $\mathsf{L}_{\mathfrak{B}}$, respectively, they are not compact since they contain the scalars, and thus Proposition (2.3.5.4) does not apply. The argument for why the reach is finite is provided between [46, Notation 3.13] and [46, Definition 3.14].*

We thus choose a natural quantity to synthesize the information given by the height and the reach of a bridge:

Definition 2.3.12 ([46, Definition 3.17]). *Let* $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ *and* $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ *be two quasi-Leibniz quantum compact metric spaces. The* length $\lambda(\gamma|\mathsf{L}_\mathfrak{A}, \mathsf{L}_\mathfrak{B})$ *of a bridge* $\gamma = (\mathfrak{D}, \pi_\mathfrak{A}, \pi_\mathfrak{B}, \omega)$ *from* $\mathfrak{A}$ *to* $\mathfrak{B}$*, and with respect to* $\mathsf{L}_\mathfrak{A}$ *and* $\mathsf{L}_\mathfrak{B}$*, is given by:*

$$\max\left\{\varsigma(\gamma|\mathsf{L}_\mathfrak{A}, \mathsf{L}_\mathfrak{B}), \varrho(\gamma|\mathsf{L}_\mathfrak{A}, \mathsf{L}_\mathfrak{B})\right\}.$$

While a natural approach, defining the quantum propinquity as the infimum of the length of all possible bridges between two given (C,D)-quasi-Leibniz quantum compact metric spaces, for some fixed $C \geqslant 1$ and $D \geqslant 0$, does not lead to a distance, as the triangle inequality may not be satisfied. Instead, a more subtle road must be taken. We introduce the notion of a trek.

Definition 2.3.13 ([46, Definition 3.20]). *Fix* $C \geqslant 1$ *and* $D \geqslant 0$*. Let* $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ *and* $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ *be two* (C,D)*-quasi-Leibniz quantum compact metric spaces. A* trek *from* $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ *to* $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ *is an* n*-tuple:*

$$\left((\mathfrak{A}_1, \mathsf{L}_1, \gamma_1, \mathfrak{A}_2, \mathsf{L}_2), \ldots, (\mathfrak{A}_n, \mathsf{L}_n, \gamma_n, \mathfrak{A}_{n+1}, \mathsf{L}_{n+1})\right),$$

for some $n \in \mathbb{N} \setminus \{0\}$*, where for all* $j \in \{1, \ldots, n\}$*, the pair* $(\mathfrak{A}_j, \mathsf{L}_j)$ *is a* (C,D)*-quasi-Leibniz quantum compact metric space, while* γ_j *is a bridge from* $\mathfrak{A}_j$ *to* $\mathfrak{A}_{j+1}$*, and* $(\mathfrak{A}_1, \mathsf{L}_1) = (\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{A}_{n+1}, \mathsf{L}_{n+1}) = (\mathfrak{B}, \mathsf{L}_\mathfrak{B})$.

Note that all bridges are treks since a trek is a bridge if $n = 1$. Building from bridges, we introduce the length of a trek.

Definition 2.3.14 ([46, Definition 3.22]). *Fix* $C \geqslant 1$ *and* $D \geqslant 0$*. Let* $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ *and* $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ *be two* (C,D)*-quasi-Leibniz quantum compact metric space s and let* $\Gamma = \left((\mathfrak{A}_1, \mathsf{L}_1, \gamma_1, \mathfrak{A}_2, \mathsf{L}_2), \ldots, (\mathfrak{A}_n, \mathsf{L}_n, \gamma_n, \mathfrak{A}_{n+1}, \mathsf{L}_{n+1})\right)$ *be a trek from* $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ *to*

$(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. *The length* $\lambda(\Gamma)$ *of* Γ *is:*

$$\lambda(\Gamma) = \sum_{j=1}^{n} \lambda\left(\gamma_j | \mathsf{L}_j, \mathsf{L}_{j+1}\right).$$

Now, we define the quantum Gromov-Hausdorff propinquity.

Definition 2.3.15 ([46, Definition 4.2])**.** *Fix* $C \geqslant 1$ *and* $D \geqslant 0$. *The* quantum Gromov-Hausdorff propinquity *between two* (C, D)*-quasi-Leibniz quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *is the quantity:*

$$\Lambda_{C,D}\left((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right) = \inf\left\{\lambda(\Gamma) : \Gamma \textit{ is a trek from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \textit{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

The following theorem provides a summary of the conclusions of [46] relevant for our work. One can in some sense take the following as a definition of the quantum propinquity due to part *6.*, which is why we use the term Theorem-Definition. In part *5.*, the following also introduces the natural notion of an isomorphism between two quantum compact metric spaces, a quantum isometry. This notion is natural because a quantum isometry provides the natural notion of isomorphisms at the C*-algebra level and the metric space level on the state space with a suitable compatiblity condition between the isomorphisms.

Theorem-Definition 2.3.16 ([46, 45])**.** *Fix* $C \geqslant 1$ *and* $D \geqslant 0$. *Let* $\mathcal{QQCMS}_{C,D}$ *be the class of all* (C, D)*-quasi-Leibniz quantum compact metric spaces. There exists a class function* $\Lambda_{C,D}$ *from* $\mathcal{QQCMS}_{\mathcal{C},\mathcal{D}} \times \mathcal{QQCMS}_{C,D}$ *to* $[0, \infty) \subseteq \mathbb{R}$ *such that:*

1. *[46, Proposition 4.6] for any* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ *we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \max\left\{\operatorname{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right), \operatorname{diam}\left(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}\right)\right\},$$

where diam *denotes the diameter of a metric space,*

2. *[46, Theorem 6.1] for any* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ *we have:*

$$0 \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}))$$

3. *[46, Theorem 6.1] for any* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}}) \in \mathcal{QQCMS}_{C,D}$ *we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})) \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})),$$

4. *(Definition (2.3.15)) for all for any* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ *and for any bridge* γ *from* $\mathfrak{A}$ *to* $\mathfrak{B}$*, we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \lambda\left(\gamma | \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right),$$

5. *[46, Theorem 6.1] for any* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$*, we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

   if and only if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *are quantum isometric, i.e. if and only if there exists a unital *-isomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ *whose dual map* π^* *is an isometry from* $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ *into* $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*, or equivalently, there exists a unital *-isomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ *with* $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$*,*

6. *(Definition (2.3.15)) if* Ξ *is a class function from* $\mathcal{QQCMS}_{C,D} \times \mathcal{QQCMS}_{C,D}$ *to* $[0, \infty)$ *which satisfies Properties* 2., 3., *and* 4. *above, then:*

$$\Xi((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

   for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *in* $\mathcal{QQCMS}_{C,D}$

Thus, for a fixed choice of $C \geqslant 1$ and $D \geqslant 0$, the quantum Gromov-Hausdorff propinquity is the largest pseudo metric on the class of (C, D)-quasi-Leibniz quantum compact metric spaces which is bounded above by the length of any bridge between its arguments. Furthermore, by part 5., the quantum propinquity is a metric up to quantum isometry.

Moreover, it was shown in [46] that the quantum Gromov-Hausdorff propinquity is a noncommutative analogue to the Gromov-Hausdorff distance. Before we present this, we introduce the classic Gromov-Hausdorff distance on the class of compact metric spaces, which is built from the Hausdorff distance introduced above in Defintion (2.3.4).

Definition 2.3.17 ([31])**.** *Let $\mathfrak{C}$ denote the class of compact metric spaces. The Gromov-Hausdorff distance between two compact metric spaces $(X, \mathsf{d}_X), (Y, \mathsf{d}_Y) \in \mathfrak{C}$ denoted by* $\mathrm{GH}((X, \mathsf{d}_X), (Y, \mathsf{d}_Y))$ *is the quantity:*

$$\inf \left\{ \mathsf{Haus}_{\mathsf{d}_Z}(f_X(X), f_Y(Y)) : \begin{array}{c} (Z, \mathsf{d}_Z) \text{ is a metric space such that} \\ f_X : X \longrightarrow Z, f_Y : Y \longrightarrow Z \\ \text{are isometries.} \end{array} \right\}$$

Note that the quantity defining the Gromov-Hausdorff distance always exists and is finite since for any two compact metric spaces there always exists a metric space for which the two metric spaces isometrically embed into. Indeed, if $(X, \mathsf{d}_X), (Y, \mathsf{d}_Y)$ are two compact metric spaces, then if $C = \max\{\mathrm{diam}(X, \mathsf{d}_X), \mathrm{diam}(Y, \mathsf{d}_Y)\}$, then the following defines a map on $Z \times Z$ where $Z = X \sqcup Y$ is the disjoint union:

$$\mathsf{d}_Z(a, b) = \begin{cases} \mathsf{d}_X(a, b) & : a, b \in X \\ \mathsf{d}_Y(a, b) & : a, b \in Y \\ C & : \text{ otherwise} \end{cases},$$

which is a metric on Z for which the canonical inclusions for X and Y into Z are isometries by construction, and their Hausdorff distance in Z is finite by Proposition (2.3.5.4).

The next theorem, due to Gromov, shows that the Gromov-Hausdorff distance is a metric up to the natural notion of isomorphism between metric spaces.

Theorem 2.3.18 ([14, Theorem (Gromov) 7.3.30],[31]). *The Gromov-Hausdorff distance is a pseudo metric on the class of compact metric spaces $\mathfrak{C}$ such that for two compact metric spaces $(X, \mathsf{d}_X), (Y, \mathsf{d}_Y) \in \mathfrak{C}$, the quantity $\mathrm{GH}\left((X, \mathsf{d}_X), (Y, \mathsf{d}_Y)\right) = 0$ if and only if there exists an isometry from X onto Y.*

Hence, the Gromov-Hausdorff distance is a metric on the class of compact metric spaces $\mathfrak{C}$ up to the equivalence relation of isometry.

We now compare the quantum propinquity to natural metrics including Rieffel's quantum distance and the Gromov-Hausdorff distance.

Theorem 2.3.19 ([46, Corollary 6.4 and Theorem 6.6]). *Fix $C \geqslant 1, D \geqslant 0$. If dist_q is Rieffel's quantum Gromov-Hausdorff distance [61], then for any pair $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ of (C, D)-quasi-Leibniz quantum compact metric spaces, we have:*

$$\mathsf{dist}_q((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \leqslant 2\mathsf{\Lambda}_{C,D}((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})). \tag{2.3.1}$$

Moreover, for any compact metric space (X, d_X), let $\mathsf{L}_{\mathsf{d}_X}$ be the Lipschitz seminorm induced on the C^*-algebra $C(X)$ of $\mathbb{C}$-valued continuous functions on X by d_X defined in Equation (2.2.1). Note that $(C(X), \mathsf{L}_{\mathsf{d}_X})$ is a Leibniz quantum compact metric space by Theorem (2.2.10) and is thus a (C, D)-quasi-Leibniz quantum compact metric space. Let $\mathfrak{C}$ be the class of all compact metric spaces. For any $(X, \mathrm{d}_X), (Y, \mathrm{d}_\mathrm{Y}) \in \mathfrak{C}$, we have:

$$\mathsf{\Lambda}_{C,D}\left((C(X), \mathsf{L}_{\mathsf{d}_X}), (C(Y), \mathsf{L}_{\mathsf{d}_Y})\right) \leqslant \mathrm{GH}((X, \mathsf{d}_X), (Y, \mathsf{d}_Y)) \tag{2.3.2}$$

where GH *is the Gromov-Hausdorff distance on the class of compact metric spaces* $\mathfrak{C}$ *defined in Definition (2.3.17).*

Furthermore, the class function $\Upsilon : (X, \mathsf{d}_X) \in \mathfrak{C} \mapsto (C(X), \mathsf{L}_{\mathsf{d}_X}) \in \mathcal{QQCMS}_{C,D}$ *is a homeomorphism onto its image, where the topology on* $\mathfrak{C}$ *is given by the Gromov-Hausdorff distance* GH *with respect to the equivalence relation of isometry, and the topology on the image of* Υ *is given by the quantum propinquity* $\Lambda_{C,D}$ *with respect to the equivalence relation of quantum isometry of Theorem-Definition (2.3.16.5)*

Proof. Inequality (2.3.1) is provided by [46, Corollary 6.4], and Inequality (2.3.2) is provided by [46, Theorem 6.6].

The map Υ is well-defined up to the associated equivalence relations and is continuous by Inequality (2.3.2) and Theorem (2.3.18). Next, we show that Υ is injective with respect to the associated equivalence relations. Therefore, assume that there are two compact metric spaces $(X, \mathsf{d}_X), (Y, \mathsf{d}_Y)$ such that there is a quantum isometry $\pi : (C(X), \mathsf{L}_{\mathsf{d}_X}) \longrightarrow (C(Y), \mathsf{L}_{\mathsf{d}_Y})$. Now, the dual map $\pi^* : C(Y)' \longrightarrow C(X)'$ is linear. Thus, since π is a quantum isometry and the state spaces $\mathscr{S}(C(Y)), \mathscr{S}(C(X))$ are convex, the dual map when restricted to $\mathscr{S}(C(Y))$ is an affine isometry from $\left(\mathscr{S}(C(Y)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_Y}}\right)$ onto $\left(\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}\right)$. Now, recall that the pure states of a C*-algebra are defined to be the extreme points of the state space (Definition (2.1.36)), and note that affine bijections preserve extreme points of convex sets. In particular, this implies that the dual map π^* restricted to the pure states is an isometry from the pure states $\left(\mathscr{P}(C(Y)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_Y}}\right)$ onto the pure states $\left(\mathscr{P}(C(X)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}\right)$. Next, by Theorem (2.2.10) and [36, Proposition 4.4.1], we have isometries from (X, d_X) onto $\left(\mathscr{P}(C(X)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_X}}\right)$ and from (Y, d_Y) onto $\left(\mathscr{P}(C(Y)), \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_Y}}\right)$. Therefore, we conclude that (X, d_X) is isometric onto (Y, d_Y). Hence, the map Υ is injective up to the associated equivalence relations.

Therefore, we may now verify continuity of the inverse up to the associated equivalence relations. Assume there exists a sequence of compact metric space

$((X_n, \mathsf{d}_{X_n}))_{n\in\mathbb{N}}$ and a compact metric space (X, d_X) such that the sequence of Leibniz quantum comapct metric spaces $\left((C(X_n), \mathsf{L}_{\mathsf{d}_{X_n}})\right)_{n\in\mathbb{N}}$ converge to the Leibniz quantum compact metric space $(C(X), \mathsf{L}_{\mathsf{d}_X})$ in the quauntum propinquity. By Inequality (2.3.1), these quantum spaces converge in dist_q. By [61, Theorem 13.16] and its proof, the spaces $((X_n, \mathsf{d}_{X_n}))_{n\in\mathbb{N}}$ converge to (X, d_X) in GH. $\square$

Now, the class of quasi-Leibniz quantum compact metric spaces forms a natural category where isomorphism is provided by quantum isometry (see [44, Section 2.2.2] for more details). Thus, although Theorem (2.3.19) does not necessarily provide an equivalence of categories between the category of compact metric spaces and the category of quasi-Leibniz quantum compact metric spaces like Gelfand duality does for unital commutative C*-algebras and compact Hausdorff spaces (see Theorems (2.1.30, 2.1.34)), we have a somewhat more suitable connection to classical case, which is that the quantum propinquity topology recovers the Gromov-Hausdorff topology on compact metric spaces.

Before we continue with the next notion, we provide the following basic lemma for motivation, which shows that finite sets in the classical setting is equivalent to finite-dimensionality in the C*-algebra setting.

Lemma 2.3.20. *Let X be a non-empty compact Hausdorff space. The C*-algebra $C(X)$ is finite-dimensional if and only if X has finite cardinality. Moreover, if X is finite, then the dimension of $C(X)$ is equal to the cardinality of X. And, if $C(X)$ is finite dimensional, then the dimension of $C(X)$ is equal to the cardinality of X.*

Proof. Assume that X has finite cardinality, so there exists $N \in \mathbb{N}$ such that $X = \{x_0, \ldots, x_N\}$. Since X is Hausdorff, the topology on X is discrete. For each $j \in \{0, \ldots, N\}$, define the function:

$$f_j(x) = \begin{cases} 1 & : x = x_j \\ 0 & : \text{ otherwise.} \end{cases}$$

Since X has the discrete topology, the function $f_j \in C(X)$ for each $j \in \{0, \ldots, N\}$. It is routine to check that $\{f_j \in C(X) : j \in \{0, \ldots, N\}\}$ is a vector space basis for $C(X)$. Thus, the dimension $\dim C(X) = N + 1 = |X|$, which is the cardinality of X.

Before moving on to the forward direction, we prove the following claim.

Claim 2.3.21. *Let X be any non-empty compact Hausdorff space. For each $x \in X$, let $\delta_x : f \in C(X) \longmapsto f(x) \in \mathbb{C}$ denote the Dirac point mass of x, and note that $\delta_x \in \mathscr{S}(C(X))$ for each $x \in X$.*

Any finite set of distinct Dirac point masses is a linearly independent set in the dual space $C(X)'$.

Proof of claim. Let $N \in \mathbb{N}$ and let $\{\delta_{x_0}, \ldots, \delta_{x_N}\}$ be a finite set of distinct Dirac point masses. Assume by way of contradiction that the set $\{\delta_{x_0}, \ldots, \delta_{x_N}\}$ is linearly dependent. Thus, there exists $j \in \{0, \ldots, N\}$ and there exists $\lambda_k \in \mathbb{C}$ for every $k \in \{0, \ldots, N\} \setminus \{j\}$ such that $\sum_{k \in \{0,\ldots,N\} \setminus \{j\}} \lambda_k \delta_{x_k} = \delta_{x_j}$. By [71, Urysohn's Lemma 15.6], there exists a function $f \in C(X)$ such that $f(x_j) \neq 0$ and $f(x_k) = 0$ for all $k \in \{0, \ldots, N\} \setminus \{j\}$. However, we then have that:

$$\begin{aligned} \sum_{k \in \{0,\ldots,N\} \setminus \{j\}} \lambda_k \delta_{x_k}(f) = \delta_{x_j}(f) &\implies \sum_{k \in \{0,\ldots,N\} \setminus \{j\}} \lambda_k f(x_k) = f(x_j) \\ &\implies \sum_{k \in \{0,\ldots,N\} \setminus \{j\}} \lambda_k \cdot 0 = f(x_j) \\ &\implies 0 = f(x_j), \end{aligned}$$

which is a contradiction. Thus, the set $\{\delta_{x_0}, \ldots, \delta_{x_N}\}$ is linearly independent. $\square$

Now, for the forward direction, assume that $\dim C(X) = N$, where $N \in \mathbb{N}$. Thus, the dual space $C(X)'$ is finite dimensional and $\dim C(X)' = N$. Therefore, there can only exists at most N distinct Dirac point masses by the claim lest there be a linearly independent set of $C(X)'$ of cardinality greater than N. However, by the homeomorphism of Proposition (2.1.32), we have that the cardinality of X is at most N, and is thus finite. Now, assume by way of contradiction that the cardinality of X is less than N, then the reverse direction of the statement of this Lemma, which was already proven, would imply that the dimension of $C(X)$ would be less than N, which is a contradction. $\square$

Next, we introduce a noncommutative analogue of the Gromov Compactness Theorem in Theorem (2.3.23). The Gromov Compactness Theorem [14, Theorem 7.4.15] informally states that a set K is compact in the Gromov-Hausdorff topology if there is a uniform bound on the diameter of all the compact metric spaces K as well as a uniform bound on the cardinality of minimal finite ε-nets for all $\varepsilon > 0$ for all the compact metric spaces in the set K. Now, by the map Υ in Theorem (2.3.19) and by Claim (2.2.11), the notion of diameter of a compact metric spaces translates to the diameter of the state space with the Monge-Kantorovich metric of a quantum compact metric space, and approximations of compact metric spaces by finite sets in the Gromov-Hausdorff distance translates to finite-dimensional approximations of quantum compact metric space in the quantum propinquity by Lemma (2.3.20). Therefore, one would hope that a noncommutative analogue of the Gromov Compactness would arise from controlling the diameter of the states spaces and the dimension of finite-dimensional approximations in the quantum propinquity. This is provided by Theorem (2.3.23).

Definition 2.3.22 ([45, Definition 4.1])**.** *Let $C \geqslant 1$ and $D \geqslant 0$. The* covering number $\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}|\varepsilon)$ *of a (C,D)-quasi-Leibniz quantum compact metric space*

$(\mathfrak{A}, \mathsf{L})$, *for some* ε, *is:*

$$\inf \left\{ \dim \mathfrak{B} : \begin{array}{l} (B, \mathsf{L}_{\mathfrak{B}}) \textit{ is a} \\ (C,D)\textit{-quasi-Leibniz quantum compact metric space, and} \\ \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon \end{array} \right\}.$$

Theorem 2.3.23 ([45, Theorem 4.2])**.** *Let $\mathcal{A}$ be a class of (C,D)-quasi-Leibniz quantum compact metric spaces, with $C \geqslant 1$ and $D \geqslant 0$, such that $\mathrm{cov}_{(C,D)}((\mathfrak{A}, \mathsf{L})|\varepsilon) < \infty$ for all $\varepsilon > 0$ and $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$. The class $\mathcal{A}$ is totally bounded for the quantum propinquity $\Lambda_{C,D}$ if and only if the conjunction of the the following two assertions hold:*

1. *there exists $\Delta > 0$ such that for all $(A, \mathsf{L}) \in \mathcal{A}$:*

$$\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}\right) \leqslant \Delta,$$

2. *there exists $G : (0, \infty) \to \mathbb{N}$ such that for all $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$ and all $\varepsilon > 0$, we have:*

$$\mathrm{cov}_{(C,D)}\left(\mathfrak{A}, \mathsf{L}|\varepsilon\right) \leqslant G(\varepsilon).$$

Remark 2.3.24. *Although we did not state the classic Gromov Compactness Theorem ([14, Theorem 7.4.15]) explicitly, it can be recovered from Theorem (2.3.23). Indeed, in Theorem (2.3.23), consider only the quantum compact metric spaces of the form $(C(X), \mathsf{L}_{\mathsf{d}_X})$ associated to a compact metric space (X, d_X), then apply the inverse of the homeomorphism Υ from Theorem (2.3.19). One then deduces that condition 1. of Theorem (2.3.23) provides a uniform bound on the diameter of compact metric spaces via Theorem (2.2.10), and condition 2. provides the uniform bound on minimal cardinalities of finite ε-nets for every $\varepsilon > 0$ via Lemma (2.3.20).*

As we noted, much more information on the quantum Gromov-Hausdorff propinquity can be found in [46] on this topic, as well as in the survey [44]. The extension of the quantum propinquity to the quasi-Leibniz setting can be found in [45]. Two very important examples of nontrivial convergences for the quantum propinquity are given by quantum tori and their finite dimensional approximations, as well as certain metric perturbations [37, 40, 42] and by matrix approximations of the C*-algebras of coadjoint orbits for semisimple Lie groups [66, 67, 69]. Furthermore, we will present other nontrivial convergences of AF algebras in Chapters 4 and 5.

Moreover, the quantum propinquity is, in fact, a special form of the dual Gromov-Hausdorff propinquity [43, 41, 45], which is a complete metric, up to quantum isometry, on the class of Leibniz quantum compact metric spaces, and which extends the topology of the Gromov-Hausdorff distance as well. Thus, as the dual propinquity is dominated by the quantum propinquity [43, Theorem 5.5], we conclude that *all the convergence results in this dissertation are valid for the dual Gromov-Hausdorff propinquity as well.*

Chapter 3

AF algebras as quasi-Leibniz quantum compact metric spaces

Before we can prove some classes of AF algebras are (nontrivial) continuous families with respect to quantum Gromov-Hausdorff propinquity, we must first show that AF algebras are points in the quantum propinquity space. That is, we must provide AF algebras with quasi-Leibniz quantum compact metric structure as displayed in Theorem-Definition (2.3.16). Thus, the purpose of this chapter is to provide several candidates for quantum compact metric structure on AF algebras and study certain aspects of these constructions to prepare for our continuity results of Chapters 4 and 5.

In [56], Ozawa and Rieffel utilized finite-dimensional filtrations, which are a weakening of the AF structure, i.e. the subspaces of the filtration need not be subalgebras, of certain group C*-algebras to provide quantum metric structure. However, in the case of AF algerbas, one may equip AF algebras with filtrations that are determined by finite-dimensional C*-subalgebras. This will prove advantageous to us in Theorem (3.1.3), in that we will be able to use the quantum Gromov-Hausdorff propinquity to show that AF algebras are metric limits of any inductive sequence of finite-dimensional C*-algebras that determine the AF algebra as the inductive limit since the quantum Gromov-Hausdorff propinquity distinguishes algebraic structure by Theorem-Defintion (2.3.16). This will be covered in Section (3.1) using conditional expectations and in Section (3.2) using quotient norms. We also show that the conditional expectation construction of Theorem (3.1.3) recovers the classical case of continuous functions on the Cantor set in Section (3.1.1). In Section (3.3), we give certain sufficient conditions for when two AF algebras are quantum isometric (see Theorem-Definition (2.3.16.5) for the definition of quantum isometry) and thus are a single point in the quantum propinquity space, which will assist us in Chapter 5 with certian convergence results (see Theorem (5.2.1) and Theorem (5.2.2)).

This chapter contains original results. We make a note of the publications for which they were obtained. Sections (3.2, 3.3), Lemma (3.1.12), and Proposition (3.1.13) are taken from the author's work in [1]. The rest of this chapter is taken from [3], which we co-authored with F. Latrémolière and brought AF algebras into the realm of Noncommutative Metric Geometry.

3.1 quasi-Leibniz Lip-norms from conditional expectations

We begin by observing that conditional expectations allow us to define $(2,0)$-quasi-Leibniz seminorms on C*-algebras defined in Definition (2.2.9).

Definition 3.1.1. *A* conditional expectation $\mathbb{E}(\cdot|\mathfrak{B}) : \mathfrak{A} \to \mathfrak{B}$ *onto* $\mathfrak{B}$*, where* $\mathfrak{A}$ *is a C*-algebra and* $\mathfrak{B}$ *is a C*-subalgebra of* $\mathfrak{A}$*, is a linear positive map of norm* 1 *such that for all* $b, c \in \mathfrak{B}$ *and* $a \in \mathfrak{A}$ *we have:*

$$\mathbb{E}(bac|\mathfrak{B}) = b\mathbb{E}(a|\mathfrak{B})c.$$

Lemma 3.1.2. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B} \subseteq \mathfrak{A}$ be a C^*-subalgebra of $\mathfrak{A}$. If $\mathbb{E}\left(\cdot|\mathfrak{B}\right) : \mathfrak{A} \mapsto \mathfrak{B}$ is a conditional expectation onto $\mathfrak{B}$, then the seminorm:*

$$\mathrm{S} : a \in \mathfrak{A} \mapsto \|a - \mathbb{E}\left(a|\mathfrak{B}\right)\|_{\mathfrak{A}}$$

is a $(2,0)$-quasi-Leibniz seminorm.

Proof. Let $a, b \in \mathfrak{A}$. We have:

$$\begin{aligned}
\mathrm{S}(ab) &= \|ab - \mathbb{E}\left(ab|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\leqslant \|ab - a\mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} + \|a\mathbb{E}\left(b|\mathfrak{B}\right) - \mathbb{E}\left(ab|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\quad + \|a\mathbb{E}\left(b|\mathfrak{B}\right) - \mathbb{E}\left(a\mathbb{E}\left(b|\mathfrak{B}\right)|\mathfrak{B}\right) + \mathbb{E}\left(a(\mathbb{E}\left(b|\mathfrak{B}\right) - b)|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} + \|a - \mathbb{E}\left(a|\mathfrak{B}\right))\|_{\mathfrak{A}}\|\mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\quad + \|\mathbb{E}\left(a(b - \mathbb{E}\left(b|\mathfrak{B}\right))|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} + \|a - \mathbb{E}\left(a|\mathfrak{B}\right)\|_{\mathfrak{A}}\|\mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\quad + \|a\|_{\mathfrak{A}}\|b - \mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} \\
&\leqslant 2\|a\|_{\mathfrak{A}}\|b - \mathbb{E}\left(b|\mathfrak{B}\right)\|_{\mathfrak{A}} + \|a - \mathbb{E}\left(a|\mathfrak{B}\right))\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}} \\
&\leqslant 2\left(\|a\|_{\mathfrak{A}}\mathrm{S}(b) + \|b\|_{\mathfrak{A}}\mathrm{S}(a)\right).
\end{aligned}$$

This proves our lemma. □

Note that the seminorms defined by Lemma (3.1.2) are zero exactly on the range of the conditional expectation. Now, our purpose is to define quasi-Leibniz Lip-norms on AF C*-algebras using Lemma (3.1.2) and a construction familiar in Von Neumann theory, which we recall here for our purpose.

We shall work with unital AF algebras (Definition (2.1.72) and [13]) endowed with a faithful tracial state. Any unital AF algebra admits at least one tracial state

[50, Proposition 3.4.11], and thus simple AF algebras admit at least one faithful tracial state. In fact, the space of tracial states of unital simple AF algebras can be any Choquet simplex [29, 9]. On the other hand, a unital AF algebra has a faithful trace if, and only if it is a C*-subalgebra of a unital simple AF algebra [51, Corollary 4.3]. Examples of unital AF algebras without a faithful trace can be obtained as essential extensions of the algebra of compact operators of a separable Hilbert space by some full matrix algebra. Thus, one way to state our main assumption for the construction of the Lip-norms of Theorem (3.1.3) is that we work on unital AF algebras which can be embedded into unital simple AF algebras.

Our main construction of Lip-norms on unital AF algebras with a faithful tracial state is summarized in the following theorem.

Theorem 3.1.3. *Using Definition (2.1.64) and Proposition (2.1.66), let* $\mathfrak{A}$ *be a unital AF algebra endowed with a faithful tracial state* μ *such that* $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ *is an inductive sequence of finite dimensional C*-algebras with C*-inductive limit* $\mathfrak{A}$*, with* $\mathfrak{A}_0 = \mathbb{C}$ *and where* α_n *is a unital *-monomorphism for all* $n \in \mathbb{N}$*.*

Let π *be the GNS representation of* $\mathfrak{A}$ *constructed from* μ *on the space* $L^2(\mathfrak{A}, \mu)$ *of Theorem (2.1.40). For all* $n \in \mathbb{N}$*, let:*

$$\mathbb{E}\left(\cdot \middle| \underrightarrow{\alpha}^n(\mathfrak{A}_n)\right) : \mathfrak{A} \to \mathfrak{A}$$

be the unique conditional expectation of $\mathfrak{A}$ *onto the canonical image* $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ *of* $\mathfrak{A}_n$ *in* $\mathfrak{A}$*, and such that* $\mu \circ \mathbb{E}\left(\cdot \middle| \underrightarrow{\alpha}^n(\mathfrak{A}_n)\right) = \mu$*.*

Let $\beta : \mathbb{N} \to (0, \infty)$ *have limit* 0 *at infinity. If, for all* $a \in \mathfrak{sa}(\mathfrak{A})$*, we set:*

$$\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) = \sup\left\{ \frac{\left\| a - \mathbb{E}\left(a \middle| \underrightarrow{\alpha}^n(\mathfrak{A}_n)\right)\right\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\},$$

then $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)$ *is a* $(2,0)$*-quasi-Leibniz quantum compact metric space of Definition*

(2.2.9). Moreover for all $n \in \mathbb{N}$:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right) \leqslant \beta(n)$$

and thus:

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right) = 0,$$

where $\mathsf{\Lambda}_{2,0}$ is the quantum Gromov-Hausdorff propinquity from Theorem-Definition (2.3.16).

Proof. To begin with, we note that, from the standard GNS construction presented in Theorem (2.1.40) where we will use ξ instead of q_μ, we have the following:

1. since μ is faithful, the map $\xi : a \in \mathfrak{A} \mapsto a + N_\mu \in L^2(\mathfrak{A}, \mu)$ is injective since $N_\mu = \{0_{\mathfrak{A}}\}$, and π is faithful and thus a unital *-monomorphism by Proposition (2.1.39),

2. since $\|\xi(a)\|_{L^2(\mathfrak{A},\mu)} = \sqrt{\mu(a^*a)} \leqslant \|a\|_{\mathfrak{A}}$ for all $a \in \mathfrak{A}$, the map ξ is a continuous (weak) contraction,

3. by construction, $\xi(ab) = \pi(a)\xi(b)$ for all $a, b \in \mathfrak{A}$,

4. if ω is $\xi(1_{\mathfrak{A}})$, then ω is cyclic and $\xi(a) = \pi(a)\omega$.

Let $n \in \mathbb{N}$. We denote the canonical unital *-monomorphism from $\mathfrak{A}_n$ into $\mathfrak{A}$ by $\underrightarrow{\alpha^n}$. Thus $\xi \circ \underrightarrow{\alpha^n} : \mathfrak{A}_n \to L^2(\mathfrak{A}, \mu)$ is a linear, weakly contractive injection. Since $\mathfrak{A}_n$ is finite dimensional, $\xi \circ \underrightarrow{\alpha^n}(\mathfrak{A}_n)$ is a closed subspace of $L^2(\mathfrak{A}, \mu)$ and ξ restricts to a linear homeomorphism of $\mathfrak{A}_n$ onto $\xi(\underrightarrow{\alpha^n}(\mathfrak{A}_n))$. Let P_n be the orthogonal projection from $L^2(\mathfrak{A}, \mu)$ onto $\xi \circ \underrightarrow{\alpha^n}(\mathfrak{A}_n)$.

We thus note that for all $a \in \mathfrak{A}$, we have $P_n(\xi(a)) \in \xi \circ \underrightarrow{\alpha^n}(\mathfrak{A}_n)$, thus, since ξ is injective, there exists a unique $\mathbb{E}_n(a) \in \underrightarrow{\alpha^n}(\mathfrak{A}_n)$ with $\xi(\mathbb{E}_n(a)) = P_n(\xi(a))$.

Step 1. *We begin by checking that the map* $\mathbb{E}_n : \mathfrak{A} \to \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ *is the conditional expectation* $\mathbb{E}\left(\cdot\middle|\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)\right)$ *of* $\mathfrak{A}$ *onto* $\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ *which preserves the state* μ.

To begin with, if $a \in \mathfrak{A}_n$ then $P_n\xi(\underset{\longrightarrow}{\alpha^n}(a)) = \xi(\underset{\longrightarrow}{\alpha^n}(a))$ so $\mathbb{E}_n(a) = \underset{\longrightarrow}{\alpha^n}(a)$. Thus $\mathbb{E}_n$ is onto $\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$, and restricts to the identity on $\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$.

We now prove that P_n commutes with $\pi(a)$ for all $a \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$. Let $a \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$. We note that if $b \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ then $\pi(a)\xi(b) = \xi(ab) \in \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))$ since $\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ is a subalgebra of $\mathfrak{A}$. Thus $\pi(a)\left(\xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))\right) \subseteq \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))$. Since $\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ is closed under the adjoint operation, and π is a *-representation, we have $\pi(a^*)\xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)) \subseteq \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))$. Thus, if we let $x \in \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))^\perp$ and $y \in \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))$, we then have:

$$\langle \pi(a)x, y\rangle = \langle x, \pi(a^*)y\rangle = 0,$$

i.e. $\pi(a)(\xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))^\perp) \subseteq \xi(\underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n))^\perp$. Consequently, if $x \in L^2(\mathfrak{A}, \mu)$, writing $x = P_nx + P_n^\perp x$, we have:

$$P_n\pi(a)x = P_n\pi(a)P_nx + P_n\pi(a)P_n^\perp x = \pi(a)P_nx.$$

In other words, P_n commutes with $\pi(a)$ for all $a \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$.

As a consequence, for all $a \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ and $b \in \mathfrak{A}$:

$$\xi(\mathbb{E}_n(ab)) = P_n\pi(a)\xi(b) = \pi(a)P_n\xi(b) = \pi(a)\xi(\mathbb{E}_n(b)) = \xi(a\mathbb{E}_n(b)).$$

Thus $\mathbb{E}_n(ab) = a\mathbb{E}_n(b)$ for all $a \in \underset{\longrightarrow}{\alpha^n}(\mathfrak{A}_n)$ and $b \in \mathfrak{A}$.

We now wish to prove that $\mathbb{E}_n$ is a *-linear map. Let $J : \xi(x) \mapsto \xi(x^*)$. The key observation is that, since μ is a trace:

$$\langle J\xi(x), J\xi(y)\rangle = \mu(yx^*) = \mu(x^*y) = \langle x, y\rangle$$

Hence J is a conjugate-linear isometry and can be extended to $L^2(\mathfrak{A}, \mu)$. It is easy to check that J is surjective, as it has a dense range and is isometric, in fact $J = J^* = J^{-1}$. This is the only point where we use that μ is a trace.

We now check that P_n and J commute. To begin with, we note that:

$$(JP_nJ)(JP_nJ) = JP_nJ$$

and thus the self-adjoint operator JP_nJ is a projection. Let $a \in \mathfrak{A}$. Then:

$$JP_nJ\xi(a) = JP_n\xi(a^*) = J\xi(\mathbb{E}_n(a^*)) = \xi(\mathbb{E}_n(a^*)^*) \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n)).$$

Now, if $\xi(a) \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$, then since $\xi(a^*) \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$, we have:

$$JP_nJ\xi(a) = JP_n\xi(a^*) = J\xi(a^*) = \xi(a).$$

Therefore, the projection JP_nJ surjects onto $\xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$. Thus $JP_nJ = P_n$, so P_n and J commute since $J^2 = 1_{\mathfrak{B}(L^2(\mathfrak{A},\mu))}$.

Consequently for all $a \in \mathfrak{A}$:

$$\xi(\mathbb{E}_n(a^*)) = P_n\xi(a^*) = P_nJ\xi(a) = JP_n\xi(a) = J\xi(\mathbb{E}_n(a)) = \xi(\mathbb{E}_n(a)^*),$$

so $\mathbb{E}_n(a^*) = \mathbb{E}_n(a)^*$.

In particular, we note that for all $a \in \mathfrak{A}$ and $b, c \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ we have:

$$\mathbb{E}_n(bac) = b\mathbb{E}_n(ac) = b\mathbb{E}_n(c^*a^*)^* = b(c^*\mathbb{E}_n(a)^*)^*) = b\mathbb{E}_n(a)c.$$

To prove that $\mathbb{E}_n$ is a positive map, we begin by checking that it preserves the state μ. First note that $1_{\mathfrak{A}} \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ so $\omega \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$, and thus $P_n\omega = \omega$. Thus for all $a \in \mathfrak{A}$:

$$\begin{aligned}\mu(\mathbb{E}_n(a)) &= \langle \pi(\mathbb{E}_n(a))\omega, \omega\rangle \\ &= \langle \xi(\mathbb{E}_n(a)), \omega\rangle = \langle P_n\xi(a), \omega\rangle \\ &= \langle \xi(a), P_n\omega\rangle = \langle \pi(a)\omega, P_n\omega\rangle \\ &= \langle \pi(a)\omega, \omega\rangle = \mu(a).\end{aligned}$$

Hence $\mathbb{E}_n$ preserves the state μ. More generally, using the conditional expectation property, for all $b, c \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ and $a \in \mathfrak{A}$:

$$\mu(b\mathbb{E}_n(a)c) = \mu(bac).$$

We now prove that $\mathbb{E}_n$ is positive. First, μ restricts to a faithful state of $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ and $L^2(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mu)$ is given canonically by $\xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$ as $\xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n))$ is closed by finite-dimensionality. Next, fix $a \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$, define:

$$\pi_n(a) : \xi(x) \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n)) \longmapsto \xi(ax) \in \xi(\underrightarrow{\alpha}^n(\mathfrak{A}_n)),$$

which is well-defined since $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ is a subalgebra and ξ is injective. Also, we have that $\pi_n(a) \in \mathfrak{B}\left(L^2(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mu)\right)$ since $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ is a finite-dimensional subalgebra and ξ is linear. Now, define:

$$\pi_n : a \in \underrightarrow{\alpha}^n(\mathfrak{A}_n) \longmapsto \pi_n(a) \in \mathfrak{B}\left(L^2(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mu)\right), \tag{3.1.1}$$

which is a unital *-homomorphism by definition of π_n and the fact that π is a unital *-homomorphism. Now, assume that $a, b \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ such that $\pi_n(a) = \pi_n(b)$. Then:

$$\xi(a) = \xi(a1_{\mathfrak{A}}) = \pi_n(a)\xi(1_{\mathfrak{A}}) = \pi_n(a)\omega = \pi_n(b)\omega = \xi(b).$$

Since ξ is injective, we have $a = b$. Therefore, the map π_n is injective. Let now $a \in \mathfrak{sa}(\mathfrak{A})$ with $a \geqslant 0$, and so there exists $c \in \mathfrak{A}$ such that $a = c^*c$. We now have for all $b \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ that:

$$\begin{aligned}
\langle \pi_n(\mathbb{E}_n(a))\,\xi(b), \xi(b)\rangle &= \langle \xi(\mathbb{E}_n(a)b), \xi(b)\rangle \\
&= \mu(b^*\mathbb{E}_n(a)b) \\
&= \mu(b^*ab) \\
&= \mu(b^*c^*cb) \\
&= \mu((cb)^*cb) \geqslant 0.
\end{aligned}$$

Thus, the operator $\pi_n(\mathbb{E}_n(a)) \in \mathfrak{B}\left(L^2(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mu)\right)$ is positive and so $\mathbb{E}_n(a)$ is positive in $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ since π_n is a *-monomorphism. Hence $\mathbb{E}_n$ is positive.

Since $\mathbb{E}_n$ restricts to the identity on $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$, this map is of norm at least one. Now, let $a \in \mathfrak{sa}(\mathfrak{A})$ and $\varphi \in \mathscr{S}(\mathfrak{A})$. Then $\varphi \circ \mathbb{E}_n$ is a state of $\mathfrak{A}$ since $\mathbb{E}_n$ is positive and unital. Thus $|\varphi \circ \mathbb{E}_n(a)| \leqslant \|a\|_{\mathfrak{A}}$. As $\mathbb{E}_n(\mathfrak{sa}(\mathfrak{A})) \subseteq \mathfrak{sa}(\mathfrak{A})$, we have:

$$\forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \|\mathbb{E}_n(a)\|_{\mathfrak{A}} = \sup\{|\varphi \circ \mathbb{E}_n(a)| : \varphi \in \mathscr{S}(\mathfrak{A})\} \leqslant \|a\|_{\mathfrak{A}}. \tag{3.1.2}$$

Thus $\mathbb{E}_n$ restricted to $\mathfrak{sa}(\mathfrak{A})$ is a linear map of norm 1.

On the other hand, for all $a \in \mathfrak{A}$, we have:

$$\begin{aligned}
0 &\leqslant \mathbb{E}_n\left((a - \mathbb{E}_n(a))^*(a - \mathbb{E}_n(a))\right) \\
&= \mathbb{E}_n(a^*a) - \mathbb{E}_n(\mathbb{E}_n(a)^*a) - \mathbb{E}_n(a^*\mathbb{E}_n(a)) + \mathbb{E}_n(\mathbb{E}_n(a)^*\mathbb{E}_n(a)) \\
&= \mathbb{E}_n(a^*a) - \mathbb{E}_n(a)^*\mathbb{E}_n(a).
\end{aligned}$$

Thus for all $a \in \mathfrak{A}$ we have:

$$\|\mathbb{E}_n(a)\|_{\mathfrak{A}}^2 = \|\mathbb{E}_n(a)^*\mathbb{E}_n(a)\|_{\mathfrak{A}}$$

$$
\begin{aligned}
&\leqslant \|\mathbb{E}_n(a^*a)\|_{\mathfrak{A}} \\
&\leqslant \|a^*a\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2 \text{ by Inequality (3.1.2).}
\end{aligned}
$$

Hence $\mathbb{E}_n$ has norm 1. We conclude that $\mathbb{E}_n$ is a conditional expectation onto $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ which preserves μ.

Now, assume $T : \mathfrak{A} \to \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ is a unital conditional expectation such that $\mu \circ T = \mu$. As before, we have:

$$\mu(bT(a)c) = \mu(bac)$$

for all $a \in \mathfrak{A}$ and $b, c \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$. Thus, for all $x, y \in \underrightarrow{\alpha}^n(\mathfrak{A}_n)$ and for all $a \in \mathfrak{A}$, we compute:

$$
\begin{aligned}
\langle \pi_n(T(a))\xi(x), \xi(y)\rangle &= \langle \xi(T(a)x), \xi(y)\rangle \\
&= \mu(y^*T(a)x) \\
&= \mu(y^*ax) \\
&= \mu(y^*\mathbb{E}_n(a)x) \\
&= \langle \xi(\mathbb{E}_n(a)x), \xi(y)\rangle = \langle \pi_n(\mathbb{E}_n(a))\xi(x), \xi(y)\rangle,
\end{aligned}
$$

where π_n was defined in Expression (3.1.1), and thus $\pi_n(\mathbb{E}_n(a)) = \pi_n(T(a))$. Hence, we have that $\mathbb{E}_n(a) = T(a)$ since π_n is a *-monomorphism. As $a \in \mathfrak{A}$ was arbitrary, the map $\mathbb{E}_n$ is the unique conditional expectation from $\mathfrak{A}$ onto $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ which preserves μ.

Step 2. *The seminorm $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is a $(2,0)$-quasi-Leibniz Lip-norm on $\mathfrak{A}$, and $\mathbb{E}_n$ is weakly contractive for $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ and for all $n \in \mathbb{N}$.*

We conclude from Lemma (3.1.2) and from Step 1 that $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is a $(2,0)$-quasi-Leibniz seminorm.

If $a \in \mathfrak{sa}(\mathfrak{A})$ and $\mathsf{L}_{\mathcal{I},\mu}^{\beta}(a) = 0$ then $\|a - \mathbb{E}_0(a)\|_{\mathfrak{A}} = 0$ and thus $a \in \mathfrak{sa}\left(\underrightarrow{\alpha}^0(\mathbb{C})\right) = \mathbb{R}1_{\mathfrak{A}}$.

We also note that if $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathcal{I},\mu}^{\beta}(a) \leqslant 1$ then $\|a - \mathbb{E}_0(a)\|_{\mathfrak{A}} \leqslant \beta(0)$. Note that $\mathbb{E}_0(a) = \mu(a)1_{\mathfrak{A}}$ as $\mathbb{E}_0$ preserves μ.

For all $n, p \in \mathbb{N}$ we have $\mathbb{E}_p \circ \mathbb{E}_n = \mathbb{E}_{\min\{n,p\}}$ by construction (since $P_n P_p = P_{\min\{n,p\}}$). Thus, if $n \leqslant p$ and $a \in \mathfrak{sa}(\mathfrak{A})$ then:

$$\|\mathbb{E}_n(a) - \mathbb{E}_p(\mathbb{E}_n(a))\|_{\mathfrak{A}} = 0. \tag{3.1.3}$$

In particular, we conclude that the dense Jordan-Lie subalgebra $\mathfrak{sa}\left(\bigcup_{n\in\mathbb{N}} \underrightarrow{\alpha}^n(\mathfrak{A}_n)\right)$ of $\mathfrak{sa}(\mathfrak{A})$ is included in the domain $\mathrm{dom}\left(\mathsf{L}_{\mathcal{I},\mu}^{\beta}\right)$ of $\mathsf{L}_{\mathcal{I},\mu}^{\beta}$ and thus $\mathrm{dom}\left(\mathsf{L}_{\mathcal{I},\mu}^{\beta}\right)$ is dense in $\mathfrak{sa}(\mathfrak{A})$.

On the other hand, if $p \leqslant n \in \mathbb{N}$ and $a \in \mathfrak{sa}(\mathfrak{A})$, then:

$$\|\mathbb{E}_n(a) - \mathbb{E}_p(\mathbb{E}_n(a))\|_{\mathfrak{A}} = \|\mathbb{E}_n(a - \mathbb{E}_p(a))\|_{\mathfrak{A}} \leqslant \|a - \mathbb{E}_p(a)\|_{\mathfrak{A}}. \tag{3.1.4}$$

Thus, by Expressions (3.1.3) and (3.1.4), for all $a \in \mathfrak{sa}(\mathfrak{A})$,

$$\mathsf{L}_{\mathcal{I},\mu}^{\beta}(\mathbb{E}_n(a)) \leqslant \mathsf{L}_{\mathcal{I},\mu}^{\beta}(a). \tag{3.1.5}$$

Last, let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\beta(n) < \frac{\varepsilon}{2}$. Let:

$$\mathfrak{B}_N = \left\{a \in \mathfrak{sa}(\mathfrak{A}_N) : \mathsf{L}_{\mathcal{I},\mu}^{\beta}(\underrightarrow{\alpha}^N(a)) \leqslant 1, \mu(a) = 0\right\}.$$

Since $\mathbb{E}_0 = \mu(\cdot)1_{\mathfrak{A}}$, we conclude:

$$\mathfrak{B}_N \subseteq \{a \in \mathfrak{sa}(\mathfrak{A}_N) : \|a\|_{\mathfrak{A}} \leqslant \beta(0)\},$$

and since a closed ball in $\mathfrak{sa}(\mathfrak{A}_N)$ is compact as $\mathfrak{A}_N$ is finite dimensional, we conclude that $\mathfrak{B}_N$ is totally bounded. Let $\mathfrak{F}_N$ be a finite $\frac{\varepsilon}{2}$-dense subset of $\mathfrak{B}_N$. Let now $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mu(a) = 0$ and $\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) \leqslant 1$. By definition of $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ we have $\|a - \mathbb{E}_N(a)\|_{\mathfrak{A}} \leqslant \beta(N) < \frac{\varepsilon}{2}$. Moreover, there exists $a' \in \mathfrak{F}_N$ such that $\|\mathbb{E}_N(a) - a'\|_{\mathfrak{A}} \leqslant \frac{\varepsilon}{2}$. Thus:

$$\|a - a'\|_{\mathfrak{A}} \leqslant \varepsilon,$$

and so:

$$\left\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) \leqslant 1, \mu(a) = 0\right\}$$

is totally bounded. Thus $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is a Lip-norm on $\mathfrak{A}$.

We conclude with the observation that as the pointwise supremum of continuous real-valued functions, $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is lower semi-continuous on $\mathfrak{sa}(\mathfrak{A})$ with respect to $\|\cdot\|_{\mathfrak{A}}$ since $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is the pointwise supremum of the continuous functions E^n for all $n \in \mathbb{N}$, where E^n is defined by $E^n(a) = \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta(n)}$ for all $a \in \mathfrak{sa}(\mathfrak{A})$ and for all $n \in \mathbb{N}$.

Step 3. *If $n \in \mathbb{N}$, then $\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right)$ is a $(2,0)$-quasi-Leibniz quantum compact metric space and $\Lambda_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right) \leqslant \beta(n)$.*

The restriction of $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ to $\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ is a $(2,0)$-quasi-Leibniz lower semi-continuous Lip-norm on $\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. We now prove our estimate on $\Lambda_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right)$.

The spaces $\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right)$ and $\left(\underrightarrow{\alpha^n}(\mathfrak{A}_n), \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)$ are quantum isometric via the unital *-isomorphism $\underrightarrow{\alpha^n} : \mathfrak{A}_n \longrightarrow \underrightarrow{\alpha^n}(\mathfrak{A}_n)$ and thus at distance zero for $\Lambda_{2,0}$. Therefore:

$$\Lambda_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right), \left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha^n}\right)\right) = \Lambda_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right), \left(\underrightarrow{\alpha^n}(\mathfrak{A}_n), \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right).$$

Let $\mathrm{id} : \mathfrak{A} \to \mathfrak{A}$ be the identity and let $\iota_n : \underrightarrow{\alpha^n}(\mathfrak{A}_n) \to \mathfrak{A}$ be the inclusion map. The quadruple $\gamma = (\mathfrak{A}, 1_{\mathfrak{A}}, \iota_n, \mathrm{id})$ is a bridge from $\underrightarrow{\alpha^n}(\mathfrak{A}_n)$ to $\mathfrak{A}$ by Definition (2.3.2). We note that by definition, the height of γ is 0 since the pivot of γ is $1_{\mathfrak{A}}$. Thus, the length of γ is the reach of γ. If $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) \leqslant 1$, then:

$$\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}} \leqslant \beta(n).$$

Since $\mathbb{E}_n$ is *-linear, we thus have $\mathbb{E}_n(a) \in \mathfrak{sa}\left(\underrightarrow{\alpha}^n(\mathfrak{A}_n)\right)$. By Equation (3.1.5):

$$\mathsf{L}^{\beta}_{\mathcal{I},\mu}(\mathbb{E}_n(a)) \leqslant 1.$$

Since $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ is contained in $\mathfrak{A}$, we conclude that the reach of γ is no more than $\beta(n)$.

We thus conclude, by Theorem-Definition (2.3.16):

$$\mathsf{\Lambda}_{2,0}\left(\left(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right) \leqslant \beta(n).$$

As $(\beta(n))_{n\in\mathbb{N}}$ converges to 0, we conclude that:

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha}^n\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right)\right) = 0,$$

and thus our theorem is proven. □

Remark 3.1.4. *We may employ similar techniques as used in the proof of Theorem (3.1.3) to show that AF algebras, equipped with the Lip-norms defined from spectral triples in [5], are limits of finite dimensional C*-subalgebras. We shall see in this dissertation, however, that the Lip-norms we introduce in Theorem (3.1.3) provide a very natural framework to study the quantum metric properties of AF algebras.*

It will also be useful for us to present Theorem (3.1.3) in the setting of the definition of AF algebras given in Definition (2.1.72). This is the following.

Theorem 3.1.5. *Using Definition (2.1.72), let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$ endowed with a faithful tracial state μ. Let $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ be an increasing* *sequence of unital finite dimensional C*-subalgebras such that* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *with* $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$.

Let π *be the GNS representation of* $\mathfrak{A}$ *constructed from* μ *on the space* $L^2(\mathfrak{A}, \mu)$ *Theorem (2.1.40). For all* $n \in \mathbb{N}$*, let:*

$$\mathbb{E}\left(\cdot|\mathfrak{A}_n\right) : \mathfrak{A} \to \mathfrak{A}$$

be the unique conditional expectation of $\mathfrak{A}$ *onto* $\mathfrak{A}_n$ *, and such that* $\mu \circ \mathbb{E}\left(\cdot|\mathfrak{A}_n\right) = \mu$.

Let $\beta : \mathbb{N} \to (0, \infty)$ *have limit* 0 *at infinity. If, for all* $a \in \mathfrak{sa}(\mathfrak{A})$*, we set:*

$$\mathsf{L}^{\beta}_{\mathcal{U},\mu}(a) = \sup\left\{\frac{\|a - \mathbb{E}\left(a|\mathfrak{A}_n\right)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N}\right\},$$

then $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)$ *is a* $(2,0)$*-quasi-Leibniz quantum compact metric space of Definition (2.2.9). Moreover, for all* $n \in \mathbb{N}$*:*

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)\right) \leqslant \beta(n)$$

and thus:

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)\right) = 0,$$

where $\mathsf{\Lambda}_{2,0}$ *is the quantum Gromov-Hausdorff propinquity of Theorem-Definition (2.3.16).*

Proof. The proof follows the same process of the proof of Theorem (3.1.3). $\square$

The Lip-norms of Theorem (3.1.5) are compatible with the Lip-norms in the inductive limit case of Theorem (3.1.3). The next Proposition (3.1.6) establishes what we mean by this.

Proposition 3.1.6. *Let* $\mathfrak{A}$ *be a unital AF algebra endowed with a faithful tracial state* μ*. Let* $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ *is an inductive sequence of finite dimensional*

*C^*-algebras with C^*-inductive limit $\mathfrak{A}$, with $\mathfrak{A}_0 \cong \mathbb{C}$ and where α_n is unital *-monomorphism for all $n \in \mathbb{N}$. Let $\beta : \mathbb{N} \to (0, \infty)$ have limit 0 at infinity.*

If we define $\mathcal{U} = (\underrightarrow{\alpha}^n(\mathfrak{A}_n))_{n\in\mathbb{N}}$, then the sequence $\mathcal{U}$ is an increasing sequence of unital finite dimensional C^*-subalgebras of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\underrightarrow{\alpha}^n(\mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$ with $\underrightarrow{\alpha}^0(\mathfrak{A}_0) = \mathbb{C}1_{\mathfrak{A}}$, and the Lip-norms $\mathsf{L}^{\beta}_{\mathcal{I},\mu} = \mathsf{L}^{\beta}_{\mathcal{U},\mu}$, where $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ is defined by Theorem (3.1.3) and $\mathsf{L}^{\beta}_{\mathcal{U},\mu}$ is defined by (3.1.5).

Proof. By Proposition (2.1.66), the sequence $\mathcal{U}$ is an increasing sequence of unital finite dimensional C*-subalgebras of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\underrightarrow{\alpha}^n(\mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$ with $\underrightarrow{\alpha}^0(\mathfrak{A}_0) = \mathbb{C}1_{\mathfrak{A}}$. The equality of the Lip-norms $\mathsf{L}^{\beta}_{\mathcal{I},\mu} = \mathsf{L}^{\beta}_{\mathcal{U},\mu}$ follows by definition. $\square$

Theorem (3.1.3) provides infinitely many Lip-norms on any given unital AF-algebra $\mathfrak{A}$, parametrized by a choice of an inductive sequence converging to $\mathfrak{A}$ and a sequence with positive entries which converges to 0. A natural choice of a Lip-norm for a given AF algebra, which will occupy a central role in our current work, is described in the following notation.

Notation 3.1.7. *Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be a unital inductive sequence of finite dimensional algebras whose inductive limit $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ has a faithful tracial state μ. Assume that $\mathfrak{A}$ is infinite dimensional. Let $k \in \mathbb{N}$, $k > 0$ and $\beta = \left(\frac{1}{\dim(\mathfrak{A}_n)^k}\right)_{n\in\mathbb{N}}$. We note that $\lim_\infty \beta = 0$. We denote the Lip-norm $\mathsf{L}^{\beta}_{\mathcal{I},\mu}$ constructed in Theorem (3.1.3) by $\mathsf{L}^{k}_{\mathcal{I},\mu}$. If $k = 1$, then we simply write $\mathsf{L}_{\mathcal{I},\mu}$ for $\mathsf{L}^{1}_{\mathcal{I},\mu}$.*

Our purpose is the study of various classes of AF algebras, equipped with Lip-norms constructed in Theorem (3.1.3). The following notation will prove useful.

Notation 3.1.8. *The class of all $(2,0)$-quasi-Leibniz quantum compact metric spaces constructed in Theorem (3.1.3) is denoted by $\mathcal{AF}$. We shall endow $\mathcal{AF}$ with the topology induced by the quantum propinquity $\mathsf{\Lambda}_{2,0}$.*

Furthermore, for any $k \in (0,\infty)$, let:

$$\mathcal{AF}^k := \left\{ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{AF} \;\middle|\; \begin{array}{l} \exists \mathcal{I} \in \mathscr{Inductive\text{-}f\text{-}d} \;\; \mathfrak{A} = \varinjlim \mathcal{I} \\ \exists \mu \textit{ faithful trace on } \mathfrak{A} \textit{ such that } \mathsf{L}_{\mathfrak{A}} = \mathsf{L}^k_{\mathcal{I},\mu} \\ \mathfrak{A} \textit{ is infinite dimensional} \end{array} \right\}$$

where $\mathscr{Inductive\text{-}f\text{-}d}$ is the class of all unital inductive sequences of finite dimensional C^*-algebras whose limit has at least one faithful tracial state.

A first corollary of Theorem (3.1.3) concerns some basic geometric properties of the class $\mathcal{AF}^k$.

Corollary 3.1.9. *Let $\mathcal{I}, \mathcal{J} \in \mathscr{Inductive\text{-}f\text{-}d}$ and β, β' be two sequences of strictly positive real numbers, converging to 0. Let μ, ν be faithful tracial states, respectively, on $\varinjlim \mathcal{I}$ and $\varinjlim \mathcal{J}$. Then:*

$$\operatorname{diam}\left(\mathscr{S}\left(\varinjlim \mathcal{I}\right), \mathsf{mk}_{\mathsf{L}^{\beta}_{\mathcal{I},\mu}} \right) \leqslant 2\beta(0),$$

where $\operatorname{diam}(,)$ *is the diameter of a metric space, and:*

$$\Lambda_{2,0}\left(\left(\varinjlim \mathcal{I}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right), \left(\varinjlim \mathcal{J}, \mathsf{L}^{\beta'}_{\mathcal{J},\nu}\right) \right) \leqslant \max\{\beta(0), \beta'(0)\}.$$

In particular, for all $k \in (0,\infty)$:

$$\operatorname{diam}\left(\mathcal{AF}^k, \Lambda_{2,0}\right) \leqslant 1.$$

Proof. Let $\mathfrak{A} = \varinjlim \mathcal{I}$ and $\mathfrak{B} = \varinjlim \mathcal{J}$.

Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) \leqslant 1$. Then $\|a - \mu(a)\|_{\mathfrak{A}} \leqslant \beta(0)$. Thus for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we have:

$$|\varphi(a) - \psi(a)| = |\varphi(a - \mu(a)1_{\mathfrak{A}}) - \psi(a - \mu(a)1_{\mathfrak{A}})| \leqslant 2\beta(0).$$

Now, let $\mathfrak{D} = \mathfrak{A} \otimes \mathfrak{B}$ be any C*-algebra formed over the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$, which exists by [55, Chapter 6.3]. Let $\pi : a \in \mathfrak{A} \longmapsto a \otimes 1_{\mathfrak{B}} \in \mathfrak{D}$ and $\rho : b \in \mathfrak{B} \longmapsto 1_{\mathfrak{A}} \otimes b \in \mathfrak{D}$ be the canonical unital *-monomorphisms. The quadruple $\gamma = (\mathfrak{D}, 1_{\mathfrak{D}}, \pi, \rho)$ is a bridge from $\mathfrak{A}$ to $\mathfrak{B}$.

Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) \leqslant 1$. Then:

$$\|\pi(a)1_{\mathfrak{D}} - 1_{\mathfrak{D}}\rho(\mu(a)1_{\mathfrak{B}})\|_{\mathfrak{D}} = \|a - \mu(a)1_{\mathfrak{A}}\|_{\mathfrak{A}} \leqslant \beta(0).$$

The result is symmetric in $\mathfrak{A}$ and $\mathfrak{B}$. Thus the reach of γ is no more than $\max\{\beta(0), \beta'(0)\}$. As the height of γ is zero, we have proven that:

$$\mathsf{\Lambda}_{2,0}\left(\left(\varinjlim \mathcal{I}, \mathsf{L}^{\beta}_{\mathcal{I},\mu}\right), \left(\varinjlim \mathcal{J}, \mathsf{L}^{\beta'}_{\mathcal{J},\nu}\right)\right) \leqslant \max\{\beta(0), \beta'(0)\},$$

by Theorem-Definition (2.3.16). Note that this last estimate is slightly better than what we would obtain with [46, Proposition 4.6].

We conclude our proof noting that if $(\mathfrak{A}, \mathsf{L}_{\mathcal{I}}) \in \mathcal{AF}^k$ then $\beta(0) = 1$. $\square$

We complete this section of taking note of the fact that the conditional expectations of Theorem (3.1.3) can be expressed explicitly in terms of matrix units, and we provide some useful continuity results associated to these conditional expectations. This valuable tool will be used throughout this dissertation.

Notation 3.1.10. *For all $d \in \mathbb{N}$, we denote the full matrix algebra of $d \times d$ matrices over $\mathbb{C}$ by $\mathfrak{M}(d)$. Let $\mathfrak{B} = \oplus_{j=1}^{N} \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \ldots, n(N) \in \mathbb{N} \setminus \{0\}$. For each $k \in \{1, \ldots, N\}$ and for each $j, m \in \{1, \ldots, n(k)\}$, we denote the matrix $((\delta^{j,m}_{u,v}))_{u,v=1,\ldots,n(k)}$ by $e_{k,j,m}$, a* matrix unit*, where we used the Kronecker symbol:*

$$\delta^{b}_{a} = \begin{cases} 1 \ \textit{if } a = b, \\ 0 \ \textit{otherwise.} \end{cases}$$

We note that for all $j, m, j', m' \in \{1, \ldots, n(k)\}$ we have:

$$\mathrm{tr}\left(e_{k,j,m}^* e_{k,j',m'}\right) = \begin{cases} \frac{1}{n(k)} \text{ if } j = j' \text{ and } m = m', \\ 0 \text{ otherwise} \end{cases}$$

when tr is the unique tracial state of $\mathfrak{M}(n(k))$.

Now, let μ be a faithful tracial state on $\mathfrak{B}$ of the above Notation (3.1.10). Then μ is a convex combination with positive coefficients of the unique tracial states on $\mathfrak{M}(n(0)), \ldots, \mathfrak{M}(n(N))$ by [19, Example IV.5.4]. We thus deduce that:

$$\{e_{k,j,m} : k \in \{1, \ldots, N\}, j, m \in \{1, \ldots, n(k)\}\}$$

is an orthogonal basis of $L^2(\mathfrak{B}, \mu)$.

Let us further assume that we are given a unital *-monomorphism $\alpha : \mathfrak{B} \hookrightarrow \mathfrak{A}$ into a unital C*-algebra $\mathfrak{A}$ with a faithful tracial state. The restriction of μ to $\alpha(\mathfrak{B})$ is thus a faithful tracial state on $\alpha(\mathfrak{B})$ and $\mu \circ \alpha$ is a faithful tracial state on $\mathfrak{B}$. We will use the notations of the proof of Theorem (3.1.3): let π be the GNS representation of $\mathfrak{A}$ defined by μ on the Hilbert space $L^2(\mathfrak{A}, \mu)$ and let $\xi : a \in \mathfrak{A} \to a + \{0_{\mathfrak{A}}\} \in L^2(\mathfrak{A}, \mu)$.

We then can regard $L^2(\alpha(\mathfrak{B}), \mu)$ as a subspace of $L^2(\mathfrak{A}, \mu)$ (as noted in the proof of Theorem (3.1.3), $L^2(\alpha(\mathfrak{B}), \mu)$ is $\alpha(\mathfrak{B}) + \{0_{\mathfrak{A}}\}$, endowed with the Hermitian norm from the inner product defined by μ). Let P be the projection of $L^2(\mathfrak{A}, \mu)$ on $L^2(\alpha(\mathfrak{B}), \mu)$. Then for all $a \in \mathfrak{A}$, we have:

$$\begin{aligned} P\xi(a) &= \sum_{k=1}^{N} \sum_{j=1}^{n(k)} \sum_{m=1}^{n(k)} \frac{\langle \xi(a), \xi\left(\alpha(e_{k,j,m})\right)\rangle}{\langle \xi\left(\alpha(e_{k,j,m}), \xi\left(\alpha(e_{k,j,m})\right)\right)\rangle} \xi\left(\alpha(e_{k,j,m})\right) \\ &= \sum_{k=1}^{N} \sum_{j=1}^{n(k)} \sum_{m=1}^{n(k)} \frac{\mu(\alpha(e_{k,j,m}^*)a)}{\mu(\alpha(e_{k,j,m}^* e_{k,j,m}))} \xi\left(\alpha(e_{k,j,m})\right) \\ &= \xi\left(\sum_{k=1}^{N} \sum_{j=1}^{n(k)} \sum_{m=1}^{n(k)} \frac{\mu(\alpha(e_{k,j,m}^*)a)}{\mu(\alpha(e_{k,j,m}^* e_{k,j,m}))} \alpha(e_{k,j,m})\right) \end{aligned} \tag{3.1.6}$$

since P is an orthogonal projection and ξ is linear. Next, if $\mathbb{E}(\cdot|\alpha(\mathfrak{B}))$ is the conditional expectation of $\mathfrak{A}$ onto $\alpha(\mathfrak{B})$ which preserves μ constructed from the Jones' projection P as in Theorem (3.1.3), then $\xi(\mathbb{E}(a|\alpha(\mathfrak{B}))) = P\xi(a)$ for all $a \in \mathfrak{A}$. Hence, by injectivity of ξ and Expression (3.1.6), we have that:

$$\mathbb{E}(a|\alpha(\mathfrak{B})) = \sum_{k=1}^{N}\sum_{j=1}^{n(k)}\sum_{m=1}^{n(k)} \frac{\mu(\alpha(e^*_{k,j,m})a)}{\mu(\alpha(e^*_{k,j,m}e_{k,j,m}))}\alpha(e_{k,j,m}). \tag{3.1.7}$$

Now, we present some preliminary continuity results of these conditional expectations which will prove crucial in the continuity results of AF algebras, which is Theorem (4.5.6).

Notation 3.1.11. *Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ denote the Alexandroff compactification of $\mathbb{N}$ with respect to the discrete topology of $\mathbb{N}$. For $N \in \mathbb{N}$, let $\mathbb{N}_{\geqslant N} = \{k \in \mathbb{N} : k \geqslant N\}$, and similarly, for $\overline{\mathbb{N}}_{\geqslant N}$.*

Lemma 3.1.12. *Let $\mathfrak{A} = \oplus_{j=1}^{N}\mathfrak{M}(d(j))$ for some $N \in \mathbb{N} \setminus \{0\}$ and $d(1), \ldots, d(N) \in \mathbb{N}$. Let $\{\tau^n : \mathfrak{A} \longrightarrow \mathbb{C}\}_{n\in\overline{\mathbb{N}}}$ be a family of tracial states. Since τ^n is a tracial state for all $n \in \overline{\mathbb{N}}$, for each $(n, j) \in \overline{\mathbb{N}} \times \{1, \ldots, N\}$, let $\lambda_{n,j} \in [0, 1]$ such that:*

$$\tau^n(a_1, \ldots, a_N) = \sum_{j=1}^{N} \lambda_{n,j}\mathsf{tr}_{d(j)}(a_j), \ \forall(a_1, \ldots, a_N) \in \mathfrak{A},$$

where $\mathsf{tr}_{d(j)}$ is the unique normalized trace on $\mathfrak{M}(d(j))$.

Then, $(\tau^n)_{n\in\mathbb{N}}$ converges to τ^∞ in the weak topology on $\mathscr{S}(\mathfrak{A})$ if and only if $((\lambda_{n,1}, \lambda_{n,2}, \ldots \lambda_{n,N}))_{n\in\mathbb{N}}$ converges to $(\lambda_{\infty,1}, \lambda_{\infty,2}, \ldots, \lambda_{\infty,N})$ in the product topology on $\mathbb{R}^N$.*

Proof. We begin with the forward implication. For $j \in \{1, 2, \ldots, N\}$, let $I^j = (b_1, b_2, \ldots, b_N) \in \mathfrak{A}$ such that $b_l = 0$ for $l \neq j$ and $b_j = 1_{\mathfrak{M}(d(j))}$. By assumption, for each $j \in \{1, 2, \ldots, N\}$, we have that:

$$\lim_{n\to\infty} \lambda_{n,j} = \lim_{n\to\infty} \lambda_{n,j}\mathsf{tr}_{d(j)}\left(1_{\mathfrak{M}(d(j))}\right) = \lim_{n\to\infty} \tau^n(I^j) = \tau^\infty(I^j) = \lambda_{\infty,j},$$

which also provides convergence in the product topology since the product is finite.

For the reverse implication, fix $b = (b_1, b_2, \ldots, b_N) \in \mathfrak{A}$. Fix $n \in \mathbb{N}$.

$$\begin{aligned}
|\tau^n(b) - \tau^\infty(b)| &= \left| \left(\sum_{j=1}^N \lambda_{n,j}\mathsf{tr}_{d(j)}(b_j) \right) - \left(\sum_{j=1}^N \lambda_{\infty,j}\mathsf{tr}_{d(j)}(b_j) \right) \right| \\
&= \left| \sum_{j=1}^N (\lambda_{n,j} - \lambda_{\infty,j})\mathsf{tr}_{d(j)}(b_j) \right| \\
&\leqslant \left(\sum_{j=1}^N |\lambda_{n,j} - \lambda_{\infty,j}| \right) \|b\|_{\mathfrak{A}}.
\end{aligned}$$

By convergence in the product topology, the sequence $\left(\sum_{j=1}^N |\lambda_{n,j} - \lambda_{\infty,j}|\right)_{n\in\mathbb{N}}$ converges to 0. Hence, $\lim_{n\to\infty} |\tau^n(b) - \tau^\infty(b)| = 0$. As $b \in \mathfrak{A}$ was arbitrary, our result is proven. $\square$

Next, we consider convergence of conditional expectations on finite-dimensional C*-algebras. We note that in the hypothesis of Proposition (3.1.13), we now impose that our tracial states are faithful.

Proposition 3.1.13. *Let $\mathfrak{B}$ be a unital C^*-algebra. Let $\mathfrak{A}$ be a finite-dimensional unital C^*-subalgebra of $\mathfrak{B}$ such that $\mathfrak{A} \cong \oplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \ldots, n(N) \in \mathbb{N} \setminus \{0\}$ with *-isomorphism $\alpha : \oplus_{j=1}^N \mathfrak{M}(n(j)) \longrightarrow \mathfrak{A}$. Let E be the set of matrix units for $\oplus_{j=1}^N \mathfrak{M}(n(j))$ of Notation (3.1.10).*

If $\{\tau^n : \mathfrak{B} \to \mathbb{C}\}_{n\in\overline{\mathbb{N}}}$ is a family of faithful tracial states, then for all $n \in \overline{\mathbb{N}}, b \in \mathfrak{B}$:

$$\mathbb{E}^n(b) = \sum_{e\in E} \frac{\tau^n(\alpha(e^*)b)}{\tau^n(\alpha(e^*e))}\alpha(e),$$

where $\mathbb{E}^n : \mathfrak{B} \to \mathfrak{A}$ *is the unique* τ^n*-preserving conditional expectation onto* $\mathfrak{A}$*. Furthermore, if* $(\tau^n)_{n\in\mathbb{N}}$ *converges to* τ^∞ *in the weak-* topology on* $\mathscr{S}(\mathfrak{B})$*, then the map:*

$$(n, b) \in \overline{\mathbb{N}} \times \mathfrak{B} \longmapsto \|b - \mathbb{E}^n(b)\|_{\mathfrak{B}} \in \mathbb{R},$$

is continuous with respect to the product topology on $\overline{\mathbb{N}} \times (\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$*.*

Proof. For $n \in \overline{\mathbb{N}}$, by Expression (3.1.7), we have that for each $n \in \overline{\mathbb{N}}$:

$$\mathbb{E}^n(b) = \sum_{e\in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))}\alpha(e)$$

since τ^n is a faithful tracial state on $\mathfrak{B}$. By faithfulness, for $e \in E$, we have $\lim_{n\to\infty} \tau^n(\alpha(e^*e)) = \tau^\infty(\alpha(e^*e)) > 0$ by weak-* convergence. Since our sum is finite by finite dimensionality, again by weak-* convergence:

$$\lim_{n\to\infty} \sum_{e\in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} = \sum_{e\in E} \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \tag{3.1.8}$$

Furthermore, if we let $C = \max_{e\in E} \{\|\alpha(e)\|_{\mathfrak{B}}\}$, then:

$$\begin{aligned}
\|\mathbb{E}^n(b) - \mathbb{E}^\infty(b)\|_{\mathfrak{B}} &= \left\| \sum_{e\in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))}\alpha(e) - \sum_{e\in E} \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))}\alpha(e) \right\|_{\mathfrak{B}} \\
&= \left\| \sum_{e\in E} \left(\frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} - \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \right) \alpha(e) \right\|_{\mathfrak{B}} \\
&\leqslant \sum_{e\in E} \left\| \left(\frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} - \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \right) \alpha(e) \right\|_{\mathfrak{B}} \\
&= \sum_{e\in E} \left| \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} - \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \right| \|\alpha(e)\|_{\mathfrak{B}} \\
&\leqslant \left(\sum_{e\in E} \left| \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} - \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \right| \right) C,
\end{aligned}$$

and $\lim_{n\to\infty} \|\mathbb{E}^n(b) - \mathbb{E}^\infty(b)\|_{\mathfrak{B}} = 0$ by Expression (3.1.8).

Fix $n, m \in \overline{\mathbb{N}}$ and $b, b' \in \mathfrak{B}$. Then, as conditional expectations are contractive:

$$\begin{aligned}\left| \|b - \mathbb{E}^n(b)\|_{\mathfrak{B}} - \left\|b' - \mathbb{E}^m(b')\right\|_{\mathfrak{B}} \right| &\leqslant \left\|(b - \mathbb{E}^n(b)) - (b' - \mathbb{E}^m(b'))\right\|_{\mathfrak{B}} \\ &\leqslant \left\|\mathbb{E}^n(b) - \mathbb{E}^n(b') + \mathbb{E}^n(b') - \mathbb{E}^m(b')\right\|_{\mathfrak{B}} \\ &\quad + \left\|b - b'\right\|_{\mathfrak{B}} \\ &\leqslant 2\left\|b - b'\right\|_{\mathfrak{B}} + \left\|\mathbb{E}^n(b') - \mathbb{E}^m(b')\right\|_{\mathfrak{B}},\end{aligned}$$

and continuity follows. □

3.1.1 Continuous functions on the Cantor Set

As is standard practice in noncommutative geometry, we first look at the commutative case of our construction in the previous section to verify that we recover a natural classical structure. Since our focus is on AF algebras, we note that by [12, Proposition 3.1], the C*-algebra $C(X)$ is AF if and only if X is a totally disconnected compact metric space. The canonical case of this is when $X = \mathcal{C}$ is the Cantor space, the space of sequences of 0's and 1's from Example (2.1.76). We call this the canonical case since every totally disconnected compact metric space is homeomorphic to a closed subspace of $\mathcal{C}$ [71, Section 30]. Now, the Cantor space comes equipped with many standard ultrametrics [34, Proposition 9]. Namely, for each $r \in (1, \infty) \subseteq \mathbb{R}$, the following is an ultrametric on $\mathcal{C}$ that metrizes its topology given in Example (2.1.76):

$$\mathsf{d}_{\mathcal{C},r}(x, y) = \begin{cases} 0 & : \text{ if } x = y \\ r^{-n} & : \text{ otherwise , where } n = \min\{n \in \mathbb{N} : x_n \neq y_n\}. \end{cases} \tag{3.1.9}$$

Next, with this ultrametric on $\mathcal{C}$, the C*-algebra $C(\mathcal{C})$ already comes equipped with a Lip-norm, which is the Lipschitz seminorm $\mathsf{L}_{\mathsf{d}_{\mathcal{C},r}}$ induced by $\mathsf{d}_{\mathcal{C},r}$, and its associated Monge-Kantorovich metric $\mathsf{mk}_{\mathsf{L}_{\mathsf{d}_{\mathcal{C},r}}}$ recovers the metric $\mathsf{d}_{\mathcal{C},r}$ via the Dirac point masses δ_x by Theorem (2.2.10). Thus, in this section, we show that we can recover this classical case by using the Lip-norms of Theorem (3.1.5) with suitable choices of the β sequences and a fixed choice of sequence of subalgebras and fixed faithful tracial state. This will be Corollary (3.1.19).

By Example (2.1.76), we have our standard description of $C(\mathcal{C})$ as an AF algebra, and thus a specific increasing sequence of finite-dimensional unital C*-subalgebras to use in Theorem (3.1.5). However, in order to construct our Lip-norm of Theorem (3.1.5), we also require a particular choice of a faithful tracial state; as $C(\mathcal{C})$ is Abelian, we have quite some choice of such states. We will focus our attention on a specific construction, which comes from a classic measure.

Lemma 3.1.14. *The set $\mathcal{C} = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ is a group for the pointwise addition modulo 1. As $\mathcal{C}$ is compact with its natural topology given by the product topology on $\prod_{n\in\mathbb{N}} \mathbb{Z}_2$ in which each copy of $\mathbb{Z}_2$ is given the discrete topology, there exists a unique Haar probability measure $\mu_{\mathcal{C}}$ on $\mathcal{C}$.*

Furtheremore, if $\emptyset \neq F \subset \mathbb{N}$ is finite and $x = (x_j)_{j\in F}$, where $x_j \in \{0,1\}$ for all $j \in F$, and we define $F_x = \{z \in \mathcal{C} : \forall j \in F, z_j = x_j\}$, then we have that $\mu_{\mathcal{C}}(F_x) = 2^{-\#F}$, where $\#F$ is the cardinality of F.

Proof. The fact that there exists a unique Haar probability measure on any compact group is [18, Haar's Theorem V.11.4]. Fix a finite, nonempty $F \subset \mathbb{N}$ and let the cardinality of F be $\#F = n$. Let X_F denote the set of all n-length vectors of the form $x = (x_j)_{j\in F}$ where $x_j \in \{0,1\}$ for all $j \in F$, and thus, the cardinality of X_F is 2^n. Furthermore, for each $x, y \in X_F, x \neq y$, we have that $F_x \cap F_y = \emptyset$. Also, the union $\cup_{x\in X_F} F_x = \mathcal{C}$. Next, we note that for each $x, y \in X_F$, there exists $z \in \mathcal{C}$ such that:

$$z + F_x = F_y, \tag{3.1.10}$$

where $+$ is the pointwise addition modulo 1. Indeed, for all $j \in F$ such that $x_j = y_j$, we let $z_j = 0$, and for all $j \in F$ such that $x_j \neq y_j$, we let $z_j = 1$, and if $l \in \mathbb{N} \setminus F$, we let $z_l = 0$.

Note that F_x is open since it is the union of basic open sets (cylinder sets), and thus F_x is measurable and $\mu_{\mathcal{C}}(F_x) > 0$ by [18, Haar's Theorem V.11.4]. Thus, as Haar measures are translation invariant under the group operation, by Expression (3.1.10), we have that for each $x, y \in X_F$, the measure $\mu_{\mathcal{C}}(F_y) = \mu_{\mathcal{C}}(F_x + z) = \mu_{\mathcal{C}}(F_x)$. Fix $x \in X_F$. Hence, as $\mu_{\mathcal{C}}$ is a probability measure, we summarize that:

$$\begin{aligned}
1 = \mu_{\mathcal{C}}(\mathcal{C}) &= \mu_{\mathcal{C}}\left(\cup_{y \in X_F} F_y\right) \\
&= \sum_{y \in X_F} \mu_{\mathcal{C}}(F_y) \\
&= \sum_{y \in X_F} \mu_{\mathcal{C}}(F_x) \\
&= 2^n \mu_{\mathcal{C}}(F_x) \implies \mu_{\mathcal{C}}(F_x) = 2^{-n} = 2^{-\# F},
\end{aligned}$$

which completes the proof. $\square$

Notation 3.1.15. *The set $\mathcal{C} = \prod_{n \in \mathbb{N}} \mathbb{Z}_2$ is a group for the pointwise addition modulo 1. By Lemma (3.1.14), as $\mathcal{C}$ is compact, there exists a unique Haar probability measure $\mu_{\mathcal{C}}$. By the Riesz Representation Theorem [18, Appendix C.18] and [18, Theorem V.11.5], the measure $\mu_{\mathcal{C}}$ defines by integration a faithful tracial state λ on $C(\mathcal{C})$.*

Recall the evalutation maps $\eta_j : (z_n)_{n \in \mathbb{N}} \in \mathcal{C} \longmapsto z_j$ for all $j \in \mathbb{N}$ of Example (2.1.76). For any finite, nonempty $F \subset \mathbb{N}$, we have that $\prod_{j \in F} \eta_j$ is simply the indicator function of the subset:

$$F_{(1,\ldots,1)} = \left\{ (z_n)_{n \in \mathbb{N}} \in \mathcal{C} : \forall j \in F \quad z_j = 1 \right\}.$$

Therefore, by Lemma (3.1.14), we have:

$$\lambda\left(\prod_{j\in F}\eta_j\right)=\int_{\mathcal{C}}\prod_{j\in F}\eta_j\ d\mu_{\mathcal{C}}=\mu_{\mathcal{C}}\left(F_{(1,\ldots,1)}\right)=2^{-\#F}$$

where $\#F$ is the cardinal of F .

The primary advantage of our choice of tracial state is illustrated in the following lemma.

Lemma 3.1.16. *We shall use Notations (2.1.77) and (3.1.15), and for each $j\in\mathbb{N}$, recall the self-adjoint unitary $u_j=2\eta_j-1_{C(\mathcal{C})}$ defined in Example (2.1.76). If we endow $C(\mathcal{C})$ with the inner product:*

$$(f,g)\in C(\mathcal{C})\mapsto\lambda(f\overline{g}),$$

then $u_n\in\mathfrak{A}_n^{\perp}$ for all $n\in\mathbb{N}$, where $^{\perp}$ is provided by the orthogonality induced by the inner product. Moreover, we have that $\left(\prod_{j\in F}u_j\right)_{F\in\mathcal{F}}$, where $\mathcal{F}$ is the set of nonempty finite subsets of $\mathbb{N}$, is an orthonormal family of $L^2(C(\mathcal{C}),\lambda)$, which is the Hilbert space of the GNS construction associated to λ of Theorem (2.1.40).

Proof. We let, for all $n\in\mathbb{N}\setminus\{0\}$:

$$\mathfrak{B}_n=\left\{1_{C(\mathcal{C})},\prod_{j\in F}u_j : F\text{ is a nonempty subset of }\{0,\ldots,n-1\}\right\}.$$

We note that $\mathfrak{B}_n$ is a basis for $\mathfrak{A}_n$. We also note that $\left(\prod_{j\in F}u_j\right)_{F\in\mathcal{F}}$ is a Hamel basis of the space $\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n$.

Now, let $n\in\mathbb{N}$ and $F\subseteq\{0,\ldots,n-1\}$ be nonempty. We have:

$$\begin{aligned}\lambda\left(u_n^*\prod_{j\in F}u_j\right)&=\lambda\left(\left(2\eta_n-1_{C(\mathcal{C})}\right)\prod_{j\in F}\left(2\eta_j-1_{C(\mathcal{C})}\right)\right)\\&=\lambda\left(\prod_{j\in F\cup\{n\}}\left(2\eta_j-1_{C(\mathcal{C})}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{G \subseteq F \cup \{n\}} (-1)^{\#F+1-\#G} 2^{\#G} \lambda \left(\prod_{j \in G} \eta_j \right) \\
&= \sum_{G \subseteq F \cup \{n\}} (-1)^{\#F+1-\#G} 2^{\#G} 2^{-\#G} \\
&= \sum_{j \in F \cup \{n\}} \binom{\#F+1}{j} (-1)^j \\
&= (1-1)^{\#F+1} = 0,
\end{aligned}
$$

and thus:

$$
\lambda \left(u_n^* \prod_{j \in F} u_j \right) = 0 \tag{3.1.11}
$$

Since $\mathfrak{B}_n$ is a basis for $\mathfrak{A}_n$, we conclude that indeed, $u_n \in \mathfrak{A}_n^{\perp}$.

Moreover, we note that Expression (3.1.11) also proves that $\mathfrak{B} = \cup_{n \in \mathbb{N}} \mathfrak{B}_n$ is an orthogonal family in $L^2(C(\mathcal{C}), \lambda)$. As the product of unitaries is unitary, our definition of the inner product then shows that the family $\mathfrak{B}$ is orthonormal. □

The primary advantage to our choice of increasing sequence of finite-dimensional unital C*-subalgebras of $C(\mathcal{C})$ is illustrated in the following lemma.

Lemma 3.1.17. *Let $\mathcal{T} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ be the increasing sequence of finite-dimensional unital C*-subalgebras of $C(\mathcal{C})$ such that $C(\mathcal{C}) = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{C(\mathcal{C})}}$ defined in Example (2.1.76).*

Let $x, y \in \mathcal{C}, x \neq y$ and $N = \min\{n \in \mathbb{N} : x_n \neq y_n\}$. If $j \in \{0, \ldots, N\}$ and $g \in \mathfrak{A}_j$, then $g(x) = g(y)$.

Proof. Let $x, y \in \mathcal{C}, x \neq y$ and $N = \min\{n \in \mathbb{N} : x_n \neq y_n\}$. First assume that $N = 0$. If $g \in \mathfrak{A}_0 = \mathbb{C}1_{C(\mathcal{C})}$, then g is constant and $g(x) = g(y)$.

Next, assume that $N > 0$. If $j = 0$, then the same argument for $N = 0$ implies that for $g \in \mathfrak{A}_j$ we have that $g(x) = g(y)$. Thus, let $j \in \{1, \ldots, N\}$. Since $u_k = 2\eta_k - 1_{C(\mathcal{C})}$ for all $k \in \mathbb{N}$, we have that $\mathfrak{A}_j$ is the finite-dimensional unital

C*-subalgebra of $C(\mathcal{C})$ generated by the evaluation maps η_k for $k \in \{0, \ldots, j-1\}$ and $1_{C(\mathcal{C})}$. However, since $j-1 < N$ and thus $x_k = y_k$ for $k \in \{0, \ldots, j-1\}$, we have that $\eta_k(x) = x_k = y_k = \eta_k(y)$ for $k \in \{0, \ldots, j-1\}$ by definition. Thus, if $g \in \mathfrak{A}_j$, then as g is a finite linear combination of finite products of η_k for $k \in \{0, \ldots, j-1\}$ and $1_{C(\mathcal{C})}$ by finite dimensionality, we have that $g(x) = g(y)$. □

We now have the tools needed to state our main theorem for this section: Lip-norms defined using Theorem (3.1.5) with the ingredients described in this section naturally lead to ultrametrics on the Cantor space via the associated Monge-Kantorovich metric by simply requiring the natural condition that the sequence β be a decreasing sequence. We call this condition natural since we will see throughout this dissertation that all β sequences that lead to desired results are decreasing.

Theorem 3.1.18. *Let $\beta : \mathbb{N} \to (0, \infty)$ be a decreasing sequence with $\lim_\infty \beta = 0$ and let $\mathcal{T} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ be the increasing sequence of unital finite-dimensional C*-subalgberas of $C(\mathcal{C})$ of Example (2.1.76). Using Notations (2.1.77) and (3.1.15) and notation from Theorem (3.1.5), we have, for all $x, y \in \mathcal{C}$:*

$$\mathsf{mk}_{\mathsf{L}^\beta_{\mathcal{T},\lambda}}(\delta_x, \delta_y) = \begin{cases} 0 & : \text{ if } x = y, \\ 2\beta\left(\min\{n \in \mathbb{N} : x_n \neq y_n\}\right) & : \text{ otherwise} \end{cases}$$

induces an ultrametric on $\mathcal{C}$ via the homeomorphism $x \in \mathcal{C} \longmapsto \delta_x \in M_{C(\mathcal{C})}$ of Proposition (2.1.32), where $M_{C(\mathcal{C})}$ is the maximal ideal space of $C(\mathcal{C})$ and δ_x are the Dirac point masses defined by $\delta_x(f) = f(x)$ for all $f \in C(\mathcal{C})$.

Proof. In this proof, we will denote by $\mathbb{E}(\cdot|\mathfrak{A}_n)$ the conditional expectation from $C(\mathcal{C})$ onto $\mathfrak{A}_n$, which leave λ invariant.

Fix $x, y \in \mathcal{C}$. Note that if $x = y$, then by definition $\delta_x = \delta_y$ and $\mathsf{mk}_{\mathsf{L}^\beta_{\mathcal{T},\lambda}}(\delta_x, \delta_y) = 0$. Thus, for the remainder of the proof, assume that $x \neq y$. By Definition (2.2.5) and definition of the Dirac point masses, we have:

$$\begin{aligned}\mathsf{mk}_{\mathsf{L}^{\beta}_{\mathcal{T},\lambda}}(\delta_x,\delta_y) &= \sup\left\{|\delta_x(f)-\delta_y(f)| : f\in\mathfrak{sa}\left(C(\mathcal{C})\right), \mathsf{L}^{\beta}_{\mathcal{T},\lambda}(f)\leqslant 1\right\}\\ &= \sup\left\{|f(x)-f(y)| : f\in\mathfrak{sa}\left(C(\mathcal{C})\right), \mathsf{L}^{\beta}_{\mathcal{T},\lambda}(f)\leqslant 1\right\}.\end{aligned}$$

Our computation relies on the following observation. Let $n\geqslant k\in\mathbb{N}$. Since $u_n\in\mathfrak{A}_k^{\perp}$ in $L^2(C(\mathcal{C}),\lambda)$ by Lemma (3.1.16), we conclude that $\mathbb{E}\left(u_n|\mathfrak{A}_k\right)=0$. Of course, if $k>n\in\mathbb{N}$ then $\mathbb{E}\left(u_n|\mathfrak{A}_k\right)=u_n$. Thus we have for all $n\in\mathbb{N}$:

$$\begin{aligned}\mathsf{L}^{\beta}_{\mathcal{T},\lambda}(u_n) &= \max\left\{\frac{\|u_n\|_{C(\mathcal{C})}}{\beta(k)} : k\leqslant n\right\}\\ &= \max\left\{\frac{1}{\beta(k)} : k\leqslant n\right\} \text{ as } u_n \text{ is unitary,}\\ &= \frac{1}{\beta(n)} \text{ as } \beta \text{ is decreasing.}\end{aligned}$$

We thus have $\mathsf{L}^{\beta}_{\mathcal{T},\lambda}(\beta(n)u_n)\leqslant 1$ for all $n\in\mathbb{N}$.

Since $x\neq y$, let $N=\min\{n\in\mathbb{N} : x_n\neq y_n\}$. Then, by definition of u_N, we have that:

$$\begin{aligned}|u_N(x)-u_N(y)| &= |2\eta_N(x)-1-(2\eta_N(y)-1)|\\ &= 2\left|\eta_N(x)-\eta_N(y)\right| = 2\cdot 1 = 2\end{aligned}$$

by $\eta_N(x)=x_N\neq y_N=\eta_N(y)$, and therefore, since $\beta(N)u_N\in\mathfrak{sa}\left(C(\mathcal{C})\right)$ and $\mathsf{L}^{\beta}_{\mathcal{T},\lambda}\left(\beta(N)u_N\right)\leqslant 1$:

$$\mathsf{mk}_{\mathsf{L}^{\beta}_{\mathcal{T},\lambda}}(\delta_x,\delta_y)\geqslant\beta(N)\left|u_N(x)-u_N(y)\right| = 2\beta(N).$$

On the other hand, if $f\in C(\mathcal{C})$, then $\mathbb{E}\left(f|\mathfrak{A}_k\right)\in\mathfrak{A}_k$ for all $k\in\mathbb{N}$. Hence, by Lemma (3.1.17), we have that $\mathbb{E}\left(f|\mathfrak{A}_n\right)(x)=\mathbb{E}\left(f|\mathfrak{A}_n\right)(y)$ for all $n\leqslant N$. Thus, if

$f \in C(\mathcal{C})$ with $\mathsf{L}^{\beta}_{\mathcal{T},\lambda}(f) \leqslant 1$, then:

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - \mathbb{E}\left(f|\mathfrak{A}_n\right)(x) - (f(y) - \mathbb{E}\left(f|\mathfrak{A}_n\right)(y))| \\ &\leqslant 2\|f - \mathbb{E}\left(f|\mathfrak{A}_n\right)\|_{C(\mathcal{C})} \\ &\leqslant 2\beta(n)\end{aligned}$$

for all $n \leqslant N$. Since β is decreasing, we thus get:

$$|f(x) - f(y)| \leqslant 2\min\{\beta(n) : n \leqslant N\} = 2\beta(N).$$

We thus conclude that:

$$\mathsf{mk}_{\mathsf{L}^{\beta}_{\mathcal{T},\lambda}}(\delta_x, \delta_y) = 2\beta(N),$$

as desired, and it is routine to check that $\mathsf{mk}_{\mathsf{L}^{\beta}_{\mathcal{T},\lambda}}$ induces an ultrametric on $\mathcal{C}$ since β is decreasing. $\square$

We thus recognize standard ultrametrics on the Cantor set using the Monge-Kanorovich metric.

Corollary 3.1.19. *Let $r \in (1,\infty) \subset \mathbb{R}$, and set $\beta_r : n \in \mathbb{N} \mapsto \frac{1}{2}r^{-n}$. Then, for any two $x, y \in \mathcal{C}$, using the notations of Theorem (3.1.18), we have:*

$$\mathsf{mk}_{\mathsf{L}^{\beta_r}_{\mathcal{T},\lambda}}(\delta_x, \delta_y) = \begin{cases} 0 & : \text{ if } x = y, \\ r^{-\min\{n \in \mathbb{N} : x_n \neq y_n\}} & : \text{ otherwise.}\end{cases}$$

In particular, if we equip $\mathcal{C}$ with the ultrametric $\mathsf{d}_{\mathcal{C},r}$ of Expression (3.1.9), then using the associated Lipschitz seminorm $\mathsf{L}_{\mathsf{d}_{\mathcal{C},r}}$ of Theorem (2.2.10), we have that for all $x, y \in \mathcal{C}$:

$$\mathsf{mk}_{\mathsf{L}^{\beta_r}_{\mathcal{T},\lambda}}(\delta_x, \delta_y) = \mathsf{d}_{\mathcal{C},r}(x, y) = \mathsf{mk}_{\mathsf{L}_{\mathsf{d}_{\mathcal{C},r}}}(\delta_x, \delta_y),$$

and the map $x \in (\mathcal{C}, \mathsf{d}_{\mathcal{C},r}) \longmapsto \delta_x \in \left(\mathscr{S}(C(\mathcal{C})), \mathsf{mk}_{\mathsf{L}^{\beta_r}_{\mathcal{T},\lambda}}\right)$ *is an isometry onto its image, which is the maximal ideal space* $M_{C(\mathcal{C})}$ *of* $C(\mathcal{C})$*.*

3.2 Leibniz Lip-norms from quotient norms and finite-dimensional approximations

Our work in [3] relied on the hypothesis of the existence of faithful tracial state for a unital AF algebra. Of course, every simple unital AF algebra has a faithful tracial state, but in the non-simple case, there exist unital AF algebras without faithful tracial states. For example, consider the unitization of the compact operators on an infinite-dimensional separable Hilbert space.

To remedy this, in Theorem (3.2.11), we introduce Leibniz Lip-norms that exist on any unital AF algebra built from quotient norms and the work of Rieffel in [67], in which he established the Leibniz property for certain quotient norms. Another consequence of this is that any unital AF algebra, $\mathfrak{A}$, has finite dimensional approximations in propinquity provided by any increasing sequence of unital finite dimensional subalgebras $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Furthermore, in Proposition (3.2.4), we show that any Lip-norm whose domain is the dense subspace $\mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ proves this fact of finite-dimensional approximations in propinquity.

We note that the introduction of these Lip-norms from quotient norms in Theorem (3.2.11) does not replace or diminish the importance of the Lip-norms from conditional expectations of Theorem (3.1.3). The conditional expectation Lip-norms give us explicit projections onto the C*-subalgebras, while also providing key estimates in quantum propinquity that are crucial to our continuity results about AF algebras (see Theorem (4.2.12) and Theorem (4.5.6)).

We begin by providing some known examples of finite dimensional approximations for the quantum propinquity to gather a better understanding of the concept.

What is especially enlightening is that there are non-AF algebras that have natural finite-dimensional approximations in the sense of the propinquity. We note for part *1.*, *2.*, and *3.* of Example (3.2.1) that for a compact metric space X, the C*-algebra $C(X)$ is AF if and only if X is totally disconnected by [12, Proposition 3.1].

Example 3.2.1. *We provide some examples of finite-dimensional approximations in the sense of quantum propinquity.*

1. *For any $C \geqslant 1, D \geqslant 0$, all C*-algebras of the form $C(X)$, where (X, d_X) is a compact metric space, have finite dimensional approximations in quantum propinquity induced by finite ε-nets, $X_\varepsilon \subseteq X$ and $C(X_\varepsilon)$. Indeed, the Gromov-Hausdorff distance* $\mathrm{GH}(X_\varepsilon, X) \leqslant \varepsilon$ *by [14, Example 7.3.11] and Theorem (2.3.19) imply that $\Lambda_{C,D}\left(\left(C(X_\varepsilon), \mathsf{L}_{\mathsf{d}_X}\right), \left(C(X), \mathsf{L}_{\mathsf{d}_X}\right)\right) \leqslant \varepsilon$.*

2. *Motivated by Mathematical Physics and using a different approach [69] than that of 1., the commutative C*-algebra $C\left(S^2\right)$ — continuous functions on the sphere — has finite dimensional approximations in quantum propinquity provided by noncommutative finite-dimensional simple C*-algebras, where $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, i.e. approximations by full matrix algebras.*

3. *The quantum (noncommutative) tori — including $C\left(\mathbb{T}^2\right)$, the C*-algebra of continuous functions on the torus —, which are non-AF, as presented in [40] have finite dimensional approximations in quantum propinquity provided by fuzzy tori.*

4. *Every nuclear quasi-diagonal C*-algebra $\mathfrak{A}$ such that $(\mathfrak{A}, \mathsf{L})$ is a Leibniz quantum compact metric space has finite-dimensional approximations in quantum propinquity by quasi-Leibniz quantum compact metric spaces [45, Section 5].*

5. *Any unital AF algebra $\mathfrak{A}$ that can be equipped with faithful tracial state has finite dimensional approximations in propinquity provided by any inductive*

sequence of finite dimensional C-algebras with inductive limit $\mathfrak{A}$ by Theorem (3.1.3).*

One thing in common with all of these examples is that the existence of finite dimensional approximations in propinquity are proven using specific Lip-norms. We shall see in Proposition (3.2.4) that in the case of unital AF algebras, the existence of a Lip-norm finite on the obvious dense subspace is all that is required to provide finite dimensional approximations in propinquity. In some sense, this means that the C*-algebra structure of an AF algebra is enough to provide finite dimensional approximations in propinquity.

Notation 3.2.2. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a quasi-Leibniz quantum compact metric space. Let $\mu \in \mathscr{S}(\mathfrak{A})$. Denote:*

$$\mathfrak{Lip}_1(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}$$
$$\mathfrak{Lip}_1(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mu) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1, \mu(a) = 0\}.$$

Lemma 3.2.3 ([46])**.** *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. If $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a bridge of Definition (2.3.2), then for any two states $\varphi_{\mathfrak{A}} \in \mathscr{S}(\mathfrak{A}), \varphi_{\mathfrak{B}} \in \mathscr{S}(\mathfrak{B})$, we have that:*

$$\begin{aligned}&\mathsf{Haus}_{\mathfrak{D}}(\pi_{\mathfrak{A}}(\mathfrak{Lip}_1(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}))\omega, \omega\pi_{\mathfrak{B}}(\mathfrak{Lip}_1(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))) \leqslant \\ &\quad \mathsf{Haus}_{\mathfrak{D}}(\pi_{\mathfrak{A}}(\mathfrak{Lip}_1(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \varphi_{\mathfrak{A}}))\omega, \omega\pi_{\mathfrak{B}}(\mathfrak{Lip}_1(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \varphi_{\mathfrak{B}}))) < \infty.\end{aligned}$$

Proof. The proof is the argument in between [46, Notation 3.13] and [46, Definition 3.14]. □

To make for easier notation we will begin presenting results in the closure of the union case rather than the inductive limit case. We have seen that this causes no issue in Proposition (3.1.6). We note that in the following result, the proof does not require any notion of quasi-Leibniz. We only only include it to utilize the full power of the quantum propinquity. The proof of Proposition (3.2.4) also does not require that the subalgebras be finite-dimensional, but since no such example of a lip-norm exists yet of this form outside the AF case, we leave this assumption there. Also, thank you to F. Latrémolière for pointing out an error of a previously incorrect version of Proposition (3.2.4) and for offering advice on a fix to this error, which resulted in this current version of Proposition (3.2.4)

Proposition 3.2.4. *Fix $C \geqslant 1, D \geqslant 0$. Let $\mathfrak{A}$ be a unital AF algebra such that $(\mathfrak{A}, \mathsf{L})$ is a (C,D)-quasi-Leibniz quantum compact metric space, in which the domain of L contains $\mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a sequence of unital finite dimensional C^*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Define a seminorm on $\mathfrak{sa}(\mathfrak{A})$ by:*

$$\mathsf{L}_f(a) = \begin{cases} \mathsf{L}(a) & : \text{ if } a \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right) \\ \infty & : \text{ otherwise .} \end{cases}$$

Let $L_{f,1} = \overline{\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_f(a) \leqslant 1\}}^{\|\cdot\|_{\mathfrak{A}}}$.

If we let $\overline{\mathsf{L}}$ be the Minkowski functional of $L_{\mathfrak{A},1}$ on $\mathfrak{sa}(\mathfrak{A})$, i.e.

$$\overline{\mathsf{L}}(a) = \inf\left\{r > 0 : \frac{1}{r}a \in L_{f,1}\right\}$$

for all $a \in \mathfrak{sa}(\mathfrak{A})$, then:

$$\begin{aligned} &\mathsf{L}(a) = \mathsf{L}_f(a) = \mathsf{L}(a) < \infty \quad \text{for all} \quad a \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right) \text{ and,} \\ &\left\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) \leqslant 1\right\} = \overline{\left\{a \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right) : \overline{\mathsf{L}}(a) \leqslant 1\right\}}^{\|\cdot\|_{\mathfrak{A}}} \\ &\qquad\qquad\qquad\qquad\qquad = \overline{\left\{a \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right) : \mathsf{L}(a) \leqslant 1\right\}}^{\|\cdot\|_{\mathfrak{A}}}, \end{aligned} \tag{3.2.1}$$

and $(\mathfrak{A}, \overline{\mathsf{L}})$ and $(\mathfrak{A}_n, \overline{\mathsf{L}}) = (\mathfrak{A}_n, \mathsf{L})$ are (C,D)-quasi-Leibniz quantum compact metric space for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \Lambda_{C,D}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}}\right), \left(\mathfrak{A}, \overline{\mathsf{L}}\right)\right) = 0$.

Proof. By [18, Proposition IV.1.14], the map $\overline{\mathsf{L}}$ is a seminorm on $\mathfrak{sa}(\mathfrak{A})$ such that $\left\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) \leqslant 1\right\} = L_{f,1}$ since $L_{f,1}$ is closed. Indeed, the proof of [18, Proposition IV.1.14] shows that $\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) < 1\} \subseteq L_{f,1}$ and $\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) \leqslant 1\} \supseteq L_{f,1}$. Now, if $a \in \mathfrak{sa}(\mathfrak{A})$ such that $\overline{\mathsf{L}}(a) = 1$, then by definition of $\overline{\mathsf{L}}$, we have that there exists a sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that converges to 1 such that $r_n > 0$ and $\frac{1}{r_n}a \in L_{f,1}$ for all $n \in \mathbb{N}$. In particular, we have that the sequence $\left(\frac{1}{r_n}a\right)_{n\in\mathbb{N}} \subset L_{f,1}$ converges to a with respect to $\|\cdot\|_{\mathfrak{A}}$. As $L_{f,1}$ is closed, we have that $a \in L_{f,1}$, which establishes the equality of the sets $L_{f,1}$ and $\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) \leqslant 1\}$.

Next, by construction, we have that $\mathsf{L}_f(a) \leqslant 1 < \infty$ implies that $L_{f,1} = \overline{\left\{a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \overline{\mathsf{L}}(a) \leqslant 1\right\}}^{\|\cdot\|_{\mathfrak{A}}}$. The fact that $\overline{\mathsf{L}}(a) = \mathsf{L}_f(a) = \mathsf{L}(a) < \infty$ for all $a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n)$ is routine to check. This establishes Expression (3.2.1). Also, one can easily deduce that $\overline{\mathsf{L}}_{\mathfrak{A}}$ is a lower semi-continuous seminorm dense domain such that $\overline{\mathsf{L}}^{-1}(\{0\}) = \mathbb{R}1_{\mathfrak{A}}$ by Expression (3.2.1).

Next, we show that $(\mathfrak{A}, \overline{\mathsf{L}})$ is a quantum compact metric space, and we use equivalence *3.* of Theorem (2.2.6) to accomplish this. Let $q : a \in \mathfrak{sa}(\mathfrak{A}) \longmapsto a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}$ denote the quotient map, which is continuous with respect to the assoicated norms. Now, since $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, we have that the set $q(\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1\}) = \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : \mathsf{L}(a) \leqslant 1\}$ is totally bounded with respect to $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$ by Theorem (2.2.6). Hence, by containment and Expression (3.2.1), the set $q(\{a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \overline{\mathsf{L}}(a) \leqslant 1\}) = q(\{a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \mathsf{L}(a) \leqslant 1\})$ is totally bounded with respect to the norm $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$.

Now, since q is continuous, we have that

$$\begin{aligned} E &= q\left(\overline{\{a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \overline{\mathsf{L}}(a) \leqslant 1\}}^{\|\cdot\|_{\mathfrak{A}}}\right) \\ &\subseteq \overline{q(\{a \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \overline{\mathsf{L}}(a) \leqslant 1\})}^{\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}}, \end{aligned}$$

which implies that E is totally bounded by containment and since the set on the right of the containment is the closure of a totally bounded set. However, by Expression (3.2.1):

$$E = q(\{a \in \mathfrak{sa}(\mathfrak{A}) : \overline{\mathsf{L}}(a) \leqslant 1\}) = \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : \overline{\mathsf{L}}(a) \leqslant 1\},$$

and therefore, the pair $(\mathfrak{A}, \overline{\mathsf{L}})$ is a quantum compact metric space of Defintion (2.2.5) by Theorem (2.2.6).

Now, we prove that $\overline{\mathsf{L}}$ is (C, D)-quasi Leibniz. Our proof follows similarly to the proof of [66, Proposition 3.1], which is the case of $C = 1, D = 0$. Another similar result is [40, Lemma 3.1], which does involve a more general case than the quasi-Leibniz case. But, rather than just reference the proofs of these results, we verify that these results still apply in our situation as there are some subtle differences with our construction with regard to the closedness of certain sets and conditions on the seminorm L.

Claim 3.2.5. *The seminorm $\overline{\mathsf{L}}$ is (C, D)-quasi-Leibniz.*

Proof of claim. First, assume that $a, b \in \mathfrak{sa}(\mathfrak{A})$ such that $\overline{\mathsf{L}}(a) = 1 = \overline{\mathsf{L}}(b)$. By definition of $\overline{\mathsf{L}}$, there exists a sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that converges to $1 = \overline{\mathsf{L}}(a)$ such that $r_n \geqslant 1 = \overline{\mathsf{L}}(a)$ and $\frac{1}{r_n}a \in L_{f,1}$ for all $n \in \mathbb{N}$. In particular, we have that the sequence $\left(\frac{1}{r_n}a\right)_{n\in\mathbb{N}} \subset L_{f,1}$ converges to a with respect to $\|\cdot\|_{\mathfrak{A}}$. Now, by definition of $L_{f,1}$, for each $n \in \mathbb{N}$, choose $a_n \in \{c \in \mathfrak{sa}(\cup_{n\in\mathbb{N}}\mathfrak{A}_n) : \mathsf{L}(a) \leqslant 1\}$ such that $\left\|a_n - \frac{1}{r_n}a\right\|_{\mathfrak{A}} < \frac{1}{n}$. Therefore, for $n \in \mathbb{N}$, we have:

$$\|a_n - a\|_{\mathfrak{A}} \leqslant \left\|a_n - \frac{1}{r_n}a\right\|_{\mathfrak{A}} + \left\|\frac{1}{r_n}a - a\right\|_{\mathfrak{A}} < \frac{1}{n} + \left\|\frac{1}{r_n}a - a\right\|_{\mathfrak{A}}.$$

Thus, the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a with respect to $\|\cdot\|_{\mathfrak{A}}$ and $\mathsf{L}(a_n) \leqslant 1 = \overline{\mathsf{L}}(a)$ for all $n \in \mathbb{N}$. Since $a \neq 0_{\mathfrak{A}}$ as $\overline{\mathsf{L}}(a) \neq 0$, up to dropping to a subsequence, we have that $\|a_n\| > 0$ for all $n \in \mathbb{N}$. Repeat the same process for b to find a sequence

$(b_n)_{n\in\mathbb{N}} \subseteq \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ of non-zero terms such that $(b_n)_{n\in\mathbb{N}}$ converges to b with respect to $\|\cdot\|_{\mathfrak{A}}$, while $0 < \mathsf{L}(b_n) \leqslant 1 = \overline{\mathsf{L}}(b)$ for all $n \in \mathbb{N}$. Now, for all $n \in \mathbb{N}$ we have that $a_nb_n + b_na_n \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ such that $(a_nb_n + b_na_n)_{n\in\mathbb{N}}$ converges to $ab + ba$. Also, we gather that since L is quasi-Leibniz:

$$\begin{aligned}\mathsf{L}\left(\frac{a_nb_n + b_na_n}{2}\right) &\leqslant C(\mathsf{L}(a_n)\|b_n\|_{\mathfrak{A}} + \mathsf{L}(b_n)\|a_n\|_{\mathfrak{A}}) + D\mathsf{L}(a_n)\mathsf{L}(b_n)\\ &\leqslant C(\overline{\mathsf{L}}(a)\|b_n\|_{\mathfrak{A}} + \overline{\mathsf{L}}(b)\|a_n\|_{\mathfrak{A}}) + D\overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)\\ &\leqslant C(\|b_n\|_{\mathfrak{A}} + \|a_n\|_{\mathfrak{A}}) + D,\end{aligned}$$

and since the right-hand side of the last inequality is non-zero, we have:

$$\mathsf{L}\left(\frac{a_nb_n + b_na_n}{2\left(C(\|b_n\|_{\mathfrak{A}} + \|a_n\|_{\mathfrak{A}}) + D\right)}\right) \leqslant 1 \text{ for all } n \in \mathbb{N}.$$

Now, the sequence $\left(\frac{a_nb_n+b_na_n}{2(C(\|b_n\|_{\mathfrak{A}}+\|a_n\|_{\mathfrak{A}})+D)}\right)_{n\in\mathbb{N}}$ converges to $\frac{ab+ba}{2(C(\|b\|_{\mathfrak{A}}+\|a\|_{\mathfrak{A}})+D)}$ with respect to $\|\cdot\|_{\mathfrak{A}}$ as all the scalars in the denominator are postive and converge to a positive scalar. Thus, by Expression (3.2.1), we have that:

$$\begin{aligned}\frac{ab + ba}{2\left(C(\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}) + D\right)} &\in \overline{\left\{c \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right) : \mathsf{L}(c) \leqslant 1\right\}}^{\|\cdot\|_{\mathfrak{A}}}\\ &= \left\{c \in \mathfrak{sa}\left(\mathfrak{A}\right) : \overline{\mathsf{L}}(c) \leqslant 1\right\},\end{aligned}$$

and thus:

$$\begin{gathered}\overline{\mathsf{L}}(a \circ b) = \overline{\mathsf{L}}\left(\frac{ab + ba}{2}\right) \leqslant C(\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}) + D\\ \text{for all } a, b \in \mathfrak{sa}\left(\mathfrak{A}\right) \text{ such that } \overline{\mathsf{L}}(a) = 1 = \overline{\mathsf{L}}(b).\end{gathered} \tag{3.2.2}$$

The same holds true for the Lie product $\{a, b\} = \frac{ab-ba}{2i}$.

Next, assume that $a, b \in \mathfrak{sa}\left(\mathfrak{A}\right)$ such that $\overline{\mathsf{L}}(a), \overline{\mathsf{L}}(b) \in (0, \infty) \subset \mathbb{R}$. Hence, we have $\overline{\mathsf{L}}\left(\frac{1}{\overline{\mathsf{L}}(a)}a\right) = 1 = \overline{\mathsf{L}}\left(\frac{1}{\overline{\mathsf{L}}(b)}b\right)$. Thus:

$$
\begin{aligned}
\frac{1}{\overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)}\overline{\mathsf{L}}(a\circ b) &= \frac{1}{\overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)}\overline{\mathsf{L}}\left(\frac{ab+ba}{2}\right)\\
&= \overline{\mathsf{L}}\left(\frac{1}{\overline{\mathsf{L}}(a)}a\circ\frac{1}{\overline{\mathsf{L}}(b)}b\right)\\
&\leqslant C\left(\left\|\frac{1}{\overline{\mathsf{L}}(b)}b\right\|_{\mathfrak{A}}+\left\|\frac{1}{\overline{\mathsf{L}}(a)}a\right\|_{\mathfrak{A}}\right)+D\\
&= C\left(\frac{1}{\overline{\mathsf{L}}(b)}\|b\|_{\mathfrak{A}}+\frac{1}{\overline{\mathsf{L}}(a)}\|a\|_{\mathfrak{A}}\right)+D.
\end{aligned}
$$

where Expression (3.2.2) was used in the last inequality. Therefore:

$$
\begin{aligned}
\overline{\mathsf{L}}(a\circ b) &\leqslant \overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)\left(C\left(\frac{1}{\overline{\mathsf{L}}(b)}\|b\|_{\mathfrak{A}}+\frac{1}{\overline{\mathsf{L}}(a)}\|a\|_{\mathfrak{A}}\right)+D\right)\\
&= C\left(\overline{\mathsf{L}}(a)\|b\|_{\mathfrak{A}}+\overline{\mathsf{L}}(b)\|a\|_{\mathfrak{A}}\right)+D\overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)\\
&\text{for all } a,b\in\mathfrak{sa}(\mathfrak{A}) \text{ such that } \overline{\mathsf{L}}(a),\overline{\mathsf{L}}(b)\in(0,\infty),
\end{aligned}
$$

and the same holds for the Lie product $\{a,b\}$.

Now, for $a,b\in\mathfrak{sa}(\mathfrak{A})$, if either $\overline{\mathsf{L}}(a)=\infty$ or $\overline{\mathsf{L}}(b)=\infty$, then the conclusion is clear. Next, if $a,b\in\mathfrak{sa}(\mathfrak{A})$ such that $\overline{\mathsf{L}}(a)=0=\overline{\mathsf{L}}(b)$, then $a,b\in\mathbb{R}1_{\mathfrak{A}}$ and thus $a\circ b\in\mathbb{R}1_{\mathfrak{A}}$ and $\overline{\mathsf{L}}(a\circ b)=0=\overline{\mathsf{L}}(\{a,b\})$, which concludes this case. Finally, assume that $a,b\in\mathfrak{sa}(\mathfrak{A})$, $\overline{\mathsf{L}}(a)=0$ and $\overline{\mathsf{L}}(b)\in(0,\infty)\subset\mathbb{R}$. Thus, $a=r1_{\mathfrak{A}}$ for some $r\in\mathfrak{A}$ and so $\|a\|_{\mathfrak{A}}=|r|$. Now:

$$
\begin{aligned}
\overline{\mathsf{L}}(a\circ b) &= \overline{\mathsf{L}}\left(\frac{rb+rb}{2}\right)\\
&= |r|\overline{\mathsf{L}}(b)\\
&= \|a\|_{\mathfrak{A}}\overline{\mathsf{L}}(b)\\
&\leqslant C\left(\|a\|_{\mathfrak{A}}\overline{\mathsf{L}}(b)+\|b\|_{\mathfrak{A}}\overline{\mathsf{L}}(a)\right)+D\overline{\mathsf{L}}(a)\overline{\mathsf{L}}(b)
\end{aligned}
$$

since $C\geqslant 1, D\geqslant 0$. Also, we have $\overline{\mathsf{L}}(\{a,b\})=\overline{\mathsf{L}}\left(\frac{rb-rb}{2i}\right)=0$. The same holds if the roles of a and b are switched. Therefore, all cases are exhausted and the proof of the claim is complete $\square$

Therefore, the pair $(\mathfrak{A}, \overline{\mathsf{L}})$ is a (C, D)-quasi-Leibniz quantum compact metric space of Definition (2.2.9).

For the C*-subalgebras. Fix $n \in \mathbb{N}$. Since $\overline{\mathsf{L}}$ is a quasi-Leibniz Lip-norm defined on $\mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$, it is routine to check that $\overline{\mathsf{L}}$ satisfies all but property *2.* of Definition (2.2.5) on $\mathfrak{sa}\left(\mathfrak{A}_n\right)$. To show property *2.*, we begin by noting that by Theorem (2.2.6) there exists some state $\mu \in \mathscr{S}(\mathfrak{A})$ such that the set $\mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$ is totally bounded for $\|\cdot\|_{\mathfrak{A}}$. However, since the set:

$$\{a \in \mathfrak{sa}\left(\mathfrak{A}_n\right) : \overline{\mathsf{L}}(a) \leqslant 1, \mu(a) = 0\} \subseteq \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$$

and $\mu \in \mathscr{S}(\mathfrak{A}_n)$, then by Theorem (2.2.6), the seminorm $\overline{\mathsf{L}}$ is a quasi-Leibniz Lip-norm on $\mathfrak{sa}\left(\mathfrak{A}_n\right)$.

Now, we prove convergence. Let $\varepsilon > 0$. The fact that $\mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$ is totally bounded by Theorem (2.2.6) implies that there exist $a_1, \ldots, a_k \in \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$ such that $\mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right) \subseteq \cup_{j=1}^{k} B_{\|\cdot\|_{\mathfrak{A}}}\left(a_j, \varepsilon/3\right).$

By Expression (3.2.1), for each $j \in \{1, \ldots, k\}$, there exist $a'_j \in \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}\right) \cap \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ such that $\|a_j - a'_j\|_{\mathfrak{A}} < \varepsilon/3$, and so $|\mu(a'_j)| = |\mu(a_j - a'_j)| \leqslant \|a_j - a'_j\|_{\mathfrak{A}} < \varepsilon/3$ since states are contractive by definition. Hence:

$$\mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right) \subseteq \cup_{j=1}^{k} B_{\|\cdot\|_{\mathfrak{A}}}\left(a'_j, 2\varepsilon/3\right). \tag{3.2.3}$$

Next, since $a'_1, \ldots, a'_k \in \mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$, let $N = \min\{m \in \mathbb{N} : \{a'_1, \ldots, a'_k\} \subseteq \mathfrak{A}_m\}$.

Fix $n \geqslant N$. Let $a \in \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$. By Expression (3.2.3), there exists $b \in \mathfrak{sa}\left(\mathfrak{A}_N\right) \subseteq \mathfrak{sa}\left(\mathfrak{A}_n\right)$ such that $b \in \mathfrak{Lip}_1\left(\mathfrak{A}_n, \overline{\mathsf{L}}\right)$, $\|a - b\|_{\mathfrak{A}} < 2\varepsilon/3$, and $|\mu(b)| < \varepsilon/3$, where μ is seen as a state of $\mathfrak{A}_n$. Now, we have that $\mu(b) \in \mathbb{R}$ by Lemma (2.1.21) since μ is positive, and so $b - \mu(b)1_{\mathfrak{A}} \in \mathfrak{Lip}_1\left(\mathfrak{A}_n, \overline{\mathsf{L}}, \mu\right)$ since Lip-norms vanish on scalars. Therefore:

$$\|a - (b - \mu(b)1_{\mathfrak{A}})\|_{\mathfrak{A}} \leqslant \|a - b\|_{\mathfrak{A}} + \|\mu(b)1_{\mathfrak{A}}\|_{\mathfrak{A}} < \varepsilon. \tag{3.2.4}$$

In summary, for each $a \in \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$, there exists $c \in \mathfrak{Lip}_1\left(\mathfrak{A}_n, \overline{\mathsf{L}}, \mu\right)$ such that $\|a - c\|_{\mathfrak{A}} < \varepsilon$ for $n \geqslant N$. Now, if $a \in \mathfrak{Lip}_1\left(\mathfrak{A}_n, \overline{\mathsf{L}}, \mu\right)$, then $a \in \mathfrak{Lip}_1\left(\mathfrak{A}, \overline{\mathsf{L}}, \mu\right)$ and $\|a - a\|_{\mathfrak{A}} = 0 < \varepsilon$.

Consider the bridge $\gamma = (\mathfrak{A}, 1_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}, \iota_n)$ in the sense of Definition (2.3.2), where $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is identity and $\iota_n : \mathfrak{A}_n \to \mathfrak{A}$ is inclusion. But, since the pivot is $1_{\mathfrak{A}}$, the height is 0. Now, combining Lemma (3.2.3), Inequality (3.2.4) and the subsequent two sentences, we gather that the reach of the bridge is bounded by ε. Thus, by definition of length and Theorem-Definition (2.3.16), we conclude:

$$\Lambda_{C,D}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}}\right), \left(\mathfrak{A}, \overline{\mathsf{L}}\right)\right) \leqslant \varepsilon,$$

which establishes convergence. The fact that $(\mathfrak{A}_n, \mathsf{L}) = (\mathfrak{A}_n, \overline{\mathsf{L}})$ for all $n \in \mathbb{N}$ is clear by Expression (3.2.1), which completes the proof. □

Remark 3.2.6. *Another nice consequence of Proposition (3.2.4) is that it utilizes the notion of "closing" a Lip-norm in a non-trivial way. This notion was introduced by Rieffel in [60] in the comments preceding [60, Proposition 4.4] to extend a Lip-norm onto the completion of a space. Whereas, we use this notion to restrict our attention to a particular dense subspace to allow for finite-dimensional approximations.*

In order for Proposition (3.2.4) to have a powerful impact, we need to show that all unital AF algebras may be equipped with quasi-Leibniz Lip-norms. In Theorem (3.2.11), we show that this can be accomplished by using quotient norms and Rieffel's work on Leibniz seminorms and best approximations [67]. However, to accomplish this, we first prove a basic fact about certain Lip-norms in Proposition (3.2.10). This fact is motivated by the observation that it can be the case that a candidate for a Lip-norm, L, to be naturally defined on a unital dense subspace $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{A}$ such that $\mathrm{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$ is dense in $\mathfrak{sa}(\mathfrak{A})$. Proposition (3.2.10) will allow us to verify a condition in this candidates natural setting of $\mathrm{dom}(\mathsf{L})$ to induce a Lip-norm on $\mathrm{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$. An example of an application of Proposition (3.2.10) will be seen immediately in Theorem (3.2.11). But, first, a definition and some basic results about best approximations.

Definition 3.2.7. *Let $\mathfrak{A}$ be a Banach space with norm $\|\cdot\|_{\mathfrak{A}}$. We say that a norm closed subspace $\mathfrak{B} \subseteq \mathfrak{A}$ satisfies* best approximation *if for all $a \in \mathfrak{A}$, there exists a $b_a \in \mathfrak{B}$ such that* $\inf\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\} = \|a-b_a\|_{\mathfrak{A}}$*, where* $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \inf\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\}$ *is the quotient norm.*

The following result is well-known. However, we provide a proof.

Lemma 3.2.8. *Let $\mathfrak{A}$ be a Banach space with norm $\|\cdot\|_{\mathfrak{A}}$. If $\mathfrak{B}$ is a finite-dimensional subspace of $\mathfrak{A}$, then $\mathfrak{B}$ satisifes best approximation.*

Proof. Let $a \in \mathfrak{A}$. Consider the set $\mathfrak{B}_a = \{b \in \mathfrak{B} : \|a-b\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\}$, which is non-empty since $0_{\mathfrak{A}} \in \mathfrak{B}_a$ since $0_{\mathfrak{A}} \in \mathfrak{B}$. Now, the sets $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\}$ and $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$ are both bounded below by 0 and we claim that they have the same infimum. Indeed, first, since $\mathfrak{B}_a \subseteq \mathfrak{B}$, then $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \inf\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\}$ is a lower bound of $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$. Assume by way of contradiction that there is a lower bound l of $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$ such that $\|a\|_{\mathfrak{A}/\mathfrak{B}} < l$. Now, since $\|a\|_{\mathfrak{A}/\mathfrak{B}}$ is the greatest lower bound of $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\}$, we have that l is not a lower bound of $\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\}$. Hence, there exists $c \in \mathfrak{B}$ such that $\|a-c\|_{\mathfrak{A}} < l$. Now, by definition of $\mathfrak{B}_a$, we have that $l \leqslant \|a\|_{\mathfrak{A}}$. However, this implies that $\|a-c\|_{\mathfrak{A}} < l \leqslant \|a\|_{\mathfrak{A}}$, which shows that $c \in \mathfrak{B}_a$. Yet, l is a lower bound for

$\mathfrak{B}_a$ and thus $l \leqslant \|a - c\|_{\mathfrak{A}} < l$, which is a contradiction. Thus, no such lower bound for $\{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$ exists, and so $\|a\|_{\mathfrak{A}/\mathfrak{B}}$ is the greatest lower bound of $\{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$. Therefore:

$$\|a\|_{\mathfrak{A}/\mathfrak{B}} = \inf\left\{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}\right\} = \inf\left\{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\right\}. \tag{3.2.5}$$

Next, assume that $b \in \mathfrak{B}_a$. Then, we have that:

$$\|b\|_{\mathfrak{A}} \leqslant \|a - b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} = 2\|a\|_{\mathfrak{A}}.$$

Therefore, the set $\mathfrak{B}_a = \{b \in \mathfrak{B} : \|a - b\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\}$ is bounded and is closed by continuity of the norm and that $\mathfrak{B}$ is closed by finite-dimensionality. By finite-dimensionality, the set $\mathfrak{B}_a$ is compact. Now, define $f_a : c \in \mathfrak{A} \longmapsto \|a - c\|_{\mathfrak{A}} \in \mathbb{R}$. Again by continuity of norm, the map f_a is continuous. However, since $\mathfrak{B}_a$ is compact, we have that $f_a(\mathfrak{B}_a) = \{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}$ is compact in $\mathbb{R}$. Thus, there exists $b_a \in \mathfrak{B}_a$ such that:

$$\|a - b_a\|_{\mathfrak{A}} = \inf f_a(\mathfrak{B}_a) = \inf\{\|a - b\|_{\mathfrak{A}} : b \in \mathfrak{B}_a\}.$$

But, by Expression (3.2.5), we have that $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \inf\{\|a - c\|_{\mathfrak{A}} : c \in \mathfrak{B}\} = \inf\{\|a - c\|_{\mathfrak{A}} : c \in \mathfrak{B}_b\} = \|a - b_a\|_{\mathfrak{A}}$, which completes the proof as $a \in \mathfrak{A}$ was arbitrary. $\square$

Lemma 3.2.9. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{B} \subseteq \mathfrak{A}$ is a norm closed self-adjoint subspace of $\mathfrak{A}$ that satisfies best approximation, then for all $a \in \mathfrak{sa}(\mathfrak{A})$ there exists $b_a \in \mathfrak{sa}(\mathfrak{B})$ such that the quotient norm $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \|a - b_a\|_{\mathfrak{A}}$.*

Moreover, for all $a \in \mathfrak{sa}(\mathfrak{A})$, the quotient norms $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \|a\|_{\mathfrak{sa}(\mathfrak{A})/\mathfrak{sa}(\mathfrak{B})}$.

Proof. Let $a \in \mathfrak{sa}(\mathfrak{A})$. By assumption, there exists $b \in \mathfrak{B}$ such that $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \|a - b\|_{\mathfrak{A}}$. Now, set $b_a = \frac{b+b^*}{2} \in \mathfrak{sa}(\mathfrak{B})$ and:

$$\begin{aligned}
\left\| a - \frac{b+b^*}{2} \right\|_{\mathfrak{A}} &= \left\| \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}b^* \right\|_{\mathfrak{A}} \\
&\leqslant \frac{1}{2}\|a-b\|_{\mathfrak{A}} + \frac{1}{2}\|(a-b)^*\|_{\mathfrak{A}} \\
&= \frac{1}{2}\|a-b\|_{\mathfrak{A}} + \frac{1}{2}\|a-b\|_{\mathfrak{A}} \\
&= \|a-b\|_{\mathfrak{A}} \\
&= \|a\|_{\mathfrak{A}/\mathfrak{B}} \\
&= \inf\{\|a-c\|_{\mathfrak{A}} : c \in \mathfrak{B}\} \leqslant \left\| a - \frac{b+b^*}{2} \right\|_{\mathfrak{A}}
\end{aligned}$$

since $*$ is an isometry by Lemma (2.1.9). Therefore, we gather that $\|a\|_{\mathfrak{A}/\mathfrak{B}} = \left\|a - \frac{b+b^*}{2}\right\|_{\mathfrak{A}} = \|a - b_a\|_{\mathfrak{A}}$.

Next, let $a \in \mathfrak{sa}(\mathfrak{A})$, then by the above, there exists $b_a \in \mathfrak{sa}(\mathfrak{B})$ such that:

$$\begin{aligned}
\|a - b_a\|_{\mathfrak{A}} &= \|a\|_{\mathfrak{A}/\mathfrak{B}} \\
&= \inf\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{B}\} \\
&\leqslant \inf\{\|a-b\|_{\mathfrak{A}} : b \in \mathfrak{sa}(\mathfrak{B})\} \\
&= \|a\|_{\mathfrak{sa}(\mathfrak{A})/\mathfrak{sa}(\mathfrak{B})} \leqslant \|a-b_a\|_{\mathfrak{A}},
\end{aligned}$$

which completes the proof. □

Proposition 3.2.10. *Let $\mathfrak{A}$ be a unital C^*-algebra and let L be a seminorm defined on some dense unital subspace* $\operatorname{dom}(\mathsf{L})$ *of $\mathfrak{A}$ such that* $\operatorname{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$ *is a dense subspace of* $\mathfrak{sa}(\mathfrak{A})$ *and* $\{a \in \operatorname{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{C}1_{\mathfrak{A}}$.

If the set $\{a + \mathbb{C}1_{\mathfrak{A}} \in \mathfrak{A}/\mathbb{C}1_{\mathfrak{A}} : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$ *is totally bounded in* $\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$ *for* $\|\cdot\|_{\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}}$*, then the pair* $(\mathfrak{A}, \mathsf{L})$ *formed by the dense unital subspace* $\operatorname{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$ *of* $\mathfrak{sa}(\mathfrak{A})$ *and the restriction of* L *to* $\operatorname{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$ *is a quantum compact metric space.*

Proof. First, the set $\mathrm{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A})$ is a dense subspace of $\mathfrak{sa}(\mathfrak{A})$ and $\{a \in \mathrm{dom}(\mathsf{L}) \cap \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$. Next, let $(a_n + \mathbb{R}1_{\mathfrak{A}})_{n\in\mathbb{N}} \subseteq \{a + \mathbb{R}1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$. The sequence $(a_n + \mathbb{C}1_{\mathfrak{A}})_{n\in\mathbb{N}} \subseteq \{a + \mathbb{C}1_{\mathfrak{A}} \in \mathfrak{A}/\mathbb{C}1_{\mathfrak{A}} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$. Hence, by assumption and total boundedness, there exists some Cauchy subsequence $(a_{n_k} + \mathbb{C}1_{\mathfrak{A}})_{k\in\mathbb{N}}$ with respect to $\|\cdot\|_{\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}}$.

The space $\mathbb{C}1_{\mathfrak{A}}$ is finite dimensional and therefore satisfies best approximation in $\mathfrak{A}$ by Lemma (3.2.8). Also, we have that $\mathfrak{sa}(\mathbb{C}1_{\mathfrak{A}}) = \mathbb{R}1_{\mathfrak{A}}$. Note that $a_n \in \mathfrak{sa}(\mathfrak{A})$ for each $n \in \mathbb{N}$. Hence, by Lemma (3.2.9) , the subsequence $(a_{n_k} + \mathbb{R}1_{\mathfrak{A}})_{n\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{\mathfrak{sa}(\mathfrak{A})/\mathbb{R}1_{\mathfrak{A}}}$, which completes the proof. □

Theorem 3.2.11. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. For each $n \in \mathbb{N}$, we denote the quotient norm of $\mathfrak{A}/\mathfrak{A}_n$ with respect to $\|\cdot\|_{\mathfrak{A}}$ by S_n. Let $\beta : \mathbb{N} \to (0, \infty)$ have limit 0 at infinity.*

If, for all $a \in \mathfrak{A}$, we set:

$$\mathsf{L}_{\mathcal{U}}^{\beta}(a) = \sup\left\{\frac{S_n(a)}{\beta(n)} : n \in \mathbb{N}\right\},$$

then the domain of $\mathsf{L}_{\mathcal{U}}^{\beta}$ contains $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$, and

1. *using notation from Proposition (3.2.4), we have that $\left(\mathfrak{A}, \mathsf{L}_{\mathcal{U}}^{\beta}\right)$, $\left(\mathfrak{A}_n, \mathsf{L}_{\mathcal{U}}^{\beta}\right)$, $\left(\mathfrak{A}, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right)$, and $\left(\mathfrak{A}_n, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right)$ for all $n \in \mathbb{N}$ are Leibniz quantum compact metric spaces where we view $\mathsf{L}_{\mathcal{U}}^{\beta}$ restricted to $\mathfrak{sa}(\mathfrak{A})$ such that*
2. $\lim_{n\to\infty} \Lambda_{1,0}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right), \left(\mathfrak{A}, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right)\right) = 0$ *and* $\lim_{n\to\infty} \Lambda_{1,0}\left(\left(\mathfrak{A}_n, \mathsf{L}_{\mathcal{U}}^{\beta}\right), \left(\mathfrak{A}, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right)\right) = 0$.

Proof. We begin by proving *1.* By [67, Theorem 3.1], for all $n \in \mathbb{N}$, we have that since $\mathfrak{A}_n$ is unital, the quotient norm S_n satisfies condition *2.* of Definition (2.2.9) for $C = 1$, $D = 0$, and therefore, so does $\mathsf{L}_{\mathcal{U}}^{\beta}$. Thus $\mathsf{L}_{\mathcal{U}}^{\beta}$ is a Leibniz seminorm.

To show that the seminorm only vanishes on scalars, note that $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}} \subsetneq \mathfrak{A}_n$ for each $n \in \mathbb{N} \setminus \{0\}$ implies that ${\mathsf{L}_{\mathcal{U}}^{\beta}}^{-1}(\{0\}) = \mathbb{C}1_{\mathfrak{A}}$.

For the domain, let $a \in \cup_{n\in\mathbb{N}}\mathfrak{A}_n$, then there exists $N \in \mathbb{N}$ such that $a \in \mathfrak{A}_k$ for all $k \geqslant N$. Therefore, the seminorm $S_k(a) = 0$ for all $k \geqslant N$, and hence, the seminorm $\mathsf{L}_{\mathcal{U}}^{\beta}$ evaluated at a is a supremum over finitely many terms, and is thus finite. Therefore, the domain of $\mathsf{L}_{\mathcal{U}}^{\beta}$ contains $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$.

Since quotient norms are continuous, we have that $\mathsf{L}_{\mathcal{U}}^{\beta}$ is lower semi-continuous as a supremum of continuous real-valued maps.

Now, note that since β is convergent, we have $K = \sup\{\beta(n) : n \in \mathbb{N}\} < \infty$. Let $q_0 : \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}_0 = \mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$ denote the quotient map. Define:

$$\mathsf{L}_1 = \left\{ a \in \cup_{n\in\mathbb{N}}\mathfrak{A}_n : \mathsf{L}_{\mathcal{U}}^{\beta}(a) \leqslant 1 \right\}.$$

By way of Proposition (3.2.10), we now show that $q_0(\mathsf{L}_1)$ totally bounded with respect to the quotient norm on $\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$, in which the quotient norm is simply S_0 since $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\varepsilon > 0$. By definition of $\mathsf{L}_{\mathcal{U}}^{\beta}$, there exists $N \in \mathbb{N}$ such that $\beta(N) < \varepsilon/3$, so that $S_N(a) \leqslant \beta(N) < \varepsilon/3$ for all $a \in \mathsf{L}_1$. Since $\mathfrak{A}_N$ is a finite dimensional subspace, there exists a best approximation to a in $\mathfrak{A}_N$ for all $a \in \mathsf{L}_1$ by Lemma (3.2.8). Thus, for all $a \in \mathsf{L}_1$, by axiom of choice, set $b_N(a) \in \mathfrak{A}_N$ to be one best approximation of a. Define:

$$B_N = \{b_N(a) \in \mathfrak{A}_N : a \in \mathsf{L}_1\}.$$

If $a \in \mathsf{L}_1$, then since $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$:

$$\begin{aligned} S_0(b_N(a)) &= \inf\{\|b_N(a) - \lambda 1_{\mathfrak{A}}\|_{\mathfrak{A}} : \lambda \in \mathbb{C}\} \\ &= \inf\{\|b_N(a) - a + a - \lambda 1_{\mathfrak{A}}\|_{\mathfrak{A}} : \lambda \in \mathbb{C}\} \\ &\leqslant \|b_N(a) - a\|_{\mathfrak{A}} + \inf\{\|a - \lambda 1_{\mathfrak{A}}\| : \lambda \in \mathbb{C}\} \end{aligned}$$

$$
\begin{aligned}
&= S_N(a) + S_0(a)\\
&\leqslant \beta(N) + \beta(0) \leqslant 2K.
\end{aligned}
$$

Hence, the set $q_0(B_N) \subset \mathfrak{A}_N/\mathbb{C}1_{\mathfrak{A}}$ is bounded with respect S_0 on $\mathfrak{A}_N/\mathbb{C}1_{\mathfrak{A}}$, and therefore totally bounded with respect to S_0 on $\mathfrak{A}_N/\mathbb{C}1_{\mathfrak{A}}$ since $\mathfrak{A}_N$ is finite dimensional. Let F_N be a finite $\varepsilon/3$-net of $q_0(B_N)$, so let $f_N = \{b_N(a_1), \ldots, b_N(a_n) \in \mathfrak{A}_N : a_j \in \mathsf{L}_1, 1, \leqslant j \leqslant n < \infty\}$ such that $F_N = q_0(f_N)$.

We claim that $q_0(\{a_1, \ldots, a_n\})$ is a finite ε-net for $q_0(\mathsf{L}_1)$. Indeed, let $a \in \mathsf{L}_1$, then $b_N(a) \in B_N$, so there exists $b_N(a_j) \in f_N$ such that $S_0(b_N(a) - b_N(a_j)) < \varepsilon/3$. Therefore:

$$
\begin{aligned}
S_0(a - a_j) &\leqslant S_0(a - b_N(a)) + S_0(b_N(a) - b_N(a_j)) + S_0(b_N(a_j) - a_j)\\
&\leqslant \|a - b_N(a)\|_{\mathfrak{A}} + \varepsilon/3 + \|b_N(a_j) - a_j\|_{\mathfrak{A}}\\
&= S_N(a) + \varepsilon/3 + S_N(a_j) < \varepsilon.
\end{aligned}
$$

Hence, the set $q_0(\{a_1, \ldots, a_n\})$ serves as a finite ε-net for $q_0(\mathsf{L}_1)$. Therefore, by Proposition (3.2.10), the pair $\left(\mathfrak{A}, \mathsf{L}_{\mathcal{U}}^{\beta}\right)$ is a Leibniz quantum compact metric space, where we view $\mathsf{L}_{\mathcal{U}}^{\beta}$ restricted to $\mathfrak{sa}(\mathfrak{A})$.

The remaining conclusions follow by Proposition (3.2.4). $\square$

Remark 3.2.12. *We note that 2. of Theorem (3.2.11) is not obtained from an inequality like that of 1., 2. of Theorem (3.1.3), and we suspect that in general, 2. of Theorem (3.2.11) cannot be obtained from an inequality. This is because it is unlikely that for $a \in \mathfrak{A}, \mathsf{L}_{\mathcal{U}}^{\beta}(a) \leqslant 1$ we have that $\mathsf{L}_{\mathcal{U}}^{\beta}(b_n(a)) \leqslant 1$ for any best approximation of a in $\mathfrak{A}_n$ for all $n \in \mathbb{N}$, which was a crucial step for the the inequality of Theorem (3.1.3) achieved by conditional expectations rather than best approximations. This highlights a vital strength of the faithful tracial state case with the Lip-norms from Theorem (3.1.3) since the inequality of Theorem (3.1.3) is crucial for our convergence results* *of AF algebras as we will see in the proof of Theorem (4.5.6). But, the Lip-norms of Theorem (3.2.11) are vital for the general theory of AF algebras as quantum metric spaces to provide natural finite dimensional approximations in propinquity for all unital AF algebras.*

Remark 3.2.13. *Proposition (3.2.4) and Theorem (3.2.11) can be easliy translated to the inductive limit setting of AF algebras.*

3.3 Quantum isometries between AF algebras

We find conditions that provide quantum isometries (Theorem-Definition (2.3.16)) between AF algebras with the Lip-norms from Theorem (3.1.3), or equivalently, when their distance is 0 in the quantum propinquity, or equivalently, when they produce the same equivlance classes that form the quantum propinqtuiy metric space. First, this is motivated by Bratteli's conditions for *-isomorphisms for AF algebras [11, Theorem 2.7]. Second, Inequality (4.5.8) of Theorem (4.5.6) along with the convergence results of [3] display the importance of providing quantum isometries not only at the level of the entire AF algebra, but also at the level of the finite-dimensional C*-subalgebras.

We now present conditions for quantum isometries for AF algebras in the faithful tracial state case. We note that the hypotheses of the theorem are natural since they are chosen specifically to preserve the trace and the finite-dimensional structure of the AF algebra, which are the ingredients used to construct the Lip-norms. Also, Theorem (3.3.1) will be used in Theorem (5.2.1) to find appropriate inductive limits that are quantum isometric to quotients. This is vital for convergence results since the inductive limit setting is more appropriate to provide convergence as seen in Section (4.5) and since most of our examples thus far are presented in the inductive limit setting.

Theorem 3.3.1. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$ endowed with a faithful tracial state μ. Let $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\mathfrak{B}$ be a unital AF algebra with unit $1_{\mathfrak{B}}$ endowed with a faithful tracial state ν and $\mathcal{V} = (\mathfrak{B}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{B} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$ with $\mathfrak{B}_0 = \mathbb{C}1_{\mathfrak{B}}$. Let $\beta : \mathbb{N} \to (0,\infty)$ have limit 0 at infinity. Let $\mathsf{L}^{\beta}_{\mathcal{U},\mu}, \mathsf{L}^{\beta}_{\mathcal{V},\nu}$ denote the associated $(2,0)$-quasi-Leibniz Lip-norms from Theorem (3.1.5) on $\mathfrak{A}, \mathfrak{B}$ respectively.*

*If $\phi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is a unital *-monomorphism such that the following hold:*

1. *$\phi(\mathfrak{A}_n) = \mathfrak{B}_n$ for all $n \in \mathbb{N}$, and*
2. *$\mu = \nu \circ \phi$,*

then:

$$\phi : \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right) \longrightarrow \left(\mathfrak{B}, \mathsf{L}^{\beta}_{\mathcal{V},\nu}\right)$$

is a quantum isometry of Theorem-Defintion (2.3.16) and:

$$\Lambda_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{B}, \mathsf{L}^{\beta}_{\mathcal{V},\nu}\right)\right) = 0.$$

Moreover, for all $n \in \mathbb{N}$, we have:

$$\Lambda_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{B}_n, \mathsf{L}^{\beta}_{\mathcal{V},\nu}\right)\right) = 0.$$

Proof. Fix $a \in \mathfrak{A}$. Let $n \in \mathbb{N}$. By Example (2.1.13), since $\mathfrak{B}_n$ is finite dimensional, the C*-algebra $\mathfrak{B}_n \cong \oplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \ldots, n(N) \in \mathbb{N} \setminus \{0\}$ with *-isomorphism $\pi : \oplus_{j=1}^N \mathfrak{M}(n(j)) \longrightarrow \mathfrak{B}_n$. Let E be the set of matrix units for $\oplus_{j=1}^N \mathfrak{M}(n(j))$ given in Notation (3.1.10). Define $E_\pi = \{\pi(b) \in \mathfrak{B}_n : b \in E\}$. . Furthermore, since $\phi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is a *-monomorphism that satisfies hypothesis *1.*, the map $\phi : \mathfrak{A} \to \mathfrak{B}$ is a *-isomorphism by [11, Theorem 2.7]. Hence, by Proposition

(3.1.13) and $\mu = \nu \circ \phi \iff \mu \circ \phi^{-1} = \nu$, we gather that:

$$
\begin{aligned}
\|\phi(a) - \mathbb{E}\left(\phi(a)|\mathfrak{B}_n\right)\|_{\mathfrak{B}} &= \left\| \phi(a) - \sum_{e \in E_\pi} \frac{\nu\left(e^*\phi(a)\right)}{\nu\left(e^*e\right)} e \right\|_{\mathfrak{B}} \\
&= \left\| \phi(a) - \sum_{e \in E_\pi} \frac{\mu \circ \phi^{-1}\left(e^*\phi(a)\right)}{\mu \circ \phi^{-1}\left(e^*e\right)} e \right\|_{\mathfrak{B}} \\
&= \left\| \phi(a) - \sum_{e \in E_\pi} \frac{\mu\left(\phi^{-1}(e^*)\phi^{-1}(\phi(a))\right)}{\mu\left(\phi^{-1}\left(e^*e\right)\right)} e \right\|_{\mathfrak{B}} \\
&= \left\| \phi^{-1}\left(\phi(a) - \sum_{e \in E_\pi} \frac{\mu\left(\phi^{-1}(e^*)a)\right)}{\mu\left(\phi^{-1}\left(e^*e\right)\right)} e \right) \right\|_{\mathfrak{A}} \\
&= \left\| a - \sum_{e \in E_\pi} \frac{\mu\left(\phi^{-1}(e^*)a)\right)}{\mu\left(\phi^{-1}\left(e^*e\right)\right)} \phi^{-1}(e) \right\|_{\mathfrak{A}} \\
&= \left\| a - \sum_{e' \in E} \frac{\mu\left(\phi^{-1} \circ \pi(e'^*)a)\right)}{\mu\left(\phi^{-1} \circ \pi\left(e'^*e'\right)\right)} \phi^{-1} \circ \pi(e') \right\|_{\mathfrak{A}} \\
&= \|a - \mathbb{E}\left(a|\mathfrak{A}_n\right)\|_{\mathfrak{A}},
\end{aligned}
$$

where the last equality follows from Proposition (3.1.13) and the fact that $\phi^{-1} \circ \pi : \oplus_{j=1}^N \mathfrak{M}(n(j)) \to \mathfrak{A}_n$ is a *-isomorphism by assumption.

Thus, since $n \in \mathbb{N}$ was arbitrary, we have:

$$
\mathsf{L}^{\beta}_{\mathcal{V},\nu} \circ \phi(a) = \mathsf{L}^{\beta}_{\mathcal{U},\mu}(a) \tag{3.3.1}
$$

for all $a \in \mathfrak{A}$. Hence:

$$
\phi : \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right) \longrightarrow \left(\mathfrak{B}, \mathsf{L}^{\beta}_{\mathcal{V},\nu}\right)
$$

is a quantum isometry by Theorem-Definition (2.3.16).

Also, we have $\left(\mathfrak{A}_m, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)$ is quantum isometric to $\left(\mathfrak{B}_m, \mathsf{L}^{\beta}_{\mathcal{V},\nu}\right)$ by the map ϕ restricted to $\mathfrak{A}_m$ for all $m \in \mathbb{N}$ by hypothesis *1.*, which completes the proof. □

Now, in Theorem (3.3.2), we provide quantum isometries in the case of the Leibniz Lip-norms from Theorem (3.2.11) of the form $\mathsf{L}^{\beta}_{\mathcal{U}}$, and as a corollary, we will do the same for the Leibniz Lip-norms of the form $\overline{\mathsf{L}^{\beta}_{\mathcal{U}}}$ with the same hypotheses. Now, since neither of these Lip-norms require information about a faithful tracial state, the conditions to provide quantum isometries are weaker than for Theorem (3.3.1). Indeed:

Theorem 3.3.2. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$. Let $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\mathfrak{B}$ be a unital AF algebra with unit $1_{\mathfrak{B}}$ and $\mathcal{V} = (\mathfrak{B}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{B} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$ with $\mathfrak{B}_0 = \mathbb{C}1_{\mathfrak{B}}$. Let $\beta : \mathbb{N} \to (0,\infty)$ have limit 0 at infinity. Let $\mathsf{L}^{\beta}_{\mathcal{U}}, \mathsf{L}^{\beta}_{\mathcal{V}}$ denote the associated Lip-norms from Theorem (3.2.11) on $\mathfrak{A}, \mathfrak{B}$ respectively.*

*If $\phi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is a unital *-monomorphism such that $\phi(\mathfrak{A}_n) = \mathfrak{B}_n$ for all $n \in \mathbb{N}$, then*

$$\phi : \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U}}\right) \longrightarrow \left(\mathfrak{B}, \mathsf{L}^{\beta}_{\mathcal{V}}\right)$$

is a quantum isometry of Theorem-Defintion (2.3.16) and:

$$\Lambda_{1,0}\left(\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U}}\right), \left(\mathfrak{B}, \mathsf{L}^{\beta}_{\mathcal{V}}\right)\right) = 0.$$

Moreover, for all $n \in \mathbb{N}$, we have:

$$\Lambda_{1,0}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U}}\right), \left(\mathfrak{B}_n, \mathsf{L}^{\beta}_{\mathcal{V}}\right)\right) = 0.$$

Proof. For each $n \in \mathbb{N}$, let $S^{\mathfrak{A}}_n : \mathfrak{A}/\mathfrak{A}_n \longrightarrow \mathbb{R}$ denote the quotient norm and similarly denote $S^{\mathfrak{B}}_n$. Fix $a \in \mathfrak{A}$. Since $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a *-isomorphism by [11, Theorem 2.7], we have for all $n \in \mathbb{N}$:

$$S^{\mathfrak{B}}_n(\phi(a)) = \inf\left\{\|\phi(a) - b\|_{\mathfrak{B}} : b \in \mathfrak{B}_n\right\}$$

$$
\begin{aligned}
&= \inf\left\{ \|\phi^{-1}\left(\phi(a) - b\right)\|_{\mathfrak{A}} : b \in \mathfrak{B}_n \right\} \\
&= \inf\left\{ \|a - \phi^{-1}(b)\|_{\mathfrak{A}} : b \in \mathfrak{B}_n \right\} \\
&= \inf\left\{ \|a - a'\|_{\mathfrak{A}} : a' \in \mathfrak{A}_n \right\} \\
&= S_n^{\mathfrak{A}}(a),
\end{aligned}
$$

where in the second to last equality we use the fact that $\phi^{-1}(\mathfrak{B}_n) = \mathfrak{A}_n$. The rest of the proof follows exactly as the rest of the proof of Theorem (3.3.1) starting at Equation (3.3.1). □

We will now provide quantum isometries for the Lip-norms that provide the desirable convergence of finite-dimensional spaces as seen in Theorem (3.2.11), and we see a direct application of the importance of having quantum isometries that preserve finite-dimensional approximations.

Corollary 3.3.3. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$. Let $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\mathfrak{B}$ be a unital AF algebra with unit $1_{\mathfrak{B}}$ and $\mathcal{V} = (\mathfrak{B}_n)_{n\in\mathbb{N}}$ be an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{B} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$ with $\mathfrak{B}_0 = \mathbb{C}1_{\mathfrak{B}}$. Let $\beta : \mathbb{N} \to (0,\infty)$ have limit 0 at infinity. Let $\overline{\mathsf{L}_{\mathcal{U}}^{\beta}}, \overline{\mathsf{L}_{\mathcal{V}}^{\beta}}$ denote the associated Lip-norms from Theorem (3.2.11) on $\mathfrak{A}, \mathfrak{B}$ respectively.*

*If $\phi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is a unital *-monomorphism such that $\phi(\mathfrak{A}_n) = \mathfrak{B}_n$ for all $n \in \mathbb{N}$, then there exists a quantum isometry (not necessarily ϕ) from $\left(\mathfrak{A}, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right)$ to $\left(\mathfrak{B}, \overline{\mathsf{L}_{\mathcal{V}}^{\beta}}\right)$ and thus:*

$$\mathsf{\Lambda}_{1,0}\left(\left(\mathfrak{A}, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right), \left(\mathfrak{B}, \overline{\mathsf{L}_{\mathcal{V}}^{\beta}}\right)\right) = 0.$$

Moreover:

$$\mathsf{\Lambda}_{1,0}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}_{\mathcal{U}}^{\beta}}\right), \left(\mathfrak{B}_n, \overline{\mathsf{L}_{\mathcal{V}}^{\beta}}\right)\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Proof. By Proposition (3.2.4), we have that $\left(\mathfrak{A}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right) = \left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U}}\right)$ and $\left(\mathfrak{B}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right) = \left(\mathfrak{B}_n, \mathsf{L}^{\beta}_{\mathcal{V}}\right)$ for all $n \in \mathbb{N}$. Thus, by Theorem (3.3.2), we have that:

$$\Lambda_{1,0}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right) = 0$$

for all $n \in \mathbb{N}$ by the triangle inequality.

Next, by the triangle inequality, we have:

$$\begin{aligned}
\Lambda_{1,0}\left(\left(\mathfrak{A}, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right) &\leqslant \Lambda_{1,0}\left(\left(\mathfrak{A}, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{A}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right)\right) \\
&\quad + \Lambda_{1,0}\left(\left(\mathfrak{A}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right) \\
&\quad + \Lambda_{1,0}\left(\left(\mathfrak{B}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right) \\
&= \Lambda_{1,0}\left(\left(\mathfrak{A}, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{A}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right)\right) + 0 \\
&\quad + \Lambda_{1,0}\left(\left(\mathfrak{B}_n, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right).
\end{aligned}$$

Hence, we have $\Lambda_{1,0}\left(\left(\mathfrak{A}, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right), \left(\mathfrak{B}, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)\right) = 0$ by part *2.* of Theorem (3.2.11). Therefore, by Theorem-Defintion (2.3.16), there exists a quantum isometry from $\left(\mathfrak{A}, \overline{\mathsf{L}^{\beta}_{\mathcal{U}}}\right)$ to $\left(\mathfrak{B}, \overline{\mathsf{L}^{\beta}_{\mathcal{V}}}\right)$. ☐

Remark 3.3.4. *The reason we state "(not necessarily ϕ)" in the above corollary is that in the case of the Lip-norms $\overline{\mathsf{L}^{\beta}_{\mathcal{U}}}$, we do not know explicitly how they are defined outside $\mathfrak{sa}\left(\cup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ and therefore on their entire domains. Hence, the proof of Theorem (3.3.2) cannot be used in this case. Thus, we see that the proof of this corollary as a consequence of the quantum propinquity and the importance of preserving finite-dimensional approximations. The map ϕ worked as a quantum isometry in Theorem (3.3.2) since we have an explicit definition of $\mathsf{L}^{\beta}_{\mathcal{U}}$ on all of $\mathfrak{A}$.*

Chapter 4

Continuous families of AF algebras with respect to Gromov-Hausdorff Propinquity

Now that quantum metric structure has been established for AF algebras, just as compact metric spaces are studied in the classical Gromov-Hausdorff topology of Definition (2.3.17), we seek to study quantum metric spaces in the quantum Gromov-Hausdorff propinquity topology of Theorem-Definition (2.3.16).

The main classes of AF algebras we study in this chapter are the UHF algebras of Glimm [28] and the Effros-Shen algebras of Effros and Shen [22]. Both of these classes are deeply rooted in the history of C*-algebras as discussed in Section (2.1.2). Thus, we felt it necessary to begin our study of AF algebras in the quantum Gromov-Hausdorff propinquity topology with these classes.

In Section (4.1), we show that the class UHF algebras equipped with quantum metric structure from Theorem (3.1.3) form a continuous image of the Baire space via their defining multiplicity sequences. In Section (4.2), we do the same for the Effros-Shen algebras, which also establishes this class as a continuous family with respect to there defining irrational parameters. Due to these results and a characterization of compact subsets of the Baire space, we establish some nontrivial compact classes of UHF and Effros-Shen algebras in Section (4.3). Section (4.4), shows how one may continuously vary the Lip-norms of Theorem (3.1.3) on a fixed AF algebra. Lastly, Section (4.5) utilizes the examples of convergence established in Sections (4.1,4.2) to provide general criteria of convergence of AF algebras. This criteria not only provides the continuity results of Sections (4.1,4.2), but also gives a valuable tool to provide the convergence results of Chapter 5.

As a final note, Section (4.5) is taken from the author's work in [1], and the remaining sections of Chapter 4 are due the work of F. Latrémolière and the author in [3], in which they established the first examples of convergence of AF algebras.

4.1 UHF algebras

As introduced in Example (2.1.79), a uniform, hyperfinite (UHF) algebra is a particular type of AF algebra obtained as the limit of unital, simple finite dimensional C*-algebras. UHF algebras were classified by Glimm [28] and, as AF algebras, they are also classified by their Elliott invariant [19]. In this section, we will study UHF algebras in the context of Noncommutative Metric Geometry. To accomplish this, one must first provide quantum metric structure for UHF algebras. By Lemma (2.1.80), UHF algebras are always unital simple AF algebras, and thus they admit a faithful tracial state. Moreover, the tracial state of a UHF algebra $\mathfrak{A}$ is unique. Therefore, we have two choices for quantum metric structure via conditional expectations from Theorem (3.1.3) or via quotient norms from Theorem (3.2.11). It will be evident in Theorem (4.1.7) that the right choice is the conditional expectation construciton from Theorem (3.1.3) and the inequality it provides on the distance from the finite-dimensional C*-subalgebras.

By Theorem (2.1.18), up to unitary conjugation, a unital *-monomorphism $\alpha : \mathfrak{B} \to \mathfrak{A}$ between two unital simple finite dimensional C*-algebras, i.e. two nonzero full matrix algebras $\mathfrak{A}$ and $\mathfrak{B}$, exists if and only if $\dim \mathfrak{A} = k^2 \dim \mathfrak{B}$ for $k \in \mathbb{N}$, and α must be of the form:

$$a \in \mathfrak{B} \longmapsto \begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix} \in \mathfrak{A}, \tag{4.1.1}$$

in which there are k-copies of a on the diagonal and 0's elsewhere.

It is thus sufficient, in order to characterize a unital inductive sequence of full matrix algebras, to give a sequence of positive integers:

Definition 4.1.1. *Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence of unital, simple finite dimensional C*-algebras with α_n a unital *-monomorphism for each $n \in \mathbb{N}$ and $\mathfrak{A}_0 = \mathbb{C}$.*

The multiplicity sequence *of $\mathcal{I}$ is the sequence $\left(\sqrt{\frac{\dim \mathfrak{A}_{n+1}}{\dim \mathfrak{A}_n}}\right)_{n\in\mathbb{N}}$ of positive integers, where $\sqrt{\frac{\dim \mathfrak{A}_{n+1}}{\dim \mathfrak{A}_n}}$ is the multiplicity of α_n for each $n \in \mathbb{N}$ by Definition (2.1.14) and Example (2.1.79).*

A multiplicity sequence is any sequence in $\mathbb{N} \setminus \{0\}$. A UHF algebra is always obtained as the limit of an inductive sequence in the following class:

Notation 4.1.2. *Let $\mathscr{StrictFullInductive}$ be the set of all unital inductive sequences of full matrix algebras whose multiplicity sequence lies in $(\mathbb{N} \setminus \{0, 1\})^{\mathbb{N}}$ and which starts with $\mathbb{C}$.*

UHF algebras have a unique tracial state, which is faithful since UHF algebras are simple. We make a simple observation relating multiplicity sequences and tracial states of the associated UHF algebras, which will be important for the main result of this section.

Lemma 4.1.3. *Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ in $\mathscr{StrictFullInductive}$. Let $\mathfrak{A} = \varinjlim \mathcal{I}$ and let $\mu_{\mathfrak{A}}$ be the unique tracial state of $\mathfrak{A}$, which is faithful. Let ϑ be the multiplicity sequence of $\mathcal{I}$.*

1. *If $a \in \mathfrak{A}_n$, then:*
$$\mu_{\mathfrak{A}}(\underrightarrow{\alpha}^n(a)) = \frac{1}{\prod_{j=0}^{n-1}\vartheta(j)}\mathrm{Tr}(a)$$
where Tr is the unique trace on $\mathfrak{A}_n$ which maps the identity to $\dim\mathfrak{A}_n$.

2. *Let $\mathcal{J} = (\mathfrak{B}_n, \alpha'_n)_{n\in\mathbb{N}}$ in $\mathscr{StrictFullInductive}$ and set $\mathfrak{B} = \varinjlim \mathcal{J}$. Let $\mu_{\mathfrak{B}}$ the unique tracial state of $\mathfrak{B}$. If the multiplicity sequences of $\mathcal{I}$ and $\mathcal{J}$ agree up to some $N \in \mathbb{N}$, then for all $n \in \{0, \ldots, N\}$, we have $\mathfrak{A}_n = \mathfrak{B}_n$ and moreover, for all $a \in \mathfrak{A}_n = \mathfrak{B}_n$, we have:*
$$\mu_{\mathfrak{A}} \circ \underrightarrow{\alpha}^n(a) = \mu_{\mathfrak{B}} \circ \underrightarrow{\alpha'}^n(a).$$

Proof. Every UHF algebra has a unique faithful tracial state by Lemma (2.1.80). Assertion *1.* follows from the uniqueness of the tracial state on $\mathfrak{A}_n$ for all $n \in \mathbb{N}$, which follows from the characterization of tracial states finite-dimensional C*-algebras in [19, Example IV.5.4].

Assertion *2.* follows directly from Assertion *1.* $\square$

The set $\mathscr{N}$ of sequences of positive integers is thus a natural parameter space for the classes $\mathcal{UHF}^k$ of Notation (3.1.8). $\mathscr{N}$ can be endowed with a natural metric d, and we thus can investigate the continuity of maps from $(\mathscr{N}, \mathsf{d})$ to $\left(\mathcal{UHF}^k, \Lambda_{2,0}\right)$.

Definition 4.1.4. *The* Baire space $\mathscr{N}$ *is the set* $(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *endowed with the metric* d *defined, for any two $(x(n))_{n\in\mathbb{N}}$, $(y(n))_{n\in\mathbb{N}}$ in $\mathscr{N}$, by:*
$$\mathsf{d}\left((x(n))_{n\in\mathbb{N}}, (y(n))_{n\in\mathbb{N}}\right) = \begin{cases} 0 & : \textit{if } x(n) = y(n) \textit{ for all } n \in \mathbb{N}, \\ 2^{-\min\{n\in\mathbb{N}: x(n)\neq y(n)\}} & : \textit{otherwise.} \end{cases}$$

Remark 4.1.5. *We note that it is common, in the literature on descriptive set theory, to employ the metric defined on* $\mathscr{N}$ *by setting on* $(x(n))_{n\in\mathbb{N}}, (y(n))_{n\in\mathbb{N}} \in \mathscr{N}$*:*

$$\mathsf{d}'\left((x(n))_{n\in\mathbb{N}}, (y(n))_{n\in\mathbb{N}}\right) = \begin{cases} 0 & : \textit{if } x(n) = y(n) \textit{ for all } n \in \mathbb{N}, \\ \frac{1}{1+\min\{n\in\mathbb{N}:x(n)\neq y(n)\}} & : \textit{otherwise.} \end{cases}$$

It is however easy to check that d *and* d' *are topologically, and in fact uniformly equivalent as metrics. Our choice will make certain statements in our paper more natural.*

We now prove the result of this section: there exists a natural continuous surjection from the Baire space $\mathscr{N}$ onto $\mathcal{UHF}^k$ for all $k \in (0, \infty)$. We recall:

Definition 4.1.6. *A function* $f : X \to Y$ *between two metric spaces* (X, d_X) *and* (Y, d_Y) *is* (c, r)*-Hölder, for some* $c \geqslant 0$ *and* $r > 0$*, when:*

$$\mathsf{d}_Y(f(x), f(y)) \leqslant c\mathsf{d}_X(x, y)^r$$

for all $x, y \in X$*.*

Theorem 4.1.7. *For any* $\beta = (\beta(n))_{n\in\mathbb{N}} \in \mathscr{N}$*, we define the sequence* $\boxtimes\beta$ *by:*

$$\boxtimes\beta = n \in \mathbb{N} \longmapsto \begin{cases} 1 \textit{ if } n = 0, \\ \prod_{j=0}^{n-1}(\beta(j) + 1) \textit{ otherwise.} \end{cases}$$

We then define, for all $\beta \in \mathscr{N}$*, the unital inductive sequence:*

$$\mathcal{I}(\beta) = \left(\mathfrak{M}\left(\boxtimes\beta(n)\right), \alpha_n\right)_{n\in\mathbb{N}}$$

where $\mathfrak{M}(d)$ *is the algebra of* $d \times d$ *matrices and for all* $n \in \mathbb{N}$*, the unital *-monomorphism* α_n *is of the form given in Expression (4.1.1).*

The map $\mathfrak{u}$ *from* $\mathscr{N}$ *to the class of UHF algebras is now defined by:*

$$(\beta(n))_{n\in\mathbb{N}} \in \mathscr{N} \longmapsto \mathfrak{u}((\beta(n))_{n\in\mathbb{N}}) = \varinjlim \mathcal{I}(\beta).$$

Let $k \in (0,\infty)$ *and* $\beta \in \mathscr{N}$. *Let* L^k_β *be the Lip-norm* $\mathsf{L}^{\vartheta}_{\mathcal{I}(\beta),\mu}$ *on* $\mathfrak{u}(\beta)$ *given by Theorem (3.1.3), the sequence* $\vartheta : n \in \mathbb{N} \mapsto \boxtimes\beta(n)^k$ *and the unique faithful trace* μ *on* $\mathfrak{u}(\beta)$.

The $(2,0)$*-quasi-Leibniz quantum compact metric space* $\left(\mathfrak{u}(\beta), \mathsf{L}^k_\beta\right)$ *will be denoted simply by* $\mathfrak{uhf}(\beta,k)$.

For all $k \in (0,\infty)$, *the map:*

$$\mathfrak{uhf}(\cdot,k) : (\mathscr{N},\mathsf{d}) \longrightarrow \left(\mathcal{UHF}^k, \Lambda_{2,0}\right)$$

is a $(2,k)$*-Hölder surjection, where* $\mathcal{UHF}^k$ *is defined in Notation (3.1.8).*

Proof. We fix $k \in (0,\infty)$. Let $\beta \in \mathscr{N}$ and write $\mathcal{I}(\beta) = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$. Note that $\mathfrak{A}_n = \mathfrak{M}(\boxtimes\beta(n))$ for all $n \in \mathbb{N}$. Moreover, we denote $\mathfrak{uhf}(\beta,k)$ by $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$.

We begin with a uniform estimate on the propinquity.

Fix $n \in \mathbb{N}$. By definition, $\boxtimes\beta(n) \geqslant 2^n$. By Theorem (3.1.3), we conclude:

$$\Lambda_{2,0}\left((\mathfrak{A},\mathsf{L}_{\mathfrak{A}}), \left(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mathsf{L}_{\mathfrak{A}}\right)\right) \leqslant \boxtimes\beta(n)^{-k} \leqslant 2^{-nk}.$$

Now, $\left(\underrightarrow{\alpha}^n(\mathfrak{A}_n), \mathsf{L}_{\mathfrak{A}}\right)$ and $\left(\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}} \circ \underrightarrow{\alpha}^n\right)$ are quantum isometric of Theorem-Defintion (2.3.16) via the quantum isometry $\underrightarrow{\alpha}^n : \mathfrak{A}_n \longrightarrow \underrightarrow{\alpha}^n(\mathfrak{A}_n)$, so:

$$\Lambda_{2,0}\left((\mathfrak{A},\mathsf{L}_{\mathfrak{A}}), \left(\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}} \circ \underrightarrow{\alpha}^n\right)\right) \leqslant 2^{-nk}. \tag{4.1.2}$$

Let now $\eta \in \mathscr{N}$ and write $\mathcal{I}(\eta) = (\mathfrak{B}_n, \alpha'_n)_{n\in\mathbb{N}}$. Note that $\mathfrak{B}_n = \mathfrak{M}(\boxtimes\eta(n))$ for all $n \in \mathbb{N}$. Moreover, we denote $\mathfrak{uhf}(\eta,k)$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

Let $N = -\log_2 \mathsf{d}(\beta, \eta) \in \mathbb{N}$. If $N = 0$, then the best estimate at our disposal is given by Corollary (3.1.9), and we conclude:

$$\Lambda_{2,0}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \max\{\boxtimes\beta(0), \boxtimes\eta(0)\} = 1 = \mathsf{d}(\eta, \beta).$$

Assume now that $N \geqslant 1$. By definition, $\boxtimes\beta(j) = \boxtimes\eta(j)$ for all $j \in \{0, \ldots, N\}$. By Lemma (4.1.3), we note that $\mathfrak{A}_N = \mathfrak{B}_N = \mathfrak{M}(\boxtimes\beta(N))$, and moreover:

$$\mu_{\mathfrak{A}} \circ \underrightarrow{\alpha^j} = \mu_{\mathfrak{B}} \circ \underrightarrow{\alpha'^j}$$

for all $j \in \{0, \ldots, N\}$.

We now employ the notations of Notation (3.1.10). For all $j \in \{0, \ldots, N\}$, we thus fix the canonical set $\{e_{k,m} \in \mathfrak{M}(\boxtimes\beta(j)) : k, m \in I_j\}$ of $\mathfrak{M}(\boxtimes\beta(j))$, where:

$$I_j = \left\{(k, m) \in \mathbb{N}^2 : 1 \leqslant k, m \leqslant \boxtimes\beta(j)\right\}.$$

Next, for all $j \in \{0, \ldots, N\}$, we have that $(\mathfrak{A}_j, \alpha_j) = (\mathfrak{B}_j, {\alpha'}_j)$. Therefore, if $j \in \{0, \ldots, N-1\}$, then $\alpha_{j,N-1} = \alpha_{N-1} \circ \cdots \circ \alpha_j = {\alpha'}_{N-1} \circ \cdots \circ {\alpha'}_j = {\alpha'}_{j,N-1}$. Also, by definition of the canonical maps $\underrightarrow{\alpha^n}$ and Proposition (2.1.66), we have that if $c \in \mathfrak{A}_j$, then $\underrightarrow{\alpha^j}(c) = \underrightarrow{\alpha^N}(\alpha_{j,N-1}(c)) = \underrightarrow{\alpha^N}({\alpha'}_{j,N-1}(c))$ for $j \in \{0, \ldots, N-1\}$. Thus, from Expression (3.1.7) for all $a \in \mathfrak{M}(\boxtimes\beta(N)), j \in \{0, \ldots, N-1\}$ we note:

$$\begin{aligned}
&\left\|\underrightarrow{\alpha^N}(a) - \mathbb{E}\left(\underrightarrow{\alpha^N}(a)\middle|\underrightarrow{\alpha^j}(\mathfrak{A}_j)\right)\right\|_{\mathfrak{A}} \\
&= \left\|\underrightarrow{\alpha^N}(a) - \sum_{l \in I_j} \frac{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^j}(e_l^*)\underrightarrow{\alpha^N}(a)\right)}{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^j}(e_l^* e_l)\right)} \underrightarrow{\alpha^j}(e_l)\right\|_{\mathfrak{A}} \\
&= \left\|\underrightarrow{\alpha^N}(a) - \sum_{l \in I_j} \frac{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^N}(\alpha_{j,N-1}(e_l^*))\underrightarrow{\alpha^N}(a)\right)}{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^N}(\alpha_{j,N-1}(e_l^* e_l))\right)} \underrightarrow{\alpha^N}(\alpha_{j,N-1}(e_l))\right\|_{\mathfrak{A}}
\end{aligned}$$

$$
\begin{aligned}
&= \left\| a - \sum_{l \in I_j} \frac{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^N}(\alpha_{j,N-1}(e_l^*)a)\right)}{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha^N}(\alpha_{j,N-1}(e_l^* e_l))\right)} \alpha_{j,N-1}(e_l) \right\|_{\mathfrak{M}(\boxtimes\beta(N))} \\
&= \left\| a - \sum_{l \in I_j} \frac{\mu_{\mathfrak{A}} \circ \underrightarrow{\alpha^N}\left({\alpha'}_{j,N-1}(e_l^*)a\right)}{\mu_{\mathfrak{A}} \circ \underrightarrow{\alpha^N}\left({\alpha'}_{j,N-1}(e_l^* e_l)\right)} {\alpha'}_{j,N-1}(e_l) \right\|_{\mathfrak{M}(\boxtimes\beta(N))} \\
&= \left\| a - \sum_{l \in I_j} \frac{\mu_{\mathfrak{B}} \circ \underrightarrow{\alpha'^N}\left({\alpha'}_{j,N-1}(e_l^*)a\right)}{\mu_{\mathfrak{B}} \circ \underrightarrow{\alpha'^N}\left({\alpha'}_{j,N-1}(e_l^* e_l)\right)} {\alpha'}_{j,N-1}(e_l) \right\|_{\mathfrak{M}(\boxtimes\beta(N))} \\
&= \left\| \underrightarrow{\alpha'^N}(a) - \sum_{l \in I_j} \frac{\mu_{\mathfrak{B}}\left(\underrightarrow{\alpha'^N}({\alpha'}_{j,N-1}(e_l^*)a)\right)}{\mu_{\mathfrak{B}}\left(\underrightarrow{\alpha'^N}({\alpha'}_{j,N-1}(e_l^* e_l))\right)} \underrightarrow{\alpha'^N}({\alpha'}_{j,N-1}(e_l)) \right\|_{\mathfrak{B}} \\
&= \left\| \underrightarrow{\alpha'^N}(a) - \sum_{l \in I_j} \frac{\mu_{\mathfrak{B}}\left(\underrightarrow{\alpha'^j}(e_l^*)\underrightarrow{\alpha'^N}(a)\right)}{\mu_{\mathfrak{A}}\left(\underrightarrow{\alpha'^j}(e_l^* e_l)\right)} \underrightarrow{\alpha'^j}(e_l) \right\|_{\mathfrak{B}} \\
&= \left\| \underrightarrow{\alpha'^N}(a) - \mathbb{E}\left(\underrightarrow{\alpha'^N}(a) \middle| \underrightarrow{\alpha'^j}(\mathfrak{B}_j)\right) \right\|_{\mathfrak{B}}.
\end{aligned}
\tag{4.1.3}
$$

If $j \geqslant N$, then $\mathbb{E}\left(\underrightarrow{\alpha^N}(a)\middle|\underrightarrow{\alpha^j}(\mathfrak{A}_j)\right) = \underrightarrow{\alpha^N}(a)$ and $\mathbb{E}\left(\underrightarrow{\alpha'^N}(a)\middle|\underrightarrow{\alpha'^j}(\mathfrak{B}_j)\right) = \underrightarrow{\alpha'^N}(a)$ by definition of conditional expectation. Consequently, by definition:

$$
\mathsf{L}_{\mathfrak{A}} \circ \underrightarrow{\alpha^N} = \mathsf{L}_{\mathfrak{B}} \circ \underrightarrow{\alpha'^N},
$$

so:

$$
\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_N, \mathsf{L}_{\mathfrak{A}} \circ \underrightarrow{\alpha^N}\right), \left(\mathfrak{B}_N, \mathsf{L}_{\mathfrak{B}} \circ \underrightarrow{\alpha'^N}\right)\right) = 0. \tag{4.1.4}
$$

Hence, by the triangle inequality applied to Inequalities (4.1.2) and (4.1.4):

$$
\mathsf{\Lambda}_{2,0}(\mathfrak{uhf}(\beta,k), \mathfrak{uhf}(\eta,k)) \leqslant \frac{2}{2^{Nk}} \leqslant 2\mathsf{d}(\beta,\eta)^k.
$$

Last, we show that the map $\mathfrak{uhf}(\cdot,k)$ is a surjection. By Example (2.1.79), if $\mathfrak{U}$ is a UHF algebra, then there exists an inductive sequence $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ of full matrix algebras whose limit is $\mathfrak{U}$ and such that $\mathfrak{A}_0 = \mathbb{C}$, while the multiplicity sequence β of $\mathcal{I}$ is in $\mathbb{N} \setminus \{0,1\}$. Thus $\mathfrak{u}((\beta(n)-1)_{n\in\mathbb{N}}) = \mathfrak{U}$. Moreover, any Lip-norm L on $\mathfrak{U}$ such that $(\mathfrak{U}, \mathsf{L}) \in \mathcal{UHF}^k$ can be obtained, by definition, from such a multiplicity sequence. $\square$

Remark 4.1.8. *Inequality (4.1.2) is sharp, as it becomes an inequality for the sequence $c = (1,1,1,\ldots) \in \mathscr{N}$, and we note that the UHF algebra $\mathfrak{u}(c)$ is the CAR algebra presented in Example (2.1.79).*

Remark 4.1.9. *Since d is an ultrametric on $\mathscr{N}$, we conclude that d^k is a topologically equivalent ultrametric on $\mathscr{N}$ as well. Hence, we could reformulate the conclusion of Theorem (4.1.7) by stating that $\mathfrak{uhf}\,(\cdot, k)$ is 2-Lipschitz for d^k.*

4.2 Effros-Shen algebras

The original classification of irrational rotation algebras, due to Pimsner and Voiculescu [58], relied on certain embeddings into the AF algebras constructed from continued fraction expansions by Effros and Shen [22]. In [40], Latrémolière proved, in particular, that the irrational rotational algebras vary continuously in quantum propinquity with respect to their irrational parameter. It is natural to wonder whether the AF algebras constructed by Pimsner and Voiculescu vary continuously with respect to the quantum propinquity if parametrized by the irrational numbers at the root of their construction. We shall provide a positive answer to this problem in this section.

In [58], Pimsner and Voiculescu construct, for any $\theta \in (0,1)\setminus\mathbb{Q}$, a unital *-monomorphism from the irrational rotation C*-algebra $\mathfrak{A}_\theta$, i.e. the universal C*-algebra generated by two unitaries U and V subject to $UV = \exp(2i\pi\theta)VU$, into an AF algebra. These AF algebras, denoted $\mathfrak{AF}_\theta$, were the AF algebras defined in Example (2.1.72). This was a crucial step in their classification of irrational rotation algebras and started a long and fascinating line of investigation about AF embeddings of various C*-algebras.

In order to apply our Theorem (3.1.3), we need to find a faithful tracial state on $\mathfrak{AF}_\theta$, for all $\theta \in (0,1) \setminus \mathbb{Q}$. We shall prove that for all $\theta \in (0,1) \setminus \mathbb{Q}$, there exists a unique tracial state on $\mathfrak{AF}_\theta$ which will be faithful as $\mathfrak{AF}_\theta$ is simple. The source of our tracial state will be the K-theory of $\mathfrak{AF}_\theta$.

We refer to [19, Section VI.3] for the computation of the Elliott invariant of $\mathfrak{AF}_\theta$, which reads:

Theorem 4.2.1 ([22])**.** *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and let $C_\theta = \{(x,y) \in \mathbb{Z}^2 : \theta x + y \geqslant 0\}$. Then $K_0(\mathfrak{AF}_\theta) = \mathbb{Z}^2$ with positive cone C_θ and order unit $(0,1)$. Thus the only state of the ordered group $(K_0(\mathfrak{AF}_\theta), C_\theta, (0,1))$ is given by the map:*

$$(x,y) \in \mathbb{Z}^2 \longmapsto \theta x + y.$$

Thus $\mathfrak{AF}_\theta$ has a unique faithful tracial state, denoted by σ_θ.

Proof. By [19, Section VI.3], we only check that σ_θ is faithful. However, the C*-algebra $\mathfrak{AF}_\theta$ is simple by its diagram in Example (2.1.87) the diagramatic characterization of unital simple AF algebras in [19, Corollary III.4.3]. Therefore, by Lemma (2.1.43), the tracial state σ_θ is faithful. $\square$

Therfore, we have all the ingredients to define our quantum metric on $\mathfrak{AF}_\theta$.

Notation 4.2.2. *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and $k \in (0,\infty)$. The Lip-norm L^k_θ on $\mathfrak{AF}_\theta$ is the lower semi-continuous, $(2,0)$-quasi Leibniz Lip-norm $\mathsf{L}^k_{\mathcal{I}_\theta,\sigma_\theta}$ defined in Notation (3.1.8) based on Theorem (3.1.3), where $\mathcal{I}_\theta = (\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n})_{n\in\mathbb{N}}$ as in Notation (2.1.82).*

As Theorem (3.1.3) provides Lip-norms based, in part, on the choice of a faithful tracial state, a more precise understanding of the unique faithful tracial state of $\mathfrak{AF}_\theta$

is required. We summarize our observations in the following Lemma (4.2.3) and Lemma (4.2.5).

Lemma 4.2.3. *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and let σ_θ be the unique faithful tracial state of $\mathfrak{AF}_\theta$, and fix $n \in \mathbb{N} \setminus \{0\}$. Using Notation (2.1.82), let:*

$$\sigma_{\theta,n} = \sigma_\theta \circ \underrightarrow{\alpha_\theta^n}.$$

Let tr_d be the unique tracial state on $\mathfrak{M}(d)$ for any $d \in \mathbb{N}$. Then, if $(p_n^\theta)_{n\in\mathbb{N}}$ and $(q_n^\theta)_{n\in\mathbb{N}}$ are defined by Expression (2.1.12), then:

$$\sigma_{\theta,n} : a \oplus b \in \mathfrak{AF}_{\theta,n} \longmapsto t(\theta,n)\mathrm{tr}_{q_n^\theta}(a) + (1 - t(\theta,n))\mathrm{tr}_{q_{n-1}^\theta}(b),$$

where

$$t(\theta,n) = (-1)^{n-1} q_n^\theta \left(\theta q_{n-1}^\theta - p_{n-1}^\theta\right) \in (0,1).$$

Proof. The map $\sigma_{\theta,n}$ is a tracial state on $\mathfrak{AF}_{\theta,n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta)$, and thus there exists $t(n,\theta) \in [0,1]$ such that for all $a \oplus b \in \mathfrak{AF}_{\theta,n}$:

$$\sigma_{\theta,n}(a \oplus b) = t(\theta,n)\mathsf{tr}_{q_n^\theta}(a) + (1 - t(\theta,n))\mathsf{tr}_{q_{n-1}^\theta}(b).$$

Let $\sigma_* : K_0(\mathfrak{AF}_\theta) \to \mathbb{R}$ be the state induced by σ_θ on the K_0 group of $\mathfrak{AF}_\theta$. We then have:

$$\begin{aligned} t(\theta,n) &= \sigma_{\theta,n}(1_{\mathfrak{M}(q_n^\theta)} \oplus 0) \\ &= \sigma_\theta \circ \underrightarrow{\alpha_\theta^n}(1_{\mathfrak{M}(q_n^\theta)} \oplus 0) \\ &= \sigma_* \circ K_0\left(\underrightarrow{\alpha_\theta^n}\right)\left(\begin{pmatrix} q_n^\theta \\ 0 \end{pmatrix}\right), \end{aligned} \tag{4.2.1}$$

where $K_0\left(\underset{\longrightarrow}{\alpha_\theta^n}\right)$ is the map from $K_0(\mathfrak{AF}_{\theta,n}) = \mathbb{Z}^2$ to $K_0(\mathfrak{AF}_\theta) = \mathbb{Z}^2$ induced by $\underset{\longrightarrow}{\alpha^n}$. By construction, following [19, Section VI.3], we have:

$$K_0\left(\underset{\longrightarrow}{\alpha_\theta^n}\right)\begin{pmatrix} z1 \\ z2 \end{pmatrix} = (-1)^{n-1}\begin{pmatrix} q_{n-1}^\theta & -q_n^\theta \\ -p_{n-1}^\theta & p_n^\theta \end{pmatrix}\begin{pmatrix} z1 \\ z2 \end{pmatrix}$$

for all $(z1, z2) \in \mathbb{Z}^2$. Therefore:

$$\begin{aligned} t(\theta, n) &= (-1)^{n-1}\sigma_*\left(\begin{pmatrix} q_{n-1}^\theta & -q_n^\theta \\ -p_{n-1}^\theta & p_n^\theta \end{pmatrix}\begin{pmatrix} q_n^\theta \\ 0 \end{pmatrix}\right) \\ &= (-1)^{n-1}\sigma_*\left(\begin{pmatrix} q_{n-1}^\theta q_n^\theta \\ -p_{n-1}^\theta q_n^\theta \end{pmatrix}\right) \\ &= (-1)^{n-1} q_n^\theta\left(\theta q_{n-1}^\theta - p_{n-1}^\theta\right). \end{aligned}$$

Since θ is irrational, $t(\theta, n) \neq 0$. Since $1_{\mathfrak{M}(q_n^\theta)} \oplus 0$ is positive in $\mathfrak{AF}_{\theta,n}$ and less than $1_{\mathfrak{AF}_{\theta,n}}$, we conclude $t(\theta, n) \in (0, 1]$.

To prove that $t(\theta, n) < 1$, we may proceed following two different routes. Applying a similar computation as in Expression (4.2.1), we get:

$$\sigma_{\theta,n}\left(0 \oplus 1_{\mathfrak{M}(q_{n-1}^\theta)}\right) = (-1)^n q_{n-1}^\theta\left(\theta q_n^\theta - p_n^\theta\right),$$

and again as θ is irrational, this quantity is nonzero. As $1 = \sigma_{\theta,n}\left(1_{\mathfrak{M}(q_n^\theta)} \oplus 1_{\mathfrak{M}(q_{n-1}^\theta)}\right)$, our lemma would thus be proven.

Instead, we employ properties of continued fraction expansions and note that since $p_n^\theta q_{n-1}^\theta - p_{n-1}^\theta q_n^\theta = (-1)^{n-1}$:

$$\begin{aligned} 1 - t(\theta, n) &= 1 - (-1)^{n-1} q_n^\theta(\theta q_{n-1}^\theta - p_{n-1}^\theta) \\ &= (-1)^{n-1}\left((-1)^{n-1} - q_n^\theta(\theta q_{n-1}^\theta - p_{n-1}^\theta)\right) \end{aligned}$$

$$= (-1)^{n-1}\left(p_n^\theta q_{n-1}^\theta - p_{n-1}^\theta q_n^\theta - q_n^\theta(\theta q_{n-1}^\theta - p_{n-1}^\theta)\right)$$
$$= (-1)^n\left(q_n^\theta(\theta q_{n-1}^\theta) - p_n^\theta q_{n-1}^\theta\right)$$
$$= (-1)^n q_{n-1}^\theta\left(\theta q_n^\theta - p_n^\theta\right),$$

which is nonzero as θ is irrational, and is less than one since $t(\theta, n) > 0$. This concludes our proof. □

Remark 4.2.4. *We may also employ properties of continued fractions expansions to show that $t(\theta, n) > 0$ for all $n \in \mathbb{N}$. We shall use the notations of the proof of Lemma (4.2.3). We have:*

$$\frac{p_{2n}^\theta}{q_{2n}^\theta} < \theta < \frac{p_{2n+1}^\theta}{q_{2n+1}^\theta}$$

and thus $\theta q_{2n}^\theta - p_{2n}^\theta > 0$ and $p_{2n+1}^\theta - \theta q_{2n+1}^\theta > 0$, which shows that $t(\theta, n) > 0$ for all $n \in \mathbb{N}$ (note that $q_n^\theta \in \mathbb{N} \setminus \{0\}$ for all $n \in \mathbb{N}$ since $\theta > 0$).

We wish to employ Expression (3.1.7) and thus, we will find the following computation helpful:

Lemma 4.2.5. *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and let $n \in \mathbb{N} \setminus \{0\}$. Let $\{e_{1,j,m} \in \mathfrak{AF}_{\theta,n} : 1 \leqslant j, m \leqslant q_n^\theta\}$ and $\{e_{2,j,m} \in \mathfrak{AF}_{\theta,n} : 1 \leqslant j, m \leqslant q_{n-1}^\theta\}$ be the standard family of matrix units in, respectively, $\mathfrak{M}(q_n^\theta)$ and $\mathfrak{M}(q_{n-1}^\theta)$ inside $\mathfrak{AF}_{\theta,n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta)$ via $\alpha_{\theta,n}$, as in Notation (3.1.10) and with $(p_n^\theta)_{n\in\mathbb{N}}$ and $(q_n^\theta)_{n\in\mathbb{N}}$ defined by Expression (2.1.12).*

For $1 \leqslant j, m \leqslant q_n^\theta$, we compute:

$$\sigma_\theta\left(\underrightarrow{\alpha}^n(e_{1,j,m}^* e_{1,j,m})\right) = (-1)^{n-1}(\theta q_{n-1}^\theta - p_{n-1}^\theta)$$

while, for $1 \leqslant j, m \leqslant q_{n-1}^\theta$:

$$\sigma_\theta(\underrightarrow{\alpha}^n(e_{2,j,m}^* e_{2,j,m})) = (-1)^n(\theta q_n^\theta - p_n^\theta).$$

Proof. Let $1 \leqslant j, m \leqslant q_n^\theta$. By Lemma (4.2.3), we have:

$$\begin{aligned}\sigma_\theta\left(\underrightarrow{\alpha}^n(e_{1,j,m}^* e_{1,j,m})\right) &= t(\theta,n)\mathsf{tr}_{q_n^\theta}(e_{1,j,m}^* e_{1,j,m}) + (1-t(\theta,n))\cdot 0 \\ &= \frac{t(\theta,n)}{q_n^\theta} \\ &= (-1)^{n-1}(\theta q_{n-1}^\theta - p_{n-1}^\theta).\end{aligned}$$

And, a similar argument proves the result for the other matrix units. □

Our proof that the map $\theta \in (0,1)\setminus\mathbb{Q} \mapsto (\mathfrak{AF}_\theta, \mathsf{L}_\theta)$ is continuous for the quantum propinquity relies on a homeomorphism between the Baire space of Definition (4.1.4) and $(0,1)\setminus\mathbb{Q}$, endowed with its topology as a subspace of $\mathbb{R}$. Indeed, the map which associates, to an irrational number in $(0,1)$, its continued fraction expansion is a homeomorphism (see, for instance, [53]). We include a brief proof of this fact as, while it is well-known, the proof is often skipped in references. Moreover, this will serve as a means to set some other useful notations for our work.

Notation 4.2.6. *Define* $\mathsf{cf} : (0,1)\setminus\mathbb{Q} \to \mathscr{N}$ *by setting* $\mathsf{cf}(\theta) = (b_n)_{n\in\mathbb{N}}$ *if and only if* $\theta = [0, b_0, b_1, \ldots]$. *We note that* cf *is a bijection from* $(0,1)\setminus\mathbb{Q}$ *onto* $\mathscr{N}$, *where* $\mathscr{N}$ *is the Baire space defined in Definition (4.1.4). The inverse of* cf *is denote by* $\mathrm{ir} : \mathscr{N} \to (0,1)\setminus\mathbb{Q}$.

Notation 4.2.7. *We will denote the closed ball in* $(\mathscr{N}, \mathsf{d})$ *of center* $x \in \mathscr{N}$ *and radius* 2^{-N} *by* $\mathscr{N}[x, N]$ *for* $N > 0$. *It consists of all sequences in* $\mathscr{N}$ *whose* N *first entries are the same as the* N *first entries of* x.

Proposition 4.2.8. *The bijection:*

$$\mathsf{cf} : ((0,1)\setminus\mathbb{Q}, |\cdot|) \longrightarrow (\mathscr{N}, \mathsf{d})$$

is a homeomorphism.

Proof. The basic number theory facts used in this proof can be found in [32]. Since every irrational in $(0,1)$ has a unique continued fraction expansion of the form given by Expression (2.1.11), and every sequence of positive integers determines the continued fraction expansion of an irrational via the same expression, cf is a bijection.

We now show that cf is continuous. Let $b = (b_n)_{n\in\mathbb{N}} \in \mathscr{N}$ and let:

$$\theta = \lim_{n\to\infty} \cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}} \in (0,1) \setminus \mathbb{Q}.$$

Let $V = \mathscr{N}[b, N]$ for some $N \in \mathbb{N} \setminus \{0\}$.

Let $\eta \in \mathsf{cf}^{-1}(V)$ and let $(x_n)_{n\in\mathbb{N}} = \mathsf{cf}(\eta)$. Thus, for all $j \in \{0, \dots, N-1\}$, we have $x_n = b_n$. Define $I_{N,\eta}$ as the open interval with end points:

$$\cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \frac{1}{b_{N-1}}}}} \quad \text{and} \quad \cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \frac{1}{b_{N-1}+1}}}},$$

and let $\Theta_{N,\eta} = I_{N,\eta} \setminus \mathbb{Q}$.

By construction, $\Theta_{N,\eta}$ is open in the relative topology on $(0,1)\setminus\mathbb{Q}$, and since η is irrational, we conclude $\eta \in \Theta_{N,\eta} \setminus \mathbb{Q}$. Furthermore, $\mathsf{cf}(\Theta_{N,\eta}) \subseteq V$, which concludes the argument since the set of open balls in $\mathscr{N}$ is a topological basis for $\mathscr{N}$.

Next, we show continuity of ir by sequential continuity. Let $(b^n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{N}$, where, for all $n \in \mathbb{N}$, we write $b^n = (b^n_m)_{m\in\mathbb{N}}$. Assume $(b^n)_{n\in\mathbb{N}}$ converges to some $b \in \mathscr{N}$ for d. Denote $\theta = \mathsf{ir}(b) \in (0,1) \setminus \mathbb{Q}$ and $\theta_n = \mathsf{ir}(b^n) \in (0,1) \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$, using Notation (2.1.12), there exists $N_1 \in \mathbb{N}$ such that $2/\left(q^\theta_{N_1}\right)^2 < \varepsilon$. Next, By Definition (4.1.4) of our metric d on $\mathscr{N}$, we have that there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $b^n_m = b_m$ for all $m \in \{0, \ldots, N_1\}$, and thus $q^{\theta_n}_m = q^\theta_m$ and $p^{\theta_n}_m = p^\theta_m$ for all $m \in \{0, \ldots, N_1\}$.

Let $n \geqslant N$, then standard estimates for continued fraction expansions lead to:

$$\begin{aligned}
|\mathsf{ir}(b^n) - \theta| &= \left|\mathsf{ir}(b^n) - p^\theta_{N_1}/q^\theta_{N_1} + p^\theta_{N_1}/q^\theta_{N_1} - \theta\right| \\
&\leqslant \left|\mathsf{ir}(b^n) - p^\theta_{N_1}/q^\theta_{N_1}\right| + \left|p^\theta_{N_1}/q^\theta_{N_1} - \theta\right| \\
&= \left|\theta_n - p^{\theta_n}_{N_1}/q^{\theta_n}_{N_1}\right| + \left|p^\theta_{N_1}/q^\theta_{N_1} - \theta\right| \\
&< 1/\left(q^{\theta_n}_{N_1}\right)^2 + 1/\left(q^\theta_{N_1}\right)^2 \\
&= 2/\left(q^\theta_{N_1}\right)^2 < \varepsilon.
\end{aligned}$$

Thus, we conclude that $\lim_{n\to\infty} \mathsf{ir}(b^n) = \theta = \mathsf{ir}(b)$ as desired, and our proof is complete. $\square$

Our main result will be proven in four steps. We begin by observing that the tracial states of $\mathfrak{AF}_\theta$ provide a continuous field of states on various finite dimensional algebras.

Lemma 4.2.9. *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and $N \in \mathbb{N}$. Let $(p^\theta_n)_{n\in\mathbb{N}}$ and $(q^\theta_n)_{n\in\mathbb{N}}$ be defined from $\mathsf{cf}(\theta)$ using Expression (2.1.12). For all $n \in \{0, \ldots, N\}$, the map:*

$$\mathrm{s}_n : (x, a) \in \mathscr{N}[\mathsf{cf}(\theta), N+1] \times \mathfrak{AF}_{\theta,n} \longmapsto \sigma_{\mathsf{ir}(x)}\left(\underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a)\right) \tag{4.2.2}$$

is well-defined and continuous from $\mathscr{N}[\mathsf{cf}(\theta), N+1] \times (\mathfrak{AF}_{\theta,n}, \|\cdot\|_{\mathfrak{AF}_{\theta,n}})$ to $\mathbb{R}$.

Proof. We note that the result is trivial for $n = 0$ since s_0 is the identity on $\mathbb{C} = \mathfrak{AF}_{x,0}$ for all $x \in \mathscr{N}$.

Let $x, y \in \mathscr{N}[\mathsf{cf}(\theta), N]$ and set $\eta = \mathsf{ir}(x)$ and $\xi = \mathsf{ir}(y)$. Since d is an ultrametric on $\mathscr{N}$, we note that $\mathsf{d}(x, y) \leqslant \frac{1}{2^N}$.

We now use the notation of Expression (2.1.12). The key observation from Expression (2.1.12) is that the functions:

$$z \in \mathscr{N}[\mathsf{cf}(\theta), N+1] \mapsto \left(q_n^{\mathrm{ir}(z)}, p_n^{\mathrm{ir}(z)}\right)$$

are constant for all $n \in \{0, \ldots, N\}$, equal to (q_n^θ, p_n^θ) — since $\mathsf{d}(x, \mathsf{cf}(\theta)) \leqslant \frac{1}{2^{N+1}}$ implies that the sequences x and $\mathsf{cf}(\theta)$ agree on their first N entries.

Thus, setting $\mathfrak{B}_n = \mathfrak{AF}_{\theta,n}$, we have:

$$\mathfrak{M}(q_n^x) \oplus \mathfrak{M}(q_{n-1}^x) = \mathfrak{B}_n$$

for all $n \in \{0, \ldots, N\}$, and the maps defined by Expression (4.2.2) are well-defined.

Let now $n \in \{1, \ldots, N\}$ be fixed. Let $a \in \mathfrak{B}_n$ and write $a = a' \oplus a'' \in \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta)$. By Lemma (4.2.3), we compute:

$$\begin{aligned}
\left|\sigma_\xi \circ \underrightarrow{\alpha_\xi^n}(a) - \sigma_\eta \circ \underrightarrow{\alpha_\eta^n}(a)\right| &= |(t(\xi, n) - t(\eta, n))(\mathrm{tr}_{q_n^\theta}(a') - \mathrm{tr}_{q_{n-1}^\theta}(a''))| \\
&\leqslant 2|t(\xi, n) - t(\eta, n)| \|a\|_{\mathfrak{B}_n} \\
&= 2|q_n^\theta(\xi q_{n-1}^\theta - p_{n-1}^\theta) - q_n^\theta(\eta q_{n-1}^\theta - p_{n-1}^\theta)| \|a\|_{\mathfrak{B}_n} \\
&= 2|q_n^\theta q_{n-1}^\theta||\xi - \eta| \|a\|_{\mathfrak{B}_n} \\
&= 2|q_n^\theta q_{n-1}^\theta||\mathrm{ir}(y) - \mathrm{ir}(x)| \|a\|_{\mathfrak{B}_n}.
\end{aligned}$$

As $n < N$ is fixed, and ir is a homeomorphism, we conclude that if $(y_m)_{m\in\mathbb{N}}$ is a sequence in $\mathscr{N}[\theta, N+1]$ converging to x then:

$$\lim_{m\to\infty}\left|\sigma_{\mathsf{ir}(y_m)}\circ\underrightarrow{\alpha^n_{\mathrm{ir}(y_m)}}(a)-\sigma_\eta\circ\underrightarrow{\alpha^n_\eta}(a)\right|=0.$$

Thus we have established that the partial function $s_n(\cdot,a)$ are continuous for all $a\in\mathfrak{B}_n$.

We now prove the joint continuity of our maps. Let $a,b\in\mathfrak{B}_n$ and η,ξ as above. Then:

$$\begin{aligned}
&\left|\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(a)\right)-\sigma_\xi\left(\underrightarrow{\alpha^n_\xi}(b)\right)\right|\\
&\qquad=\left|\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(a)\right)-\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(b)\right)+\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(b)\right)-\sigma_\xi\left(\underrightarrow{\alpha^n_\xi}(b)\right)\right|\\
&\qquad\leqslant\left|\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(a)\right)-\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(b)\right)\right|\\
&\qquad\quad+\left|\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(b)\right)-\sigma_\xi\left(\underrightarrow{\alpha^n_\xi}(b)\right)\right|\\
&\qquad\leqslant\|a-b\|_{\mathfrak{A}_n}+\left|\sigma_\eta\left(\underrightarrow{\alpha^n_\eta}(b)\right)-\sigma_\xi\left(\underrightarrow{\alpha^n_\xi}(b)\right)\right|.
\end{aligned}$$

It follows immediately that the map s_n defined by Expression (4.2.2) is continuous as desired. □

Our second step is to prove that, thanks to Lemma (4.2.9), the Lip-norms induced from $\mathfrak{AF}_\theta$ on their finite dimensional C*-subalgebras form a continuous field of Lip-norms [61]. Moreover, we obtain a joint continuity result for these Lip-norms, which are thus in particular continuous rather than only lower semi-continuous.

Lemma 4.2.10. *Let $\theta\in(0,1)\setminus\mathbb{Q}$ and $N\in\mathbb{N}$. Let $(p^\theta_n)_{n\in\mathbb{N}}$ and $(q^\theta_n)_{n\in\mathbb{N}}$ be defined from $\mathsf{cf}(\theta)$ using Expression (2.1.12). For all $n\in\{0,\ldots,N\}$ and $k\in(0,\infty)$, the map:*

$$\mathsf{l}_n:(x,a)\in\mathscr{N}[\mathsf{cf}(\theta),N+1]\times\mathfrak{AF}_{\theta,n}\longmapsto\mathsf{L}^k_{\mathsf{ir}(x)}\left(\underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a)\right)\tag{4.2.3}$$

defined using Notation (4.2.2), is well-defined and continuous from $\mathscr{N}[\mathsf{cf}(\theta), N+1] \times (\mathfrak{B}_n, \|\cdot\|_{\mathfrak{B}_n})$ *to* $\mathbb{R}$.

Proof. We note that the proof of Lemma (4.2.9) also establishes, by a similar argument, that the maps l_n are well-defined for all $n \in \{0, \ldots, N\}$. We also note that l_0 is constantly 0, and thus the result is trivial for $n = 0$.

Fix $n \in \{1, \ldots, N\}$. Let $x, y \in \mathscr{N}[\mathsf{cf}(\theta), N+1]$ and write $\eta = \mathrm{ir}(x)$ and $\xi = \mathrm{ir}(y)$. As within the proof of Lemma (4.2.9), we note that for all $M \in \{0, \ldots, n\}$, we have that and set $q_M = q_M^\theta = q_M^\eta = q_M^\xi$ and similarly, $p_M = p_M^\theta = p_M^\eta = p_M^\xi$ (using the notations of Expression (2.1.12)). Furthermore, for all $M \in \{0, \ldots, n\}$, we set $(\mathfrak{B}_M, \alpha_M) = (\mathfrak{AF}_{\theta,M}, \alpha_{\theta,M}) = (\mathfrak{AF}_{\eta,M}, \alpha_{\eta,M}) = (\mathfrak{AF}_{\xi,M}, \alpha_{\xi,M})$. Note further that $\alpha_{M,n-1} = \alpha_{n-1} \circ \cdots \circ \alpha_M = \alpha_{\theta,M,n-1} = \alpha_{\eta,M,n-1} = \alpha_{\xi,M,n-1}$ for all $M \in \{0, \ldots, n-1\}$.

Fix $M \in \{0, \ldots, n-1\}$, we employ the notations of Notation (3.1.10) and thus, we have a set $\{e_{1,j,m} \in \mathfrak{B}_M : 1 \leqslant j, m \leqslant q_M\}$ of matrix units of $\mathfrak{M}(q_M) \subseteq \mathfrak{B}_M$ and a set $\{e_{2,j,m} \in \mathfrak{B}_M : 1 \leqslant j, m \leqslant q_{M-1}\}$ of matrix units for $\mathfrak{M}(q_{M-1}) \subseteq \mathfrak{B}_M$.

To lighten our notations in this proof, let:

$$I_1 = \{(1, j, m) \in \mathbb{N}^3 : 1 \leqslant j, m \leqslant q_M\},\ I_2 = \{(2, j, m) \in \mathbb{N}^3 : 1 \leqslant j, m \leqslant q_{M-1}\}$$

and $I = I_1 \cup I_2$.

Let $a \in \mathfrak{B}_n$. By Expression (3.1.7) and the same argument provided by Equation (4.1.3) in the proof of Theorem (4.1.7), we conclude that:

$$\begin{aligned}
&\left\| \underrightarrow{\alpha_\eta^n}(a) - \mathbb{E}\left(\underrightarrow{\alpha_\eta^n}(a) \middle| \underrightarrow{\alpha_\eta^M}(\mathfrak{B}_M) \right) \right\|_{\mathfrak{AF}_\eta} \\
&= \left\| a - \sum_{j \in I} \frac{\sigma_\eta\left(\underrightarrow{\alpha_\eta^n}(\alpha_{M,n-1}(e_j^*) a) \right)}{\sigma_\eta\left(\underrightarrow{\alpha_\eta^M}(e_j^* e_j) \right)} \alpha_{M,n-1}(e_j) \right\|_{\mathfrak{B}_n}
\end{aligned}$$

and

$$\left\|\underset{\longrightarrow}{\alpha_\xi^n}(a)-\mathbb{E}\left(\underset{\longrightarrow}{\alpha_\xi^n}(a)\middle|\underset{\longrightarrow}{\alpha_\xi^M}(\mathfrak{B}_M)\right)\right\|_{\mathfrak{A}\mathfrak{F}_\xi}$$

$$=\left\|a-\sum_{j\in I}\frac{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^n}(\alpha_{M,n-1}(e_j^*)a)\right)}{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^M}(e_j^*e_j)\right)}\alpha_{M,n-1}(e_j)\right\|_{\mathfrak{B}_n}.$$

Next, let $a,b\in\mathfrak{B}_n$, we have:

$$\left|\left\|\underset{\longrightarrow}{\alpha_\eta^n}(a)-\mathbb{E}\left(\underset{\longrightarrow}{\alpha_\eta^n}(a)\middle|\underset{\longrightarrow}{\alpha_\eta^M}(\mathfrak{B}_M)\right)\right\|_{\mathfrak{A}\mathfrak{F}_\eta}-\left\|\underset{\longrightarrow}{\alpha_\xi^n}(b)-\mathbb{E}\left(\underset{\longrightarrow}{\alpha_\xi^n}(b)\middle|\underset{\longrightarrow}{\alpha_\xi^M}(\mathfrak{B}_M)\right)\right\|_{\mathfrak{A}\mathfrak{F}_\xi}\right|$$

$$\leqslant\left|\left\|a-\sum_{j\in I}\frac{\sigma_\eta\left(\underset{\longrightarrow}{\alpha_\eta^n}(\alpha_{M,n-1}(e_j^*)a)\right)}{\sigma_\eta\left(\underset{\longrightarrow}{\alpha_\eta^M}(e_j^*e_j)\right)}\alpha_{M,n-1}(e_j)\right\|_{\mathfrak{B}_n}\right.$$

$$\left.-\left\|b-\sum_{j\in I}\frac{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^n}(\alpha_{M,n-1}(e_j^*)b)\right)}{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^M}(e_j^*e_j)\right)}\alpha_{M,n-1}(e_j)\right\|_{\mathfrak{B}_n}\right|$$

$$\leqslant\|a-b\|_{\mathfrak{B}_n}$$

$$+\left\|\sum_{j\in I_1}\left(\frac{\sigma_\eta\left(\underset{\longrightarrow}{\alpha_\eta^n}(\alpha_{M,n-1}(e_j^*)a)\right)}{q_{M-1}^\eta\eta-p_{M-1}^\eta}-\frac{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^n}(\alpha_{M,n-1}(e_j^*)b)\right)}{q_{M-1}^\xi\xi-p_{M-1}^\xi}\right)e_j\right\|_{\mathfrak{M}(q_M)}$$

$$+\left\|\sum_{j\in I_2}\left(\frac{\sigma_\eta\left(\underset{\longrightarrow}{\alpha_\eta^n}(\alpha_{M,n-1}(e_j^*)a)\right)}{q_M^\eta\eta-p_M^\eta}-\frac{\sigma_\xi\left(\underset{\longrightarrow}{\alpha_\xi^n}(\alpha_{M,n-1}(e_j^*)b)\right)}{q_M^\xi\xi-p_M^\xi}\right)e_j\right\|_{\mathfrak{M}(q_{M-1})}$$

$$=\|a-b\|_{\mathfrak{B}_n}+\left\|\sum_{j\in I_1}\left(\frac{\mathrm{s}_n\left(x,\alpha_{M,n-1}(e_j^*)a\right)}{q_{M-1}\mathrm{ir}(x)-p_{M-1}}-\frac{\mathrm{s}_n\left(y,\alpha_{M,n-1}(e_j^*)b\right)}{q_{M-1}\mathrm{ir}(y)-p_{M-1}}\right)e_j\right\|_{\mathfrak{M}(q_M)}$$

$$+\left\|\sum_{j\in I_2}\left(\frac{\mathrm{s}_n\left(x,\alpha_{M,n-1}(e_j^*)a\right)}{q_M\mathrm{ir}(x)-p_M}-\frac{\mathrm{s}_n\left(y,\alpha_{M,n-1}(e_j^*)b\right)}{q_M\mathrm{ir}(y)-p_M}\right)e_j\right\|_{\mathfrak{M}(q_{M-1})},$$

where we used Lemma (4.2.5) in the second inequality above, and s_n is defined by Expression (4.2.2). Now, since ir is a homeomorphism from $\mathscr{N}$ to the irrationals in $(0,1)$, and the map s_n is continuous by Lemma (4.2.9), we conclude that as $I = I_1 \cup I_2$ is finite:

$$\begin{aligned}(x,a) &\in \mathscr{N}[\mathsf{cf}(\theta), N+1] \times \mathfrak{B}_n \\ &\longmapsto \frac{1}{\beta(M)} \left\| \underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a) - \mathbb{E}\left(\underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a) \middle| \underrightarrow{\alpha^M_{\mathrm{ir}(x)}}(\mathfrak{B}_M) \right) \right\|_{\mathfrak{AF}_{\mathrm{ir}(x)}}\end{aligned} \tag{4.2.4}$$

is continuous, where $\beta(M) = \frac{1}{((q_M)^2 + (q_{M-1})^2)^k}$.

Last, we note that since for all $j \geqslant n$ we have:

$$\mathbb{E}\left(\underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a) \middle| \underrightarrow{\alpha^j_{\mathrm{ir}(x)}}(\mathfrak{AF}_{\theta,j}) \right) = \underrightarrow{\alpha^n_{\mathrm{ir}(x)}}(a)$$

by definition of conditional expectation, and therefore, the function l_n is the maximum of the functions given in Expression (4.2.4) with M ranging over $\{0, \ldots, n-1\}$.

As the maximum of finitely many continuous functions is continuous, our lemma is proven. $\square$

Our third step establishes a bound for the propinquity between finite dimensional quantum compact metric spaces which constitute the building blocks of the C*-algebras $\mathfrak{AF}_\theta$.

Lemma 4.2.11. *Let $\theta \in (0,1) \setminus \mathbb{Q}$ and $N \in \mathbb{N}$. Let $(p^\theta_n)_{n\in\mathbb{N}}$ and $(q^\theta_n)_{n\in\mathbb{N}}$ be defined from $\mathsf{cf}(\theta)$ using Expression (2.1.12). For all $n \in \{0, \ldots, N\}$ and $k \in (0,\infty)$, setting $\mathfrak{B}_n = \mathfrak{AF}_{\theta,n}$, the map:*

$$\mathsf{q}_n : x \in \mathscr{N}[\mathsf{cf}(\theta), N+1] \longmapsto \left(\mathfrak{B}_n, \mathsf{L}^k_{\mathsf{ir}(x)} \circ \underrightarrow{\alpha^n_{\mathrm{ir}(x)}} \right) \tag{4.2.5}$$

defined using Notation (4.2.2), is well-defined and continuous from $(\mathscr{N}, \mathsf{d})$ to the class of $(2,0)$-quasi-Leibniz quantum compact metric spaces metrized by $\Lambda_{2,0}$.

Proof. The statement is obvious for $n = 0$. Thus, let $n \in \{1, \ldots, N\}$. Let $\mathfrak{W}$ be any complementary subspace of $\mathbb{R}1_{\mathfrak{A}}$ in $\mathfrak{sa}(\mathfrak{B}_n)$ — which exists since $\mathfrak{sa}(\mathfrak{B}_n)$ is finite dimensional. We shall denote by $\mathfrak{S}$ the unit sphere $\{a \in \mathfrak{W} : \|a\|_{\mathfrak{B}_n} = 1\}$ in $\mathfrak{W}$. Note that since $\mathfrak{W}$ is finite dimensional, $\mathfrak{S}$ is a compact set.

We let $x \in \mathscr{N}[\mathsf{cf}(\theta), N+1]$. Let $(y_m)_{m \in \mathbb{N}}$ be a sequence in $\mathscr{N}[\mathsf{cf}(\theta), N+1]$ converging to x. Let:

$$S = \{x, y_m : m \in \mathbb{N}\} \times \mathfrak{S}$$

which is a compact subset of $\mathscr{N} \times \mathfrak{W}$. Since the function:

$$\mathsf{l}_n : (u, a) \in \mathscr{N}[\mathsf{cf}(\theta), N] \times \mathfrak{B}_n \longmapsto \mathsf{L}^k_{\mathsf{ir}(u)}\left(\underrightarrow{\alpha^n_{\mathsf{ir}(u)}}(a)\right)$$

is continuous by Lemma (4.2.10), l_n reaches a minimum on S: thus there exists $(z, c) \in S$ such that $\min_S \mathsf{l}_n = \mathsf{l}_n(z, c)$. In particular, since Lip-norms are zero only on the scalars, we have $\mathsf{l}_n(z, c) > 0$ as $\|c\|_{\mathfrak{W}} = 1$ yet the only scalar multiple of $1_{\mathfrak{B}_n}$ in $\mathfrak{W}$ is 0. We denote $m_S = \mathsf{l}_n(z, c) > 0$ in the rest of this proof. Moreover, l_n is continuous on the compact S so it is uniformly continuous.

Let $\varepsilon > 0$. As l_n is uniformly continuous on S, there exists $M \in \mathbb{N}$ such that for all $m \geqslant M$ and for all $a \in \mathfrak{S}$ we have:

$$|\mathsf{l}_n(y_m, a) - \mathsf{l}_n(x, a)| \leqslant m_S^2 \varepsilon.$$

We then have, for all $a \in \mathfrak{S}$ and $m \geqslant M$:

$$\begin{aligned}\left\| a - \frac{\mathsf{l}_n(y_m, a)}{\mathsf{l}_n(x, a)} a \right\|_{\mathfrak{B}_n} &= \frac{|\mathsf{l}_n(y_m, a) - \mathsf{l}_n(x, a)|}{\mathsf{l}_n(x, a)} \|a\|_{\mathfrak{B}_n} \\ &\leqslant \frac{\varepsilon m_S^2}{m_S} \leqslant m_S \varepsilon.\end{aligned}$$

Similarly:

$$\left\| a - \frac{\mathsf{l}_n(x,a)}{\mathsf{l}_n(y_m,a)} a \right\|_{\mathfrak{B}_n} \leqslant m_S \varepsilon, \tag{4.2.6}$$

by switching the roles of y_m and x.

We are now ready to provide an estimate for the quantum propinquity. Let $m \geqslant M$ be fixed. Writing id for the identity of $\mathfrak{B}_n$, the quadruple:

$$\gamma = (\mathfrak{B}_n, 1_{\mathfrak{B}_n}, \mathrm{id}, \mathrm{id})$$

is a bridge from $\left(\mathfrak{B}_n, \mathsf{L}^k_{\mathrm{ir}(y_m)} \circ \underrightarrow{\alpha^n_{\mathrm{ir}(y_m)}}\right)$ to $\left(\mathfrak{B}_n, \mathsf{L}^k_{\mathrm{ir}(x)} \circ \underrightarrow{\alpha^n_{\mathrm{ir}(x)}}\right)$.

As the pivot of γ is the unit, the height of γ is null. We are left to compute the reach of γ.

Step 1. *Assume that $a \in \mathbb{R}1_{\mathfrak{B}_n}$.*

We then have that $\mathsf{l}_n(y_m, a) = 0$ as well, and that $\|a - a\|_{\mathfrak{B}_n} = 0$.

Step 2. *Assume that $a \in \mathfrak{S}$.*

We note again that $\mathsf{l}_n(x,a) \geqslant m_S > 0$. By Inequality (4.5.4), we note that:

$$\left\| a - \frac{\mathsf{l}_n(x,a)}{\mathsf{l}_n(y_m,a)} a \right\|_{\mathfrak{B}_n} \leqslant \varepsilon m_S \leqslant \varepsilon \mathsf{l}_n(x,a),$$

while $\mathsf{l}_n\left(y_m, \frac{\mathsf{l}_n(x,a)}{\mathsf{l}_n(y_m,a)} a\right) = \mathsf{l}_n(x,a)$.

Step 3. *Assume that $a = b + t1_{\mathfrak{B}_n}$ with $b \in \mathfrak{S}$.*

Note that $\mathsf{l}_n(x,b) = \mathsf{l}_n(x,a)$. Therefore, let $b' \in \mathfrak{sa}(\mathfrak{B}_n)$ be constructed as in Step 2. We then check easily that:

$$\|a - (b' + t1_{\mathfrak{B}_n})\|_{\mathfrak{B}_n} = \|b - b'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n(x,a)$$

while $\mathsf{l}_n(y_m, b' + t1_{\mathfrak{B}_n}) = \mathsf{l}_n(y_m, b') \leqslant \mathsf{l}_n(x,a)$.

Step 4. *Let $a \in \mathfrak{sa}(\mathfrak{B}_n)$.*

By definition of $\mathfrak{S}$ there exists $r, t \in \mathbb{R}$ such that $a = rb + t1_{\mathfrak{B}_N}$ with $b \in \mathfrak{S}$. Let $b' \in \mathfrak{sa}(\mathfrak{A})$ be constructed from b as in Step 3. Then set $a' = rb'$. By Step 3, we have $\mathsf{l}_n(y_m, b') \leqslant \mathrm{l}_n(x, a')$ and $\|a' - b'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n(x, a')$.

Thus by homogeneity, we conclude that:

$$\forall a \in \mathfrak{sa}(\mathfrak{B}_n) \quad \exists a' \in \mathfrak{sa}(\mathfrak{B}_n) \quad \|a - a'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n(x, a) \text{ and } \mathsf{l}_n(y_m, a') \leqslant \mathsf{l}_n(x, a). \tag{4.2.7}$$

By symmetry in the roles of x and y_m we can conclude as well that:

$$\forall a \in \mathfrak{sa}(\mathfrak{B}_n) \quad \exists a' \in \mathfrak{sa}(\mathfrak{B}_n) \quad \|a - a'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n(y_m, a) \text{ and } \mathsf{l}_n(x, a') \leqslant \mathsf{l}_n(y_m, a). \tag{4.2.8}$$

Now, Expressions (4.5.5) and (4.5.6) together imply that the reach, and hence the length of the bridge γ is no more than ε. Therefore, for all $m \geqslant M$, we have:

$$\mathsf{\Lambda}_{2,0}\left((\mathfrak{B}_n, \mathsf{l}_n(x, \cdot)), (\mathfrak{B}_n, \mathsf{l}_n(y_m, \cdot))\right) \leqslant \varepsilon,$$

which concludes our proof. □

We are now able to establish the main result of this section.

Theorem 4.2.12. *For all $k \in (0, \infty)$ and using Notations (2.1.82) and (4.2.2), the function:*

$$\theta \in (0,1) \setminus \mathbb{Q} \longmapsto \left(\mathfrak{AF}_\theta, \mathsf{L}_\theta^k\right) \in \mathcal{AF}^k$$

is continuous from $(0,1) \setminus \mathbb{Q}$, with its topology as a subset of $\mathbb{R}$, to the class of $(2,0)$-quasi-Leibniz quantum compact metric spaces metrized by $\mathsf{\Lambda}_{2,0}$.

Proof. The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ and $\Phi = \phi - 1 = \frac{1}{\phi}$ be its reciprocal. The continued fraction expansion of Φ is given by:

$$\Phi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

and $\mathfrak{AF}_\Phi$ is sometimes called the Fibonacci C*-algebra [19]. Its importance for our work is that the associated sequence $(q_n^\Phi)_{n\in\mathbb{N}}$ defined by Expression (2.1.12) is the least possible sequence of the form $(q_n^\theta)_{n\in\mathbb{N}}$ given by the same expression, over all possible $\theta \in (0,1)\setminus\mathbb{Q}$ (where the order is defined entry-wise).

Let $\theta \in (0,1)\setminus\mathbb{Q}$. By Theorem (3.1.3), we have for all $n\in\mathbb{N}$:

$$\mathsf{\Lambda}_{2,0}\left((\mathfrak{AF}_\theta, \mathsf{L}_\theta^k), (\mathfrak{AF}_{\theta,n}, \mathsf{l}_n(\theta,\cdot))\right) \leqslant \left(\frac{1}{(q_n^\theta)^2+(q_{n-1}^\theta)^2}\right)^k \leqslant \left(\frac{1}{(q_n^\Phi)^2+(q_{n-1}^\Phi)^2}\right)^k, \tag{4.2.9}$$

where l_n is defined in Lemma (4.2.10).

Let $(\theta_m)_{m\in\mathbb{N}}$ be a sequence in $(0,1)\setminus\mathbb{Q}$ converging to θ. Let $\varepsilon>0$. To begin with, let $N\in\mathbb{N}$ such that for all $n\geqslant N$, we have:

$$\left(\frac{1}{(q_n^\Phi)^2+(q_{n-1}^\Phi)^2}\right)^k \leqslant \frac{\varepsilon}{2}.$$

We thus have, for all $m\in\mathbb{N}$, that:

$$\mathsf{\Lambda}_{2,0}\left((\mathfrak{AF}_\theta, \mathsf{L}_\theta^k), (\mathfrak{AF}_{\theta_m}, \mathsf{L}_{\theta_m}^k)\right) \leqslant \varepsilon + \mathsf{\Lambda}_{2,0}\left((\mathfrak{AF}_{\theta,N}, \mathsf{l}_N(\theta,\cdot)), (\mathfrak{AF}_{\theta_m,N}, \mathsf{l}_N(\theta_m,\cdot))\right). \tag{4.2.10}$$

Now, let $x_m = \mathsf{cf}(\theta_m)$ for all $m\in\mathbb{N}$ and $x=\mathsf{cf}(\theta)$. Since cf is a continuous, the sequence $(x_m)_{m\in\mathbb{N}}$ converges to x in $\mathscr{N}$. Thus there exists $M_1\in\mathbb{N}$ such that, for all $m\geqslant M_1$, we have $\mathsf{d}(x,x_m)\leqslant \frac{1}{2^{N+1}}$, i.e. $x_m\in\mathscr{N}[x,N+1]$.

We thus apply Lemma (4.2.11) to obtain from Expression (4.2.10) that:

$$\mathsf{\Lambda}_{2,0}\left((\mathfrak{AF}_\theta, \mathsf{L}^k_\theta), (\mathfrak{AF}_{\theta_m}, \mathsf{L}^k_{\theta_m})\right) \leqslant \varepsilon + \mathsf{\Lambda}_{2,0}\left(\mathsf{q}_N(\theta), \mathsf{q}_N(\theta_m)\right).$$

Now, Lemma (4.2.11) establishes that q_N is continuous. Hence:

$$\limsup_{m\to\infty} \mathsf{\Lambda}_{2,0}\left((\mathfrak{AF}_\theta, \mathsf{L}^k_\theta), (\mathfrak{AF}_{\theta_m}, \mathsf{L}^k_{\theta_m})\right) \leqslant \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, our Theorem is proven. □

4.3 Some compact families for AF algebras

The search for compact classes of quantum compact metric spaces for the quantum propinquity is a delicate yet interesting challenge. The main result on this topic is given by an analogue of the Gromov compactness theorem, proven in [45] by Latrémolière.

Our construction in Theorem (3.1.3) is designed so that AF algebras with faithful tracial states are indeed limits of finite dimensional quasi-Leibniz quantum metric spaces, so we may apply Theorem (2.3.23) to obtain:

Theorem 4.3.1. *If $U, L : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ are two sequences in $\mathbb{N} \setminus \{0\}$ such that $\lim_\infty L = \lim_\infty U = \infty$ while $L(n) \leqslant U(n)$ for all $n \in \mathbb{N}$, and if $k \in (0, \infty)$, then the class:*

$$\mathcal{AF}^k(L,U) = \left\{ (\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \in \mathcal{AF}^k \;\middle|\; \begin{array}{l} \exists \mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}} \quad \mathfrak{A} = \varinjlim \mathcal{I} \\ \mathfrak{A}_0 = \mathbb{C} \\ \forall n \in \mathbb{N} \quad L(n) \leqslant \dim \mathfrak{A}_n \leqslant U(n) \\ \exists \mu \textit{ faithful tracial state on } \mathfrak{A} \quad \mathsf{L}_\mathfrak{A} = \mathsf{L}^k_{\mathcal{I},\mu} \end{array} \right\}$$

is totally bounded for the quantum propinquity $\mathsf{\Lambda}_{2,0}$.

Proof. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $L(n) \geqslant \sqrt[k]{\frac{1}{\varepsilon}}$. If $(\mathfrak{A}, \mathsf{L}) \in \mathcal{AF}^k(L,U)$ then by definition, $\mathfrak{A} = \varinjlim \mathcal{I}$ where $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)$ such that $U(n) \geqslant \dim_{\mathbb{C}} \mathfrak{A}_n \geqslant L(n)$ for all $n \in \mathbb{N}$ and $\mathsf{L} = \mathsf{L}^k_{\mathcal{I},\mu}$ for some faithful tracial state μ of $\mathfrak{A}$.

Therefore, by Theorem (3.1.3):

$$\Lambda_{2,0}((\mathfrak{A}, \mathsf{L}), (\mathfrak{A}_N, \mathsf{L} \circ \underrightarrow{\alpha^N})) \leqslant \frac{1}{\dim(\mathfrak{A}_N)^k} \leqslant \frac{1}{L(N)^k} \leqslant \varepsilon.$$

Thus $\mathrm{cov}_{(2,0)}(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant U(N)$. Moreover, $\mathrm{diam}^*(\mathfrak{A}, \mathsf{L}) \leqslant 2$, and thus by Theorem (2.3.23), the class $\mathcal{AF}^k(L,U)$ is totally bounded for $\Lambda_{2,0}$. $\square$

The quantum propinquity is not known to be complete. The dual propinquity [43], introduced and studied by Latrémolière, is a complete metric and the proper formulation of Theorem (2.3.23) can thus be used to characterized compactness of certain classes of quasi-Leibniz compact quantum metric spaces. However, we face a few challenges when searching for compact subclasses of $\mathcal{AF}^k$.

As the quantum propinquity dominates the dual propinquity, Theorems (3.1.3), (4.1.7) and (4.2.12) are all valid for the dual propinquity, as is Theorem (4.3.1). However, we do not know what is the closure of the classes described in Theorem (4.3.1) for the dual propinquity, and thus we may not conclude whether these classes are, in general, compact. It should be noted that, as shown by Latrémolière in [45], there are many quasi-Leibniz quantum compact metric spaces which are limits of finite dimensional quasi-Leibniz quantum compact metric spaces for the dual propinquity.

Moreover, we do not know what the completion of the classes in Theorem (4.3.1) are for the quantum propinquity either. Thus it is again difficult to describe compact classes from Theorem (4.3.1).

Yet, the situation is actually quite interesting if looked at from a somewhat different perspective. Indeed, Theorems (4.1.7) and (4.2.12) provide us with continuous maps from the Baire space to subclasses of $\mathcal{AF}^k$. Thus, knowledge about the compact subsets of $\mathscr{N}$ provides actual knowledge of some compact subclasses of $\mathcal{AF}^k$ for the quantum propinquity.

To illustrate this point, we begin by giving a theorem characterizing closed, totally bounded, and compact subspaces of the Baire space. This theorem is well-known in descriptive set theory; however the proofs of these results seem scattered in the literature and, maybe more importantly, rely on a more complex framework and terminology than is needed for our purpose. We thus include a short proof for the convenience of our readers.

Notation 4.3.2. *If $x \in \mathscr{N}$ and $n \in \mathbb{N}$ then we denote the finite sequence $(x_0, \ldots, x_n)$ by $x|_n$.*

Theorem 4.3.3. *The Baire Space $\mathscr{N}$ is complete for the ultrametric d, defined for all $x, y \in \mathscr{N}$ by:*

$$\mathsf{d}(x,y) = 2^{-\min\{n \in \mathbb{N} \cup \{\infty\} : x|_n \neq y|_n\}}.$$

Thus the compact subsets of $\mathscr{N}$ are its closed, totally bounded subsets. Moreover, for any $X \subseteq \mathscr{N}$:

1. *the closure of X is the set:*

$$\{x \in \mathscr{N} : \forall n \in \mathbb{N} \; \exists y \in X \quad x|_n = y|_n\}$$

2. *X is totally bounded if and only for all $n \in \mathbb{N}$:*

$$\{x|_n : x \in X\}$$

   is finite.

Proof. We prove each assertion of our theorem in each of the following step.

Step 1. *The space $(\mathscr{N}, \mathsf{d})$ is complete.*

Let $(x^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{N}, \mathsf{d})$. For all $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that, if $p, q \geqslant M$, we have $\mathsf{d}(x^p, x^q) < \frac{1}{2^n}$. Since d is an ultra-metric, we have equivalently that $\mathsf{d}(x^M, x^p) < \frac{1}{2^n}$ for all $p \geqslant M$: thus for all $m \geqslant M$ we have $x^M|_n = x^p|_n$. In particular, $(x_n^m)_{m\in\mathbb{N}}$ is an eventually constant function for all $n \in \mathbb{N}$. It is then trivial to check that the sequence $(\lim_{m\to\infty} x_n^m)_{n\in\mathbb{N}}$ is the limit of $(x^m)_{m\in\mathbb{N}}$.

Step 2. *The closure of $X \subseteq \mathscr{N}$ is:*

$$Y = \{x \in \mathscr{N} : \forall n \in \mathbb{N}\ \exists y \in X \quad x|_n = y|_n\}$$

Note that by definition, $X \subseteq Y$. We now check that Y is closed. Let $(z^m)_{m\in\mathbb{N}}$ be a sequence in Y converging to some $z \in \mathscr{N}$. By definition of d, for all $N \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for all $m \geqslant M$ we have $\mathsf{d}(z^m, z) < \frac{1}{2^N}$. Thus $z^M|_N = z|_N$ by definition. So $z \in Y$ as desired, and thus Y is closed.

Let now $y \in Y$. Let $n \in \mathbb{N}$. By definition, there exists $x^n \in X$ such that $x^n|_n = y|_n$, i.e. $\mathsf{d}(x^n, y) < \frac{1}{2^n}$. Thus $(x^n)_{n\in\mathbb{N}}$ converges to y. Thus Y is contained in the closure of X. Since Y is closed, it follows from the minimality of closures that Y is indeed the closure of X.

Step 3. *A characterization of totally bounded subsets of the Baire Space.*

Assume now that X is totally bounded. Then for all $n \in \mathbb{N}$ there exists a finite subset X_n of X such that for all $x \in X$ there exists $y \in X_n$ with $\mathsf{d}(x, y) < \frac{1}{2^n}$, or equivalently, such that $x|_n = y|_n$. Thus $\{x|_n : x \in X\} = \{x|_n : x \in X_n\}$, the latter being finite. Conversely, note that X_n converges to X for the Hausdorff distance

$\mathsf{Haus_d}$, and thus if $(X_n)_{n\in\mathbb{N}}$ is finite for all $n \in \mathbb{N}$, we conclude easily that X is totally bounded. $\square$

Remark 4.3.4. *Theorem (4.3.3) is well-known in descriptive set theory, though the proof is often presented within a much more elaborate framework. Our assertion about the closure of sets is often phrased by noting that a subset of $\mathscr{N}$ is closed if and only if it is given as all infinite paths in a pruned tree. In this context, a tree over the Baire Space is a subset of the collection of all* finite *sequences valued in $\mathbb{N} \setminus \{0\}$ with a simple hereditary property: if a finite sequence is in our tree, then so is its sub-sequence obtained by dropping the last entry. A pruned tree is a tree T such that every sequence in it is a proper sub-sequence of another element of T. Last, a path is simply a sequence $x \in \mathscr{N}$ such that $x|_n \in T$ for all N. This relation makes the translation between Theorem (4.3.3) and the terminology of certain branches of set theory.*

Moreover, a tree is finitely branching when given a finite sequence x of length n in the tree, there are only finitely many possible finite sequences of length $n+1$ whose n first entries coincide with x. It is easy to see that Theorem (4.3.3) exactly states that a subset of the Baire space is compact if and only if it consists of all infinite paths through a pruned tree with finite branching (and our theorem makes the tree explicit)

We now consider the Effros-Shen AF algebras.

Corollary 4.3.5. *For all $k \in \mathbb{N}$ and all sequence $B : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ with $\sqrt{\frac{B(n+1)}{B(n)}} \in \mathbb{N} \setminus \{0,1\}$ for all $n \in \mathbb{N}$, the class:*

$$\mathcal{UHF}^k \cap \mathcal{AF}^k((2^n)_{n\in\mathbb{N}}, B)$$

is compact for the quantum propinquity $\Lambda_{2,0}$.

Proof. Let:

$$X = \left\{ x \in \mathscr{N} : \forall n \in \mathbb{N} \quad x_n + 1 \leqslant \sqrt{\frac{B(n+1)}{B(n)}} \right\}.$$

By construction, $\mathfrak{uhf}\,(X,k) = \mathcal{UHF}^k \cap \mathcal{AF}^k((2^n)_{n\in\mathbb{N}}, B)$ (the lower bound on the dimension of the matrix algebras was observed in the proof of Theorem (4.1.7)). On the other hand, by Theorem (4.3.3), the set X is compact and by Theorem (4.1.7), the map $\mathfrak{uhf}\,(\cdot,k)$ is continuous. So $\mathcal{UHF}^k \cap \mathcal{AF}^k((2^n)_{n\in\mathbb{N}}, B)$ is compact. $\square$

We also obtain:

Corollary 4.3.6. *Let $C, B \in \mathscr{N}$, and set:*

$$X = \left\{ \theta \in (0,1) \setminus \mathbb{Q} : \theta = \lim_{n\to\infty} \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\cdots + \cfrac{1}{r_n}}}} \text{ and } \forall n \in \mathbb{N} \quad C(n) \leqslant r_n \leqslant B(n) \right\}$$

Then the set:

$$\left\{ (\mathfrak{A}, \mathsf{L}) \in \mathcal{AF}^k : \mathfrak{A} \in \mathfrak{AF}_X \right\}$$

is compact for the quantum propinquity $\Lambda_{2,0}$.

Proof. This follows from Theorem (4.3.3) and the continuity established in Theorem (4.2.12). $\square$

We were thus able to obtain several examples of compact classes of quasi-Leibniz quantum compact metric spaces for the quantum propinquity and consisting of infinitely many AF algebras, which is a rather notable result. We also note that since the dual propinquity [43] is also a metric up to isometric isomorphism and is dominated by the quantum propinquity, the topology induced by the quantum propinquity and the dual propinquity on these compact classes must agree.

4.4 Family of Lip-norms for a fixed AF algebra

In this section, we consider the situation in which we fix a unital AF-algebra with faithful tracial state and consider the construction of the Lip-norm from Theorem (3.1.3), in which we vary our choices of the sequence β. From this, we describe convergence in quantum propinquity with respect to this notion. We note that Section (4.2) essentially provides an outline for the process.

Notation 4.4.1. *Let $\beta : \mathbb{N} \to (0, \infty)$ be a positive sequence that tends to 0 at infinity. Denote the space of real-valued sequences that converge to 0 as $c_0(\mathbb{N}, \mathbb{R})$. Define:*

$$c_\beta = \{x \in c_0(\mathbb{N}, \mathbb{R}) : \forall n \in \mathbb{N},\ 0 < x(n) \leqslant \beta(n)\}.$$

Theorem 4.4.2. *Let $\mathfrak{A}$ be an AF algebra endowed with a faithful tracial state μ such that $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ is an inductive sequence of finite dimensional C^*-algebras with C^*-inductive limit $\mathfrak{A}$, with $\mathfrak{A}_0 = \mathbb{C}$ and where α_n is a unital *-monomorphism for all $n \in \mathbb{N}$. If $\beta : \mathbb{N} \to (0, \infty)$ is a positive sequence that tends to 0 at infinity and $\left(x^k\right)_{k\in\mathbb{N}} \cup \{x\} \subset c_\beta$ such that x^k converges point-wise to x, then using the notations of Theorem (3.1.3):*

$$\lim_{k\to\infty} \Lambda_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x^k}\right), \left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x}\right)\right) = 0.$$

Proof. The proof follows the procedure from Section (4.2).

Let $\beta : \mathbb{N} \to (0, \infty)$ be a positive sequence that tends to 0 at infinity. Assume that $\left(x^k\right)_{k\in\mathbb{N}} \cup \{x\} \subset c_\beta$ such that x^k converges point-wise to x. Next, we show convergence of the finite dimensional spaces $\mathfrak{A}_n$ for all $n \in \mathbb{N}$. Thus, fix $N \in \mathbb{N}$. Let $y \in c_\beta$, so that $y(n) > 0$ for all $n \in \mathbb{N}$, and let $a \in \mathfrak{A}_N$. Then:

$$\mathsf{L}^y_{\mathcal{I},\mu} \circ \underrightarrow{\alpha}^N(a) = \max\left\{ \frac{\left\| \underrightarrow{\alpha}^N(a) - \mathbb{E}\left(\underrightarrow{\alpha}^N(a) \middle| \underrightarrow{\alpha}^n(\mathfrak{A}_n) \right) \right\|_{\mathfrak{A}}}{y(n)} : n \in \mathbb{N}, n \leqslant N \right\}.$$

Define $R^N_+ = \{y = (y(0), y(1), \ldots, y(N)) \in \mathbb{R}^{N+1} : \forall n \in \{0, 1, \ldots, N\}, y(n) > 0\}$. For $x, y \in R^N_+$, we define $\mathsf{d}_\infty(x, y) = \max\{|x(n) - y(n)| : n \in \{0, 1, \ldots, N\}\}$. Thus, $\left(R^N_+, \mathsf{d}_\infty\right)$ is a metric space. Define $g : R^N_+ \times \mathfrak{A}_N \to \mathbb{R}$ by:

$$g(y, a) = \max\left\{ \frac{\left\| \underrightarrow{\alpha}^N(a) - \mathbb{E}\left(\underrightarrow{\alpha}^N(a) \middle| \underrightarrow{\alpha}^n(\mathfrak{A}_n) \right) \right\|_{\mathfrak{A}}}{y(n)} : n \in \mathbb{N}, n \leqslant N \right\},$$

which is finite by definition of R^N_+. Therefore, it follows that:

$$g : \left(R^N_+, \mathsf{d}_\infty\right) \times (\mathfrak{A}_N, \|\cdot\|_{\mathfrak{A}_N}) \to \mathbb{R}$$

is continuous. Denote the class of all $(2, 0)$-quasi-Leibniz quantum compact metric spaces by $\mathcal{QQCMS}_{2,0}$. Next, define $G : R^N_+ \to \mathcal{QQCMS}_{2,0}$ by:

$$G(y) = (\mathfrak{A}_N, g(y, \cdot)),$$

which is well-defined by definition of g. Thus, following the proof of Theorem (4.2.11), we conclude that $G : \left(R^N_+, \mathsf{d}_\infty\right) \to (\mathcal{QQCMS}_{2,0}, \mathsf{\Lambda}_{2,0})$ is continuous. If $y \in \mathbb{R}^{\mathbb{N}}$, then we denote $y|_N = (y(0), y(1), \ldots, y(N))$. Since $\left(x^k\right)_{k \in \mathbb{N}} \cup \{x\} \subset c_\beta$, we have that $\left(x^k|_N\right)_{k \in \mathbb{N}} \cup \{x|_N\} \subset R^N_+$. Furthermore, the assumption that x^k converges pointwise to x implies that $\lim_{k \to \infty} \mathsf{d}_\infty\left(x^k|_N, x|_N\right) = 0$. Therefore:

$$\lim_{k \to \infty} \mathsf{\Lambda}_{2,0}\left(G\left(x^k|_N\right), G\left(x|_N\right)\right) = 0.$$

But, for all $k \in \mathbb{N}$:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_N, \mathsf{L}_{\mathcal{I},\mu}^{x^k} \circ \underrightarrow{\alpha^N}\right), \left(\mathfrak{A}_N, \mathsf{L}_{\mathcal{I},\mu}^{x} \circ \underrightarrow{\alpha^N}\right)\right) = \mathsf{\Lambda}_{2,0}\left(G\left(x^k|_N\right), G\left(x|_N\right)\right).$$

We thus have:

$$\lim_{k\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_N, \mathsf{L}_{\mathcal{I},\mu}^{x^k} \circ \underrightarrow{\alpha^N}\right), \left(\mathfrak{A}_N, \mathsf{L}_{\mathcal{I},\mu}^{x} \circ \underrightarrow{\alpha^N}\right)\right) = 0. \tag{4.4.1}$$

As $N \in \mathbb{N}$ was arbitrary, we conclude that Equation (4.4.1) is true for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $M \in \mathbb{N}$ such that for all $n \geqslant M$, $\beta(n) < \varepsilon/2$. Hence, if $n \geqslant M$, then by Theorem (3.1.3) and definition of c_β:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}_{\mathcal{I},\mu}^{x^k} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x^k}\right)\right) \leqslant x^k(n) \leqslant \beta(n) < \varepsilon/2$$

for all $k \in \mathbb{N}$ and:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_n, \mathsf{L}_{\mathcal{I},\mu}^{x} \circ \underrightarrow{\alpha^n}\right), \left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x}\right)\right) \leqslant x(n) \leqslant \beta(n) < \varepsilon/2.$$

By the triangle inequality and Equation (4.4.1), we thus get:

$$\limsup_{k\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x^k}\right), \left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x}\right)\right) \leqslant \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $\lim_{k\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x^k}\right), \left(\mathfrak{A}, \mathsf{L}_{\mathcal{I},\mu}^{x}\right)\right) = 0.$ $\square$

In particular, for the Cantor set, we can use this result to discuss continuity in quantum propinquity of the continuous functions on the Cantor set with respect to the quantum ultrametrics discussed in Section (3.1.1). All that is required is a sequence in c_β, which converges point-wise to some element in c_β. We present this in the case of the standard ultrametrics, and note that although we are using the same C^*-algebra, $C(\mathcal{C})$, if $r \neq s$, then the associated standard ultrametrics on the

Cantor set are not isometric. This implies that the function defined in the following Corollary (4.4.3) is not constant up to quantum isometry since the dual map of a quantum isometry provides an isometry between pure states as seen in the proof of Theorem (2.3.19).

Corollary 4.4.3. *Let $r > 1$, and set $\beta_r : n \in \mathbb{N} \mapsto \frac{1}{2}r^{-n}$. Using the notations of Theorem (3.1.5) along with Notations (2.1.77) and (3.1.15), the function:*

$$\mathsf{u} : r \in (1, \infty) \longmapsto \left(C(\mathcal{C}), \mathsf{L}^{\beta_r}_{\mathcal{T},\lambda} \right)$$

is continuous from $(1, \infty)$ to the class of $(2, 0)$-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity $\mathsf{\Lambda}_{2,0}$.

Proof. Let $(r_n)_{n\in\mathbb{N}} \cup \{r\} \subset (1, \infty)$ such that $\lim_{n\to\infty} |r_n - r| = 0$. Since $(r_n)_{n\in\mathbb{N}} \cup \{r\}$ is a compact set, there exists some $a > 1$ such that for all $n \in \mathbb{N}$, $r_n, r \in [a, \infty)$. Therefore, for all $n \in \mathbb{N}$, we have that $\beta_{r_n}, \beta_r \in c_{\beta_a}$. The sequence $(\beta_{r_n})_{n\in\mathbb{N}}$ converges point-wise to β_r since:

$$\lim_{n\to\infty} |\beta_{r_n}(m) - \beta_r(m)| = \lim_{n\to\infty} \left| \tfrac{1}{2} r_n^{-m} - \tfrac{1}{2} r^{-m} \right| = 0$$

for all $m \in \mathbb{N}$. Hence, by the Theorem (4.4.2),

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}(\mathsf{u}(r_n), \mathsf{u}(r)) = 0.$$

Thus, sequential continuity provides the desired result. $\square$

4.5 Criteria for Convergence of AF algebras in the quantum Gromov-Hausdorff propinquity

Taking stock of our construction of Lip-norms for unital AF algebras with faithful tracial state in Theorem (3.1.3), it is apparent that the construction relies on the inductive sequence, faithful tracial state, and some real-valued positive sequence converging to 0. Thus, this section provides suitable notions of convergence for all 3 of these structures, which in turn produce convergence of AF algebras in the quantum propinquity. This is motivated by our arguments of continuity in Section (4.1) of UHF algebras and in Section (4.2) of Effros-Shen AF algebras, and in fact, we can reproduce these continuity results as a consequence of Theorem (4.5.6).

We now introduce an appropriate notion of merging inductive sequences together in Definition (4.5.1).

Definition 4.5.1. *We consider 2 cases of inductive sequences in this definition.*

Case 1. *Closure of union*

For each $k \in \overline{\mathbb{N}}$, let $\mathfrak{A}^k$ be a C^*-algebras with $\mathfrak{A}^k = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_{k,n}}^{\|\cdot\|_{\mathfrak{A}^k}}$ such that $\mathcal{U}^k = (\mathfrak{A}_{k,n})_{n\in\mathbb{N}}$ is a non-decreasing sequence of C^*-subalgebras of $\mathfrak{A}^k$, then we say $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is a fusing family *if:*

1. *There exists $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ non-decreasing such that $\lim_{n\to\infty} c_n = \infty$, and*
2. *for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $\mathfrak{A}_{k,n} = \mathfrak{A}_{\infty,n}$ for all $n \in \{0, 1, \ldots, N\}$.*

Case 2. *Inductive limit*

For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$ be an inductive sequence of C^*-algebras with inductive limit, $\mathfrak{A}^k$. We say that the family of C^*-algebras $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family *of C^*-algebras if:*

1. *There exists $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ non-decreasing such that $\lim_{n\to\infty} c_n = \infty$, and*

2. for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $(\mathfrak{A}_{k,n}, \alpha_{k,n}) = (\mathfrak{A}_{\infty,n}, \alpha_{\infty,n})$ for all $n \in \{0, 1, \ldots, N\}$.

In either case, we call the sequence $(c_n)_{n\in\mathbb{N}}$ the fusing sequence*.*

Remark 4.5.2. *The results in this section are phrased in terms of* IL-fusing families *since our propinquity convergence results are all in terms of inductive limits. But, we note that all the results of this section are valid for the closure of union case as well with appropriate translations, but most convergence results are convergence results are more easily fulfilled in the inductive limit case. Note that any IL-fusing family may be viewed as a fusing family via the canonical *-homomorphisms of Notation (2.1.65) and Proposition (2.1.66), which is why we don't decorate the term* fusing family *in the closure of union case.*

Hypotheses *2.* and *3.* in the following Lemma (4.5.3) introduce the remaining notions of convergence that together with fusing families will imply convergence of quantum propinquity of AF algebras in Theorem (4.5.6). Indeed, *2.* is simply an appropriate use of weak-* convergence for the faithful tracial states in relation to fusing families, and *3.* is an appropriate use of pointwise convergence of the sequences that provide convergence of the finite dimensional subspaces in Theorem (3.1.3).

Furthermore, Lemma (4.5.3) provides that the Lip-norms induced on the finite dimensional subspaces form a continuous field of Lip-norms, a notion introduced by Rieffel in [61].

Lemma 4.5.3. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$ be an inductive sequence of finite dimensional C^*-algebras with inductive limit $\mathfrak{A}^k$, such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is a unital *-monomorphism for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$.*

If:

1. $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$,

2. $\{\tau^k : \mathfrak{A}^k \to \mathbb{C}\}_{k\in\overline{\mathbb{N}}}$ *is a family of faithful tracial states such that for each* $N \in \mathbb{N}$, *we have that* $\left(\tau^k \circ \underrightarrow{\alpha_k^N}\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ *converges to* $\tau^\infty \circ \underrightarrow{\alpha_\infty^N}$ *in the weak-* topology on* $\mathscr{S}(\mathfrak{A}_{\infty,N})$, *and*

3. $\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k\in\overline{\mathbb{N}}}$ *is a family of convergent sequences such that for all* $N \in \mathbb{N}$ *if* $k \in \mathbb{N}_{\geqslant c_N}$, *then* $\beta^k(n) = \beta^\infty(n)$ *for all* $n \in \{0,1,\ldots,N\}$ *and there exists* $B : \overline{\mathbb{N}} \to (0,\infty)$ *with* $B(\infty) = 0$ *and* $\beta^m(l) \leqslant B(l)$ *for all* $m, l \in \overline{\mathbb{N}}$,

then for all $N \in \mathbb{N}$, *if* $n \in \{0,1,\ldots,N\}$, *then the map:*

$$\mathsf{l}_n^N : (k,a) \in \overline{\mathbb{N}}_{\geqslant c_N} \times \mathfrak{A}_{\infty,n} \longmapsto \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k} \circ \underrightarrow{\alpha_k^n}(a) \in \mathbb{R}$$

is well-defined and continuous with respect to the product topology on $\overline{\mathbb{N}} \times \left(\mathfrak{A}_{\infty,n}, \|\cdot\|_{\mathfrak{A}_{\infty,n}}\right)$, *where* $\mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}$ *is given by Theorem (3.1.3).*

Proof. First, we establish a weak-* convergence result implied by (2). Let $N \in \overline{\mathbb{N}}$.

Claim 4.5.4. $\left(\tau^k \circ \underrightarrow{\alpha_k^m}\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ *converges to* $\tau^\infty \circ \underrightarrow{\alpha_\infty^m}$ *in the weak* topology on* $\mathscr{S}(\mathfrak{A}_{\infty,m})$ *for each* $m \in \{0,1,\ldots,N\}$.

Proof of claim. Let $m \in \{0,1,\ldots,N\}$. The case $m = N$ is given by assumption. So, assume that $N \geqslant 1$ and $m \in \{0,\ldots,N-1\}$. Fix $a \in \mathfrak{A}_{\infty,m}$, we have by Proposition (2.1.66) and definition of IL-fusing family:

$$\tau^k \circ \underrightarrow{\alpha_k^m}(a) = \tau^k \circ \underrightarrow{\alpha_k^N}(\alpha_{k,N-1} \circ \cdots \circ \alpha_{k,m}(a)) = \tau^k \circ \underrightarrow{\alpha_k^N}(\alpha_{\infty,N-1} \circ \cdots \circ \alpha_{\infty,m}(a))$$

for $k \in \overline{\mathbb{N}}_{\geqslant c_N}$, which proves our claim since $\left(\tau^k \circ \underrightarrow{\alpha_k^N}\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\tau^\infty \circ \underrightarrow{\alpha_\infty^N}$ in the weak* topology on $\mathscr{S}(\mathfrak{A}_{\infty,N})$. $\square$

Next, we establish a more explicit form of our Lip-norms on the finite-dimensional subspaces. Fix $N \in \mathbb{N}$ and $n \in \{0,1,\ldots,N\}$. l_n^N is well-defined by definition of a

IL-fusing family. Furthermore, as $\mathbb{E}\left(\cdot\middle|\underrightarrow{\alpha}_{k}^{j}(\mathfrak{A}_{k,j})\right)$ is a conditional expectation for all $k \in \overline{\mathbb{N}}, j \in \mathbb{N}$, we have that:

$$\mathbb{E}\left(\underrightarrow{\alpha}_{k}^{n}(a)\middle|\underrightarrow{\alpha}_{k}^{j}(\mathfrak{A}_{k,j})\right) = \underrightarrow{\alpha}_{k}^{n}(a)$$

for $j \geqslant n, a \in \mathfrak{A}_{\infty,n}$.

Therefore:

$$\mathsf{l}_n^N(k,a) = \max\left\{ \frac{\left\|\underrightarrow{\alpha}_{k}^{n}(a) - \mathbb{E}\left(\underrightarrow{\alpha}_{k}^{n}(a)\middle|\underrightarrow{\alpha}_{k}^{m}(\mathfrak{A}_{\infty,m})\right)\right\|_{\mathfrak{A}^k}}{\beta^\infty(m)} : m \in \{0,\ldots,n-1\} \right\}, \tag{4.5.1}$$

which will allow use to apply Proposition (3.1.13).

Fix $m \in \{0,\ldots,n-1\}, k \geqslant N, a \in \mathfrak{A}_{\infty,n}$. Since $\mathfrak{A}_{\infty,m}$ is finite dimensional, the C*-algebra $\mathfrak{A}_{\infty,m} \cong \oplus_{j=1}^{N}\mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1),\ldots,n(N) \in \mathbb{N}\setminus\{0\}$ with *-isomorphism $\gamma : \oplus_{j=1}^{N}\mathfrak{M}(n(j)) \to \mathfrak{A}_{\infty,m}$. Let E be the set of matrix units for $\oplus_{j=1}^{N}\mathfrak{M}(n(j))$. Now, define $\alpha_{k,m\to n} = \alpha_{k,n-1}\circ\cdots\circ\alpha_{k,m}$, and by definition of IL-fusing family, we have that $\alpha_{k,m\to n} = \alpha_{\infty,m\to n}$. Therefore, by Proposition (2.1.66) and Proposition (3.1.13):

$$\begin{aligned}
&\left\|\underrightarrow{\alpha}_{k}^{n}(a) - \mathbb{E}\left(\underrightarrow{\alpha}_{k}^{n}(a)\middle|\underrightarrow{\alpha}_{k}^{m}(\mathfrak{A}_{\infty,m})\right)\right\|_{\mathfrak{A}^k}\\
&= \left\|\underrightarrow{\alpha}_{k}^{n}(a) - \sum_{e\in E}\frac{\tau^k(\underrightarrow{\alpha}_{k}^{m}\circ\gamma(e^*)\underrightarrow{\alpha}_{k}^{n}(a))}{\tau^k(\underrightarrow{\alpha}_{k}^{m}\circ\gamma(e^*e))}\underrightarrow{\alpha}_{k}^{m}\circ\gamma(e)\right\|_{\mathfrak{A}^k}\\
&= \left\|\underrightarrow{\alpha}_{k}^{n}(a) - \sum_{e\in E}\frac{\tau^k(\underrightarrow{\alpha}_{k}^{n}(\alpha_{k,m\to n}(\gamma(e)^*)a))}{\tau^k(\underrightarrow{\alpha}_{k}^{n}(\alpha_{k,m\to n}(\gamma(e^*e)))}\underrightarrow{\alpha}_{k}^{n}(\alpha_{k,m\to n}(\gamma(e)))\right\|_{\mathfrak{A}^k}\\
&= \left\|\underrightarrow{\alpha}_{k}^{n}\left(a - \sum_{e\in E}\frac{\tau^k(\underrightarrow{\alpha}_{k}^{n}(\alpha_{\infty,m\to n}(\gamma(e)^*)a))}{\tau^k(\underrightarrow{\alpha}_{k}^{n}(\alpha_{\infty,m\to n}(\gamma(e^*e)))}\alpha_{\infty,m\to n}(\gamma(e))\right)\right\|_{\mathfrak{A}^k}\\
&= \left\|a - \sum_{e\in E}\frac{\tau^k\circ\underrightarrow{\alpha}_{k}^{n}(\alpha_{\infty,m\to n}\circ\gamma(e^*)a)}{\tau^k\circ\underrightarrow{\alpha}_{k}^{n}(\alpha_{\infty,m\to n}\circ\gamma(e^*e))}\alpha_{\infty,m\to n}\circ\gamma(e)\right\|_{\mathfrak{A}_{\infty,n}}
\end{aligned} \tag{4.5.2}$$

Hence, by Claim (4.5.4) and Proposition (3.1.13), the map:

$$(k,a) \in \overline{\mathbb{N}}_{\geqslant c_N} \times \mathfrak{A}_{\infty,n} \mapsto \frac{\left\| \underrightarrow{\alpha_k^n}(a) - \mathbb{E}\left(\underrightarrow{\alpha_k^n}(a) \middle| \underrightarrow{\alpha_k^m}(\mathfrak{A}_{\infty,m}) \right) \right\|_{\mathfrak{A}^k}}{\beta^\infty(m)} \in \mathbb{R}$$

is continuous for each $m \in \{0, \ldots, n-1\}$. As the maximum of finitely many continuous real-valued functions is continuous, our lemma is proven by Expression (4.5.1). □

This next Theorem (4.5.5) establishes conditions for the convergence of the finite dimensional subspaces of an AF algebra.

Theorem 4.5.5. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C^*-algebras with inductive limit $\mathfrak{A}^k$, such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is a unital *-monomorphism for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$.*

If:

1. *$\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$,*

2. *$\{\tau^k : \mathfrak{A}^k \to \mathbb{C}\}_{k \in \overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$, we have that $\left(\tau^k \circ \underrightarrow{\alpha_k^N}\right)_{k \in \mathbb{N}_{\geqslant c_N}}$ converges to $\tau^\infty \circ \underrightarrow{\alpha_\infty^N}$ in the weak-* topology on $\mathscr{S}(\mathfrak{A}_{\infty,N})$, and*

3. *$\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k \in \overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geqslant c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \ldots, N\}$ and there exists $B : \overline{\mathbb{N}} \to (0,\infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leqslant B(l)$ for all $m, l \in \overline{\mathbb{N}}$,*

then for every $N \in \mathbb{N}$ and $n \in \{0, \ldots, N\}$, the map:

$$F_n^N : k \in \overline{\mathbb{N}}_{\geqslant c_N} \longmapsto \left(\mathfrak{A}_{k,n}, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k} \circ \underrightarrow{\alpha_k^n}\right) \in (\mathcal{QQCMS}_{2,0}, \Lambda_{2,0})$$

is well-defined and continuous, and therefore:

$$\lim_{k\to\infty} \Lambda_{2,0}\left(\left(\mathfrak{A}_{k,n}, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k} \circ \underrightarrow{\alpha^n_k}\right), \left(\mathfrak{A}_{\infty,n}, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty} \circ \underrightarrow{\alpha^n_\infty}\right)\right) = 0,$$

where $\mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}$ *is given by Theorem (3.1.3).*

Proof. Fix $N \in \overline{\mathbb{N}}$ and $n \in \{0, \ldots, N\}$. If $n = 0$, then $\mathfrak{A}_{k,0} = \mathfrak{A}_{\infty,0} \cong \mathbb{C}$ and since Lip-norms vanish only on scalars by Definition (2.2.9), the map F^N_0 is constant up to quantum isometry and therefore continuous.

Assume that $n \in \{1, \ldots, N\}$ and $k \geqslant c_N$. Set $\mathfrak{B}_n = \mathfrak{A}_{k,n} = \mathfrak{A}_{\infty,n}$ by definition of IL-fusing family. Let $\mathfrak{W}$ be any complementary subspace of $\mathbb{R}1_{\mathfrak{A}}$ in $\mathfrak{sa}(\mathfrak{B}_n)$ — which exists since $\mathfrak{sa}(\mathfrak{B}_n)$ is finite dimensional. We shall denote by $\mathfrak{S}$ the unit sphere $\{a \in \mathfrak{W} : \|a\|_{\mathfrak{B}_n} = 1\}$ in $\mathfrak{W}$. Note that since $\mathfrak{W}$ is finite dimensional, $\mathfrak{S}$ is a compact set. Set $S = \overline{\mathbb{N}}_{\geqslant c_N} \times \mathfrak{S}$, which is a compact set in the product topology. Since the function l^N_n is continuous by Lemma (4.5.3), it reaches a minimum on S. Thus, there exists $(K, c) \in S$ such that: $\min_S \mathsf{l}^N_n = \mathsf{l}^N_n(K, c)$. In particular, since Lip-norms are zero only on the scalars, we have $\mathsf{l}^N_n(K, c) > 0$ as $\|c\|_{\mathfrak{W}} = 1$ yet the only scalar multiple of $1_{\mathfrak{B}_n}$ in $\mathfrak{W}$ is 0. We denote $m_S = \mathsf{l}^N_n(K, a) > 0$ in the rest of this proof.

Moreover, the function l^N_n is continuous on the compact set S, and thus, it is uniformly continuous with respect to any metric that metrizes the product topology. In particular, consider the max metric, denoted by m, with respect to the norm on $\mathfrak{S}$ and the metric on $\overline{\mathbb{N}}$ defined by $\mathsf{d_A}(n, m) = |1/(n+1) - 1/(m+1)|$ for all $n, m \in \overline{\mathbb{N}}$ with the convention that $1/(\infty+1) = 0$, in which the metric $\mathsf{d_A}$ metrizes the topology on $\overline{\mathbb{N}}$.

Let $\varepsilon > 0$. As l^N_n is uniformly continuous on the metric space (S, m), there exists $\delta > 0$ such that if $\mathsf{m}(s, s') < \delta$, then $|\mathsf{l}^N_n(s) - \mathsf{l}^N_n(s')| \leqslant m_S^2 \varepsilon$. Now, there exsits $M \in \mathbb{N}_{\geqslant c_N}$ such that $1/M < \delta$. Let $m \geqslant M$ and $a \in \mathfrak{S}$, then by definition of the metrics m and $\mathsf{d_A}$:

$$\mathsf{m}((m,a),(\infty,a)) = 1/(m+1) < 1/m \leqslant 1/M < \delta.$$

Thus, for all $m \geqslant M$ and for all $a \in \mathfrak{S}$ we have:

$$|\mathsf{l}_n^N(m,a) - \mathsf{l}_n^N(\infty,a)| \leqslant m_S^2 \varepsilon.$$

We then have, for all $a \in \mathfrak{S}$ and $m \geqslant M$, since l_n^N is positive on S:

$$\begin{aligned} \left\| a - \frac{\mathsf{l}_n^N(m,a)}{\mathsf{l}_n^N(\infty,a)} a \right\|_{\mathfrak{B}_n} &= \frac{|\mathsf{l}_n^N(m,a) - \mathsf{l}_n^N(\infty,a)|}{\mathsf{l}_n^N(\infty,a)} \|a\|_{\mathfrak{B}_n} \\ &\leqslant \frac{\varepsilon m_S^2}{m_S} \leqslant m_S \varepsilon. \end{aligned} \tag{4.5.3}$$

Similarly:

$$\left\| a - \frac{\mathsf{l}_n^N(\infty,a)}{\mathsf{l}_n^N(m,a)} a \right\|_{\mathfrak{B}_n} \leqslant m_S \varepsilon. \tag{4.5.4}$$

We are now ready to provide an estimate for the quantum propinquity. Let $m \geqslant M$ be fixed. Writing id for the identity of $\mathfrak{B}_n$, the quadruple:

$$\gamma = (\mathfrak{B}_n, 1_{\mathfrak{B}_n}, \mathrm{id}, \mathrm{id})$$

is a bridge in the sense of Definition (2.3.2) from $\left(\mathfrak{B}_n, \mathsf{L}_{\mathcal{I}(m),\tau^m}^{\beta^m} \circ \underrightarrow{\alpha_m^n}\right)$ to $\left(\mathfrak{B}_n, \mathsf{L}_{\mathcal{I}(\infty),\tau^\infty}^{\beta^\infty} \circ \underrightarrow{\alpha_\infty^n}\right)$.

As the pivot of γ is the unit, the height of γ is null. We are left to compute the reach of γ.

Let $a \in \mathfrak{sa}(\mathfrak{B}_n)$. We proceed with three cases.

Case 1. *Assume that $a \in \mathbb{R}1_{\mathfrak{B}_n}$.*

We then have that $\mathsf{l}_n^N(\infty,a) = \mathsf{l}_n^N(m,a) = 0$, and that $\|a - a\|_{\mathfrak{B}_n} = 0$.

Case 2. *Assume that $a \in \mathfrak{S}$.*

We note again that $\mathsf{l}_n^N(\infty, a) \geqslant m_S > 0$. Thus, we may define $a' = \frac{\mathsf{l}_n^N(\infty,a)}{\mathsf{l}_n^N(m,a)} a$. By Inequality (4.5.4), we have:

$$\left\|a - a'\right\|_{\mathfrak{B}_n} = \left\|a - \frac{\mathsf{l}_n^N(\infty, a)}{\mathsf{l}_n^N(m, a)} a\right\|_{\mathfrak{B}_n} \leqslant \varepsilon m_S \leqslant \varepsilon \mathsf{l}_n^N(\infty, a),$$

while $\mathsf{l}_n^N(m, a') = \mathsf{l}_n^N\left(m, \frac{\mathsf{l}_n^N(\infty,a)}{\mathsf{l}_n^N(m,a)} a\right) = \mathsf{l}_n^N(\infty, a)$.

Case 3. *Assume that $a \in \mathfrak{sa}(\mathfrak{B}_n)$.*

By definition of $\mathfrak{S}$ there exists $r, t \in \mathbb{R}$ such that $a = rb + t1_{\mathfrak{B}_n}$ with $b \in \mathfrak{S}$. We may assume $r \neq 0$ since the case $r = 0$ would be Case 1. If $r < 0$, then $-r > 0, -b \in \mathfrak{S}$ and $a = -r(-b) + t1_{\mathfrak{B}_n}$. Hence, we may assume that $r > 0$.

Note that $\mathsf{l}_n^N(\infty, a) = \mathsf{l}_n^N(\infty, rb)$. Let $b' \in \mathfrak{sa}(\mathfrak{B}_n)$ be constructed from $b \in \mathfrak{S}$ as in Case 2. Now, consider $a' = rb' + t1_{\mathfrak{B}_n}$. Thus, by Case 2 and $r > 0$:

$$\begin{aligned}
\left\|a - a'\right\|_{\mathfrak{B}_n} &= \left\|rb + t1_{\mathfrak{B}_n} - \left(rb' + t1_{\mathfrak{B}_n}\right)\right\|_{\mathfrak{B}_n} \\
&= r\left\|b - b'\right\|_{\mathfrak{B}_n} \\
&\leqslant r\varepsilon \mathsf{l}_n^N(\infty, b) \\
&= \varepsilon \mathsf{l}_n^N(\infty, rb) = \varepsilon \mathsf{l}_n^N(\infty, a),
\end{aligned}$$

while $\mathsf{l}_n^N(m, a') = \mathsf{l}_n^N(m, rb') = r\mathsf{l}_n^N(m, b') \leqslant r\mathsf{l}_n^N(\infty, b) = \mathsf{l}_n^N(\infty, rb) = \mathsf{l}_n^N(\infty, a)$ by Case 2, $r > 0$, and since Lip-norms vanish on scalars. Thus, from Case 3:

$$\forall a \in \mathfrak{sa}(\mathfrak{B}_n),\ \exists a' \in \mathfrak{sa}(\mathfrak{B}_n) \text{ with } \|a - a'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n^N(\infty, a),\ \mathsf{l}_n^N(m, a') \leqslant \mathsf{l}_n^N(\infty, a). \tag{4.5.5}$$

By symmetry in the roles of ∞ and m and Inequality (4.5.3), we can conclude as well that:

$$\forall a \in \mathfrak{sa}(\mathfrak{B}_n),\ \exists a' \in \mathfrak{sa}(\mathfrak{B}_n)\text{with}\|a - a'\|_{\mathfrak{B}_n} \leqslant \varepsilon \mathsf{l}_n^N(m, a),\ \mathsf{l}_n^N(\infty, a') \leqslant \mathsf{l}_n^N(m, a). \tag{4.5.6}$$

Now, Expressions (4.5.5) and (4.5.6) together imply that the reach, and hence the length of the bridge γ is no more than ε.

Thus, by definition of length and Theorem-Definition (2.3.16), we gather:

$$\Lambda_{2,0}\left((\mathfrak{B}_n, \mathsf{l}_n^N(\infty, \cdot)), (\mathfrak{B}_n, \mathsf{l}_n^N(m, \cdot))\right) \leqslant \varepsilon$$

for all $m \geqslant M$, which concludes our proof. □

Next, we are now in a position to provide criteria for convergence of AF algebras in quantum propinquity.

Theorem 4.5.6. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$ be an inductive sequence of finite dimensional C^*-algebras with inductive limit $\mathfrak{A}^k$, such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is a unital *-monomorphism for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$.*

If:

1. *$\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$,*

2. *$\{\tau^k : \mathfrak{A}^k \to \mathbb{C}\}_{k\in\overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$, we have that $\left(\tau^k \circ \underrightarrow{\alpha_k^N}\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\tau^\infty \circ \underrightarrow{\alpha_\infty^N}$ in the weak-* topology on $\mathscr{S}(\mathfrak{A}_{\infty,N})$, and*

3. *$\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k\in\overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geqslant c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \ldots, N\}$ and there exists $B : \overline{\mathbb{N}} \to (0,\infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leqslant B(l)$ for all $m, l \in \overline{\mathbb{N}}$*

then, for each $N \in \mathbb{N}$, we have for all $k \geqslant c_N$:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) \leqslant 2B(N) + \mathsf{\Lambda}\left(F_N^N(k), F_N^N(\infty)\right), \quad (4.5.7)$$

where $\mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}$ *is given by Theorem (3.1.3) and* $F_N^N(k)$ *is given by Theorem (4.5.5).*

Furthermore:

$$\lim_{k\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) = 0.$$

Proof. Fix $N \in \mathbb{N}$. Then, for all $k \in \overline{\mathbb{N}}$:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}_{k,N}, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k} \circ \underrightarrow{\alpha_k^N}\right)\right) \leqslant \beta^k(N) \leqslant B(N)$$

by assumption and Theorem (3.1.3). And, by the triangle inequality:

$$\begin{aligned}
&\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right)\\
&\quad \leqslant 2B(N) + \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}_{k,N}, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k} \circ \underrightarrow{\alpha_k^N}\right), \left(\mathfrak{A}_{\infty,N}, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty} \circ \underrightarrow{\alpha_\infty^N}\right)\right)
\end{aligned}$$

Now, assume $k \geqslant c_N$. Then, we have:

$$\mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) \leqslant 2B(N) + \mathsf{\Lambda}_{2,0}\left(F_N^N(k), F_N^N(\infty)\right),$$

and:

$$\limsup_{\substack{k\to\infty\\ k\in\mathbb{N}_{\geqslant c_N}}} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) \leqslant 2B(N),$$

since F_N^N is continuous by Theorem (4.5.5). And, thus:

$$\limsup_{k\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) \leqslant 2B(N). \quad (4.5.8)$$

Hence, as the left hand side of Inequality (4.5.8) does not depend on N, we gather:

$$\limsup_{k\to\infty} \Lambda_{2,0}\left(\left(\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}\right), \left(\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty}\right)\right) \leqslant \lim_{N\to\infty} 2B(N) = 0,$$

which concludes the proof. □

Theorem (4.5.6) provides a satisfying insight to the quantum metric structure of the Lip-norms of Theorem (3.1.3). Indeed, hypotheses *1.*, *2.*, and *3.* of Theorem (4.5.6) are simply appropriate notions of convergence relying on the criteria used to construct the Lip-norms of Theorem (3.1.3) and nothing more.

Another powerful and immediate consequence of Theorem (4.5.6) is that, in the Effros-Shen AF algebra case, since the proof of Theorem (4.2.12) uses sequential continuity and convergence of irrationals in the Baire Space , it is not difficult to see how one may use this Theorem (4.5.6) to achieve the same result and we present a proof of this here in Theorem (4.5.7). For the UHF case Theorem (4.1.7), one could also apply Theorem (4.5.6) to achieve continuity, but although Theorem (4.5.6) does not directly provide the fact that the map in Theorem (4.1.7) is Hölder, one may use Inequality (4.5.7) in the statement of Theorem (4.5.6), to deduce such a result.

We present the Effros-Shen AF Algebra case here as Theorem (4.5.7) to display how one may use the results of this section to prove Theorem (4.2.12) with ease. We note that another substantial application of Theorem (4.5.6) is used in [2] to provide convergence of quotients via convergence of ideals in a suitable setting, which is presented as Theorem (5.2.21) in this dissertation.

Although the following proof of Theorem (4.5.7) cites results from Section (4.2), the results used from Section (4.2) only pertain to the metric structure of the Baire space and the definition of the faithful tracial states on the finite dimensional sub-algebras and are not the convergence results themselves. Now, we display a new proof of Theorem (4.2.12) using the power of Theorem (4.5.6).

Theorem 4.5.7. *Using Notation (2.1.82) and notation from Theorem (3.1.3), the function:*

$$\theta \in ((0,1) \setminus \mathbb{Q}, |\cdot|) \longmapsto \left(\mathfrak{AF}_\theta, \mathsf{L}^{\beta_\theta}_{\mathcal{I}_\theta,\sigma_\theta}\right) \in (\mathcal{QQCMS}_{2,0}, \Lambda_{2,0})$$

is continuous from $(0,1)\setminus\mathbb{Q}$*, with its topology as a subset of* $\mathbb{R}$*, to the class of* $(2,0)$*-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity* Λ*, where* σ_θ *is the unique faithful tracial state of Theorem (4.2.1), and* β_θ *is the sequence of the reciprocal of dimensions of the inductive sequence,* $\mathcal{I}_\theta$*.*

Proof. Let $(\theta^n)_{n\in\overline{N}} \subset (0,1)\setminus\mathbb{Q}$ such that $\lim_{n\to\infty}\theta^n = \theta^\infty$. For each $n \in \overline{\mathbb{N}}$, let $\mathsf{cf}(\theta^n) = [a_j^{\theta^n}]_{j\in\mathbb{N}}$ denote the continued fraction expansion of θ^n. By Proposition (4.2.8), the sequence $(\mathsf{cf}(\theta^n))_{n\in\mathbb{N}}$ converges to $\mathsf{cf}(\theta^\infty)$ in the Baire space metric defined in Definition (4.1.4). By definition of convergence, there exists a non-decreasing sequence $(c_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{n\to\infty} c_n = \infty$, and if $k \geqslant c_N$, then

$$\mathsf{d}_{\mathscr{N}}(\mathsf{cf}(\theta^k), \mathsf{cf}(\theta^\infty)) < \frac{1}{2^N}$$

for each $N \in \mathbb{N}$. By definition of the metric $\mathsf{d}_{\mathscr{N}}$, this implies that for each $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $a_n^{\theta^k} = a_n^{\theta^\infty}$ for all $n \in \{0,\ldots,N\}$ and thus the same holds for $p_n^{\theta^k}$ and $q_n^{\theta^k}$ by Equation (2.1.12). Therefore by Notation (2.1.82) and Definition (4.5.1), the family $\{\mathfrak{AF}_{\theta^n} : n \in \overline{\mathbb{N}}\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$. Therefore, hypothesis *1.* of Theorem (4.5.6) is satisfied.

For hypothesis *2.* of Theorem (4.5.6), fix $N \in \mathbb{N}$ and assume $k \in \mathbb{N}_{\geqslant c_N}$. By Lemma (4.2.3) and Lemma (3.1.12), we only need to show that $(t(\theta^k, N))_{n\in\mathbb{N}} \subset \mathbb{R}$ converges to $t(\theta^\infty, N)$, where $t(\theta, n) = (-1)^{n-1}q_n^\theta(\theta q_{n-1}^\theta - p_{n-1}^\theta)$ for all $\theta \in (0,1)\setminus\mathbb{Q}$ and $n \in \mathbb{N}\setminus\{0\}$. Now, by our fusing sequence $(c_n)_{n\in\mathbb{N}}$, if $k \geqslant c_N$, then:

$$t(\theta^k, N) = (-1)^{N-1}q_N^{\theta^\infty}(\theta^k q_{N-1}^{\theta^\infty} - p_{N-1}^{\theta^\infty}).$$

Therefore, since $\lim_{n\to\infty}\theta^n = \theta^\infty$, we have that $(t(\theta^k, N))_{n\in\mathbb{N}} \subset \mathbb{R}$ converges to $t(\theta^\infty, N)$, which establishes hypothesis *2.* of Theorem (4.5.6).

For hypothesis *3.* of Theorem (4.5.6), consider the continued fraction $\mathsf{cf}(\Phi) = [1]_{j\in\mathbb{N}}$, which is given by $\Phi = 1 - \phi$, where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. By definition of the rational approximations defined by Equation (2.1.12), we have that $q_n^\theta \geqslant q_n^\phi$ for all $\theta \in (0,1)\setminus\mathbb{Q}$ and $n \in \mathbb{N}$. Now, if we define:

$$\beta_\theta(n) = \frac{1}{\dim(\mathfrak{AF}_{\theta,n})} = \frac{1}{(q_n^\theta)^2 + (q_{n-1}^\theta)^2}$$

for all $\theta \in (0,1)\setminus\mathbb{Q}$ and $n \in \mathbb{N}\setminus 0$, then $\beta_\theta(n) \leqslant \beta_\Phi(n)$ for all $\theta \in (0,1)\setminus\mathbb{Q}$ and $n \in \mathbb{N}\setminus\{0\}$. Therefore, the family of sequences $\{\beta_{\theta^n} : n \in \overline{\mathbb{N}}\}$ along with the sequence $B(n) = \beta_\Phi(n)$ for all $n \in \mathbb{N}$ satisfy hypothesis *3.* of Theorem (4.5.6) with the fusing sequence $(c_n)_{n\in\mathbb{N}}$, which completes the proof. $\square$

Chapter 5

Convergence of quotients of AF algebras in the quantum Gromov-Hausdorff propinquity by convergence of ideals

The Fell topology as shown in Section (2.1.1) is a natural topology on the ideal space of a C*-algebra constructed from the Jacobson topology on primitive ideals. Now, a natural map can be created from the ideal space of a C*-algebra to the quotients, which are also C*-algebras by Theorem (2.1.44), and this map has the Fell topology on its domain. Thus, this sparks the question of whether the operation of taking a quotient is continuous. However, this question is not well-defined unless the range of this map comes equipped with a topology. If the quotient C*-algebras are equipped with quantum metric structure, then this codomain may be equipped with the quantum Gromov-Hausdorff propinquity topology, and thus providing a new possible application of Noncommutative Metric Geometry.

The goal of this chapter is to provide criteria for when this map is continuous, while also providing a concrete example of a continuous family of quotients of the Boca-Mundici AF algebra. The first task is to provide a metric for the Fell topology on ideals in the AF case, which is Section (5.1). Next, we look at convergence of quotients with respect to this metric in Section (5.2), which also presents a concrete example of such a convergence. The entire content of this chapter is from the author's work in [2].

5.1 A metric on the ideal space of C*-algebras

For a fixed C*-algebra, the ideal space may be endowed with various natural topologies. We may identify each ideal with a quotient, which is a C*-algebra itself. Now, this defines a function from the ideal space, which has natural topologies, to the class of C*-algebras. But, if each quotient has a quasi-Leibniz Lip-norm, then this function becomes much more intriguing as we may now discuss its continuity or lack thereof since we now have topology on the codomain provided by quantum propinquity. Towards this, we develop a metric topology on ideals of any C*-inductive limit. The purpose of this is to allow fusing families of ideals to provide fusing families of quotients in Proposition (5.1.12) — a first step in providing convergence of quotients in quantum propinquity. But, our metric is greatly motivated by the Fell topology (seet Definition (2.1.58)) on the ideal space and is stronger than the Fell topology.

Now, the Fell topology induces a topology on $\mathrm{Prim}(\mathfrak{A})$ via its relative topology. But, the set $\mathrm{Prim}(\mathfrak{A})$ can also be equipped with the Jacobson topology (see Definition (2.1.52)). Thus, a comparison of both topologies is in order in Proposition (5.1.1), which can be proven using Lemma (2.1.59).

Proposition 5.1.1. *The relative topology induced by the Fell topology of Definition (2.1.58) on* $\text{Prim}(\mathfrak{A})$ *contains the Jacobson topology of Definition (2.1.52) on* $\text{Prim}(\mathfrak{A})$.

Proof. Let $F \subseteq \text{Prim}(\mathfrak{A})$ be closed in the Jacobson topology. Then, there exists a unique $I_F \in \text{Ideal}(\mathfrak{A})$ such that $F = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ by Definition (2.1.58).

Let $J \in \text{Prim}(\mathfrak{A})$ such that there exists a convergent net $(J^\mu)_{\mu\in\Delta} \subseteq F$ that converges to $J \in \text{Prim}(\mathfrak{A})$ in the Fell topology. Let $x \in I_F$, then $x \in J^\mu$ for all $\mu \in \Delta$. Thus, the net $\left(\|x + J^\mu\|_{\mathfrak{A}/J^\mu}\right)_{\mu\in\Delta} = (0)_{\mu\in\Delta}$, which is a net that converges to $\|x + J\|_{\mathfrak{A}/J}$ by Lemma (2.1.59). Thus, the limit $\|x + J\|_{\mathfrak{A}/J} = 0$, which implies that $x \in J$. Hence, $J \supseteq I_F$ and since $J \in \text{Prim}(\mathfrak{A})$, we have $J \in F$.

Thus, F is closed in the relative topology on $\text{Prim}(\mathfrak{A})$ induced by the Fell topology, which verifies the containment of the topologies. □

The next two Lemmas concern the question of how the Jacobson and Fell topologies behave with respect to *-isomorphic C*-algebras. This is more or less evident by construction, but we still present it here to familiarize ourselves with these structures. First, we discuss the Jacobson topology.

Lemma 5.1.2. *If* $\mathfrak{A}, \mathfrak{B}$ *are C*-algebras that are *-isomorphic, then using notation from Definition (2.1.52), the topological spaces* $(\text{Prim}(\mathfrak{A}), \boldsymbol{Jacobson})$ *and* $(\text{Prim}(\mathfrak{B}), \boldsymbol{Jacobson})$ *are homeomorphic.*

In particular, if $\alpha : \mathfrak{A} \to \mathfrak{B}$ *is a *-isomorphism, then:*

$$\alpha_i : I \in \text{Prim}(\mathfrak{A}) \longmapsto \alpha(I) \in \text{Prim}(\mathfrak{B})$$

is well-defined and a homeomorphism from $(\text{Prim}(\mathfrak{A}), \boldsymbol{Jacobson})$ *to* $(\text{Prim}(\mathfrak{B}), \boldsymbol{Jacobson})$.

Proof. Let $\alpha : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a *-isomorphism. We begin by establishing that α_i is well-defined. Let $I \in \text{Prim}(\mathfrak{A})$. By Definition (2.1.52), there exists a non-zero irreducible *-representation $\pi_I : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ such that $\ker \pi_I = I$, where $\mathfrak{B}(\mathcal{H})$ denotes the C*-algebra of bounded operators on some Hilbert space, $\mathcal{H}$. But, the composition $\pi_I \circ \alpha^{-1} : \mathfrak{B} \longrightarrow \mathfrak{B}(\mathcal{H})$ is a non-zero irreducible *-representation on $\mathfrak{B}$ since α^{-1} is a *-isomorphism and π_I is a non-zero irreducible *-representation. We show that the kernel of $\pi_I \circ \alpha^{-1}$ is $\alpha(I)$.

Consider $\alpha(I) \subseteq \mathfrak{A}$. The set $\alpha(I) \in \mathrm{Ideal}(\mathfrak{A})$ since α is a *-isomorphism. However:

$$\begin{aligned} a \in \alpha(I) &\iff \alpha(a)^{-1} \in I \\ &\iff \alpha(a)^{-1} \in \ker \pi_I \\ &\iff a \in \ker \pi_I \circ \alpha^{-1}, \end{aligned}$$

and thus, the ideal $\alpha(I) = \ker \pi_I \circ \alpha^{-1} \in \mathrm{Prim}(\mathfrak{A})$ by Definition (2.1.52).

Therefore, the following map is well-defined:

$$\alpha_i : I \in \mathrm{Prim}(\mathfrak{A}) \longmapsto \alpha(I) \in \mathrm{Prim}(\mathfrak{B}),$$

and is injective since α is a *-isomorphism. For surjectivity, let $I \in \mathrm{Prim}(\mathfrak{B})$. The fact that $\alpha^{-1}(I) \in \mathrm{Prim}(\mathfrak{A})$ follows the same argument for proving that α_i is well-defined. Also, the image $\alpha_i(\alpha^{-1}(I)) = \alpha(\alpha^{-1}(I)) = I$ since α is a bijection. Hence, the map α_i is a well-defined bijection.

Now, we establish continuity. Let $F \subseteq \mathrm{Prim}(\mathfrak{B})$ be closed. By Definition (2.1.52), there exists an $I_F \in \mathrm{Ideal}(\mathfrak{B})$ such that $F = \{J \in \mathrm{Prim}(\mathfrak{B}) : J \supseteq I_F\}$. Consider $\alpha_i^{-1}(F) = \{I \in \mathrm{Prim}(\mathfrak{A}) : \alpha_i(I) \in F\}$. Assume that $I \in \alpha_i^{-1}(F)$. Then, we have that $\alpha_i(I) \in \mathrm{Prim}(\mathfrak{A})$ by well-defined, and moreover:

$$\alpha(I) \supseteq I_F = \alpha(\alpha^{-1}(I_F)) \implies I \supseteq \alpha^{-1}(I_F) \in \mathrm{Ideal}(\mathfrak{A})$$

since α is a bijection and α^{-1} is a *-isomorphism. Next, assume that $I \in \mathrm{Prim}(\mathfrak{A})$ such that $I \supseteq \alpha^{-1}(I_F)$, then $\alpha(I) \supseteq I_F$ since α is a bijection, which implies that $\alpha_i(I) \in F$ and $I \in \alpha_i^{-1}(F)$ by well-defined. Combing the inclusions, the set $\alpha_i^{-1}(F) = \{I \in \mathrm{Prim}(\mathfrak{A}) : I \supseteq \alpha^{-1}(I_F)\}$, which is closed by Definition (2.1.52). The continuity argument for α_i^{-1} follows similarly, which completes the proof. $\square$

Let's continue by proving that the Fell topology also satisfies the conclusions of Lemma (5.1.2), which will prove useful later in Corollary (5.1.24) by showing that the metric topology we develop is preserved homeomorphically by *-isomorphisms in the case of AF algebras.

Lemma 5.1.3. *If $\mathfrak{A}, \mathfrak{B}$ are C*-algebras that are *-isomorphic, then using notation from Definition (2.1.58), the topological spaces* $(\mathrm{Ideal}(\mathfrak{A}), \boldsymbol{Fell})$ *and* $(\mathrm{Ideal}(\mathfrak{B}), \boldsymbol{Fell})$ *are homeomorphic.*

*In particular, if $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a *-isomorphism, then:*

$$\alpha_i : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto \alpha(I) \in \mathrm{Ideal}(\mathfrak{B})$$

is well-defined and a homeomorphism from $(\mathrm{Prim}(\mathfrak{A}), \boldsymbol{Fell})$ *to* $(\mathrm{Prim}(\mathfrak{B}), \boldsymbol{Fell})$.

Proof. Let $\alpha : \mathfrak{A} \to \mathfrak{B}$ be a *-isomorphism, then the map $\alpha_i : I \in \mathrm{Ideal}(\mathfrak{A}) \mapsto \alpha(I) \in \mathrm{Ideal}(\mathfrak{B})$ is a well-defined bijection. Assume that $\left(I_{\mathfrak{A}}^{\mu}\right)_{\mu \in \Delta} \subset \mathrm{Ideal}(\mathfrak{A})$ is a net that converges with respect to the Fell topology to $I_{\mathfrak{A}} \in \mathrm{Ideal}(\mathfrak{A})$. We show that $\left(\alpha_i\left(I_{\mathfrak{A}}^{\mu}\right)\right)_{\mu \in \Delta} \subset \mathrm{Ideal}(\mathfrak{B})$ converges with respect to the Fell topology to $\alpha_i\left(I_{\mathfrak{A}}\right) \in \mathrm{Ideal}(\mathfrak{B})$. Let $b \in \mathfrak{B}$, then $\alpha^{-1}(b) \in \mathfrak{A}$. Thus, by Lemma (2.1.59), we have $\left(\left\|\alpha^{-1}(b) + I_{\mathfrak{A}}^{\mu}\right\|_{\mathfrak{A}/I_{\mathfrak{A}}^{\mu}}\right)_{\mu \in \Delta}$ converges to $\left\|\alpha^{-1}(b) + I_{\mathfrak{A}}\right\|_{\mathfrak{A}/I_{\mathfrak{A}}}$. But, fix $\mu \in \Delta$, then since α is a *-isomorphism:

$$\begin{aligned}\left\|\alpha^{-1}(b) + I_{\mathfrak{A}}^{\mu}\right\|_{\mathfrak{A}/I_{\mathfrak{A}}^{\mu}} &= \inf\left\{\left\|\alpha^{-1}(b) + a\right\|_{\mathfrak{A}} : a \in I_{\mathfrak{A}}^{\mu}\right\} \\ &= \inf\left\{\|b + \alpha(a)\|_{\mathfrak{B}} : a \in I_{\mathfrak{A}}^{\mu}\right\}\end{aligned}$$

$$= \inf\left\{\left\|b+b'\right\|_{\mathfrak{B}} : b' \in \alpha(I^{\mu}_{\mathfrak{A}})\right\}$$
$$= \left\|b + \alpha_i\left(I^{\mu}_{\mathfrak{A}}\right)\right\|_{\mathfrak{B}/\alpha_i\left(I^{\mu}_{\mathfrak{A}}\right)},$$

and similarly, the limit $\left\|\alpha^{-1}(b) + I_{\mathfrak{A}}\right\|_{\mathfrak{A}/I_{\mathfrak{A}}} = \|b + \alpha_i\left(I_{\mathfrak{A}}\right)\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}})}$.

Hence, the net $\left(\left\|b + \alpha_i\left(I^{\mu}_{\mathfrak{A}}\right)\right\|_{\mathfrak{B}/\alpha_i\left(I^{\mu}_{\mathfrak{A}}\right)}\right)_{\mu\in\Delta}$ converges to $\|b + \alpha_i\left(I_{\mathfrak{A}}\right)\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}})}$. Therefore, since $b \in \mathfrak{B}$ was arbitrary, the net $\left(\alpha_i\left(I^{\mu}_{\mathfrak{A}}\right)\right)_{\mu\in\Delta} \subset \mathrm{Ideal}(\mathfrak{B})$ converges with respect to the Fell topology to $\alpha_i\left(I_{\mathfrak{A}}\right) \in \mathrm{Ideal}(\mathfrak{B})$ by Lemma (2.1.59). Thus, α_i is continuous, and since both topologies are compact Hausdorff, the proof is complete. ☐

As stated earlier, it is with the Fell topology for which we will provide a notion of convergence of quotients from ideals of AF algebras. But, it seems that a metric notion is in order to move from fusing family of ideals to a fusing family of quotients as we will see in Proposition (5.1.12).

Next, we develop a metric on the ideal space on any inductive limit in the sense of Definition (2.1.64), and the following Proposition (5.1.5) is key for defining our metric. But, first, a remark on our change in the language of inductive limits for some of the following results.

Remark 5.1.4. *By [55, Chapter 6.1], if $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ is an inductive sequence with inductive limit $\mathfrak{A} = \varinjlim \mathcal{I}$ as in Definition (2.1.64), then $(\underrightarrow{\alpha}^{n}(\mathfrak{A}_n))_{n\in\mathbb{N}}$ is a non-decreasing sequence of C^*-subalgebras of $\mathfrak{A}$, in which $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\underrightarrow{\alpha}^{n}(\mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$ by Proposition (2.1.66). Thus, in some of the following definitions and results, when we say, "Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$," we are also including the case of inductive limits. The purpose of this will be to avoid notational confusion later on if we were to work with multiple inductive limits (see for example Proposition (5.1.12)), and the purpose of this remark is to note that this does not weaken our results.*

Proposition 5.1.5. *Let $\mathfrak{A}$ be a C*-algebra with a non-decreasing sequence of C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. The map:*

$$i(\cdot,\mathcal{U}) : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto (I \cap \mathfrak{A}_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$$

is a well-defined injection.

Proof. Since $I \in \mathrm{Ideal}(\mathfrak{A})$ and $\mathfrak{A}_n$ is a C*-subalgebra for all $n \in \mathbb{N}$, we have that $I \cap \mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$. Thus, the map $i(\cdot,\mathcal{U})$ is well-defined.

Next, for injectivity, assume that $I, J \in \mathrm{Ideal}(\mathfrak{A})$ such that $i(I,\mathcal{U}) = i(J,\mathcal{U})$. Hence, the sets $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for all $n \in \mathbb{N}$, which implies that $\cup_{n\in\mathbb{N}}(I \cap \mathfrak{A}_n) = \cup_{n\in\mathbb{N}}(J \cap \mathfrak{A}_n)$. Therefore, the closures $\overline{\cup_{n\in\mathbb{N}}(I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\cup_{n\in\mathbb{N}}(J \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$. But, by Proposition (2.1.69), we conclude $I = J$. □

With this injection, we may define a metric.

Definition 5.1.6. *Let $\mathfrak{A}$ be a C*-algebra with a non-decreasing sequence of C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. We define a map from* $\mathrm{Ideal}(\mathfrak{A}) \times \mathrm{Ideal}(\mathfrak{A})$ *to* $[0, 1]$ *such that for all* $I, J \in \mathrm{Ideal}(\mathfrak{A})$*:*

$$\mathsf{m}_{i(\mathcal{U})}(I, J) = \begin{cases} 0 & \text{if } \forall n \in \mathbb{N}, I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n \\ 2^{-n} & \text{otherwise, where } n = \min\{m \in \mathbb{N} : I \cap \mathfrak{A}_m \neq J \cap \mathfrak{A}_m\} \end{cases}$$

Proposition 5.1.7. *If $\mathfrak{A}$ is a C*-algebra with a non-decreasing sequence of C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then:*

$$\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$$

is a zero-dimensional ultrametric space, where $\mathsf{m}_{i(\mathcal{U})}$ is given by Definition (5.1.6).

Proof. Consider the metric on $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$ defined by:

$$\mathsf{m}\left((I_n)_{n\in\mathbb{N}}, (J_n)_{n\in\mathbb{N}}\right) = \begin{cases} 0 & \text{if } \forall n \in \mathbb{N}, I_n = J_n \\ 2^{-n} & \text{otherwise, where } n = \min\{m \in \mathbb{N} : I_m \neq J_m\} \end{cases}.$$

Thus, $\left(\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n), \mathsf{m}\right)$ is a zero-dimensional metric space since it metrizes the product topology on $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$, in which $\mathrm{Ideal}(\mathfrak{A}_n)$ is given the discrete topology for all $n \in \mathbb{N}$. But, the identification $\mathsf{m}_{i(\mathcal{U})} = \mathsf{m} \circ (i(\cdot,\mathcal{U}) \times i(\cdot,\mathcal{U}))$ implies that $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ is a zero-dimensional metric space since $i(\cdot,\mathcal{U})$ is injective by Proposition (5.1.5). $\square$

Remark 5.1.8. *If $\mathfrak{A}$ is any C^*-algebra, then, $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, where $\mathfrak{A}_n = \mathfrak{A}$ for all $n \in \mathbb{N}$. If we set $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$. then, the metric $\mathsf{m}_{i(\mathcal{U})}$ of Proposition (5.1.7) is a metric on the ideal space of any C^*-algebra, but we see in this case that this metric simply metrizes the discrete topology. However, the metric of Proposition (5.1.7) is not always trivial as we shall see in the case of AF algebras (Theorem (5.1.21)), in which the metric spaces will always be compact. In particular, if an AF algebra were to contain at least infinitely many ideals (see Section (5.2.1) for an example of such an AF algebra), then the metric of Proposition (5.1.7) could not be discrete. Furthermore, this implies that the conclusion of Theorem (5.1.13) is not trivial.*

Remark 5.1.9. *The metric of Proposition (5.1.7) can be seen as an explicit presentation of a metric on a metrizable topology on ideals presented in [7], where this metrizable topology is presented only in the case of AF algebras and metrizes the Fell topology in the AF case, which we also prove for the metric of Proposition (5.1.7) via a different approach in Theorem (5.1.21). But, we note that the metric of Proposition (5.1.7) is more general as it exists on the ideal space of any C^*-inductive limit — and on any C^*-algebra by Remark (5.1.8)—, and in the AF case (Section (5.1.1)), we define a metric entirely in the graph setting of a Bratteli diagram on the* *space of directed and hereditary subsets of the diagram (Theorem (5.1.21)), which in turn is isometric to the metric of Proposition (5.1.7). This allows us to explicitly calculate distances between ideals in Remark (5.2.13), and therefore, make interesting comparisons with certain classical metrics on irrationals. And, in Proposition (5.1.12), the metric of Proposition (5.1.7) will explcitly provide fusing families of quotients.*

Before we move to fusing families of quotients, we show that being a fusing family of ideals is equivalent to convergence in the metric on ideals of Proposition (5.1.7).

Lemma 5.1.10. *Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, and let $\left(I^k\right)_{k\in\overline{\mathbb{N}}} \in \mathrm{Ideal}(\mathfrak{A})$.*

Then, using notation of Proposition (5.1.7), the sequence $\left(I^k\right)_{k\in\mathbb{N}}$ converges to I^∞ with respect to the metric $\mathsf{m}_{i(\mathcal{U})}$ if and only if the family $\left\{I^k = \overline{\cup_{n\in\mathbb{N}}I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ is a fusing family of Definition (4.5.1).

Proof. We begin with the forward direction. Assume that $\left(I^k\right)_{k\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$. Thus, we have $\lim_{k\to\infty} \mathsf{m}_{i(\mathcal{U})}\left(I^k, I^\infty\right) = 0$. From this, construct an increasing sequence $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}\setminus\{0\}$ such that:

$$\mathsf{m}_{i(\mathcal{U})}\left(I^k, I^\infty\right) \leqslant 2^{-(n+1)}$$

for all $k \geqslant c_n$. In particular, fix $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ for all $n \in \{0, \ldots, N\}$, which implies that $\left\{I^k = \overline{\cup_{n\in\mathbb{N}}I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$ by Definition (4.5.1).

For the other direction, assume that $\left\{I^k = \overline{\cup_{n\in\mathbb{N}}I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$. Therefore, for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ for all $n \in \{0, \ldots, N\}$. Hence, let $\varepsilon > 0$. There exists $N \in \mathbb{N}$

such that $2^{-N} < \varepsilon$. If $k \geqslant c_N \in \mathbb{N}$, then

$$\mathsf{m}_{i(\mathcal{U})}\left(I^k, I^\infty\right) \leqslant 2^{-(N+1)} < 2^{-N} < \varepsilon,$$

which completes the proof. $\square$

In the context of this paper, the main motivation for the metric of Proposition (5.1.7) is to provide a fusing family of quotients via convergence of ideals. First, for a fixed ideal of an inductive limit of the form $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, we provide an inductive limit in the sense of Definition (2.1.64) that is *-isomorphic to the quotient. The reason for this is that given $I \in \mathrm{Ideal}(\mathfrak{A})$, then $\mathfrak{A}/I$ has a canonical closure of union form as $\mathfrak{A}/I = \overline{\cup_{n\in\mathbb{N}}((\mathfrak{A}_n + I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$ (see Proposition (5.1.12)), but if two ideals satisfy $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for some $n \in \mathbb{N}$, then even though this provides that $(\mathfrak{A}_n + I)/I$ is *-isomorphic to $(\mathfrak{A}_n + J)/J$ as they are both *-isomorphic to $\mathfrak{A}_n/(I\cap\mathfrak{A}_n)$ (see Proposition (5.1.12)) , the two algebras $(\mathfrak{A}_n+J)/J$ and $(\mathfrak{A}_n+I)/I$ are not equal in any way if $I \neq J$, yet, equality is a requirement for fusing families (see Definition (4.5.1)). Thus, Notation (5.1.11) will allow us to present, up to *-isomorphism, quotients as IL-fusing families as we will see in Proposition (5.1.12) from convergence of ideals in the metric of Proposition (5.1.7). Note that the next Proposition (5.1.12) is in the case of AF algebras.

Notation 5.1.11. *Let $\mathfrak{A}$ be a C*-algebra with a non-decreasing sequence of C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $I \in \mathrm{Ideal}(\mathfrak{A})$. For $n \in \mathbb{N}$:*

$$\gamma_{I,n} : a + I\cap\mathfrak{A}_n \in \mathfrak{A}_n/(I\cap\mathfrak{A}_n) \longmapsto a + (I\cap\mathfrak{A}_{n+1}) \in \mathfrak{A}_{n+1}/(I\cap\mathfrak{A}_{n+1}),$$

is a-monomorphism by the same argument of Claim (5.1.14) and $\mathcal{U}$ is non-decreasing. Let $\mathcal{I}(\mathfrak{A}/I) = (\mathfrak{A}_n/(I\cap\mathfrak{A}_n), \gamma_{I,n})_{n\in\mathbb{N}}$, and denote the C*-inductive limit by $\varinjlim \mathcal{I}(\mathfrak{A}/I)$. Let $\mathfrak{B} \subseteq \mathfrak{A}$ be a C*-subalgebra and $I \in \mathrm{Ideal}(\mathfrak{A})$. Let $\mathfrak{B}+I = \overline{\{b+c : b \in \mathfrak{B}, c \in I\}}^{\|\cdot\|_{\mathfrak{A}}}$.*

Proposition 5.1.12. *Let $\mathfrak{A}$ be a C*-algebra with a non-decreasing sequence of finite-dimensional C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Using Notation (5.1.11), if $I \in \mathrm{Ideal}(\mathfrak{A})$, then there exists a *-isomorphism $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$ such that for all $n \in \mathbb{N}$ the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{A}_n/(I\cap\mathfrak{A}_n) \arrow[r, "\overset{\gamma_I^n}{\to}"] \arrow[dr, "\phi_I^n"'] & \varinjlim \mathcal{I}(\mathfrak{A}/I) \arrow[d, "\phi_I"] \\
 & \mathfrak{A}/I
\end{tikzcd},$$

*where for all $n \in \mathbb{N}$, the maps $\phi_I^n : a+(I\cap\mathfrak{A}_n) \in \mathfrak{A}_n/(I\cap\mathfrak{A}_n) \longmapsto a+I \in (\mathfrak{A}_n+I)/I \subseteq \mathfrak{A}/I$ are *-monomorphisms onto $(\mathfrak{A}_n + I)/I$, in which $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ is a C*-subalgebra of $\mathfrak{A}$ containing I as an ideal and $\cup_{n\in\mathbb{N}}((\mathfrak{A}_n + I)/I)$ is a dense *-subalgebra of $\mathfrak{A}/I$ with $((\mathfrak{A}_n + I)/I)_{n\in\mathbb{N}}$ non-decreasing.*

Furthermore, if $(I^k)_{k\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ of Proposition (5.1.7), then using Definition (4.5.1), we have $\left\{I^k = \overline{\cup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ is a fusing family with respect to some fusing sequence $(c_n)_{n\in\mathbb{N}}$ such that $\left\{\varinjlim \mathcal{I}\left(\mathfrak{A}/I^k\right) : k \in \overline{\mathbb{N}}\right\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$.

Proof. Let $I \in \mathrm{Ideal}(\mathfrak{A})$. Fix $n \in \mathbb{N}$. Note that $\mathfrak{A}_n + I$ is a C*-subalgebra of $\mathfrak{A}$ since $I \in \mathrm{Ideal}(\mathfrak{A})$, and furthermore $I \in \mathrm{Ideal}(\mathfrak{A}_n + I)$. Now, we have $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ since $\mathfrak{A}_n$ and I are both closed in $\mathfrak{A}$ and $\mathfrak{A}_n$ is finite dimensional. Next, we have ϕ_I^n is an injective *-homomorphism by Claim (5.1.14). If $a \in \mathfrak{A}_n$, then $\phi_I^n(a + \mathfrak{A}_n/(I \cap \mathfrak{A}_n)) = a + I$ and the composition $\phi_I^{n+1}(\gamma_{I,n}(a + (I \cap \mathfrak{A}_n))) = \phi_I^{n+1}(a+(I\cap\mathfrak{A}_{n+1})) = a+I$. Hence, for all $n \in \mathbb{N}$, the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A}_n/(I\cap\mathfrak{A}_n) \arrow[r, "\gamma_{I,n}"] \arrow[dr, "\phi_I^n"'] & \mathfrak{A}_{n+1}/(I\cap\mathfrak{A}_{n+1}) \arrow[d, "\phi_I^{n+1}"] \\
 & \mathfrak{A}/I
\end{tikzcd}.$$

Hence, by Theorem (2.1.67), there exists a unique *-monomorphism $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$ such that for all $n \in \mathbb{N}$ the diagram in the statement of this theorem commutes. Furthermore, ϕ_I is an isometry by Proposition (2.1.11).

Next, fix $n \in \mathbb{N}$. Let $x \in (\mathfrak{A}_n + I)/I$, and so $x = a + b + I$, where $a \in \mathfrak{A}_n, b \in I$. Thus, we have $a + b - a = b \in I \implies x - (a + I) = 0 + I \implies x = a + I$. But, then, the image $\phi_I^n(a + (I \cap \mathfrak{A}_n)) = x$. Hence, the map ϕ_I^n is onto $(\mathfrak{A}_n + I)/I$. We thus have:

$$\phi_I \left(\cup_{n\in\mathbb{N}} \underrightarrow{\gamma_I^n}(\mathfrak{A}_n/(I \cap \mathfrak{A}_n)) \right) = \cup_{n\in\mathbb{N}} \left((\mathfrak{A}_n + I)/I \right),$$

in which the right-hand side is a dense *-subalgebra of $\mathfrak{A}/I$ by continuity of the quotient map and the assumption that $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ is dense in $\mathfrak{A}$. Hence, since $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ is complete and ϕ_I is a linear isometry on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$, we have ϕ_I surjects onto $\mathfrak{A}/I$. Thus, the function $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$ is a *-isomorphism.

Next, assume that $\left(I^k\right)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$. By Lemma (5.1.10), the family $\left\{ I^k = \overline{\cup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$ is a fusing family with fusing sequence $(b_n)_{n\in\mathbb{N}}$ by Definition (4.5.1).

Let $c_n = b_{n+1}$ for all $n \in \mathbb{N}$. Then, the sequence $(c_n)_{n\in\mathbb{N}}$ is a fusing sequence for $\left\{ I^k = \overline{\cup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$. Fix $N \in \mathbb{N}, n \in \{0, \ldots, N\}$, and $k \in \mathbb{N}_{\geqslant c_N}$. Then, the equality $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ implies that $\mathfrak{A}_n/(I^k \cap \mathfrak{A}_n) = \mathfrak{A}_n/(I^\infty \cap \mathfrak{A}_n)$. But, also, we gather $\gamma_{I^k,n} = \gamma_{I^\infty,n}$ since $\mathfrak{A}_{n+1}/(I^k \cap \mathfrak{A}_{n+1}) = \mathfrak{A}_{n+1}/(I^\infty \cap \mathfrak{A}_{n+1})$ as $c_n = b_{n+1}$. Hence, the familiy of inductive limits $\left\{ \varinjlim \mathcal{I}\left(\mathfrak{A}/I^k\right) : k \in \overline{\mathbb{N}} \right\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$. □

For the ideal space, Proposition (5.1.7) provides a zero-dimensional Hausdorff space metrized by an ultrametric. We will see that if the sequence of C*-subalgebras $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ are all assumed to be finite dimensional (or if $\mathfrak{A}$ is AF), then the metric space of Proposition (5.1.7) will be compact in Theorem (5.1.21). But, we will approach this by first providing a compact metric on the directed hereditary subsets of a Bratteli diagram in Proposition (5.1.18), and then translating this metric back to the setting of Proposition (5.1.7), which will provide compactness with ease. This provides another in the line of many applications of the novel Bratteli diagram. But, before we continue in this path, we see that in the very least, the metric of Proposition (5.1.7) can be utilized as a tool to provide convergence in the Fell topology as the metric topology is stronger. This is the content of following Theorem (5.1.13). Later on, this will show in the AF algebra case that the Fell and metric topologies agree by maximal compactness in Theorem (5.1.21).

Theorem 5.1.13. *If $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ is a C^*-algebra in which $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of C^*-subalgebras of $\mathfrak{A}$, then on* $\mathrm{Ideal}(\mathfrak{A})$*, the Fell topology is contained in the metric topology of $\mathsf{m}_{i(\mathcal{U})}$.*

Proof. First, we prove the following claim to provide norm calculations.

Claim 5.1.14. *Let $J \in \mathrm{Ideal}(\mathfrak{A})$. For each $k \in \mathbb{N}$, the map:*

$$\phi_J^k : a + (J \cap \mathfrak{A}_k) \in \mathfrak{A}_k/(J \cap \mathfrak{A}_k) \longmapsto a + J \in \mathfrak{A}/J. \qquad (5.1.1)$$

*is a *-monomorphism.*

Proof of claim. Assume that $a, b \in \mathfrak{A}_k$ such that $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which implies that $a - b \in J \cap \mathfrak{A}_k \subseteq J \implies a + J = b + J$, and thus, ϕ_J^k is well-defined. Next, assume that $a, b \in \mathfrak{A}_k$ such that $a + J = b + J$, which implies that $a - b \in J$. But, we have $a - b \in \mathfrak{A}_k \implies a - b \in J \cap \mathfrak{A}_k$ and $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which provides injectivity. Thus, for each $k \in \mathbb{N}$, we have ϕ_J^k is a well-defined injective *-homomorphism since J is an ideal. □

Let $F \subseteq \mathrm{Ideal}(\mathfrak{A})$ be closed with respect to ***Fell***. We show that F is closed with respect to the metric topology of $\mathsf{m}_{i(\mathcal{U})}$. Since the topology of $\mathsf{m}_{i(\mathcal{U})}$ is metric, we may use sequences. Thus, let $\left(I^l\right)_{l\in\mathbb{N}} \subseteq F$ and $I \in \mathrm{Ideal}(\mathfrak{A})$ such that

$\lim_{l\to\infty} \mathsf{m}_{i(\mathcal{U})}\left(I^l, I\right) = 0$. Now, we claim that this sequence converges with respect to the Fell topology, and thus, we will approach by Lemma (2.1.59).

Let $\varepsilon > 0, a \in \mathfrak{A}$. By density of $\cup_{n\in\mathbb{N}}\mathfrak{A}_n$ in $\mathfrak{A}$, there exists $N \in \mathbb{N}$ such that $a_N \in \mathfrak{A}_N$ and $\|a - a_N\|_{\mathfrak{A}} < \varepsilon/2$. By convergence in $\mathsf{m}_{i(\mathcal{U})}$, there exists $k_N \in \mathbb{N}$ such that $I^l \cap \mathfrak{A}_N = I \cap \mathfrak{A}_N$ for all $l \geqslant k_N$. Furthermore, since $\phi^N_{I^l}$ is an isometry by Claim (5.1.14) and Proposition (2.1.11), we have that $\left\|a_N + I^l \cap \mathfrak{A}_N\right\|_{\mathfrak{A}_N/(I^l\cap\mathfrak{A}_N)} = \left\|a_N + I^l\right\|_{\mathfrak{A}/I^l}$ for all $l \geqslant k_N$. But, we have:

$$\left\|a_N + I^l \cap \mathfrak{A}_N\right\|_{\mathfrak{A}_N/(I^l\cap\mathfrak{A}_N)} = \|a_N + I \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(I\cap\mathfrak{A}_N)} = \|a_N + I\|_{\mathfrak{A}/I}$$

for all $l \geqslant k_N$ since $I^l \cap \mathfrak{A}_N = I \cap \mathfrak{A}_N$. Therefore, for $l \geqslant k_N$, we conclude:

$$\left\|a_N + I^l\right\|_{\mathfrak{A}/I^l} = \|a_N + I\|_{\mathfrak{A}/I}\,. \tag{5.1.2}$$

Now, let $l \geqslant k_N$, then by Expression (5.1.2) and the fact that any quotient norm of $\mathfrak{A}$ with respect to $\|\cdot\|_{\mathfrak{A}}$ is bounded above by $\|\cdot\|_{\mathfrak{A}}$, we gather:

$$\begin{aligned}
\left|\left\|a + I^l\right\|_{\mathfrak{A}/I^l} - \|a + I\|_{\mathfrak{A}/I}\right| &\leqslant \left|\left\|a + I^l\right\|_{\mathfrak{A}/I^l} - \left\|a_N + I^l\right\|_{\mathfrak{A}/I^l}\right| \\
&\quad + \left|\left\|a_N + I^l\right\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I}\right| \\
&\quad + \left|\|a_N + I\|_{\mathfrak{A}/I} - \|a + I\|_{\mathfrak{A}/I}\right| \\
&\leqslant \left|\left\|a_N + I^l\right\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I}\right| \\
&\quad + \left\|a - a_N + I^l\right\|_{\mathfrak{A}/I^l} + \|a - a_N + I\|_{\mathfrak{A}/I} \\
&\leqslant 2\|a - a_N\|_{\mathfrak{A}} + \left|\left\|a_N + I^l\right\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I}\right| \\
&< \varepsilon + 0
\end{aligned}$$

Therefore, we may conclude $\lim_{l\to\infty}\left\|a + I^l\right\|_{\mathfrak{A}/I^l} = \|a + I\|_{\mathfrak{A}/I}$, and by Lemma

(2.1.59), since $a \in \mathfrak{A}$ was arbitrary, the net $\left(I^l\right)_{l\in\mathbb{N}}$ converges with respect to the Fell topology to I. But, as F is closed in ***Fell***, we have that $I \in F$. Thus, F is closed with respect to $\mathsf{m}_{i(\mathcal{U})}$. This completes the containment argument. $\square$

5.1.1 Metric on Ideal Space of C*-Inductive Limits: AF case

In this section, the ultrametric of Proposition (5.1.7) is greatly strengthened in the AF case. For instance, its induced topology will be compact. The notion of a Bratteli diagram will prove quite useful in providing these advantages. Thus, for the moment, we introduce a new metric based entirely on the diagram structure. And, we will see in Theorem (5.1.21) that, when AF algebras are reintroduced, the inductive limit metric and diagram metrics are isometric and form a topology that equals the Fell topology on ideals. We begining by defining what an ideal of a Bratteli diagram diagram is, where Bratteli diagram was defined in Definition (2.1.83).

Definition 5.1.15. *Let $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ be a Bratteli diagram as defined in Definition (2.1.83). We call $\mathcal{D}(I) = (V^I, E^I)$ an* ideal diagram *of $\mathcal{D}$ if $V^I \subseteq V^{\mathcal{D}}, E^I \subseteq E^{\mathcal{D}}$ and:*

(i) (directed) if $(n,k) \in V^I$ and $((n,k),(n+1,q)) \in E^{\mathcal{D}}$, then $(n+1,q) \in V^I$.

(ii) (hereditary) if $(n,k) \in V^{\mathcal{D}}$ and $R^{\mathcal{D}}_{(n,k)} \subseteq V^I$, then $(n,k) \in V^I$.

(iii) (edges) If $(n,k),(n+1,q) \in V^I$ such that $((n,k),(n+1,q)) \in E^{\mathcal{D}}$, then $((n,k),(n+1,q)) \in E^I$.

Furthermore, if $(n,k) \in V^{\mathcal{D}} \cap V^I$, then $[n,k]_{\mathcal{D}} = [n,k]_{\mathcal{D}(I)}$. And, if $((n,k),(n+1,q)) \in E^{\mathcal{D}} \cap E^I$, then $[(n,k),(n+1,q)]_{\mathcal{D}} = [(n,k),(n+1,q)]_{\mathcal{D}(I)}$.

Also, for $n \in \mathbb{N}$, denote $V^I_n = V^{\mathcal{D}}_n \cap V^I$ and $E^I_n = E^{\mathcal{D}}_n \cap E^I$ with $I_n = (V^I_n, E^I_n)$ to also include all associated labels and number of edges, and we will refer to V^I_n as the vertices at level n of the diagram. Let $\mathrm{Ideal}(\mathcal{D})$ *denote the set of ideals of $\mathcal{D}$.*

Lemma 5.1.16. *Using Definition (2.1.83), let $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ be a Bratteli diagram. Using Definition (5.1.15), if $I, J \in \mathrm{Ideal}(\mathcal{D})$ such that there exists $n \in \mathbb{N}$ with $V_n^I \neq \emptyset$ and $V_n^I = V_n^J$, then $I_m = J_m$ for all $m \leqslant n$.*

Proof. Assume that $n \in \mathbb{N} \setminus \{0\}$ and $V_n^I = V_n^J \neq \emptyset$. Let $(n-1, k) \in V_{n-1}^I$. By directed, for all $(n, q) \in R_{(n-1,k)}^{\mathcal{D}}$, we have that $(n, q) \in V_n^I = V_n^J$. Therefore, the set $R_{(n-1,k)}^{\mathcal{D}} \subset V_n^J$. Hence, by hereditary, we have that $(n-1, k) \in V_{n-1}^J$. Thus, the set $V_{n-1}^I \subseteq V_{n-1}^J$ and the fact that the argument is symmetric in the other direction implies that $V_{n-1}^I = V_{n-1}^J$. We may continue in this fashion to show that vertices of the ideals agree up to n. By the edge axiom in Definition (5.1.15), we thus have that $E_m^I = E_m^J$ for all $m \leqslant n-1$, but by the directed property, we also have that $E_n^I = E_n^J$. As the labels of vertices and number of edges for both I and J are both inherited from $\mathcal{D}$, our proof is finished. □

We now define a metric on ideals of a Bratteli diagram.

Definition 5.1.17. *Using Defintion (2.1.83), let $\mathcal{D} \in \mathscr{BD}$ be a Bratteli diagram and for each $n \in \mathbb{N}$, let $Z_{v_n^{\mathcal{D}}} = \prod_{k=0}^{v_n^{\mathcal{D}}} \mathbb{Z}_2$.*

Let $\mathcal{C}_{\mathcal{D}} = \prod_{n \in \mathbb{N}} Z_{v_n^{\mathcal{D}}}$. Denote an element in $x \in \mathcal{C}_{\mathcal{D}}$ by $x = (x(0), x(1), \ldots)$, where $x(n) = (x(n)_0, x(n)_1, \ldots, x(n)_{v_n^{\mathcal{D}}}) \in Z_{v_n^{\mathcal{D}}}$ for all $n \in \mathbb{N}$. Define a metric on $\mathcal{C}_{\mathcal{D}}$ by:

$$\mathsf{m}_{\mathcal{C}}(x, y) = \begin{cases} 0 & \text{if } x(n) = y(n), \forall n \in \mathbb{N} \\ 2^{-n} & \text{otherwise, where } n = \min\{m \in \mathbb{N} : x(n) \neq y(n)\}. \end{cases}$$

We note that it is a routine argument that $\mathsf{m}_{\mathcal{C}}$ is a metric. Furthermore, if each $Z_{v_n^{\mathcal{D}}}$ is given the discrete topology and $\mathcal{C}_{\mathcal{D}}$ is given the product topology, then $\mathsf{m}_{\mathcal{C}}$ metrizes this topology. As each $Z_{v_n^{\mathcal{D}}}$ is finite and nonempty, $(\mathcal{C}_{\mathcal{D}}, \mathsf{m}_{\mathcal{C}})$ is a Cantor space, a nonempty perfect zero-dimensional compact ultrametric space.

Proposition 5.1.18. *Using Definition (2.1.83), let $\mathcal{D}$ be a Bratteli diagram. Using Definitions (5.1.15, 5.1.17), if we define:*

$$i_m(\cdot, \mathcal{D}) : \mathrm{Ideal}(\mathcal{D}) \to \mathcal{C}_{\mathcal{D}}$$

coordinate-wise in the following way:

$$i_m(I, \mathcal{D})(n)_k = \begin{cases} 1 & \textit{if } (n,k) \in V^I \\ 0 & \textit{if } (n,k) \in V^{\mathcal{D}} \setminus V^I \end{cases},$$

then $i_m(\cdot, \mathcal{D})$ is a well-defined injection such that $(i_m(\mathrm{Ideal}(\mathcal{D}), \mathcal{D}), \mathsf{m}_{\mathcal{C}})$ is a zero-dimensional compact ultrametric space.

Furthermore, let $\mathsf{m}_{i_m(\mathcal{D})} = \mathsf{m}_{\mathcal{C}} \circ (i_m(\cdot, \mathcal{D}) \times i_m(\cdot, \mathcal{D}))$. Then, the metric space $\left(\mathrm{Ideal}(\mathcal{D}), \mathsf{m}_{i_m(\mathcal{D})}\right)$ is a zero-dimensional compact ultrametric space.

Proof. The map $i_m(\cdot, \mathcal{D})$ is well-defined by construction. For injectivity, assume that there exist $I, J \in \mathrm{Ideal}(\mathcal{D})$ such that $i_m(I, \mathcal{D}) = i_m(J, \mathcal{D})$. By definition, this implies that $i_m(I, \mathcal{D})(n) = i_m(J, \mathcal{D})(n)$ for each $n \in \mathbb{N}$, and therefore, the vertices $V_n^I = V_n^J$ for each $n \in \mathbb{N}$. Thus, applying Lemma (5.1.16), we have that $I = J$.

For compactness of $(i_m(\mathrm{Ideal}(\mathcal{D}), \mathcal{D}), \mathsf{m}_{\mathcal{C}})$, we need only to show that $i_m(\mathrm{Ideal}(\mathcal{D}), \mathcal{D})$ is closed as $(\mathcal{C}_{\mathcal{D}}, \mathsf{m}_{\mathcal{C}})$ is compact. Thus, assume $j \in \mathcal{C}_{\mathcal{D}}$ such that there exists $(J^n)_{n \in \mathbb{N}} \subseteq \mathrm{Ideal}(\mathcal{D})$ with $\lim_{n \to \infty} \mathsf{m}_{\mathcal{C}}(i_m(J^n, \mathcal{D}), j) = 0$. With this, construct an increasing sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ such that for fixed $n \in \mathbb{N}$, we have:

$$\mathsf{m}_{\mathcal{C}}(i_m(J^l, \mathcal{D}), j) \leqslant 2^{-(n+1)}$$

for all $l \geqslant c_n$. By definition of $\mathsf{m}_{\mathcal{C}}$, for each $n \in \mathbb{N}, p \in \{0, \ldots, n\}$, we gather that $j(p) = i_m(J^l, \mathcal{D})(p)$ for all $l \geqslant c_n$. In particular, we also have that:

$$j(p) = i_m(J^{c_n}, \mathcal{D})(p) = i_m(J^{c_{n+q}}, \mathcal{D})(p) \tag{5.1.3}$$

for all $n \in \mathbb{N}, q \in \mathbb{N}, p \in \{0, \ldots n\}$ since $(c_n)_{n\in\mathbb{N}}$ is increasing. Thus:

$$V_p^{J^{c_n}} = V_p^{J^{c_{n+q}}} \tag{5.1.4}$$

for all $q \in \mathbb{N}, n \in \mathbb{N}, p \in \{0, \ldots n\}$ by definition of $i_m(\cdot, \mathcal{D})$. Thus, for each $n \in \mathbb{N}$, define $V_n^J = V_n^{J^{c_n}}$. Now, let:

$$V^J = \bigcup_{n\in\mathbb{N}} V_n^{J^{c_n}}.$$

Form E^J by imposing the edge axiom (iii) from Definition (5.1.15). For $J = (V^J, E^J)$, inherit the vertex labels and number of edges from $\mathcal{D}$ as done in Definition (5.1.15). We claim that $J \in \text{Ideal}(\mathcal{D})$ and that $i_m(J, \mathcal{D}) = j$.

First, let $(n, k) \in V^J$ such that $((n, k), (n + 1, q)) \in E^{\mathcal{D}}$. But, $(n, k) \in V_n^{J^{c_n}} = V_n^{J^{c_{n+1}}}$ by Equation (5.1.4). Since $V^{J^{c_{n+1}}}$ is an ideal, by the directed axiom (i), we have that $(n+1, q) \in V^{J^{c_{n+1}}} \cap V_{n+1}^{\mathcal{D}} = V_{n+1}^{J^{c_{n+1}}} \subseteq V^J$, which provides directed axiom (i) for J.

Next, for the hereditary axiom (ii), let $(n, k) \in V^{\mathcal{D}}$ and $R_{(n,k)}^{\mathcal{D}} \subseteq V^J$. Now, the set $R_{(n,k)}^{\mathcal{D}} \subseteq V_{n+1}^{J^{c_{n+1}}}$. Thus, since $V^{J^{c_{n+1}}}$ is an ideal, then $(n, k) \in V_n^{J^{c_{n+1}}} = V_n^{J^{c_n}} \subseteq V^J$ by Equation (5.1.4), which proves the hereditary axiom (ii) for J. Axiom (iii) for edges is given by construction. Furthermore, as the labels of vertices and number of edges are inherited from $\mathcal{D}$, we have that $J \in \text{Ideal}(\mathcal{D})$ by Definition (5.1.15).

Next, fix $n \in \mathbb{N}, k \in \{0, \ldots, v_n^{\mathcal{D}}\}$, then by Equation (5.1.3), we have $j(n)_k = 1 \iff i_m(J^{c_n}, \mathcal{D})(n)_k = 1 \iff (n, k) \in V^{J^{c_n}} \iff (n, k) \in V_n^{J^{c_n}} = V_n^J \subseteq V^J \iff i_m(J, \mathcal{D})(n)_k = 1$.

Now, assume that $j(n)_k = 0$. Then, by Equation (5.1.3), we have $0 = j(n)_k = i_m(J^{c_n}, \mathcal{D})(n)_k$ implies that $(n, k) \in V^{\mathcal{D}} \setminus V^{J^{c_n}} = \cap_{l\in\mathbb{N}} \left(V^{\mathcal{D}} \setminus V_l^{J^{c_n}}\right)$. Thus, the vertex $(n,k) \in V^{\mathcal{D}} \setminus V_n^{J^{c_n}} = V^{\mathcal{D}} \setminus V_n^J$. However, for all $m \in \mathbb{N} \setminus \{n\}$, the set V_m^J does not contain a vertex of the form (n,l) for any l, and thus the vertex $(n,k) \notin V_m^J$ for all $m \in \mathbb{N} \setminus \{n\}$ as well. Hence, the vertex $(n,k) \in V^{\mathcal{D}} \setminus V^J \iff i_m(J,\mathcal{D})(n)_k = 0$.

For the reverse implication, assume that $i_m(J,\mathcal{D})(n)_k = 0$, then $(n,k) \in V^{\mathcal{D}} \setminus V^J = \cap_{l \in \mathbb{N}} \left(V^{\mathcal{D}} \setminus V_l^{J^n}\right)$. Hence, it must be the case that $(n,k) \notin V_n^{J^{c_n}} = V_n^J$. Again, the vertex $(n,k) \notin V_m^{J^{c_n}}$ for all $m \in \mathbb{N} \setminus \{n\}$ as well. Thus, the vertex $(n,k) \in V^{\mathcal{D}} \setminus V^{J^{c_n}}$, which implies that $j(n)_k = i_m(J^{c_n}, \mathcal{D})(n)_k = 0$ by Equation (5.1.3). Hence, we conclude $j(n)_k = 0 \iff i_m(J,\mathcal{D})(n)_k = 0$ for $n \in \mathbb{N}, k \in \{0, \ldots, v_n^{\mathcal{D}}\}$.

Therefore, we have $i_m(J,\mathcal{D})(n) = j(n)$ for all $n \in \mathbb{N}$. Hence, the space $(i_m(\mathrm{Ideal}(\mathcal{D}), \mathcal{D}), \mathsf{m}_{\mathcal{C}})$ is a compact metric space. Zero-dimensional is inherited from $(\mathcal{C}_{\mathcal{D}}, \mathsf{m}_{\mathcal{C}})$. The fact that the metric space $\left(\mathrm{Ideal}(\mathcal{D}), \mathsf{m}_{i_m(\mathcal{D})}\right)$ is a zero-dimensional compact metric space follows from the fact that $i_m(\cdot, \mathcal{D})$ is injective and that $i_m(\mathrm{Ideal}(\mathcal{D}), \mathcal{D})$ is compact in $(\mathcal{C}_{\mathcal{D}}, \mathsf{m}_{\mathcal{C}})$. □

The metric of Proposition (5.1.18) is stated entirely in the setting of Bratteli diagram without reference to an AF algebra. But, we would like utilize Proposition (5.1.18) to provide compactness of the metric of Proposition (5.1.7) in the case of AF algebras. Thus, we now begin this transition.

Notation 5.1.19. *Let $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be an AF algebra where $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite dimensional C*-subalgebras of $\mathfrak{A}$. Let $\mathcal{D}_b(\mathfrak{A})$ be the diagram given by Definition (2.1.85).*

Let $I \in \mathrm{Ideal}(\mathfrak{A})$ be a norm closed two-sided ideal of $\mathfrak{A}$, then by [11, Lemma 3.2], the subset Λ of $\mathcal{D}_b(\mathfrak{A})$ formed by I is an ideal in the sense of Definition (5.1.15), and denote this by $\mathcal{D}_b(\mathfrak{A})(I) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$, where $\mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$ is the set of ideals of $\mathcal{D}_b(\mathfrak{A})$ from Definition (5.1.15).

Proposition 5.1.20. *[11, Lemma 3.2] Let* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *be an AF algebra where* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence of finite dimensional C*-subalgebras of* $\mathfrak{A}$ *and Bratteli diagram* $\mathcal{D}_b(\mathfrak{A})$ *from Definition (2.1.85). Using Notation (5.1.19) and Definition (5.1.15), the map:*

$$i(\cdot, \mathcal{D}_b(\mathfrak{A})) : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto \mathcal{D}_b(\mathfrak{A})(I) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$$

given by [11, Lemma 3.2] is a well-defined bijection, where the vertices of $V_n^{\mathcal{D}_b(\mathfrak{A})(I)}$ *are determined by* $I \cap \mathfrak{A}_n$ *for each* $n \in \mathbb{N}$.

We are now prepared to strengthen Proposition (5.1.7) in the case of AF algebras.

Theorem 5.1.21. *If* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *is C*-algebra where* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence of finite dimensional C*-subalgebras of* $\mathfrak{A}$*, then using Definition (5.1.6), we have that the map* $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ *of Proposition (5.1.20) is an isometry from the metric space* $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ *of Proposition (5.1.7) onto the metric space* $\left(\mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A})), \mathsf{m}_{i_m(\mathcal{D}_b(\mathfrak{A}))}\right)$ *using notation of Definition (2.1.85) and Proposition (5.1.18).*

Therefore, the space $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ *is a zero-dimensional compact ultrametric space, and moreover, the topology induced by* $\mathsf{m}_{i(\mathcal{U})}$ *on* $\mathrm{Ideal}(\mathfrak{A})$ *is the Fell topology of Definition (2.1.58).*

Proof. The isometry is given by Proposition (5.1.20). Indeed, since the vertices of $V_n^{\mathcal{D}_b(\mathfrak{A})(I)}$ are determined by $I \cap \mathfrak{A}_n$ for each $n \in \mathbb{N}$ for any $I \in \mathrm{Ideal}(\mathfrak{A})$, if $I, J \in \mathrm{Ideal}(\mathfrak{A})$, then $i(I, \mathcal{D}_b(\mathfrak{A}))(n) = i(J, \mathcal{D}_b(\mathfrak{A}))(n)$ if and only if $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ by Lemma (5.1.16). Thus:

$$i(\cdot, \mathcal{D}_b(\mathfrak{A})) : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto \mathcal{D}_b(\mathfrak{A})(I) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$$

is an isometry from $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ onto the metric space

$\left(\mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A})), \mathsf{m}_{i_m(\mathcal{D}_b(\mathfrak{A}))}\right)$. Therefore, $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ is a zero-dimensional compact ultrametric space. But, by Theorem (5.1.13), the metric topology of $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ is a compact Hausdorff topology that contains the compact Hausdorff topology, ***Fell***. Therefore, by maximal compactness, the two topologies equal, which completes the proof. □

We now begin a sequence of Corollaries that highlight the consequences of Theorem (5.1.21). All of these following Corollaries are phrased in terms of $\mathsf{m}_{i(\mathcal{U})}$, but can be translated in terms of the diagram metric $\mathsf{m}_{i_m(\mathcal{D})}$ by Theorem (5.1.21), and we note that $\mathsf{m}_{i_m(\mathcal{D})}$ will prove useful in its own right in the proof of Theorem (5.1.28), Proposition (5.2.10), and the main result of Section (5.2.1), which is Theorem (5.2.21), since many results and constructions with regard to AF algebras are phrased diagramatically. First, Theorem (5.1.21) provides that the notion of fusing family of ideals is a topological and metric notion, which motivates the definition of fusing family (Definition (4.5.1)).

Corollary 5.1.22. *Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of finite dimensional C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.*

If $\left(I^k\right)_{k\in\overline{\mathbb{N}}} \subseteq \mathrm{Ideal}(\mathfrak{A})$, then the following are equivalent:

1. $\left\{I^k = \overline{\cup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ *is a fusing family of Definition (4.5.1),*
2. $\left(I^k\right)_{k\in\mathbb{N}}$ *converges to I^∞ with respect to the metric $\mathsf{m}_{i(\mathcal{U})}$,*
3. $\left(I^k\right)_{k\in\mathbb{N}}$ *converges to I^∞ in the Fell topology.*

Proof. Apply Theorem (5.1.21) to Lemma (5.1.10). □

Next, the metric topology has same comparison with the Jacobson topology as the Fell topology.

Corollary 5.1.23. *If $\mathfrak{A}$ is a C^*-algebra with a non-decreasing sequence of finite dimensional C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then using* *notation from Proposition (5.1.7), the space* $\left(\mathrm{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ *is a totally bounded zero-dimensional ultrametric space in which the relative topology on* $\mathrm{Prim}(\mathfrak{A})$ *induced by the metric topology* $\mathsf{m}_{i(\mathcal{U})}$ *or the Fell topology contains the Jacobson topology on* $\mathrm{Prim}(\mathfrak{A})$*.*

Proof. Apply Theorem (5.1.21) to Proposition (5.1.1). And, since total boundedness and zero-dimensionality are hereditary properties, the proof is complete. □

Another immediate consequence of Theorem (5.1.21) is that, although the metric is built using a fixed inductive sequence, the metric topology with respect to an inductive sequence is homeomorphic to the metric topology on the same AF algebra with respect to any other inductive sequence. In particular, concerning continuity or convergence results, Corollary (5.1.24) provides that one need not worry about the possibility of choosing the wrong inductive sequence, and therefore, one may choose any inductive sequence without worry to suit the needs of the problem at hand.

Corollary 5.1.24. *Let* $\mathfrak{A}, \mathfrak{B}$ *be* C^**-algebras with non-decreasing sequences of finite dimensional* C^**-subalgebras* $\mathcal{U}_{\mathfrak{A}} = (\mathfrak{A}_n)_{n\in\mathbb{N}}, \mathcal{U}_{\mathfrak{B}} = (\mathfrak{B}_n)_{n\in\mathbb{N}}$*, respectively, such that* $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *and* $\mathfrak{B} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$*.*

If $\mathfrak{A}$ *and* $\mathfrak{B}$ *are *-isomorphic, then the metric spaces* $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_{\mathfrak{A}})}\right)$ *and* $\left(\mathrm{Ideal}(\mathfrak{B}), \mathsf{m}_{i(\mathcal{U}_{\mathfrak{B}})}\right)$ *are homeomorphic.*

In particular, if $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_{1,n}}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_{2,n}}^{\|\cdot\|_{\mathfrak{A}}}$*, where* $\mathcal{U}_1 = (\mathfrak{A}_{1,n})_{n\in\mathbb{N}}, \mathcal{U}_2 = (\mathfrak{A}_{2,n})_{n\in\mathbb{N}}$ *are non-decreasing sequences of finite dimensional* C^**-subalgebras of* $\mathfrak{A}$*, then the metric spaces* $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_1)}\right)$ *and* $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_2)}\right)$ *are homeomorphic.*

Proof. Apply Lemma (5.1.3) to Theorem (5.1.21). □

Furthermore, as another consequence of Theorem (5.1.21), we may strengthen Proposition (5.1.11) with the Fell topology in the case of AF algebras.

Corollary 5.1.25. *Let $\mathfrak{A}$ be a C*-algebra with a non-decreasing sequence of finite dimensional C*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.*

If $(I^n)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ or the Fell topology, then using Definition (4.5.1), the family:

$$\left\{ I^n = \overline{\cup_{k\in\mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}} \right\}$$

is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$ such that $\left\{ \varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}} \right\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$.

Proof. Apply Theorem (5.1.21) to Proposition (5.1.11). □

Now, that we have this identification with the Fell topology, we finish our discussion of the metric topology by considering it in the commutative case. The reason for this is because if $\mathfrak{A}$ is a commutative C*-algebra, then the Jacobson topology on the primitive ideals of $\mathfrak{A}$ is homeomorphic to the maximal ideal space with its weak* topology, which is a classic result for which we provided a proof of in the unital case as Theorem (2.1.55). Furthermore, in the unital case, we will show that the relative topology on the primitive ideals induced by the metric topology will be compact, which will provide that this metric topology agrees with the Jacobson topology since it is compact in the unital case. This result rests on a characterization of Bratteli diagrams associated to unital commutative AF algebras provided by Bratteli as [12, Expression 3.1] along with his diagrammatic characterization of primitive ideals found as [11, Theorem 3.8], [12, Expression 2.7]. We return to diagrams and some common notation with respect to Bratteli diagrams and vertices that are connected by a sequence of edges.

Notation 5.1.26. *Let $\mathcal{D} \in \mathscr{BD}$ be a Bratteli diagram of Definition (2.1.83). For $(n,k),(m,r) \in V^{\mathcal{D}}, m \geqslant n$, we write:*

$$(n,k) \Downarrow (m,r)$$

if there exists a sequence $((n,k_p))_{p=n}^m \subset V^{\mathcal{D}}$ *such that* $(n,k_n) = (n,k)$ *and* $(m,r) = (m,k_m)$ *and* $((n,k_p),(p+1,k_{p+1})) \in E^{\mathcal{D}}$ *for all* $p \in \{n,\ldots,m-1\}$.

We require more information for the diagram associated to the AF algebra as a quotient of an ideal of an AF algebra. This is Remark (5.1.27).

Remark 5.1.27. *Let* $\mathfrak{A}$ *be a unital* C*-*algebra with a non-decreasing sequence of finite dimensional unital* C*-*subalgebras* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *such that* $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. *Let* $I \in \mathrm{Ideal}(\mathfrak{A}))$. *Recall the map,* $i(\cdot,\mathcal{D}_b(\mathfrak{A})$ *defined in Proposition (5.1.20). We define the graph* $\mathcal{D}(\mathfrak{A}/I) = \left(V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))}, E^{\mathfrak{A}/I}\right)$, *where* $E^{\mathfrak{A}/I}$ *is all edges from* $E^{\mathcal{D}_b(\mathfrak{A})}$ *between vertices in* $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))}$ *along with the induced labels and number of edges from* $\mathcal{D}_b(\mathfrak{A})$. *By [11, Proposition 3.7], the diagram* $\mathcal{D}(\mathfrak{A}/I)$ *satisfies axioms (i),(ii),(iii) of Definition (2.1.83). Furthermore, the diagram* $\mathcal{D}(\mathfrak{A}/I)$ *forms the diagram associated to the Bratteli diagram* $\mathcal{D}_b(\mathfrak{A}/I)$ *from Definition (2.1.85) up to shifting the placement of vertices as done in [11, Proposition 3.7].*

Thus, we are now in a position to compare the relative metric topology with the Jacobson topology on the primitive ideals of a unital commutative C*-algebra.

Theorem 5.1.28. *Let* $\mathfrak{A}$ *be a unital* C*-*algebra with a non-decreasing sequence of unital finite dimensional* C*-*subalgebras* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *such that* $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.

If $\mathfrak{A}$ *is commutative, then the metric space* $\left(\mathrm{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ *with relative topology induced by the metric topology of* $\mathsf{m}_{i(\mathcal{U})}$ *(Proposition (5.1.7)):*

1. *is a zero-dimensional compact ultrametric space,*

2. *has the same topology as the Jacobson topology or the relative Fell topology on* $\mathrm{Prim}(\mathfrak{A})$, *and*

3. *is homeomorphic to the space of non-zero multiplicative linear functionals on* $\mathfrak{A}$ *denoted* $M_{\mathfrak{A}}$ *with its weak-* topology, in which the homeomorphism is given by:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \mathrm{Prim}(\mathfrak{A}).$$

Proof. We start by verifying conclusion *1.*. For this, we show that $\mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$ is open in $\big(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\big)$. Note that $\mathfrak{A} \in \mathrm{Ideal}(\mathfrak{A})$ is not primitive by Definition (2.1.52). Thus, we approach the proof in two cases.

Case 1. *Assume* $I = \mathfrak{A}$.

We show that $\mathfrak{A}$ is isolated. Note that $\big\{P \in \mathrm{Ideal}(\mathfrak{A}) : \mathsf{m}_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\big\}$ is a basic open set such that $\{\mathfrak{A}\} \subseteq \big\{P \in \mathrm{Ideal}(\mathfrak{A}) : \mathsf{m}_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\big\}$. However, let $K \in \big\{P \in \mathrm{Ideal}(\mathfrak{A}) : \mathsf{m}_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\big\}$, then by definition of $\mathsf{m}_{i(\mathcal{U})}$, we have $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{A}_0 = K \cap \mathfrak{A}_0$ and thus K would be unital, which implies that $K = \mathfrak{A}$. Hence, the ideal $I \in \{\mathfrak{A}\} = \big\{P \in \mathrm{Ideal}(\mathfrak{A}) : \mathsf{m}_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\big\} \subseteq \mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$.

Case 2. *Assume* $I \in \mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$ *such that* $I \neq \mathfrak{A}$

Recall the map $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ defined in Proposition (5.1.20). By [12, Expression 2.7], since I is not primitive:

$$\begin{aligned} &\text{there exists } N_I \in \mathbb{N} \text{ such that for all } m \geqslant N_I, (m,r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))} \\ &\text{there exists } (N_I, k) \in V^{\mathcal{D}} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))} \text{ such that } (N_I, k) \not\Downarrow (m,r) \end{aligned} \tag{5.1.5}$$

using Notation (5.1.26).

Next, we consider the vertices of $i(I, \mathcal{D}_b(\mathfrak{A}))$, where $I \neq \mathfrak{A}$. Assume by way of contradiction that there exists $k \in \mathbb{N}$ such that $V_k^{i(I,\mathcal{D}_b(\mathfrak{A}))} = V_k^{\mathcal{D}_b(\mathfrak{A})}$, then by definition of $i(I, \mathcal{D}_b(\mathfrak{A}))$, this would imply that $I \cap \mathfrak{A}_k = \mathfrak{A}_k$. Since $\mathfrak{A}_k$ is unital, then I would contain the unit, and thus, the ideal $I = \mathfrak{A}$, a contradiction to our assumption that $I \neq \mathfrak{A}$ of Case 2. Therefore, we have that $\emptyset \subsetneq V_M^{\mathcal{D}_b(\mathfrak{A})} \setminus V_M^{i(I,\mathcal{D}_b(\mathfrak{A}))} \subsetneq V_M^{\mathcal{D}_b(\mathfrak{A})}$ for all $M \in \mathbb{N}$. Therefore, at $N_I + 1$, there exists $(N_I + 1, r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))}$. By Expression (5.1.5), since I is not primitive, there exists $(N_I, k) \in V^{\mathcal{D}} \setminus V^{i(I,\mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, k) \not\Downarrow (N_I+1, r)$. Let $\mathcal{D}(\mathfrak{A}/I)$ denote the diagram associated to $\mathfrak{A}/I$ defined in Remark (5.1.27). Thus, since $\mathcal{D}(\mathfrak{A}/I)$ satisfies axiom (iii) of Definition (2.1.83) and $(N_I, k) \not\Downarrow (N_I+1, r)$, we have that there must exist $(N_I, l) \in V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I,\mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, l) \Downarrow (N_I+1, r)$, and so the cardinality of $V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I,\mathcal{D}_b(\mathfrak{A}))}$ is greater than or equal to 2.

Thus, consider the basic open set
$B_{\mathsf{m}_{i(\mathcal{U})}}\left(I, 2^{-(N_I+2)}\right) = \left\{J \in \mathrm{Ideal}(\mathfrak{A}) : \mathsf{m}_{i(\mathcal{U})}(I, J) < 2^{-(N_I+2)}\right\}$. Let $J \in B_{\mathsf{m}_{i(\mathcal{U})}}\left(I, 2^{-(N_I+2)}\right)$. Therefore, since $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ is an isometry by Theorem (5.1.21), we have $V_{N_I}^{i(I,\mathcal{D}_b(\mathfrak{A}))} = V_{N_I}^{i(J,\mathcal{D}_b(\mathfrak{A}))}$ and $V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I,\mathcal{D}_b(\mathfrak{A}))} = V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(J,\mathcal{D}_b(\mathfrak{A}))}$, which thus has cardinality greater than or equal to 2, and so there exists $(N_I, k), (N_I, l) \in V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(J,\mathcal{D}_b(\mathfrak{A}))}$ such that $k \neq l$.

We claim that $J \in \mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$. Assume by way of contradiction that $J \in \mathrm{Prim}(\mathfrak{A})$. Thus by [12, Expression 2.7], there exist $m > N_I$ and $(m, r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(J,\mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, k) \Downarrow (m, r)$ and $(N_I, l) \Downarrow (m, r)$.

Let $((N_I, k_p))_{p=N_I}^m \subseteq V^{\mathcal{D}_b(\mathfrak{A})}$ and $((N_I, k_l))_{p=N_I}^m \subseteq V^{\mathcal{D}_b(\mathfrak{A})}$ be the sequences defined by $(N_I, k) \Downarrow (m, r)$ and $(N_I, l) \Downarrow (m, r)$, respectively, and Notation (5.1.26). Thus, the vertices $(m, k_m) = (m, r)$ and $(m, l_m) = (m, r)$. Hence, since $(N_I, k) \neq (N_I, l)$, there exists $p \in \{N_I + 1, \ldots, m\}$ such that $(p-1, k_{p-1}) \neq (p-1, l_{p-1})$ and $(p, k_p) = (p, l_p)$ lest the condition $\Downarrow$ not be satisfied. But, then, the edges $((p-1, k_{p-1}), (p, k_p)), ((p-1, l_{p-1}), (p, k_p)) \in E^{\mathcal{D}_b(\mathfrak{A})}$. Since the diagram $\mathcal{D}_b(\mathfrak{A})$ is a Bratteli diagram of a unital commutative AF algebra, by Bratteli's characterization of Bratteli diagrams of unital commutative AF algebras as [12, Expression 3.1], we have reached a contradiction since $(p-1, k_{p-1}) \neq (p-1, l_{p-1})$. Therefore, the ideal $J \in \mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$ and $I \in B_{\mathsf{m}_{i(\mathcal{U})}}\left(I, 2^{-(N_I+2)}\right) \subseteq \mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$.

Combining Case 1 and Case 2, the set $\mathrm{Ideal}(\mathfrak{A}) \setminus \mathrm{Prim}(\mathfrak{A})$ is open, and thus $\mathrm{Prim}(\mathfrak{A})$ is closed in the zero-dimensional compact metric space $\left(\mathrm{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$, which is compact by Theorem (5.1.21). Therefore, the space $\left(\mathrm{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ is a zero-dimensional compact metric space with its relative topology.

For conclusion *2.*, the comment about the relative Fell topology is already established by Theorem (5.1.21). By Theorem (2.1.55) and Corollary (5.1.23), we have that the Jacobson topology on $\mathrm{Prim}(\mathfrak{A})$ is a compact Hausdorff topology contained in the compact Hausdorff topology given by $\left(\mathrm{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$, which is compact Hausdorff by part *1.* By maximal compactness, the topologies equal.

For conclusion *3.*, by Theorem (2.1.55), the set $\mathrm{Prim}(\mathfrak{A})$ with its Jacobson topology is homeomorphic to $M_{\mathfrak{A}}$ with its weak-* topology. Thus, by part *2.*, we conclude that $\left(\mathrm{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})}\right)$ is homeomorphic to $M_{\mathfrak{A}}$ with its weak-* topology by the described homeomorphism. □

5.2 Criteria for convergence of quotients of AF algebras

In the case of unital AF algebras, we provide criteria for when convergence of ideals in the Fell topology provides convergence of quotients in the quantum propinquity topology, when the quotients are equipped with faithful tracial states. But, first, as we saw in Corollary (5.1.25) and Proposition (5.1.12), it seems that an inductive limit is suitable for describing fusing families with regard to convergence of ideals. Thus, in order to avoid the notational trouble of too many inductive limits, we will phrase many results in this section in terms of closure of union.

Now, when a quotient has a faithful tracial state, it turns out that the *-isomorphism provided in Proposition (5.1.12) is a quantum isometry (Theorem-Definition (2.3.16)) between the induced quantum compact metric spaces of Theorem (3.1.3) and Theorem (3.1.5), which preserves the finite-dimensional structure as well in Theorem (5.2.1). The purpose of this is to apply Theorem

(4.5.6) directly to the quotient spaces. This utilizes our criteria for quantum isometries between AF algebras in Section (3.3) as Theorem (3.3.1).

Theorem 5.2.1. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C^*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $I \in \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$. By Proposition (5.1.12), the C^*-algebra $\mathfrak{A}/I = \overline{\cup_{n\in\mathbb{N}}((\mathfrak{A}_n + I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$ and denote $\mathcal{U}/I = ((\mathfrak{A}_n + I)/I)_{n\in\mathbb{N}}$, and note that $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$.*

If $\mathfrak{A}/I$ is equipped with a faithful tracial state, μ, then using notation from Proposition (5.1.12), the map $\mu \circ \phi_I$ is a faithful traical state on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

Furthermore, let $\beta : \mathbb{N} \to (0, \infty)$ have limit 0 at infinity. If $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$ is the $(2,0)$-quasi-Leibniz Lip norm on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ given by Theorem (3.1.3) and $\mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}$ is the $(2,0)$-quasi-Leibniz Lip norm on $\mathfrak{A}/I$ given by Theorem (3.1.5), then:

$$\phi_I^{-1} : \left(\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right) \to \left(\varinjlim \mathcal{I}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}\right)$$

is a quantum isometry of Theorem-Definition (2.3.16) and:

$$\Lambda_{2,0}\left(\left(\varinjlim \mathcal{I}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}\right), \left(\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right)\right) = 0$$

Moreover, for all $n \in \mathbb{N}$, we have:

$$\Lambda_{2,0}\left(\left(\mathfrak{A}_n/(I \cap \mathfrak{A}_n), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I} \circ \underrightarrow{\gamma_I^n}\right), \left((\mathfrak{A}_n + I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right)\right) = 0.$$

Proof. Since $I \neq \mathfrak{A}$, the AF algebra $\mathfrak{A}/I$ is unital and $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$ as $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Since μ is faithful on $\mathfrak{A}/I$, we have $\mu \circ \phi_I$ is faithful on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ since ϕ_I is a *-isomorphism by Proposition (5.1.12).

Using Notation (5.1.11), define $\mathcal{U}(\mathfrak{A}/I) = \left(\underrightarrow{\gamma_I^m}\left(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)\right)\right)_{m\in\mathbb{N}}$. By Proposition (2.1.66), the sequence $\mathcal{U}(\mathfrak{A}/I) = \left(\underrightarrow{\gamma_I^m}\left(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)\right)\right)_{m\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C*-subalgebras of $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ such that $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I) = \overline{\cup_{m\in\mathbb{N}}\gamma^m_{\underrightarrow{I}}\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right)}^{\|\cdot\|_{\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)}}$ and $\gamma^0_{\underrightarrow{I}}\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_0)\right) = \mathbb{C}1_{\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)}$.

Thus, we may define $\mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}$ on $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ from Theorem (3.1.5), and $\mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}$ on $\mathfrak{A}/I$ from Theorem (3.1.5).

Now, fix $m\in\mathbb{N}$, since $\phi_I\circ\gamma^m_{\underrightarrow{I}} = \phi^m_I$ by Proposition (5.1.12), we thus have:

$$\gamma^m_{\underrightarrow{I}}\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right) = \phi_I^{-1}\circ\phi^m_I\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right) = \phi_I^{-1}\left((\mathfrak{A}_m+I)/I\right).$$

Also, since the chosen faithful tracial state on $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ is $\mu\circ\phi_I$, we have by Theorem (3.3.1) that $\left(\gamma^m_{\underrightarrow{I}}\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}\right)$ is quantum isometric to $\left((\mathfrak{A}_m+I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right)$ by the map ϕ_I^{-1} restricted to $(\mathfrak{A}_m+I)/I$ for all $m\in\mathbb{N}$. However, the space $\left(\gamma^m_{\underrightarrow{I}}\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}\right)$ is quantum isometric to $\left(\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)\right), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}\circ\gamma^m_{\underrightarrow{I}}\right)$ by the map $\gamma^m_{\underrightarrow{I}}$. Since quantum isometry is an equivlance relation, we conclude that:

$$\Lambda_{2,0}\left(\left(\mathfrak{A}_m/(I\cap\mathfrak{A}_m), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}\circ\gamma^m_{\underrightarrow{I}}\right), \left((\mathfrak{A}_m+I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right)\right) = 0$$

by Theorem-Definition (2.3.16).

Moreover, Theorem (3.3.1) also implies that:

$$\phi_I^{-1} : \left(\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}\right) \to \left(\underrightarrow{\lim}\mathcal{U}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}\right)$$

is a quantum isometry. Next, define $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$ from Theorem (3.1.3). By Proposition (3.1.6), we may replace $\mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}$ with $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$, which completes the proof. □

Thus, the quantum isometry, ϕ_I, of Theorem (5.2.1) is in some sense the best one could hope for since it preserves the finite-dimensional approximations in the quantum propinquity. Next, we give criteria for when a family of quotients converge in the quantum propinquity with respect to ideal convergence.

Theorem 5.2.2. *Let $\mathfrak{A}$ be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is an increasing sequence of unital finite dimensional C*-subalgebras such that $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $(I^n)_{n\in\overline{\mathbb{N}}} \subseteq \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$ such that $\{\mu_k : \mathfrak{A}/I^k \to \mathbb{C} : k \in \overline{\mathbb{N}}\}$ is a family of faithful tracial states. Let $Q^k : \mathfrak{A} \to \mathfrak{A}/I^k$ denote the quotient map for all $k \in \overline{\mathbb{N}}$. If:*

1. *$(I^n)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ of Definition (5.1.6) or the Fell topology (Definition (2.1.58)) with fusing sequence $(c_n)_{n\in\mathbb{N}}$ for the fusing family $\left\{ I^n = \overline{\cup_{k\in\mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}} \right\}$,*
2. *for each $N \in \mathbb{N}$, we have that $\left(\mu_k \circ Q^k\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ Q^\infty$ in the weak-* topology on $\mathscr{S}(\mathfrak{A}_N)$, and*
3. *$\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k\in\overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geqslant c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \ldots, N\}$ and there exists $B : \overline{\mathbb{N}} \to (0,\infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leqslant B(l)$ for all $m, l \in \overline{\mathbb{N}}$,*

then using notation from Theorem (5.2.1):

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}/I^n, \mathsf{L}^{\beta^n}_{\mathcal{U}/I^n,\mu_n}\right), \left(\mathfrak{A}/I^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{U}/I^\infty,\mu_\infty}\right)\right) = 0$$

Proof. By Corollary (5.1.25), the assumption that $(I^n)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ of Definition (5.1.6) or the Fell topology implies that:

$$\left\{ I^n = \overline{\cup_{k\in\mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}} \right\}$$

is a fusing family with some fusing sequence $(c_n)_{n\in\mathbb{N}}$ such that $\left\{ \varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}} \right\}$ is an IL-fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$.

Fix $N \in \mathbb{N}$ and $k \in \mathbb{N}_{\geqslant c_N}$. Let $x \in \mathfrak{A}_N$, and let $Q_N^k : \mathfrak{A}_N \to \mathfrak{A}_N/(I^k \cap \mathfrak{A}_N)$ and $Q_N^\infty : \mathfrak{A}_N \to \mathfrak{A}_N/(I^\infty \cap \mathfrak{A}_N)$ denote the quotient maps, and let Let $\phi_{I^k} : \varinjlim \mathcal{I}(\mathfrak{A}/I^k) \to \mathfrak{A}/I^k$ denote the *-isomorphism given in Proposition (5.1.12) and recall that $\mathcal{I}(\mathfrak{A}/I^k) = \left(\mathfrak{A}_n/(I^k \cap \mathfrak{A}_n), \gamma_{I^k,n}\right)_{n\in\mathbb{N}}$ from Notation (5.1.11). Now, by Proposition (5.1.12) and its commuting diagram, we gather:

$$\begin{aligned}\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma_{I^k}^N} \circ Q_N^k(x) &= \mu_k \circ \phi_{I^k}^N \circ Q_N^k(x)\\ &= \mu_k \circ \phi_{I^k}^N(x + I^k \cap \mathfrak{A}_N)\\ &= \mu_k(x + I^k)\\ &= \mu_k \circ Q^k(x).\end{aligned}$$

Therefore, by hypothesis *2.*, the sequence $\left(\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma_{I^k}^N} \circ Q_N^k\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \underrightarrow{\gamma_{I^\infty}^N} \circ Q_N^\infty$ in the weak-* topology on $\mathfrak{A}_N$. Hence, the sequence $\left(\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma_{I^k}^N}\right)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \underrightarrow{\gamma_{I^\infty}^N}$ in the weak-* topology on $\mathscr{S}\left(\mathfrak{A}_N/(I^\infty \cap \mathfrak{A}_N)\right)$ by [18, Theorem V.2.2]. Thus, by hypothesis *3.* and by Theorem (4.5.6), we have that:

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\varinjlim \mathcal{I}(\mathfrak{A}/I^n), \mathsf{L}^{\beta^n}_{\mathcal{I}(\mathfrak{A}/I^n),\mu_n\circ\phi_{I^n}}\right), \left(\varinjlim \mathcal{I}(\mathfrak{A}/I^\infty), \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\mathfrak{A}/I^\infty),\mu_\infty\circ\phi_{I^\infty}}\right)\right) = 0.$$

But, as $\phi_{I^n}^{-1}$ is an isometric isomorphism for all $n \in \overline{\mathbb{N}}$ by Theorem (5.2.1), we conclude:

$$\lim_{n\to\infty} \mathsf{\Lambda}_{2,0}\left(\left(\mathfrak{A}/I^n, \mathsf{L}^{\beta^n}_{\mathcal{U}/I^n,\mu_n}\right), \left(\mathfrak{A}/I^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{U}/I^\infty,\mu_\infty}\right)\right) = 0,$$

which completes the proof. □

5.2.1 Continuous families of quotients of the Boca-Mundici algebra

The Boca-Mundici AF algebra arose in [10] and [54] independently and is constructed from the Farey tessellation (see [10] for a definition). In both [10] and

[54], it was shown that the all Effros-Shen AF algebras (Notation (2.1.82)) arise as quotients up to *-isomorphism of certain primitive ideals of the Boca-Mundici AF algebra, which is the main motivation for our convergence result of this section, Theorem (5.2.21). In both [10] and [54], it was also shown that the center of the Boca-Mundici AF algebra is *-isomorphic to C([0,1]), which provided the framework for C. Eckhardt to introduce a noncommutative analogue to the Gauss map in [21].

We present the construction of this algebra as presented in the paper by F. Boca [10] due to its diagrammatic approach. As in [10], the definition of the Boca-Mundici AF algebra in Definition (5.2.6) begins with the following Relations (5.2.1).

$$\begin{cases} q(n,0) = q(n,2^{n-1}) = 1, \quad p(n,0) = 0, \quad p(n,2^{n-1}) = 1 & n \in \mathbb{N} \setminus \{0\}; \\ q(n+1,2k) = q(n,k), \quad p(n+1,2k) = p(n,k), & n \in \mathbb{N} \setminus \{0\}, \\ & \quad k \in \{0,\ldots,2^{n-1}\}; \\ q(n+1,2k+1) = q(n,k) + q(n,k+1), & n \in \mathbb{N} \setminus \{0\}, \\ & \quad k \in \{0,\ldots,2^{n-1}-1\}; \\ p(n+1,2k+1) = p(n,k) + p(n,k+1), & n \in \mathbb{N} \setminus \{0\}, \\ & \quad k \in \{0,\ldots,2^{n-1}-1\}; \\ r(n,k) = \frac{p(n,k)}{q(n,k)}, & n \in \mathbb{N} \setminus \{0\}, \\ & \quad k \in \{0,\ldots,2^{n-1}-1\}. \end{cases} \tag{5.2.1}$$

We note that the above relations presented here are the same as in [10, Section 1], but instead of starting at $n = 0$, these relations begin at $n = 1$. We now define the finite dimensional algebras which determine the inductive limit $\mathfrak{F}$ that defines the Boca-Mundici AF algebra of Definition (5.2.6).

Definition 5.2.3. *For $n \in \mathbb{N} \setminus \{0\}$, define the finite dimensional C*-algebras:*

$$\mathfrak{F}_n = \bigoplus_{k=0}^{2^{n-1}} \mathfrak{M}(q(n,k)) \text{ and } \mathfrak{F}_0 = \mathbb{C}.$$

Next, we define *-homomorphisms to complete the inductive limit recipe. We utilize partial multiplicity matrices by Theorem (2.1.18).

Definition 5.2.4. *For $n \in \mathbb{N} \setminus \{0\}$, let F_n be the $(2^n+1) \times (2^{n-1}+1)$ matrix with entries in $\{0,1\}$ determined entry-wise by:*

$$(F_n)_{h,j} = \begin{cases} 1 & \text{if } \big(h = 2k+1, k \in \{0, \ldots, 2^{n-1}\}, \wedge j = k+1\big) \\ & \quad \vee \big(h = 2k, k \in \{1, \ldots, 2^{n-1}\} \wedge (j = k \vee j = k+1)\big); \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We would like these matrices to determine unital *-monomorphisms, so that our inductive limit is a unital C*-algebra. By Theorem (2.1.18), this motivates the following Lemma (5.2.5).

Lemma 5.2.5. *Using Definition (5.2.4), if $n \in \mathbb{N} \setminus \{0\}$, then:*

$$F_n \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q\left(n, 2^{n-1}\right) \end{pmatrix} = \begin{pmatrix} q(n+1,0) \\ q(n+1,1) \\ \vdots \\ q\left(n+1, 2^{n}\right) \end{pmatrix}.$$

Proof. Let $n \in \mathbb{N} \setminus \{0\}$. Let $k \in \{1, \dots, 2^{n-1}\}$ and consider $q(n+1, 2k-1)$. Now, by Definition (5.2.4), row $2k-1+1 = 2k$ of F_n has 1 in entry k and $k+1$, and 0 elsewhere. Thus:

$$\begin{aligned} \left((F_n)_{2k,1}, \dots, (F_n)_{2k,2^{n-1}+1}\right) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q\left(n, 2^{n-1}\right) \end{pmatrix} &= q(n, k-1) + q(n, k-1+1) \\ &= q(n+1, 2k-1) \end{aligned}$$

by Relations (5.2.1). Next, let $k \in \{0, \dots, 2^{n-1}\}$ and consider $q(n+1, 2k)$. By Definition (5.2.4), row $2k+1$ of F_n has 1 in entry $k+1$ and 0 elsewhere. Thus:

$$\left((F_n)_{2k+1,1}, \dots, (F_n)_{2k+1,2^{n-1}+1}\right) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q\left(n, 2^{n-1}\right) \end{pmatrix} = q(n, 2k) = q(n+1, 2k)$$

by Relations (5.2.1). Hence, by matrix multiplication, the proof is complete. □

Definition 5.2.6 ([10, 54])**.** *Define $\varphi_0 : \mathfrak{F}_0 \to \mathfrak{F}_1$ by $\varphi_0(a) = a \oplus a$. For $n \in \mathbb{N} \setminus \{0\}$, by Theorem (2.1.18) and Lemma (5.2.5), we let $\varphi_n : \mathfrak{F}_n \to \mathfrak{F}_{n+1}$ be a unital *-monomorphism determined by F_n of Definition (5.2.4). Using Definition (5.2.3),* *we let the unital C*-inductive limit (Definition (2.1.64)):*

$$\mathfrak{F} = \varinjlim (\mathfrak{F}_n, \varphi_n)_{n\in\mathbb{N}}$$

denote the Boca-Mundici AF algebra.

By Proposition (2.1.66), let $\mathfrak{F}^n = \underrightarrow{\varphi}^n(\mathfrak{F}_n)$ *for all* $n \in \mathbb{N}$ *and* $\mathcal{U}_{\mathfrak{F}} = (\mathfrak{F}^n)_{n\in\mathbb{N}}$*, which is a non-decreasing sequence of unital C*-subalgebras of* $\mathfrak{F}$ *such that* $\mathfrak{F} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{F}^n}^{\|\cdot\|_{\mathfrak{F}}}$*, where* $\mathfrak{F}^0 = \mathbb{C}1_{\mathfrak{F}}$*.*

We note that in [10], the AF algebra $\mathfrak{F}$ is constructed by a diagram displayed as [10, Figure 2], so in order to utilize the results of [10], we verify that we have the same diagram up to adding one vertex of label 1 at level $n = 0$ satisfying the conditions at the beginning of [10, Section 1].

Proposition 5.2.7. *The Bratteli diagram of* $\mathfrak{F}$*, denoted* $\mathcal{D}_b(\mathfrak{F}) = \left(V^{\mathcal{D}_b(\mathfrak{F})}, E^{\mathcal{D}_b(\mathfrak{F})}\right)$ *of Definition (2.1.85) satisfies for all* $n \in \mathbb{N}\setminus\{0\}$*:*

(i) $V_n^{\mathcal{D}_b(\mathfrak{F})} = \left\{(n,k) : k \in \{0,\ldots,2^{n-1}\}\right\}$

(ii) $((n,k),(n+1,l)) \in E_n^{\mathcal{D}_b(\mathfrak{F})}$ *if and only if* $|2k-l| \leqslant 1$*. And, there exists only one edge between any two vertices for which there is an edge.*

Proof. Property (i) is clear by Definition (5.2.3). By Definition (2.1.85), an edge exists from (n,s) to $(n+1,t)$ if and only if its associated entry in the partial multiplicity matrix $(F_n)_{t+1,s+1}$ is non-zero.

Now, assume that $|2s-t| \leqslant 1$. Assume $t = 2k+1$ for some $k \in \{0,\ldots,2^{n-1}-1\}$. We thus have $|2s-t| \leqslant 1 \iff k \leqslant s \leqslant k+1 \iff s \in \{k,k+1\}$, since $s \in \mathbb{N}$.

Next, assume that $t = 2k$ for some $k \in \{0,\ldots,2^{n-1}\}$. We thus have $|2s-t| \leqslant 1 \iff -1/2+k \leqslant s \leqslant 1/2+k \iff |s-k| \leqslant 1/2 \iff s = k$ since $s \in \mathbb{N}$. But, considering both t odd and even, these equivalences are equivalent to the conditions for $(F_n)_{t+1,s+1}$ to be non-zero by Definition (5.2.4), which determine the edges of $\mathcal{D}_b(\mathfrak{F})$. Furthermore, since the non-zero entries of F_n are all 1, only one edge exists between vertices for which there is an edge by Definition (2.1.85). $\square$

Next, we describe the ideals of $\mathfrak{F}$, whose quotients are *-isomorphic to the Effros-Shen AF algebras.

Definition 5.2.8 ([10])**.** *Let $\theta \in (0,1) \setminus \mathbb{Q}$. We define the ideal $I_\theta \in \mathrm{Ideal}(\mathfrak{F})$ diagrammatically by the one-to-one correspondence of Proposition (5.1.20).*

By [10, Proposition 4.i], for each $n \in \mathbb{N} \setminus \{0\}$, there exists a unique $j_n(\theta) \in \{0, \ldots, 2^{n-1} - 1\}$ such that $r(n, j_n(\theta)) < \theta < r(n, j_n(\theta) + 1)$ of Relations (5.2.1). The set of vertices of the diagram of the ideal $\mathcal{D}(I_\theta)$ of Definition (5.1.15) is defined by:

$$V^{\mathcal{D}_b(\mathfrak{F})} \setminus (\{(n, j_n(\theta)), (n, j_n(\theta) + 1) : n \in \mathbb{N} \setminus \{0\} \cup \{(0,0)\}\})$$

and we denote this set by $V^{\mathcal{D}(I_\theta)}$. Let $E^{\mathcal{D}(I_\theta)}$ be the set of edges of $\mathcal{D}_b(\mathfrak{F})$, which are between the vertices in $V^{\mathcal{D}(I_\theta)}$ and let $\mathcal{D}(I_\theta) = \left(V^{\mathcal{D}(I_\theta)}, E^{\mathcal{D}(I_\theta)}\right)$. By [10, Proposition 4.i], the diagram $\mathcal{D}(I_\theta) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{F}))$ is an ideal diagram of Definition (5.1.15).

Using Proposition (5.1.20), define:

$$I_\theta = i(\cdot, \mathcal{D}_b(\mathfrak{F}))^{-1} \left(\mathcal{D}(I_\theta)\right) \in \mathrm{Ideal}(\mathfrak{F}).$$

By [10, Proposition 4.i], if $n \in \mathbb{N} \setminus \{0,1\}$ and $1 \leqslant j_n(\theta) \leqslant 2^{n-1} - 2$, then:

$$I_\theta \cap \mathfrak{F}^n = \underrightarrow{\varphi}^n \left(\left(\oplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n,k)) \right) \oplus \{0\} \oplus \{0\} \oplus \left(\oplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n,k)) \right) \right).$$

If $j_n(\theta) = 0$, then:

$$I_\theta \cap \mathfrak{F}^n = \underrightarrow{\varphi}^n \left(\{0\} \oplus \{0\} \oplus \left(\oplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n,k)) \right) \right).$$

If $j_n(\theta) = 2^{n-1} - 1$, then:

$$I_\theta \cap \mathfrak{F}^n = \underrightarrow{\varphi}^n \left(\left(\oplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n,k)) \right) \oplus \{0\} \oplus \{0\} \right),$$

and if $n \in \{0,1\}$, then $I_\theta \cap \mathfrak{F}^n = \{0\}$. We note that $I_\theta \in \mathrm{Prim}(\mathfrak{F})$ by [10, Proposition 4.i].

Before we move on to describing the quantum metric structure of quotients of the ideals of Definition (5.2.8), let's first capture more properties of the structure of the ideals introduced in Definition (5.2.8), which are sufficient for later results.

Lemma 5.2.9. *Using notation from Definition (5.2.8), if $n \in \mathbb{N} \setminus \{0\}, \theta \in (0,1) \setminus \mathbb{Q}$, then $j_{n+1}(\theta) \in \{2j_n(\theta), 2j_n(\theta)+1\}$.*

Proof. We first note that the vertices $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)}$ determine a Bratteli diagram associated to the AF algebra $\mathfrak{F}/I_\theta$, which we will denote $\mathcal{D}_b(\mathfrak{A}/I_\theta)$, as in Definition (2.1.85) by [11, Proposition 3.7] up to shifting vertices, in which the edges for $\mathcal{D}_b(\mathfrak{A}/I_\theta)$ are given by all the edges from $E^{\mathcal{D}_b(\mathfrak{A})}$ between vertices all vertices in $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)}$. Thus, by Defintion (5.2.8), the vertex set for $\mathcal{D}_b(\mathfrak{A}/I_\theta)$ is:

$$V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)} = \left\{(n, j_n(\theta)), (n, j_n(\theta)+1) \in \mathbb{N}^2 : n \in \mathbb{N} \setminus \{0\}\right\} \cup \{(0,0)\}, \quad (5.2.2)$$

and in particular, this vertex set along with the edges between the vertices satisfy axioms (i),(ii), (iii) of Definition (2.1.83).

Consider $n = 1$. Since there are only 3 vertices at level $n = 2$, the conclusion is satisfied since $j_2(\theta), j_2(\theta)+1 \in \{0,1,2\}$ and $j_1(\theta) = 0$ since there are only 2 vertices at level $n = 1$.

Furthemore, note by definition, we have $j_n(\theta) \leqslant 2^{n-1} - 1$ since $j_n(\theta) + 1 \in \{0, \ldots, 2^{n-1}\}$.

Step 1. *For $n \geqslant 2$, we show that $j_{n+1}(\theta) \geqslant 2j_n(\theta)$.*

We note that if $j_n(\theta) = 0$, then clearly $j_{n+1}(\theta) \geqslant 0 = 2j_n(\theta)$. Thus, we may assume that $j_n(\theta) \geqslant 1$. Hence, we may assume by way of contradiciton that $j_{n+1}(\theta) \leqslant 2j_n(\theta) - 1$. Consider $j_n(\theta)+1$. By Expression (5.2.2), the only vertices at level $n+1$ of the diagram of $\mathfrak{F}/I_\theta$ are $(n+1, j_{n+1}(\theta))$ and $(n+1, j_{n+1}(\theta)+1)$. Consider $j_{n+1}(\theta)+1$. Now:

$$|2(j_n(\theta)+1) - (j_{n+1}(\theta)+1)| = |2j_n(\theta) - j_{n+1}(\theta) + 1|.$$

But, by our contradiction assumption, we have $2j_n(\theta) - j_{n+1}(\theta) + 1 \geqslant 2j_n(\theta) + 1 - 2j_n(\theta) + 1 = 2$. Thus, by Proposition (5.2.7), there is no edge from $(n, j_n(\theta)+1)$ to $(n+1, j_{n+1}(\theta)+1)$. Next, consider $j_{n+1}(\theta)$. Similarly, we have $|2(j_n(\theta)+1) - j_{n+1}(\theta)| = |2j_n(\theta) - j_{n+1}(\theta) + 2|$. However, the indices $2j_n(\theta) - j_{n+1}(\theta) + 2 \geqslant 2j_n(\theta) + 1 - 2j_n(\theta) + 2 = 3$. And, again by Proposition (5.2.7), there is no edge from $(n, j_n(\theta)+1)$ to $(n+1, j_{n+1}(\theta))$. But, by Expression (5.2.2), this implies that $(n, j_{n+1}(\theta)+1)$ is a vertex in the quotient diagram $\mathfrak{F}/I_\theta$ in which there does not exist a vertex $(n+1, l)$ in the diagram of $\mathfrak{F}/I_\theta$ such that $((n, j_{n+1}(\theta)+1), (n+1, l))$ is an edge in the diagram of $\mathfrak{F}/I_\theta$, which is a contradiction since the quotient diagram is a Bratteli diagram that would not satisfy axiom (ii) of Definition (2.1.83). Therefore, we conclude $j_{n+1}(\theta) \geqslant 2j_n(\theta)$.

Step 2. *For $n \geqslant 2$, we show that $j_{n+1}(\theta) \leqslant 2j_n(\theta) + 1$.*

Now, if $j_n(\theta) = 2^{n-1} - 1$, then $j_{n+1}(\theta) + 1 \leqslant 2^n = 2(2^{n-1} - 1) + 2$ and thus $j_{n+1}(\theta) \leqslant 2(2^{n-1} - 1) + 1 = 2j_n(\theta) + 1$ and we would be done. Thus, we may assume that $j_n(\theta) \leqslant 2^{n-1} - 2$ and we note that this can only occur in the case that $n \geqslant 3$, which implies that the case of $n = 2$ is complete. Thus, we may assume by way of contradiction that $j_{n+1}(\theta) \geqslant 2j_n(\theta) + 2$. Consider $j_n(\theta)$. As in Step 1, we provide a contradiction via a diagram approach. Consider $j_{n+1}(\theta)+1$. Now, we have $|2j_n(\theta) - (j_{n+1}(\theta)+1)| = |2j_n(\theta) - j_{n+1}(\theta) - 1|$. But, by our contradiction assumption, we gather that $2j_n(\theta) - j_{n+1}(\theta) - 1 \leqslant 2j_n(\theta) - 2j_n(\theta) - 2 - 1 = -3$ and $|2j_n(\theta) - (j_{n+1}(\theta) + 1)| \geqslant 3$. Thus, by Proposition (5.2.7), there is no edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta) + 1)$. Next, consider $j_{n+1}(\theta)$. Similarly, we have $2j_n(\theta) - j_{n+1}(\theta) \leqslant 2j_n(\theta) - 2j_n(\theta) - 2 = -2$ and $|2j_n(\theta) - j_{n+1}(\theta)| \geqslant 2$. Thus, by Proposition (5.2.7), there is no edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta))$. Thus, by Expression (5.2.2) and the same diagram argument of Step 1, we have reached a contradiction. Hence, $j_{n+1}(\theta) \leqslant 2j_n(\theta) + 1$.

Thus, combining Step 1 and Step 2, the proof is complete. $\square$

Next, on the subspace of ideals of Definition (5.2.8), we provide a useful topological result about the metric on ideals of Proposition (5.1.7), in which the equivalence of *1.* and *3.* is a consequence of [10, Corollary 12], which is unique to Boca's work on the AF algebra, $\mathfrak{F}$.

Proposition 5.2.10. *If $(\theta_n)_{n\in\overline{\mathbb{N}}} \subseteq (0,1) \setminus \mathbb{Q}$, then using notation from Definition (5.2.6) and Definition (5.2.8), the following are equivalent:*

1. *$(\theta_n)_{n\in\mathbb{N}}$ converges to θ_∞ with respect to the usual topology on $\mathbb{R}$;*
2. *$(\mathsf{cf}(\theta_n))_{n\in\mathbb{N}}$ converges to $\mathsf{cf}(\theta_\infty)$ with respect to the Baire space, $\mathscr{N}$ and its metric from Definition (4.1.4), where cf denotes the bijection determined by the unique continued fraction expansion of an irrational;*
3. *$(I_{\theta_n})_{n\in\mathbb{N}}$ converges to I_{θ_∞} with respect to the Jacobson topology (Definition (2.1.52)) on $\mathrm{Prim}(\mathfrak{F})$;*
4. *$(I_{\theta_n})_{n\in\mathbb{N}}$ converges to I_{θ_∞} with respect to the metric topology of $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ of Proposition (5.1.7) or the Fell topology of Definition (2.1.58).*

Proof. The equivalence between *1.* and *2.* is a classic result, in which a proof can be found in [3, Proposition 5.10]. The equivalence between *1.* and *3.* is immediate from [10, Corollary 12]. And, therefore, *2.* is equivalent to *3.*. Thus, it remains to prove that *3.* is equivalent to *4.*

4. implies *3.* is an immediate consequence of Corollary (5.1.23) as the Fell topology is stronger. Hence, assume *3.*, then since we have already established *3.* implies *2.*, we may assume *2.* to prove *4.*. For each $n \in \overline{\mathbb{N}}$, let $\mathsf{cf}(\theta_n) = [a_j^n]_{j \in \mathbb{N}}$. By assumption, the coordinates $a_0^n = 0$ for all $n \in \overline{\mathbb{N}}$. Now, assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $a_j^n = a_j^\infty$ for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, N\}$. Assume without loss of generality, assume that N is odd. Thus, using [10, Figure 5], we have that:

$$L_{a_1^n - 1} \circ R_{a_2^n} \circ \cdots \circ L_{a_N^n} = L_{a_1^\infty - 1} \circ R_{a_2^\infty} \circ \cdots \circ L_{a_N^\infty} \tag{5.2.3}$$

for all $n \in \mathbb{N}$. But, Equation (5.2.3) determines the verties for the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ for all $n \in \overline{\mathbb{N}}$ by [10, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [10, Figure 5] in arXiv v6). But, the vertices of the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ are simply the complement of the vertices of the diagram of I_{θ_n} by [19, Theorem III.4.4]. Now, primitive ideals must have the same vertices at level 0 of the diagram since they cannot equal $\mathfrak{A}$ by Definition (2.1.52) and are thus non-unital. But, for any $\eta \in (0,1) \setminus \mathbb{Q}$, the ideals I_η must always have the same vertices at level 1 of the diagram as well since the only two vertices are $(1,0), (1,1)$ and $r(1,0) = 0 < \theta < 1 = r(1,1)$ by Relations (5.2.1) for all $\theta \in (0,1) \setminus \mathbb{Q}$. Thus, Equation (5.2.3) and the isometry of Theorem (5.1.21), we gather that $I_{\theta_n} \cap \mathfrak{F}^j = I_{\theta_\infty} \cap \mathfrak{F}^j$ for all $n \in \mathbb{N}$ and:

$$j \in \left\{0, \ldots, \max\left\{1, a_1^\infty - 1 + \left(\sum_{k=2}^{N} a_k^N\right)\right\}\right\},$$

where $\max\left\{1, a_1^\infty - 1 + \left(\sum_{k=2}^{N} a_k^N\right)\right\} \geqslant N$ as the terms of the continued fraction expansion are all positive integers for terms after the first term. Thus, by the definition of the metric on the Baire Space and the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$, we conclude that convergence in the the Baire space metric of $(\mathsf{cf}(\theta_n))_{n\in\mathbb{N}}$ to $\mathsf{cf}(\theta_\infty)$ implies convergence of $(I_{\theta_n})_{n\in\mathbb{N}}$ to I_{θ_∞} with respect to the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ or the Fell topology by Theorem (5.1.21). $\square$

The next result follows from Proposition (5.2.10) and the proof of [10, Proposition 4.i]. For $\theta \in (0,1) \setminus \mathbb{Q}$, the idea of the proof of Proposition (5.2.11) is to show that the ideals I_θ with their unique diagram capture the standard rational approximations of θ (see Example (2.1.81)) in a suitable manner.

Proposition 5.2.11. *The map:*

$$\theta \in (0,1) \setminus \mathbb{Q} \longmapsto I_\theta \in \operatorname{Prim}(\mathfrak{A})$$

is a homeomorphism onto its image when $(0,1) \setminus \mathbb{Q}$ is equipped with the topology induced by the usual topology on $\mathbb{R}$ and $\operatorname{Prim}(\mathfrak{A})$ is equipped with either the Jacobson topology, Fell topology, or the metric topology of $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ of Proposition (5.1.7).

Proof. By Proposition (5.2.10), the fact that the Jacobson topology of a separable C*-algebra is second countable (see [57, Corollary 4.3.4]), and the Fell topology of an AF algebra is metrizable (see Theorem (5.1.21)), we only need to to verify that the map defined in this proposition is a bijection onto its image.

Claim 5.2.12. *If $\theta \in (0,1) \setminus \mathbb{Q}$, then:*

$$\lim_{n\to\infty} r(n, j_n(\theta)) = \theta,$$

where for all $n \in \mathbb{N} \setminus \{0\}$, the quantity $r(n, j_n(\theta))$ is defined in Relations (5.2.1) and Definition (5.2.8).

Proof of claim. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Let $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ denote the standard rational approximations of θ that converge to θ from Example (2.1.81). Now, by the proof of

[10, Proposition 4.i], there exists an increasing sequence $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N} \setminus \{0\}$ such that:

$$(r(k_n, j_{k_n}(\theta)), r(k_n, j_{k_n}(\theta)+1)) \in \left\{ \left(\frac{p_n^\theta}{q_n^\theta}, \frac{p_{n-1}^\theta}{q_{n-1}^\theta} \right), \left(\frac{p_{n-1}^\theta}{q_{n-1}^\theta}, \frac{p_n^\theta}{q_n^\theta} \right) \right\} \text{ for all } n \in \mathbb{N}\setminus\{0\}. \tag{5.2.4}$$

Next, fix $n \in \mathbb{N} \setminus \{0\}$. Consider $r(n, j_n(\theta))$. By Lemma (5.2.9), first assume that $j_{n+1}(\theta) = 2j_n(\theta)$. Then, we have:

$$r(n+1, j_{n+1}(\theta)) = \frac{p(n+1, 2j_n(\theta))}{q(n+1, 2j_n(\theta))} = r(n, j_n(\theta))$$

by Relations (5.2.1). Also, we have:

$$\begin{aligned} r(n+1, j_{n+1}(\theta)+1) &= \frac{p(n+1, 2j_n(\theta)+1)}{q(n+1, 2j_n(\theta)+1)} \\ &= \frac{p(n, j_n(\theta)) + p(n, j_n(\theta)+1)}{p(n, j_n(\theta)) + p(n, j_n(\theta)+1)} \\ &\leqslant r(n, j_n(\theta)+1) \end{aligned}$$

by Relations (5.2.1) and the fact that $p(n, j_n(\theta)+1)q(n, j_n(\theta)) - p(n, j_n(\theta))q(n, j_n(\theta)+1) = 1 > 0$ from [10, Section 1]. For the case $j_{n+1}(\theta) = 2j_n(\theta)+1$, a similar argument shows that $r(n+1, j_{n+1}(\theta)) \geqslant r(n, j_n(\theta))$ and $r(n+1, j_{n+1}(\theta)+1) = r(n, j_n(\theta)+1)$. Hence, for all $n \in \mathbb{N} \setminus \{0\}$, we gather that:

$$r(n+1, j_{n+1}(\theta)+1) - r(n+1, j_{n+1}(\theta)) \leqslant r(n, j_n(\theta)+1) - r(n, j_n(\theta)). \tag{5.2.5}$$

Let $n \in \mathbb{N} \setminus \{0\}$ such that $n \geqslant k_1$. Now, let $N_n = \max\{k_m : k_m \leqslant n\}$. Note that since $(k_n)_{n\in\mathbb{N}}$ is increasing, we have that $\lim_{n\to\infty} N_n = \infty$. Now, fix $n \in \mathbb{N} \setminus \{0\}$, combining Expression (5.2.4) and (5.2.5), we have:

$$0 < \theta - r(n, j_n(\theta)) < r(n, j_n(\theta)+1) - r(n, j_n(\theta))$$

$$\begin{aligned} &\leqslant r(N_n, j_{N_n}(\theta)+1) - r(N_n, j_{N_n}(\theta)) \\ &= \left| \frac{p^\theta_{N_n}}{q^\theta_{N_n}} - \frac{p^\theta_{N_n-1}}{q^\theta_{N_n-1}} \right|, \end{aligned}$$

and therefore $\lim_{n\to\infty} r(n, j_n(\theta)) = \theta$ since $\lim_{n\to\infty} \frac{p^\theta_n}{q^\theta_n} = \theta$. □

Next, let $\theta, \eta \in (0,1) \setminus \mathbb{Q}$. Assume that $I_\theta = I_\eta$ and thus their diagrams agree as well as their complementary diagrams. Hence, we have that $j_n(\theta) = j_n(\eta)$ for all $n \in \mathbb{N}$, and thus, we have that $r(n, j_n(\theta)) = r(n, j_n(\eta))$ for all $n \in \mathbb{N} \setminus \{0\}$. Therefore, by the claim:

$$\theta = \lim_{n\to\infty} r(n, j_n(\theta)) = \lim_{n\to\infty} r(n, j_n(\eta)) = \eta,$$

which completes the proof. □

Remark 5.2.13. *An immediate consequence of Proposition (5.2.11) is that if:* $(0,1) \setminus \mathbb{Q}$ *is equipped with its relative topology from the usual topology on* $\mathbb{R}$*, the set* $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ *is equipped with its relative topology induced by the Jacobson topology, and the set* $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ *is equipped with its relative topology induced by the metric topology of* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *of Definition (5.1.7) or the Fell topology of Definition (2.1.58), then all these spaces are homeomorphic to the Baire space* $\mathscr{N}$ *with its metric topology from Definition (4.1.4). In particular, from Corollary (5.1.23), the totally bounded metric* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *topology on the set of ideals* $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ *is homeomorphic to* $(0,1) \setminus \mathbb{Q}$ *with its totally bounded metric topology inherited from the usual topology on* $\mathbb{R}$*. Hence, in some sense, the metric* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *topology shares more metric information with* $(0,1) \setminus \mathbb{Q}$ *and its metric than the Baire space metric topology as the Baire space is not totally bounded [3, Theorem 6.5]. This can also be displayed in metric calculations as well.*

Indeed, consider $\theta, \mu \in (0,1) \setminus \mathbb{Q}$ with continued fraction expansions $\theta = [a_j]_{j\in\mathbb{N}}$ and $\mu = [b_j]_{j\in\mathbb{N}}$, in which $a_0 = 0, a_1 = 1000, a_j = 1 \forall j \geqslant 2$ and $b_0, b_1 = 1, b_j = 1 \forall j \geqslant 2$, and thus $\theta \approx 0.001, \mu \approx 0.618, |\theta - \mu| \approx 0.617$. In the Baire metric $\mathsf{d}(\mathsf{cf}(\theta), \mathsf{cf}(\mu)) = 0.5$, and, in the ideal metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}(I_\theta, I_\mu) = 0.25$ by Theorem (5.1.21) since at level $n = 1$ the diagram for $\mathfrak{F}/I_\theta$ begins with L_{999} and for $\mathfrak{F}/I_\mu$ begins with R_{b_2} by [10, Proposition 4.i], so the ideal diagrams differ first at $n = 2$. Now, assume that for μ we have instead $b_1 = 999, b_j = 1 \forall j \geqslant 2$, and thus $|\theta - \mu| \approx 0.000000998$, but in the Baire metric, we still have that $\mathsf{d}(\mathsf{cf}(\theta), \mathsf{cf}(\mu)) = 0.5$, while $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}(I_\theta, I_\mu) = 2^{-1000}$ by Theorem (5.1.21) since at level $n = 1$ the diagram for $\mathfrak{F}/I_\theta$ begins with L_{999} and for $\mathfrak{F}/I_\mu$ begins with L_{998} and then transitions to R_{b_2} by [10, Proposition 4.i], so the ideal diagrams differ first at $n = 1000$. In conclusion, in this example, the absolute value metric $|\cdot|$ behaves much more like the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ than the Baire metric.

Fix $\theta \in (0,1) \setminus \mathbb{Q}$, we present a *-isomorphism from $\mathfrak{F}/I_\theta$ to the Effros-Shen algebra $\mathfrak{AF}_\theta$ of Notation (2.1.82) as a proposition to highlight a useful property for our purposes. Of course, [10, Proposition 4.i] already established that $\mathfrak{F}/I_\theta$ and $\mathfrak{AF}_\theta$ are *-isomorphic, but here we simply provide an explicit detail of such a *-isomorphism, which will serve us in the results pertaining to tracial states in Lemma (5.2.20).

Proposition 5.2.14. *If $\theta \in (0,1) \setminus \mathbb{Q}$ with continued fraction expansion $\theta = [a_j]_{j\in\mathbb{N}}$ as in Expression (2.1.11), then using Notation (2.1.82) and Definition (5.2.8), there exists a *-isomorphism $\mathfrak{af}_\theta : \mathfrak{F}/I_\theta \to \mathfrak{AF}_\theta$ such that if $x = x_0 \oplus \cdots \oplus x_{2^{a_1 - 1}} \in \mathfrak{F}_{a_1}$, then:*

$$\mathfrak{af}_\theta \left(\underrightarrow{\varphi}^{a_1}(x) + I_\theta \right) = \underrightarrow{\alpha_\theta}^1 \left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} \right) \in \underrightarrow{\alpha_\theta}^1 \left(\mathfrak{AF}_{\theta,1} \right).$$

Proof. By [10, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [10, Figure 5] in arXiv v6), the Bratteli diagram of $\mathfrak{F}/I_\theta$ begins with the diagram $L_{a_1 - 1}$ of [10, Figure 5] at level $n = 1$. Now, the diagram $C_{a_1 - 1} \circ C_{a_2}$ of [10, Figure 6] is a section of the diagram of Example (2.1.87), in which the left column of $C_{a_1-1} \circ C_{a_2}$ is the bottom row of the first two levels from left to right after level $n=0$ of Example (2.1.87). Therefore, by the placement of $\circledast$ at level a_1 in [10, Figure 6], define a map $f : (\mathfrak{F}^{a_1} + I_\theta)/I_\theta \to \underset{\longrightarrow}{\alpha_\theta^1}\left(\mathfrak{A}\mathfrak{F}_{\theta,1}\right)$ by:

$$f : \underset{\longrightarrow}{\varphi^{a_1}}(x) + I_\theta \longmapsto \underset{\longrightarrow}{\alpha_\theta^1}\left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)}\right),$$

where $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. We show that f is a *-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ onto $\underset{\longrightarrow}{\alpha_\theta^1}(\mathfrak{A}\mathfrak{F}_{\theta,1})$.

We first show that f is well-defined. Let $c, e \in (\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ such that $c = e$. Now, we have $c = \underset{\longrightarrow}{\varphi^{a_1}}(c') + I_\theta, e = \underset{\longrightarrow}{\varphi^{a_1}}(e') + I_\theta$ where $c' = c'_0 \oplus \cdots \oplus c'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $e' = e'_0 \oplus \cdots \oplus e'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. But, the assumption $c = e$ implies that $\underset{\longrightarrow}{\varphi^{a_1}}(c' - e') \in I_\theta \cap \mathfrak{F}^{a_1}$. Thus, by Definition (5.2.8) of I_θ, we have that $c'_{j_{a_1}(\theta)+1} \oplus c'_{j_{a_1}(\theta)} = e'_{j_{a_1}(\theta)+1} \oplus e'_{j_{a_1}(\theta)}$, and since $j_{a_1}(\theta) = q_0^\theta$ and $j_{a_1}(\theta) + 1 = q_1^\theta$ by [10, Proposition 4.i] and the discussion at the start of the proof, we gather that f is a well-defined *-homomorphism since the canonical maps $\underset{\longrightarrow}{\alpha_\theta^1}$ and $\underset{\longrightarrow}{\varphi^{a_1}}$ are *-homomorphisms.

For surjectivity of f, let $x = \underset{\longrightarrow}{\alpha_\theta^1}\left(x_{q_1^\theta} \oplus x_{q_0^\theta}\right)$, where $x_{q_1^\theta} \oplus x_{q_0^\theta} \in \mathfrak{A}\mathfrak{F}_{\theta,1}$. Define $y = y_0 \oplus \cdots y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $y_{j_{a_1}(\theta)} = x_{q_0^\theta}$ and $y_{j_{a_1}(\theta)+1} = x_{q_1^\theta}$ with $y_k = 0$ for all $k \in \{0, \ldots, 2^{a_1-1}\} \setminus \{j_{a_1}(\theta), j_{a_1}(\theta)+1\}$. Hence, the image $f\left(\underset{\longrightarrow}{\varphi^{a_1}}(y) + I_\theta\right) = x$.

For injectivity of f, let $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $y = y_0 \oplus \cdots \oplus y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $f\left(\underset{\longrightarrow}{\varphi^{a_1}}(x) + I_\theta\right) = f\left(\underset{\longrightarrow}{\varphi^{a_1}}(y) + I_\theta\right)$. Thus, since $\underset{\longrightarrow}{\alpha_\theta^1}$ is injective, we have that $x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} = y_{j_{a_1}(\theta)+1} \oplus y_{j_{a_1}(\theta)}$. But, this then implies that $\underset{\longrightarrow}{\varphi^{a_1}}(x - y) \in I_\theta \cap \mathfrak{F}^{a_1} \subseteq I_\theta$ by Definition (5.2.8), and therefore, the terms $\underset{\longrightarrow}{\varphi^{a_1}}(x) + I_\theta = \underset{\longrightarrow}{\varphi^{a_1}}(y) + I_\theta$, which completes the argument that f is a *-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ onto $\underset{\longrightarrow}{\alpha_\theta^1}(\mathfrak{A}\mathfrak{F}_{\theta,1})$.

Lastly, using Definition (2.1.85), consider the Bratteli diagram of $\mathfrak{F}/I_\theta$ given by the sequence of unital C*-subalgebras $((\mathfrak{F}^{x_{j+1}} + I_\theta)/I_\theta)_{j \in \mathbb{N}}$, where $x_{j+1} = \sum_{k=1}^{j+1} a_k$ for all $j \in \mathbb{N}$. With respect to this diagram, the proof of [10, Proposition 4.i] and [10, Figure 6] provide that this diagram of $\mathfrak{F}/I_\theta$ is equivalent to the Bratteli diagram of $\mathfrak{AF}_\theta$ beginning at $\mathfrak{AF}_{\theta,1}$ given by Example (2.1.87), where this equivalence of Bratteli diagrams is given by [8, Section 23.3 and Theorem 23.3.7]. Therefore, combining the equivalence relation of [8, Section 23.3 and Theorem 23.3.7] and Theorem (2.1.88), we conclude that there exists a *-isomorphism $\mathfrak{af}_\theta : \mathfrak{F}/I_\theta \to \mathfrak{AF}_\theta$ such that $\mathfrak{af}_\theta(z) = f(z)$ for all $z \in (\mathfrak{F}^{a_1} + I_\theta)/I_\theta$, which completes the proof. $\square$

From the *-isomorphism of Proposition (5.2.14), we may provide a faithful tracial state for the quotient $\mathfrak{F}/I_\theta$ from the unique faithful tracial state of $\mathfrak{AF}_\theta$. Indeed:

Notation 5.2.15. *Fix $\theta \in (0,1) \setminus \mathbb{Q}$. There is a unique faithful tracial state on $\mathfrak{AF}_\theta$ denoted σ_θ of Theorem (4.2.1) and Lemma (4.2.3). Thus,*

$$\tau_\theta = \sigma_\theta \circ \mathfrak{af}_\theta$$

is a unique faithful tracial state on $\mathfrak{F}/I_\theta$ with $\mathfrak{af}_\theta$ from Proposition (5.2.14).

Let $Q_\theta : \mathfrak{F} \to \mathfrak{F}/I_\theta$ denote the quotient map. Thus, by [18, Theorem V.2.2], there exists a unique linear functional on $\mathfrak{F}$ denoted, ρ_θ, such that $\ker \rho_\theta \supseteq I_\theta$ and $\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$ for all $x \in \mathfrak{F}$. Since τ_θ is a tracial state and:

$$\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$$

for all $x \in \mathfrak{F}$, we conclude that ρ_θ is also a tracial state that vanishes on I_θ. Furthermore, ρ_θ is faithful on $\mathfrak{F} \setminus I_\theta$ since τ_θ is faithful on $\mathfrak{F}/I_\theta$.

By Theorem (3.1.5), One more ingredient remains before we define the quantum metric structure for the quotient spaces $\mathfrak{F}/I_\theta$.

Lemma 5.2.16. *Let $\theta \in (0,1) \setminus \mathbb{Q}$. Using notation from Definition (5.2.6) and Definition (5.2.8), if we define:*

$$\beta^\theta : n \in \mathbb{N} \longmapsto \frac{1}{\dim((\mathfrak{F}^n + I_\theta)/I_\theta)} \in (0, \infty),$$

then $\beta^\theta(n) = \frac{1}{q(n,j_n(\theta))^2+q(n,j_n(\theta)+1)^2} \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N} \setminus \{0\}$ and $\beta^\theta(0) = 1$.

Proof. First, the quotient $(\mathfrak{F}^0 + I_\theta)/I_\theta = \mathbb{C}1_{\mathfrak{F}/I_\theta}$. Hence, the term $\beta^\theta(0) = 1$.

Fix $n \in \mathbb{N} \setminus \{0\}$. Since $(\mathfrak{F}^n + I_\theta)/I_\theta$ is *-isomorphic to $\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)$ (see Proposition (5.1.12)), we have that $\dim((\mathfrak{F}^n + I_\theta)/I_\theta) = \dim(\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)) = q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2$ by Definition (5.2.8) and the dimension of the quotient is the difference in dimensions of $\mathfrak{F}^n$ and $I_\theta \cap \mathfrak{F}^n$. Therefore, the term $\beta^\theta(n) = \frac{1}{q(n,j_n(\theta))^2+q(n,j_n(\theta)+1)^2}$.

Next, we claim that for all $n \in \mathbb{N}\setminus\{0\}$, we have $q(n, j_n(\theta)) \geqslant n$ or $q(n, j_n(\theta)+1) \geqslant n$. We proceed by induction. If $n = 1$, then $q(1, j_1(\theta)) = 1$ and $q(1, j_1(\theta) + 1) = 1$ by Relations (5.2.1). Next assume the statement of the claim is true for $n = m$. Thus, we have that $q(m, j_m(\theta)) \geqslant m$ or $q(m, j_m(\theta) + 1) \geqslant m$. First, assume that $q(m, j_m(\theta)) \geqslant m$. By Lemma (5.2.9), assume that $j_{m+1}(\theta) = 2j_m(\theta)$. Thus, we gather $q(m+1, j_{m+1}(\theta)+1) = q(m+1, 2j_m(\theta)+1) = q(m, j_m(\theta))+q(m, j_m(\theta)+1) \geqslant m + 1$ by Relations (5.2.1) and since $q(m, j_m(\theta) + 1) \in \mathbb{N} \setminus \{0\}$. The case when $j_{m+1}(\theta) = 2j_m(\theta)+1$ follows similarly as well as the case when $q(m, j_m(\theta)+1) \geqslant m$, which completes the induction argument.

In particular, for all $n \in \mathbb{N} \setminus \{0\}$, we have $q(n, j_n(\theta)) \geqslant n$ or $q(n, j_n(\theta) + 1) \geqslant n$, which implies that $q(n, j_n(\theta))^2 \geqslant n^2$ or $q(n, j_n(\theta) + 1)^2 \geqslant n^2$. And thus, the term:

$$\frac{1}{q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2} \leqslant \frac{1}{n^2}$$

for all $n \in \mathbb{N} \setminus \{0\}$. $\square$

Hence, we have all the ingredients to define the quotient quantum metric spaces associated to the ideals of Definition (5.2.8).

Notation 5.2.17. *Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Using Definition (5.2.6), Definition (5.2.8), Notation (5.2.15), and Lemma (5.2.16), let:*

$$\left(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}/I_\theta,\tau_\theta}}\right)$$

denote the $(2,0)$-quasi-Leibniz quantum compact metric space given by Theorem (5.2.1) associated to the ideal I_θ, faithful tracial state τ_θ, and $\beta^\theta : \mathbb{N} \to (0,\infty)$ having limit 0 at infinity by Lemma (5.2.16).

Remark 5.2.18. *Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Although $\mathfrak{F}/I_\theta$ and $\mathfrak{AF}_\theta$ are *-isomorphic, it is unlikely that $\left(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}/I_\theta,\tau_\theta}}\right)$ is quantum isometric to $\left(\mathfrak{AF}_\theta, \mathsf{L}^{\beta_\theta}_{\mathcal{I}_\theta,\sigma_\theta}\right)$ of Theorem (4.2.12) based on the Lip-norm constructions. Thus, one could not simply apply Proposition (5.2.10) to Theorem (4.2.12) to achieve our main result of this section, Theorem (5.2.21).*

In order to provide our continuity results via Theorem (4.5.6), we describe the faithful tracial states on the quotients in sufficient detail through Lemma (5.2.19) and Lemma (5.2.20).

Lemma 5.2.19. *Fix $\theta \in (0,1)\setminus\mathbb{Q}$. Let tr_d be the unique tracial state of $\mathfrak{M}(d)$. Using notation from Definitions (5.2.6, 5.2.8), if $n \in \mathbb{N}\setminus\{0\}$ and $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$, then using Notation (5.2.15):*

$$\rho_\theta \circ \underrightarrow{\varphi}^n(a) = c(n,\theta)\mathsf{tr}_{q(n,j_n(\theta))}\left(a_{j_n(\theta)}\right) + (1-c(n,\theta))\mathsf{tr}_{q(n,j_n(\theta)+1)}\left(a_{j_n(\theta)+1}\right),$$

where $c(n,\theta) \in (0,1)$ and $\rho_\theta \circ \underrightarrow{\varphi}^0(a) = a$ for all $a \in \mathfrak{F}_0$.

Furthermore, let $n \in \mathbb{N} \setminus \{0\}$, then:

$$c(n+1,\theta) = \begin{cases} \frac{(q(n,j_n(\theta))+q(n,j_n(\theta)+1))c(n,\theta)-q(n,j_n(\theta))}{q(n,j_n(\theta)+1)} & \text{if } j_{n+1}(\theta) = 2j_n(\theta) \\ \\ \left(1+\frac{q(n,j_n(\theta)+1)}{q(n,j_n(\theta))}\right)c(n,\theta) & \text{if } j_{n+1}(\theta) = 2j_n(\theta)+1 \end{cases}.$$

Proof. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. If $n = 0$, then $\rho_\theta \circ \underrightarrow{\varphi}^0(a) = a$ for all $a \in \mathfrak{F}_0$ since $\mathfrak{F}_0 = \mathbb{C}$. Let $n \in \mathbb{N} \setminus \{0\}$ and $a = a_0 \oplus \cdots \oplus a_{2^{n-1}} \in \mathfrak{F}_n$. Now, ρ_θ is a tracial state on $\mathfrak{F}$, and thus, the composition $\rho_\theta \circ \underrightarrow{\varphi}^n$ is a tracial state on $\mathfrak{F}_n$. Hence, by [19, Example IV.5.4]:

$$\rho_\theta \circ \underrightarrow{\varphi}^n(a) = \sum_{k=0}^{2^{n-1}} c_k \mathsf{tr}_{q(n,k)}(a_k),$$

where $\sum_{k=0}^{2^{n-1}} c_k = 1$ and $c_k \in [0,1]$ for all $k \in \{0,\ldots,2^{n-1}\}$. But, since ρ_θ vanishes on I_θ by definition of ρ_θ in Notation (5.2.15), we conclude that $c_k = 0$ for all $k \in \{0,\ldots,2^{n-1}\} \setminus \{j_n(\theta), j_n(\theta)+1\}$. Also, the fact that ρ_θ is faithful on $\mathfrak{F} \setminus I_\theta$ implies that $c_{j_n(\theta)}, c_{j_n(\theta)+1} \in (0,1)$ and $c_{j_n(\theta)} + c_{j_n(\theta)+1} = 1$. Define $c(n,\theta) = c_{j_n(\theta)}$ and clearly $c_{j_n(\theta)+1} = 1 - c(n,\theta)$.

Next, let $n \in \mathbb{N} \setminus \{0\}$ and let $j_{n+1}(\theta) = 2j_n(\theta)$. Combining Lemma (5.2.9) and Proposition (5.2.7), there is one edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta))$ and one edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta)+1)$ with no other edges from $(n, j_n(\theta))$ to either $(n, j_n(\theta))$ or $(n+1, j_{n+1}(\theta)+1)$. Also, there is one edge from $(n, j_n(\theta)+1)$ to $(n+1, j_{n+1}(\theta)+1)$ with no other edges from $(n, j_n(\theta)+1)$ to either $(n, j_n(\theta))$ or $(n+1, j_{n+1}(\theta)+1)$.

Hence, consider an element $a = a_0 \oplus \cdots \oplus a_{2^{n-1}} \in \mathfrak{F}_n$ such that $a_k = 0$ for all $k \in \{0,\ldots,2^{n-1}\} \setminus \{j_n(\theta), j_n(\theta)+1\}$. Since the edges determine the partial multiplicities of φ_n, we have that $\varphi_n(a) = b_0 \oplus \cdots \oplus b_{2^n}$ such that:

$$b_{j_{n+1}(\theta)} = Ua_{j_n(\theta)}U^* \text{ and } b_{j_{n+1}(\theta)+1} = V \begin{pmatrix} a_{j_n(\theta)} & \\ & a_{j_n(\theta)+1} \end{pmatrix} V^*, \tag{5.2.6}$$

where $U \in \mathfrak{M}(q(n+1, j_{n+1}(\theta))), V \in \mathfrak{M}(q(n+1, j_{n+1}(\theta)+1))$ are unitary by Theorem (2.1.18). Also, the terms $b_k = 0$ for all $k \in \{0, \ldots, 2^{n-1}\} \setminus \{j_{n+1}(\theta), j_{n+1}(\theta)+1\}$. But, by Proposition (2.1.66), we have that $\underrightarrow{\varphi^n}(a) = \underrightarrow{\varphi^{n+1}}(\varphi_n(a))$.

Now, assume that $a_{j_n(\theta)} = 1_{\mathfrak{M}(q(n,j_n(\theta)))}$ and $a_{j_n(\theta)+1} = 0_{\mathfrak{M}(q(n,j_n(\theta)+1))}$. Therefore, by Expression (5.2.6):

$$\begin{aligned}
c(n,\theta) &= \rho_\theta \circ \underrightarrow{\varphi^n}(a) \\
&= \rho_\theta \circ \underrightarrow{\varphi^{n+1}}(\varphi_n(a)) \\
&= c(n+1,\theta)\mathsf{tr}_{q(n+1,j_{n+1}(\theta))}\left(Ua_{j_n(\theta)}U^*\right) \\
&\quad + (1-c(n+1,\theta))\mathsf{tr}_{q(n+1,j_{n+1}(\theta)+1)}\left(V \begin{pmatrix} a_{j_n(\theta)} & \\ & 0_{\mathfrak{M}(q(n,j_n(\theta)+1))} \end{pmatrix} V^*\right) \\
&= c(n+1,\theta)\cdot 1 \\
&\quad + (1-c(n+1,\theta))\mathsf{tr}_{q(n+1,j_{n+1}(\theta)+1)} \begin{pmatrix} 1_{\mathfrak{M}(q(n,j_n(\theta)))} & \\ & 0_{\mathfrak{M}(q(n,j_n(\theta)+1))} \end{pmatrix} \\
&= c(n+1,\theta) + (1-c(n+1,\theta))\frac{1}{q(n+1,j_{n+1}(\theta)+1)}q(n,j_n(\theta)).
\end{aligned} \tag{5.2.7}$$

Thus, since $q(n+1, 2j_n(\theta)+1) = q(n, j_n(\theta)) + q(n, j_n(\theta)+1)$ from Relations (5.2.1) and $j_{n+1}(\theta)+1 = 2j_n(\theta)+1$, we conclude that:

$$c(n+1,\theta) = \frac{(q(n,j_n(\theta)) + q(n,j_n(\theta)+1))c(n,\theta) - q(n,j_n(\theta))}{q(n,j_n(\theta)+1)}.$$

Lastly, assume that $j_{n+1}(\theta) = 2j_n(\theta) + 1$. Let $a = a_0 \oplus \cdots \oplus a_{2^{n-1}} \in \mathfrak{F}_n$ such that $a_k = 0$ for all $k \in \{0, \ldots, 2^{n-1}\} \setminus \{j_n(\theta), j_n(\theta) + 1\}$. A similiar argument shows that $\varphi_n(a) = b_0 \oplus \cdots \oplus b_{2^n}$ such that:

$$b_{j_{n+1}(\theta)} = Y \begin{pmatrix} a_{j_n(\theta)} & \\ & a_{j_n(\theta)+1} \end{pmatrix} Y^* \text{ and } b_{j_{n+1}(\theta)+1} = Z a_{j_n(\theta)+1} Z^*,$$

where $Y \in \mathfrak{M}(q(n+1, j_{n+1}(\theta))), Z \in \mathfrak{M}(q(n+1, j_{n+1}(\theta)+1))$ are unitary. Now, assume that $a_{j_n(\theta)} = 1_{\mathfrak{M}(q(n,j_n(\theta)))}$ and $a_{j_n(\theta)+1} = 0_{\mathfrak{M}(q(n,j_n(\theta)+1))}$. Therefore, similarly to Expression (5.2.7), we gather that:

$$c(n, \theta) = c(n+1, \theta) \frac{1}{q(n+1, j_{n+1}(\theta))} q(n, j_n(\theta)),$$

and therefore:

$$c(n+1, \theta) = \left(1 + \frac{q(n, j_n(\theta) + 1)}{q(n, j_n(\theta))}\right) c(n, \theta)$$

by Relations (5.2.1). By Lemma (5.2.9), this exhausts all possibilities for $c(n+1, \theta)$, and the proof is complete. $\square$

Lemma 5.2.20. *Using notation from Lemma (5.2.19), if $\theta \in (0,1) \setminus \mathbb{Q}$, then:*

$$c(1, \theta) = 1 - \theta.$$

Moreover, using notation from Definition (5.2.8), if $\theta, \mu \in (0,1) \setminus \mathbb{Q}$ such that there exists $N \in \mathbb{N} \setminus \{0\}$ with $I_\theta \cap \mathfrak{F}^N = I_\mu \cap \mathfrak{F}^N$, then there exists $a, b \in \mathbb{R}, a \neq 0$ such that:

$$c(N, \theta) = a\theta + b, \; c(N, \mu) = a\mu + b.$$

Proof. Let $\theta \in (0,1)\setminus\mathbb{Q}$, and denote its continued fraction expansion by $\theta = [a_j]_{j\in\mathbb{N}}$. Recall, by Proposition (5.2.14), we have for all $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$:

$$\mathfrak{af}_\theta\left(\underrightarrow{\varphi^{a_1}}(x) + I_\theta\right) = \underrightarrow{\alpha^1_\theta}\left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)}\right). \tag{5.2.8}$$

Next, by Notation (5.2.15), we note that:

$$\rho_\sigma \circ \underrightarrow{\varphi^{a_1}} = \tau_\theta \circ Q_\theta \circ \underrightarrow{\varphi^{a_1}} = \sigma_\theta \circ \mathfrak{af}_\theta \circ Q_\theta \circ \underrightarrow{\varphi^{a_1}} \tag{5.2.9}$$

Now, consider $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $x_{j_{a_1}(\theta)+1} = 1_{\mathfrak{M}\left(q^\theta_1\right)}$ and $x_k = 0$ for all $k \in \{0, \ldots, 2^{a_1-1}\} \setminus \{j_{a_1}(\theta)\}$. Then, by Lemma (5.2.19) and Expressions (5.2.8,5.2.9), we have that $(1 - c(a_1,\theta)) = \rho_\theta \circ \underrightarrow{\varphi^{a_1}}(x) = \sigma_\theta \circ \underrightarrow{\alpha^1_\theta}\left(1_{\mathfrak{M}\left(q^\theta_1\right)} \oplus 0\right) = a_1\theta$ by Lemma (4.2.3). And, thus:

$$c(a_1,\theta) = 1 - a_1\theta. \tag{5.2.10}$$

Thus, if $a_1 = 1$, then we would be done.

Assume that $a_1 \geqslant 2$. By [10, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [10, Figure 5] in arXiv v6), the Bratteli diagram of $\mathfrak{F}/I_\theta$ begins with the diagram L_{a_1-1} of [10, Figure 5] at level $n = 1$. Thus, the term $j_m(\theta) = 0$ for all $m \in \{1, \ldots, a_1\}$. Hence, if $m \in \{1, \ldots, a_1 - 1\}$, then $j_{m+1}(\theta) = 2j_m(\theta)$.

We claim that for all $m \in \{1, \ldots, a_1\}$ we have that:

$$c(m,\theta) = mc(1,\theta) - (m-1). \tag{5.2.11}$$

We proceed by induction. The cases $m = 1$ and $a_1 = 1$ are clear. So, assume that $a_1 \geqslant 2$. Assume true for $m \in \{1, \ldots, a_1 - 1\}$. Consider $m + 1$. Since $j_{m+1}(\theta) = 2j_m(\theta)$, by Lemma (5.2.19), we have that:

$$c(m+1,\theta) = \frac{(q(m,0)+q(m,1))c(m,\theta) - q(m,0)}{q(m,1)} \\ = \frac{c(m,\theta) + q(m,1)c(m,\theta) - 1}{q(m,1)}. \tag{5.2.12}$$

By Relations (5.2.1), we gather that $q(m,1) = m$. Hence, by induction hypothesis and Expression (5.2.12), we have:

$$\begin{aligned} c(m+1,\theta) &= \frac{mc(1,\theta) - (m-1) + m(mc(1,\theta) - (m-1)) - 1}{m} \\ &= c(1,\theta) - 1 + 1/m + mc(1,\theta) - (m-1) - 1/m \\ &= (m+1)c(1,\theta) - ((m+1)-1), \end{aligned}$$

which completes the induction argument. Hence, by Expression (5.2.11), we conclude $c(a_1,\theta) = a_1 c(1,\theta) - (a_1 - 1)$, which implies that:

$$c(1,\theta) = 1 - \theta \tag{5.2.13}$$

by Equation (5.2.10).

Lastly, let $\theta, \mu \in (0,1) \setminus \mathbb{Q}$. We prove the remaining claim in the Lemma by induction. Assume $N = 1$. Then, by Equation (5.2.13), the coefficients $c(1,\mu) = 1-\mu$ and $c(1,\theta) = 1-\theta$, which completes the base case.

Assume true for $N \in \mathbb{N} \setminus \{0,1\}$. Assume that $I_\mu \cap \mathfrak{F}^{N+1} = I_\theta \cap \mathfrak{F}^{N+1}$. Now, since $\mathfrak{F}^N \subseteq \mathfrak{F}^{N+1}$, we thus have $I_\mu \cap \mathfrak{F}^N = I_\theta \cap \mathfrak{F}^N$. Hence, by the induction hypothesis, there exists $a, b \in \mathbb{R}, a \neq 0$ such that $c(N,\mu) = a\mu + b$ and $c(N,\theta) = a\theta + b$. But, as $I_\mu \cap \mathfrak{F}^{N+1} = I_\theta \cap \mathfrak{F}^{N+1}$, the vertices a level $N+1$ agree in the ideal diagrams by Proposition (5.1.20). By Definition (5.2.8), we have $j_{N+1}(\theta) = j_{N+1}(\mu)$, and similarly, the term $j_N(\theta) = j_N(\mu)$ by $I_\mu \cap \mathfrak{F}^N = I_\theta \cap \mathfrak{F}^N$. Therefore, the conclusion follows by Lemma (5.2.19). $\square$

We can now prove the main result of this section.

Theorem 5.2.21. *Using Definition (5.2.8) and Notation (5.2.17), the map:*

$$I_\theta \in (\mathrm{Prim}(\mathfrak{F}), \tau) \longmapsto \left(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta}\right) \in (\mathcal{QQCMS}_{2,0}, \Lambda_{2,0})$$

is continuous to the class of $(2,0)$*-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity* $\Lambda_{2,0}$*, where* τ *is either the Jacobson topology, the relative metric topology of* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *(Proposition (5.1.7)), or the relative Fell topology (Definition (2.1.58)).*

Proof. By Proposition (5.2.10) and Proposition (5.2.11), we only need to show continuity with respect to the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ with sequential continuity. Thus, let $(I_{\theta_n})_{n\in\overline{\mathbb{N}}} \subset \mathrm{Prim}(\mathfrak{F})$ be a sequence, in which I_{θ_n} is uniquely determined by $\theta_n \in (0,1) \setminus \mathbb{Q}$ for all $n \in \overline{\mathbb{N}}$ by Proposition (5.2.11), such that $(I_{\theta_n})_{n\in\mathbb{N}}$ converges to I_{θ_∞} with respect to $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$. Therefore, by Corollary (5.1.25), this implies that $\left\{ I_{\theta_n} = \overline{\cup_{k\in\mathbb{N}} I_{\theta_n} \cap \mathfrak{F}^k}^{\|\cdot\|_{\mathfrak{F}}} : n \in \overline{\mathbb{N}} \right\}$ is a fusing family with some fusing sequence $(c_n)_{n\in\mathbb{N}}$. Thus, condition *1.* of Theorem (5.2.2) is satisfied.

For condition *2.* of Theorem (5.2.2), let $N \in \mathbb{N}$, then by definition of fusing sequence, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I_{\theta_k} \cap \mathfrak{F}^N = I_{\theta_\infty} \cap \mathfrak{F}^N$. Now, let $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. Consider ρ_{θ_k} on $\mathfrak{F}^N$. By Lemma (5.2.20), there exists $a, b \in \mathbb{R}, a \neq 0$, such that $c(N, \theta_k) = a\theta_k + b$ for all $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. But, by Proposition (5.2.10), we obtain $(\theta_n)_{n\in\mathbb{N}}$ converges to θ_∞ with respect to the usual topology on $\mathbb{R}$. Hence, the sequence $(c(N, \theta_k))_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $c(N, \theta_\infty)$ with respect to the usual topology on $\mathbb{R}$ and the same applies to $(1 - c(N, \theta_k))_{k\in\mathbb{N}_{\geqslant c_N}}$. However, by Lemma (5.2.19), the coefficient $c(N, \theta_k)$ determines ρ_k for all $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. Hence, Lemma (3.1.12) provides that $(\rho_{\theta_k})_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to ρ_{θ_∞} in the weak* topology on $\mathscr{S}\left(\mathfrak{F}^N\right)$.

Condition *3.* of Theorem (5.2.2) follows a similar argument as in the proof of condition *2.* since the sequences β^θ of Lemma (5.2.16) are determined by the terms

$j_n(\theta)$. Also, by Lemma (5.2.16), all β^θ are uniformly bounded by the sequence $\left(1/n^2\right)_{n\in\mathbb{N}}$ which converges to 0. Therefore, the proof is complete. $\square$

As an aside to Remark (5.2.18), we obtain the following analogue to Theorem (4.2.12) in terms of quotients.

Corollary 5.2.22. *Using Notation (5.2.17), the map:*

$$\theta \in ((0,1)\setminus\mathbb{Q}, |\cdot|) \longmapsto \left(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}/I_\theta},\tau_\theta}\right) \in (\mathcal{QQCMS}_{2,0}, \mathsf{\Lambda}_{2,0})$$

is continuous from $(0,1)\setminus\mathbb{Q}$*, with its topology as a subset of* $\mathbb{R}$ *to the class of* $(2,0)$*-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity* $\mathsf{\Lambda}$*.*

Proof. Apply Proposition (5.2.10) and Proposition (5.2.11) to Theorem (5.2.21). $\square$

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