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Abstract

We construct new Banach spaces using barriers in high dimensional Ellentuck spaces [9] following the classical framework under which a Tsirelson type norm is defined from a barrier in Ellentuck space [3]. It is shown that these spaces contain arbitrary large copies of ℓ_∞^n and specific block subspaces isomorphic to ℓ_p. We also prove that they are ℓ_p-saturated and not isomorphic to each other. Finally, a study of alternative norms for our spaces is presented.

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Introduction

Several natural questions about the linear structure of infinite-dimensional Banach spaces, that were asked since the early days of the theory, remained without answer for many years:

1. Does every Banach space contain a subspace isomorphic to some ℓ_p or c_0?
2. Does every Banach space contain an infinite unconditional basic sequence?
3. Is ℓ_p, with $1 < p < \infty$, distortable?
4. Is it true that every Banach space is isomorphic to its closed hyperplanes?
5. If a Banach space X is isomorphic to every infinite-dimensional closed subspace of itself, does it follow that X is isomorphic to ℓ_2?
6. Is it possible to decompose every Banach space as a topological direct sum of two infinite-dimensional closed subspaces?

All these questions were answered during the period from 1990 to 2005, with the exception of the first question. That question was settled negatively in 1974 by the famous example of Tsirelson [17], who constructed a reflexive Banach space T that does not contain any ℓ_p or c_0. Nowadays, it is obvious when we look back that the first giant step in the direction of all solutions to the questions above was done by Tsirelson. His space was the first example of a Banach space where the norm is defined implicitly as opposed to explicitly.

Almost 20 years later, Schlumprecht introduced his space S as a descendant of T and as the first example of an arbitrarily distortable Banach space [15]. It turned out that S provided the necessary framework for the fundamental Gowers-Maurey construction [12] that led to the solutions to questions 2-6 [14].

The idea of Tsirelson's construction became apparent after Figiel and Johnson [11] showed that the norm of the dual space of Tsirelson space satisfies the following equation:

$$\left\|\sum_n a_n e_n\right\| = \max\left\{\sup_n |a_n|\,,\, \frac{1}{2}\sup \sum_{i=1}^{m}\left\|E_i\left(\sum_n a_n e_n\right)\right\|\right\},$$

where the sequences $(E_i)_{i=1}^m$ considered above consist of successive finite subsets of positive integers with the property that $m \leq \min(E_1)$ and $E_i\left(\sum_n a_n e_n\right) = \sum_{n\in E_i} a_n e_n$. Interestingly, this dual space is what nowadays is understood in Banach space theory as Tsirelson space.

Banach space theory offers many applications of fronts and barriers within the framework of Ellentuck space (the set of infinite subsets of $\mathbb{N}$ endowed with the exponential topology); see [8], [14] and Part B of [3]. However, experts in this area prefer to think about them as compact (under the topology of pointwise convergence) families of finite subsets of $\mathbb{N}$ by considering in fact not fronts or barriers themselves but their downwards closures.

The first systematic abstract study of Tsirelson's construction was given by Argyros and Deliyanni [1]. Their construction starts with a real number $0 < \theta < 1$ and an arbitrary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ that is the downwards closure of a barrier in Ellentuck space. Then, one defines the *Tsirelson type space* $T(\mathcal{F}, \theta)$ as the completion of $c_{00}(\mathbb{N})$ with the implicitly given norm above replacing $1/2$ by θ and using sequences $(E_i)_{i=1}^m$ of finite subsets of positive integers which are $\mathcal{F}$-admissible, i.e., there is some $\{k_1, k_2, \ldots, k_m\} \in \mathcal{F}$ such that

$$k_1 \leq \min(E_1) \leq \max(E_1) < k_2 \leq \cdots < k_m \leq \min(E_m) \leq \max(E_m).$$

In this notation, Tsirelson space is denoted by $T(\mathcal{S}, 1/2)$, where

$$\mathcal{S} = \{F \subset \mathbb{N} : |F| \leq \min(F)\}$$

is the so called *Schreier family*. Besides $\mathcal{S}$, among all compact families, the *low complexity hierarchy* $\{\mathcal{A}_d\}_{d=1}^{\infty}$ with

$$\mathcal{A}_d := \{F \subset \mathbb{N} : |F| \leq d\}$$

is of utmost importance in the realm of Tsirelson type spaces. In fact, Bellenot proved in [4] the following remarkable theorem:

Theorem (Bellenot [4]). *If $d\theta > 1$, then for every $x \in T(\mathcal{A}_d, \theta)$,*

$$\frac{1}{2d} \|x\|_p \leq \|x\|_{T(\mathcal{A}_d,\theta)} \leq \|x\|_p \, ,$$

where $d\theta = d^{1/p}$ and $\|\cdot\|_p$ denotes the ℓ_p-norm.

Infinite-dimensional Ramsey theory is a branch of Ramsey Theory initiated by Nash-Williams in the course of developing his theory of better-quasi-ordered sets in the early 60's. This theory introduced the notions of *fronts* and *barriers* that turned out to be very important in the context of Tsirelson type norms. During the 70's, Nash-Williams' theory was reformulated and strengthened by the work of Galvin, Prikry, Silver, and specially Ellentuck by introducing the topological Ramsey theory.

Recently, Todorcevic has distilled the key properties of Ellentuck space into four axioms that determine a *topological Ramsey space* (Chapter 5 of [16]). He has shown that the theory of fronts and barriers in the case of Ellentuck space allows extension to the context of general topological Ramsey spaces. Judging on the basis of the applicability of the original theory, it is reasonable to expect that this extension will find interesting applications.

In this dissertation we define new Banach spaces using barriers in high dimensional Ellentuck spaces (a hierarchy of topological Ramsey spaces which generalize the Ellentuck space [9]). The motivation for our construction and many ideas behind our results came from the Tsirelson type spaces $T(\mathcal{A}_d, \theta)$. It is shown that these spaces contain arbitrary large copies of ℓ_∞^n and specific block subspaces isomorphic to ℓ_p. We also prove that they are ℓ_p-saturated and not isomorphic to each other. Finally, a study of alternative norms for our spaces is presented.

Chapter 1

Banach Space Theory

1.1 Fundamental Notions

A *normed space* $(X, \|\cdot\|)$ is a vector space X endowed with a nonnegative function $\|\cdot\| : X \to \mathbb{R}$ called norm satisfying for all $x, y \in X$ and $c \in \mathbb{R}$:

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|cx\| = |c| \, \|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$.

A *Banach space* is a normed space $(X, \|\cdot\|)$ that is complete in the metric defined by $d(x, y) = \|x - y\|$.

A vector subspace Y of a Banach space $(X, \|\cdot\|)$ is closed in X if and only if $(Y, \|\cdot\|_Y)$ is a Banach space, where $\|\cdot\|_Y$ denotes the restriction of $\|\cdot\|$ to Y. If Y is a subspace of X, so is its closure $\overline{Y}$.

Two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are *equivalent* if there exist positive constants c and C such that for all $x \in X$ we have

$$c \|x\|_0 \leq \|x\| \leq C \|x\|_0 .$$

Let $T : X \to Y$ be a linear map between two Banach spaces X and Y. The *continuity* of T with respect to the norm topologies of X and Y can be characterized by the following condition: there is a constant $C > 0$ such that $\|Tx\| \leq C \|x\|$ for all $x \in X$. We say that T is *bounded* whenever it satisfies the preceding condition. Therefore, T is continuous if and only if T is bounded.

T is called an *isomorphism* if T is a continuous bijection whose inverse T^{-1} is also continuous. That is, an isomorphism between normed spaces is a linear homeomorphism. Equivalently, T is an isomorphism if and only if T is onto and there exist positive constants c and C so that

$$c \|x\| \leq \|Tx\| \leq C \|x\|$$

for all $x \in X$. In such a case the spaces X and Y are said to be *isomorphic* and we write $X \approx Y$. T is an *isometric isomorphism* when $\|Tx\| = \|x\|$ for all $x \in X$.

T is an *embedding of X into Y* if T is an isomorphism onto its image $T(X)$. In this case we say that X *embeds in* Y or that Y contains an isomorphic copy of X.

The dual X^* of a Banach space X is the space of all linear maps $f : X \to \mathbb{R}$ which are continuous, or equivalently bounded. The elements of X^* are called *functionals*. X^* is a Banach space under the norm

$$\|f\| = \sup \{|f(x)| : \|x\| \leq 1\} .$$

1.2 The Sequence Spaces ℓ_p and c_0

Arguably the first infinite-dimensional Banach spaces to be studied were the sequence spaces ℓ_p and c_0. In this section we introduce these spaces. Define s to be the vector space of all real sequences $x = (x_n)_{n=1}^{\infty}$.

For each $1 \leq p < \infty$, we define

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p},$$

and take ℓ_p to be the collection of those $x \in s$ for which $\|x\|_p < \infty$. Applying Hölder's and Minkowski's inequalities we obtain that ℓ_p is a normed space; from there is not difficult to see that ℓ_p is actually a Banach space.

It is important to point out that the space ℓ_p is defined exactly in the same way for $0 < p < 1$, but in such a case $\|\cdot\|_p$ defines a complete quasi-norm.

For $p = \infty$, we define ℓ_∞ to be the collection of all bounded sequences; that is, ℓ_∞ consist of those $x \in s$ for which

$$\|x\|_\infty = \sup_n |x_n| < \infty.$$

Since convergence in ℓ_∞ is the same as uniform convergence on $\mathbb{N}$, we conclude that ℓ_∞ is complete. There are two very natural closed subspaces of ℓ_∞:

1. c, the space consisting of all convergent sequences.
2. c_0, the space consisting of all sequences converging to 0.

As subsets of s we have

$$\ell_1 \subset \ell_p \subset \ell_q \subset c_0 \subset c \subset \ell_\infty$$

for any $1 < p < q < \infty$. Moreover, each of this inclusions is of norm one:

$$\|x\|_1 \geq \|x\|_p \geq \|x\|_q \geq \|x\|_\infty .$$

On the other hand, we will also consider the finite-dimensional versions of the ℓ_p spaces. We write ℓ_p^n to denote $\mathbb{R}^n$ under the ℓ_p norm. That is, ℓ_p^n is the space of all sequences $x = (x_1, \ldots, x_n)$ supplied with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} ,$$

for $1 \leq p < \infty$, and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

for $p = \infty$.

Recall that all norms on $\mathbb{R}^n$ are equivalent. In particular, given any norm $\|\cdot\|$ on $\mathbb{R}^n$, we can find a positive constant C such that

$$C^{-1} \|x\|_1 \leq \|x\| \leq C \|x\|_1 .$$

for all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$.

1.3 Bases in Banach Spaces

In this section we present the fundamental notion of a Schauder basis of a Banach space and the correspoding notion of a basic sequence. One of the key ideas in the isomorphic theory of Banach spaces is to use the properties of bases and basic sequences as a tool to understanding the differences and similarities between spaces.

Definition 1.3.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. Then:

1. $(x_n)_{n=1}^{\infty}$ is *normalized* if $\|x_n\| = 1$ for every n.
2. The *linear span* of $(x_n)_{n=1}^{\infty}$ is the subspace

$$\text{span}\{x_n\} := \left\{ \sum_{i=1}^{k} a_i x_i : a_1, \ldots, a_k \in \mathbb{R}, k \in \mathbb{N} \right\}$$

   consisting of all finite linear combinations of elements of $(x_n)_{n=1}^{\infty}$. The *closed linear span* of $(x_n)_{n=1}^{\infty}$ is denoted by $[x_n]$ or $\overline{\text{span}}\{x_n\}$.
3. $(x_n)_{n=1}^{\infty}$ is a *Schauder basis* or just a *basis* of X if for every $x \in X$ there is a unique sequence of scalaras $(a_n)_{n=1}^{\infty}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$.
4. If $(x_n)_{n=1}^{\infty}$ is a basis of $[x_n]$, then $(x_n)_{n=1}^{\infty}$ is called a *basic sequence*.

Definition 1.3.2. Let X be a Banach space with basis $(e_n)_{n=1}^{\infty}$. The elements of the sequence $(e_n^*)_{n=1}^{\infty}$ in X^* defined by

1. $e_i^*(e_j) = 1$ if $i = j$, and $e_i^*(e_j) = 0$ otherwise, and
2. $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$ for each $x \in X$,

are called the *biorthogonal functionals* associated with $(e_n)_{n=1}^{\infty}$.

For $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$, the *support of* x is the set of all positive integers n such that $e_n^*(x) \neq 0$. We denote it by supp (x). If $|\text{supp}\,(x)| < \infty$ we say that x is *finitely supported*.

If we have a basis $(e_n)_{n=1}^{\infty}$ of a Banach space X, then we can specify $x \in X$ by its coordinates $(e_n^*(x))_{n=1}^{\infty}$. Clearly, not every scalar sequence $(a_n)_{n=1}^{\infty}$ defines an element of X. Therefore, X is coordinatized by a particular sequence space, a vector subspace of the vector space of all sequences s. Consequently, this leads naturally to the following definition.

Definition 1.3.3. Two bases (basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in respective Banach spaces X and Y are *equivalent* if whenever we take a sequence of scalars $(a_n)_{n=1}^{\infty}$, then $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n x_n$ converges.

Hence, if the bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent, then the corresponding sequence spaces associated to X by $(x_n)_{n=1}^{\infty}$ and to Y by $(y_n)_{n=1}^{\infty}$ coincide. Thus, applying the Closed Graph Theorem, we have:

Theorem 1.3.4. *Two bases (basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if and only if there is an isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$ for each n.*

Corollary 1.3.5. *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two bases for Banach spaces X and Y, respectively. Then, $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if and only if there exists a constant $C > 0$ such that for all finitely nonzero sequences of scalars $(a_n)_{n=1}^{\infty}$ we have*

$$C^{-1} \left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n y_n \right\| . \tag{1.3.1}$$

Definition 1.3.6. Whenever equation (1.3.1) holds, we say that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are *C-equivalent*.

We now introduce a very special type of basic sequence:

Definition 1.3.7. Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X. Suppose that $(p_n)_{n=1}^{\infty}$ is a strictly increasing sequence of integers with $p_0 = 0$ and that $(a_n)_{n=1}^{\infty}$ are scalars. Then, a sequence of nonzero vectors $(x_n)_{n=1}^{\infty}$ in X of the form

$$x_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i$$

is called a *block basic sequence* or briefly a *block basis* of $(e_n)_{n=1}^{\infty}$. A subspace of X generated by a block basis is called a *block subspace*.

The following stability result dates back to 1940. Roughly speaking, it says that if $(x_n)_{n=1}^{\infty}$ is a basic sequence in a Banach space X and $(y_n)_{n=1}^{\infty}$ is another sequence in X so that $\|x_n - y_n\| \to 0$ fast enough, then $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent.

Theorem 1.3.8 (Principle of Small Perturbations [13])**.** *Let $(x_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space X with corresponding biorthogonal functionals $(x_n^*)_{n=1}^{\infty}$. Suppose that $(y_n)_{n=1}^{\infty}$ is a sequence in X with $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n - y_n\| = \delta$. If $\delta < 1$, then $(y_n)_{n=1}^{\infty}$ is a basic sequence equivalent to $(x_n)_{n=1}^{\infty}$.*

An application of the Principle of Small Perturbations provides us with the following very useful result known as the Bessaga-Pelczynski Selection Principle. It allows us to restrict our attention to block bases and block subspaces when studying the differences and similarities between Banach spaces.

Theorem 1.3.9 (Bessaga-Pelczynski Selection Principle [6])**.** *Let X be a Banach space with basis $(e_n)_{n=1}^{\infty}$, let Y be a subspace of X, and let $\varepsilon > 0$. Then, Y has a subspace Z generated by a basis which is $(1+\varepsilon)$-equivalent to a block basis of $(e_n)_{n=1}^{\infty}$.*

In fact, much more is true. For every sequence $(\delta)_{n=1}^{\infty}$ of positive real numbers, we can find a subspace Z with a basis $(z_n)_{n=1}^{\infty}$ such that there is a normalized block basis $(x_n)_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ with $\|x_n - z_n\| \leq \delta$ for every n. In other words, Z can be chosen to be an arbitrarily small perturbation of a block subspace of X. Moreover, given any basic sequence in Y, we can choose Z to be spanned by a subsequence.

Finally, we present a special class of bases: unconditional bases. This important concept was developed by R. C. James in the early 1950s.

Definition 1.3.10. A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called *unconditional* if for each $x \in X$ the series $\sum_{n=1}^{\infty} e_n^*(x)e_n$ converges unconditionally.

Obviously, $(e_n)_{n=1}^{\infty}$ is an unconditional basis of X if and only if $(e_{\pi(n)})_{n=1}^{\infty}$ is a basis of X for all permutations π of the positive integers. The canonical bases of c_0 and ℓ_p with $1 \leq p < \infty$ are unconditional.

The following is a useful characterization of unconditional bases:

Proposition 1.3.11. *A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is unconditional if and only if there is a constant $K \geq 1$ such that for all $N \in \mathbb{N}$, whenever $a_1, \dots, a_N, b_1, \dots, b_N$ are scalars satisfying $|a_n| \leq |b_n|$ for $n = 1, \dots, N$, then the following inequality holds:*

$$\left\| \sum_{n=1}^{N} a_n e_n \right\| \leq K \left\| \sum_{n=1}^{N} b_n e_n \right\| . \tag{1.3.2}$$

Definition 1.3.12. Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of a Banach space X. The *unconditional basis constant* K_u of $(e_n)_{n=1}^{\infty}$ is the least constant K so that equation (1.3.2) holds. We then say that $(e_n)_{n=1}^{\infty}$ is K-*unconditional* whenever $K \geq K_u$.

Chapter 2

Infinite Dimensional Ramsey Theory

2.1 Topological Ramsey Theory

We provide an overview of topological Ramsey spaces. Building on earlier work of Carlson and Simpson [7], Todorcevic distilled the key properties of the Ellentuck space into four axioms which guarantee that a space is a topological Ramsey space. For further background, we refer the reader to Chapter 5 of [16].

The axioms **A.1** - **A.4** are defined for triples $(\mathcal{R}, \leq, r)$ of objects with the following properties. $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings, where $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, set

$$[a, B] = \{A \in \mathcal{R} : A \leq B \text{ and } (\exists n)\, r_n(A) = a\}.$$

For $a \in \mathcal{AR}$, let $|a|$ denote the length of the sequence a. Thus, $|a|$ equals the integer k for which $a = r_k(A)$ for some $A \in \mathcal{R}$. For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ if and only if $a = r_m(b)$ for some $m \leq |b|$. $a \sqsubset b$ if and only if $a = r_m(b)$ for some $m < |b|$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$.

A.1 *For each $A, B \in \mathcal{R}$,*

(a) $r_0(A) = \emptyset$.

(b) $A \neq B$ *implies* $r_n(A) \neq r_n(B)$ *for some* n.

(c) $r_n(A) = r_m(B)$ *implies* $n = m$ *and* $r_k(A) = r_k(B)$ *for all* $k < n$.

A.2 *There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that*

(a) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ *is finite for all* $b \in \mathcal{AR}$,

(b) $A \leq B$ *iff* $(\forall n)(\exists m)\ r_n(A) \leq_{\text{fin}} r_m(B)$,

(c) $\forall a, b, c \in \mathcal{A}R[a \sqsubseteq b \wedge b \leq_{\text{fin}} c \to \exists d \sqsubseteq c \quad a \leq_{\text{fin}} d]$.

For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, $\text{depth}_B(a)$ is the least n, if it exists, such that $a \leq_{\text{fin}} r_n(B)$. If such an n does not exist, then we write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = n < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_n(a), B]$.

A.3 *For each $A, B \in \mathcal{R}$ and each $a \in \mathcal{AR}$,*

(a) *If* $\text{depth}_B(a) < \infty$ *then* $[a, A] \neq \emptyset$ *for all* $A \in [\text{depth}_B(a), B]$.

(b) $A \leq B$ *and* $[a, A] \neq \emptyset$ *imply that there is an* $A' \in [\text{depth}_B(a), B]$ *such that* $\emptyset \neq [a, A'] \subseteq [a, A]$.

If $n > |a|$, then $r_n[a, A]$ denotes the collection of all $b \in \mathcal{AR}_n$ such that $a \sqsubset b$ and $b \leq_{\text{fin}} A$.

A.4 *For each $B \in \mathcal{R}$ and each $a \in \mathcal{AR}$, if* $\text{depth}_B(a) < \infty$ *and* $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, *then there is* $A \in [\text{depth}_B(a), B]$ *such that*

$$r_{|a|+1}[a, A] \subseteq \mathcal{O} \quad \textit{or} \quad r_{|a|+1}[a, A] \subseteq \mathcal{O}^c.$$

The *Ellentuck topology* on $\mathcal{R}$ is the topology generated by the basic open sets $[a, B]$; it extends the usual metrizable topology on $\mathcal{R}$ when we consider $\mathcal{R}$ as a subspace of the Tychonoff cube $\mathcal{AR}^{\mathbb{N}}$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. We say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* iff $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 2.1.1 ([16])**.** A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the property of Baire is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

The following result can be found as Theorem 5.4 in [16]:

Theorem 2.1.2 (Abstract Ellentuck Theorem)**.** *If $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $\mathcal{AR}^{\mathbb{N}}$) and satisfies axioms* **A.1**, **A.2**, **A.3**, *and* **A.4**, *then every subset of $\mathcal{R}$ with the property of Baire is Ramsey, and every meager subset is Ramsey null; in other words, the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.*

Infinite-dimensional Ramsey theory introduced the notions of *front* and *barrier* that turned out to be very important in the context of Tsirelson type norms. Barriers, in particular, play a key role in the work presented in this dissertation.

Definition 2.1.3 ([16])**.** A family $\mathcal{F} \subseteq \mathcal{AR}$ of finite approximations is

(1) *Nash-Williams* if $a \not\sqsubseteq b$ for all $a \neq b \in \mathcal{F}$;

(2) *Sperner* if $a \not\leq_{\text{fin}} b$ for all $a \neq b \in \mathcal{F}$.

Definition 2.1.4 ([16])**.** Suppose $(\mathcal{R}, \leq, r)$ is a closed subspace of $\mathcal{AR}^{\omega}$ and satisfies **A.1, A.2, A.3** and **A.4**. A family $\mathcal{F} \subseteq \mathcal{AR}$ is a *barrier* (*front*) on $[\emptyset, X]$ if

(1) For each $Y \in [\emptyset, X]$, there is an $a \in \mathcal{F}$ such that $a \sqsubset Y$, and

(2) $\mathcal{F}$ is Sperner (Nash-Williams).

2.2 Ellentuck Space

A prototype example of a triple $(\mathcal{R}, \leq, r)$ satisfying axioms **A.1 - A.4** is the *Ellentuck Space* $([\omega]^{\omega}, \subseteq, r)$, where

$$[\omega]^{\omega} = \{M \subseteq \omega : M \text{ is infinite}\}$$

is the family of all infinite subsets of ω, $r_n(A)$ is the initial segment of A formed by taking the first n elements of A, and the relation $\subseteq_{\text{fin}}$ is defined on the family of all finite subsets ω as follows:

$$a \subseteq_{\text{fin}} b \quad \text{iff} \quad a = b = 0 \quad \text{or} \quad a \subseteq b \text{ and } \max(a) = \max(b).$$

Note that the topology of the prototype example is equal to the topology that $[\omega]^{\omega}$ gets as a subset of the exponential space $\exp(\omega)$.

Banach space theory offers many applications of fronts and barriers within the framework of Ellentuck space (see [8], [14] and Part B of [3]). However, experts in this area prefer to think about them as compact (under the topology of pointwise convergence) families of finite subsets of $\mathbb{N}$ by considering in fact not fronts or barriers themselves but their downwards closures. The following are concrete examples of barriers that are important in the realm of Tsirelson type spaces:

$$\mathcal{S}' = \{F \subset \mathbb{N} : |F| = \min(F)\} \qquad \text{and} \qquad \mathcal{A}'_d := \{F \subset \mathbb{N} : |F| = d\}.$$

$\mathcal{S}'$ is the so called *Schreier barrier*, and for each positive integer $d, \mathcal{A}'_d$ is the barrier used to define the d-th member of the *low complexity hierarchy*.

The motivation for the construction of the Banach spaces explored in the following chapters and many ideas behind our results came from the Tsirelson type spaces defined from the downward closure of each $\mathcal{A}'_d$.

Chapter 3

High Dimensional Ellentuck Spaces

Recently, Dobrinen introduced a new hierarchy $(\mathcal{E}_k)_{k<\omega}$ of topological Ramsey spaces which generalize the Ellentuck space in a natural manner [9]. We shall let $\mathcal{E}_1$ denote the Ellentuck space.

3.1 Construction Framework

We now begin the process of defining the new class of spaces $\mathcal{E}_k$. The definition presented here is slightly different than the one in [9]. We have chosen to do so in order to simplify the construction of the Banach spaces presented in the following chapter. We start by defining a well-ordering on non-decreasing sequences of members of ω which forms the backbone for the structure of the members in the spaces.

Definition 3.1.1. For $k \geq 2$, denote by $\omega^{\not\downarrow \leq k}$ the collection of all non-decreasing sequences of members of ω of length less than or equal to k.

Definition 3.1.2 (The well-order $<_{\text{lex}}$)**.** Let $(s_1, \ldots, s_i)$ and $(t_1, \ldots, t_j)$, with $i, j \geq 1$, be in $\omega^{\not\downarrow \leq k}$. We say that $(s_1, \ldots, s_i)$ is lexicographically below $(t_1, \ldots, t_j)$, written $(s_1, \ldots, s_i) <_{\text{lex}} (t_1, \ldots, t_j)$, if and only if there is a non-negative integer m with the following properties:

(i) $m \leq i$ and $m \leq j$;

(ii) for every positive integer $n \leq m$, $s_n = t_n$; and

(iii) either $s_{m+1} < t_{m+1}$, or $m = i$ and $m < j$.

This is just a generalization of the way the alphabetical order of words is based on the alphabetical order of their component letters.

Example 3.1.3. Consider the sequences $(1, 2)$, (2), and $(2, 2)$ in $\omega^{\not\downarrow \leq 2}$. Following the preceding definition we have $(1, 2) <_{\text{lex}} (2) <_{\text{lex}} (2, 2)$. Let us look at this carefully: we conclude that $(1, 2) <_{\text{lex}} (2)$ by setting $m = 0$ in Definition 3.1.2; similarly, $(2) <_{\text{lex}} (2, 2)$ follows by setting $m = 1$ (notice that any proper initial segment of a sequence is lexicographically below that sequence).

Definition 3.1.4 (The well-ordered set $(\omega^{\not\downarrow \leq k}, \prec)$)**.** Set the empty sequence $()$ to be the $\prec$-minimum element of $\omega^{\not\downarrow \leq k}$; so, for all nonempty sequences s in $\omega^{\not\downarrow \leq k}$, we have $() \prec s$. In general, given $(s_1, \ldots, s_i)$ and $(t_1, \ldots, t_j)$ in $\omega^{\not\downarrow \leq k}$ with $i, j \geq 1$, define $(s_1, \ldots, s_i) \prec (t_1, \ldots, t_j)$ if and only if either

(i) $s_i < t_j$, or

(ii) $s_i = t_j$ and $(s_1, \ldots, s_i) <_{\text{lex}} (t_1, \ldots, t_j)$.

Notation. Since $\prec$ well-orders $\omega^{\not\downarrow \leq k}$ in order-type ω, we fix the notation of letting $\vec{s}_m$ denote the m-th member of $(\omega^{\not\downarrow \leq k}, \prec)$. Let $\omega^{\not\downarrow k}$ denote the collection of all non-decreasing sequences of length k of members of ω. Note that $\prec$ also well-orders $\omega^{\not\downarrow k}$ in order type ω. Fix the notation of letting $\vec{u}_n$ denote the n-th member of $(\omega^{\not\downarrow k}, \prec)$.

Definition 3.1.5 (The spaces $(\mathcal{E}_k, \leq, r)$, $k \geq 2$)**.** Let $s, t \in \omega^{\not\downarrow \leq k}$ and denote by $|s|$ the length of the sequence s. We say that $\widehat{X}$ is an $\mathcal{E}_k$*-tree* if $\widehat{X}$ is a function from $\omega^{\not\downarrow \leq k}$ into $\omega^{\not\downarrow \leq k}$ such that

(i) $\left|\widehat{X}(s)\right| = |s|$.

(ii) $s \prec t \Rightarrow \widehat{X}(s) \prec \widehat{X}(t)$.

(iii) $s \sqsubset t \Leftrightarrow \widehat{X}(s) \sqsubset \widehat{X}(t)$.

For $\widehat{X}$ an $\mathcal{E}_k$-tree, let X denote the restricition of $\widehat{X}$ to $\omega^{\not\downarrow k}$. Define the space $\mathcal{E}_k$ to be the collection of all X such that $\widehat{X}$ is an $\mathcal{E}_k$-tree. Thus, $\mathcal{E}_k$ is the space of all functions X from $\omega^{\not\downarrow k}$ into $\omega^{\not\downarrow k}$ which induce an $\mathcal{E}_k$-tree.

Notation. We identify X with its range and write $X = \{v_1, v_2, \ldots\}$, where $v_1 = X(\vec{u}_1) \prec v_2 = X(\vec{u}_2) \prec \cdots$. Notice then that we identify the identity function with $\omega^{\not\downarrow k}$. We think of $\omega^{\not\downarrow k}$ as the prototype for all members of $\mathcal{E}_k$ in the sense that every member of $\mathcal{E}_k$ will be a subset of $\omega^{\not\downarrow k}$ which has the same structure as $\omega^{\not\downarrow k}$ (Definition 3.1.5 simply generalizes the key points about the structure of the identity function on $\omega^{\not\downarrow \leq k}$). For each $n < \omega$, define $r_n(X) = \{v_1, v_2, \ldots, v_n\}$ to be the n-th finite approximation of X. As usual, we set

$$\mathcal{AR}^k := \{r_n(X) : n < \omega, X \in \mathcal{E}_k\} \qquad \text{and} \qquad \mathcal{AR}^k_n := \{r_n(X) : X \in \mathcal{E}_k\}.$$

The family of all non-empty finite subsets of $\omega^{\not\downarrow k}$ will be denoted by $\mathrm{FIN}(\omega^{\not\downarrow k})$. Clearly, $\mathcal{AR}^k \subset \mathrm{FIN}(\omega^{\not\downarrow k})$. If $E \in \mathrm{FIN}(\omega^{\not\downarrow k})$, then we denote the minimal and maximal elements of E with respect to $\prec$ by $\min_{\prec}(E)$ and $\max_{\prec}(E)$, respectively.

Example 3.1.6 (The space $\mathcal{E}_2$)**.** The members of $\mathcal{E}_2$ look like ω many copies of the Ellentuck space; that is, each member has order-type $\omega \times \omega$, under the lexicographic order. The well-order $(\omega^{\not\leq 2}, \prec)$ begins as follows:

$$() \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec \cdots$$

The tree structure of $\omega^{\not\leq 2}$, under lexicographic order, looks like ω copies of ω, and has order type the countable ordinal ω^2. Here, we picture the finite tree $\{\vec{s}_m : 1 \leq m \leq 21\}$, which indicates how the rest of the tree $\omega^{\not\leq 2}$ is formed. This is the same as the tree formed by taking all initial segments of the set $\{\vec{u}_n : 1 \leq n \leq 15\}$.

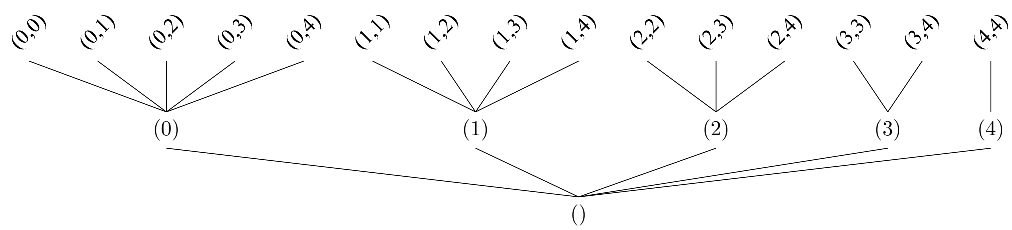

Figure 3.1: Initial structure of $\omega^{\not\leq 2}$.

Next we present the specifics of the structure of the space $\mathcal{E}_3$.

Example 3.1.7 (The space $\mathcal{E}_3$)**.** The well-order $(\omega^{\not\leq 3}, \prec)$ begins as follows:

$$\begin{aligned} &() \prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \\ &\quad \prec (1,1) \prec (1,1,1) \prec (0,0,2) \prec (0,1,2) \prec (0,2) \prec (0,2,2) \\ &\quad \prec (1,1,2) \prec (1,2) \prec (1,2,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (0,0,3) \prec \cdots \end{aligned}$$

The set $\omega^{\not\leq 3}$ is a tree of height three with each non-maximal node branching into ω many nodes. The maximal nodes in the following figure are technically the set $\{\vec{u}_n : 1 \leq n \leq 20\}$, which indicates the structure of $\omega^{\not\leq 3}$.

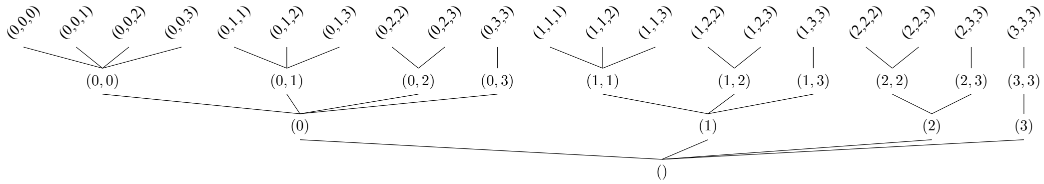

Figure 3.2: Initial structure of $\omega^{\nearrow\leq 3}$.

3.2 Upper Triangular Representation

We now present an alternative and very useful way to visualize elements of $\mathcal{E}_2$. This turned out to be fundamental to develop some intuition and to understand the Banach spaces that we have constructed. We refer to it as the *upper triangular representation of* $\omega^{\nearrow 2}$:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	...
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	
																$\ddots$

Figure 3.3: Upper triangular representation of $\omega^{\nearrow 2}$.

The well-order $(\omega^{\nearrow 2}, \prec)$ begins as follows: $(0,0) \prec (0,1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2,2) \prec \cdots$. In comparison with the tree representation shown in Figure 3.1, the upper triangular representation makes it simpler to visualize this well-order:

starting at $(0,0)$ we move from top to bottom throughout each column, and then to the right to the next column.

The following figure shows the initial part of an $\mathcal{E}_2$-tree $\widehat{X}$. The highlighted pieces represent the restriction of $\widehat{X}$ to $\omega^{\not{2}}$.

$$\begin{tikzcd}[column sep=small] () \arrow[d] & \prec & (0) \arrow[d] & \prec & \boxed{(0,0)} \arrow[d] & \prec & \boxed{(0,1)} \arrow[d] & \prec & (1) \arrow[d] & \prec & \boxed{(1,1)} \arrow[d] & \prec & \boxed{(0,2)} \arrow[d] & \prec & \boxed{(1,2)} \arrow[d] & \prec & (2) \arrow[d] & \prec & \boxed{(2,2)} \arrow[d] & \prec & \boxed{(0,3)} \arrow[d] & \prec & \cdots \\ () & \prec & (2) & \prec & \boxed{(2,4)} & \prec & \boxed{(2,6)} & \prec & (6) & \prec & \boxed{(6,6)} & \prec & \boxed{(2,8)} & \prec & \boxed{(6,8)} & \prec & (9) & \prec & \boxed{(9,9)} & \prec & \boxed{(2,10)} & \prec & \cdots \end{tikzcd}$$

Figure 3.4: Initial part of an $\mathcal{E}_2$-tree.

Under the identification discussed after Definition 3.1.5, we have that

$$X = \{(2,4),(2,6),(6,6),(2,8),(6,8),(9,9),(2,10),\ldots\}$$

is an element of $\mathcal{E}_2$. Using the upper triangular representation of $\omega^{\not{2}}$ we can visualize $r_{10}(X)$:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	...
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	
																⋱

Figure 3.5: $r_{10}(X)$ for a typical $X \in \mathcal{E}_2$.

Observe that when we map $(0,0)$ to $(2,4)$ we are forcing the first row of X to be a subset of the third row of $\omega^{\not<2}$; similarly, when we map $(1,1)$ to $(6,6)$ we are forcing the second row of X to be a subset of the seventh row of $\omega^{\not<2}$. In general, mapping (n,n) to (l,m) forces the $(n+1)$-th row of X to be a subset of the $(l+1)$-th row of $\omega^{\not<2}$.

Next figure clearly shows us how $\omega^{\not<2}$ is the prototype for all members of $\mathcal{E}_2$ in the sense that every member of $\mathcal{E}_2$ is a subset of $\omega^{\not<2}$ which has the same structure as $\omega^{\not<2}$.

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	…
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	…
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	…
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	⋱

Figure 3.6: $r_{28}(X)$ for a typical $X \in \mathcal{E}_2$.

3.3 Special Maximal Elements

There are special elements in $\mathcal{E}_k$ that are useful to describe the structure of some subspaces of the Banach spaces that we study in the following chapter. Given $v \in \omega^{\not<k}$ we want to construct a special $X_v^{\max} \in \mathcal{E}_k$ that has v as its first element and that is as large as possible (with respect to $\subseteq$).

For example, if $v = (0,4) \in \omega^{\not 2}$, then a finite approximation of $X_v^{\max} \in \mathcal{E}_2$ looks like this:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	
													(13,13)	(13,14)	(13,15)	...
														(14,14)	(14,15)	
															(15,15)	
																⋱

Figure 3.7: A finite approximation of $X_{(0,4)}^{\max} \in \mathcal{E}_2$.

Now let us illustrate this with $k = 3$ and $v = (0,2,7)$. Since we want v as the first element of $X_v^{\max}$, we identify it with $(0,0,0)$ and then we choose the next elements as small as possible following Definition 3.1.5:

$$
\begin{aligned}
&() \prec (0) \prec (0,2) \prec (0,2,7) \prec (0,2,8) \prec (0,8) \prec (0,8,8) \prec (8) \\
&\quad \prec (8,8) \prec (8,8,8) \prec (0,2,9) \prec (0,8,9) \prec (0,9) \prec (0,9,9) \\
&\quad \prec (8,8,9) \prec (8,9) \prec (8,9,9) \prec (9) \prec (9,9) \prec (9,9,9) \prec (0,2,10) \prec \cdots
\end{aligned}
$$

Therefore, under the identification discussed after Definition 3.1.5, we have

$$X_v^{\max} = \{(0,2,7), (0,2,8), (0,8,8), (8,8,8), (0,2,9), \ldots\}.$$

In general, $X_v^{\max}$ is constructed as follows. Suppose $v = (n_1, n_2, \ldots, n_k)$. First we define the $\mathcal{E}_k$-tree $\widehat{X}_v$ that will determine $X_v^{\max}$. $\widehat{X}_v$ must be a function from $\omega^{\not\downarrow \leq k}$ to $\omega^{\not\downarrow \leq k}$ satisfying Definition 3.1.5 and such that $\widehat{X}_v(0, 0, \ldots, 0) = v$. So, for $m, j \in \mathbb{Z}^+, j \leq k$ define the following auxiliary functions: $f_j(0) := n_j$ and $f_j(m) := n_k + m$. Then, for $t = (m_1, m_2, \ldots, m_l) \in \omega^{\not\downarrow \leq k}$ set $\widehat{X}_v(t) := (f_1(m_1), f_2(m_2), \ldots, f_l(m_l))$. Finally, define $X_v^{\max}$ to be the restriction of $\widehat{X}_v$ to $\omega^{\not\downarrow k}$.

Lemma 3.3.1. *Let $v, w \in \omega^{\not\downarrow k}$ be such that $v \neq w, v = (n_1, n_2, \ldots, n_k)$ and $w = (m_1, m_2, \ldots, m_k)$. Then, $w \in X_v^{\max}$ if and only if either $m_1 > n_k$, or there is $1 \leq l < k$ such that $(m_1, m_2, \ldots, m_l) = (n_1, n_2, \ldots, n_l)$ and $m_{l+1} > n_k$.*

Proof. Suppose $w \in X_v^{\max}$. Then, by definition of $X_v^{\max}$, there is $t = (j_1, j_2, \ldots, j_k)$ in $\omega^{\not\downarrow \leq k}$ such that $m_1 = f_1(j_1), m_2 = f_2(j_2), \ldots, m_k = f_k(j_k)$. Now, by definition of the auxiliary functions $f_1, f_2, \ldots, f_k$, either $m_1 > n_k$, or there is $1 \leq l < k$ such that $(m_1, m_2, \ldots, m_l) = (n_1, n_2, \ldots, n_l)$ and $m_{l+1} > n_k$.

On the other hand, suppose first that $m_1 > n_k$. For $i = 1, 2, \ldots, k$, set $j_i := m_i - n_k$ and $t = (j_1, j_2, \ldots, j_k)$. Since $m_1 \leq m_2 \leq \ldots \leq m_k$, we have that $j_i > 0$ and $t \in \omega^{\not\downarrow k}$. Moreover, $\widehat{X}_v(t) := (f_1(j_1), f_2(j_2), \ldots, f_k(m_k)) = w$. Therefore, $w \in X_v^{\max}$.

Assume now that there is $1 \leq l < k$ such that $(m_1, m_2, \ldots, m_l) = (n_1, n_2, \ldots, n_l)$ and $m_{l+1} > n_k$. Set $j_1 := 0, \ldots, j_l := 0, j_{l+1} := m_{l+1} - n_l, \ldots, j_k := m_k - n_l$, and $t = (j_1, \ldots, j_l, j_{l+1}, \ldots, j_k)$. Once again, $\widehat{X}_v(t) = w$, and consequently $w \in X_v^{\max}$. $\square$

We easily check that:

Corollary 3.3.2. *If $w \in X_v^{\max}$, then $X_w^{\max} \subseteq X_v^{\max}$. Particularly, if $E \in \mathcal{AR}^k$ and* $\min_{\prec}(E) \in X_v^{\max}$*, then $E \subset X_v^{\max}$.*

Since the only elements in the range of $\widehat{X}_v$ that are $\prec$-smaller than v are the initial segments of v, we conclude that:

Corollary 3.3.3. *If $s \prec v$ and $s \sqsubset v$, then $s \in$* $\mathrm{ran}(\widehat{X}_v)$.

Chapter 4

The Banach Spaces $T(\mathcal{A}_d^k, \theta)$

We begin this chapter with the construction of new Banach spaces using barriers in high dimensional Ellentuck spaces [9]. The motivation for our construction and many ideas behind our results came from the Tsirelson type spaces $T(\mathcal{A}_d, \theta)$. Then, we show that these spaces contain arbitrary large copies of ℓ_∞^n and specific block subspaces isomorphic to ℓ_p. Moreover, we prove that they are ℓ_p-saturated and not isomorphic to each other.

4.1 Construction Framework

4.1.1 Preliminary Notation

Set $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{Z}^+ := \mathbb{N} \setminus \{0\}$. For the rest of this chapter, fix $d, k \in \mathbb{Z}^+$ with $d, k \geq 2$ and $\theta \in \mathbb{R}$ with $0 < \theta < 1$. Given $E, F \in \text{FIN}(\omega^{\not\downarrow k})$, we write $E < F$ (resp. $E \leq F$) to denote that $\max_\prec(E) \prec \min_\prec(F)$ (resp. $\max_\prec(E) \preceq \min_\prec(F)$), and in this case we say that E and F are successive. Similarly, for $v \in \omega^{\not\downarrow k}$, we write $v < E$ (resp. $v \leq E$) whenever $\{v\} < E$ (resp. $\{v\} \leq E$).

By $c_{00}(\omega^{\not\angle k})$ we denote the vector space of all functions $x : \omega^{\not\angle k} \to \mathbb{R}$ such that the set $\mathrm{supp}\,(x) := \{v \in \omega^{\not\angle k} : x(v) \neq 0\}$ is finite. Usually we write x_v instead of $x(v)$. We can extend the orders defined above to vectors $x, y \in c_{00}(\omega^{\not\angle k})$: $x < y$ (resp. $x \leq y$) iff $\mathrm{supp}\,(x) < \mathrm{supp}\,(y)$ (resp. $\mathrm{supp}\,(x) \leq \mathrm{supp}\,(y)$).

Remember that $\omega^{\not\angle k} = \{\vec{u}_1, \vec{u}_2, \ldots\}$. Denote by $(e_{\vec{u}_n})_{n=1}^{\infty}$ the canonical basis of $c_{00}(\omega^{\not\angle k})$. To simplify notation, we will usually write e_n instead of $e_{\vec{u}_n}$. So, if $x \in c_{00}(\omega^{\not\angle k})$, then $x = \sum_{n=1}^{\infty} x_{\vec{u}_n} e_{\vec{u}_n} = \sum_{n=1}^{m} x_{\vec{u}_n} e_{\vec{u}_n}$ for some $m \in \mathbb{Z}^+$. Using the above convention, we will write $x = \sum_{n=1}^{\infty} x_n e_n = \sum_{n=1}^{m} x_n e_n$. If $E \in \mathcal{AR}^k$, we put $Ex := \sum_{v \in E} x_v e_v$.

4.1.2 Definition of $T(\mathcal{A}_d^k, \theta)$

The Banach spaces that we introduce in this section have their roots (as all subsequent constructions [4], [15], [1], [12], [2]) in Tsirelson's fundamental discovery of a reflexive Banach space T with an unconditional basis not containing c_0 or ℓ_p with $1 \leq p < \infty$ [17]. Based on these constructions, we present the following definitions.

Definition 4.1.1. Set $\mathcal{A}_d^k := \bigcup_{i=1}^{d} \mathcal{AR}_i^k$ and let $m \in \{1, 2, \ldots, d\}$. We say that $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ is $\mathcal{A}_d^k$*-admissible*, or simply *admissible*, if and only if there exists $\{v_1, v_2, \ldots, v_m\} \in \mathcal{A}_d^k$ such that $v_1 \leq E_1 < v_2 \leq E_2 < \cdots < v_m \leq E_m$.

Notice that $\mathcal{A}_d^1$ can be identified with the member $\mathcal{A}_d$ of the classical low complexity hierarchy. In general, $\mathcal{A}_d^k$ is the downward closure of the barrier $\mathcal{AR}_d^k$ on $\omega^{\not\angle k} \in \mathcal{E}_k$.

Before proceeding with the definition of the Banach space $T(\mathcal{A}_d^k, \theta)$, we provide an example of an $\mathcal{A}_7^2$-admissible sequence:

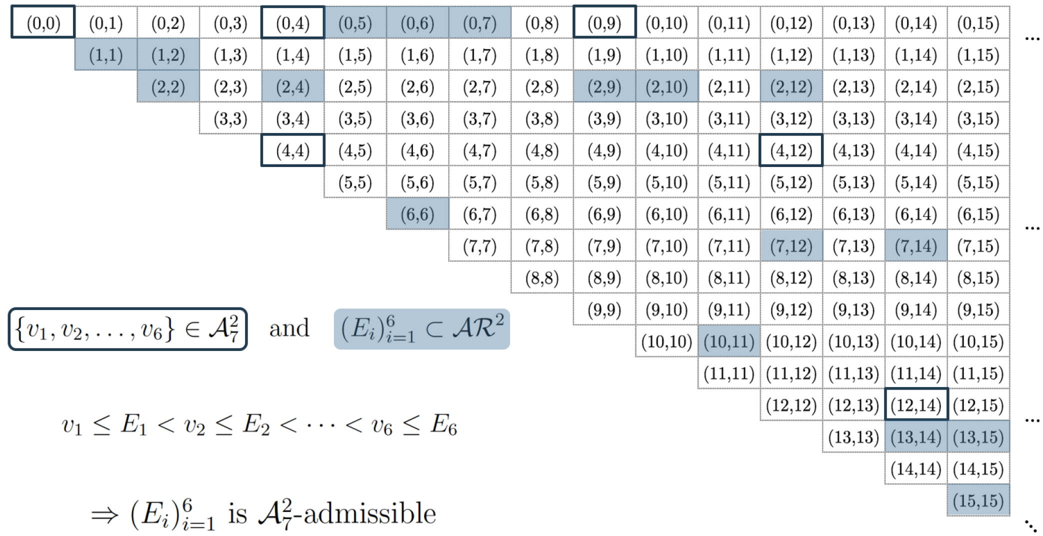

Figure 4.1: An $\mathcal{A}_7^2$-admissible sequence.

For $x = \sum_{n=1}^{\infty} x_n e_n \in c_{00}(\omega^{\not{l}k})$ and $j \in \mathbb{N}$, we define a non-decreasing sequence of norms on $c_{00}(\omega^{\not{l}k})$ as follows:

- $|x|_0 := \max_{n \in \mathbb{Z}^+} |x_n|$,

- $|x|_{j+1} := \max \left\{ |x|_j, \theta \max \left\{ \sum_{i=1}^{m} |E_i x|_j : 1 \leq m \leq d, (E_i)_{i=1}^m \ \mathcal{A}_d^k\text{-admissible} \right\} \right\}$.

For fixed $x \in c_{00}(\omega^{\not{l}k})$, the sequence $(|x|_j)_{j \in \mathbb{N}}$ is bounded above by the $\ell_1(\omega^{\not{l}k})$-norm of x. Therefore, we can set

$$\|x\|_{(\mathcal{A}_d^k, \theta)} := \sup_{j \in \mathbb{N}} |x|_j .$$

Clearly, $\|\cdot\|_{(\mathcal{A}_d^k, \theta)}$ is a norm on $c_{00}(\omega^{\not{l}k})$.

Definition 4.1.2. The completion of $c_{00}(\omega^{\not< k})$ with respect to the norm $\|\cdot\|_{(\mathcal{A}_d^k,\theta)}$ is denoted by $(T(\mathcal{A}_d^k,\theta),\|\cdot\|)$.

Notice that $T(\mathcal{A}_d^1,\theta)$ denotes the Tsirelson type space $T(\mathcal{A}_d,\theta)$. For $v \in \omega^{\not< k}$ and $x \in T(\mathcal{A}_d^k,\theta)$ we also write $v < x$ whenever $v < \operatorname{supp}(x)$. From the preceding definition we have the following:

Proposition 4.1.3. *If $x \in c_{00}(\omega^{\not< k})$ and $|\operatorname{supp}(x)| = n$, then $|x|_n = |x|_{n+1} = \cdots$.*

Therefore, we conclude that for every $x \in c_{00}(\omega^{\not< k})$ we have

$$\|x\|_{(\mathcal{A}_d^k,\theta)} = \max_{j\in\mathbb{N}} |x|_j \,.$$

The following propositions follow by standard arguments (see Proposition 2 in [15]):

Proposition 4.1.4. *$(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis of $T(\mathcal{A}_d^k,\theta)$.*

Proposition 4.1.5. *For $x = \sum_{n=1}^{\infty} x_n e_n \in T(\mathcal{A}_d^k,\theta)$ it follows that*

$$\|x\| = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^{m} \|E_i x\| : 1 \le m \le d, (E_i)_{i=1}^{m}\ \mathcal{A}_d^k\text{-admissible}\right\}\right\},$$

where $\|x\|_\infty := \sup_{n\in\mathbb{Z}^+} |x_n|$.

On Part A, Chapter 1 of [3] the Tsirelson type space $T(\mathcal{A}_d,\theta)$ is defined, where $\mathcal{A}_d := \{F \subset \mathbb{Z}^+ : |F| \le d\}$ is a member of the low complexity hierarchy. Moreover, the following remarkable theorem is presented:

Theorem 4.1.6 (Bellenot [4]). *If $d\theta > 1$, then for every $x \in T(\mathcal{A}_d, \theta)$,*

$$\frac{1}{2d} \|x\|_p \leq \|x\|_{T(\mathcal{A}_d,\theta)} \leq \|x\|_p \,,$$

where $d\theta = d^{1/p}$ and $\|\cdot\|_p$ denotes the ℓ_p-norm.

4.2 Subspaces of $T(\mathcal{A}_d^k, \theta)$ isomorphic to ℓ_∞^N

Let $s \in \omega^{\not\uparrow \leq k}$. The tree generated by s and the Banach space associated to it are given by

$$\tau^k[s] := \left\{v \in \omega^{\not\uparrow k} : s \sqsubseteq v\right\} \qquad \text{and} \qquad T^k[s] := \overline{\text{span}}\{e_v : v \in \tau^k[s]\},$$

respectively. Along this section, let $N \in \mathbb{Z}^+$ and $s_1, \ldots, s_N \in \omega^{\not\uparrow \leq k}$ be such that $|s_1| = \cdots = |s_N| < k$ and $s_1 \prec \cdots \prec s_N$. The following is a straightforward consequence of Corollary 3.3.3

Corollary 4.2.1. *If $v \in \omega^{\not\uparrow k}$ satisfies $s_N \prec v$, then there is at most one $i \leq N$ such that $X_v^{\max} \cap \tau^k[s_i] \neq \emptyset$.*

Proof. Since $|s_1| = \cdots = |s_N|$, at most one s_i can be an initial segment of v. □

It is useful to have an analogous result to the preceding corollary but related to approximations $E \in \mathcal{AR}^k$ instead of special maximal elements of $\mathcal{E}_k$:

Lemma 4.2.2. *Suppose $E \in \mathcal{AR}^k$ and set $v := \min_\prec(E)$. If $s \prec v$ and $s \sqsubset v$, then $E \cap \tau^k[s] \neq \emptyset$.*

We will study the Banach space structure of the subspaces of $T(\mathcal{A}_d^k, \theta)$ of the form $Z := T^k[s_1] \oplus T^k[s_2] \oplus \cdots \oplus T^k[s_N]$. Since $(e_{\vec{u}_n})_{n=1}^{\infty}$ is 1-unconditional, we can decompose Z as $F \oplus C$, where

$$F = \overline{\operatorname{span}}\{e_v \in Z : v \in \omega^{\not< k}, v \preceq s_N\} \quad \text{and} \quad C = \overline{\operatorname{span}}\{e_v \in Z : v \in \omega^{\not< k}, s_N \prec v\}.$$

By setting $k = 2, N = 4, s_1 = (4), s_2 = (6), s_3 = (8)$, and $s_4 = (10)$, the following figure shows the elements of $\omega^{\not< 2}$ used to generate the subspaces F (blue, dashed outline) and C (green, thicker outline) in which we decompose the subspace $T^2[(4)] \oplus T^2[(6)] \oplus T^2[(8)] \oplus T^2[(10)]$:

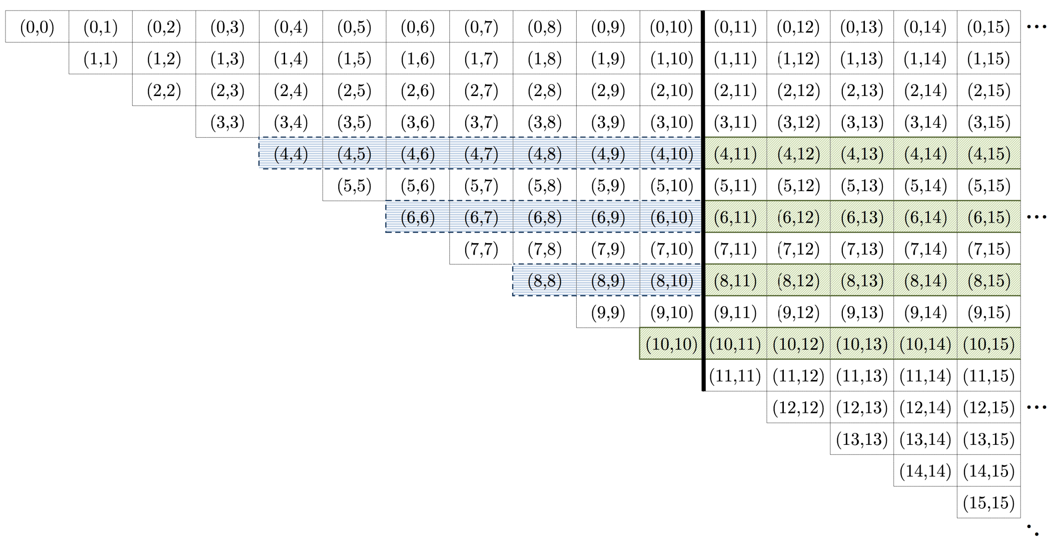

Figure 4.2: Elements of $\omega^{\not< 2}$ used to generate $T^2[(4)] \oplus T^2[(6)] \oplus T^2[(8)] \oplus T^2[(10)]$.

Applying Corollary 4.2.1 and Lemma 4.2.2 we have:

Lemma 4.2.3. *Let $E \in \mathcal{AR}^k$ be such that $s_N \prec \min_{\prec}(E)$. If $E[T(\mathcal{A}_d^k, \theta)] := \overline{\operatorname{span}}\{e_w : w \in E\}$, then either $E[T(\mathcal{A}_d^k, \theta)] \cap C = \emptyset$, or there is exactly one $i \leq N$ such that $E[T(\mathcal{A}_d^k, \theta)] \cap C \subset T^k[s_i]$.*

Proof. Suppose that $E[T(\mathcal{A}_d^k,\theta)]\cap C \neq \emptyset$ and set $v := \min_{\prec}(E)$. Then, by Corollary 4.2.1, there is exactly one $i \in \{1,\ldots,N\}$ such that $X_v^{\max} \cap \tau^k[s_i] \neq \emptyset$; consequently, $s_i \sqsubset v$ and $E \cap \tau^k[s_j] = \emptyset$ for any $j \in \{1,\ldots,N\}, j \neq i$. By hypothesis, $s_i \preceq s_N \prec v$. Applying Lemma 4.2.2 we conclude that $E \cap \tau^k[s_i] \neq \emptyset$. Hence, $E[T(\mathcal{A}_d^k,\theta)] \cap C \subset T^k[s_i]$. $\square$

This lemma helps us establish the presence of arbitrary large copies of ℓ_∞^N inside $T(\mathcal{A}_d^k,\theta)$:

Theorem 4.2.4. *Suppose that $s_1 \prec s_2 \prec \cdots \prec s_N$ belong to $\omega^{\not\downarrow<k}$ and that $|s_1| = \cdots = |s_N| < k$. Let $v \in \omega^{\not\downarrow k}$ with $s_N \prec v$ and suppose that $x \in \sum_{i=1}^N \oplus T^k[s_i]$ satisfies $v < x$. If we decompose x as $x_1 + \cdots + x_N$ with $x_i \in T^k[s_i]$, then*

$$\max_{1\le i\le N} \|x_i\| \le \|x\| \le \frac{\theta(d-1)}{1-\theta} \max_{1\le i\le N} \|x_i\|.$$

In particular, if $\|x_1\| = \cdots = \|x_N\| = 1$, span$\{x_1,\ldots,x_N\}$ is isomorphic to ℓ_∞^N in a canonical way and the isomorphism constant is independent of N and of the x_i's.

Proof. Suppose $x \in C$ is finitely supported. Since the basis of $T(\mathcal{A}_d^k,\theta)$ is unconditional, we have the lower bound $\max_{1\le i\le N} \|x_i\| \le \|x\|$. We will check the upper bound. Let $m \in \{1,\ldots,d\}$ and $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ be an admissible sequence such that $\|x\| = \theta\sum_{i=1}^m \|E_i x\|$.

Without loss of generality we assume that $E_1 x \neq 0$, so that $s_N \prec \min_{\prec}(E_2)$. By Lemma 4.2.3, when $j \ge 2$, we have $E_j x = E_j x_i$ for some $i \in \{1,\ldots,N\}$. Then it follows that $\|E_j x\| \le \max_{1\le i\le N} \|x_i\|$. Consequently,

$$\|x\| \le \theta\|E_1 x\| + \theta(d-1)\max_{1\le i\le N}\|x_i\|$$

Repeat the argument for E_1x. Find $m' \in \{1, \ldots, d\}$ and an admissible sequence $(F_i)_{i=1}^{m'} \subset \mathcal{AR}^k$ such that $\|E_1x\| = \theta \sum_{i=1}^{m'} \|F_i(E_1x)\|$. We can assume that $F_1(E_1x) \neq 0$, and applying Lemma 4.2.3 once again we conclude that for $j \geq 2$, $\|F_j(E_1x)\| \leq \max_{1\leq i\leq N} \|x_i\|$. Then,

$$\|x\| \leq \theta \left(\theta \|F_1(E_1x)\| + \theta(d-1) \max_{1\leq i\leq N} \|x_i\| \right) + \theta(d-1) \max_{1\leq i\leq N} \|x_i\| .$$

Iterating this process we conclude that

$$\|x\| \leq \sum_{n=1}^{\infty} \theta^n (d-1) \max_{1\leq i\leq N} \|x_i\| \leq \frac{\theta(d-1)}{1-\theta} \max_{1\leq i\leq N} \|x_i\| .$$

□

4.3 Block Subspaces of $T(\mathcal{A}_d^k, \theta)$ isomorphic to ℓ_p

For the rest of this chapter suppose that $d\theta > 1$ and let $p \in (1, \infty)$ be determined by the equation $d\theta = d^{1/p}$. Bellenot proved that $T(\mathcal{A}_d^1, \theta)$ is isomorphic to ℓ_p (see Theorem 4.1.6). The same result was then proved by Argyros and Deliyanni in [1] with different arguments which can be extended to more general cases like ours. In this section we show that we can find many copies of ℓ_p spaces inside $T(\mathcal{A}_d^k, \theta)$ for $k \geq 2$:

Theorem 4.3.1. *Suppose that $(x_i)_{i=1}^{\infty}$ is a normalized block sequence in $T(\mathcal{A}_d^k, \theta)$ and that we can find a sequence $(v_i)_{i=1}^{\infty} \subset \omega^{\not{\,}k}$ such that:*

(1) $\operatorname{supp}(x_i) \subset X_{v_i}^{\max}$ *and* $v_{i+1} \in X_{v_i}^{\max}$.

(2) $v_i \leq x_i < v_{i+1}$.

(3) *If* $u_i = \max_{\prec}(\operatorname{supp}(x_i))$, *then* $\max(u_i) < \max(v_{i+1})$.

Then, (x_i) *is equivalent to the basis of* ℓ_p.

Corollary 4.3.2. *If* $s \in \omega^{\not<\leq k}$ *and* $|s| = k-1$, *then* $T^k[s]$ *is isomorphic to* ℓ_p.

The proof of Theorem 4.3.1 is established in the following subsections. Throughout these subsections, we introduce additional definitions and prove some auxiliary results that will be used beyond this section.

Before doing so, we illustrate the restrictions that hypotheses (1)-(3) in Theorem 4.3.1 are imposing on the block sequence $(x_i)_{i=1}^{\infty}$. The figure shows, in the case $k = 2$, the support of the first three elements of such a block sequence (thicker outline) together with possible v_1, v_2, v_3 (dashed outline) and their corresponding special maximal elements. Informally, these hypotheses are forcing the support of each x_i to be inside nested special maximal elements and to be nicely separated from one another.

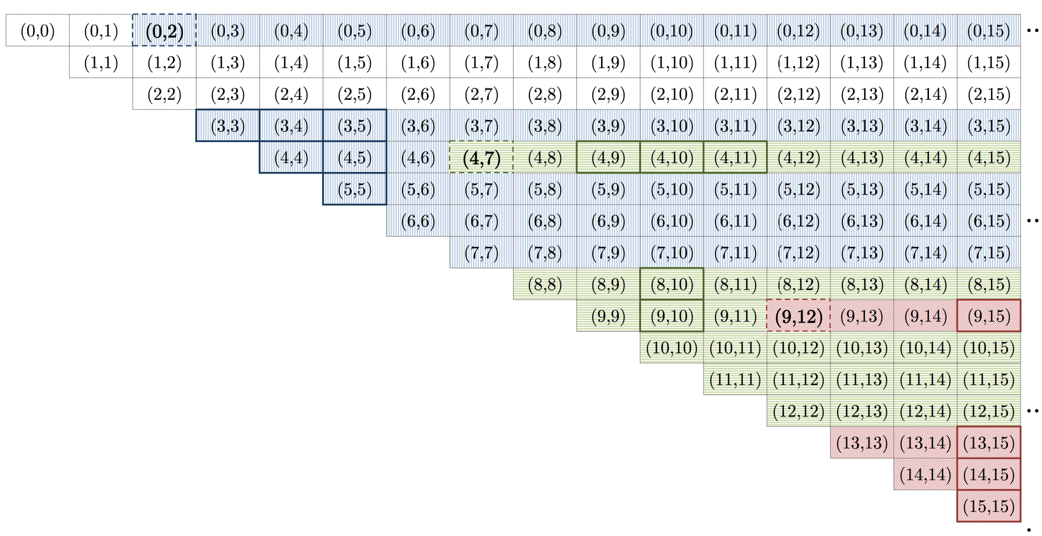

Figure 4.3: Hypotheses (1)-(3) in Theorem 4.3.1.

4.3.1 Lower Bound for Theorem 4.3.1

The following lemmas are essential to establish a lower ℓ_p-estimate.

Lemma 4.3.3. *Suppose that $(q_i)_{i=1}^{\infty} \subset \mathbb{N}$ is such that $q_1 < q_2 < \cdots$. Then, there exists $X = \{w_1, w_2, \ldots\} \in \mathcal{E}_k$ such that for every $i \in \mathbb{Z}^+, \max(w_i) = q_i$ and all the terms of w_i are in $(q_i)_{i=1}^{\infty}$.*

Proof. Following Definition 3.1.5 we will construct inductively an $\mathcal{E}_k$-tree $\widehat{X}$ that determines X. First, set $\widehat{X}((0)) := (q_1)$. Next, having defined $\widehat{X}(\vec{s}_m)$, we define $\widehat{X}(\vec{s}_{m+1})$ based on the following cases:

Case 1: $|\vec{s}_{m+1}| = |\vec{s}_m| + 1$ and $|\vec{s}_{m+1}| < k$. If $\widehat{X}(\vec{s}_m) = (n_1, \ldots, n_{|\vec{s}_m|})$, set $\widehat{X}(\vec{s}_{m+1}) := (n_1, \ldots, n_{|\vec{s}_m|}, n_{|\vec{s}_m|})$.

Case 2: $|\vec{s}_{m+1}| = |\vec{s}_m| + 1$ and $|\vec{s}_{m+1}| = k$. If $\widehat{X}(\vec{s}_m) = (n_1, \ldots, n_{k-2}, q_j)$ for some $j \in \mathbb{Z}^+$, set $\widehat{X}(\vec{s}_{m+1}) := (n_1, \ldots, n_{k-2}, q_j, q_{j+1})$.

Case 3: $|s_{m+1}| = |s_m| = k$. If $\widehat{X}(\vec{s}_m) = (n_1, \ldots, n_{k-1}, q_j)$ for some $j \in \mathbb{Z}^+$, and $\vec{s}_{m+1} = (l_1, \ldots, l_{k-1}, l_k)$, then set $\widehat{X}(\vec{s}_{m+1}) := \widehat{X}((l_1, \ldots, l_{k-1})) \frown (q_{j+1})$.

Case 4: $|\vec{s}_{m+1}| < |\vec{s}_m|$. This case only happens when $|\vec{s}_m| = k$. Find the common initial segment s of $\vec{s}_m$ and $\vec{s}_{m+1}$ (might be the empty sequence). Notice that $|s| \leq |\vec{s}_{m+1}| - 1$ by definition of $\prec$. Then, if $\widehat{X}(\vec{s}_m) = (n_1, \ldots, n_k)$, set $\widehat{X}(\vec{s}_{m+1}) := \widehat{X}(s) \frown (n_k)$. $\square$

Lemma 4.3.4. *Assume that all the hypotheses of Theorem 4.3.1 are satisfied. If $m \in \{1, \ldots, d\}$ and $E_1 < E_2 < \cdots < E_m$ are finite subsets of $\mathbb{Z}^+$, then there exists an $\mathcal{A}_d^k$-admissible sequence $(F_i)_{i=1}^{m} \subset \mathcal{AR}^k$ such that*

$$E_i \subset \left\{ j \in \mathbb{Z}^+ : \operatorname{supp}(x_{2j}) \subseteq F_i \right\}.$$

Proof. For $i \in \{1, \ldots, m\}$, set $n_i := \min(E_i)$ and $n_{m+1} := \max(E_m) + 1$. It is helpful to keep the following picture in mind throughout this proof:

$$\cdots \leq x_{2(n_i-1)} < v_{2n_i-1} \leq x_{2n_i-1} < v_{2n_i} \leq x_{2n_i} < \cdots \leq x_{2(n_{i+1}-1)} < \cdots .$$

Set $q_i := \max(v_{2n_i-1})$. By hypothesis (3) in Theorem 4.3.1 it is the case that $q_1 < q_2 < \cdots < q_m$. Then, applying Lemma 4.3.3, we can find $\{w_1, w_2, \ldots, w_m\}$ in $\mathcal{AR}_d^k$ such that $\max(w_i) = q_i$ and all the terms of w_i are in $\{q_1, q_2, \ldots, q_m\}$. Consequently,

$$\cdots \leq x_{2(n_i-1)} < w_i \prec v_{2n_i} \leq x_{2n_i} < \cdots . \tag{4.3.1}$$

Now, from hypotheses (1) and (2) in Theorem 4.3.1, we know that $X_{v_{2n_i}}^{\max}$ contains the support of x_j for any $j \geq 2n_i$. Therefore, define F_i as the initial segment of $X_{v_{2n_i}}^{\max}$ for which

$$\max_{\prec}(F_i) = \max_{\prec}\left(\text{supp}\left(x_{2(n_{i+1}-1)}\right)\right). \tag{4.3.2}$$

By construction, $F_i \in \mathcal{AR}^k$ and $\min_{\prec}(F_i) = v_{2n_i}$. Moreover, F_i also contains the supports of $x_{2n_i}, x_{2(n_i+1)}, x_{2(n_i+2)}, \ldots, x_{2(n_{i+1}-1)}$. Thus, from equations (4.3.1) and (4.3.2), we have:

$$w_1 \prec v_{2n_1} \leq F_1 < w_2 \prec v_{2n_2} \leq F_2 < \cdots \leq F_{m-1} < w_m \prec v_{2n_m} \leq F_m.$$

Hence, $F_1, F_2, \ldots, F_m \in \mathcal{AR}^k$ is the desired admissible sequence. $\square$

The following proposition provides a lower ℓ_p-estimate for Theorem 4.3.1.

Proposition 4.3.5. *Under the same hypotheses of Theorem 4.3.1, we have:*

$$\frac{1}{2d}\left(\sum\nolimits_i |a_i|^p\right)^{1/p} \leq \left\|\sum\nolimits_i a_i x_i\right\|.$$

Proof. Denote by (t_i) the canonical basis of $T(\mathcal{A}_d^1, \theta)$. In order to avoid confusion, we will write $\|\cdot\|_1$ to denote the norm on $T(\mathcal{A}_d^1, \theta)$.

We will prove a lower ℓ_p-estimate for the sequences (x_{2n}) and (x_{2n-1}). Since the closed span of (x_{2n}) and (x_{2n+1}) are complemented in the closed span of (x_n), the general result follows. We will obtain the estimate for (x_{2n}). The other case is similar. The goal is to show that $\|\sum_i a_i t_i\|_1 \leq \|\sum_i a_i x_{2i}\|$ for any sequence $(a_i) \subset \mathbb{R}$ with finitely many non-zero elements.

Let $x = \sum_i a_i t_i$. Following Bellenot [4], either $\|x\|_1 = \max_i |a_i|$, or there exist $m \in \{1, \ldots, d\}$ and $E_1 < E_2 < \cdots < E_m$ such that $\|x\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1$. For each $j \in \{1, \ldots, m\}$, either $\|E_j x\|_1 = \max_i\{|a_i| : i \in E_j\}$, or there exist $m' \in \{1, \ldots, d\}$ and $E_{j1} < E_{j2} < \cdots < E_{jm'}$ subsets of E_j such that $\|E_j x\|_1 = \sum_{l=1}^{m'} \theta \|E_{jl} x\|_1$. Since the sequence (a_i) has only finitely many non-zero terms, this process ends and x is normed by a tree.

We will prove the result by induction on the height of the tree. If $\|x\|_1 = \max_i |a_i|$, the height of the norming tree is zero, and since (x_{2n}) is unconditional, $\max_i |a_i| \leq \|\sum_i a_i x_{2i}\|$.

Suppose that the result is proved for elements of $T(\mathcal{A}_d^1, \theta)$ that are normed by trees of height less than or equal to h and that x is normed by a tree of height $h+1$. Then, there exist $m \in \{1, \ldots, d\}$ and $E_1 < E_2 < \cdots < E_m$ such that $\|x\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1$ and each $E_j x$ is normed by a tree of height less than or equal to h.

Now, for each $j \in \{1, \ldots, m\}$, set $n_j := \min(E_j)$ and $n_{m+1} := \max(E_m) + 1$. Then, apply Lemma 4.3.4 to find an $\mathcal{A}_d^k$-admissible sequence $(F_j)_{j=1}^m$ such that each F_j contains the supports of $x_{2n_j}, x_{2(n_j+1)}, x_{2(n_j+2)}, \ldots, x_{2(n_{j+1}-1)}$.

By the induction hypothesis,

$$\|E_j x\|_1 = \left\| \sum\nolimits_{l \in E_j} a_l t_l \right\|_1 \leq \left\| \sum\nolimits_{l \in E_j} a_l x_{2l} \right\|.$$

Since $E_j \subseteq \{n_j, n_j + 1, \ldots, n_{j+1} - 1\}$, it follows that

$$\|E_j x\|_1 \leq \left\| \sum\nolimits_{l=n_j}^{n_{j+1}-1} a_l x_{2l} \right\| = \|F_j z\|,$$

where $z = \sum_i a_i x_{2i}$. Therefore,

$$\left\| \sum\nolimits_i a_i t_i \right\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1 \leq \sum_{j=1}^m \theta \|F_j z\| \leq \|z\| = \left\| \sum\nolimits_i a_i x_{2i} \right\|.$$

The result follows now applying Theorem 4.1.6. $\square$

4.3.2 Dual Norm

To establish a upper ℓ_p-estimate we will adapt an alternative and useful description of the norm on $T(\mathcal{A}_d^1, \theta)$ introduced by Argyros and Deliyanni [1] to our spaces. In that regard, the following definition plays a key role.

Definition 4.3.6. Let $m \in \{1, \ldots, d\}$. A sequence $(F_i)_{i=1}^m \subset \text{FIN}(\omega^{\not<k})$ is called *almost admissible* if there exists an $\mathcal{A}_d^k$-admissible sequence $(E_n)_{n=1}^d$ such that $F_i \subseteq E_{n_i}$, where $n_1, \ldots, n_m \in \{1, \ldots, d\}$ are such that $n_1 < n_2 < \cdots < n_m$.

A standard alternative description of the norm of the space $T(\mathcal{A}_d^k, \theta)$, closer to the spirit of Tsirelson space, is as follows. Let $K_0 := \{\pm e_i^* : i \in \mathbb{Z}^+\}$, and for $n \in \mathbb{N}$,

$$K_{n+1} := K_n \bigcup \{\theta(f_1 + \cdots + f_m) : m \leq d, (f_i)_{i=1}^m \subset K_n\},$$

where (supp $(f_i))_{i=1}^m$ is almost admissible. Then, set $K := \bigcup_{n\in\mathbb{N}} K_n$. Now, for each $n \in \mathbb{N}$ and fixed $x \in c_{00}(\omega^{\not\downarrow k})$, define the following non-decreasing sequence of norms:

$$|x|_n^* := \max\{f(x) : f \in K_n\}.$$

Lemma 4.3.7. *For every $n \in \mathbb{N}$ and $x \in c_{00}(\omega^{\not\downarrow k})$ we have $|x|_n = |x|_n^*$.*

Proof. Clearly, $|x|_0 = |x|_0^*$ for every $x \in c_{00}(\omega^{\not\downarrow k})$. So, let $n \in \mathbb{Z}^+$ and suppose that $|y|_j = |y|_j^*$ for every $j \in \mathbb{N}, j < n$ and $y \in c_{00}(\omega^{\not\downarrow k})$.

If $|x|_n = |x|_{n-1}$, then $|x|_n = |x|_{n-1}^* \leq |x|_n^*$. Suppose $|x|_n \neq |x|_{n-1}$. Let $m \in \{1, \ldots, d\}$ and $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ be an admissible sequence such that $|x|_n = \theta \sum_{i=1}^m |E_i x|_{n-1}$. Then, $|x|_n = \theta \sum_{i=1}^m |E_i x|_{n-1}^* = \theta \sum_{i=1}^m f_i(E_i x)$ for some $(f_i)_{i=1}^m \subset K_{n-1}$. Define, for each $i \in \{1, \ldots, m\}$, a new functional f_i' satisfying $f_i'(y) = f_i(E_i y)$ for every $y \in c_{00}(\omega^{\not\downarrow k})$. This implies that supp $(f_i') =$ supp $(f_i) \cap E_i$. Then, $(f_i')_{i=1}^m \subset K_{n-1}$ with (supp $(f_i'))_{i=1}^m$ almost admissible and $f_i'(E_i x) = f_i(E_i x)$. So,

$$\theta \sum_{i=1}^m f_i(E_i x) = \theta \sum_{i=1}^m f_i'(E_i x) \leq |E_i x|_n^* \leq |x|_n^*;$$

therefore, $|x|_n \leq |x|_n^*$.

Now, let $f = \theta(f_1 + \cdots + f_m)$ for some $m \in \{1, \ldots, d\}$ and $(f_i)_{i=1}^m \subset K_{n-1}$ with (supp $(f_i))_{i=1}^m$ almost admissible. Then,

$$f(x) = \theta \sum_{i=1}^{m} f_i(x) \leq \theta \sum_{i=1}^{m} |\text{supp}\,(f_i)\,x|_{n-1}^* = \theta \sum_{i=1}^{m} |\text{supp}\,(f_i)\,x|_{n-1}\,.$$

Since $(\text{supp}\,(f_i))_{i=1}^m$ is almost admissible, there exists an admissible sequence $(E_i)_{i=1}^d \subset \mathcal{AR}^k$ such that $\text{supp}\,(f_i) \subseteq E_{n_i}$, where $n_1, \ldots, n_m \in \{1, \ldots, m\}$ and $n_1 < \cdots < n_m$. So,

$$\theta \sum_{i=1}^{m} |\text{supp}\,(f_i)\,x|_{n-1} \leq \theta \sum_{i=1}^{m} |E_{n_i} x|_{n-1} \leq |x|_n\,;$$

hence, by definition of $|\cdot|_n^*$, we conclude that $|x|_n^* \leq |x|_n$. $\square$

Consequently, an alternative description of the norm on $T(\mathcal{A}_d^k, \theta)$ is:

Proposition 4.3.8. *For every $x \in T(\mathcal{A}_d^k, \theta)$,*

$$\|x\| = \text{sup}\,\{f(x) : f \in K\}\,.$$

4.3.3 Upper Bound for Theorem 4.3.1

For $m \in \{1, \ldots, d\}$ we say that $f_1, \ldots, f_m \in K$ are successive if $\text{supp}\,(f_1) < \text{supp}\,(f_2) < \cdots < \text{supp}\,(f_m)$.

If $f \in K$, then there exists $n \in \mathbb{N}$ such that $f \in K_n$. The "complexity" of f increases as n increases. That is to say, for example, that the complexity of $f \in K_1$ is less than that of $g \in K_{10}$. This is captured in the following definition.

Definition 4.3.9. Let $n \in \mathbb{Z}^+$ and $\phi \in K_n \setminus K_{n-1}$. An *analysis of* ϕ is a sequence $(K_l(\phi))_{l=0}^n$ of subsets of K such that:

1. $K_l(\phi)$ consists of successive elements of K_l and $\bigcup_{f \in K_l(\phi)} \text{supp}\,(f) = \text{supp}\,(\phi)$.

2. If $f \in K_{l+1}(\phi)$, then either $f \in K_l(\phi)$ or there exist $m \in \{1, \ldots, d\}$ and successive $f_1, \ldots, f_m \in K_l(\phi)$ with $(\text{supp}\,(f_i))_{i=1}^m$ almost admissible and $f = \theta(f_1 + \cdots + f_m)$.
3. $K_n(\phi) = \{\phi\}$.

Note that, by definition of the sets K_n, each $\phi \in K$ has an analysis. Moreover, if $f_1 \in K_l(\phi)$ and $f_2 \in K_{l+1}(\phi)$, then either $\text{supp}\,(f_1) \subseteq \text{supp}\,(f_2)$ or $\text{supp}\,(f_1) \cap \text{supp}\,(f_2) = \emptyset$.

Let $\phi \in K_n \setminus K_{n-1}$ and let $(K_l(\phi))_{l=0}^n$ be a fixed analysis of ϕ. Suppose $(x_j)_{j=1}^N$ is a finite block sequence on $T(\mathcal{A}_d^k, \theta)$.

Following [1], for each $j \in \{1, \ldots, N\}$, set $l_j \in \{0, \ldots, n-1\}$ as the smallest integer with the property that there exists at most one $g \in K_{l_j+1}(\phi)$ with $\text{supp}\,(x_j) \cap \text{supp}\,(g) \neq \emptyset$.

Then, define the *initial part* and *final part* of x_j with respect to $(K_l(\phi))_{l=0}^n$, and denote them respectively by x_j' and x_j'', as follows. Let

$$\left\{f \in K_{l_j}(\phi) : \text{supp}\,(f) \cap \text{supp}\,(x_j) \neq \emptyset\right\} = \{f_1, \ldots, f_m\},$$

where $f_1, \ldots, f_m$ are successive. Set

$$x_j' = (\text{supp}\,(f_1))x_j \qquad \text{and} \qquad x_j'' = (\cup_{i=2}^m \text{supp}\,(f_i))x_j.$$

The following is an useful property of the sequence $(x_j')_{j=1}^N$ (see [5]). The analogous property is true for $(x_j'')_{j=1}^N$.

Proposition 4.3.10. *For $l \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, N\}$, set*

$$A_l(x'_j) := \left\{ f \in K_l(\phi) : \text{supp}\,(f) \cap \text{supp}\,\left(x'_j\right) \neq \emptyset \right\}.$$

Then, there exists at most one $f \in A_l(x'_j)$ such that supp $(f) \cap$ supp $(x'_i) \neq \emptyset$ *for some $i \neq j$.*

Proof. Let $A_l(x'_j) = \{f_1, \ldots, f_m\}$, where $m \geq 2$ and $f_1, \ldots, f_m$ are successive. Obviously, only supp (f_1) and supp (f_m) could intersect supp (x'_i) for some $i \neq j$. We will prove that it is not possible for f_m.

Suppose, towards a contradiction, that supp $(f_m) \cap$ supp $(x'_i) \neq \emptyset$ for some $i > j$. Given that $m \geq 2$, we must have $l \leq l_j$. Consequently, there exists $g \in K_{l_j}(\phi)$ such that supp $(f_m) \subseteq$ supp (g). Since supp $(g) \cap$ supp $(x_j) \neq \emptyset$ and supp $(g) \cap$ supp $(x_i) \neq \emptyset$ for some $i > j$, the definition of x''_j implies that supp $(g) \cap$ supp $(x_j) \subseteq$ supp $\left(x''_j\right)$. Therefore, supp $(f_m) \cap$ supp $\left(x'_j\right) = \emptyset$, a contradiction. □

Following [3] and [5] we now provide an upper ℓ_p-estimate:

Proposition 4.3.11. *Let $(x_j)_{j=1}^N$ be a finite normalized block basis on $T(\mathcal{A}_d^k, \theta)$. Denote by $(t_n)_{n=1}^\infty$ the canonical basis of $T(\mathcal{A}_d^1, \theta)$. Then, for any $(a_j)_{j=1}^N \subset \mathbb{R}$, we have:*

$$\left\| \sum_{j=1}^N a_j x_j \right\| \leq \frac{2}{\theta} \left(\sum_{j=1}^N |a_j|^p \right)^{1/p}.$$

Proof. In order to avoid confusion, we will write $\|\cdot\|_1$ to denote the norm on $T(\mathcal{A}_d^1, \theta)$. By Proposition 4.3.8 and Theorem 4.1.6 it suffices to show that for every $\phi \in K$,

$$\left| \phi \left(\sum_{j=1}^N a_j x_j \right) \right| \leq \frac{2}{\theta} \left\| \sum_{j=1}^N a_j t_j \right\|_1.$$

By unconditionality we can assume that $x_1, \dots, x_N$ and ϕ are positive. Suppose $\phi \in K_n \setminus K_{n-1}$ for some $n \in \mathbb{Z}^+$, and let $(K_l(\phi))_{l=0}^n$ be an analysis of ϕ (see Definition 4.3.9). Next, split each x_j into its initial and final part, x_j' and x_j'', with respect to $(K_l(\phi))_{l=0}^n$.

We will show by induction on $l \in \{0, 1, \dots, n\}$ that for all $J \subseteq \{1, \dots, N\}$ and all $f \in K_l(\phi)$ we have

$$\left| f\left(\sum_{j\in J} a_j x_j' \right) \right| \leq \frac{1}{\theta} \left\| \sum_{j\in J} a_j t_j \right\|_1 \quad \text{and} \quad \left| f\left(\sum_{j\in J} a_j x_j'' \right) \right| \leq \frac{1}{\theta} \left\| \sum_{j\in J} a_j t_j \right\|_1 .$$

We prove the first inequality given that the other one is analogous. Let $J \subseteq \{1, \dots, N\}$ and set $y = \sum_{j\in J} a_j x_j'$.

If $f \in K_0(\phi)$, then $f = e_i^*$ for some $i \in \mathbb{Z}^+$. We want to prove that

$$|e_i^*(y)| \leq \frac{1}{\theta} \left\| \sum\nolimits_{j\in J} a_j t_j \right\|_1 .$$

Suppose that $e_i^*(y) \neq 0$. So, there exists exactly one $j_i \in J$ such that $e_i^*(x_{j_i}') \neq 0$. Applying Proposition 4.3.8 we have

$$|e_i^*(y)| = \left|e_i^*(a_{j_i} x_{j_i}')\right| \leq \left\|a_{j_i} x_{j_i}'\right\| \leq |a_{j_i}| \, \|x_{j_i}\| \leq \left\| \sum\nolimits_{j\in J} a_j t_j \right\|_1$$

since the basis of $T(\mathcal{A}_d^k, \theta)$ is unconditional, $\|x_{j_i}\| = 1$, and by definition

$$\max_{j\in J} |a_j| \leq \left\| \sum\nolimits_{j\in J} a_j t_j \right\|_1 .$$

Now suppose that the desired inequality holds for any $g \in K_l(\phi)$. We will prove it for $K_{l+1}(\phi)$. Let $f \in K_{l+1}(\phi)$ be such that $f = \theta(f_1 + \cdots + f_m)$, where

$f_1, \dots, f_m$ are successive elements in $K_l(\phi)$ with $(\operatorname{supp}(f_i))_{i=1}^m$ almost admissible. Then, $1 \leq m \leq d$. Without loss of generality assume that $f_i(y) \neq 0$ for each $i \in \{1, \dots, m\}$. Define the following sets:

$$I' := \left\{i \in \{1, \dots, m\} : \exists j \in J \text{ with } f_i(x'_j) \neq 0 \text{ and } \operatorname{supp}(f) \cap \operatorname{supp}\left(x'_j\right) \subseteq \operatorname{supp}(f_i)\right\}$$

and

$$J' := \left\{j \in J : \exists i \in \{1, \dots, m-1\} \text{ such that } f_i(x'_j) \neq 0 \text{ and } f_{i+1}(x'_j) \neq 0\right\}.$$

We claim that $|I'| + |J'| \leq m$. Indeed, if $j \in J'$, there exists $i \in \{1, \dots, m-1\}$ such that $f_i(x'_j) \neq 0$, $f_{i+1}(x'_j) \neq 0$. From the proof of Proposition 4.3.10 it follows that $f_{i+1}(x'_h) = 0$ for every $h \neq j$, which implies that $i+1 \notin I'$. Hence, we can define an injective map from J' to $\{1, \dots, m\} \setminus I'$ and we conclude that $|I'| + |J'| \leq m$.

Finally, for each $i \in I'$, set $D_i := \left\{j \in J : \operatorname{supp}(f) \cap \operatorname{supp}\left(x'_j\right) \subseteq \operatorname{supp}(f_i)\right\}$. Notice that for all $i \in I'$ we have $D_i \cap J' = \emptyset$. Then,

$$f(y) = \theta \left[\sum\nolimits_{i \in I'} f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j\right) + \sum\nolimits_{j \in J'} f(a_j x'_j)\right],$$

and consequently

$$|f(y)| \leq \theta \left[\sum\nolimits_{i \in I'} \left|f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j\right)\right| + \sum\nolimits_{j \in J'} \left|f(a_j x'_j)\right|\right].$$

However, by induction hypothesis,

$$\left|f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j\right)\right| \leq \frac{1}{\theta} \left\|\sum\nolimits_{j \in D_i} a_j t_j\right\|_1.$$

Moreover, for each $j \in J'$, we have $\left|f(a_j x'_j)\right| \leq \left\|a_j x'_j\right\| \leq \|a_j x_j\| = |a_j| = \|a_j t_j\|_1$. Hence,

$$\begin{aligned}|f(y)| &\leq \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| \sum\nolimits_{j \in D_i} a_j t_j \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right] \\ &= \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| D_i \left(\sum\nolimits_{j \in J} a_j t_j \right) \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right].\end{aligned}$$

Given that for every $i \in I'$, $D_i \cap J' = \emptyset$ and $|I'| + |J'| \leq m \leq d$, the family $\{D_i\}_{i \in I'} \cup \{\{j\}\}_{j \in J'}$ is $\mathcal{A}_d$-admissible. So, by definition of $\|\cdot\|_1$, we have

$$|f(y)| \leq \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| D_i \left(\sum\nolimits_{j \in J} a_j t_j \right) \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right] \leq \frac{1}{\theta} \left\| \sum\nolimits_{j \in J} a_j t_j \right\|_1.$$

□

4.4 $T(\mathcal{A}_d^k, \theta)$ is ℓ_p-saturated

In this section we prove that every infinite dimensional subspace of $T(\mathcal{A}_d^k, \theta)$ has a subspace isomorphic to ℓ_p.

Recall that the subspaces $T^k[s]$ for $s \in \omega^{\not\leq k}$ with $|s| < k$ decompose naturally into countable sums. Namely, if $s = (a_1, a_2, \ldots, a_l) \in \omega^{\not\leq k}$ and $l < k$, then $\tau^k[s] = \bigcup_{j=a_l}^{\infty} \tau^k[s \frown j]$, and therefore $T^k[s] = \sum_{j=a_l}^{\infty} \oplus T^k[s \frown j]$.

The next lemma tells us that we can find elements $v \in \tau^k[s]$ such that $X_v^{\max}$ contains arbitrary tails of the decomposition of $\tau^k[s]$. Its proof follows from the definition of the $\mathcal{E}_k$-tree $\widehat{X}_v$ that determines $X_v^{\max}$ (see paragraph preceding Lemma 3.3.1).

Lemma 4.4.1. *Let* $s = (a_1, a_2, \ldots, a_l) \in \omega^{\not\leq k}$ *with* $l < k$. *If* $m \in \mathbb{N}$ *with* $m \geq a_l$ *and* $v = s \frown (m, m, \ldots, m) \in \omega^{\not\leq k}$, *then* $X_v^{max} \cap \tau^k[s] = \bigcup_{j=m}^{\infty} \tau^k[s \frown j]$.

We now present the main result of this section:

Theorem 4.4.2. *Suppose that* Z *is an infinite dimensional subspace of* $T(\mathcal{A}_d^k, \theta)$. *Then, there exists* $Y \subseteq Z$ *isomorphic to* ℓ_p.

Proof. Let Z be an infinite dimensional subspace of $T(\mathcal{A}_d^k, \theta)$. After a standard perturbation argument, we can assume that Z has a normalized block basic sequence (x_n).

We will show that a subsequence of (x_n) is isomorphic to ℓ_p. From Proposition 4.3.11 we have that

$$\left\|\sum\nolimits_n a_n x_n\right\| \leq \frac{2}{\theta}\left(\sum\nolimits_n |a_n|^p\right)^{1/p}.$$

To obtain the lower bound we will find a subsequence and a projection Q onto a subspace of the form $T^k[s]$ such that $\left(Q\left(x_{n_j}\right)\right)$ has a lower ℓ_p-estimate.

To this end, assume that $Z \subset T^k[s]$ for some $s \in \omega^{\not\leq k}$ with $|s| < k$. Decompose $T^k[s] = \sum_{j=1}^{\infty} \oplus T^k[s_j]$, where for each $j \in \mathbb{Z}^+$, $s \sqsubset s_j$, $|s_j| = |s|+1$, and $s_j \prec s_{j+1}$. For each $j \in \mathbb{Z}^+$ let $Q_j : T^k[s] \to T^k[s_j]$ be the projection onto $T^k[s_j]$. Then we have the two cases:

<u>Case 1:</u> $\forall j \in \mathbb{Z}^+, Q_j x_n \to 0$.

<u>Case 2:</u> $\exists j_0 \in \mathbb{Z}^+$ such that $Q_{j_0} x_n \not\to 0$.

Let us look at Case 1 first. Let v_1 be the first element of $\tau^k[s]$. Since there exists p_1 such that supp $(x_1) \subset \bigcup_{j=1}^{p_1} \tau^k[s_j]$, applying Lemma 4.4.1 we can find $q_1 > p_1$ and $v_2 \in \tau^k[s]$ such that $v_1 \leq x_1 < v_2$ and $X_{v_2}^{\max} \cap \tau^k[s] = \bigcup_{j=q_1}^{\infty} \tau^k[s_j]$. Since

$Q_j x_n \to 0$ for $1 \le j \le q_1$ we can find $n_2 > 1$ and $y_2 \in T^k[s]$ such that $y_2 \approx x_{n_2}$ and $Q_j y_2 = 0$ for $1 \le j \le q_1$. Then we have

$$v_1 \le x_1 < v_2 < y_2 \text{ and } \operatorname{supp}(y_2) \subset X_{v_2}^{\max}.$$

We now repeat the argument. Since there exists p_2 such that $\operatorname{supp}(y_2) \subset \bigcup_{j=1}^{p_2} \tau^k[s_j]$, applying Lemma 4.4.1 we can find $q_2 > p_2$ and $v_3 \in \tau^k[s]$ such that $v_2 < y_2 < v_3$ and $X_{v_3}^{\max} \cap \tau^k[s] = \bigcup_{j=q_2}^{\infty} \tau^k[s_j]$. Since $Q_j x_n \to 0$ for $1 \le j \le q_2$, we can find $n_3 > n_2$ and $y_3 \in T^k[s]$ such that $y_3 \approx x_{n_3}$ and $Q_j y_3 = 0$ for $1 \le j \le q_2$. Then we have

$$v_1 \le x_1 < v_2 < y_2 < v_3 < y_3 \text{ and } \operatorname{supp}(y_2) \subset X_{v_2}^{\max}, \operatorname{supp}(y_3) \subset X_{v_3}^{\max}.$$

Proceeding this way we find a subsequence (x_{n_i}) and a sequence (y_i) such that y_i is close enough to x_{n_i}. Consequently, $\overline{\operatorname{span}}\{y_i\} \approx \overline{\operatorname{span}}\{x_{n_i}\}$ and

$$v_1 \le x_1 < v_2 < y_2 < v_3 < y_3 < \cdots \text{ and } \operatorname{supp}(y_i) \subset X_{v_i}^{\max} \text{ for } i > 1.$$

By Proposition 4.3.5, there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\left\| \sum_i a_i x_{n_i} \right\| \ge C_1 \left\| \sum_i a_i y_{n_i} \right\| \ge C_2 \left(\sum_i |a_i|^p \right)^{1/p}.$$

Let us look at Case 2 now. Find a subsequence (n_i) and $\delta > 0$ such that $\delta \le \|Q_{j_0} x_{n_i}\| \le 1$.

Let $W = \overline{\operatorname{span}}\{Q_{j_0} x_{n_i}\}$. We now apply the argument in Case 1 to the sequence $Q_{j_0} x_{n_1} < Q_{j_0} x_{n_2} < Q_{j_0} x_{n_3} < \cdots$. That is, first decompose $T^k[s_{j_0}] =$

$\sum_{j=1}^{\infty} \oplus T^k[t_j]$, where for every $j \in \mathbb{Z}^+$, $s_{j_0} \sqsubset t_j, |t_j| = |s_{j_0}| + 1, t_j \prec t_{j+1}$. Then, look at the two cases for the sequence $(Q_{j_0} x_{n_i})$. If Case 1 is true, $(Q_{j_0} x_{n_i})$ has a subsequence with a lower ℓ_p estimate, and therefore (x_{n_i}) has a subsequence with a lower ℓ_p estimate; and if Case 2 is true, we can repeat the argument for some t_j that has length strictly larger than the length of s_{j_0}. If Case 1 continues to be false, after a finite number of iterations of the same argument, the length of t_j will be equal to $k-1$, and therefore, applying Corollary 4.3.2, $T^k[t_j]$ would be isomorphic to ℓ_p. The result follows. $\square$

4.5 The spaces $T(\mathcal{A}_d^k, \theta)$ are not isomorphic to each other

In what follows, we establish that the Banach spaces that we have defined are not isomorphic to each other:

Theorem 4.5.1. *If $k_1 \neq k_2$, then $T(\mathcal{A}_d^{k_1}, \theta)$ is not isomorphic to $T(\mathcal{A}_d^{k_2}, \theta)$.*

We will prove by induction that when $k_1 > k_2$, $T(\mathcal{A}_d^{k_1}, \theta)$ does not embed in $T(\mathcal{A}_d^{k_2}, \theta)$. The idea behind our argument is that if we had an isomorphic embedding, we would map an ℓ_∞^N-sequence into an ℓ_p^N-sequence for arbitrarily large N. Proposition 4.5.4 below is a stronger and more technical statement from which Theorem 4.5.1 follows. The following lemmas are needed to establish this proposition.

Lemma 4.5.2. *If $s \in \omega^{\not\emptyset \leq k}$, and $|s| < k$, there exist $s_1 \prec s_2 \prec s_3 \prec \cdots$ such that $|s_i| = |s| + 1$ and $\tau^k[s] = \bigcup_{i=1}^{\infty} \tau^k[s_i]$. Consequently, we decompose $T^k[s] = \sum_{i=1}^{\infty} \oplus T^k[s_i]$ and for $m \in \mathbb{Z}^+$, there is a canonical projection $P_m : T^k[s] \to \sum_{i=1}^{m} \oplus T^k[s_i]$.*

Proof. If $s = (a_1, \dots, a_l)$, then $s_1 = (a_1, \dots, a_l, a_l)$, $s_2 = (a_1, \dots, a_l, a_l + 1)$, $s_3 = (a_1, \dots, a_l, a_l + 2), \cdots$. $\square$

Lemma 4.5.3. *Let $s \in \omega^{\not\leq k}$ with $|s| < k$. Let $v = \min \tau^k[s]$. Then $\tau^k[s] \subset X_v^{\max}$.*

Proof. If $s = (a_1, \dots, a_l)$, then we have that $v = (a_1, \dots, a_l, a_l, \dots, a_l)$ and the result follows from Lemma 3.3.1. $\square$

We are ready to state and prove the main proposition.

Proposition 4.5.4. *Let $s \in \omega^{\not\leq k_1}$ with $|s| < k_1$ and decompose $T^{k_1}[s] = \sum_{i=1}^{\infty} \oplus T^{k_1}[s_i]$ according to Lemma 4.5.2. Let $M \in \mathbb{Z}^+$ and $t_1, \dots, t_M \in \omega^{\not\leq k_2}$ such that $|t_1| = \cdots = |t_M| < k_2$.*

If $k_1 - |s| > k_2 - |t_1|$, then for every $n \in \mathbb{Z}^+$, $\sum_{i=n}^{\infty} \oplus T^{k_1}[s_i]$ does not embed into $T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$.

Proof. We proceed by induction. First assume that $k_2 - |t_1| = 1 < k_1 - |s|$. By Corollary 4.3.2, $T^{k_2}[t_i]$ is isomorphic to ℓ_p, and consequently so is $T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$. On the other hand, Theorem 4.2.4 guarantees that $T^{k_1}[s]$ has arbitrarily large copies of ℓ_∞^N. Hence, for every $n \in \mathbb{Z}^+$, there cannot be an embedding from $\sum_{i=n}^{\infty} \oplus T^{k_1}[s_i]$ into $T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$.

Suppose now that the result is true for $m \in \mathbb{Z}^+$ and let $k_2 - |t_1| = m+1 < k_1 - |s|$. We will show a simpler case first, when $M = 1$. Suppose, towards a contradiction, that there exists $n \in \mathbb{Z}^+$ and an isomorphism

$$\Phi : \sum_{i=n}^{\infty} \oplus T^{k_1}[s_i] \to T^{k_2}[t_1].$$

Decompose $T^{k_2}[t_1] = \sum_{j=1}^{\infty} \oplus T^{k_2}[r_j]$ according to Lemma 4.5.2. Find N large enough and $v \in \omega^{\not k_1}$ such that $s_n \prec s_{n+1} \prec \cdots \prec s_{n+N-1} \prec v$. We will find normalized $x_1 \in T^{k_1}[s_n], x_2 \in T^{k_1}[s_{n+1}], \ldots, x_N \in T^{k_1}[s_{n+N-1}]$ such that $v < x_i$ for $i \leq N$. Under this conditions Theorem 4.2.4 implies that span$\{x_1, \ldots, x\} \approx \ell_\infty^N$, with the isomorphism constant independent of N and of the x_i's.

Let v_1 be the first element of $\tau^{k_2}[t_1]$, and let $x_1 \in T^{k_1}[s_n]$ be such that $\|x_1\| = 1$ and $v < x_1$. Find a finitely supported $y_1 \in T^{k_2}[t_1]$ such that $y_1 \approx \Phi(x_1)$. Applying Lemma 4.4.1 we can find $v_2 \in \tau^{k_2}[t_1]$ such that $v_1 \leq y_1 < v_2$ and $X_{v_2}^{\max} \cap \tau^{k_2}[t_1] = \bigcup_{j=m_1+1}^{\infty} \tau^{k_2}[r_j]$ for some $m_1 \in \mathbb{N}$.

Since $k_1 - |s_{n+1}| > k_2 - |r_1| = m$ we can apply the induction hypothesis. In particular, the map

$$P_{m_1}\Phi_{|T^{k_1}[s_{n+1}]} : T^{k_1}[s_{n+1}] \to T^{k_2}[r_1] \oplus \cdots \oplus T^{k_2}[r_{m_1}].$$

is not an isomorphism. As a result, there exists $x_2 \in T^{k_1}[s_{n+1}]$ such that $\|x_2\| = 1$ and $P_{m_1}\Phi(x_2) \approx 0$. To add the property $v < x_2$, we decompose $T^{k_1}[s_{n+1}] = \sum_{i=1}^{\infty} \oplus T^{k_1}[u_i]$ as in Lemma 4.5.2 and apply the induction hypothesis to $\sum_{i=p}^{\infty} \oplus T^{k_1}[u_i]$ for p large enough.

Now that we have a normalized $x_2 \in T^{k_1}[s_{n+1}]$ that satisfies $v < x_2$ and $P_m\Phi(x_2) \approx 0$, we find a finitely supported $y_2 \in T^{k_2}[t_1]$ such that $y_2 \approx \Phi(x_2)$ and $P_{m_1}y_2 = 0$. Notice that $v_1 \leq y_1 < v_2 < y_2$ and that Lemma 4.5.3 gives that supp$\,(y_2) \subset X_{v_2}^{\max}$.

We now repeat the argument. Use Lemma 4.4.1 to find $v_3 \in \tau^{k_2}[t_1]$ such that $y_2 < v_3$ and $X_{v_3}^{\max} \cap \tau^{k_2}[t_1] = \bigcup_{j=m_2+1}^{\infty} \tau^{k_2}[r_j]$. Then we find a normalized $x_3 \in T^{k_1}[s_{n+2}]$ such that $v < x_3$ and $P_{m_2}\Phi(x_3)$ is essentially zero. Finally, we find a finitely supported $y_3 \in T^{k_2}[t_1]$ such that $y_3 \approx \Phi(x_3)$ and $P_{m_2}y_3 = 0$.

Proceeding this way, for every $i \leq N$, we find normalized $x_i \in T^{k_1}[s_{n+i-1}]$ with $v < x_i$, $v_i \in \omega^{\not k_2}$, and $y_i \in T^{k_2}[t_1]$ such that $y_i \approx \Phi(x_i)$ and

$$v_1 \leq y_1 < v_2 < y_2 < \cdots < v_N < y_N \text{ and } v_{i+1}, \text{supp}\,(y_i) \subset X^{\max}_{v_i}.$$

By Theorem 4.3.1, $(y_i)_{i=1}^N$ is isomorphic to the canonical basis of ℓ_p^N. Hence, Φ maps ℓ_∞^N isomorphically into ℓ_p^N. Since N is arbitrary, this contradicts that Φ is continuous (see equation (4.5.1) below). This concludes the proof for the case $M = 1$.

Now let $M > 1$ and suppose, towards a contradiction, that there exists $n \in \mathbb{Z}^+$ and an isomorphism

$$\Phi : \sum_{i=n}^{\infty} \oplus T^{k_1}[s_i] \to T^{k_2}[t_1] \oplus T^{k_2}[t_2] \oplus \cdots \oplus T^{k_2}[t_M].$$

For each $j \in \mathbb{Z}^+$ let $Q_j : \sum_{i=1}^M T^{k_2}[t_i] \to T^{k_2}[t_j]$ be the canonical projection. Decompose $T^{k_2}[t_j] = \sum_{i=1}^\infty T^{k_2}[r_i^j]$ as in Lemma 4.5.2 and for each $m \in \mathbb{Z}^+$, let $P_m^j : T^{k_2}[t_j] \to \sum_{i=1}^m T^{k_2}[r_i^j]$ be the canonical projection onto the first m blocks.

The proof is similar to the case $M = 1$. Find N large enough and $v \in \omega^{\not k_1}$ such that $s_n \prec s_{n+1} \prec \cdots \prec s_{n+N-1} \prec v$. Find $x_1 \in T^{k_1}[s_n]$ such that $\|x_1\| = 1$ and $v < x_1$ and find a finitely supported $y_1 \in T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$ such that $y_1 \approx \Phi(x_1)$.

For each $j \leq M$, let $v_1^j = \min_\prec(\tau^{k_2}[t_j])$. Use Lemma 4.4.1 to find $v_2^j \in \tau^{k_2}[t_j]$ such that $Q_j(y_1) < v_2^j$ and $X^{\max}_{v_2} \cap \tau^{k_2}[t_j] = \bigcup_{i=m_1^j+1}^\infty \tau^{k_2}[r_i^j]$ for some $m_1^j \in \mathbb{N}$. Let $P_1 = \sum_{j=1}^M P^j_{m_1^j}$ be the projection onto the first blocks of each of the $T^{k_2}[t_j]$'s.

Since $k_1 - |s_{n+1}| > k_2 - |r_1| = m$ we can apply the induction hypothesis. In particular, the map $P_1\Phi_{|T^{k_1}[s_{n+1}]}$ is not an isomorphism and we can find $x_2 \in T^{k_2}[s_{n+1}]$ such that $\|x_2\| = 1$ and $P_1\Phi(x_2) \approx 0$. Arguing as in the case $M = 1$, we can also assume that $v < x_2$. We then find a finitely supported $y_2 \in \sum_{j=1}^{M} \oplus T^{k_2}[t_j]$ such that $y_2 \approx \Phi(x_2)$ and $P_1 y_2 = 0$.

Proceeding this way, for every $i \leq N$, we find normalized $x_i \in T^{k_1}[s_{n+i-1}]$ with $v < x_i$ and $y_i \in \sum_{j=1}^{M} \oplus T^{k_2}[t_j]$ such that $y_i \approx \Phi(x_i)$. Moreover, for every $j \leq M$, we can find $v_i^j \in \omega^{\not{}k_2}$ such that

$$v_1^j \leq Q_j(y_1) < v_2^j < Q_j(y_2) < \cdots < v_N^j < Q_j(y_N) \text{ and } v_{i+1}^j, \operatorname{supp}(Q_j(y_i)) \subset X_{v_i^j}^{\max}.$$

By Theorem 4.3.1, there exists $C_1 > 0$ independent of N such that for every $j \leq M$,

$$\frac{1}{C_1}\left(\sum_{i=1}^{N} \|Q_j(y_i)\|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{i=1}^{N} Q_j(y_i)\right\| \leq C_1\left(\sum_{i=1}^{N} \|Q_j(y_i)\|^p\right)^{\frac{1}{p}}.$$

Using the triangle inequality for $y_i = \sum_{j=1}^{M} Q_j(y_i)$, Holder's inequality $\left(\sum_{i=1}^{N} |a_i| \leq N^{1/q}\left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p}, \frac{1}{p} + \frac{1}{q} = 1\right)$, Theorem 4.3.1, and the fact that the projections Q_j are contractive, we get

$$\begin{aligned}
\sum_{i=1}^{N} \|y_i\| &\leq \sum_{i=1}^{N}\sum_{j=1}^{M} \|Q_j(y_i)\| \leq N^{1/q} \sum_{j=1}^{M}\left(\sum_{i=1}^{N} \|Q_j(y_i)\|^p\right)^{1/p} \\
&\leq C_1 N^{1/q} \sum_{j=1}^{M}\left\|\sum_{i=1}^{N} Q_j(y_i)\right\| \leq C_1 N^{1/q} \sum_{j=1}^{M}\left\|\sum_{i=1}^{N} y_i\right\| \\
&= C_1 N^{1/q} M \left\|\sum_{i=1}^{N} y_i\right\| \approx C_1 N^{1/q} M \left\|\Phi\left(\sum_{i=1}^{N} x_i\right)\right\|
\end{aligned}$$

$$\leq C_1 N^{1/q} M \|\Phi\| \left\| \sum_{i=1}^{N} x_i \right\|. \tag{4.5.1}$$

Since N is arbitrary, $\sum_{i=1}^{N} \|y_i\|$ is of order N, and $\| \sum_{i=1}^{N} x_i\|$ stays bounded, we see that Φ cannot be bounded, contradicting our assumption. $\square$

Chapter 5

Alternative Norms

We explore here the consequences of alternative definitions of the norm presented in Chapter 4. Throughout the sections of this chapter the notation used may overlap with the one previously used. Moreover, the definition of *admissible sequence* will be different in each section, even though we always use phrases like "admissible" or "admissible sequence". We have done so in order to simplify as much as possible the exposition and discussion of the Banach spaces constructed. Therefore, we warn the reader that the notation and the definition of admissible sequence are relative to each section.

In this chapter we always assume $d, k \in \mathbb{Z}^+$ are such that $d, k \geq 2$, and $\theta \in \mathbb{R}$ is such that $0 < \theta \leq 1$. Unless specified otherwise, we will denote the basis of the Banach spaces defined here by $(e_i)_{i=1}^{\infty}$. Remember that we write e_i instead of $e_{\vec{u}_i}$. Also, $\|\cdot\|'$ will denote the norm studied in the preceding chapter. For each $n \in \mathbb{Z}^+$, the following subspaces will be studied:

The subspace

$$R[n] := \overline{\text{span}}\{e_v : v = (n-1, m) \in \omega^{\not\angle 2}\}$$

is generated by all the basis elements whose support lies on the n-th row of the upper triangular representation of $\omega^{\not\angle 2}$; we refer to $R[n]$ simply as the *n-th row subspace*. Similarly, the subspace

$$C[n] := \overline{\text{span}}\{e_v : v = (m, n-1) \in \omega^{\not\angle 2}\}$$

is generated by all the basis elements whose support lies on the n-th column of the upper triangular representation of $\omega^{\not\angle 2}$; we refer to $C[n]$ simply as the *n-th column subspace*. Finally, the subspace

$$D := \overline{\text{span}}\{e_v : \exists m \in \mathbb{N} \text{ such that } v = (m, m)\}$$

is generated by all the basis elements whose support lies on the main diagonal of the upper triangular representation of $\omega^{\not\angle 2}$; we refer to D simply as the *diagonal subspace*.

Finally, we say that $(v_i)_{i=1}^n \subset \omega^{\not\angle 2}$, with $v_i = (l_i, m_i)$, is a *generalized column of* $\omega^{\not\angle 2}$ whenever $l_1 < \cdots < l_n < m_1 < \cdots < m_n$. The subspace

$$GC[(v_i)_{i=1}^n] := \overline{\text{span}}\{e_{v_i}\}$$

is called the *generalized column subspace generated by* $(v_i)_{i=1}^n$.

5.1 Case I

Definition 5.1.1. Let $\mathcal{M} \subset \mathcal{AR}^k$ be a family of finite approximations. We say that $\mathcal{M}$ is *compact* if the set $\{\chi_E : E \in \mathcal{M}\}$ is a compact subset of the set $\{0,1\}^{\omega^{\not{\, }k}}$ endowed with the product topology. Moreover, $\mathcal{M}$ is *hereditary* if $E \in \mathcal{M}$ and $F = r_n(E)$ for some $n \in \mathbb{Z}^+$ implies that $F \in \mathcal{M}$.

In the preceding definition, $\{0,1\}^{\omega^{\not{\, }k}}$ is identified with the space of all functions $f : \omega^{\not{\, }k} \to \{0,1\}$ and χ_E is the characteristic function of E. In $\{0,1\}^{\omega^{\not{\, }k}}$, the convergence under the product topology is the pointwise convergence. Consequently, if $E \in \mathcal{AR}^k$ and χ_{E_n} converges to χ_E pointwise, there exists $N \in \mathbb{Z}^+$ such that $E \subseteq E_n$ for all $n \geq N$.

Notice that $\mathcal{A}_d^k$, the downward closure of the barrier $\mathcal{AR}_d^k$ on $\omega^{\not{\, }k} \in \mathcal{E}_k$, is compact and hereditary.

Definition 5.1.2. Let $\mathcal{M} \subset \mathcal{AR}^k$ be compact hereditary. We say that a sequence $(E_i)_{i=1}^m$ of finite subsets of $\omega^{\not{\, }k}$ is *$\mathcal{M}$-admissible* if and only if there exists $\{v_1, v_2, \ldots, v_m\} \in \mathcal{M}$ such that $v_1 \leq E_1 < v_2 \leq E_2 < \cdots < v_m \leq E_m$.

Definition 5.1.3. Let $\mathcal{M} \subset \mathcal{AR}^k$ be compact hereditary. We denote by $T(\mathcal{M}, \theta)$ the completion of $c_{00}(\omega^{\not{\, }k})$ with respect to the norm defined by

$$\|x\| = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^m \|E_i x\| : (E_i)_{i=1}^m \ \mathcal{M}\text{-}admissible\right\}\right\},$$

where $x = \sum_{n=1}^\infty x_n e_n$ and $\|x\|_\infty := \sup_{n \in \mathbb{Z}^+} |x_n|$. We call $T(\mathcal{M}, \theta)$ the *high dimensional Tsirelson type space defined by* $(\mathcal{M}, \theta)$.

The following proposition follows by standard arguments:

Proposition 5.1.4. $(e_n)_{n=1}^{\infty}$ *is a 1-unconditional basis of* $T(\mathcal{M}, \theta)$.

In this section, we will explore the case $\theta = 1$. We will see that the space ℓ_1 plays an important role.

Proposition 5.1.5. *If* $\mathcal{M} \subset \mathcal{AR}^k$ *is a compact hereditary family of finite approximations with the property that there is* $v \in \omega^{\not\downarrow k}$ *for which there are infinitely many* $w \in \omega^{\not\downarrow k}$ *such that* $\{v, w\} \in \mathcal{M}$*, then* $T(\mathcal{M}, 1)$ *is* ℓ_1*-saturated.*

Proof. By the Bessaga-Pelczynski Selection Principle [6], it suffices to prove that every block subspace contains a further subspace isomorphic to ℓ_1. Let (x_n) be a normalized block basic sequence. We will extract a subsequence (x_{n_i}) of (x_n) equivalent to the ℓ_1 basis.

Since there is $\{v\} \in \mathcal{M}$, we can choose x_{n_1} such that $v < x_{n_1}$ (remember that this means that $v < \operatorname{supp}(x_{n_1})$). By hypothesis, there exists $w_1 \in \omega^{\not\downarrow k}$ such that $x_{n_1} < w_1$ and $\{v, w_1\} \in \mathcal{M}$. Then, we can choose x_{n_2} such that $w_1 < x_{n_2}$. Repeating this process, we can extract a subsequence (x_{n_i}) of (x_n) with the property that for each $i \in \mathbb{Z}^+$ there exists $w_i \in \omega^{\not\downarrow k}$ such that $v < x_{n_i} < w_i < x_{n_{i+1}}$ and $\{v, w_i\} \in \mathcal{M}$.

Now, for each $M \in \mathbb{Z}^+$, let us estimate $\left\|\sum_{i=1}^{M} a_i x_{n_i}\right\|$ for any $(a_i) \subset \mathbb{R}$. By definition, $\left\|\sum_{i=1}^{M} a_i x_{n_i}\right\| \leq \left\|\sum_{i=1}^{M} a_i x_{n_i}\right\|_1$ (remember that $\|\cdot\|_p$ denotes the ℓ_p-norm).

Given that there are $j, j_1 \in \mathbb{Z}^+$ such that $v = \vec{u}_j$ and $w_1 = \vec{u}_{j_1}$, we can set $E_1 := \{v, \vec{u}_{j+1}, \ldots, \vec{u}_{j_1-1}\}$ and $E_2 := \{w_1, \vec{u}_{j_1+1}, \ldots, w_M\}$. By construction, (E_1, E_2) is an $\mathcal{M}$-admissible sequence, and consequently

$$\begin{aligned}\left\|\sum_{i=1}^{M} a_i x_{n_i}\right\| &\geq \left\|E_1\left(\sum_{i=1}^{M} a_i x_{n_i}\right)\right\| + \left\|E_2\left(\sum_{i=1}^{M} a_i x_{n_i}\right)\right\| \\ &= \|a_1 x_{n_1}\| + \left\|\sum_{i=2}^{M} a_i x_{n_i}\right\| \\ &= |a_1| + \left\|\sum_{i=2}^{M} a_i x_{n_i}\right\|.\end{aligned}$$

Next, let j_1 be such that $w_2 = \vec{u}_{j_1}$ and set $E_{21} := \{v, \vec{u}_{j+1}, \ldots, \vec{u}_{j_1-1}\}$ and $E_{22} := \{w_2, \vec{u}_{j_1+1}, \ldots, w_M\}$. By construction, (E_{21}, E_{22}) is an $\mathcal{M}$-admissible sequence, and consequently

$$\begin{aligned}\left\|\sum_{i=2}^{M} a_i x_{n_i}\right\| &\geq \left\|E_{21}\left(\sum_{i=2}^{M} a_i x_{n_i}\right)\right\| + \left\|E_{22}\left(\sum_{i=2}^{M} a_i x_{n_i}\right)\right\| \\ &= \|a_2 x_{n_2}\| + \left\|\sum_{i=3}^{M} a_i x_{n_i}\right\| \\ &= |a_2| + \left\|\sum_{i=2}^{M} a_i x_{n_i}\right\|.\end{aligned}$$

Therefore,

$$\left\|\sum_{i=1}^{M} a_i x_{n_i}\right\| \geq |a_1| + |a_2| + \left\|\sum_{i=3}^{M} a_i x_{n_i}\right\|.$$

Continuing in this way we conclude that

$$\left\|\sum_{i=1}^{M} a_i x_{n_i}\right\| \geq |a_1| + |a_2| + \cdots + |a_M|.$$

□

Corollary 5.1.6. *$T(\mathcal{A}_d^k, 1)$ is ℓ_1-saturated.*

Proof. We already know that $\mathcal{A}_d^k$ is compact hereditary. Moreover, $\mathcal{A}_d^k$ satisfies the other hypotheses of Proposition 5.1.5 by setting $v = \vec{u}_1$ and noticing that there are infinitely many $E \in \mathcal{A}_d^k$ of length 2 with v as their first element. $\square$

The following two results allow us to see that spaces satisfying the hypotheses of Proposition 5.1.5 can have different structure.

Proposition 5.1.7. *If $d > 2$, then $T(\mathcal{A}_d^2, 1)$ is isomorphic to ℓ_1.*

Proof. We will prove that the basis $(e_i)_{i=1}^{\infty}$ of $T(\mathcal{A}_d^2, 1)$ is equivalent to the standard basis of ℓ_1. Let $x := \sum_{i=1}^{\infty} a_i e_i$, where $(a_i)_{i=1}^{\infty} \subset \mathbb{R}$. By definition, $\|x\| \leq \|x\|_1$.

On the other hand, given that $\{\vec{u}_1, \vec{u}_2\} \in \mathcal{A}_d^2$ and $d > 2$, we know that for every $N \in \mathbb{Z}^+$ it is the case that $E_1 := \{\vec{u}_1\}$ and $E_2 := \{\vec{u}_2, \vec{u}_3, \ldots, \vec{u}_N\}$ define an $\mathcal{A}_d^2$-admissible sequence. Therefore,

$$\|x\| \geq \|E_1 x\| + \|E_2 x\| = |a_1| + \left\| \sum_{i=2}^{N} a_i e_i \right\|.$$

We now proceed to prove that if $y := \left\| \sum_{i=M}^{N} a_i e_i \right\|$ with $M < N$, then $\|y\| \geq |a_M| + \left\| \sum_{i=M+1}^{N} a_i e_i \right\|$. There are three mutually exclusive cases for $\vec{u}_M$: for some $h, l \in \mathbb{Z}^+, \vec{u}_M = (h-1, h)$ or $\vec{u}_M = (h, h)$ or $\vec{u}_M = (h, l)$ with $l - h > 1$. We construct a suitable $\mathcal{A}_d^2$-admissible sequence for each of these cases as follows:

Case $\vec{u}_M = (h-1, h)$. Set $v := (h-1, h-1)$. Then, since $d > 2$, $\{v, \vec{u}_M, \vec{u}_{M+1}\} \in \mathcal{A}_d^2$ and consequently $E_1 := \{v\}, E_2 := \{\vec{u}_M\}, E_3 := \{\vec{u}_{M+1}, \vec{u}_{M+2}, \ldots, \vec{u}_N\}$ define an $\mathcal{A}_d^2$-admissible sequence.

Case $\vec{u}_M = (h, h)$. Notice that $\vec{u}_{M+1} = (0, h+1)$. Then, $\{\vec{u}_1, \vec{u}_{M+1}\} \in \mathcal{A}_d^2$ and consequently $E_1 := \{\vec{u}_M\}, E_2 := \{\vec{u}_{M+1}, \vec{u}_{M+2}, \ldots, \vec{u}_N\}$ define an $\mathcal{A}_d^2$-admissible sequence.

Case $\vec{u}_M = (h, l)$ with $l - h > 1$. Set $v := (h+1, h+1)$ and notice that $\vec{u}_{M+1} = (h+1, l)$. Then, $\{v, \vec{u}_{M+1}\} \in \mathcal{A}_d^2$ and consequently $E_1 := \{\vec{u}_M\}, E_2 := \{\vec{u}_{M+1}, \vec{u}_{M+2}, \ldots, \vec{u}_N\}$ define an $\mathcal{A}_d^2$-admissible sequence.

Therefore, given $\vec{u}_M$, we can construct an $\mathcal{A}_d^2$-admissible sequence which guarantees that

$$\|y\| \geq |a_M| + \left\| \sum_{i=M+1}^{N} a_i e_i \right\|.$$

Hence, for every $N \in \mathbb{Z}^+$ and $(a_i)_{i=1}^N \subset \mathbb{R}$, it is the case that

$$\|x\| \geq |a_1| + |a_2| + \cdots + |a_N|.$$

$\square$

Proposition 5.1.8. *There is a high dimensional Tsirelson type space that is ℓ_1-saturated but not isomorphic to ℓ_1.*

Proof. Let $n \in \mathbb{Z}^+$. If $\vec{u}_{2^n} = (l', l)$, set $w_n := (0, l)$. Under this notation, let

$$\mathcal{M} := \left\{ E \in \mathcal{AR}^2 : E = \{\vec{u}_1\} \text{ or } \exists n \in \mathbb{Z}^+ \text{ such that } E = \{\vec{u}_1, w_n\} \right\}.$$

Then, $T(\mathcal{M}, 1)$ is ℓ_1-saturated but not isomorphic to ℓ_1. Indeed, $T(\mathcal{M}, 1)$ is ℓ_1-saturated by Proposition 5.1.5. So, suppose that $T(\mathcal{M}, 1)$ is isomorphic to ℓ_1. Then, there is $C > 1$ such that for all $n \in \mathbb{Z}^+$ and all $(a_i) \subset \mathbb{R}$,

$$\frac{1}{C}\sum_{i=1}^{n}|a_i| \leq \left\|\sum_{i=1}^{n}a_ie_i\right\| \leq C\sum_{i=1}^{n}|a_i|\,. \tag{5.1.1}$$

Let $j \in \mathbb{Z}^+$. Consider $x := \sum_{i=2^j+1}^{2^{j+1}} a_ie_i$ and remember that we write e_i instead of $e_{\vec{u}_i}$. We know that $w_{j+1} = \vec{u}_m$ for some $m \in \mathbb{Z}^+$. Then, since $w_{j+1} \prec \vec{u}_{2^{j+1}}$,

$$\left\|\sum_{i=2^j+1}^{2^{j+1}} e_i\right\| \leq \left\|\sum_{i=2^j+1}^{m-1} e_i\right\| + \left\|\sum_{i=m}^{2^{j+1}-1} e_i\right\| + \|e_{2^{j+1}}\|\,.$$

Letting $y = \sum_{i=2^j+1}^{m-1} e_i$ and $z = \sum_{i=m}^{2^{j+1}-1} e_i$, we see that $w_j < \operatorname{supp}(y) < w_{j+1}$ and $w_{j+1} \leq \operatorname{supp}(z) < w_{j+2}$. By definition of $\mathcal{M}$, it is not possible to split up the support of y or z, so that we cannot make their norm bigger than their $\|\cdot\|_\infty$-norm. Therefore, $\|x\| \leq \|y\| + \|z\| + \|e_{2^{j+1}}\| = 1+1+1 = 3$.

From (5.1.1) we conclude that $2^j/C \leq 3$. Hence, $2^j \leq 3C$ for every $j \in \mathbb{Z}^+$, a contradiction. □

5.1.1 n-th Row and Diagonal Subspaces

Notice that Proposition 5.1.7 clearly implies the following:

Corollary 5.1.9. *If $d > 2$, then any n-th row subspace and the diagonal subspace of $T(\mathcal{A}_d^2, 1)$ are isomorphic to ℓ_1.*

However, when $0 < \theta < 1$ and $d\theta > 1$, the n-th row subspaces and the diagonal subspace of $T(\mathcal{A}_d^2, \theta)$ have quite a different structure:

Proposition 5.1.10. *Suppose that $0 < \theta < 1$ and $d\theta > 1$. Then, $R[n]$ and D are isomorphic to ℓ_p, where $\theta = \frac{1}{d^{1/q}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

To establish Proposition 5.1.10 we need to present two useful alternative ways to define the norm of $T(\mathcal{A}_d^2, \theta)$ that are just a slight variation of the ones presented in the previous chapter:

(i) Given $x = \sum_{j=1}^{\infty} x_j e_j \in c_{00}(\omega^{\not<2})$, define the following non-decreasing sequence of norms: $|x|_0 := \max_{j \in \mathbb{Z}^+} |x_j|$, and for $n \in \mathbb{N}$,

$$|x|_{n+1} := \max \left\{ |x|_n , \theta \max \left\{ \sum_{i=1}^{m} |E_i x|_n : 1 \leq m \leq d, (E_i)_{i=1}^m \ \mathcal{A}_d^2\textit{-admissible} \right\} \right\}.$$

Then,

$$\|x\| := \sup_{n \in \mathbb{N}} |x|_n .$$

(ii) Let $K_0 := \{\pm e_i^* : i \in \mathbb{Z}^+\}$. For $n \in \mathbb{N}$, let

$$K_{n+1} := K_n \bigcup \{\theta(f_1 + \cdots + f_m) : 1 \leq m \leq d, (f_i)_{i=1}^m \subset K_n\},$$

where $(\mathrm{supp}\,(f_i))_{i=1}^m$ is an $\mathcal{A}_d^2$-admissible sequence. Set $K := \bigcup_{n \in \mathbb{N}} K_n$. For each $n \in \mathbb{N}$ and fixed $x \in c_{00}(\omega^{\not<2})$, define the following non-decreasing sequence of norms:

$$|x|_n^* := \max \{f(x) : f \in K_n\}.$$

As in the previous chapter, for every $n \in \mathbb{N}$ and $x \in c_{00}(\omega^{\not<2})$ we have $|x|_n = |x|_n^*$. Then,

$$\|x\| = \sup \{f(x) : f \in K\}.$$

Proposition 5.1.10 follows from the next two lemmas. We will prove the second one only for $R[n]$ since the proof for D is analogous.

Lemma 5.1.11. *For any finitely supported $x \in T(\mathcal{A}_d^2, \theta)$ we have $\|x\| \leq \|x\|_p$.*

Proof. We will prove by induction on $m \in \mathbb{N}$ that for every $f \in K_m$ we have $|f(x)| \leq \|x\|_p$. This is trivially the case when $m = 0$. Suppose that the result holds for $m \in \mathbb{N}$ and let $f = \theta(f_1 + \cdots + f_j) \in K_{m+1}$. By definition of K_{m+1} we know that $j \leq d$ and $(f_i)_{i=1}^j \subset K_m$. If $E_i :=$ supp (f_i), then $|f_i(x)| = |f_i(E_i x)| \leq \|E_i x\|_p$ by induction hypothesis. Therefore,

$$|f(x)| \leq \theta \sum_{i=1}^{j} |f_i(x)| \leq \theta \sum_{i=1}^{j} \|E_i x\|_p = \frac{1}{d^{1/q}} \sum_{i=1}^{j} \|E_i x\|_p .$$

Applying Hölder's inequality and the fact that $j/d \leq 1$ we get,

$$\frac{1}{d^{1/q}} \sum_{i=1}^{j} \|E_i x\|_p \leq \left(\frac{j}{d}\right)^{1/q} \left(\sum_{i=1}^{j} \|E_i x\|_p^p\right)^{1/p} \leq \|x\|_p .$$

□

The next lemma give us the necessary lower ℓ_p-estimate. Its proof exploits the fact that the norm of $T(\mathcal{A}_d^2, \theta)$ as defined in this section is always bigger than or equal to the norm $\|\cdot\|'$ of the corresponding Banach space studied in the previous chapter.

We will use special notation for the basis elements of the n-th row subspace. For $i \in \mathbb{Z}^+$, set $e_i^n := e_{(n-1,n+i-2)}$. Then, $(e_i^n)_{i=1}^\infty$ denotes the canonical basis of $R[n]$.

Lemma 5.1.12. *For every $x \in R[n]$ we have $\frac{1}{2d} \|x\|_p \leq \|x\|$.*

Proof. Set $v_i :=$ supp (e_i^n) , $x_i := e_i^n$ for each $i \in \mathbb{Z}^+$. Then, applying Proposition 4.3.5, we conclude that $\frac{1}{2d} \|x\|_p \leq \|x\|' \leq \|x\|$ for every $x \in R[n]$. □

5.1.2 n-th Column and Generalized Column Subspaces

We analize here the finite dimensional n-th column subspaces of $T(\mathcal{A}_d^2, \theta)$, as well as its generalized column subspaces. For the rest of this subsection suppose that $0 < \theta < 1, d\theta > 1$, and let $p \in (1, \infty)$ be determined by the equation $d\theta = d^{1/p}$ (which is equivalent to $\theta = \frac{1}{d^{1/q}}$ whenever $\frac{1}{p} + \frac{1}{q} = 1$).

As before, special notation for the basis elements of the n-th column subspace is needed. For $i \in \{1, 2, \ldots, n\}$, set $e_{i:n} := e_{(i-1,n-1)}$. Then, $(e_{i:n})_{i=1}^n$ denotes the canonical basis of $C[n]$. Also, for any $n \in \mathbb{Z}^+$, set

$$\text{supp}\,(C[n]) := \{(0, n-1), (1, n-1), \ldots, (n-1, n-1)\}\,;$$

clearly, supp $(x) \subseteq$ supp $(C[n])$ for every $x \in C[n]$.

In order to study $C[n]$ we need to set up an specific framework. Consider the approximation $X_d^{\text{is}} := \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d\}$ in the upper triangular representation of $\omega^{\not\downarrow 2}$. X_d^{is} is just the initial segment of $\omega^{\not\downarrow 2}$ of length d. Define l to be the length of the largest column of X_d^{is}; e.g., when $d = 7$, we have $l = 3$. In general, if $\vec{u}_d = (m', m)$, then $l = m$ if $m' \neq m$, or $l = m + 1$ if $m' = m$.

Remark 5.1.13. It is clear that we always have $l < d$. Moreover, by definition, if $E \in \mathcal{A}_d^2$, then $|E \cap \text{supp}\,(C[n])| \leq l$ for any $n \in \mathbb{Z}^+$. Therefore, given any $\mathcal{A}_d^2$-admissible sequence $(F_i)_{i=1}^m$, at most $l + 1$ of these F_i's have non-empty intersection with supp $(C[n])$.

Under this framework, we have:

Proposition 5.1.14. *If $l\theta > 1$, then $C[n]$ is isomorphic to $\ell_{p'}^n$, where p' is determined by $l\theta = l^{1/p'}$.*

We will prove this proposition by establishing that the basis $(e_{i:n})_{i=1}^n$ is equivalent to the canonical basis of $\ell_{p'}^n$. The following two lemmas are based on Bellenot's original argument for the Tsirelson type space $T(\mathcal{A}_l, \theta)$, where $\mathcal{A}_l$ is a member of the low complexity hierarchy introduced in Chapter 2 (see [4]).

Lemma 5.1.15. *For every $x \in C[n]$ we have $\|x\| \leq M \|x\|_{p'}$ for some $M \geq 1$.*

Proof. Denote the norm of $T(\mathcal{A}_l, \theta)$ by $\|\cdot\|_{\mathcal{A}_l}$. Bellenot proved that for every $y \in T(\mathcal{A}_l, \theta)$, there is $M' \geq 1$ such that $\|y\|_{\mathcal{A}_l} \leq M' \|y\|_{p'}$. His fundamental idea was to exploit the fact that in $T(\mathcal{A}_l, \theta)$ we can arbitrarily arrange up to l successive subsets of $\mathbb{N}$ to produce an $\mathcal{A}_l$-admissible sequence.

We can follow a similar argument in $C[n]$. Set $V := \text{supp}\,(C[n]) \backslash \{(n-1, n-1)\}$ and let $i \in \{2, 3, \dots, l\}$. Suppose that $E_1, E_2, \dots, E_i$ are subsets of V such that $E_1 < E_2 < \cdots < E_i$. It is clear that it is always possible to construct an $X \in \mathcal{A}_d^2$ whose last $i-1$ elements are $\min_\prec(E_2), \min_\prec(E_3), \dots, \min_\prec(E_i)$ and such that it makes $E_1, E_2, \dots, E_i$ an $\mathcal{A}_d^2$-admissible sequence. Consequently, following Bellenot, $C'[n-1] := \overline{\text{span}}\{e_v : v \in V\}$ has an upper $\ell_{p'}^{n-1}$-estimate.

Therefore, given that $C'[n-1]$ and $\overline{\text{span}}\{e_{(n-1,n-1)}\}$ are complemented in $C[n]$, there must be $M \geq 1$ such that $\|x\| \leq M \|x\|_{p'}$ for all $x \in C[n]$. $\square$

To prove the next lemma we will use the Tsirelson type space $T(\mathcal{A}_l, \theta)$. Denote its norm by $\|\cdot\|_{\mathcal{A}_l}$ and by $(t_i)_{i=1}^\infty$ its standard basis. It will also be useful to remember that the sum of the first l positive integers is equal to $\frac{1}{2}l(l+1)$. Following ideas developed by Bellenot in [4] we have:

Lemma 5.1.16. *For every $x \in C[n]$ we have $\frac{1}{2l}\|x\|_{p'} \leq \|x\|$.*

Proof. Let $x := \sum_{i=1}^{n} a_i e_{i:n}$ and $y := \sum_{i=1}^{n} a_i t_i$ for $(a_i)_{i=1}^{n} \subset \mathbb{R}$. We will prove that $\|y\|_{\mathcal{A}_l} \leq \|x\|$. This inequality will establish the desired result given that Bellenot proved in [4] that $\frac{1}{2l}\|y\|_{p'} \leq \|y\|_{\mathcal{A}_l}$.

We know that either $\|y\|_{\mathcal{A}_l} = \|y\|_\infty$, or there exist $l' \in \{1, \ldots, l\}$ and $F_1 < \cdots < F_{l'}$ finite subsets of $\mathbb{Z}^+$ such that $\|y\|_{\mathcal{A}_l} = \theta \sum_{j=1}^{l'} \|F_j y\|_{\mathcal{A}_l}$. For each $j \in \{1, \ldots, l'\}$, either $\|F_j y\|_{\mathcal{A}_l} = \|F_j y\|_\infty$, or there exist $l'' \in \{1, \ldots, l\}$ and $F_{j1} < \cdots < F_{jl''}$ subsets of F_j such that $\|F_j y\|_{\mathcal{A}_l} = \theta \sum_{s=1}^{l''} \|F_{js} y\|_{\mathcal{A}_l}$. Since y has finite support, this process will end and y is normed by a tree.

We will prove that $\|y\|_{\mathcal{A}_l} \leq \|x\|$ by induction on the height h of the tree. If $\|y\|_{\mathcal{A}_l} = \|y\|_\infty$, then $h = 0$. It is clear that by definition of $\|\cdot\|$ we have $\|y\|_\infty = \max_{1 \leq i \leq n} |a_i| \leq \|x\|$.

Suppose now that the result is proved for elements of $T(\mathcal{A}_l, \theta)$ that are normed by trees of height less than or equal to h and that y is normed by a tree of height $h+1$. Then, there exist $l' \in \{1, \ldots, l\}$ and $F_1 < \cdots < F_{l'}$ such that $\|y\|_{\mathcal{A}_l} = \theta \sum_{j=1}^{l'} \|F_j y\|_{\mathcal{A}_l}$ and with each $F_j y$ normed by a tree of height less than or equal to h. By induction hypothesis,

$$\|F_j y\|_{\mathcal{A}_l} = \left\| \sum\nolimits_{m \in F_j} a_m t_m \right\|_{\mathcal{A}_l} \leq \left\| \sum\nolimits_{m \in F_j} a_m e_{m:n} \right\|. \tag{5.1.2}$$

For each $j \in \{1, \ldots, l'\}$, set $m_j := \min(F_j)$ and $m_{l'+1} := n+1$. We will use $m_1, \ldots, m_{l'+1}$ to construct an $\mathcal{A}_d^2$-admissible sequence that would allow us to overestimate $\left\| \sum_{m \in F_j} a_m e_{m:n} \right\|$ appropriately.

Suppose that $m_{l'} = n$ and write $w_j := (m_j - 1, n-1)$ for $j \in \{1, \ldots, l'\}$. We will first construct an $X \in \mathcal{A}_d^2$ whose last l' elements are $w_1, \ldots, w_{l'}$.

The first element v_1 of X is determined by w_1: set $v_1 := (m_1 - 1, m_1 - 1)$; the next two elements of X (its second column) are determined by w_1 and w_2: set $v_2 := (m_1 - 1, m_2 - 1)$ and $v_3 := (m_2 - 1, m_2 - 1)$; the next three elements of X (its third column) are determined by w_1, w_2, w_3: set $v_4 := (m_1 - 1, m_3 - 1), v_5 := (m_2 - 1, m_3 - 1), v_6 := (m_3 - 1, m_3 - 1)$. We follow this process until the last l' elements of X are $w_1, \dots, w_{l'}$. By construction of X and definition of l we have that $|X| \leq \frac{1}{2}l(l+1) < d$, and consequently $X \in \mathcal{A}_d^2$.

Writing $s := |X| - l'$, let $E_1 := \{v_1\}, E_2 := \{v_2\}, \dots, E_s := \{v_s\}$. Then, for $j \in \{1, \dots, l'\}$, let

$$E_{s+j} := \{w_j, (m_j, n-1), \dots, (m_{j+1} - 2, n-1)\};$$

therefore, $E_1 < \cdots < E_s < E_{s+1} < \cdots < E_{s+l'}$ is an $\mathcal{A}_d^2$-admissible sequence.

Now, going back to equation 5.1.2, notice that $F_j \subseteq \{m_j, \dots, m_{j+1} - 1\}$. Hence,

$$\left\| \sum\nolimits_{m \in F_j} a_m e_{m:n} \right\| \leq \left\| \sum\nolimits_{r=m_j}^{m_{j+1}-1} a_r e_{r:n} \right\| = \|E_{s+j} x\|.$$

Finally,

$$\|y\|_{\mathcal{A}_l} = \theta \sum_{j=1}^{l'} \|F_j y\|_{\mathcal{A}_l} \leq \theta \sum_{j=1}^{l'} \|E_{s+j} x\| \leq \|x\|.$$

When $m_{l'} < n$ the construction of X is analogous to the one presented above. We start the construction by defining $v_1 := (m_2 - 1, m_2 - 1)$; that is, using w_2 instead of w_1. Then we set $E_1 := \{(0, n-1), (1, n-1), \dots, (m_2 - 2, n-1)\}$ and E_{s+j} for $j = 2, \dots, l'$ as above. $\square$

We now turn our attention to the finite dimensional subspaces generated by generalized columns $(v_i)_{i=1}^n$ of $\omega^{\not{}2}$. Remember that we say that $(v_i)_{i=1}^n$, with $v_i =$

(l_i, m_i), is a generalized column of $\omega^{\not{\ell}2}$ whenever $l_1 < \cdots < l_n < m_1 < \cdots < m_n$. Their associated subspace is $GC[(v_i)_{i=1}^n] := \overline{\operatorname{span}}\{e_{v_i}\}$.

It turns out that $GC[(v_i)_{i=1}^n]$ is isomorphic to ℓ_p^n, where p is determined as in Proposition 5.1.10. Observe that $p \neq p'$, so that the structure of $GC[(v_i)_{i=1}^n]$ is different than the one described above for $C[n]$.

Proposition 5.1.17. *If $d\theta > 1$, then $GC[(v_i)_{i=1}^n]$ is isomorphic to ℓ_p^n, where p is determined by $d\theta = d^{1/p}$.*

Proof. The upper ℓ_p-estimate follows immediately from Lemma 5.1.11. For the lower ℓ_p-estimate we will follow the general idea presented in the proof of Lemma 5.1.16 but this time exploiting the Tsirelson type space $T(\mathcal{A}_d, \theta)$. Denote the norm of $T(\mathcal{A}_d, \theta)$ by $\|\cdot\|_{\mathcal{A}_d}$ and by $(t_i)_{i=1}^\infty$ its standard basis.

Let $(v_i)_{i=1}^n$ be a generalized column of $\omega^{\not{\ell}2}$. For $(a_i)_{i=1}^n \subset \mathbb{R}$, write $x := \sum_{i=1}^n a_i e_{v_i}$ and $y := \sum_{i=1}^n a_i t_i$. Once again, we will prove that $\|y\|_{\mathcal{A}_d} \leq \|x\|$ by induction on the height h of the norming tree of y.

As before, the result is clear when $h = 0$. Suppose now that the result is proved for elements of $T(\mathcal{A}_d, \theta)$ that are normed by trees of height less than or equal to h and that y is normed by a tree of height $h + 1$. Then, there exist $d' \in \{1, \ldots, d\}$ and $F_1 < \cdots < F_{d'}$ such that $\|y\|_{\mathcal{A}_d} = \theta \sum_{j=1}^{d'} \|F_j y\|_{\mathcal{A}_d}$ and with each $F_j y$ normed by a tree of height less than or equal to h. By induction hypothesis,

$$\|F_j y\|_{\mathcal{A}_d} = \left\| \sum\nolimits_{m \in F_j} a_m t_m \right\|_{\mathcal{A}_d} \leq \left\| \sum\nolimits_{m \in F_j} a_m e_{v_m} \right\|. \qquad (5.1.3)$$

For each $j \in \{1, \ldots, d'\}$, set $m_j := \min(F_j)$ and $m_{d'+1} := n + 1$. We will use $m_1, \ldots, m_{d'+1}$ to construct an $\mathcal{A}_d^2$-admissible sequence that would allow us to overestimate $\left\| \sum_{m \in F_j} a_m e_{v_m} \right\|$ appropriately.

Suppose that $v_i := (l_1^i, l_2^i)$ for $i \in \{1, \ldots, n\}$. Set $w_1 := v_{m_1}$. Since by definition of generalized column we have that $l_1^{m_1} < l_1^{m_2}$ and $l_2^{m_1} < l_2^{m_2-1} < l_2^{m_2}$, then $w_2 := (l_1^{m_1}, l_2^{m_2})$ is such that $v_{m_2-1} \prec w_2 \prec v_{m_2}$ and $\{w_1, w_2\} \in \mathcal{A}_d^2$. Now, given that $l_2^{m_2} < l_2^{m_3-1} < l_2^{m_3}$, then $w_3 := (l_2^{m_3-1}, l_2^{m_3-1})$ is such that $v_{m_3-1} \preceq w_3 \prec v_{m_3}$ and $\{w_1, w_2, w_3\} \in \mathcal{A}_d^2$. In general, for each $j \in \{2, \ldots, d'\}$, the fact that $l_2^{m_j-1} < l_2^{m_j}$ guarantees the existence of $w_j \in \omega^{\not{2}}$ such that $v_{m_{j-1}} \preceq w_j \prec v_{m_j}$ and $\{w_1, \ldots, w_j\} \in \mathcal{A}_d^2$. Therefore, $X := \{w_1, \ldots, w_{d'}\} \in \mathcal{A}_d^2$.

Let E_j be the finite subset of $\omega^{\not{2}}$ that contains every w such that $v_{m_j} \preceq w \preceq v_{m_{j+1}-1}$. Then, by construction,

$$w_1 \leq E_1 < w_2 < E_2 \leq w_3 < E_3 \leq w_4 < \cdots < E_{d'-1} \leq w_{d'} < E_{d'};$$

i.e., $(E_j)_{j=1}^{d'}$ is an $\mathcal{A}_d^2$-admissible sequence.

Since $F_j \subseteq \{m_j, \ldots, m_{j+1} - 1\}$,

$$\left\| \sum\nolimits_{m \in F_j} a_m e_{v_m} \right\| \leq \left\| \sum\nolimits_{r=m_j}^{m_{j+1}-1} a_r e_{v_r} \right\| = \|E_j x\| .$$

Hence, applying equation 5.1.3, we have that

$$\|y\|_{\mathcal{A}_d} = \theta \sum_{j=1}^{d'} \|F_j y\|_{\mathcal{A}_d} \leq \theta \sum_{j=1}^{d'} \|E_j x\| \leq \|x\| .$$

$\square$

5.1.3 Thin Approximations

The key property of generalized columns that allowed us to prove Proposition 5.1.17 is that each element of such a column is always at least a column to the right of the previous one in the upper triangular representation of $\omega^{\not{<}2}$. This property motivates the concept of thin approximation:

Definition 5.1.18. Any approximation $X := \{v_1, v_2, \ldots\} \in \mathcal{A}_d^2$, with $v_i = (l_1^i, l_2^i)$, such that $l_2^i < l_2^{i+1}$ is called a *thin approximation*. The subspace $\overline{\text{span}}\{e_{v_i}\}$ generated by X is denoted by $T_2[X]$.

Surprisingly, the subspaces generated by thin approximations share the same structure with the subspaces $R[n]$:

Proposition 5.1.19. *If $X \in \mathcal{A}_d^2$ is a thin approximation, then $T_2[X]$ is isomorphic to ℓ_p, where p is determined by $d\theta = d^{1/p}$.*

Proof. Let $X := \{v_1, v_2, \ldots\}$, with $v_i = (l_1^i, l_2^i)$, be a thin approximation. This proof follows the one presented for Proposition 5.1.17. In this case, to get a lower ℓ_p-estimate, we have to show that $\|\sum_{i=1}^{\infty} a_i t_i\|_{\mathcal{A}_d} \leq \|\sum_{i=1}^{\infty} a_i e_{v_i}\|$ for any $(a_i)_{i=1}^{\infty} \subset \mathbb{R}$ with finitely many non-zero elements. We can establish this inequality by following the same construction of the $\mathcal{A}_d^2$-admissible sequence $(E_j)_{j=1}^{d'}$ since it is based only on the fact that $l_2^i < l_2^{i+1}$. $\square$

5.2 Case II

Definition 5.2.1. Let $m \in \{1, 2, \ldots, d\}$. We say that $(E_i)_{i=1}^{m} \subset \mathcal{AR}^k$ is *admissible* if and only if there exists $v_1, v_2, \ldots, v_m \in \omega^{\not{<}k}$ such that $v_1 \leq E_1 < v_2 \leq E_2 < \cdots < v_m \leq E_m$.

For $x = \sum_{n=1}^{\infty} x_n e_n \in c_{00}(\omega^{\not< k})$ and $j \in \mathbb{N}$, we define a non-decreasing sequence of norms on $c_{00}(\omega^{\not< k})$ as follows:

- $|x|_0 := \max\limits_{n \in \mathbb{Z}^+} |x_n|$,

- $|x|_{j+1} := \max \left\{ |x|_j, \theta \max \left\{ \sum\limits_{i=1}^{m} |E_i x|_j : 1 \leq m \leq d, (E_i)_{i=1}^{m} \textit{ admissible} \right\} \right\}.$

For fixed $x \in c_{00}(\omega^{\not< k})$, the sequence $(|x|_j)_{j \in \mathbb{N}}$ is bounded above by the $\ell_1(\omega^{\not< k})$-norm of x. Therefore, we can set

$$\|x\| := \sup_{j \in \mathbb{N}} |x|_j .$$

Clearly, $\|\cdot\|$ is a norm on $c_{00}(\omega^{\not< k})$.

Definition 5.2.2. The completion of $c_{00}(\omega^{\not< k})$ with respect to the norm $\|\cdot\|$ is denoted by $(T_k(d, \theta), \|\cdot\|)$.

Under standard arguments we have:

Proposition 5.2.3. *$(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis of $T_k(d, \theta)$.*

Proposition 5.2.4. *For $x = \sum_{n=1}^{\infty} x_n e_n \in T_k(d, \theta)$ it follows that*

$$\|x\| = \max \left\{ \|x\|_{\infty}, \theta \sup \left\{ \sum_{i=1}^{m} \|E_i x\| : 1 \leq m \leq d, (E_i)_{i=1}^{m} \textit{ admissible} \right\} \right\},$$

where $\|x\|_{\infty} := \sup_{n \in \mathbb{Z}^+} |x_n|$.

5.2.1 n-th Row and Diagonal Subspaces

Surprisingly, as in Case I, when $0 < \theta < 1$ and $d\theta > 1$, these subspaces remain isomorphic to ℓ_p:

Proposition 5.2.5. *Suppose that* $0 < \theta < 1$ *and* $d\theta > 1$. *Then,* $R[n]$ *and* D *are isomorphic to* ℓ_p*, where* $\theta = \frac{1}{d^{1/q}}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

This proposition follows immediately from the next two lemmas. For the first of these lemmas we will also use an alternative way to define the norm of $T_k(d, \theta)$ using functionals:

Definition 5.2.6. Let $m \in \{1, \ldots, d\}$. A sequence $(F_i)_{i=1}^m \subset \mathrm{FIN}(\omega^{\nless k})$ is called *almost admissible* if there exists an admissible sequence $(E_n)_{n=1}^d$ such that $F_i \subseteq E_{n_i}$, where $n_1, \ldots, n_m \in \{1, \ldots, d\}$ are such that $n_1 < n_2 < \cdots < n_m$.

As before, let $K_0 := \{\pm e_i^* : i \in \mathbb{Z}^+\}$. Then, for $n \in \mathbb{N}$, set

$$K_{n+1} := K_n \bigcup \{\theta(f_1 + \cdots + f_m) : 1 \leq m \leq d, (f_i)_{i=1}^m \subset K_n\},$$

where $(\mathrm{supp}\,(f_i))_{i=1}^m$ is almost admissible. Set $K := \bigcup_{n \in \mathbb{N}} K_n$. For each $n \in \mathbb{N}$ and fixed $x \in c_{00}(\omega^{\nless k})$, define the following non-decreasing sequence of norms:

$$|x|_n^* := \max\{f(x) : f \in K_n\}.$$

As in the previous chapter, for every $n \in \mathbb{N}$ and $x \in c_{00}(\omega^{\nless k})$ we have $|x|_n = |x|_n^*$. Then,

$$\|x\| = \sup\{f(x) : f \in K\}.$$

In contrast with Case I, here we will establish Proposition 5.2.5 only for the diagonal subspace D (the proof for $R[n]$ is analogous).

Lemma 5.2.7. *For any finitely supported $x \in T_2(d,\theta)$ we have $\|x\| \leq \|x\|_p$.*

Proof. It is clear that $|f(x)| \leq \|x\|_p$ for every $f \in K_0$. Suppose that this inequality holds for some $m \in \mathbb{N}$. Let $f = \theta(f_1 + \cdots + f_j) \in K_{m+1}$. By definition, $j \leq d$, $(f_i)_{i=1}^j \subset K_m$, and there exists an admissible sequence $(E_n)_{n=1}^d$ such that $\operatorname{supp}(f_i) \subseteq E_{n_i}$, where $n_1, \ldots, n_j \in \{1, \ldots, d\}$ and $n_1 < \cdots < n_j$. Then, since $|f_i(x)| = |f_i(E_{n_i}x)| \leq \|E_{n_i}x\|_p$ by induction hypothesis, we conclude that

$$|f(x)| \leq \theta \sum_{i=1}^{j} |f_i(x)| \leq \theta \sum_{i=1}^{j} \|E_{n_i}x\|_p \,.$$

The result now follows by applying Hölder's inequality as in Lemma 5.1.11. □

For each $i \in \mathbb{Z}^+$, set $e'_n := e_{(n-1,n-1)}$. Then, $(e'_n)_{i=1}^n$ denotes the canonical basis of D. Remember that it is always the case that $\|\cdot\|' \leq \|\cdot\|$.

Lemma 5.2.8. *For any $x \in D$ we have $\frac{1}{2d}\|x\|_p \leq \|x\|$.*

Proof. Setting $v_i := \operatorname{supp}(e'_i)$, $x_i := e'_i$ for each $i \in \mathbb{Z}^+$, and then applying Proposition 4.3.5 we obtain

$$\frac{1}{2d}\left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{\infty} a_i e'_i\right\|' \leq \left\|\sum_{i=1}^{\infty} a_i e'_i\right\|$$

for every $(a_i)_{i=1}^{\infty} \subset \mathbb{R}$. □

5.2.2 n-th Column and Generalized Column Subspaces

Here we study the finite dimensional n-th column and generalized column subspaces of $T_2(d, \theta)$. Remember that $(e_{i:n})_{i=1}^n$ denotes the canonical basis of $C[n]$.

Even though $R[n]$ is isomorphic to ℓ_p as a subspace of either $T(\mathcal{A}_d^2, \theta)$ or $T_2(d, \theta)$, the following results show that columns turn out to have a different structure.

Lemma 5.2.9. *Suppose $E, F \in \mathcal{AR}^2$ and $E \cap \operatorname{supp}(C[n]) \neq \emptyset$. If $E < F$, then either $F \cap \operatorname{supp}(C[n]) = \{\min_{\prec}(F)\}$ or $F \cap \operatorname{supp}(C[n]) = \emptyset$.*

Proof. If $\min_{\prec}(F) \notin \operatorname{supp}(C[n])$, then any $(m_1, m_2) \in F$ is such that $n - 1 < m_2$ by definition of approximation. Consequently, $F \cap \operatorname{supp}(C[n]) = \emptyset$.

On the other hand, if $\min_{\prec}(F) \in \operatorname{supp}(C[n])$, any other $(m_1, m_2) \in F$ is such that $n - 1 < m_2$. Therefore, $F \cap \operatorname{supp}(C[n]) = \{\min_{\prec}(F)\}$. $\square$

Proposition 5.2.10. *$C[n]$ is isomorphic to ℓ_∞^n.*

Proof. Let $x \in C[n]$. By definition, $\|x\|_\infty \leq \|x\|$. So, suppose that for some $m \in \{1, \ldots, d\}$ we have an admissible sequence $(E_i)_{i=1}^m$ such that $\|x\| = \theta \sum_{i=1}^m \|E_i x\|$. Let $j \in \{1, \ldots, m\}$ be the smallest integer such that $E_j \cap \operatorname{supp}(x) \neq \emptyset$. By Lemma 5.2.9, for any $l = j, j+1, \ldots, m$, we have $|E_l \cap \operatorname{supp}(C[n])| \leq 1$, and consequently $\|E_l x\| \leq \|x\|_\infty$. Then,

$$\begin{aligned} \|x\| = \theta \sum_{i=1}^m \|E_i x\| &= \theta \left(\|E_j x\| + \|E_{j+1} x\| + \cdots + \|E_m x\| \right) \\ &\leq \theta \left(\|E_j x\| + (m - j) \|x\|_\infty \right) \\ &\leq \theta \|E_j x\| + d\theta \|x\|_\infty . \end{aligned}$$

Since $E_j x \in C[n]$, we can apply the previous argument to $E_j x$ instead of x to conclude that

$$\|x\| \leq \theta^2 \|F(E_j x)\| + d\theta^2 \|x\|_\infty + d\theta \|x\|_\infty$$

for some $F \in \mathcal{AR}^2$.

Given that $|\text{supp}\,(x)| \leq n$, we have

$$\|x\| \leq d \|x\|_\infty \sum_{i=1}^{\infty} \theta^i = \frac{d\theta}{1-\theta} \|x\|_\infty .$$

$\square$

Proposition 5.2.11. *For any generalized column $(v_i)_{i=1}^n$ of $\omega^{\not<2}$, we have that $GC[(v_i)_{i=1}^n]$ is isomorphic to ℓ_∞^n.*

Proof. Suppose that $v_i = (l_i, m_i)$. By definition of generalized column we know that $l_1 < \cdots < l_n$. Then, for $s_i := (l_i)$, we have $s_1 \prec s_2 \prec \cdots \prec s_n$. Therefore, under the notation used in Section 4.2 with $N = n$, it is easy to see that $GC[(v_i)_{i=1}^n]$ is a subspace of C. Hence, applying Theorem 4.2.4, we conclude that

$$\|x\|_\infty \leq \|x\| \leq \frac{d\theta}{1-\theta} \|x\|_\infty$$

for every $x \in GC[(v_i)_{i=1}^n]$. $\square$

5.2.3 $T_3(d,\theta)$ is not isomorphic to $T_2(d,\theta)$

In this section we shed even more light on the similarities of the spaces $T_k(d,\theta)$ and the spaces $T(\mathcal{A}_d^k, \theta)$ constructed in the previous chapter whenever $0 < \theta < 1$ and $d\theta > 1$.

We begin pointing out a couple of results from Chapter 4 that we can transfer to our current setting. Notice that Lemma 3.3.1 and its corollaries provide, given $v \in \omega^{\not\ell k}$, a description of the elements of $X_v^{\max}$, and are therefore independent of the definition of the Banach space $T_k(d,\theta)$. As a consequence we can transfer Lemma 4.2.3 and Theorem 4.2.4 to establish the presence of arbitrary large copies of ℓ_∞^N inside $T_k(d,\theta)$.

We will take advantage of the notation developed in Section 4.2. Particularly, given $j \in \mathbb{N}$, we write $\tau^k[j]$ and $T^k[j]$ instead of $\tau^k[(j)]$ and $T^k[(j)]$, respectively.

Lemma 5.2.12. *Suppose $m \in \mathbb{Z}^+$ and $j, j_1, \ldots, j_m \in \mathbb{N}$. Then, $T^3[j]$ does not embed into $T^2[j_1] \oplus \cdots \oplus T^2[j_m]$.*

Proof. Since $T^2[n] = R[n+1]$ for every $n \in \mathbb{N}$, applying Proposition 5.2.5 we conclude that $T^2[n]$ is isomorphic to ℓ_p, and consequently so is $T^2[j_1] \oplus \cdots \oplus T^2[j_m]$. On the other hand, Theorem 4.2.4 guarantees us that $T^3[j]$ has arbitrarily large copies of ℓ_∞^N. Hence, there cannot be an embedding from $T^3[j]$ into $T^2[j_1] \oplus \cdots \oplus T^2[j_m]$. $\square$

Proposition 5.2.13. *$T_3(d,\theta)$ does not embed into $T_2(d,\theta)$.*

Proof. Suppose, towards a contradiction, that there exists an isomorphism

$$\Phi : T_3(d,\theta) \to T_2(d,\theta).$$

Throughout this proof the following decompositions will be very useful:

$$T_k(d,\theta) = \sum_{j=0}^{\infty} \oplus T^k[j] \quad \text{and} \quad \omega^{\not\ell k} = \bigcup_{j=0}^{\infty} \tau^k[j].$$

Theorem 4.2.4 implies that for every $N \in \mathbb{Z}^+$ and any $v \in \omega^{\not{<}3}$ with $(N-1) \prec v$ we have $\overline{\text{span}}\{x_1, \ldots, x_N\} \approx \ell_\infty^N$ whenever $x_i \in T^3[i-1], v < x_i$, and $\|x_i\| = 1$. Recall that in such a case the isomorphism constant is independent of N and the x_i's.

Fix $N \in \mathbb{Z}^+$ and set $v := (N-1, N-1, N-1)$.

Let $v_1 := (0,0)$ and pick $x_1 \in T^3[0]$ such that $v < x_1$ and $\|x_1\| = 1$. Find a finitely supported $y_1 \in T_2(d,\theta)$ such that $y_1 \approx \Phi(x_1)$. Setting $u_1 := \max_\prec(\text{supp}\,(y_1))$ it is clear that if $u_1 = (l_1^1, l_2^1)$, then $v_2 := (m_1, m_1)$, with $m_1 := l_2^1 + 1$, is such that $y_1 < v_2$. Moreover, $X_{v_2}^{\max} = \bigcup_{j=m_1}^\infty \tau^2[j]$ given that v_2 is an element on the diagonal of the upper triangular representation of $\omega^{\not{<}2}$.

Now let $P_m : T_2(d,\theta) \to T^2[0] \oplus \cdots \oplus T^2[m]$ denote the projection onto the first m terms of the decomposition of $T_2(d,\theta)$. Consider

$$P_{m_1}\Phi_{|T^3[1]} : T^3[1] \to T^2[0] \oplus \cdots \oplus T^2[m_1].$$

By Lemma 5.2.12 we know that $P_{m_1}\Phi_{|T^3[1]}$ is not an embedding, and therefore we can find a normalized $x_2 \in T^3[1]$ such that $v < x_2$ and $P_{m_1}\Phi(x_2)$ is essentially zero. Hence, there is a finitely supported $y_2 \approx \Phi(x_2)$ such that $P_{m_1}y_2 = 0$. Notice that this last equality implies that $\tau^2[n] \cap \text{supp}\,(y_2) = \emptyset$ for any $n = 0, 1, \ldots, m_1$, so that $v_2 < y_2$ and $\text{supp}\,(y_2) \subset X_{v_2}^{\max}$. Thus far we have:

$$v_1 \leq y_1 < v_2 < y_2 \text{ and } \text{supp}\,(y_2) \subset X_{v_2}^{\max}.$$

We now repeat the argument. Setting $u_2 := \max_\prec(\text{supp}\,(y_2))$ it is clear that if $u_2 = (l_1^2, l_2^2)$, then $v_3 := (m_2, m_2)$, with $m_2 := l_2^2 + 1$, is such that $y_2 < v_3$. Moreover, $X_{v_3}^{\max} = \bigcup_{j=m_2}^\infty \tau^2[j]$ given that v_3 is an element on the diagonal of the upper triangular representation of $\omega^{\not{<}2}$. Then, applying Lemma 5.2.12 once again, we proceed to choose $x_3 \in T^3[2]$ such that $v < x_3$ and $P_{m_2}\Phi(x_3)$ is essentially zero; so that there is a finitely supported $y_3 \approx \Phi(x_3)$ such that $P_{m_2} y_3 = 0$. Now we have

$$v_1 \leq y_1 < v_2 < y_2 < v_3 < y_3 \text{ and supp } (y_2) \subset X_{v_2}^{\max}, \text{supp } (y_3) \subset X_{v_3}^{\max}.$$

Iterating this argument, for each $i \in \{1, 2, \ldots, N\}$, we find a normalized $x_i \in T^3[i-1]$ with $v < x_i$ and $y_i \in T_2(d, \theta)$ with $y_i \approx \Phi(x_i)$ such that there is $(v_j)_{j=1}^N \subset \omega^{\not\downarrow 2}$ for which

$$v_1 \leq y_1 < v_2 < y_2 < \cdots < v_N < y_N \text{ and supp } (y_j) \subset X_{v_j}^{\max}.$$

As pointed out at the beginning of this proof, we have that $\overline{\text{span}}\{x_1, \ldots, x_N\} \approx \ell_\infty^N$; and applying Theorem 4.3.1 we conclude that $\overline{\text{span}}\{y_1, \ldots, y_N\} \approx \ell_p^N$. Since N is arbitrary, this is a contradiction. $\square$

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