1-1-2017

Mathematics Teachers' Beliefs and Mathematical Knowledge for Teaching: How Teachers' MKT Shifts in Planning and Impacts Their Beliefs and Instructional Practice

Paul Conley
University of Denver

Mathematics Teachers' Beliefs and Mathematical Knowledge for Teaching: How Teachers' MKT Shifts in Planning and Impacts Their Beliefs and Instructional Practice

Abstract

In an effort to understand how a teacher developed Mathematical Knowledge for Teaching and how that knowledge can shift teacher's beliefs and instructional practice, I worked with a teacher to deeply plan and implement six mathematical lessons.

The research suggests that planning can be a vehicle to develop a teacher's Mathematical Knowledge for Teaching. In addition, as a teacher's Mathematical Knowledge for Teaching started to develop through lesson planning, the teacher's beliefs about her own knowledge of mathematics started to increase which started to shift the teacher's instructional practice.

This combination of a stronger Mathematical Knowledge for Teaching, self confidence in her own understanding of the mathematics and shifts in her instructional practice created a new pedagogical practice I have called The Pedagogy of Knowing.

Document Type

Dissertation

Degree Name

Ph.D.

Department

Curriculum and Instruction

First Advisor

Richard Kitchen, Ph.D.

Second Advisor

Bruce Uhrmacher, Ph.D.

Third Advisor

Alvaro Arias

Keywords

Mathematical knowledge, Teaching, Lesson plan, Pedagogical practices

Subject Categories

Curriculum and Instruction | Other Teacher Education and Professional Development | Science and Mathematics Education

MATHEMATICS TEACHER'S BELIEFS AND MATHEMATICAL KNOWLEDGE FOR TEACHING: HOW TEACHER'S MKT SHIFTS IN PLANNING AND IMPACTS THEIR BELIEFS AND INSTRUCTIONAL PRACTICE

A Dissertation

Presented to

the Faculty of the Morgridge College of Education

University of Denver

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Paul Conley

November 2017

Advisor: Richard Kitchen

Author: Paul Conley
Title: MATHEMATICS TEACHER’S BELIEFS AND MATHEMATICAL KNOWLEDGE FOR TEACHING: HOW TEACHER’S MKT SHIFTS IN PLANNING AND IMPACTS THEIR BELIEFS AND INSTRUCTIONAL PRACTICE
Advisor: Richard Kitchen
Degree Date: November 2017

Abstract

In an effort to understand how a teacher developed Mathematical Knowledge for Teaching and how that knowledge can shift teacher’s beliefs and instructional practice, I worked with a teacher to deeply plan and implement six mathematical lessons.
The research suggests that planning can be a vehicle to develop a teacher’s Mathematical Knowledge for Teaching. In addition, as a teacher’s Mathematical Knowledge for Teaching started to develop through lesson planning, the teacher’s beliefs about her own knowledge of mathematics started to increase which started to shift the teacher’s instructional practice.

This combination of a stronger Mathematical Knowledge for Teaching, self confidence in her own understanding of the mathematics and shifts in her instructional practice created a new pedagogical practice I have called The Pedagogy of Knowing.

Table of Contents

List of Figures

Chapter 1: Introduction

For my research study, I worked with a teacher in the planning and implementation of six core lessons to research how a teacher's Mathematical Knowledge for Teaching (MKT) developed through lesson planning and how that impacted the teacher's beliefs and instructional practice related to classroom discourse and patterns of questioning. By working alongside a teacher in the planning and implementation of the selected lessons, I gained a better understanding of how this teacher thinks about the teaching of mathematics, about how students learn mathematics and how the teacher sees herself as a mathematician. The focus of my research, however, was on how the teacher saw herself as a mathematician and how her content knowledge and pedagogical content knowledge affected her beliefs and practice in the planning and during the course of the implementation of the lesson.

In 1986, Shulman coined the term "pedagogical content knowledge" (PCK) to try and capture the knowledge that is required of a teacher to effectively implement lessons. Deborah Ball (2002) expanded on this idea of content specific knowledge for teaching in the area of mathematics education and labeling it "Mathematical Knowledge for Teaching" (MKT). In this country, we do not prepare teachers to be successful when entering the classroom (Boerst, Shaughnessy, Ball and Farmer, 2015). The lack of preparedness is one of several reasons that 39% of mathematics teachers leave the profession after only 5 years (Ingersoll 2011). There are a lot of connections that are made between a teachers' own MKT, the teachers' beliefs and teachers' teaching practices. (Gencturk 2012). Gencturk noted that as a teachers' MKT grew, they started to change their practice to a more inquiry-based instructional approach. The inquiry-based approach started to allow students to engage in "real" mathematics (Romagnano, 1994), as the teachers allowed students the opportunity to "do mathematics" (Stein, Smith, Henningsen and Silver, 2000). Doing mathematics involves giving students opportunities to engage in problems that there is not a pathway suggested by the task to solve the problem and forces students to analyze the underlying structures of mathematics to solve. This requires features such as: requiring an explanation, using manipulatives, multi-step solution paths, using diagrams, creates opportunities for students to use symbolic and abstract reasoning and promotes the use of technology. (Stein and colleagues 2000 & Gencturk 2012) By engaging students in this way, the teacher will have the opportunity to learn how students develop and think about mathematics in a way the teacher may never have learned or been taught. The teacher, as a learner, will then be able to engage students in discourse that can help shape the teacher's thinking and beliefs about what students can and are able to do independent of being taught procedure and skills. With these ideas in mind, my research questions focused on the following:

1. How does a teacher's MKT develop through the planning of and implementation of key lessons?

2. How does a teacher's beliefs about her own pedagogy and instructional practice shift as she deepens her content knowledge and MKT expand through lesson planningReflections on My Own Mathematics Education

When I reflect on my own education in mathematics, I remember one class period in detail sitting in an 8th grade Algebra classroom. I remember sitting in the back corner of the classroom, hoping that no one would ask me any questions, and watching the teacher solve an equation through a rote procedural process he was trying to get the class to learn. I remember this vividly because this is the only mathematics class I remember from middle school and I remember the feeling I had in that particular class, thinking to myself that I had no idea what was going on in that class. Oddly enough, when I started high school, I started to find success in mathematics. Success here is loosely defined as I remembered only how to follow a procedure to come to a correct solution as a freshman all the way through my junior year in high school. I received good grades in my mathematics classes because I learned that the assessment the teacher gave me would looked exactly like the homework and the classwork the teacher assigned and went over in class. I could then repeat the procedure to solve an almost identical problem on an assessment that would allow me to get a correct solution and then get a good score on the assessment. I never once had the opportunity to make sense of the mathematics for myself, understand the underlying concepts that allowed me to use a given procedure or algorithm, or discuss how other students thought about solving the same problem I was working on. My lack of conceptual understanding and inability to see that there could be multiple solution paths to solve a problem started to create issues for me my senior year in high school. I was taking a trigonometry course and following procedures no longer worked for me to find the correct solution. I struggled my whole senior year and I just chalked it up to being a senior and blamed the teacher. The teacher had exactly the same method for teaching that I had been exposed to all through high school, why was I starting to have issues with the mathematics I was supposed to be good at? These problems started to carry over to college when I was taking Calculus courses. I really had no idea about how to think mathematically or make sense of what I was being told by my professors. I was grasping to re-apply the procedure to solve problems without success and started to lose all self-confidence in being successful in mathematics courses. I learned to persevere and push on despite, what I know now, was a lack of conceptual understanding. I was rarely given opportunities to reason about mathematics or discuss ideas in a classroom with other students, I was only expected to learn mathematics from the direct instruction I had been taught by my instructors to reproduce an answer to be successful in their respective courses. I never once remember being pushed to learn the underlying concepts to explain the algorithms and how they were derived. I finished my degree in mathematics with a minor in secondary education and moved in to the classroom as an ill-prepared teacher. As a teacher, I pushed myself to not be the teacher I had always had in school because I knew how that set me up for failure. So, from the beginning of my career, I was biased against the sort of mathematics instruction that I had received for years from my teachers.

My Journey as a Teacher

In my first year of teaching, I remember struggling to survive and the least of my concerns was the curriculum that I was given to use. My first experience as a teacher was in a charter school and the school chose to adopt Saxon Mathematics as their curricular resource. At this point in my career, the curriculum was my best friend and I tried to implement how I was told to do so in my classroom. I had students in rows and I taught them some skills and the students then practiced those skills repeatedly doing some very routine problems over and over. Halfway through the year, I decided to deviate a bit from the curriculum and would pull in more open-ended tasks for my students to tackle. I thought this would be great, we have learned so much and now we can apply this to some real-world application problems. When I gave the students the tasks, I got a lot of frustration from the students for two main reasons: 1) they were confused by the task and could not think about how to approach finding a solution to the given problem, and 2) students were unwilling to try to do the task because I had never expected them to think independently about mathematics before. I had my first "ah-ha" moment as a teacher and realized that I was doing what my teachers in high school and college had done before that set me up for failure. I was not going to let that happen. In my second year of teaching, I worked to modify the lessons and worked to create a classroom that had students working collaboratively and students sharing solutions to the tasks they were given. I learned a lot in my second year about how students think about mathematics and about myself as a teacher of mathematics. This year changed my life in how I thought about mathematics education, I was never going back to direct instruction with students sitting in rows.

The following school year, I landed where I belonged, at a diverse middle school in central Denver. I wanted to be a teacher in this school because they used the Connected Mathematics Project 2 (CMP2) as their main curricular resource. With this curriculum, I truly learned how to facilitate discourse, pose open-ended questions to students and identify misconceptions students had about specific concepts in mathematics. I spent countless hours meticulously planning the lessons and doing the mathematics for each lesson before I taught it so I would be prepared for the students in front of me each day.

When I left the classroom, I felt that I was a completely average teacher and that middle school mathematics teachers in DPS created the same learning environments for their students with students working on rich-tasks in small groups and discussing the mathematics as a whole class. When I turned the page on my career as a mathematics teacher, I was shocked to see how other teachers thought about and taught mathematics in middle school. Many teachers appeared to have low-expectations of their students as evidenced by low-level worksheets, skill and drill, and teacher-centered lessons where students were disengaged from thinking and more engaged in taking notes and repeating the procedure the teacher gave them. I wanted to learn more about why this was happening in the classrooms I was observing and I wanted to understand the reasons for the teachers' instructional practices.

Observations of Mathematics Teachers

When I left the classroom, I took a position where part of my job was to observe mathematics classrooms and teachers of mathematics to work with school leaders to support mathematics instruction. When I started this new position in the 2013-2014 school year, I had no idea of what I was getting in to in terms of support that mathematics leaders and teachers would need. At the beginning of September 2013, I observed every mathematics classroom that I served in the Middle School Network in Denver Public Schools (DPS). In the forty-three classrooms I visited during the first two weeks of the school year, only eight were using the district curriculum, CMP2. This was the case, even though the expectation for over 10 years had been that DPS teachers use this curriculum. In my previous tenure as a teacher, I spent the last eight years teaching middle school mathematics using CMP2. I was shocked that other middle school mathematics teachers had abandoned the curriculum and were replacing it with whatever else was available. I argue that this happened for three reasons (Ernest, 1991; Swan, 2006). The first reason I believe that teachers were not using CMP2 was their lack of understanding of the curriculum and the mathematics in it (Remillard, 2005; Swan, 2006; Stein & Kim, 2008). In discussing the curriculum with teachers, I would hear from teachers that the problems were too hard or that they did not make sense. The teachers struggled with the mathematics in the program and the wording of the curriculum and therefore assumed students would not be able to access the content. The wording in the curriculum was a struggle for two reasons: 1) the teachers themselves did not fully understand the task and the learning goal of the task and 2), the teachers did not believe the students would understand what they are being asked to do when solving the task. Without setting the mathematical goal and fully understanding the mathematics that students should be engaging in, teachers tend to think of the lesson as an activity that students need to complete rather than the learning that should occur from engaging in the lesson (Smith & Stein 2011). If the teachers only see the lesson as an activity, it becomes difficult to understand why the lesson asks the questions it does and that makes it more of a challenge for teachers to set students up for success when implementing the lesson (Hiebert and colleagues, 2007). When getting this feedback from teachers, I decided to work with teachers in planning upcoming lessons. It was clear to me that they were not taking the time to do the mathematics in the lessons themselves but were merely glancing over the lesson and creating superficial lesson plans. There was not a deep understanding of how the unit, investigation or even lesson unfolded and connected to the rest of the program. Horizon Content Knowledge is defined as the understanding of how mathematics unfolds over the span of mathematics and is particularly related to how mathematics unfolds within a curriculum (Ball, Thames and Phelps, 2008). This lack of knowledge in teachers has shown that they only focus on the content they are teaching, and among the reasons for this, was a lack of content knowledge of what they are teaching. Teachers not being able to see a broader horizon will only focus on the knowledge they have for what they are teaching in that grade-level, unit of study or even down to a lesson (Mosvold & Fauskanger, 2014). The second reason I believe teachers were not using the curriculum was that the program does not look like curricula they had experienced as mathematics students. When a curriculum asks students to engage in a task without direct instruction to “guide” them to a solution or there are not examples of “how” to solve the various tasks, then teachers may not understand how students are learning mathematics because that is not how they experienced learning mathematics themselves (Ernest, 1989). I have found this was especially true for new mathematics teachers that were presented with a curriculum for which they needed, but lacked, strong professional development to be prepared to implement effectively in the classroom. New classroom teachers would inform me that they tried to implement a CMP2 lesson in the classroom, but it “fell apart,” and so they went back to teaching mathematics the way they learned because that is what made sense to them (Stein & Kim, 2008). The final reason I would argue for CMP not being used in the classroom has to do with teachers' beliefs about students (Riegle-Crumb & Humphries, 2012). When planning a lesson with an 8^{th} grade teacher, I continually heard the teacher say that this would be a great lesson for her “honors” students but this program would never work for her English Language (EL) students or non-honors classes. The teacher did not implement CMP2 in classrooms where the she felt the students could not access the content due to language barriers or ability levels (Stinson, 2006). The real problem was that whenever I saw a teacher using something other than CMP2 in the classroom, I saw a wide spectrum of instructional materials from worksheets printed off the internet to textbooks that were outdated. And in all these classrooms that were not using CMP2, students were arranged in rows and the teacher was doing all the instruction in front of the classroom. There was little to no effort by the teachers to allow students to engage in the mathematics and try and make sense of it for themselves. The lack of discourse in most the classrooms created learning environments where students were sitting and getting instruction with no relevance or understanding. Students in these classrooms were merely expected to remember what the teacher said the "correct" process was and re-apply that process later in the form of an assessment (Stinson, 2006).

In classrooms where the teacher made an effort to promote discourse, the types of questions and pattern of questions was such that the teacher had an answer that they already wanted students to give them and that ended the questioning cycle. Herbel-Eisenmann and Breyfogle (2005) referred to this type of questioning as funneling. The teacher asks students questions to merely guide them to a predetermined set of solutions the teacher already had in mind when asking the question. There is little to no real discourse through the questioning cycle. This has been the pattern of questioning that has been prevalent in classrooms for over a quarter of a century (Stigler & Hiebert, 1999). The reason for the lack of questioning to push students to go deeper comes from a lack of MKT (Hill, Rowan & Ball 2005). Without thorough planning and thinking about how a student will approach solving a given task, teachers struggle to ask questions that push students' understanding and the questions tend to be close-ended and low-level (Smith & Stein 2011).

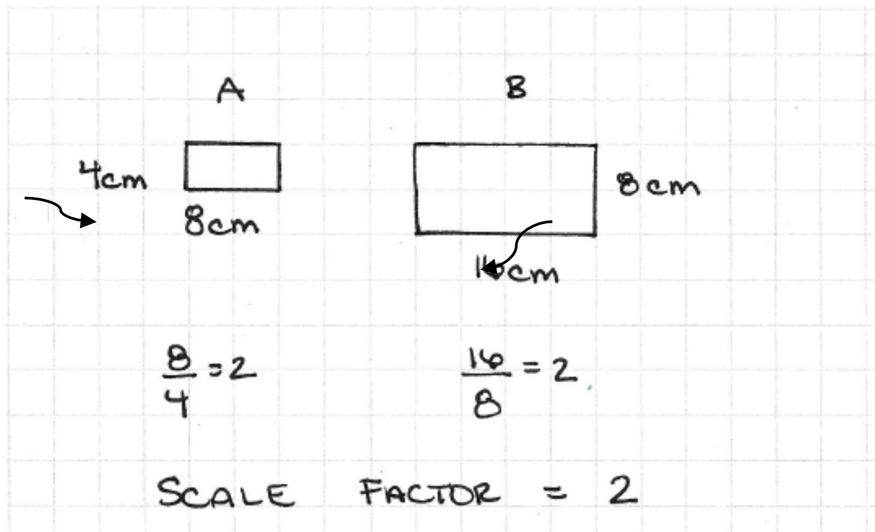
Shifts in Instruction in the Common Core State Standards

As DPS moves in to the era of the Common Core State Standards (CCSS), effective instruction and questioning should support engaging students in mathematics to mathematically reason and problem solve (Leinwand, Brahier, & Huinker, 2014). CCSS for Mathematics is composed of two distinct sets of standards, the first set is for the mathematical content that varies from grade to grade in a K-12 system. The second set of standards are called The Standards for Mathematical Practice. The content standards are written to provide a coherent strand of learning from Kindergarten through 12th grade. The shifts in the content standards clearly show that we are trying to reduce the amount of new content students learn every year and instead are designed to support students gaining a deeper understanding of the concepts (Mathematics Standards. (n.d.). Retrieved July 23, 2015). The Standards for Mathematical Practice (SMP) are unique to mathematics as they are a separate set of standards that are focused solely on how students should be engaging and learning mathematics within a classroom. There are eight SMP's and they are not specific to any one grade-level, like the content standards are, and rather were designed to be threaded through all grades so that when students graduate from high school, they will truly be college and career ready. To implement these classroom practices effectively, teachers will need strong content knowledge as well as MKT (Ball, Thames, and Phelps 2008). The reason that teachers will need strong command of content and MKT is really grounded in how the CCSSM is calling upon teachers to shift their instructional practice to push students to: make sense of problems, construct viable arguments, critique the reasoning of others, attend to precision, and model with mathematics; to name a few.

An example that I observed is when a teacher was looking at student work and the student got a correct solution but the process used was incorrect. The task was from a 7th grade mathematics lesson and the student was asked to find the scale factor from the smaller rectangle to the larger rectangle. The side lengths of the smaller rectangle were

4cm and 8cm and the side lengths of the larger rectangle were 8cm and 16cm (Figure 1.1).

Figure 1.1. Student Work Example for Finding Scale Factor



The student calculated the correct scale factor in this situation but by comparing the width and length of the same rectangle, the student did not show strong understanding of scale factor. The teacher also missed an opportunity to have students justify and critique, which could have made this correct answer/ wrong process a powerful learning moment for all of the students in the classroom. In discussing this problem later with the teacher, the teacher did not initially see the flawed mathematical process the student used. I would argue that a lack of content knowledge and MKT in this situation caused a powerful learning moment for students to be missed.

Productive vs. Unproductive Beliefs

The teachers in the middle school network are working incredibly hard and it is evident from my observations that they believe that what they are doing is what is best.

When working with teachers in planning, it has been a struggle to get teachers to change the tasks they give students and how they implement those tasks with students. I would argue that moving forward, two area of focus for a teacher need to be the quality of student-facing materials and the implementation of those materials in the classroom. What I learned from experience, and research supports, is that teachers' beliefs influence the decisions they make in selecting and implementing mathematical tasks in their classrooms (Stein, Smith, Henningsen, & Silver, 2000; Leinwand, Brahier & Huinker, 2014). Leinwand and colleagues (2014) classify these beliefs in to two categories; productive beliefs and unproductive beliefs. These are not meant to be viewed as good or bad but rather understood that unproductive beliefs promote instructional practices that only hinder the teaching and learning of mathematics (Leinwand and colleagues, 2014). Unproductive beliefs produce instructional practices where students focus on rote memorization and algorithms rather the conceptual understanding. The role of the teacher is to tell the students what they need to know and it is the role of the student to learn skills and solve problems using routine methods and procedures to get to a single correct solution. The teacher also believes that they are effective because they make learning mathematics easy for students and do not challenge students to think critically and critique others. The unproductive beliefs hinder a student's opportunities to engage in mathematics and push their own thinking about mathematics deeper. My personal biases are that I believe strongly in an inquiry-based, problem-based approach to the teaching and learning of mathematics. This comes from my own experiences as a classroom teacher and watching students engage in mathematics in a way I had not previously experienced. Students were more engaged, willing to discuss and took more ownership for their own learning when given the opportunity. I am also biased toward the notion that teachers should not just tell students the process to solving a task but rather facilitate learning through questioning and classroom discourse. However, moving forward, teachers have to ensure they are implementing tasks in a manner that will not reduce the cognitive demand of the task and allow students opportunities to make sense of the mathematics for themselves.

Stein and colleagues (2000) discussed case studies where teachers would implement tasks that required students to solve without a clear procedure for solving and make connections to other concepts in mathematics. During the first time a teacher would implement the task, the teacher would make certain moves, such as shift the emphasis to getting the correct solution or take over and turn the task in to a procedural skill, and therefore reduce the cognitive demand of the task through the implementation of the task. To maintain high cognitive demand, a teacher needs to make certain moves in the classroom, such as: scaffold student thinking and reasoning, students can monitor their own progress, have models of high-level performance, build on prior knowledge, make conceptual connections to other areas of mathematics, and give students time to engage in the mathematics. It has been well-documented in many mathematics classrooms in The United States, far too often, teachers are not engaging students in appropriate tasks that push their cognitive thinking and when they do provide appropriate tasks teachers are not implementing them to maintain the cognitive demand throughout the lesson (Stigler & Hiebert 1999; Stein, Smith, Henningsen & Silver 2000; Leinwand 2014).

The Opportunity Gap

Flores (2007) discusses obstacles that are specific to mathematics education that create opportunity gaps for students, particularly in schools that serve low-income students of color. These particular schools have a harder time hiring and retaining highly-qualified mathematics teachers. Schools find themselves having to hire out-of-field teachers, teachers that do not have at least a minor in the subject they are teaching, to fill vacancies from year to year. He paints the picture of how classrooms with these types of teachers tends to look with teachers placing emphasis on low-level problems that only have students practicing skills and solving non-routine problems, teachers not integrating technology to provide additional supports for students, and teachers that have low-expectations for students. In this sense, the opportunity gap in mathematics education is centered on too many students not having appropriate teachers in classrooms that can plan and implement lessons that allow students opportunities to engage in mathematics. By not engaging in the mathematics, students lose the opportunity to make sense of problems, critique the reasoning of others and justify their own reasoning, to name a few. This lack of critical-thinking is in stark contrast to the shifts in instruction that are called for in the CCSS. If we are to ever increase student achievement for all students and close the achievement gap, we need to close the opportunity gap that far too many students in our country are faced with each day.

Research Proposal

For my research study, I worked with Mrs. W. in planning and implementing six different tasks throughout the spring of 2016. During the planning of the tasks, the focus was on two distinct pieces: 1) anticipating student solutions and solution paths and understanding the learning goal of the task and 2) connecting students thinking and solution paths through questioning to bring the mathematical learning goal(s) out of the task (Smith & Stein 2011).

At the heart of starting to plan tasks by anticipating student responses are two critical pieces that help a teacher be better prepared to implement a given task. First, the teacher should clearly know and understand the learning goal that students should walk away with at the end of the lesson. By having a clear grasp of the mathematics to be learned, the teacher will then be able to make better decisions about the implementation of the task and will be prepared to make adjustments to the task during instruction (Leinwand and colleagues, 2014). Second, the teacher will be prepared to address misconceptions students may have when producing solutions to a given task as s/he will have taken the time to do the mathematics task as a learner. Next, a teacher needs to know how to use the information gained from doing the problem to know how s/he is going to implement the given task or lesson in their classroom. When a teacher takes the time to anticipate student outcomes, s/he develops knowledge about the pedagogy of teaching that will help him/her be more effective (Castro, 2006). There appears to be a gap in the research about how a teacher's MKT impacts their ability to effectively plan lessons. The work around MKT thus far has been situated in regards to how a teacher delivers instruction and student achievement as a result. There seems to be very little research though in regards to how a teacher actually needs this specialized knowledge in planning their daily lessons and more importantly how a teacher can develop this specialized knowledge in the planning of those lessons. Morris, Hiebert and Spitzer (2009) started down this path of connecting planning and MKT with effective instruction but their methodology involved pre-service teachers and they argued that common content knowledge (Ball and colleagues, 2008) is important for unpacking the mathematical learning goals from a task but the teaching of the task requires MKT. However, there was not a connection to effective planning for a task and a pre-service teacher's MKT. Effective planning is when a teacher has a clear understanding of the mathematics that helps them frame decisions in and be able to adjust instruction. (Leinwand and colleagues, 2014) Having clear goals and understandings of the mathematics allows teachers to leverage the goal and be better prepared to facilitate mathematical discourse, make connections between ideas and support students' learning (Hiebert and colleagues, 2007 and (Seidle, Rimmele, & ManfredPrenzel, 2005)Seidle, Rimmele, and Prenzel, 2005). These are all parts of content knowledge and pedagogical content knowledge that a teacher can obtain through lesson planning.

Hiebert, Morris, Berk and Jansen (2007) argue that the path to becoming an effective teacher requires two distinct competencies. The first competency is grounded in a teacher's mathematical content knowledge and their pedagogical content knowledge, or more specifically, their MKT. The second competency is a teacher's ability to hypothesize and connect cause-effect relationships between teaching and learning. This competency is broken down in to a set of four skills: "a) setting learning goals for students, b) assessing whether the goals are being achieved during the lesson, c) specifying hypotheses for why the lesson did or did not work well, and d) using the hypotheses to revise the lesson" (Hiebert and colleagues, 2007, p. 49).

In far too many classrooms, I have observed a lack of teachers getting to the summary and conclusion of the mathematical task or lesson. As a result, the students that I talked to after the lesson left without really being able to name what they learned in that class. Stein and Smith (2011) noticed that the lack of summary in the classrooms they observed left students without a concrete learning goal for the lesson and they argued that students were not learning the intended goal of the lesson because the teacher did not ask questions to make connections for students. This lack of questioning to make students work visible and connect mathematical relationships lead some students to believe they either understood the mathematics when they did not or that students believed they did not understand the mathematics when they did. The teachers need to help students in seeing how the connections between their various representations, models, processes and diagrams are all allowing them to get to the same learning outcome. This strategy also plays a critical role in developing students' ability to justify their own reasoning and critique the reasoning of others as called for in The Standards for Mathematical Practice ("Common Core State Standards Initiative", 2015).

When I worked with Mrs. W. during the school year, I wanted to learn how planning with a focus on the anticipating and connecting strategies supported the development of her MKT and her beliefs about her knowledge of mathematics. Through the planning and development of Mrs. W.'s MKT, I wanted to learn about her beliefs about her own mathematical content knowledge and her self-efficacy with regards to being a teacher of mathematics. The teacher's self-efficacy is her own belief about her content knowledge and MKT in mathematics. Wilhelm (2014) showed there was a clear connection between a teacher's MKT and the teacher's conceptions of teaching and student learning when implementing cognitively demanding tasks. As Mrs. W.'s MKT developed through lesson planning, I was interested in understanding how her own thinking about the teaching and learning of mathematics shifted when implementing CMP3 lessons in her classroom. The shifts in teaching and learning that I was particularly focused on were her patterns for questioning and student discourse she promoted as a result.

The anticipating strategy as part of a lesson planning process (Smith & Stein, 2011), will help the teacher to consider multiple solution paths students might take while simultaneously ensuring the teacher has a clear understanding of the learning goal of the lesson. This will have a direct link to the connecting strategy as the teacher will have time to craft questions to ask students about solution paths the teacher is more familiar. I would not anticipate any teacher ever knowing every solution path a student might take when given a task and thus cannot craft every question necessary to be successful but by spending time thinking about how students may approach a given task, teachers will be better prepared to lead students in discourse in their classroom (Smith & Stein 2011). At the end of the data collection, I have learned how a teacher thinks about students' thinking and potentially how all teachers can develop MKT through the planning of mathematical tasks and lessons. Through the development of content knowledge and MKT in planning a mathematical lesson, I wanted to learn how a teacher's beliefs about classroom instruction shifted to a more inquiry-based and problem-centered model of teaching. By inquiry-based instruction, I am wanting to see the classroom instruction shift to more opportunities for students to ask and be asked questions (i.e., classroom discourse) that pushes their thinking and helps them develop deeper conceptual understanding of the topic they are studying (Otten and Soria, 2014). Gencturk (2012) noted that as teachers' MKT developed and grew, the teachers' instructional practices shifted towards a more inquiry-based classroom instructional model. In these inquiry-based classrooms, teachers facilitated student learning more through questioning and moved away from the model of direct instruction. I hypothesize that this happens because as teachers gain confidence in their own mathematical content knowledge and MKT, they become more open to promoting a classroom where students share their thinking These areas of learning for me will support my work in planning tasks and lessons with teachers and will add to the research on the importance of effective planning for mathematical tasks and lessons that promote effective implementation of those tasks and lessons to improve teaching and learning.

My plan for working with Mrs. W. was to spend two to three hours prior to the teaching of a lesson to plan the lesson and focus on the anticipating and writing a clear mathematical goal. We started the planning by working through the task and discussing what the mathematical learning goal for the particular lesson was so that we could make sure to focus on that during the implementation of the task. Then, we worked together to create a variety of potential solution paths that students may use to get the solution to the given task in the lesson. This part of the planning helped us to think about how students learn mathematics and the potential misconceptions they may have when approaching the given task. By identifying the various solution paths and misconceptions, we then discussed possible questions the teacher could ask of students to push their thinking and support the learning in the classroom as a whole. Finally, we planned out how Mrs. W. would transition from student discourse to the summary and conclusion of the lesson to ensure that students leave being able to clearly articulate what they learned in class. The following day, I observed the lesson Mrs. W. implements in her classroom during each of the three 90-minute periods of mathematics each day. After each class period and/or at the end of the school day, Mrs. W. and I reflected on the lesson and its implementation. The reflection was centered on how Mrs. W. felt the implementation of the lesson went in each of her class periods, how understanding the mathematical content of the lesson deeply helped her be prepared to answer and ask questions, and what she learned from teaching the lesson as a result of the planning and the implementation. This type of reflection helped me gain an insight on how Mrs. W. is developing her MKT and thinking about its impact on student learning.

Chapter 2: Literature Review

This chapter begins with a discussion of mathematics teachers' beliefs with a strong focus on unproductive and productive beliefs and how they influence teaching and learning. Then I will discuss the differences between the two main beliefs mathematics teachers have about instructional practices. This chapter will then describe Mathematical Knowledge for Teaching (MKT) and its importance in mathematics education and student achievement. From there, the focus shifts to the interconnectedness of beliefs and MKT and teachers' instructional practice. This includes a detailed explanation of questioning and classroom discourse as an instructional practice. Lastly the chapter aims to outline the importance of planning as vehicle to increase MKT and ultimately shift beliefs about the teaching and learning of mathematics and instructional practice.

Mathematics Teachers Beliefs

Beliefs are assumed to act as filters through which one sees the world (Pajares, 1992). For my research, the focus on teachers' beliefs and their role they play on teacher practices. Cultural beliefs about the teaching and learning of mathematics continue to present challenges to consistent implementation of effective teaching and learning of mathematics (Handal, 2003 & Philipp, 2007). There is a general belief from educators and non-educators alike that the best practice for the teaching and learning of mathematics is grounded in teaching students the procedure, having the students memorize the procedure or skill and then having the students practice the procedure or skill over and over (Sam & Ernest, 2000). Because of this belief about the teaching and learning of mathematic, this type of classroom practice still dominates mathematics classroom instruction (Banilower, Boyd, Pasley and Weiss, 2006).

The National Council for Teachers of Mathematics (NCTM) has been promoting lessons that are student-centered and promote students solving, reasoning and discussing tasks that promote discourse. While procedural fluency is an important aspect of students learning in mathematics, it needs to be built on a foundation of conceptual understanding. Procedural fluency is more than just memorizing facts and algorithms, it involves the ability to analyze your own and other's calculations, processes and application in solving problems. Conceptual understanding and procedural fluency develop in classrooms where students have opportunities to engage in meaningful discourse with each other and share their ideas and thinking (NCTM, 2014). Teachers that hold these beliefs about instructional practice promote student discourse and questioning in their classrooms to help students make sense of mathematical concepts and their connections to mathematical procedures (Monson, 2011).

The first focus on beliefs grows from the work of Leinwand, Brahier and Huinker's (2014) thinking about beliefs as being either productive beliefs or unproductive beliefs. This is important because my research question is focusing on Mrs. W.'s beliefs and instructional practice through classroom observations and de-brief/reflection conversations. Because of this, I am focused on what she is doing and considering as a result of planning and the development of her MKT in relation to her instructional practice. Leinwand and colleagues (2014) discuss unproductive and productive beliefs as not being either good or bad but how they influence the way a teacher teaches mathematics. The table below shows the impact of each of these beliefs on the teaching and learning of mathematics:

Figure 2.1. Productive and Unproductive Beliefs Chart

Beliefs about teaching and learning mathematics	
Unproductive Beliefs	Productive Beliefs
Mathematics learning should focus on practicing procedures and memorizing basic number combinations.	Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.
Students need to only learn and use the same standards computational algorithms and the same prescribes methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can only learn to apply mathematics only after they have mastered the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.
The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.	The role of the teachers is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests	The role of the students is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the	An effective teacher provides

mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.	students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

(Adapted from Principles to Actions: Ensuring Mathematical Success for All, 2014, p. 11)

In the case study of Fran Gorman and Kevin Cooper; Stein, Smith, Henningsen and Silver (2000) share the reflection of Kevin Cooper after implementing a lesson on multiplying fractions with pattern blocks. During this reflection, Kevin openly admits that he did not think students would be able to solve the given problems using the pattern blocks. His initial plan on implementing the lesson was to teach students the algorithm for multiplying fractions and then have them use the pattern blocks to practice rather than to use the pattern blocks to develop the concept and then formalize the algorithm for themselves.

After implementing this new approach to solving a fraction multiplication problem, Kevin was surprised at how students were able to make sense of multiplication of fractions and how much they learned from having the time to struggle with the given task. In Kevin's reflection, he admitted that he was starting to embrace this new way of thinking of instruction as a result of the shift in his beliefs. These types of instructional practices that Kevin references are behavioralist practices, teaching the algorithm and then practicing the skill, and constructivist practices, having students explore the idea and constructing meaning for themselves (Gales and Yan, 2000).

Behavioralist beliefs vs. constructivist beliefs.

Teacher's beliefs about instruction are usually grounded in one of two areas of thinking: behavioralist and constructivist (Gales and Yan, 2000). The beliefs drive a teacher's instructional moves and classroom pedagogy in relation to how they view learning (Nisbet & Warren, 2000). The behavioralist belief about learning is grounded in students learning a set of skills that are accumulated and those skills lead to the learning of more complex skills (Gagne and Briggs, 1979). The learning of the skills are typically delivered by the teacher through direct instruction (Gagne, 1985). The constructivist belief about learning is grounded in students analyzing, hypothesizing, testing theories and coming to their own conclusions from what they have analyzed and tested (Rumelhart, 1980; Shapiro, 1980). In the constructivist model, students must be engaged in developing new ideas and concepts and connecting those to previous learning. The students in a constructivist classroom also work with peers to share solutions and ideas that help students further clarify and defend their own thinking (Piaget, 1929; von Glaserfeld, 1984).

Paul Ernest (1991) discusses mathematics teachers' beliefs as being situated in three main ideas: beliefs about how a teacher of mathematics learned mathematics, beliefs about how a mathematics teacher is confident in their own understanding of mathematics, and beliefs a mathematics teacher has in regards to which students can learn mathematics. These beliefs have one of three main ways of manifesting in to classroom instruction as either an instructional emphasis on rules, formulas, and procedures; extensive use of definitions and proofs as an instructional strategy; and having students "doing" mathematics (Liljedahl, 2008; Romagnano, 1994). Paul Ernest (1989) also argues that beliefs are what drives a teacher's professional practice in the classroom. Ove Drageset (2010) shows empirical evidence that there is a correlation between a teacher's mathematical knowledge and beliefs. He also noted that, teachers with a higher level of mathematical knowledge, in both common content knowledge and specialized content knowledge, had a stronger instructional emphasis on reasoning with the students they were teaching. Common Content Knowledge is defined as the mathematical knowledge that is needed to solve mathematics problems but necessary for teaching. In contrast, Specialized Content Knowledge (SCK) is mathematical knowledge needed for the teaching of mathematics (Ball, 2008).

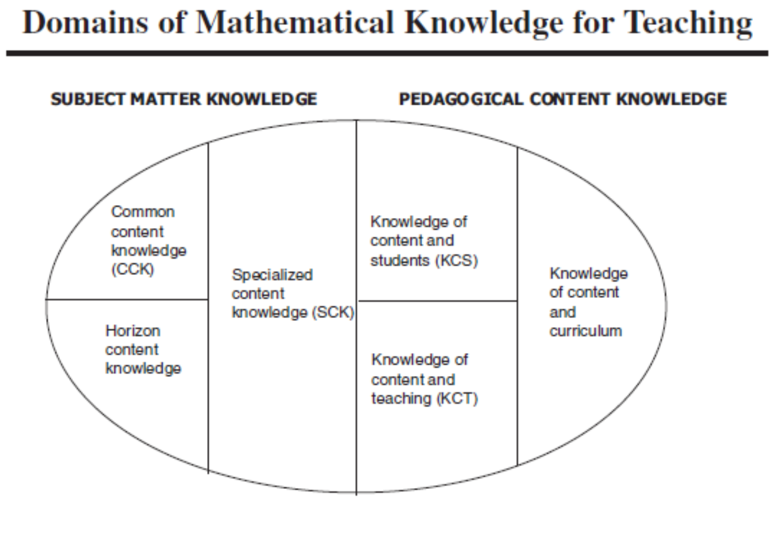
When comparing teacher's beliefs and practices, Gales and Yan (2000) used a linear regression model to compare student achievement data from teachers with behavioralist and constructivist beliefs. The Third International Mathematics and Science Study (TIMSS) data was used for comparison. The researchers found that teachers who were situated in instructional practices that were more behavioralist in nature there was a statistically strong negative relationship between student achievement and behaviorist teacher-directed instructional practice (Gales and Yan, 2000). However, when they analyzed teachers with more constructivist beliefs and practices, the researchers found a positive relationship between student achievement and constructivist teacher instructional practices.

Stigler and Hiebert (1999) analyzed TIMSS data and worked to understand why there was such a dramatic disparity between student achievement in the US and high-performing countries around the world, such as Germany and Japan. One of the main outcomes that the researchers had was that teaching was a cultural activity and for teaching to improve in the US, we have to change the culture of teaching. The culture of teaching mathematics in the US is a strong belief and practice that we have to teach skills and procedures for students to learn. While in contrast, Japan, had a strong culture of teaching that focused on students learning concepts and applying them to solve problems involving rich mathematics (Stigler and Hiebert, 1999).

Mathematical Knowledge for Teaching

Mathematical Knowledge for Teaching is a combination of two domains: subject matter knowledge and pedagogical content knowledge (Ball, Thames, Phelps, 2008). Within the domain of subject matter knowledge lies Common Content Knowledge (CCK), Horizon Content Knowledge, and Specialized Content Knowledge (SCK). CCK is the knowledge of mathematics that is useful for the purpose of solving problems or recognizing an incorrect solution. Horizon Content Knowledge is an awareness of how mathematical ideas are connected over the course of a mathematical curriculum. Specialized Content Knowledge is more specific to teaching in being able to analyze student work for correctness of a given solution path. This specialized knowledge is unique to the teaching of mathematics.

Figure 2.2. Domains of MKT



The impact of MKT on student achievement.

Research strongly suggests that a teacher's MKT is significantly related to student achievement (Hill, Rowan & Ball, 2005). This supports that to improve students' mathematics achievement we need to improve teachers' MKT. However, there is little known about how this specialized knowledge for the teaching of mathematics is developed in pre-service teacher education programs (Kleickman, Richter, Kunter, Elsner, Nesser, Krauss & Baumert, 2013). Hill and colleagues suggest that this specialized knowledge is important for student achievement and that to provide students with "highly qualified teachers" this knowledge needs to be developed in teachers. Hill (2007) also showed a strong relationship between MKT, teachers' subject matter preparation, certification type, teaching experience, and their students' poverty status. Hill collected a nationally representative sample of teachers and found that more affluent students are more likely to have teachers with more content knowledge and MKT. The implications of this for research suggest that we need to do more to support teachers in developing MKT in schools that serve students with high-levels of poverty to improve outcomes for those students that may not otherwise have the same opportunities as their more affluent counterparts in other schools (Hill, 2007 and Flores, 2007).

The connection of beliefs and MKT and teachers' instructional practice.

In 1998, Shulman argued that prospective teachers "learn to transform their own understanding of subject matter into representations and forms of presentation that make sense to students" (Shulman & Grossman, 1988, as cited in Brown & Borko, 1992, p. 217). Blanton (1998) extends Shulman's work in to mathematics education by positing that a teacher's limited knowledge of mathematical content is often revealed in their delivery of instruction. Teachers that only have procedural and rule-dependent knowledge create classrooms in which the focus of the classroom is centered around the accumulation of knowledge, not conceptual understanding.

Curriculum materials and MKT are two key instructional resources that impact the quality of classroom instruction (Charalambos & Hill, 2012). The teacher's MKT is more prevalent in setting up cognitively demanding tasks (Jackson, Garrison, Wilson, Gibbons & Shahan, 2013). In setting up the tasks, the teachers' instructional practices were examined to identify the nature of students' opportunities to learn mathematics in whole-class discussion. These whole class discussions led to higher quality opportunities for students to learn (Jackson, Garrison, Wilson, Gibbons and Shanan, 2013). MKT also contributed to richer teacher-to-student and student-to-student dialogue, mathematical language used during instruction, explanations offered by students and the teacher's capacity to summarize and connect the understandings of mathematical ideas for students (Hill & Charalambos, 2012).

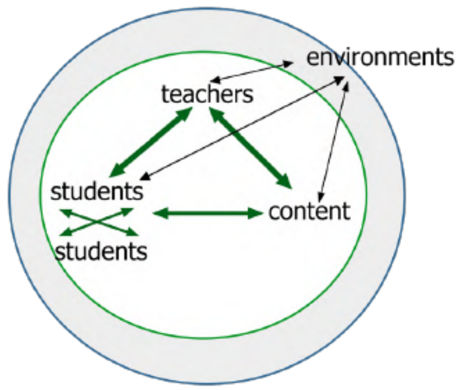
When analyzing how teachers setup tasks and opportunities for students to learn, there is a connection between a teacher's MKT, planning and implementation of the given task (Jackson, Garrison, Wilson, Gibbons and Shanan, 2013). The researchers identified three factors that influenced whether the teacher and the students maintained the high-cognitive demand of the task through the setup: the teacher's goal for instruction, the teacher's subject matter knowledge, and the teacher's knowledge of students (Jackson et al 2013; Stein, Smith, Henningsen and Silver, 1999). All three of those factors are components that require strong MKT (Jackson and colleagues, 2013). When teachers are able to setup and maintain cognitively demanding tasks, they will lead students to opportunities for thinking, reasoning, problem solving and mathematical communication (Stein, Smith, Henningsen and Silver, 2000).

Classroom Discourse and Discussion

National studies have shown that the majority of American classrooms do not have students engaged in mathematical tasks or problems that develop conceptual understanding or critical thinking (Stigler and Hiebert, 1999). In these classrooms, students solve problems where the teacher has taught a preferred method for solving the task and there is low engagement from students (Smith and Stein, 2011). Research, however, suggests that "complex learning and skills are learned through social interaction" (Smith and Stein, 2011, p.1). Learning happens through active investigation and discussion that supports communication and having students justify and critique the reasoning of others (Smith and Stein, 2011).

Deborah Ball (2011) argues that good teaching is created by high-leverage teaching practices that maximize teacher-to-student and student-to-student discourse in the classroom learning environment.

Figure 2.3. Connection Between Teacher, Students, Content and Learning Environment



In Figure 2.3 above, Ball shows the connections between teachers, students, content and environment. From this diagram, Ball (2011) states that effective teachers deliberately “maximize the quality of interactions” either amongst the students or between the students and the teacher in ways that increase the chance that all students learn mathematics. Through creating a culture of classroom discussion and student discourse, teachers are able guide students that may have misconceptions or incorrect solutions to become more proficient mathematicians (Keiser, 2012).

Classroom Discourse and Culturally Relevant Pedagogy.

Gloria Ladson-Billings (1995) argues that culturally relevant pedagogy is one that honors and hears student voices in the classroom. Geneva Gay (2009) builds on this to say that people are social and cultural beings and teachers that promote classroom discourse value and honor these cultural differences by including all voices in learning. Classroom discourse allows teachers to hear and understand their students and give them voice.

Effective classroom discourse.

Instructional practices that allow students to engage in meaningful discussions and debate ideas and thoughts centered around big mathematical ideas and concepts can have a positive impact on student learning (Brophy, 2001; Ball, Lubienski and Mewbron, 2001). Walshaw & Anthony (2008) claims that effective classroom discourse starts with a teacher's content knowledge and pedagogical content knowledge and a student's access to materials and resources to engage in meaningful discourse. Effective classroom discourse is grounded in teachers shifting the focus of student's cognitive demand from procedural skill to making sense and persevering in problem-solving (Yackel & Cobb, 1996). The process of justifying and constructive argument in classrooms also shifts the focus from doing mathematics to arrive at a set solution to students engaging in learning that helps them make sense of the mathematics. This process of engaging in meaningful classroom discourse is the only way in which students can truly understand mathematics (Carpenter, Franke and Levi, 2003).

While effective classroom discourse is critical to students' learning in mathematics, having students just talking in class does not transfer to student learning (Walshaw & Anthony, 2008). There are pedagogical instructional practices that have been shown to be effective in setting up classroom discourse that lead to students' learning.

These practices include: a) participating rights and obligations, b) articulating thinking, and c) shaping mathematical argumentation (Walshaw and Anthony, 2008).

Classrooms that have strong norms and systems to support classroom discourse that is productive and has students responding to each other demonstrated higher rates of student learning (Scribner and Cole, 1981; Yackel and Cobb, 1996). Steinberg, Empson and Carpenter (2004) showed a strong connection between classroom discourse and a student's conceptual understanding.

Through observed teacher's pedagogical practice, teachers that responded to students' responses by articulating students' thinking and valuing their ideas, whether right or wrong, and then putting the students thinking back on the class to respond had higher levels of student learning. This practice was observed in classrooms where teachers could clearly give mathematical structure to student responses and students were able to develop deeper mathematical reasoning. In classrooms where teachers were uncertain how to respond to a student's response, the teacher would simply give verbal praise to the student and as a result, student learning was not notably observed by researchers (Khisty and Chval, 2002; Knight, 2003). Cobb and colleagues (1993) noted that effective teachers valued a student's response and re-framed the response as a topic for discussion. In this way, the teacher did not close the loop in the classroom discourse but instead, created a greater sense of community within the classroom by allowing students opportunities for deeper conceptual understanding of the mathematical ideas the students were discussing.

Finally, teachers that were able to effectively press students to elaborate their thinking and/or ideas maintained higher-levels of cognitive demand and allowed teachers to have a better understanding of what the students know (Stein, Grover, and Henningsen, 1996; Morrone, Harkness, D'Ambrosio and Caulfied, 2004). In these effective classrooms where teachers continued to press students to go deeper with their reasoning and justifications, students' mathematical thinking was made visible and they promoted other students' thinking that extended the learning of the entire class (Walshaw and Anthony, 2008).

Research provides strong evidence that supports the need for effective classroom discourse. As students engage in classroom discourse, their mathematical dispositions shift from procedural skill of solving problems to arrive at a given solution, to understanding mathematics as a subject worth exploring. This shift in disposition creates students that have a deeper conceptual understanding of the mathematics being studied and prepares them to engage in higher-level mathematics (Walshaw and Anthony, 2008).

A vignette of classroom discourse.

The following classroom vignette is taken from *Patterns of Instructional Discourse that Promote the Perception of Mastery Goals in a Social Constructivist Mathematics Course* (Morrone, Harkness, D'Ambrosio and Caulfield, 2004).

In the following classroom, students were engaged in the *Condominium Problem* and the classroom transcription is from the teacher and the students engaging in classroom discourse related to the task.

The Condominium Problem

In an adult condominium complex, 2/3 of the men are married to 3/5 of the women. What part of the residents are married?

T: *"Folks, let's see what progress you've made. Let's have some conversation. Who would like to share what they did and how they, how they went about thinking?"*

Student talks about knowing they needed the same number of men and women who were married.

T: *"Do you want to draw. . ."*

Student walks to the chalkboard and shows a model of circles. Student asks a question (can't hear).

T: *"Okay. So, what did that tell you? That's right. (Laughs) What does that tell you? The fact that 6/9ths is equivalent to 2/3s and 6/10ths is equivalent to 3/5ths? What does that mean? I mean, that is an important insight to notice that. What does it mean?"*

Student says they're the same.

T: *"They're the same, the same what?"*

Student answers.

T: *"Yeah, but when she had the 3/5ths in that particular picture it would have been just five people and now I have ten people."*

T: *"Yeah, (student name)?"*

Student talks about ratio and proportion – "whatever you do to the numerator" etc.

T: *"Did you think about that, (student name), when you did that?"*

Student answers.

T: *"Married to total amount of people, you wanted to keep that same ratio, that same relationship. Okay. So, when you double the number of women, you want to make sure that you have that same relationship. "Why did you color in 3 women in the second row* *instead of 2, or 4, or 5?"*

Student answers.

T: *"Okay. So, you checked that. Okay. So, we're saying here that in this particular context that equivalent fractions mean something. It means that in a bigger population we're keeping the same ratio. Does that seem strange to you? (Student name), what prompted you to say that it was the same size? What prompted, what made you think it was the same size?"*

Student answers.

T: (Laughs) *"I'll ask (student name) through you, what prompted (student's name) to think that it was the same size?* (Laughs) *Okay. . .?"*

Student responds again.

T: *"So are women a smaller group?"*

Student responds.

T: (Laughs) *"Does that explanation match the context?"*

Student responds again.

T: *"Some of the others? What puzzled you as you started? Does everyone agree with (student name)? Does someone want to ask (student name)'s group what they did, why they did it?"*

Student asks how they can add 6/9 and 6/10. Student answers.

T: *"Yeah, (student name)?"*

Student explains how her group did it. T writes on the chalkboard.

T: *"So in this situation how did you come up with 18/27ths and 18/30ths?"*

Student answers.

T: *"Okay, so you kept thinking of what happens to the population to keep that ratio. All right."*

T: *"Oh and this (36/57) reduces to 12/19ths? Hum. . ."*

Student comments.

T: *"(Student name)'s question is still on some people's minds, I think.
How can you add numerators and denominators?"*

Student talks about the context of the problem and how this problem is different than other addition problems, "two different groups of people" and "you want the total."

T: *". . .question. . . When can you add the way we're adding, using the traditional algorithm, finding the common denominator? When does that make sense? Several of you started, the first thing you did was add, and you ended up, what did you end up with, with 19/16ths. What does that mean? When can you do that? When does it make sense to add that way?"*

In this vignette, the dialogue between the teacher and the student demonstrates how the teacher validates the student's response and re-frames the student's response and poses a new question based-on the response to probe deeper in to the student's understanding of their own solution to the task. Throughout the process, the teacher did not accept incomplete responses and continued to push for deeper-understanding. This vignette demonstrates an example of effective classroom discourse as denoted by the validation of the student response, the re-framing of the student's response and then continuing to use that re-framing to push the student's thinking deeper.

Patterns of questioning.

Early research on classroom patterns of questioning suggest that classrooms in The United States tended to have a pattern of questioning Mehan (1979) referred to as Initiation-Response-Feedback. In this pattern of questioning, the teacher asks a question, a single student provides a response and the teacher provides evaluative feedback on the response. This would close the feedback loop and the pattern of questioning would start over with the next questions. This type of questioning is still prevalent in classrooms today (Stigler and Hiebert, 1999). Another common form of questioning that occurs in classrooms is something called funneling questioning (Herbel-Eisenmann and Breyfogle, 2005). This pattern of questioning happens when a teacher asks a series of questions to get students to a pre-determined solution. In this pattern of questioning, the teacher is the one that is doing the cognitive lift and the students are merely answer simple questions that lead to the solution the teacher wants. Another type of questioning, one that Herbel-Eisenmann and Breyfogle (2005) argue that need to become more prevalent in classrooms to promote discourse and student learning, is called focusing questioning. Focusing questioning is where the teacher asks a question and listens to the student response. After the student responds, the teacher then asks questions to guide the student in their own thinking rather than the teacher's thinking.

Teachers too often do not consider the quantity and quality of their questioning and as a result, miss opportunities to deepen their students' mathematical understanding of important concepts (Tarihi, 2105). For students to become mathematicians, they need to engage in real learning and have time to struggle while being asked high-quality questions by their teachers to place the learning back on the students (Reinhart, 2000). This emphasis on questioning and the quality of questioning has shown to lead to student achievement in mathematics (Tarihi, 2015 and Reinhart, 2000).

The Importance of Planning

"The hard work of improving teaching in the United States can't succeed without changes in the culture of teacher learning" (Stigler and Hiebert, 2009, p. 32). Teaching can no longer be a profession that happens in isolation, in fact, we need to change the way in which teachers think about the purpose and value of planning a lesson before ever getting in front of students (Fernandez, 2008). Stigler and Hiebert (1999) concluded that for us to close "the teaching gap", we need to promote teachers effective planning of mathematical tasks.

There are five key practices for supporting productive classroom discussion (Smith and Stein, 2011). Those five key practices are: 1) anticipating, 2) monitoring, 3) selecting, 4) sequencing, and 5) connecting. During the planning of the lesson, Smith and Stein (2011) make the case for teachers planning the mathematical lesson by doing the mathematics and clearly stating the mathematical learning goal for students learning.

While a teacher cannot do all of these 5 practices in planning, there are several pieces and ideas that can be done in planning. A teacher can anticipate student responses and solution paths to a task within a lesson prior to the teaching of a lesson. Leinwand and colleagues (2014) also argue that if a teacher does not take the time to anticipate student's responses in the planning of the lesson, they will not be prepared to support the student's learning while the student engages in productive mathematics struggle. A teacher can consider the appropriate ways in which to present student work and the teacher can plan for how they might make connections between the different solution strategies a student presents to pull out the main ideas for all students in the classroom to learn from (Smith and Stein, 2011). However, Lewis, Perry, and Hurd (2004) suggest that teachers need to have strong MKT to understand how students think about solving various tasks. Perry and colleagues (2004) suggest that this knowledge can be developed through lesson planning in collaboration amongst teachers.

By setting up the five practices in planning and in the implementation of the lesson, teachers tended to maintain high-cognitive demand of mathematical tasks in their classrooms (Smith and Stein, 2011). But without proper planning to setup the mathematical tasks, the teachers will likely lower the cognitive demand of the task and thus reducing the learning opportunity for students (Jackson et al, 2013). Both the student-facing materials and implementation of the task have to engage students in carrying the work load throughout the entire lesson for students to develop as mathematicians and learn mathematics (Doyle, 1984; Doyle and Carter, 1984; Romaganano, 1994; Leinwand et al, 2014).

The Need for Change

Romagnano (1994) discusses a need for change in mathematics education that has to happen before our students are ready to fully participate in a "democratic and increasingly technological society" (p.7). Romagnano (1994) makes the case that teachers need to shift the role of teachers to allowing students opportunities to engage in "real mathematics". When students are given the opportunity to struggle productively with mathematics, students develop the ability to make connections to prior learning to learn mathematics in a way that allows them to make connections to new concepts (von Glaserfeld,1991; Leinwand et al, 2014).

Without giving students the opportunity to engage in productive struggle, students will only see themselves as “smart” when they are only able to provide a correct solution. This reduces student’s self-efficacy and tends to push students away from mathematics as a subject worthy of study (Leinwand and colleagues, 2014). What we need are students engaging in productive struggle, having opportunities to discuss and share their ideas and thinking, and receive praise for persevering and making an effort that makes them feel capable of doing mathematics and see mathematics as a subject of joy and passion worth pursuing (Mueller and Dweck, 1998; Boaler, 2014). Productive struggle supports students in making sense of mathematics, persevering to solve problems, and developing growth mindsets in our students.

Chapter 3: Methodology

For my research study, the research questions were as follows:

1. How did a teacher's MKT develop through the planning of and implementation of key lessons?
2. How did a teacher's beliefs about her own pedagogy and instructional practice shift as she deepened her content knowledge and MKT expand through lesson planning?

The participating teacher, Mrs. W. and I planned six different lessons from Connected Mathematics Project 3 (CMP3) over the course of six months. After we planned the various lessons together, Mrs. W. implemented each lesson in the classroom with her students. During the implementation of each lesson, I journaled and videotaped each class to learn about the questions she asked her students, patterns of questioning she engaged in with her students, and classroom discourse. The teacher's patterns of questioning can help to drive classroom discourse but the teacher has to be able to set up the task so that students engage in whole-class discussion thus promoting the deeper discourse needed to develop conceptual understanding (Jackson, Wilson, Gibbons, and Shahan, 2013). After each classroom observation, I debriefed with the teacher about how she felt planning impacted the implementation of the lesson in the class. The debrief had five guiding questions:

1. How did doing the task within the lesson as a learner prepare you to implement the lesson in your own classroom?
2. What misconceptions did students have during the lesson that you anticipated? That you did not anticipate?
3. During the implementation of the lesson, was there a certain question(s) that you asked that had an impact on how students thought about the task they were working on? Did you change or modify the question(s) for another class period?
4. If you were to implement this lesson again, what, if anything, would you do differently and why?
5. Did you feel that you modified or shifted your instructional practice as a result of the collaborative planning of the lesson?

Due to the nature of my two research questions, I approached each research question with a different methodology to answer each question. For the first research question, I worked to answer the research question with a participatory action research methodology because I was working with the teacher as a learner along with her. For the second research question, I worked to answer the research question using a case study methodology because the focus of the research was on the journaling and videotaping of classroom observations and then journaling the post-observation to learn more about the teacher's thinking in regard to the patterns of questioning and discourse. While the classroom observations and post-observation de-brief helped to support my first research question, the emphasis of the observations and de-brief was to answer the second research question.

Methodological Framework

Addressing research question #1.

How did a teacher's MKT develop through the planning of and implementation of key lessons? To answer this research question, I used a Participatory Action Research (PAR) methodology because the work involved doing research in collaboration with the teacher. The participatory action research methodology is appropriate to this study for this research question because I do not see myself, the researcher, as the expert that holds all of the knowledge. Instead, I see myself as a learner along with the teacher to better understand how student may think about solving a variety of tasks within a lesson and the misconceptions they may have along the way.

I will be working with the teacher in planning and thinking through the implementation of each lesson. Participatory Action Research is a research design that creates space for both the participants of a research project and the researcher(s) in a research project to work collaboratively to impact change. PAR is collaborative research that has multiple, more than one, participants, that work together to address a research question and make immediate changes to address the issue the researchers are engaging in studying. "PAR seeks to understand and improve the world by changing it."(Baum, MacDougall and Smith, 2006, p.1)

PAR is different from other research methodologies in three ways: 1) its purpose is to enable action or change, 2) it advocates for power to be shared, and 3) PAR advocates that people involved in the research should be actively involved in the process. In this research design, I created research questions to be explored. During the research process, the teacher and I will be working to address the research questions through collaborative planning (i.e., the action of the research), This will allow me to understand how the planning supports the development of MKT and shifts in the classroom teacher's instructional practice related to discourse and patterns of questioning.

The first phase of the research for this question will be to assess Mrs. W's baseline MKT through the Teacher Knowledge Assessment System (TKAS) that was developed at The University of Michigan to measure a teacher's MKT. The assessment is a computer-adaptive test that progresses in relation to each of the teacher's prior responses on the assessment. After assessing Mrs. W.s' baseline MKT, we will be working together to take action to develop her MKT through collaborative lesson planning. The TKAS is designed to analyze a teacher's content knowledge specific to the teaching of mathematics (Hill, Schilling and Ball, 2004). The instrument I have chosen to use for my research is a computer adaptive testing model because it is designed to measure a participant's level of MKT after each assessment. I have selected "Middle School Number Concepts and Operations" and "Middle School Patterns, Functions, and Algebra" for the purposes of my study because that is a primary focus of 8^{th} grade mathematics and is the focus of the six lessons Mrs. W. and I will be planning together.

The instrument also focuses on two areas of knowledge: content knowledge and pedagogical content knowledge (PCK). These two areas of knowledge are what creates MKT (Ball, Thames and Phelps, 2008). Content knowledge is the knowledge of the mathematics and its organizing structures (Ball et al, 2008). PCK is focused on the teacher specific knowledge as it relates to the teaching of mathematics, such as knowledge of students, knowledge of curriculum, and knowledge of various forms and representations of mathematics (Shulman, 1986).

Figure 3.1. An example of an item that would assess content knowledge.

Ms. Covey wants to use story problems during a unit on linear functions. She looks through some supplemental materials for teaching algebra, but becomes concerned that not all problems she sees represent linear functions.

Of the following stories, which does **not** represent linear behavior in the context of the story? (Select ONE answer.)

- ○ The first day Nathan gets one penny. Each day after that, he gets twice as many pennies as he did the day before. How many pennies does he get on the n^{th} day?
- ○ It takes Kara 2 pints of paint to cover 30 square yards of a wall. How many pints does Kara need to cover n square yards?
- ○ The cost of a taxicab ride is $1.00 plus 20 cents for each fifth of a mile. What is the fare for traveling n miles?
- ○ Shelly has $100.00. Each day she spends $1.25. How much money does she have on the n^{th}?
- ○ All of these stories represent linear behavior.

This item requires the participant to have content knowledge about linear equations as it relates to the context of a word problem. The focus of this item is to assess content knowledge, one of the two forms of knowledge that measures MKT. As demonstrated in Figure 1.1, the teacher understood the solution but did not see the structure of the work to know the solution was incorrect.

Figure 3.2. An example of an item that may be more aligned to measuring a participant's PCK.

Sharon was trying to solve the equation $2x^2 = 6x$. First, she divided both sides by 2.

$$x^2 = 3x$$

Then she divided both sides by x:

$$x = 3$$

Gustavo said, "You can't divide both sides by x." Sharon responded, "If you can divide both sides by 2, why can't you divide by x?" They asked their teacher, Mr. Padilla, to explain. Which of the following explanations is correct? (Select ONE answer.)

- ○ Because x is a variable, it can vary – you may not be canceling the same amount from both sides.
- ○ You can cancel x because it represents a real number.
- ○ You can only divide by whole numbers when solving equations.
- ○ It is better to take the square root of both sides after dividing by 2, that way you won't have to worry about dividing by x.
- ○ You cannot divide by zero, so you would have to consider the case of $x = 0$ separately.

This item shows how the participant is asked to consider the students' thinking and then respond to their thinking with an explanation that justifies either Gustavo or Sharon that is also mathematically sound. In Figure 3.2, the correct solution is that the teacher should respond by saying that you can cancel x from both sides because x represents a real number. The reason this is the correct answer is that it addresses the idea of a variable and how they just representations of something real. It also addresses the notion that the same variable in an equation does not change the real number that they represent within that equation. The option at the end is true in that in the case of $x = 0$, you cannot divide by zero but that does not address the question the students are asking. The teacher in this situation may add the zero case as a way to push students thinking about what that might mean in terms of possible solution sets and the graphical representations of the equation but not necessarily to address the specific question being asked. These two items also help to capture some of the complex nuances of the knowledge needed for the teaching of mathematics, it is not as simple as just knowing the content of mathematics but also considers the variety of variables that teachers deal with constantly in the classroom.

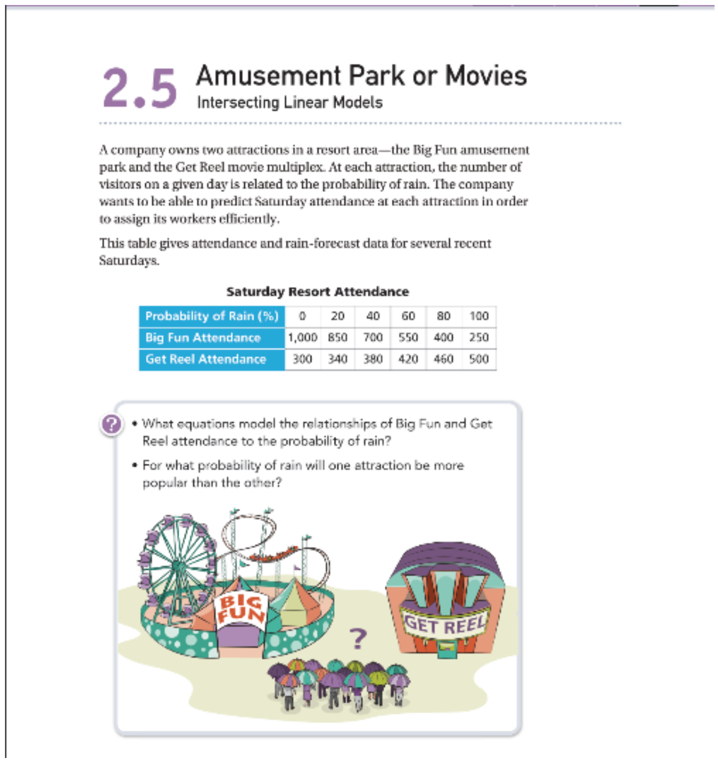
Teachers will never be fully prepared to teach any lesson because students come to the classroom with so many different ways of thinking about mathematics but, through effective planning of mathematics lessons, the teacher will be able to better understand the mathematics and prepare for student misconceptions, questions and stumbling points along the way (Castro, 2006). Which is why, during the second phase of the PAR, the planning of each lesson is critical to understanding how students will consider the given task within the lesson. During the second phase of PAR, the teacher and I will be working collaboratively to plan the lessons and through the action of planning develop a deeper MKT. This work will help to develop a body of research on how planning for the implementation of mathematical lessons can develop and support a teacher's MKT. Research shows that there is a strong correlation to a teacher's MKT and student achievement (Hill, Rowan and Ball, 2005 & Sleep and Eskelson, 2012). Research also shows that effective lesson planning has a positive effect on student achievement (Jackson et al, 2013 & Morris, Hiebert and Spitzer, 2009). My research will bridge the gap in research between developing a teacher's MKT and lesson planning.

When planning the lessons collaboratively with the teacher, we will both be developing our thinking of the mathematics, student strategies for solving and misconceptions they might have when exploring the task within the lesson. Through the collaborative effort of lesson planning, the teacher and I will work to consider the multiple solution paths students might use to arrive at a solution to the task within the lesson. As we consider how students may approach the task, we want to consider the various representations that students may take and as mathematicians, we want to be able to make connections to the various representations for students.

By focusing on various representations and possible solution paths, we will also consider solution paths students will take that bring out the student's misconceptions about the concept. This "amalgam of knowledge of content and pedagogy is central to the knowledge needed for teaching" (Ball, Thames & Phelps, 2008, p.5).

As Mrs. W. and I plan the lessons and consider the student solution strategies and potential misconceptions, we will also be working to script out questions Mrs. W. could pose to the student or the class to support the student's or the class's thinking to improve learning outcomes for students. To understand what the planning session may look like, consider the example of planning the lesson below:

Figure 3.3a. Task for finding probabilities of going to amusement park or movies.



2.5 Amusement Park or Movies

Intersecting Linear Models

A company owns two attractions in a resort area—the Big Fun amusement park and the Get Reel movie multiplex. At each attraction, the number of visitors on a given day is related to the probability of rain. The company wants to be able to predict Saturday attendance at each attraction in order to assign its workers efficiently.

This table gives attendance and rain-forecast data for several recent Saturdays.

Saturday Resort Attendance

Probability of Rain (%)	0	20	40	60	80	100
Big Fun Attendance	1,000	850	700	550	400	250
Get Reel Attendance	300	340	380	420	460	500

- What equations model the relationships of Big Fun and Get Reel attendance to the probability of rain?
- For what probability of rain will one attraction be more popular than the other?

Figure 3.2b. Student prompts for response

Problem 2.5

A Use the table to find linear functions relating the probability of rain p to the following quantities.

1. Saturday attendance F at Big Fun
2. Saturday attendance R at Get Reel

Saturday Resort Attendance

Probability of Rain (%)	0	20	40	60	80	100
Big Fun Attendance	1,000	850	700	550	400	250
Get Reel Attendance	300	340	380	420	460	500

B Use your functions from Question A to answer these questions. Show your calculations and explain your reasoning.

1. Suppose there is a 50% probability of rain this Saturday. What is the expected attendance at each attraction?
2. Suppose 475 people visited Big Fun one Saturday. Estimate the probability of rain on that day.
3. What probability of rain gives a predicted Saturday attendance of at least 360 people at Get Reel?
4. Is there a probability of rain for which the predicted attendance is the same at both attractions?
5. For what probability of rain is attendance at Big Fun likely to be greater than at Get Reel?
6. For what probability of rain is attendance at Big Fun likely to be less than at Get Reel?

ACE Homework starts on page 45.

In the example lesson above, Figure 3.3a sets up the lesson and Figure 3.3b presents the task to the students for them to consider and respond to in this situation. In planning this lesson, the teacher should start by answering the questions to the task as a student (Smith and Stein, 2015). By doing this, the teacher will have a clear understanding of the learning objective, or goal, of the lesson. For this particular lesson, the goal is to introduce the concept of linear equations in the context of intersecting graphs for linear functions. After doing the tasks within the lesson, the planning could then shift to thinking through how students would approach solving the given tasks. Let's consider solution paths to Part A1. Students may write the equation: $F_{attendance} = 1000 - 150p$, which would imply that the student only considered the decrease in attendance to Big Fun Amusement Park ($F_{attendance}$) as opposed to the decrease in attendance in relation to the probability of rain (p) increasing by 1% point. In my experience as a classroom teacher, this was a common mistake students made because too often they focused only on the change in the dependent variable without taking in to consideration that they needed to consider how the dependent variable changes in relation to the independent variable. By knowing the misconception and the mistake that students could potentially make, the teacher could plan questions that push students thinking to help them correct their own error and learn from their mistakes. For example, the teacher may plan to ask something as simple as "How did you get -150p?" or something more complex as "If it *will* rain then you are implying that -14,000 people will show up to the amusement park, does that make sense to you?". These two sample questions could help shift a student's thinking that could lead them to correct their own response. The more the teacher is able to anticipate and plan for student misconceptions and incorrect solution paths, the better the learning outcomes for students will be (Smith and Stein, 2011). This is only an example of what may occur during the planning of the six lessons with the teacher. During the planning process, I will be journaling the questions that we both have as we work together to plan out lessons. The journaling will focus on the types of questions that the teacher could ask in response to student misconceptions and questions that could be asked to push students' thinking to identify their own misconception and promote a deeper level of understanding.

During the planning of the lessons together, I videotaped each planning session to capture the discussion between Mrs. W. and I in regards to how we understand the mathematics and the mathematical goal of each lesson, how we considered the various solution paths students may take and the misconceptions we anticipated students having when approaching the task within the lesson. I also audio taped the debrief to then capture what the teacher learned about the MKT she developed through planning and what further MKT was developed after the implementation of the lesson in her classroom. The video and audio taped sessions, prior to and after the lesson, over the course of six different lessons helped me understand how Mrs. W.'s MKT developed as a result of the collaborative planning. The video tapes served as a way to understand the progression of learning the teacher and I had about planning, anticipating student work and types of questions we could ask to address those misconceptions. After the planning of the six lessons together, Mrs. W. completed a post-assessment measure of MKT to then analyze her change, if any, in proficiency level on the TKAS measure.

Data analysis for research question #1.

In this research method, a single subject collaboratively planned six various core lessons and was given a pre- and post-assessment to gauge the development of her MKT throughout the planning process. The quantitative data collected from the TKAS was not tested for statistical significance because I was not trying to ascertain if the subject's development of MKT was statistically significant but rather how a teacher's MKT developed through planning. The TKAS measure Mrs. W. participated in was a computer adaptive test (CAT). Computerized adaptive testing (CAT) is a method for administering assessments that selectively adapts to a participant's proficiency level. The CAT algorithm searches the pool of available questions and then selects the next question based upon the participant's prior responses. After each question is answered, the participant's proficiency estimate is updated. This process is repeated until the participant's responses have reached a specified level of assessment reliability. In pre- and post-assessments, CAT scores are equated and can be used to study teacher learning. The data that was collected will be the teacher's raw score from the pre-assessment to the post-assessment to look for change in Mrs. W.'s level of understanding to help explain how the planning had impacted her MKT.

Through the use of videotaping, audio recording and journaling, I summarized qualitative the data to find meaning for how Mrs. W.'s MKT developed through the lesson planning process. After each of the planning sessions, I focused on the data related to how the teacher demonstrated content knowledge of the curricular materials through her ability to solve the tasks in the lesson and how the teacher demonstrated her MKT in the way she considered the various solution paths students might take to solve the task and which possible misconceptions students may have in solving the given task. Through a series of iterations of comparing the prior session's summaries and key findings to the current session's summary and key findings, I looked for how Mrs. W. was able to articulate the mathematics that students would be doing in the lesson from her own understanding of the task as she solved each task in the lesson. From there, I then compared her thinking about student knowledge, misconceptions and potential questions to ask students as a result of their possible solution path or misconception. This analysis of the data allowed me to understand how her thinking about content and MKT developed over the course of the planning of the six lessons. Qualitative analysis is based-on inductive reasoning (Gay, Mills, Airasian, 2009). Inductive reasoning is an observation of patterns and using those patterns to make conjectures about what the pattern is saying. When observing and reflecting on the experiences of my research study, I hoped to see emerging patterns that would become themes that allowed me some insight in to the development of Mrs. W.'s MKT. These patterns and themes then allowed me to make some sense and learn how Mrs. W. developed MKT through the experience of lesson planning.

Addressing research question #2.

How does a teacher's beliefs about her pedagogy and instructional practice shift when she deepens her MKT through lesson planning? This research question was addressed through a single instrumental case study approach. "Case study research is a qualitative research approach in which researchers focus on a unit of study known as a bounded system" Gay, Mills, & Airasian, 2009, p.426). As I focused my lens on Mrs. W.'s beliefs about her own content knowledge and instructional practice, the qualitative data collected allowed me to answer the question about "how" those changes happen as she deepens her MKT. The single instrumental case study methodology is suitable for answering this research question because I was trying to gain an understanding of how practice changed, specifically patterns of questioning and classroom discourse, as a single teacher developed her MKT. The case study methodology is designed to understand an issue through a specific example or illustration (Creswell, 2007). In this methodology, the researcher is investigating a bounded system or systems such as a case or cases over time. The data is collected through observations, interviews, audiovisual materials and/or documents and reports. The single instrumental case study is appropriate because: a) I wanted to gain an in-depth understanding of how Mrs. W's beliefs about her own content knowledge and instructional practice shifted as her MKT developed through lesson planning, b) I was focused on a single individual, Mrs. W. for the study, c) the data are being collected over several months through several observations, videotaping, journaling and debrief interviews with a single subject and d) the final report focused about lessons learned from the research in regards to how Mrs. W.'s beliefs about content knowledge and instructional practice shifted and why. For these reasons, I selected a single instrumental case study design to learn and understand how Mrs. W.'s beliefs and instructional practice shifts as she develops her MKT. This research question does imply that Mrs. W.'s MKT increased through the collaborative lesson planning. If I assume that MKT would increase through deep lesson planning, the teacher's beliefs about her content knowledge and shifts in instructional practice was collected through videotaping, journaling and feedback/reflection conversations after the implementation of each of the six lessons. The classroom observations were videotaped and I then used the videotapes to journal the observation. The journaling from watching the videotapes focused on the questions the teacher asks, her pattern of questioning and the classroom discourse. These three areas are all connected in that the question the teacher asks is either a focusing or funneling question (Herbel-Eisenmann and Breyfogle, 2005). Focusing questions are types of questions that have a clear answer and they are normally guided by the teacher to get students to a predetermined solution or way of thinking by the teacher. An example that demonstrates the difference between these two types of questions follows:

Example A:

> *Teacher: What is the slope of the line through the points (1, 3) and (4, 12)?*
>
> *Teacher: What is the rise?*
>
> *Student: 9*
>
> *Teacher: Good, the rise is 9. What is the run?*
>
> *Student: 3*
>
> *Teacher: Good, so the slope is?*
>
> *Student: 3*
>
> *Teacher: Great! The slope is 3 because slope is rise over run and 9 over 3 is 3.*

In this example, the teacher gave the students a problem and then asked them specific questions to get the solution the teacher already wanted to hear. Focusing questions are designed to be more probing and expect students to justify and explain their thinking. Taking the example from above, the teacher could re-work their pattern of questioning to be focusing in the following way:

Example B:

> *Teacher: A line goes through the points (1, 3) and (4, 9). How could I find the slope of the line that goes through those points?*
>
> *Student: We could graph the two points and draw slope triangles to find the slope.*
>
> *Teacher: What do you mean slope triangles?*

Student: Slope triangles help us find the differences, or changes, between two points. We could then use that information to find the slope.

Teacher: How does the slope triangle help us find the differences and why is that important?

Student: So, if I draw the slope triangle I can find the change in the x-coordinates and the change in the y-coordinates. For this problem, the change in the x-coordinates is 3 and the change in the y-coordinates is 9. Then I can divide 9 by 3 and get a slope of 3.

Teacher: Why did you divide 9 by 3 to get the slope?

Student: Because the slope is the relationship between the independent variable and the dependent variable, the x and y. That means that every time x increases by 1, y increases by 3.

If a teacher asks a funneling question, there is very little discourse, as noted from Example A. When the teacher asks questions with a specific solution then the student response will only be to give an answer and then the discourse, or lack thereof, ends with the teacher validating the student's response and moving on. When the teacher engages students in a focusing pattern of questioning, students do not merely give an answer but are also expected to share their thinking and why. The expectation would be that several students are responding and pushing each other's thinking as well so the classroom becomes a community of learners trying to push each other to not only get a correct solution but to know that the solution they are getting is correct and why it is correct. This pattern of questioning puts the cognitive work load on the students and pushes them to a deeper level of explanation and understanding, and not just for the one student but for the whole class. Focusing questions also do not have a predetermined solution path that the teacher wants to hear, the question is open-ended so that students have freedom to think critically to arrive at their own solutions and justify their thinking (Herbel-Eisenmann and Breyfogle, 2005). This type of questioning will also promote classroom discourse as students are pushed to think beyond just getting a solution. Gencturk (2010) argues that teachers who have stronger MKT tend to open their classrooms to a more inquiry-based model where the teacher is continually probing and asking questions of students to think more critically. His research also calls out that there is very little research that shows connections between teachers' MKT and their instructional practice and thus the need for this research in the field to gain a deeper understanding of these connections. Gencturk (2010) does not specifically call out the types or patterns of questioning in classrooms where teachers have a deeper MKT but from my classroom observations and discussions with mathematics teachers, it is my belief that focusing patterns of questioning are more common in those classrooms where teachers have a deeper MKT. Therefore, the focus of the reviewing the videotaped classroom lessons were to capture the questions the teacher asks and the students' responses to these questions.

After observing the teacher implement the six lessons, videotaping each of the lessons and scribing the classroom questioning and discourse, Mrs. W. and I debriefed the implementation of the lesson. This debrief, although guided by a series of pre-planned questions, was designed to be more of a discussion between me and the Mrs. W. I was audiotaping and analyzing these reflections to try and identify trends in Mrs. W.'s thinking that show what she learned about herself as a mathematician, a teacher of mathematics and in her classroom as a learner along with the students. I would expect to hear such things as "I never thought about Student A's solution" or "Several students had this misconception about Topic A and I did not even anticipate that misconception." From those types of sentence stems, I started to understand how Mrs. W. was developing her MKT.

Data analysis for research question #2.

In my analysis of the data, I am trying to learn how a teacher's beliefs about her content knowledge and instructional practice shift when she deepens her MKT through lesson planning. This is assuming lesson planning had an impact on her beliefs about her own content knowledge and her instructional practice related to discourse and patterns of questioning. The initial lesson that I videotaped Mrs. W. teaching helped me understand a baseline for the continued research study. I analyzed the first lesson to identify the types of questions, either focusing or funneling, and then to identify the impact the question(s) had on classroom discourse.

I was looking for student engagement in response to the question and how often the teacher provided evaluative feedback to a student's response or how often the teacher opened-up the classroom to provide evaluative feedback on the student's response. I was analyzing the questions to see if the teacher was prompting the student for a pre-determined response or if the teacher was asking questions that push students to articulate their thinking and justify the reasonableness of their solution. Over the course of the next five lessons, I analyzed the data in the same manner but now also comparing the data to previously observed lessons that Mrs. W. taught. This continued process of reflection and new learning helped me to identify patterns and themes from the research.

The data collected during the debrief session was analyzed to learn how Mrs. W. thought about how her understanding of the content of the lesson made her more confident in asking questions to push students thinking and open her classroom to discourse. The data was also analyzed to learn how the teacher saw work or heard responses from students that she may not have otherwise seen or thought about if she had implemented the lesson another way.

Again, I was looking for emerging themes from the data that allowed me to identify a causal relationship between the development of Mrs. W.'s MKT and her beliefs and instructional practice. The emerging themes helped me to indicate patterns and meaning of Mrs. W.'s response to the five questions she responded to in the debrief of her lesson. As themes emerged, the meaning of the themes guided my interpretation of the categories into what I can learn from those categories as it relates to my research question.

Teacher Participant

The participant for this research study was a single 8th grade math teacher. This teacher came in to mathematics education through an alternative path for licensure. Mrs. W. was in her 4th year of teaching 8th grade mathematics at a traditional middle school in Denver Public Schools. The teacher was very willing to receive feedback on her instructional practice and continues to work to improve her pedagogy. She also admitted that she lacks deep content knowledge in mathematics but wants to learn more and have a deeper conceptual understanding of the mathematics she teaches to improve learning outcomes for students. The students in Mrs. W.'s classes reflected her school as a whole. The school is predominantly low-income Hispanic students, with a small percent of White, Black and Asian students rounding out the student body. During my research study, the school had about 85% of its students identified as receiving free and reduced lunch, about 41% of their students identified as ELA and about 16% of their students receiving special education services. The school has historically had a low average student attendance rate, approximately an 88% average student attendance rate, and is currently being phased-out because of the school's continued low-performance. The table below shows the enrollment and demographic trends at Mrs. W.'s school:

Figure 3.4. Demographic trends of Mrs. W.'s school.

Demographic Trends

		Free and Reduced Lunch		Race and Ethnicity							English Language Learner		Special Education	
School Year	Enrollment	Students Receiving FRL	% FRL	Hispanic	White	African American	Asian / Pacific Islander	American Indian	Multiple Races	% Minority	ELA Students	% ELA	SPED Students	% SPED
2010-11	948	737	77.7%	703	118	19	76	9	23	87.6%	365	38.5%	139	14.7%
2011-12	907	735	81.0%	657	128	21	80	7	14	85.9%	350	38.6%	120	13.2%
2012-13	828	693	83.7%	623	96	16	75	6	12	88.4%	310	37.4%	133	16.1%
2013-14	787	682	86.7%	603	85	19	63	5	12	89.2%	344	43.7%	116	14.7%
2014-15	723	613	84.8%	550	73	25	56	6	13	89.9%	293	40.5%	112	15.5%

Multiple: Students claiming 2 or more races (unless one ethnicity is Hispanic) FRL: Free and Reduced Lunch, one measure of socioeconomic status
ELLs: Students eligible for ELA services, including ELLs with a PPF 3 (parent opt out of ELA services). These ELLs are not necessarily receiving ELA services.

http://media.dpsk12.org/enrollmentsnapshots/ES418.PDF (pulled 10/11/2015)

Mrs. W. and I had discussed this research project at length and she was a very willing and open to collaborating on planning, having me in her classroom videotaping and debriefing the lesson. She had expressed to me that she was not worried about the time commitment and we both feel that this was a great learning opportunity for each of us. Because of Mrs. W.'s alternative route to teacher licensure and not having taken a formal mathematics methods courses in college, she was hoping to gain a deeper content knowledge in mathematics and, although she did not fully understand what MKT was, she was excited to learn more and have the opportunity to be a better teacher through the process. This willingness to participate did make the planning and conversations more open and, I would argue, allowed us to have conversations that did not feel forced and were honest and therefore more reflective of the teacher's true feelings.

Procedure

The table below shows the timeline and lessons that were implemented during the research study.

Figure 3.5. Chart outlining dates, planning, tasks, purpose and de-brief of lessons.

Date	Planning	Task	Purpose	De-Brief
1-25 to 1-29	Prior to the implementation of the lesson, I will work with the teacher in planning the lesson.	CMP 3: Looking For Pythagoras Lesson 4.2: Irrational Numbers		After the teacher has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.
2-15	Prior to the	CMP 3:	This lesson has	After the teacher

to 2-19	implementation of the lesson, I will work with the teacher in planning the lesson.	Looking for Pythagoras Lesson 5.1: Stopping Sneaky Sally	historically been one in which students have many misconceptions from my own teaching experience and takes a lot of planning and unpacking by the teacher to set this lesson up for students to develop a deeper understanding of how to apply The Pythagorean Theorem	has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.
3-7 to 3-11	Prior to the implementation of the lesson, I will work with the teacher in planning the lesson.	CMP 3: Butterflies, Pinwheels and Wallpaper Lesson 2.3: Minimum Measurement	This lesson builds upon mathematical justification and proof with asking students to provide counterexamples to disprove conjectures	After the teacher has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.

4-11 to 4-15	Prior to the implementation of the lesson, I will work with the teacher in planning the lesson	CMP3: Frogs, Fleas and Painted Cubes Lesson 2.1: Trading Land	This lesson develops students thinking in regards to converting a quadratic equation in factored form in to standard form through the use of areas models.	After the teacher has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.
4-25 to 5/2	Prior to the implementation of the lesson, I will work with the teacher in planning the lesson.	CMP3: Frogs, Fleas and Painted Cubes Lesson 3.2: Counting Handshakes	This lesson develops students thinking in regards to modeling patterns with linear and quadratic relationships to compare models	After the teacher has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.
5-9 to	Prior to the	Investigation 2:	Students will learn	After the teacher

5-13	implementation of the lesson, I will work with the teacher in planning the lesson.	Arithmetic and Geometric Sequences Lesson 2.2: Geometric Sequences	how to make sense repeated multiplication patterns and convert them in to a more formalized function called a geometric sequence	has implemented the lesson in her classroom, I will meet with the teacher after school hours to reflect on the lesson and her implementation of the lesson while it is still fresh in her memory.

Figure 3.5 lays out the plan for when Mrs. W. and I planned the lessons, when Mrs. W. was teaching each lesson, what lesson we collaboratively planned and the lesson Mrs. W. had taught in her class, the initial thinking behind the purpose of the selection of that specific lesson to plan and observe and a high-level description of the debrief of the planning and implementation of the lesson.

The collaborative planning of each lesson took approximately two hours to complete. The planning of the lesson occurred, when possible, the day prior to when Mrs. W. would be teaching the lesson in her classroom. Some extraneous circumstances prevented planning the day prior and the planning was flexible to happen as close to when the lesson was taught as possible but the planning needed to happen prior to the implementation of the lesson. The first 20-30 minutes of the planning session was focused solely on completing the task within the lesson and discussing the mathematical goal that students should take-away from completing this lesson in Mrs. W.'s classroom. After establishing a clear learning goal for students, Mrs. W. and I started to consider the different strategies and approaches students took to arrive at a solution. This took approximately 60-90 minutes. As we considered the approaches that students may take to solving the task, we were also considering approaches that students may take due to their misconceptions about either the task or the mathematics. For example, what students may not understand about the wording of the task vs. what they understand about mathematics that prevents them from arriving at the correct solution? As discussed earlier in Figure 3.3a and 3.3b, a possible student misconception noted was in regards to writing a linear function relating attendance (F) and the probability of rain (p). The possible student error was in regards to the mathematics and not necessarily the wording of the item because the student may not fully understand the concept of slope and as a result was unable to calculate the correct slope from the given table of values.

As we worked through the misconceptions, it was not enough to just identify what the students' misconceptions will be to facilitate the learning environment in the classroom. To create a learning moment for all of the students in the classroom through questioning, Mrs. W. would have students identify the mistake and support each other to learn from the misconception. In this way, during planning, Mrs. W. considered appropriate instructional practices that created the most meaningful learning for students. For example, as she considered the questions she asks to clarify and support the learning from the misconception, she was thinking through various questions she could ask, what the possible student responses to her question would be and how those responses could impact her classroom in various ways. Through this process, she was able to sharpen her patterns of questioning to pull out the learning from students and developing her MKT.

The videotaping of the classroom observations followed the planning of the lesson as soon as possible, preferably the following day. Mrs. W. taught three 90-minute classes each day. I videotaped one of the class each time following the planning of the lesson. In the journaling following the videotaping of the classrooms, I was focused on the types and patterns of questioning and how the question either opened up or closed the classroom discourse loop.

In Example B, the teacher asked specific questions to push students thinking and the questions were intentional and specific. In a lot of classrooms I observe, teachers assume that just asking the question "why?" is enough to promote good classroom discourse. In Example B, instead of asking just an open-ended "why?" question, the teacher asks a specific "why?" question, "Why did you divide 9 by 3 to get the slope?", because they wanted the student to explicitly explain how the numbers 9 and 3 are related through the operation of division. If the teacher just asked "Why?" there is potential for confusion to still linger in regards to what the teacher is focused on and sometimes there is a missed opportunity for deeper student learning because the questions are not intentional enough that are being asked of students.

Finally, the debrief of the planning and implementation of the lesson occurred, as often as possible, at the end of the school day. I allotted an hour for each of these sessions but needed to be flexible depending on how the conversation with the teacher was going and the direction the four guiding questions took us. The main objective of the debrief was to learn how the experience of planning supported her development of her MKT and how the planning shifted her instructional practice with the focus being on questions, patterns of questioning and classroom discourse. The debriefs followed a scripted set of questions as outlined in the methodology.

Data Collection

The data collected for the first research question was both quantitative and qualitative. The quantitative data was a pre- and post-assessment of Mrs. W.'s MKT through the TKAS assessment platform. This data helped to quantitatively either affirm or deny my hypotheses about the impact planning can have on Mrs. W.'s MKT. While there has been some research done on the reliability and validity of the TKAS tool (Schilling and Hill, 2007 & Gleason, 2010), with only two data points collected from a single participant, I will only be analyzing this data to identify growth or not as it relates to Mrs. W.'s MKT from prior to the research study through the conclusion of the research study.

Biases and Understandings of Research

Mrs. W. and I have worked together in Denver Public Schools and have had a positive working experience in learning from each other. In addition, my role in Denver Public Schools sometimes feels evaluative for teachers that I am working with in the district. Mrs. W. and I discussed her role in relation to the design of the research study and that there is no evaluation tied to any of the data collected or the analysis of the data. This is a collaboration but I worked to limit bias in the collection and analysis of the data as a result of the research study.

Chapter 4: Research Findings

The data collected for my research study is designed to help answer my two research questions:

1. How does a teacher's MKT develop through the planning of and implementation of key lessons?
2. How does a teacher's beliefs about her own pedagogy and instructional practice shift as she deepens her content knowledge and MKT expand through lesson planning?

This chapter will discuss the planning, implementation and reflection of the six lessons that were collaboratively planned, taught by Mrs. W. and her reflections on the planning, instruction and implementation of each lesson.

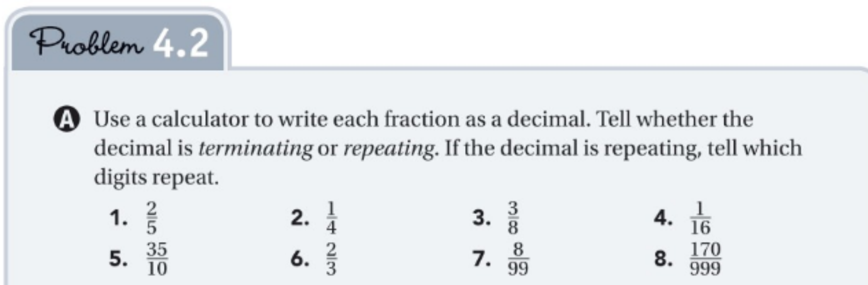
Lesson 1: Problem 4.2

The first lesson Mrs. W. and I planned was taken from Looking for Pythagoras, a unit covering the Pythagorean Theorem and introducing students to irrational numbers. The problem chosen from this unit was Problem 4.2: Representing Fractions to Decimals. Mrs. W and I started the session by doing the mathematics in the lesson to develop the mathematical learning goal we wanted students to walk-away with at the end of the lesson. In addition to understanding the mathematics and the learning goal of the lesson, we wanted to consider how students may approach solving this lesson and anticipate their thinking. By addressing these two main ideas, we were able to anticipate how students may approach the given tasks in the lesson.

Planning Problem 4.2.

We started with doing the mathematics in Part A, which was to convert various fractions to decimal numbers and determine if the decimal was terminating or repeating.

Figure 4.1. Problem 4.2 student-facing task statement.


Problem 4.2

A Use a calculator to write each fraction as a decimal. Tell whether the decimal is *terminating* or *repeating*. If the decimal is repeating, tell which digits repeat.

1. $\frac{2}{5}$ **2.** $\frac{1}{4}$ **3.** $\frac{3}{8}$ **4.** $\frac{1}{16}$

5. $\frac{35}{10}$ **6.** $\frac{2}{3}$ **7.** $\frac{8}{99}$ **8.** $\frac{170}{999}$

For each of these eight fractions, Mrs. W. and I used a calculator to determine if the fractions were terminating or repeating. After going through a few of these together, I paused and asked if she understood why the lesson was having students convert fractions to decimals and having them determine if they are terminating or repeating. Mrs. W. admitted that she did not understand why they were doing this lesson and how it was connected to what they were learning in prior lessons and would learn in future lessons. We then deviated for a while to discuss what a rational number is and what was an irrational number is. This lesson was designed to help students see that rational numbers are any number that can be written as an integer over an integer. Additionally, the lesson was designed to have students understand when a fraction would have a terminating or repeating decimal.

The importance of this lesson for future work in the unit on Pythagorean Theorem was discussed so that Mrs. W. had a better understanding of the connections between this lesson and its relevance to working with radicals and irrational numbers. With this idea, there are several misconceptions that students have when looking at decimal numbers to determine if they are rational or irrational, one of those misconceptions being that repeating decimal numbers are irrational. The lesson intentionally has students making predictions about fractions and their decimal representations to understand that some rational numbers do not have decimal values that terminate, thus having non-terminating decimal equivalents.

When planning this lesson, Mrs. W. and I decided that the goal of the lesson was for students to be able to identify if a fractional number had an equivalent decimal number that terminated or not. This would help students with the misconception that may arise later when deciding if a fraction or decimal number was rational or irrational. We also discussed the importance of students sharing their thinking and creating conjectures. This was an important piece of the lesson because the fractions either had decimal numbers that terminated or not and a student could guess and be correct but not truly understand the concept. In planning this lesson, there was an “ah-ha” moment that this would actually take students time to do and struggle with, which was something Mrs. W. allowed in her class but did not consider how much time would be needed for students to truly engage in the mathematics she was asking them to do. For struggle to be productive, teachers need to provide the time and structure for students to engage in mathematics in a meaningful manner (Leinwand et al, 2014). This was especially important for when students started working on Part B of Problem 4.2. In this part of the task, students were moving away from using a calculator and deciding if the decimal was terminating or not but were asked to consider other students' thinking and to agree or disagree with an explanation.

Figure 4.2. Problem 4.2 student-facing task statement continued.

B

1. Jose says he knows that the decimal representation of a fraction, such as $\frac{3}{8}$, will be a terminating decimal if he can scale up the denominator to make a power of 10. What scale factor would Jose need to use to rewrite $\frac{3}{8}$ as $\frac{x}{1{,}000}$? What is the decimal representation?
2. Mei says she can scale up $\frac{2}{3}$ to $\frac{66\frac{2}{3}}{100}$, but the decimal representation of $\frac{2}{3}$ is a repeating decimal. Do Jose and Mei disagree? Explain.
3. Make a conjecture about how to predict when a fraction will have a terminating decimal representation. Test your conjecture on the following fractions:

$\frac{4}{7}$ $\frac{5}{6}$ $\frac{25}{12}$ $\frac{19}{20}$

In planning this part of the lesson, Mrs. W. and I discussed the idea of the denominator needing to divide evenly in to any power of ten for the fraction to have a terminating decimal number. When looking at Jose's and Mei's thinking, students would have to consider this idea. As we discussed the implementation of this piece of the lesson, Mrs. W. and I really wanted to be able to ask questions so that students made explicit connections between Jose's idea and Mei's idea. In these two situations, both students thinking is correct but only Jose has a conjecture that will help him make a prediction while Mei's thinking doesn't make the connection to the denominator dividing evenly in to a power of ten. We ended the planning session by reviewing the goal of the lesson and where we wanted students to go with their learning. We didn't take the time to intentionally script out questions but we had some good ideas about misconceptions and anticipated student's thinking. However, at the end of the planning session, I could tell that this had impacted how Mrs. W. thought about planning and learning mathematics. An example of this is when we were wrapping up the planning of the lesson, Mrs. W. commented "Because if I am struggling with how to, well first, when we sat started with this, I was struggling with, well, how *do* we do this. And now, yeah, the question is, how do you *lead* a kid to discover this on their own. They don't always like to discover on their own, they really like to be spoon-fed, I think they have been spoon-fed a lot. This has helped me to learn the mathematics and then to realize for the students, it is all in the questions." This comment from Mrs. W. shows that she is already starting to develop a sense of the importance of asking questions to students to guide their thinking to "discover" the mathematics.

Implementation of Problem 4.2.

During the implementation of this lesson, Mrs. W. started her class with a warm-up lesson with the students, she had them jump in to exploring whether or not a fraction had a repeating or terminating decimal number. In Part A of Problem 4.2, students were allowed to use calculators to convert a fraction to a decimal and decide if the decimal equivalent was terminating or repeating. As students worked through this, Mrs. W. did not really promote discussion through questioning as her questions were grounded in a right or wrong response and did not push discourse with her questioning. When students started to engage in Part B of the task, one student started to make a connection in Part B in regards to the denominator needing to divide in to one hundred, which is a power of ten. Mrs. W. pushed back on the student by asking her to justify if that was true for a couple of other fractions. But this was a singular interaction with one student that did not get put back on the class. Mrs. W. did not re-visit this idea, however, and the lesson really became more about if a student could use a calculator to determine if a fraction had an equivalent decimal number that was either terminating or repeating by using a calculator. The real mathematics of the lesson was lost and students did not leave learning the mathematics that we had set as the goal of the task.

Debrief of Problem 4.2.

In the de-brief of this lesson, Mrs. W. admitted that the mathematics in this lesson scared her. She felt that she did not understand the mathematics well enough and, even though she said she learned a lot in the planning of the lesson, she still felt uncomfortable teaching this lesson. Here is a short vignette from the initial de-brief of our first planned lesson together:

R: "The de-brief is around five guiding questions and as you respond other questions may come up. You know yesterday we spent some time just doing the math, how did that help you prepare for this lesson?"

Mrs. W.: *"I usually do the math before the kids and I build out my flipcharts before the lesson and that helps me know what kids are supposed to know before I do the lesson. But the math of this particular lesson and doing the math and getting at what kids were supposed to know was scaring me as we did the math. It didn't help instill confidence but it helped me prepare for the struggle that kids would have."*

R: “I am curious to know if you, when you were walking around the tables and asking students questions, if any question you asked had an impact on student thinking?”

Mrs. W.: “*There was one student, Jocelyn, who was getting to the idea of the denominator being divisible or going evenly into one hundred and my question to her was if she could prove that with a couple of other fractions. Can you disprove that? So, I was looking for students to justify their thinking. So, the next time I teach this I am going to spend more time with my students in helping to make sure that they are able to create a conjecture and think about how to prove it. This took a lot longer than I expected and we were planning this for like 50-60 minutes and I thought that they would just be sitting their quietly because they wouldn't have an idea and they surprised me because they were really trying to figure this out, even though only a few students came up with a meaningful conjecture.”*

R: “So there is a lot of math that is in this lesson that students have done but this may the first time they have been asked to create a conjecture and you are trying to have students make connections and make a conjecture and that presents challenges the first time kids are asked to do this because they may not be sure that you are wanting from them in terms of a conjecture.”

Mrs. W.: *“Yes, there were a few more questions I would have liked to ask the students that I didn't think about until after the lesson was over. For example, can you continue this pattern and create some sort of rule for this.”*

R: “So it was more of like an instructional question about the patterns in relation to the problem?”

Mrs. W.: *"Yeah, because otherwise I would never have thought to ask those questions about patterns. That is a question I would ask normally but would not have asked for this lesson until our conversation in planning. I think my focus for planning alone was different than when we were planning together for this lesson, because I would have done this by myself I would have stuck solely with the questions that I saw in the book whereas with the different content or different lesson that I felt more experience or more confident in it I would have come up with questions like that but this lesson, I am not sure I would have been able to come up with any questions on my own."*

Personal reflection of Problem 4.2.

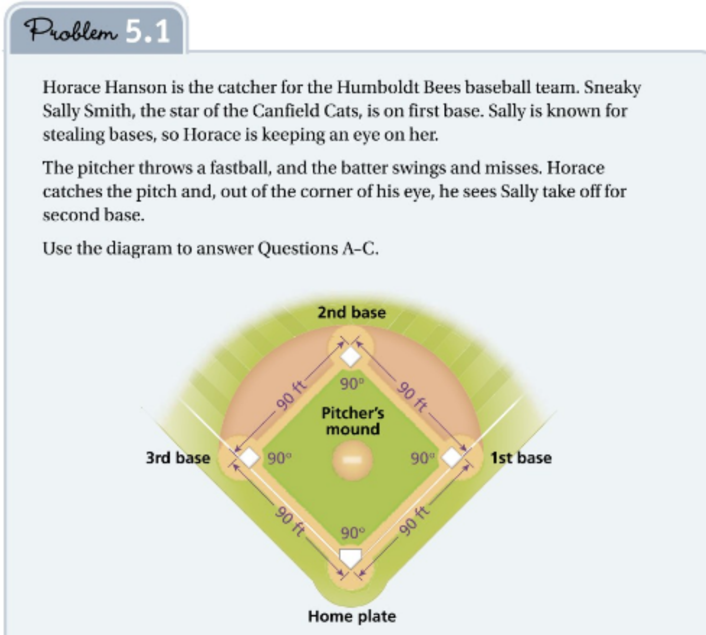
From the first lesson we planned together, I could already sense from the debrief that planning the lesson and doing the math impacted how she thought about the goal of the lesson and what kinds of questions were important for students to consider. Even though the implementation of the lesson was not as tight as Mrs. W. would have liked, she recognized how the planning helped her feel more confident in implementing the lesson and even thinking through a few intentional questions helped her to push her students' thinking in regards to the task. For example, a small group of students were struggling trying to find the pattern or relationship between the numerator and denominator to decide if a fraction had a terminal or repeating decimal. Mrs. W. asked the group of students what they notice about the denominators of fractions that had an equivalent terminating decimal. The students noticed that certain numbers, such as 1, 2, 4, 5, 8 and 10 all had equivalent decimal values with a terminating decimal. Mrs. W. then asked the students why denominators such as 3, 6, 7, and 9 did not have terminating decimal values. The students were not sure how to respond to this and were hypothesizing about prime numbers and even and odd numbers. Mrs. W. started to ask questions, for example, "when 1 is the denominator, it is always going to be terminating because we know that a number divided by itself is 1. However, what are some characteristics that the numbers 2, 4, 5, 8, and 10 have in common? I am going to give you a few minutes and the come back to you." Mrs. W. did not give away the answer, but instead she asked a specific question about characteristics of certain numbers because she knew the importance of those characteristics in relation to the learning goal of the lesson and thus, was able to ask a specific open-ended question that prompted students to think about the task in a new, meaningful way. I would argue, that in this situation, the planning helped her to understand the content and in turn, she was able to pose a purposeful question to promote student learning (Leinwand and colleagues, 2014).

As I processed this lesson more to gain a baseline of instructional practice with Mrs. W., I spent time considering the debrief of the lesson. When I asked Mrs. W. about her instructional practice and how, if any, changes were made as a result of the planning, Mrs. W. commented that she had a few more questions that came out than what she would have asked "normally". When I probed more on an example of this, she replied that she asked a group of students if they saw a pattern and what is was. Mrs. W. said that they may have been a normal question in another lesson but from the planning, she thought that was an important question that she would have never asked from students in this lesson. She admitted that this came as a result of the planning and having an opportunity to have a conversation because her focus on planning is different when planning alone. She stated, *"If I would have done it by myself, I would have stuck solely with the questions that I saw in the book. Whereas, with the different content, where, with a different lesson, where I feel more comfortable and feel more experienced, I can come up with more questions for that because I feel more comfortable. But because of my lack of comfort with this particular lesson, I would not have come up with additional questions had I not, we not had those conversations and planned together."* I did not immediately pick up on this after the debrief but as I listened to Mrs. W. over the next several lessons, this started to develop a connection for me with the teacher's comfort level with the content and her pedagogical interactions between content and students. This emerging idea is something I would call the pedagogy of knowing.

Lesson 2: Problem 5.1

The second lesson Mrs. W. and I planned together was Problem 5.1: Stopping Sneaky Sally. This lesson was focused on a baseball diamond and students needing to find the correct right triangle to apply The Pythagorean Theorem to then finding an unknown length.

Figure 4.3. Stopping Sneaky Sally Task.



Problem 5.1

Horace Hanson is the catcher for the Humboldt Bees baseball team. Sneaky Sally Smith, the star of the Canfield Cats, is on first base. Sally is known for stealing bases, so Horace is keeping an eye on her.

The pitcher throws a fastball, and the batter swings and misses. Horace catches the pitch and, out of the corner of his eye, he sees Sally take off for second base.

Use the diagram to answer Questions A–C.

Planning Problem 4.3.

The problem sets up the tasks students will be engaging in by setting up some context, as seen in Figure 4.3. As we started to engage in the mathematics together, we continually referenced the diagram to help guide our thinking and considered how students might use this diagram as well to help guide their thinking. As we started to do the mathematics, Mrs. W. was definitely more comfortable with the mathematics in this task as it was more procedural in order to get the correct solution to the task but the real mathematics that students needed to engage in was to create the appropriate right triangle in order to solve the given task.

Figure 4.4. Stopping Sneaky Sally Student-Facing Task Questions.

A 1. How far must Horace throw the baseball to get Sally out at second base? Explain.

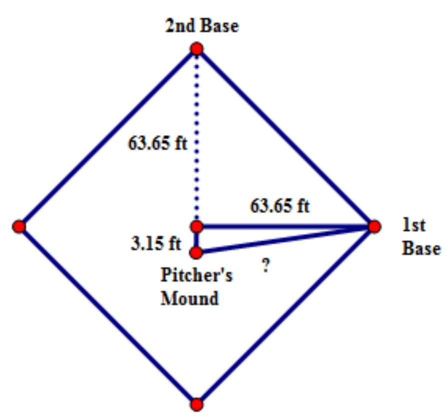
2. Jen says the distance that Horace throws the baseball is a rational number. Funda says that it is an irrational number. Explain each student's reasoning.

B The shortstop is standing on the baseline, halfway between second base and third base. How far is the shortstop from Horace?

C The pitcher's mound is 60 feet 6 inches from home plate. Use this information and your answer to Question A to find the distance from the pitcher's mound to each base.

Part A was straightforward in that the triangle needed to answer the prompt was clearer to see from the diagram. We were able to find the solution to Part A relatively easily but we did not address the question about rational or irrational in our planning. We used the distance from home plate to first base as one leg of the triangle and we used the distance from first base to second base as the length of the other leg of the triangle and the hypotenuse would be the distance from home plate to second base. By applying the Pythagorean Theorem, we got $90^2 + 90^2 = c^2$ and for c we got a value of approximately 127.28 feet. When we started to look at Part B, we noticed right away that this would take more critical thinking on the students' part to create the correct right triangle to be able to solve the task. The right triangle we ended up creating was created by using the baseline between third base and home plate and the baseline between second base and third base as the two baselines were perpendicular to each other creating the ninety degree angle needed to apply The Pythagorean Theorem. So, if the shortstop was halfway between second base and third base, we created a triangle that was 90 feet long on one leg, 45 feet long on the other leg and we could find the hypotenuse, the distance from Horace to the shortstop, with a simple calculation. Applying the Pythagorean Theorem again, we got the equation $45^2 + 90^2 = c^2$. We then got a value for c of approximately 100.62 feet. The last piece of the Problem, Part C created the greatest challenge for us to create the appropriate right triangle. We decided on an approach that allowed us to take our solution from Part A and the information from the task that the pitcher's mound was 60 feet 6 inches away from home plate, then we could take our solution to part A, which was 127.3 feet and divide it by two to get 63.65 feet. So, 63.65 feet was the midpoint from home plate to second base and by subtracting the distance from home plate to the pitcher's mound, 60.5 feet, we could find the distance from the pitcher's mound to the center of the diamond. We were then able to construct our right triangle and solve Part C. We subtracted 60.5 from 63.65 and got 3.15, which is the length of one leg of our triangle, and then we used 63.65 as the length of the other leg of our triangle since this was the distance from the center of the infield to any base on the baseball diamond (See Figure 4.5).

Figure 4.5. Diagram showing the triangle used to solve Part C.



After we completed the mathematics for Part C, we stepped back to consider this part of the task again.

Mrs. W: "*This is 63.65* (the distance from 2nd base to the center of the diamond) *which means this is 63.65* (the distance from the center of the diamond to 1st base) *that they are just going from 1st base to the center which is 63.65 feet. But they are going to have to use The Pythagorean Theorem to find this distance* (the distance from the Pitcher's Mound to 1st base) *and they* (the students) *may think they are already done because the misconception is that they are just going from 1st base to the center which is 63.65 feet away.*"

R: "So where are you anticipating students struggling with this Part?"

Mrs. W.: *"I would anticipate that students will not understand how to find the distance from the center (of the baseball diamond) to the Pitcher's Mound because they have to go back and use the information from Part A and cut that in half and then subtract 60.5 from that to find the distance from the Pitcher's Mound to the center."*

R: "What questions would you want to ask students as you anticipate seeing this mistake?"

Mrs. W.: *"I am not sure, I think I could stand right here and say I am ten feet from the door and say that the calculator is 4 feet from the door and ask the students how far I am from the door to help them think about the situation."*

R: "What if you asked students to have them tell you what they know about the distance from home plate to the Pitcher's Mound? And then, you could ask them what they remember from Part A (which was the distance from home plate to 2nd base). What may be helpful is to ask them (the students) questions here to guide get them in to a place where they understand their mistake and are able to correct based on some questions you pose"

Mrs. W. realized that there was potential for rich discussion but was not certain she wanted to open up the classroom to discussion as she did not like the way in which some of the questions were worded, in particular, the questions in Part A2. This was the part we avoided earlier in planning because Mrs. W. did not like the question and was not sure about having students doing this task. We discussed this more in depth as to considering how students may round their solutions to each part of the task but having a conversation about the "exactness" of their solutions as opposed to discussing right away if the answers were rational or irrational. Some of the questions were: "Why did you round that number?" or "What did you notice about the solution that prompted you to round it?". We were trying to ask questions that promoted the thinking that Part A2 was getting at without really doing Part A2. In this way, we still wanted to engage students with considering rational vs. irrational and why that may be important to consider with problems such as this.

We ended the planning session by reviewing the goal of the lesson and considered briefly how students may approach this lesson. Mrs. W. did not think students would have any misconceptions other than on Part C where they might confuse the center of the baseball diamond versus the Pitcher's Mound. Mrs. W. thought that this lesson would be straightforward for them in getting the solutions to each part of the problem.

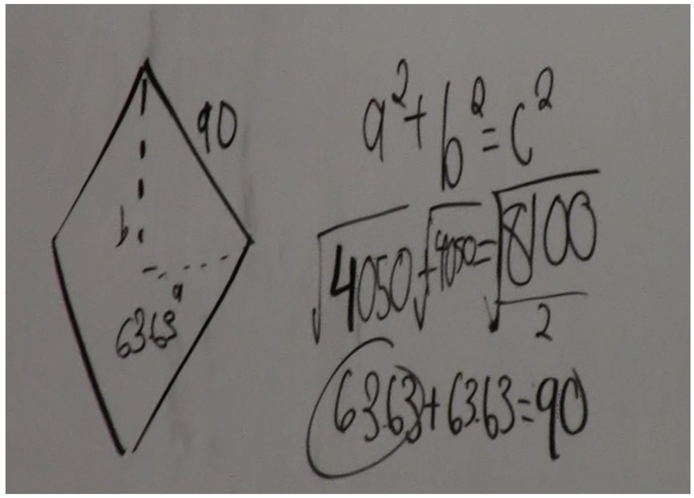
Implementation of Problem 5.1.

During the implementation of this lesson, Mrs. W. started the lesson by providing context for the lesson to make sure that every student understood the context of the tasks they would be working on. She had the students read the prompts out loud and drew a diagram on the board to model the diagram from Figure 4.3. After about ten minutes of work time, students were then asked to come up to the board to share their thinking and work for Part A, finding the distance from home plate to second base. As she brought the students back together, she looked at each any questions and Mrs. W. proceeded to move on without discussing the solutions or having students sharing their solutions. However, Mrs. W. did ask the class if they thought 127.3 was an exact solution or not and the students did recognize that the solution was not exact as they had to round up the answer on their calculator. Mrs. W. then had students read Part A2, where the students analyzed two different student's thinking about whether the solution to Part A1 was rational or irrational. Students then discussed this idea as a whole class and Mrs. W. addressed the rounding error on the calculator. This seemed to help students understand that even though the calculator may have terminated the decimal value, it does not mean that it is a rational number.

Mrs. W. moved on to Part B. in the lesson, which was finding the distance from home plate to the shortstop who was halfway between second and third base. She gave the students several minutest to engage in some productive struggle and then had students go up to the whiteboard and write down their solutions with some work to support their solutions. Again, Mrs. W. just looked at the solutions from each student and focused her feedback on the units of measure instead of the mathematics students were engaging in for this part of the task. All students that shared their work had the same solution but not one student explained or justified their thinking.

For the last part of the task, Part C, Mrs. W. set up the prompt to find the distance from the pitcher's mound to first base. Students immediately went to work on this last prompt and Mrs. W. moved around the classroom helping students with their hands raised to understand the information they are given to solve Part C. After several minutes, Mrs. W. again had selected students post their work on the whiteboard and this time she solicited student voice to explain and justify their reasoning. However, she only solicited responses from students whose work looked the same as how we did the math during planning or when the solution was correct. There was a particular student's solution that had some interesting mathematics that we did not anticipate being a misconception. This student decided to find the distance by using the length from second base to first base. Even though the student did not fully understand that the pitcher's mound was offset from the center of the diamond, the student's misconception was one that could have been a teachable moment for students.

Figure 4.6. Student's incorrect solution to finding distance from Pitcher's mound to 1st base.



The student tried to work backwards from the fact that the distance between 1st base and 2nd base is 90 feet. The student squared the 90 feet and got 8100, which they interpreted as the c^2 value in The Pythagorean Theorem. The student then divided 8100 by two and set the square root of 4050 plus the square root of 4050 equal to the square root of 8100. This student's thinking was a wonderful opportunity for Mrs. W. to prompt discourse and ask questions that would allow students to see this error in the mathematics However, student's voices were not heard and because of the lack of classroom discourse, this opportunity for learning never happened.

After reviewing student work on the board and going over only the correct solutions. The lesson ended. Mrs. W. did not address the learning goal for this lesson as the focus became more about finding the solution from what was originally planned in regards to using right triangles to apply The Pythagorean Theorem to solving problems.

Debrief of Problem 5.1.

During the de-brief, we discussed her implementation of the lesson and, in particular, her implementation of Part C. Mrs. W. admitted that planning the lesson and doing the mathematics helped her understand the importance of listening to students. I asked her about the student's work from Figure 4.6 and asked if she noticed one student's thinking and her thoughts on that piece of student work.

R: "There was a student that did Part C and the student put square root of 4050 plus the square root of 4050 equals the square root of 8100. The student then wrote the square root of 63.63 plus 63.63 equals 90."

Mrs. W.: *"Oh yeah! That student did some interesting math."*

R: "Do you remember how you handled that situation or did you ask that student some questions about their thinking?"

Mrs. W.: *"As you are bringing this up I don't even remember seeing this but I don't know how you anticipate this?"*

R: "Well, I don't think you can anticipate everything. But it seemed like there were a couple of opportunities where students were thinking something differently and I am not sure the learning was where you wanted it for those kids."

Mrs. W.: *"I like when we share our wrong answers because we learn from each other, but when it is something like this I am afraid to let them share their wrong answers. I really am because I am so afraid that if you explain your wrong answer, I am going to be 'oh yeah, that makes total sense' and I am going to be changing my right answer to yours because…"*

Personal reflection of Problem 5.1.

In that last statement of the de-brief, there was something telling that aligned to research in regards to MKT in research that suggests, not only do teachers not want to open up their classrooms to discourse because of a lack of confidence, but they may be scared to have students present wrong solutions to the class because the teacher may not understand why the solution is wrong and therefore may second guess their own understanding of the problem (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, and Ball, 2008). Not only did the lack of content knowledge make Mrs. W. feel hesitant to open her classroom to questions and student discourse, she also felt unprepared to deal with wrong student answers because she was worried she would not be able to understand why the student's solution was wrong or where the thinking by the student caused the error (Keiser, 2012).

Another point I took away from the debrief with Mrs. W. was her thoughtfulness on being a listener. When asked how planning helped her to feel prepared to implement the lesson, she commented that knowing she needed to listen to what kids were talking about during that lesson and to understand how that would connect with what they were going to do in an upcoming lesson. She noted that some of the right triangles we had discussed in planning were ones that had not considered and as a direct result of her learning, she wanted to hear what kids were thinking about as well. There was an immediate connection to knowing the mathematics and having a deeper understanding to wanting to investigate how her students were thinking about the mathematics. This is important because she is already making mental shifts in her instructional practice as a result of knowing the mathematics beyond just how to get a solution in this lesson.

Lesson 3: Problem 2.3

Lesson 3 was selected because of its focus on having students create conjectures and justify their thinking. This lesson builds on a prior lesson where students determine if two triangles are congruent by measuring all of the corresponding sides and corresponding angle measures to then state whether or not two triangles are congruent or not. Lesson 3 builds on this idea and asks the question: What is the least amount of information you would need to know about two triangles to determine if they are congruent or not?

Planning Problem 2.3.

This lesson is from the CMP3 unit called Butterflies, Pinwheels and Wallpaper. This entire unit is focused primarily on geometric concepts and having students really work to justify their thinking and prove/disprove arguments and conjectures.

Figure 4.7. Setting up Problem 2.3

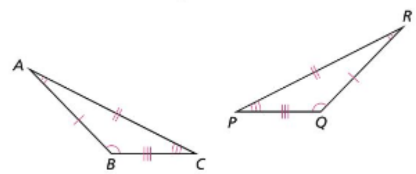
2.3 Minimum Measurement
Congruent Triangles II

In Problem 2.2, you might have noticed that it is not necessary to move one triangle onto the other to determine whether two triangles are congruent. If you know that the corresponding sides and angles are equal, you can conclude that the triangles are congruent.

The problem reviews prior learning and re-states that if you know all three corresponding sides are congruent and all three corresponding angle measures are equal then the triangles are congruent. From there it sets up the lesson by providing an example of this idea about needing all three sides and angle and sets up the question of being able to determine congruence between two triangles by knowing only one, two or three pieces of information and what is the least amount if information you would need to know to be able to determine of two triangles are congruent.

Figure 4.8. Context for Problem 2.3.

For example, in triangles *ABC* and *RQP* below, $\angle A \cong \angle R$, $\angle B \cong \angle Q$, $\angle C \cong \angle P$, $\overline{AB} \cong \overline{RQ}$, $\overline{BC} \cong \overline{QP}$, and $\overline{CA} \cong \overline{PR}$. Since all ***corresponding parts*** are congruent, the two triangles are congruent. Triangles *ABC* and *RQP* show a common way of marking congruent sides and angles. The sides with the same number of tic marks are congruent. The angles with the name number of arcs are congruent.

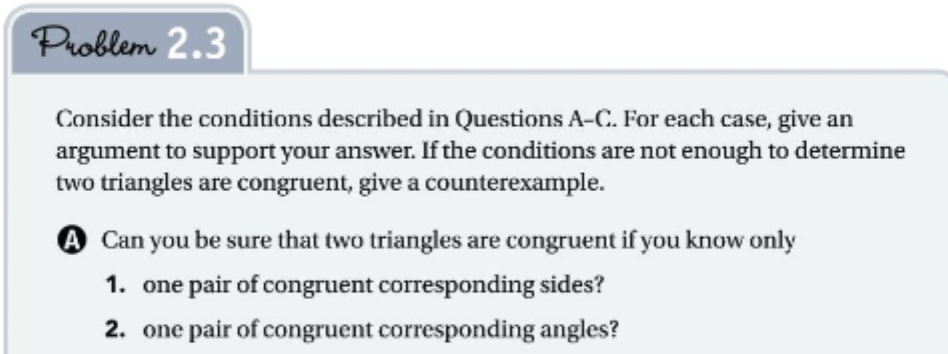


In this Problem, you will explore whether you need to know the measures of all the sides and angles of two triangles in order to determine congruence.

> ? Can you conclude that two triangles are congruent if you know the measures of only one, two, or three pairs of corresponding parts? Explain.

When Mrs. W. and I started to plan, we worked through each of the parts of the problem to know the mathematics for ourselves to then be able to implement the lesson effectively. The first part of the lesson focused only on knowing one piece of information.

Figure 4.9. Task A from Problem 2.3.



Problem 2.3

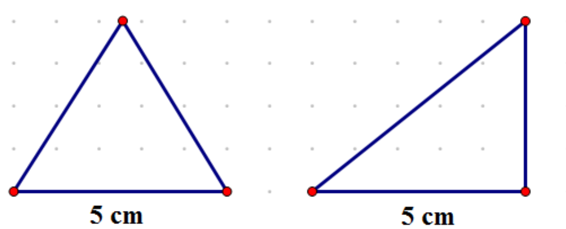
Consider the conditions described in Questions A–C. For each case, give an argument to support your answer. If the conditions are not enough to determine two triangles are congruent, give a counterexample.

A Can you be sure that two triangles are congruent if you know only

1. one pair of congruent corresponding sides?
2. one pair of congruent corresponding angles?

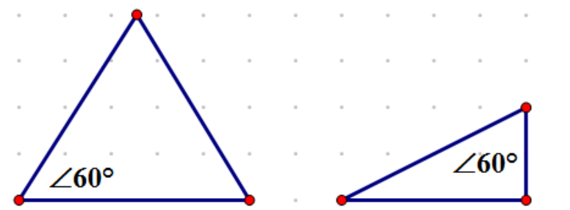
To do this, we focused on the idea of drawing counterexamples. The first question asked if it is possible to know if two triangles are congruent if you knew only one pair of corresponding sides were congruent. We were able to quickly draw a counterexample to "disprove" this conjecture. We drew an equilateral triangle with a base of 5cm and a right triangle with a base of 5cm and showed that the angle measures were not equal so the two triangles were not congruent.

Figure 4.10. Counterexample for Part A1.



This same counterexample was also used to "disprove" the second part as well, which asked if just knowing two corresponding angles of equal measure were enough to prove two triangles were congruent. We argued that the right triangle could be a 30-60-90 triangle and the equilateral had angle measures of all 60 degrees.

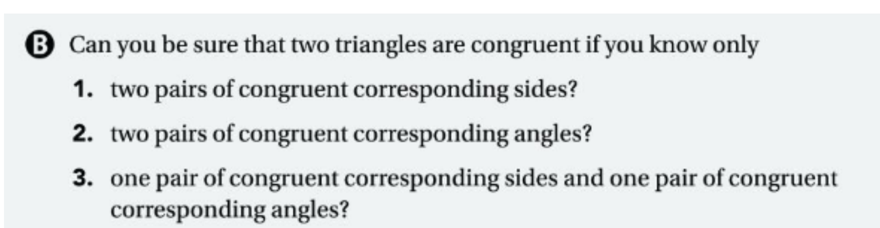
Figure 4.11. Counterexample for Part A2.



Therefore, we knew that only knowing one corresponding angle was not sufficient to prove two triangles were congruent. After we finished these two questions, Mrs. W. paused and admitted that she did not do well in Geometry in high school and had to re-take it in college to get her teaching license. I immediately knew this was a lesson she was not confident in her own knowledge of content. As we planned, she was starting to make connections to prior and future learning. This was the first time in planning that she started to make those connections.

For example, when we were investigating whether or not we could disprove two pairs of corresponding angles (Part B) as being enough to conclude that two triangles were congruent, Mrs. W. wondered if students would see that you could verify similarity but not congruence.

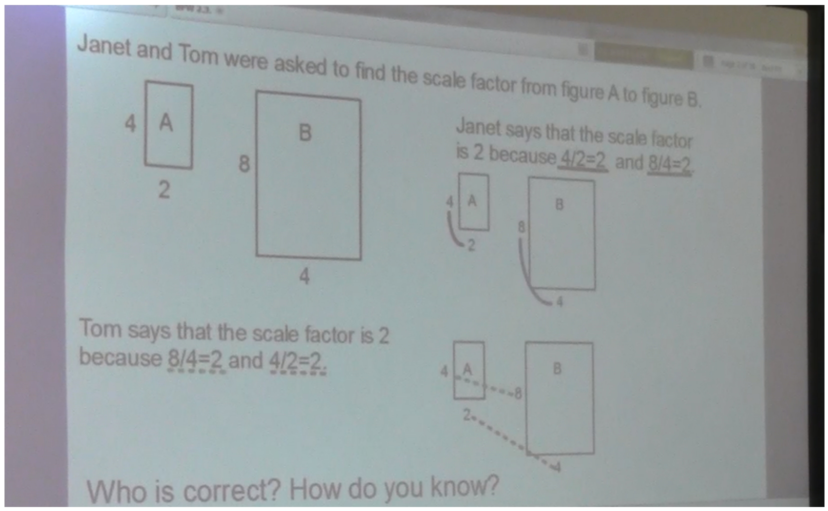
Figure 4.12. Task B from Problem 2.3.


B Can you be sure that two triangles are congruent if you know only

1. two pairs of congruent corresponding sides?
2. two pairs of congruent corresponding angles?
3. one pair of congruent corresponding sides and one pair of congruent corresponding angles?

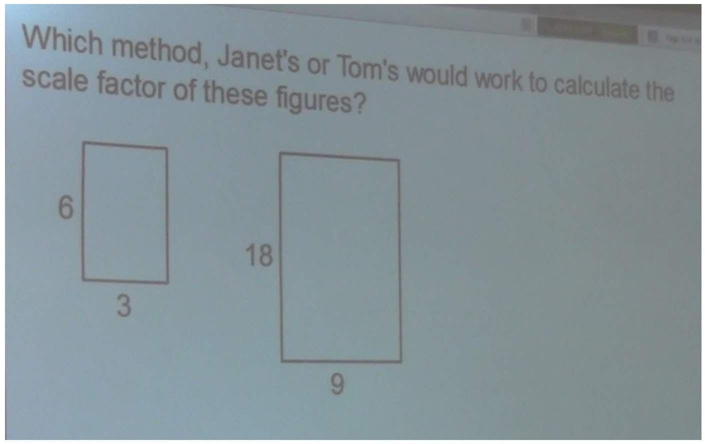
As we probed in to this idea, a misconception that 7th grade students might have is grounded in how they think about corresponding angles and sides and finding scale factor. This led us to build in an intentional activity to see where students were grounded in their understanding of corresponding sides and angles that might need to be addressed prior to the lesson. We decided on the following question to pose to students:

Figure 4.13. Warm-up activity for Problem 2.3.



By posing this question before students started in to the mathematical content of the lesson, we wanted to know how they understood the concept of similarity being between shapes rather than within a shape. When students came to a conclusion, we posed the next question to have them verify what student, Janet or Tom, had the correct understanding.

Figure 4.14. Warm-up activity for Problem 2.3 continued.



This led to a discussion about which student's thinking was correct and why. We felt this was important to have students discuss because sometimes students may not understand the idea of corresponding sides and/or angles and how those are connected to prior learning in 7th grade. This was important as we moved in to the 8th grade lesson that focused on the ideas of corresponding sides and angles and their relationships between shapes.

Implementation of Problem 2.3.

For Part A and B of Problem 2.3, Mrs. W. had students think about whether or not there was enough information to determine if two triangles would be congruent or not. Since Mrs. W. and I had done the math together, she was very confident with student responses as she received them from students in groups. In one situation, a student created two congruent triangles that had an equal corresponding angle. Because she knew that this would not guarantee that congruence, she was able to compare other triangles students had created that were counter examples and help the student come to their own conclusion that it was not enough information to be certain that the two triangles would be congruent. When students were finished with Part B, Mrs. W. had some students working on the whiteboard in the back of the classroom. She went back to the back board and had students share their counterexamples that 2 pairs of corresponding sides is not enough to determine of two triangles are congruent. This was one of the first times she had the student explain their thinking instead of her explaining the student's thinking to the class.

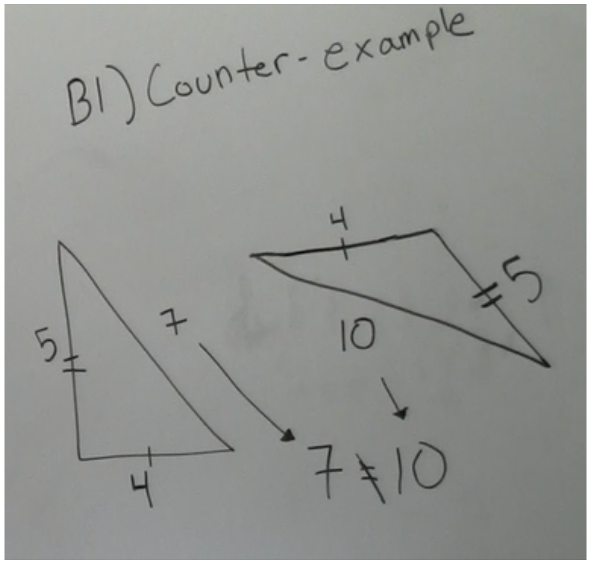
Mrs. W.: *"So (student) was answering two pairs of congruent corresponding sides. Talk to me about what you did and talk loud enough so everyone can hear you."*

S: *"Um, I um, drew two triangles and they both had 4 and 4 on both triangles and 5 and 5 but that doesn't mean they are congruent. So those two sides are congruent but the triangle on the right could have a different angles so that would make the left side a different length."*

Mrs. W.: *"So you say that the triangles on the right could be a different angle, what angle are you specifically talking about? Is there a particular one you want to point out?"*

S: *"Student goes to board and points out the angle between the lengths of 4 and 5. "This one could be a right triangle and this one could be obtuse."*

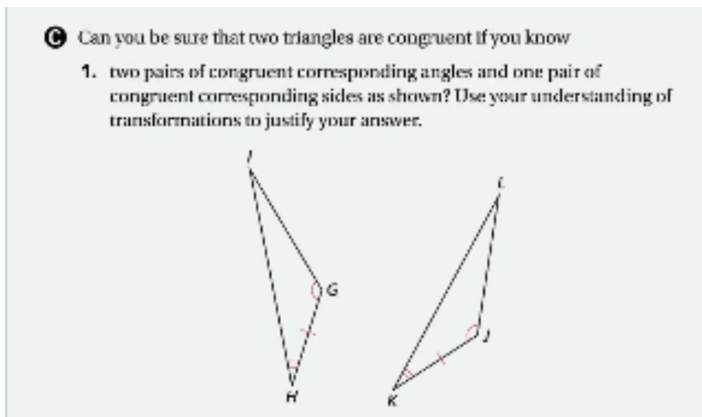
Figure 4.15. Student counterexample for Part B1 from Problem 2.3.



Mrs. W.: *"Okay, so you were able to give a counterexample so that means that it is not necessarily gonna be true if it has two congruent sides. Some of you were able to draw two triangles with two pairs of corresponding sides and you made them congruent triangles but this just proved it that it's not always the case."*

Before students moved in to Part C of Problem 2.3, Mrs. W. reviewed what students learned from Part A and B. The students learned that having only one or two pieces of information was not enough to determine congruence. Students were presented with the question of "*What would be the least amount of information you would need to be certain that two triangles would be congruent?".* She asked the students to share their ideas and students elicited several responses. One student said that you would have to know all of the lengths of every side and the measure of all corresponding angles to be certain that the two triangles would be congruent. This was an intentional pause we had discussed in the planning of the lesson to let students process what they had learned thus far in the lesson and try and make some predictions before moving in to the final part of the task. With this Mrs. W. said that we are going to move on, knowing that the students thinking would get some resolution in Part C of the lesson. Because Mrs. W. knew where the lesson was headed next, she didn't seem to feel the need to respond to the students right away, instead she knew the students would get an answer by digging in to Part C.

Figure 4.16. Task C from Problem 2.3.



As students worked on Part C, they were presented with only one scenario to wrap up the lesson, which was the case of angle-side-angle. This is the conjecture that if you have two congruent corresponding angles with a congruent corresponding side between them then the two triangles must be congruent. She had the students try and create a counterexample to see if they could disprove this idea. No student was able to create a counterexample so Mrs. W. asked what the implications of knowing these specific measurements would mean about the triangles. Students recognized that the other angle not marked had to be congruent because the angle sums are always 180 degrees and if the other two angles are the same, the third angle would have to have the same measure. Then the question about that would be enough to prove to triangles are congruent came up. If you only knew that all three angles were congruent, could you determine the triangles are congruent? Students discussed this in their groups and then proceeded to create counterexamples to disprove this notion. So, the next question was, how does knowing one side length help us guarantee congruence between the two triangles? Students saw that knowing all of the angles were the same could determine the two triangles were similar but not congruent. However, if one corresponding side was congruent then because they were similar also meant the two triangles had to be congruent. Mrs. W. wrapped up the lesson with students writing down a reflection of their learning.

Debrief of Problem 2.3.

In the de-brief of the lesson, it was obvious Mrs. W. felt like the lesson went very well by the way students were engaged. She admitted that knowing the math helped her with this lesson as well as the lesson prior to this one because she knew where the math was headed in the bigger picture of student learning. She also admitted that it was sometimes hard for her to verbalize what she expected students to do in each problem but she felt very confident in clarifying and asking students questions. We also discussed her interaction with a student where she asked him “if you knew that two corresponding angles were equal then could you prove that the two triangles were congruent?”. Mrs. W. was worried that the idea of counterexample was lost and students were going right to creating an example that worked. However, Mrs. W, didn’t consider that these were important questions for the entire class to hear as they were isolated to working with a single student. She recognized that her students were better in one-on-one situations or in small groups but didn't consider presenting these questions to the whole class to have an opportunity to discuss. We also discussed her types of questions that she asked in this lesson were different than in previous lessons because she was asking more questions that were open-ended and promoted discussion as opposed to have students just provide an answer. She said that really wanted her role to be a conversation facilitator more in this lesson. This was attributed to her understanding of the lesson and knowing what she wanted students to take-away at the end of the class.

Personal reflection of Problem 2.3.

In this lesson, I saw that Mrs. W. made a shift in her patterns of questioning and how she thought about her role in the classroom and she attributed this to her own understanding of the mathematics. I felt that for the first time, lesson planning was starting to have the desired impact on classroom instruction and that the Mrs. W. was feeling more confident and prepared to deliver the content to students. The impact I was looking for was for Mrs. W. to be able to ask students questions that pushed their thinking and promoted discourse for students to hear and learn from each other. As I reflected on what was different about the planning this time compared to the other two times, I recognized that the first two planning sessions felt rushed and we ran out of time to really get in to the mathematics a bit deeper. Mrs. W. had to run to staff meetings and our planning time was cut short. This time, planning was able to go as long as needed and we did not feel pressured to hurry or cut it short. We were able to have richer conversations about the mathematics and Mrs. W. left the planning time with a true understanding of the mathematics and what she wanted students to take-away from the lesson. I thought about the crunch for time in schools, where a teacher's planning time is already reduced and teachers are constantly being asked to cover other classes or attend a meeting, the value of planning has been reduced and appropriate amounts of time are not created and kept sacred for teachers to engage in planning on a consistent and meaningful manner.

During the implementation of this lesson, Mrs. W. started to shift her patterns of questions and admitted that planning the lesson and understanding the mathematics helped her to make those shifts in her questioning. One example of this is when Mrs. W. had a student sharing their work with the class. The students were working to provide an example or a counterexample if you can be sure that two triangles are congruent if you know only two pairs of congruent corresponding angles. Mrs. W. asked a student to explain their counterexample to the class.

Student A: I made the same shape but one is smaller and one is bigger. So, the two angles don't mean they are always congruent.

Mrs. W.: Say more.

Student A: So, what I did was I made the same shape but made one smaller so they are not exactly congruent because they are not the same size but the two angles are the same.

Mrs. W.: Okay, so if I know two angles, inside our triangle, what does that tell us about our triangles? If I know two angles.

Student B: If two angles are the same then all three are the same.

Mrs. W.: Why are three the same?

Student B: Because they have to add up to 180 degrees.

Mrs. W.: Okay, so if you know two angles are the same, and they have to add up to 180 degrees, is there anything we could say for certain about the triangles?

Student C: All the angles are the same between the two triangles

Mrs. W.: But if all of the angles are the same, does that mean that the triangles are congruent?

Student C: No, but the triangles are similar.

So even though this type of pattern is more funneling, Mrs. W. has a very intentional reason for the questions she is asking because she knew the mathematics and knew what she wanted students to take-away from looking at two triangles when two corresponding angles are the same measure. She kept prodding students to get to a place where they recognized that two corresponding angles implied similarity but not always congruence. I would argue that for this example, because she knew the mathematics she was teaching, she pushed students to make connections to prior learning while asking questions to get them to a place that she was important for them to understand mathematically.

Lesson 4: Problem 2.1

Problem 2.1 moved back in to a focus on algebraic reasoning and was grounded in quadratic expressions and the concept of equivalent expressions.

Figure 4.17. Problem 2.1 context for students.

In the last Problem, you used the lengths of rectangles with a fixed perimeter to write an expression that represents their area. *Length* was the independent variable, and *area* was the dependent variable. In this Investigation, you will continue to write expressions using area as a context.

2.1 Trading Land

Representing Areas of Rectangles

Suppose you give a friend two $1 bills, and your friend gives you eight quarters.

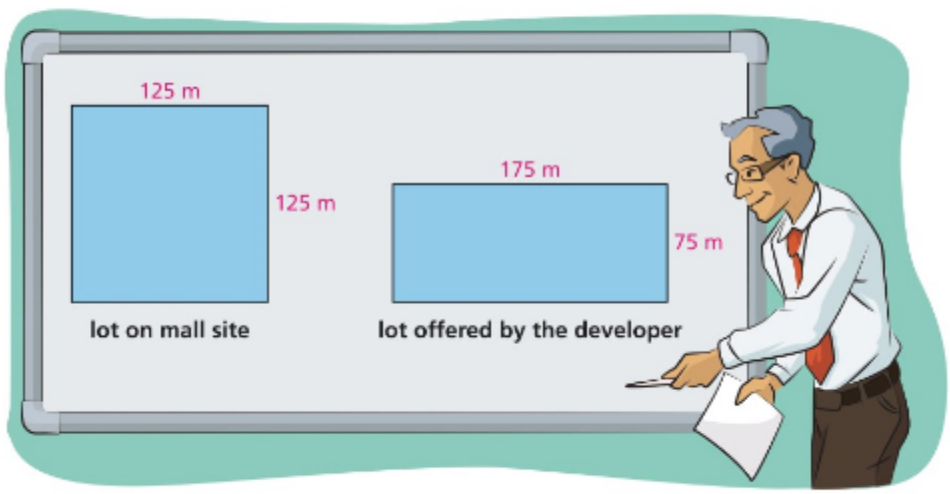
- Would you consider this a fair trade?

The context for the task was in regards to exchanging land and whether or not an offer for an exchange is fair based on the area of the land being traded.

Figure 4.18. Problem 2.1 context for students continued.

Sometimes it is not this easy to determine whether a trade is fair. Consider the following situation:

> A developer has purchased all the land on a mall site except for one square lot. The lot measures 125 meters on each side. In exchange for the lot, the developer offers its owner a lot on another site. The plan for this lot is shown below.



- Do you think this is a fair trade? Why or why not?

> How is the area of a square affected if one dimension is increased by x and the second dimension decreased by x? Explain.

In this Problem, you will look at a trade situation. See if you can find a pattern that will help you make predictions about more complex situations.

Planning Problem 2.1.

We started our planning with just thinking through this first question from the lens of a student. We calculated that the lot on the mall site had a larger area than what was being offered and we also considered the location of the land being traded. If the location of the mall lot was in a more affluent area then the lot offered by the developed it would not be fair but if the lot offered by the developer was in a more affluent area then it may be a fair trade even if the lots are different areas. So, considering these possible student responses allowed us to focus the question more about is it fair, if everything else is the same, for the person to trade their lot for the developer's lot. Students could also calculate the area of each lot and decide whether or not they thought it was fair. As students started to understand the context problem in the start of the task, we wanted to put the idea out to students that maybe the areas are the same since all you did was exchange your lot with a lot that had the same perimeter. So, how could that be an unfair trade?

Mrs. W.: *"I think they (students) will go to perimeter first because it is the easy one. If they look at perimeter they will say it fair and if they look at area, they will say unfair."*

R: "Okay, so you are going to have the why or why not, right? So, what are some questions you would want to put out there?"

Mrs. W.: *"Well, a lot of students are going to go perimeter and a few students will go area because we just discussed this at the end of this unit we were just in with scale factor and similar figures. So, I am trying to think how I would…"*

R: "Well, what would you ask a kid…?"

Mrs. W.: *"Well, I think the idea of fair and not fair, I guess it depends on the location of the lot. They could even say that the smaller lot is a better deal. I wonder if presenting that to them and having them discuss it and the move on to focus on the mathematics."*

Mrs. W. was not just demonstrating content knowledge but she was also anticipating how students may think about the question and considering how they might respond to the question.

In Part A, the task changes the dimensions but presents the same idea. Students are asked to complete a table and compare areas of the original square with a new rectangle where the length is increased by 2 meters and the width is decreased by 2 meters.

Figure 4.19. Part A from Problem 2.1.

Problem 2.1

Suppose you trade a square lot for a rectangular lot. The length of the rectangular lot is 2 meters greater than the side length of the square lot, and the width is 2 meters less.

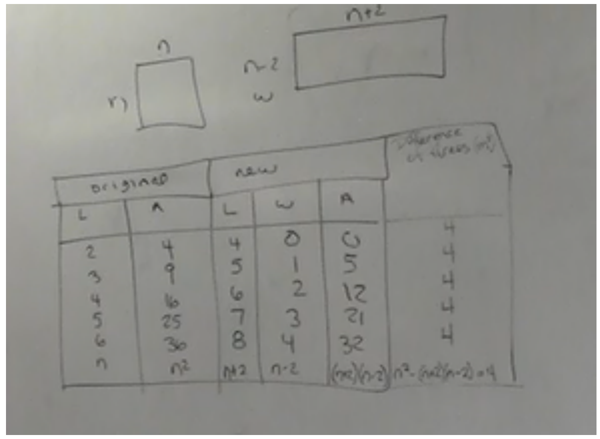
A **1.** Copy and complete the table.

Original Square		New Rectangle			Difference of Areas (m^2)
Side Length (m)	Area (m^2)	Length (m)	Width (m)	Area (m^2)	
2	4	4	0	0	4
3	9	5	1	5	4
4	■	■	■	■	■
5	■	■	■	■	■
6	■	■	■	■	■
n	■	■	■	■	■

2. Explain why the table starts with a side length of 2 meters, rather than 0 meters or 1 meter.

3. For each side length, tell how the areas of the new and original lots compare. For which side lengths, if any, is the trade fair?

Figure 4.20. Work from planning Part A of Problem 2.1.



We really started our mathematics discussion on the last row of the table, we were give *n* as the length of the original square which meant that the area of the square would be n^2. The new length would be $n+2$ and the new width would be $n-2$. Now that we had these in place, we were able to consider the new area of the rectangle and then evaluate the difference in area. We debated on one of two possible expressions that students might arrive at and their thinking that would result in either of those expressions for the area of each expression. The first expression we considered was $(n+2)(n-2)$ since students were generally familiar with area being equal to length times the width and so it was a simple case of just multiplying the new length ($n+2$) by the new width ($n-2$). The other expression we considered was in looking at the difference in areas, they can see that it is always 4. From this, they know the area of the original square, n^2, and could simply write the expression n^2-4 for the area of the new rectangle. As we considered these two cases, we were starting to understand how this task was setting students up to explore equivalent quadratic expressions.

Mrs. W.: *"Is this wrong to have four here? I think they start seeing a pattern."*

R: "Okay, I think there is a question here where $n^2 - (n\text{-}2)(n\text{+}2) = 4$. I think that might be an interesting question…"

Mrs. W.: *"Oh,* $n^2 - (n\text{-}2)(n\text{+}2) = 4$*. That is where I should have them (students) because, I should get them to get there…I can already see a couple of brains, boom!"*

R: "Well, this is the point because they will see it is all the same difference (4)."

Mrs. W.: *"So they are going to want to know how they get that (4) with this* $n^2 - (n\text{-}2)(n\text{+}2)$*. So that is going to be the struggle."*

R: "So you are going to have kids that are just going to put 4 because that is the pattern."

Mrs. W.: *"Yeah, like me."*

R: "But I think that is ok, right?"

Mrs. W.: *"But you know what I am thinking about is I may…have them show how are you getting it because then when they get down here they are forced to think about it differently. Really, algebraically."*

In this brief dialogue, I saw her being confident in the mathematics as we planned the lesson together but was also confident in pushing her students to a place of really engaging in mathematics that, perhaps, without having gone through this process pf planning, she would not have pushed with students.

Figure 4.21. Part B-D for Problem 2.1.

Problem 2.1 *continued*

B 1. Write an equation for the relationship between the side length n and the area A_1 of the original square.

2. Write an equation for the relationship between the side length n of the original square lot and the area A_2 of the new rectangular lot.

3. Carl claims there are two different expressions for the area of the new lot. Is this possible? Explain.

C 1. On the same axes, sketch graphs of the area equations for both lots. For the independent variable, show values from -10 to 10. For the dependent variable, show values from -10 to 30.

2. Tell which part of each graph makes sense for the situation.

3. Describe any similarities and differences in the two graphs.

D 1. Do either of the relationships represent quadratic functions? Explain.

2. Compare the graphs in this Problem to the graphs in Investigation 1. How are they alike? How are they different?

As we started to focus on the next part of the lesson, we noticed that Part B was really having students take the time to repeat the last row of the table from Part A and being intentional with writing out the equation and making connections to the two different looking equations as equivalent equations. Once we made this connection, we landed on the learning goal for the lesson, which was to have students be able to understand and write equivalent quadratic equations and be able to justify why two quadratic equations were equivalent.

Implementation of Problem 2.1.

When Mrs. W. started the lesson, she had the students think about the question, share with their table group and then she had the students share their thoughts with the class.

Mrs. W.: "*So I want to know if it is a fair trade and why.*"

S_1: *"I think it is a fair trade because if you add up the sides, the perimeter, it equals to the same amount. It the same size but a different shape."*

Mrs. W.: *"Okay, so (student) says 500 is the perimeter of the 1st and 500 is the perimeter of the 2nd one. Okay, (student) I haven't heard from you yet, what did you come up with?"*

S_2: *"It's not fair because the first one is a square and has more area then the 2nd one."*

Mrs. W.: *"Oh, so it's a square and it has the same perimeter. Someone is pulling in some stuff from other…stuff."*

S_3: *"They don't necessarily say if the, a certain thing, such as area or perimeter. So how do you determine if its fair or not if they haven't out that in to the equation at all?"*

Mrs. W.: *"That's very interesting, (Student)."*

S_4: *"It's not a fair trade because the area of the 1st one is larger than the 2nd one."*

Mrs. W.: *"If it was your land, would you want less space or more space?"*

S(collectively): *"More"*

From this brief dialogue, I started to hear something change. Mrs. W.'s patterns of questioning and the number of different students that she involved in the discussion. I could also observe from her body language and affect in speech that she felt more confident by how she was immediately able to respond with a question and didn't shut down any student's thinking. This was, I thought, a turning point in her instructional practice that I had not yet noticed. As students moved in to solving Part B of the task, I saw the same patterns of questioning and confidence in her body language and speech. This dialogue starts with the teacher posing the question of how you write the side length of the new rectangle of a square with a side length of *n*.

S_1: *"So, n, like for all the other ones, like 2+ 2 equals 4 and that is how we got that and 3 + 2 equals 5 and that is how we got that (the length of the new rectangle if the side length of the square was 3) for the new rectangle. So, n+2 should make the new measurement for the new rectangle."*

Mrs. W.: *"Did you guys listen to what she was saying? She went all the way back up here (pointing to the first row of the table). 2+2 equals 4 and 3+2 equals 5. If I had a square that was 125m x 125m, what would my rectangle be? If I just wanted to know the length, what would it be?"*

S_2: *"The length would be 127"*

Mrs. W.: *"So who wants to raise their hand and tell me why what they got for the width"*

S_3: *"n-2"*

Mrs. W.: *"n-2? What do you guys think?"*

S (collectively): *"Yes!"*

From this brief dialogue in the same lesson, Mrs. W. elicited responses from several students and the focus of her questioning was on thinking and sense making. She was starting to make some shifts just in how she approached her discourse with students that was previously not there.

Debrief of Problem 2.1.

During the debrief on the lesson, Mrs. W. admitted she felt well prepared to have students engage in the mathematics of the task. She also mentioned that after doing the mathematics, she was more comfortable in having students explore the various pieces of the task because she was "ready" for the students' responses and questions. I mentioned that this was probably the first lesson I observed thus far where she had several students engaging in responding to her questions and she admitted that when we planned the lesson together she really enjoyed doing the mathematics and wanted the students to have the same opportunity to enjoy and make sense of the problem as she did. She attributed this to seeing how the mathematics from the previous lesson connected to the lesson she taught and as she was planning upcoming lessons she could also see where the students were going with the concepts they were taking away from the lesson she just taught (Problem 2.1). Having the vertical curricular knowledge is part of Mathematical Knowledge for Teaching that Ball and associates (2008) discussed as being critical to a teacher demonstrating strong MKT.

Personal reflection on Problem 2.1.

After we discussed how the lesson went, I was trying to understand what was different about the planning that made Mrs. W. feel, for the first time, that she felt comfortable, confident and the lesson went well. This was the first time in our planning together where we discussed the connections between the math students were engaging in for this particular lesson to the mathematics students learned prior to the lesson and where they were going to go with their new learning.

I think there are a couple of reasons that the planning made Mrs. W. feel more confident and that confidence translated to lesson implementation engaging and pushing students to think critically. First, from the planning, I could tell Mrs. W. had a lot of "ah-ha" moments where she made sense of the mathematics in a new way that provided her a deeper understanding of, not just the content, but how students were going to think about the mathematics. For example, when we got to a discussion of why there was always a difference of 4, Mrs. W. had never taken the time to work and think through why that was the case and how that related to the equivalent expressions the students were creating. Going through the specific cases such as when the original length was 5 meters and the new length was 7 meters, was how Mrs. W. had approached this task prior when planning this lesson. She did not consider how the algebraic expressions also could provide a generalizable way to always get a difference of 4. When she realized this connection with the algebraic understanding, she was very excited and wanted students to have that same experience with the task that she had. Second, when Mrs. W. went to implement the lesson in her class, she was noticeably more confident in her delivery and execution of the task from the way in which she set the task up for students to the way she asked questions and pushed students' thinking. She invited students in to a conversation about the mathematics and was able to create a discussion because she knew, not only the learning goal, but also the mathematics.

When I watched Mrs. W. implement the lesson again after the debrief, I focused on the piece of the lesson where Mrs. W. had students focus on writing a rule for area of the original square and the new rectangle. She was very intentional about hearing from as many students as she could while making their thinking visible on the board. As she captured their thinking and heard from students, she did not, however, leave it at that. She returned to ask students how they came up with their rules and asked how the rules were connected to each other. She knew, from planning, that the focus was on having students be able to write and identify equivalent expressions. She started her questioning with the most straightforward rule, $n^2 – 4$. She then moved to the next student that had written $(n+2)(n-2)$ and then to another student that had said $n^2 – 2n + 2n – 4$. I noticed that Mrs. W. had a very intentional sequence for having students share their thinking, moving from the most concrete example to the least concrete example. When a student explained how they got their rule for the area of the new rectangle, she asked them how they knew if that rule worked. Many students connected their rule to the difference of 4 and Mrs. W. had a question she posed to the class, that if Student A’s rule was right and Student B’s rule was right and Student C’s rule was right, what does that tell us about those expressions (rules). Mrs. W. was able to get the students to a place through questioning that had them conclude that the expressions were the same, equivalent, while not being written exactly the same. Mrs. W. did some teacher moves that came out of knowing the mathematic and the goal of the lesson that we had never discussed in planning. She selected and sequenced student work and thinking in such a way that they would promote discourse and get the students to meet the learning goal of the lesson (Smith and Stein, 2011).

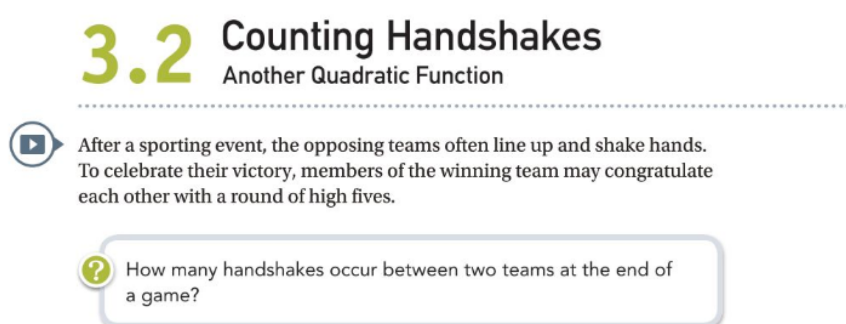
Lesson 5: Problem 3.2

As we engaged in Problem 3.2 from Frogs, Fleas and Painted Cubes, we had a conversation about how the planning was going and how Mrs. W. was feeling that really engaging in the mathematics as a learner was helping her implement the lesson in her classroom. Mrs. W. was very reflective about how this was changing the way she has started to plan on her own and wanted to know how she could do this depth of planning with the other mathematics teacher as her school. It was evident at this point, that even if she didn’t show gains in her MKT, as measured on the post-assessment measure, she was benefiting in her own professional growth of planning and truly understanding the mathematics and the mathematical goal of the lesson.

Planning Problem 3.2.

Moving in to the planning of Problem 3.2, we were again planning a lesson deeply rooted in algebraic thinking. The curriculum approached this problem in terms of cases, where each scenario was a separate case that we could analyze. In this problem, students engage in a problem considering the number of handshakes that would occur in three different cases.

Figure 4.22. Problem 3.2 context for students.


3.2 Counting Handshakes
Another Quadratic Function

After a sporting event, the opposing teams often line up and shake hands. To celebrate their victory, members of the winning team may congratulate each other with a round of high fives.

How many handshakes occur between two teams at the end of a game?

Figure 4.23. Part A from Problem 3.2.

Problem 3.2

Consider three cases of greeting team members:

A **Case 1** Two teams have the same number of players. Each player on one team shakes hands with each player on the other team.

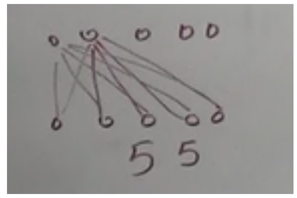
1. How many handshakes will take place between two 5-player teams? Between two 10-player teams?
2. Write an equation for the number of handshakes h between two n-player teams.

In Case 1, we discussed what this would mean and look like if students were to actually model this scenario. Since modeling with mathematics is one of The Standards for Mathematical Practice, we thought it would be important to bring this to life in the lesson as well. As we were planning, we drew diagrams to help us see how the scenario would unfold and then to consider how students may approach the task.

R: “Okay, I am going to let you, how would you approach this one?”

Mrs. W.: *“First I am going to demonstrate it and then I will ask them to draw something…on there. So, yeah, there will be crazy lines everywhere. So, I know this guy is already shaking hands with everybody and I guess I would go 1, 2, 3, 4, 5 okay, and by the time I would have gotten to here, I would say, oh I am sorry, yeah 5 and another 5 and I would have realized 5, 10, 15, 20, 25 and that’s what my brain would have done. But I can see that I am going to have students, they are gonna, this guy’s going to shake everyone’s hand this way and this guy’s going to shake everyone’s hand this way also and not realize that this line here represents a single hand shake. That we are not going to count this handshake as a separate handshake.”*

Figure 4.24. Work from planning Part A from Problem 3.2.



R: “So what if a student is struggling with this?”

Mrs. W.: *“Well I was going to say, maybe I should wait to have the kids get out of their desks and model it, but…that wouldn’t help. If I wait until they are struggling, if kids aren’t (sigh). Well, I should probably start with okay, shake my hands. Okay, how many handshakes just happened? How many people are there? Now with the table groups the sizes that they are, um, maybe I could have them demonstrate it with only a few of them at their tables.”*

In this brief dialogue, Mrs. W. was demonstrating an understanding that she may need to scaffold the task for students by using a simpler case for all students to be able to access before engaging with multiple players on multiple teams. Without really knowing it, she is engaging students in a mathematical problem solving process of simplifying the situation, not the mathematical concepts, to ensure an understanding of what is happening with this scenario. By asking her to consider “what if students struggle with this”, she was able to create a way of engaging students in the same level of mathematics through some intentional questions and modeling. Now students would have the tools to access all of the scenarios presented in Case 1.

We wrapped up the planning for Case 1 by Mrs. W. understanding the number of handshakes between two teams that had the same number of players on both teams was simply n^2, where n was the number of players on one team.

Figure 4.25. Part B from Problem 3.2.

B **Case 2** One team has one fewer player than the other. Each player on one team shakes hands with each player on the other team.

1. How many handshakes will take place between a 7-player team and a 6-player team? Between a 9-player team and an 8-player team?
2. Write an equation for the number of handshakes h between an n-player team and an $(n - 1)$-player team.

Case 2 was moving to have students consider the case where one team had one less player than the other team. In this situation, Mrs. W. thought that students would go to making a picture or a table and mentioned that she wanted students to use as many strategies as possible for this problem. This was important because she was confident enough in her ability to understand the math that she wanted students to approach solving this in a variety of ways, which meant she was not afraid of having students present work to other students and therefore comfortable enough to be able to handle those situations. Mrs. W. started to do the mathematics and was very comfortable herself in drawing a picture to model the situation and come to a solution. She then started to draw a table and as she was drawing and setting up the table, she almost immediately stopped and commented that “this is going to be confusing but that is good because I now know it will be confusing for the students.” From this, we discussed if it was ok for students to create a table with two rows or three rows to model this situation in a table. We thought about this in two ways:

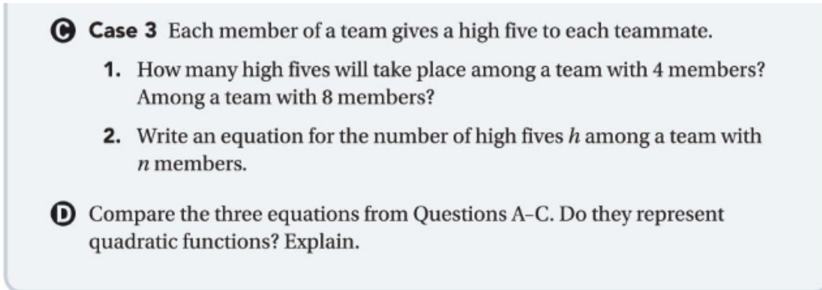
n (players on Team 1)	1	2	3	4	5	6
h (handshakes)	0	2	6	12	20	30

Or

n	1	2	3	4	5	6
n-1 (players on team 2)	0	1	2	3	4	5
H	0	2	6	12	20	30

After we thought through tables and Mrs. W. was okay with students using either table, we discussed what equations students might come up with for this case. We landed on four different equations: $n-1$, n^2-2, n^2-n, and $n(n-1)$. We anticipated students only looking at the first case where Team 1 had one player and therefore, $n-1$ would work in this situation. However, if we asked the question "what if Team 1 had two players?" then students could recognize that this equation did not work for other situations. For n^2-2, we thought students may only look at the situation where Team 1 had 2 players and this would work but, again, was not generalizable to all situations. The last two equations, n^2-n and $n(n-1)$, were both accurate but looked different and Mrs. W. wanted to make sure students saw that these two equations were equivalent, similar to Case 1.

Figure 4.26. Part C and D from Problem 3.2.



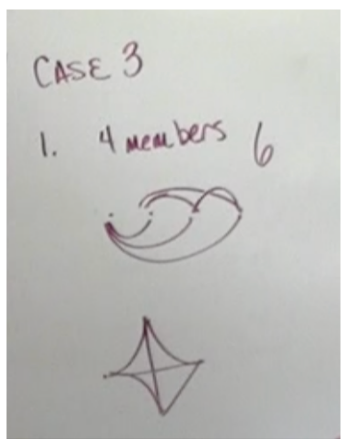
C **Case 3** Each member of a team gives a high five to each teammate.

1. How many high fives will take place among a team with 4 members? Among a team with 8 members?
2. Write an equation for the number of high fives h among a team with n members.

D Compare the three equations from Questions A–C. Do they represent quadratic functions? Explain.

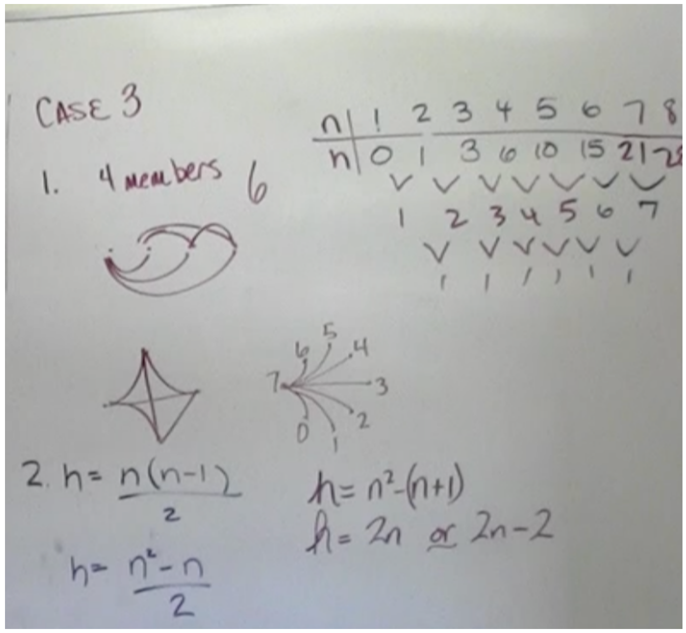
When we started on Case 3, we had to pause and recognize that there was a shift in the problem from people shaking hands with other teams to teammates within a team giving high fives. Mrs. W. noted that she wanted to have students pause here to make sure that they recognized the shift. We started to engage in the mathematics together with Mrs. W. leading the way with a diagram again.

Figure 4.27. Work from planning Part C from Problem 3.2.



After this was drawn on the board, I paused to ask what she thought was different about this model than in the other two cases. I was having her try and consider how this might look of students were to act out Case 2, for example, and then act out Case 3. We really wanted students to see how this was different than the previous two cases when it was high fives within a team as opposed to handshakes between teams. We started with the situation of looking at a team of four players. If player 1 went down they would high five three other teammates. Once he was done, player 2 would go down and only high five two teammates since they already high fived player 1. She started to recognize the difference and I asked if she thought if students would recognize the difference and if it was important if they did or not. After a pause, she thought it was important that students recognize the difference because it was not as straightforward as Case 1 or Case 2. As we shifted in to writing an equation to model the scenario, there was some pause by Mrs. W. to consider what the equation for this scenario would be. She spent several minutes on looking at the table and the diagram. As she was thinking, I asked her if she remembered what a triangular number was from Problem 3.1. She replied that she knew but didn't make the connection to the scenario and the triangular numbers from Problem 3.1. I wrote down the two "correct" equations that students may come up with and asked her to verify that these were "correct". She checked the numbers in her table against both equations and concluded that they were both accurate. I then asked her if she felt comfortable with the mathematics and she said that she was now, but was really spending time on the guess-and-check method of writing an equation and made a comment that is was "because I don't have a math degree". This recognition on her part did two things for me as a participant: 1) it made me feel like I made her feel less than by jumping in and providing the equations for her and 2) I felt that I gave the wrong impression of superiority that because I had a math degree I was somehow smarter. I recognized this right away and paused to reassure her that this was a partnership and we were learning together. She admitted that she spends time on the internet trying to get answers to understanding the mathematics sometimes and that she was a person who relied heavily on guess-and-check methods to find solutions. I assured her that this was true for lots of folks and that the guess-and-check method was perfectly fine and I use it as well as many mathematicians because it is natural to a lot of people. As we moved from the equations to non-examples, I could tell she was immediately getting back in to her comfort zone of anticipating how students may approach this and come up with incorrect solutions. She presented two different equations that she thought students may present that were incorrect, the first one was $h = n^2 - (n+1)$ and the second one was $h = 2n-2$. I asked her why she thought these two equations would come up and she responded by saying that those were the ones she was thinking about that she recognized as being incorrect after plugging the data from her table in to the equations. I asked her, if she saw these equations in her class how she would respond. She said she wanted to have students go back and verify their equations work using the same method she did, guess-and-check. So, she wanted to simply ask students to plug the values back in to the equation to check for correctness of thinking. This discussion and thinking allowed me a better understanding of Mrs. W. in that she was a teacher who admitted she was not a mathematician but was willing to work to understand the mathematics to be better prepared for her students. This is an anomaly in my experience in working with teachers as they see themselves as knowing all of the knowledge and not needing to do this work to be successful as a classroom teacher. This was not true with Mrs. W., she very much wanted to improve and was willing to do anything to get better as a teacher.

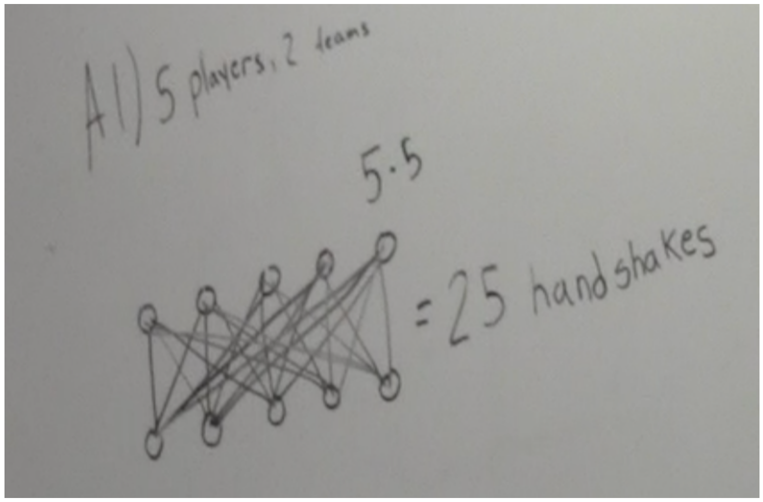
Figure 4.28. Work from planning Part C from Problem 3.2 continued.



Implementation of Problem 3.2.

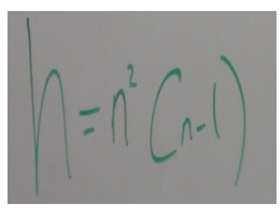
As students started to engage in Case 1, I could hear students start to try and make sense of the task by talking out loud to make sense of the scenario. Students at tables were even trying to re-create the scenario in their small groups by shaking hands. Almost all of the groups were able to get an answer of 25 handshakes through various methods and diagrams.

Figure 4.29. Student diagram for Case 1 for Problem 3.2.



After working on Case 2 for several minutes where students were finding the equation for the number of handshakes, *h*, given number of players, *n*, on a team; Mrs. W. had a few students she selected to come up to the board and write down their equations for Case 2. Mrs. W. started with the first one, $h = n(n-1)$.

Figure 4.30. Student response for Case 2 form Problem 3.2.



She asked the class if they thought this equation would work or not. One student raised their hand and responded that they did not think it would work. When Mrs. W. asked why they thought that, the student went on to explain that when there seven players on a team and six players on the other team, you would get 294 handshakes total. Instead of saying that the equation was right or wrong, Mrs. W. asked the group to weigh in on the student's response. Several students agreed and rather than validating their work, she let the students use a similar method she used in planning to guess-and-check if the equation worked or not. Mrs. W. moved on to the second and third equation, h = (n-1)n and h = (n-1)n.

Figure 4.31. Student response for Case 2 from Problem 3.2.

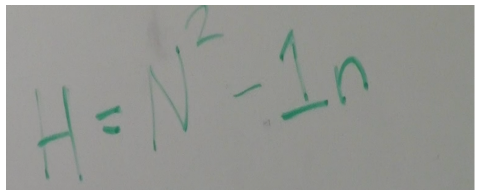
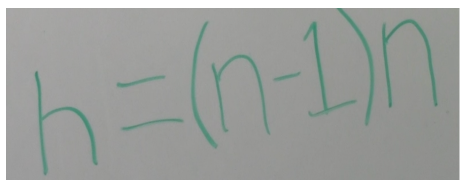


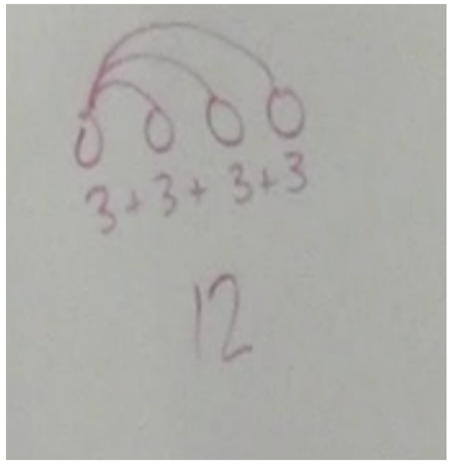
Figure 4.32. Student response for Case 2 from Problem 3.2.



She again had students consider if these two equations would work or not and students again would plug in values for the number of players on a team and get the number of handshakes that matched their diagram but they only did this for a team of size seven (n = 7). Both equations appeared to be correct and Mrs. W. moved on to Case 3.

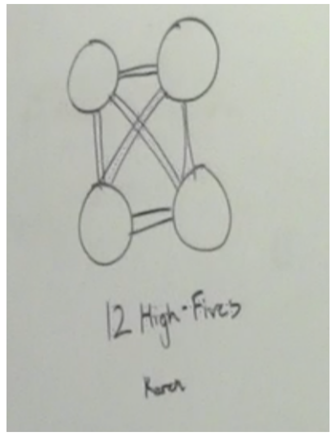
As Mrs. W. transitioned to Case 3, she started by making sure that students understood the scenario as we had discussed in planning. She gave an analogy for students to connect to and then let them work in small groups on Case 3. As students went to work, she had students put their work up on the board. Several students were putting 12 for the number of high fives between a four-player team.

Figure 4.33. Student diagram for Case 3 from Problem 3.2.



Mrs. W. asked for students to explain why they got 12 for their answer and one student stated that each of the players gave three high fives and you could just multiply 4 players times 3 high fives each and you would have 12 high fives (Figure 4.34). Mrs. W. wasn't sure about this and she was prepared from our lesson planning to tackle this incorrect solution. Mrs. W. had four students come up to the front of class to model the scenario while the other students counted high fives. She had the students line up and the first student when down the line and high fived each of the other three students and then held up three fingers to show he gave three high fives. The next student repeated this but only high fived two students and help up only two fingers. The next student high fived the remaining "team mate" and help up one finger. The last student did not high five anyone as she already had high fived each person on the team. The students added up 3 + 2 + 1. As soon as the students added up the numbers and got six, all of the students had an "ah-ha" moment. Instead of going into why six, she came back to one of the student's diagrams and asked the student to reflect on their diagram.

Figure 4.34. Student diagram for reflection for Case 3 from Problem 3.2.



The student noted that they drew two lines connecting each person to represent one high five for each line and thought that two people had two high fives between them instead of counting one high five between two people. It was clear to the students that they had double counted and now all the students appeared ready to produce a correct solution. Through planning, Mrs. W. and I considered this happening and planned how to deal with this when and if it did. Her ability to think through the mathematics, consider student approaches, and plan how to respond created a learning opportunity for all of the students around a common misconception in this lesson. The students were able to self-correct and some to a valid solution independently with the appropriate modeling and questions from Mrs. W.

Debrief of Problem 3.2.

R: "How did doing the task within the lesson as a learner prepare you to implement the lesson in your own classroom?"

Mrs. W.: *"Taking the time to do the task through the eyes of a student helped me to see and realize where my students (or myself - yikes!) would struggle.*

Anticipating their answers and questions ahead of time gave me an opportunity to process how I would respond or what I would need to do in order to help them to think deeper. I often anticipate those things but don't think deep enough about how to respond or help them more. I think another important planning of the lesson was having another person going through and planning with me. Having two brains working through the lesson, anticipating student responses, and working the math together was great. I wish that I could do that more. I think that sometimes I shorten or modify my lessons to make up for my inability to lead the students to discover what they need to know or my lack of confidence in explaining/teaching/understanding different mathematical concepts. Taking the time to plan ahead like this makes me feel a little more comfortable/confident and is helping me."

R: "What misconceptions did students have during the lesson that you anticipated? That you did not anticipate?"

Mrs. W.: *"The students struggled with visualizing the situations as I had anticipated. For those situations, I was able to help them or other students at their table was able to help them, or I was able to get the students to help me act out the situation. For the ones that I had not anticipated, like the student who had placed her work on the back board for the high fives and included an additional high five. Something didn't look right but I wasn't sure what. Her math was correct but she had included one too many high fives for the first person. I had to go get my work/answers. On my way, another student realized what the mistake was and able to offer assistance. I also didn't anticipate that the students might not be able to identify equivalent equations when they were asked to place their equations on the board. They also didn't make some of the mistakes that I had* *anticipated, but I think that the whole process of planning helped me to respond better to the other misconceptions. I'm sure that there were other misconceptions but I can't remember what they were..."*

R: "During the implementation of the lesson, was there a certain question(s) that you asked that had an impact on how students thought about the task they were working on? If you were to implement this lesson again, what, if anything, would you do differently and why?"

Mrs. W.: *"I think the most important thing that I asked was for them to justify their answer. How can you prove this? Can you show me your thinking? What strategy did you use to come up with this? Do you see a pattern? Can you think of another way to show me what you did? how you came to your answer? Can you use your equation and substitute in blank to show me that it works? Does it also work for blank?"*

R: "Did you feel that you modified or shifted your instructional practice as a result of the collaborative planning of the lesson?"

Mrs. W.: *"Yes, my teaching was modified as a result of the collaborative planning. I was more prepared to help the students with their misconceptions, had better questions that were more thoughtful and related to the work that they were doing/showing (as opposed to generic how? why?), and I also had a little better understanding of the math. As I look at the lessons coming up this week, I will be doing similar planning (even though it will be by myself). I will be the crazy lady in her classroom at lunch working on the back board and talking out loud to* *myself in an effort to mimic the collaborative planning. I think that I will really be focusing on the misconceptions and questioning."*

Personal reflection on Problem 3.2.

I felt that this lesson went better than prior lessons as Mrs. W. appeared more confident in how she interacted with students, posed questions to them, and pushed them to revise their thinking. This was especially noticeable when students were working in small groups as she moved around to check in with each group during work time. She was moving in closer to hear students and seemed better prepared to respond to questions and/or student solutions as they arose in conversation. It was obvious she felt better prepared to implement this lesson as she herself noted in the de-brief. Her own reflection on planning and implementation was impacted in her knowledge of the content, understanding possible misconceptions and anticipating student work with possible questions.

After observing the planning and hearing Mrs. W.'s reflection on the implementation, I was, however, bothered by the fact that she did not push students to check their equations against other values than the ones the book asked. For example, when students were working on Case 2, they only checked their equations with a team of seven players (n=7) and I wondered why Mrs. W. did not probe students to check with different values for the team other than seven. I reflected on the planning time together and realized I never pushed her when she was going through the guess-and-check process to determining if an equation was correct or not. As a co-planner with her, I had failed to push her thinking beyond single situations to determine if an equation worked or not and therefore she did not see it necessary to push students. This was an opportunity to push her MKT and it was the one opportunity that I missed. What I took away from this reflection was that, when planning with teachers, we have to push each other just like we would want to push students to model appropriate types and patterns of questions but to also ensure that people don't just assume that it is correct because it works for a specific situation.

Lesson 6: Problem 2.2

The final lesson of the research study was from a completely new unit in CMP3 and the focus of the unit was on functions. This particular problem involved students analyzing a geometric function. I anticipated that this would be new content for Mrs. W. as well.

Planning Problem 2.2.

Before we started in to the content, I asked Mrs. W. about her understanding of arithmetic and geometric sequences. She openly admitted that she did not know how to define either one or the differences between them. This was important because the prior lesson, Problem 2.1, was focused on analyzing arithmetic sequences. We took the time to develop an understanding geometric sequences and how they compared and contrasted to arithmetic sequences. This was important because I felt that Mrs. W. needed to be able to make connections to arithmetic sequences as appropriate in the upcoming lesson.

Figure 4.35. Context for Problem 2.2.

2.2 Geometric Sequences

The payoffs for the *Double or Nothing* quiz show are 50, 100, 200, 400, 800, . . . The sequence is an example of a **geometric sequence**. It also can be modeled with an exponential function $g(n)$ having a domain that is the whole numbers 1, 2, 3, 4, . . . As you examine situations that involve geometric sequences, ask yourself these questions:

- What are the key properties of geometric sequences?
- How are geometric sequences related to exponential functions?

In the following Problem, look for a pattern relating each term of the sequence to the next. Then find a way to calculate any term in the sequence.

Figure 4.36. Student facing task for Problem 2.2.

Problem 2.2

A **1.** Copy and complete this table of values for the function $g(n)$. Use the exponential pattern shown by the first five values of $g(n)$.

n	1	2	3	4	5	6	7	8	9	10
$g(n)$	50	100	200	400	800	■	■	■	■	■

2. Daniela answers the fifteenth question correctly. What is her payoff?

3. What is the relationship between each term in the sequence and the next? That is, what equation relates $g(n)$ and $g(n + 1)$ in every case?

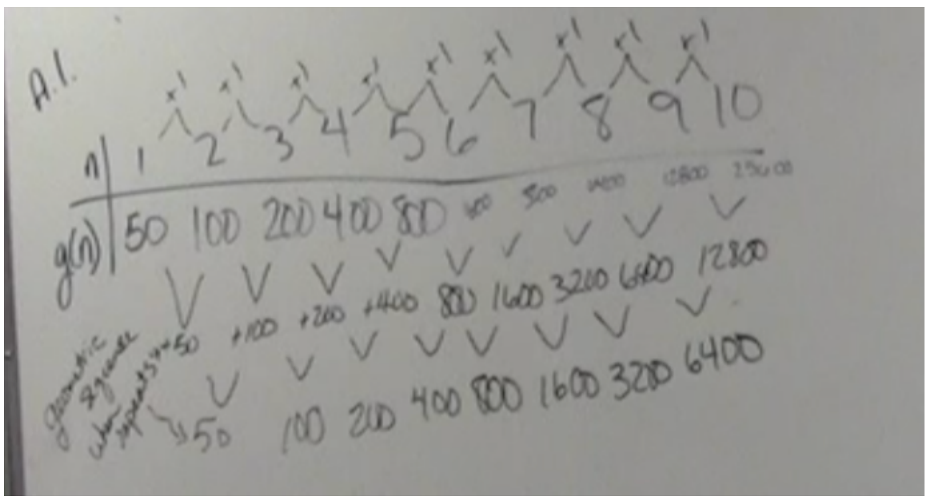
4. What expression shows how to calculate $g(n)$ for any value of n, without finding every term in the sequence?

5. How are the answers to parts (1) and (2) related to each other?

continued on the next page >

As we started to dig in to Part A., Mrs. W. wanted to jump right in to where she thought students would go and she was connecting her thinking to what students had done in a prior unit with quadratic functions. She suggested that students should try and find the differences between the 2nd row, g(n), and try to make connections between the differences and the degree of the polynomial. She started to work this out on the board to have a better understanding for herself:

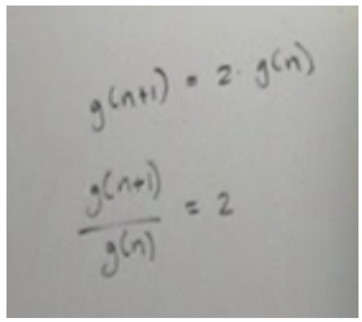
Figure 4.37. Work from planning Part A1 from Problem 2.2.



She noted that as students looked for the difference and tried to find a constant difference between the values that they would not find one. I asked her if students might make other connections to prior learning and she added that students would notice that the 2nd row doubled and that would connect back to the unit on exponential functions. Mrs. W. knew that this was an exponential relationship but when she read the prompt to part three, she paused. I asked her what g(1) was equal to and she replied 50. I then asked what g(2) was equal to and she replied 100. So, I asked her what the relationship between g(1) and g(2) was and she was able to say that two times g(1) was equal to g(2) and that

that was the relationship. As we progressed through Part A., I observed that this use of notation was new and the concept of sequences was something she understood but did not have any formal training on as we engaged in these problems together. We worked through a few examples of how students could respond to the prompt for 3:

Figure 4.38. Work from planning Part A3 from Problem 2.2.



When we were discussing these two ways to express the relationship of the functions g(n) and g(n+1), we also considered students that may reflect back on exponential functions and write $g(n) = 25(2^n)$. We decided that if that came up in her class, the question would be how does that equation relate g(n) and g(n+1). We decided collectively that $g(n) = 25(2^n)$ did not really show the relationship and was therefore not answering the question being asked.

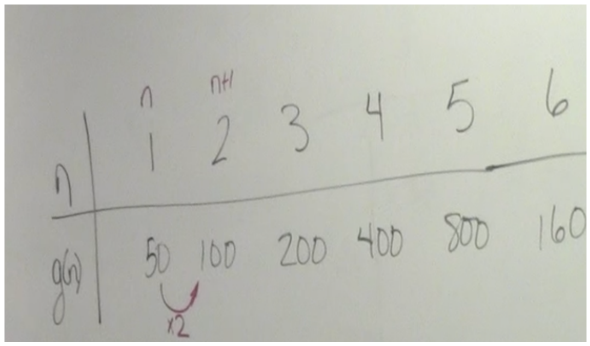
As we worked through the other parts of Problem 2.2, we decided that the most important learning for students was to have them have an understanding of what a geometric sequence was and how to identify that it was geometric and to also understand how terms in sequence relate to each other.

Implementation of Problem 2.2.

Mrs. W. launched the problem by having students read and understand the scenario being presented. Students were able to make sense of the context and started to work on the table. Mrs. W. again showed her comfort with the lesson by moving in to ask students probing questions in their small groups and actively engaged in hearing students' discussions. After several minutes of group work time, she brought the class back together to discuss Part A. Students were able to provide correct solutions when completing the table and answering that the payoff was for Daniella when she answered the fifteenth question correctly.

As students engaged with trying to make sense of how to explain the relationship between different terms, Mrs. W. went to the board where the table from Part A1 was written down and asked the students, what is happening between here, where g(2)=100, and here, where g(3) = 200. The students responded by saying that it was multiplied by 2. Mrs. W. asked "when you say it's times two, what is times two?". She was pushing the students to explain their understanding of the sequence and the relationship between the terms. Students made the connection and were able to articulate that g(n) was being multiplied by two each time. Then, Mrs. W. backed up to make sure that students understood the idea of n and (n+1) as it relates to terms in the sequence. She used the table to show how if 1 was n, then 2 was (n+1). As students processed this, she came back to her original question and that was to relate g(n) and g(n+1). This was challenging to any students because, even though they could see that to get the next term in the sequence you multiplied by 2, they could not understand how to explain how g(n) and g(n+1) were related.

Figure 4.39. Student work for Part A1 from Problem 2.2.



One student made a comment that the bottom is being multiplied by two and that is what you do each time to get the next number. Mrs. W. wrote down 2g(n) and asked if that is what she meant. The student replied that, yes, that is what she meant so Mrs. W. asked her how that relates to g(n+1). The student struggled a bit here and another student chimed in that they were the same. Mrs. W. asked what she meant by the same. The student responded that 2g(n) was equal to g(n+1) and that is how they are related. Mrs. W. paused for a second to let the class process what the student just said and asked if that was true for g(n+1) and g(n+2). As students thought for a minute, they all seemed to have a moment of understanding that it did not matter what g(n) or g(n+1) or g(n+2) was but that the next term in the sequence was twice the previous term and the notation only mattered to show the next term in the sequence, g(n) and g(n+1) for example. Students were able to take that idea from Part A and run with it for the remainder of the lesson. Once they understood the notation and the idea of a sequence, from Mrs. W.'s questions, they were able to connect the dots for themselves and access other parts of the lesson.

Debrief of Problem 2.2.

Mrs. W. admitted that doing the math had made her better prepared for the kids and to make connections to what they had already been doing. She stated that instead of just looking at an answer key, really knowing the mathematics made her feel confident. As she discussed the misconceptions she anticipated, she said that she anticipated several students struggling with notation and understanding what g(n) meant and what n and (n+1) meant when it came to a sequence and terms. Mrs. W. noted that the planning helped her make sense of the mathematics for herself and as a result she felt that she could ask questions appropriately to help the students make sense of the concepts and notation. She also admitted that she did not anticipate how challenging it would be for students to describe the relationship and having experienced that struggle herself, firsthand in planning, she understood how the students were thinking about the question and processing what that could mean.

Personal reflection on Problem 2.2.

This was the final lesson we planned and I had the privilege of observing Mrs. W. implement in her classroom. I considered her own reflection on this last lesson together and the process as a whole. I reflected on what lessons went well and what lessons Mrs. W. admittedly did not feel prepared and as a result the instruction was not as good as she would have liked. I felt like the lessons where we tried to focus only on a small piece of the lesson that got at the heart of the learning goal had the best implementation because we were able to focus more deeply on the mathematics and anticipate student thinking better with less to consider. As we had more time to plan, when time allowed, the implementation of the lesson was also more effective. These were not surprising but again I wondered how schools and districts prioritize and kept planning time for teachers sacred. Mrs. W. had been able to show tremendous growth in her instructional practice because we created the time to make planning a priority.

In this lesson, before we started in to the questions, Mrs. W. commented that she felt comfortable because she knew the mathematics. This idea was surfacing for me as a theme about instructional practice. As Mrs. W. had a better grasp of the mathematics she admittedly felt better about implementing the lesson. This notion of confidence and comfort resonates with the research that suggests how teachers deliver instructional materials in their classrooms (Ma, 1999). Ma discusses the idea that both teachers understanding of the mathematics and the teachers genuine interest in the mathematics is critical to their success in the classroom with students. This idea of thinking mathematically is one I had not thought about in the previous debriefs or lessons. As I reflected on another aspect of the time Mrs. W. and I spent planning, I could tell that she was genuinely interested in solving the problems we were working on and this is what allowed her to have a more conceptual understanding of the mathematics.

Summary

At the start of this research project, I was interested in learning about how a teacher’s instructional practice shifted, or could shift, through developing their Mathematical Knowledge for Teaching through lesson planning. Within that inquiry, was another question in regards to whether or not a teacher’s MKT could be developed by utilizing lesson planning as a vehicle for growth. These two main themes guided my research and discussion and will continue to serve as a guide for my future endeavors.

However, as I process the learning I had during the time I was able to spend working with a teacher, there were many aspect of lesson planning that I took away in addition to the research questions that guided my work. From my experience in planning with Mrs. W., there was a recurring theme of comments unrelated to MKT and instructional shifts. The recurring themes for me were really grounded in her own reflection of the planning experience as something she had never engaged in at that depth and how she wishes she had more time to plan and more time to plan with her teacher peers in the school. These themes fell into three main categories from her comments. First, lesson planning for mathematics requires the teacher to have structures for planning. Second, the time that teachers are provided to engage in lesson planning is severely lacking and teachers are not given appropriate amounts of time to engage in lesson planning at the level in which Mrs. W. and I were able to plan. Third, lesson planning can be a vehicle for growing teachers, but only if teachers are given supports needed to focus on the mathematics and understanding of how the content in the lesson they are planning for will connect to prior learning and future learning for students.

When considering structures for planning, I mean there needs to be a clear purpose and outcome for the planning of the implementation of the lesson. For Mrs. W. and I, the structure was focused on being able to understand the mathematics that the students would be engaging in, thinking about how students would respond to various tasks to prepare Mrs. W. to ask questions to support discourse and student learning, and finally knowing the learning goal of the lesson for students. Smith and Stein (2011) discuss effective practices for promoting classroom discourse. One of those practices is setting clear learning goals for instruction, and this happens in planning. In addition, Smith and Stein (2011) argue that anticipating student thinking during planning will support a teacher in promoting classroom discourse.

Merritt (2016) points out that teachers spend the majority of their time in front of students, doing the work of "teaching". However, she argues that the 45 minutes per day that most secondary teachers have to plan is not enough time to fully prepare for instruction. Morris and Hiebert (2017) suggest that lesson planning is an important task for teaching, however, most pre-teachers are not provided enough relevant content knowledge to plan effectively. Taking in to consideration that teachers are limited on time to plan and most teachers are not fully prepared to engage in the work of planning, I would argue that we need to provide time and structure to lesson planning to develop and grow our teachers.

Finally, to build on Morris and Hiebert (2017), planning is an important function in growing and developing teachers to prepare them for the work of instruction, as long as supports are in place to help teachers to plan effectively. "Most important, creating effective classrooms and learning environments for all students in every school and district will take teachers who plan and implement effective instruction…" , p.114). As teachers take the time to understand the mathematics and the learning outcomes for students, they will be poised to implement lessons that promote discourse and lead to deeper student learning (Smith and Stein, 2011).

Throughout this process, I saw Mrs. W. developing her ability to think about content knowledge, knowledge about students, knowledge about teaching and how to bring all of those together in a meaningful way for students. I believe that the development of knowledge is a life long journey and as I seek to understand how lesson planning is a vehicle for developing specialized types of knowledge, I acknowledge that there will always be more to learn. Mathematical Knowledge for Teaching can be learned and developed in many ways, and the connection I am seeking to make is how the development of that special knowledge needed for the teaching of mathematics impacts a teacher's beliefs and instructional practice. My research aligns with other researchers in that there are instructional shifts that happen for a teacher as they develop a different type of knowledge and gain confidence in being able to not only understand the content but understand how to teach the content. The main difference in my research is that I choose to focus on lesson planning as the vehicle for delivering that knowledge and to then observe for those shifts in classroom instruction. Chapter 5: Research Findings

Hiebert, Morris, Berk and Jansen (2007) make the argument that an effective framework for preparing teachers to teach is to actually have them learn from teaching. While they note, that there are a significant number of approaches to prepare teachers to teach, it is unrealistic to think that a teacher will be an expert upon graduating from a teacher education program. So, I would argue that it is the work of a school or district to prioritize extending the learning for teachers beyond their experiences in a teacher education program and to extend that learning for them while they are teaching.

Kilpatrick, Swafford, and Bradford (2001) clearly outline that effective instruction starts with planning. However, while most teachers see lesson planning as a routine activity, planning for instruction needs to focus on teachers going deep with the content and focusing on the learning goals for students. (Kilpatrick and colleagues, 2001). Through my research, I am arguing that planning is the vehicle that needs to be prioritized in schools and districts to improve teacher content and pedagogical content knowledge. The work of planning for daily instruction is not merely identifying the content that will be taught but an opportunity for the teacher to understand the learning goal of the lesson, be able to anticipate student responses, and prepare teachers to monitor student work and ask questions that promote discourse to engage all students (Kilpatrick and colleagues, 2001; Smith and Stein, 2011; Leinwand, Brahier, and Huinker, 2014). In my research, I am also arguing that if we leverage planning with our mathematics teachers, we will help them develop specialized knowledge for the teaching of mathematics and shift instructional practice that supports the shifts called for with the new state standards. In this, we will better prepare teachers to learn how to teach from teaching. Great teachers are not born, they are taught (Ball, 2012).

In this chapter, I will make a case for the importance of planning and how lesson planning that focuses on the content and the learning goal can better prepare teachers to support student learning, develop mathematical knowledge for teaching and shift instructional practice.

Research Question 1

In this section, I will focus on answering the following research question:

1. How does a teacher's MKT develop through the planning of and implementation of key lessons?

Mathematical Knowledge for Teaching.

As part of the research study, Mrs. W. participated in a pre-assessment and post-assessment of her MKT in the domains of number sense, operation and patterns, and functions and algebra. The Teacher Knowledge Assessment System (TKAS) was designed to measure a teacher's MKT to determine if a treatment influenced their MKT from the start of a treatment through to the end of the treatment. From the TKAS that Mrs. W. participated in on the pre-assessment and post-assessment, there is quantitative evidence that she grew in her MKT. Her scale score from the pre-assessment and post-assessment showed evidence that she made gains in her MKT during the course of our time planning and implementing the six lessons.

Mathematical Knowledge for Teaching four separate structures: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS) and knowledge about content and teaching (KCT) (Ball, Thames and Phelps, 2011). Each of these structures was systemically developed through the lesson planning process. First, as Mrs. W. engaged in the mathematics as a learner, she developed content knowledge and skills necessary to solve various tasks outside of the classroom setting. Second, as Mrs. W. learned the mathematics she started to apply her knowledge to the teaching of the content and developed specialized content knowledge.

Third, Mrs. W. continually developed her lens for how the students would engage in the mathematics by anticipating misconceptions and how students might approach and think about solving each of the tasks in the lesson. Finally, Mrs. W. was able to apply her knowledge in a way that was useful for the teaching of the content by preparing to ask questions and respond to students' questions during the lessons. This development of all four of the structures of MKT through the process supported Mrs. W.'s her pedagogical content knowledge for teaching of mathematics.

Analyzing MKT.

Mrs. W. was assessed through the Teacher Knowledge Assessment System (TKAS), which is an instrument designed to measure a teacher's MKT. The assessments for Mrs. W. were broken in to two main domains for middle school mathematics, middle school number concepts and operations and middle school patterns, functions and algebra. These two domains encompass a large portion of 8^{th} grade mathematics. Each domain is scored separately in the TKAS system and scores are assigned to each domain. The domains assess and measure content knowledge and pedagogical content knowledge. In the first domain, middle school number concepts and operations, Mrs. W. correctly responded to 9 out of the 17 items assessed on this domain and on the post-assessment Mrs. W. correctly scored 11 out of 17. On the domain of middle school patterns, functions and algebra; Mrs. W. correctly responded to 9 items on the pre-assessment and correctly responded to 11 items on the post-assessment. While these small gains from pre-assessment and post-assessment may not seem important, the gains made on these domains over the course of planning only six lessons show evidence that deep lesson planning can have a positive impact on a teacher's content knowledge and pedagogical content knowledge. These gains are not statistically significant, however, they show that Mrs. W. was making some progress in a short period of time.

Developing content knowledge and MKT through planning.

As we planned the six lessons together, Mrs. W and I focused our time together on doing the mathematics and then thinking through possible student misconceptions. In the first lesson we planned together, there was a steep learning curve for Mrs. W. for not only doing the mathematics but to also then consider how students will consider solving the task and possible misconceptions they may have in solving the task. The learning curve was grounded in an understanding of the mathematics in the task and how that mathematics connects to future learning for students (Hill and colleagues, 2008). During the planning of the mathematics, we paused to make sure we understood the importance of the mathematics and it was unclear to Mrs. W. about the direction of the lesson in relation to where students would apply this knowledge in upcoming concepts and ideas. For example, when planning Looking for Pythagoras Problem 4.2, Mrs. W. did not understand why the students were having to learn about rational and irrational numbers and how they would connect that learning, later on, to The Pythagorean Theorem. It was simply in the scope and sequence for what she needed to cover it and that is why she was going to teach it. She had not made those connections to the bigger picture about how this would support the students with making sense of square roots that would create irrational numbers later on when students were finding missing side lengths within right triangles. Leinwand and colleagues (2014) argue that effective teaching and learning happens when students are able to make connections to new learning from prior knowledge and experiences. It is then up to the teachers to make those connections as students are progressing through mathematical content. If teachers lack horizon content knowledge (Ball, Thames and Phelps, 2008), an awareness of how mathematical topics are related over the span of a curriculum, they may lack the ability to make decisions to talk about the content they are teaching in a meaningful manner.

When defining MKT, the focus of the work has been around the work of the teacher in the classroom and there is a gap about the MKT needed prior to the classroom. The focus of the research has been around skillful teaching and the ability of a teacher to spot if a student solution is correct or not and the appropriate response (Ball, Thames, and Phelps, 2008). The knowledge to identify incorrect solutions or thinking is what researchers would argue is content knowledge and the way in which a teacher responds is part of a bigger bucket of pedagogical content knowledge (Shulman, 1986). I argue that the work of teacher in planning the mathematics develops both content and pedagogical content knowledge. An example of this is when planning the lesson and taking the time to engage in the mathematics as a student, the teacher is immersing themselves in the actual mathematics, so rather than just looking at a teacher's guide to know what the answer is to a task or problem, the teacher has a deep understanding of not only the solution but ways in which students could approach various tasks and problems. This kind of knowledge, Ball et al (2008) and Shulman (1987) situate as knowledge of students and students' thinking. When Mrs. W. took the time to work through various tasks, she was putting herself in the place of a student and developing a lens for how students could engage in the tasks and when implementing the lesson, not only did she know what the solution was to the task but also had a lens to align student's thinking with her own. This is powerful and cannot be understated because had we not taken the time to engage in the mathematics prior to Mrs. W. implementing the lesson, the only way in which she would have developed this particular facet of MKT of student's thinking would have been on-the-fly in the classroom as she was hearing it and seeing it from students (Hiebert, Morris, Berk, and Jansen, 2007). She was poised to be able to better to respond to student's thinking and push their thinking through questions because she was developing MKT during the planning of the lessons (Keiser, 2012). If a teacher implements a lesson and has not taken the time to develop specific knowledge of the content they are presenting to the class, the only way in which the teacher can develop their MKT is through trial and error with students and, thus too late to be useful for that group of students for that given lesson or task (Kleickmann, Richter, Kunter, Elsner, Besser, Krauss, Baumert, 2013).

Research Question 2

To address my second research question, I focused on Mrs. W.'s patterns of questioning and how she reflected on the lesson after she implemented the lesson. The second research question I am trying to answer is:

2. How does a teacher's beliefs about her pedagogy and instructional practice shift when she deepens her MKT through lesson planning?

To answer this question, I started by analyzing the implementation of the first task as a baseline for patterns of questioning and to reflect on Mrs. W.'s response to the five debrief questions. The debrief had five guiding questions:

1. How did doing the task within the lesson as a learner prepare you to implement the lesson in your own classroom?
2. What misconceptions did students have during the lesson that you anticipated? That you did not anticipate?
3. During the implementation of the lesson, was there a certain question(s) that you asked that had an impact on how students thought about the task they were working on? Did you change or modify the question(s) for another class period?
4. If you were to implement this lesson again, what, if anything, would you do differently and why?
5. Did you feel that you modified or shifted your instructional practice as a result of the collaborative planning of the lesson?

Analyzing shifts in beliefs.

In reflecting on my first research question, I would argue that Mrs. W.'s MKT was showing signs of growth. As Mrs. W. started to show signs of growth in her MKT, I reflected on the debrief questions and her responses to further understand if there were some shifts happening with her beliefs as a result.

Paul Ernest (1999) holds that a mathematics teacher's beliefs are grounded in three main pillars. The first is a teacher's beliefs about students and which student can and cannot learn mathematics. Second is the belief a mathematics teacher has in regards to his/her own experience in learning mathematics. Third, and where my research is focused, is on a mathematics teacher's beliefs about their own understanding of mathematics. Drageset (2010) argues that a teacher's instructional practices can be traced back to their own knowledge of the mathematics. Teachers with a more limited knowledge of mathematics have a stronger instructional emphasis on rules and formulas. In contrast, a teacher that has a stronger command of content knowledge and specialized content knowledge (SCK) tends to focus more on student thinking rather than the solution (Nisbet & Warren, 2000). A reason that this may occur is because teachers want to avoid the risk of uncertainty in their own mathematics knowledge (Drageset, 2010). SCK is a part of MKT and is defined as the mathematical knowledge needed only for the teaching of mathematics (Ball and colleagues, 2008). Wilson and Cooney (2002) wrote:

> …regardless of whether one calls teacher thinking beliefs, knowledge, conceptions, cognitions, view, or orientations … the evidence is clear that teacher thinking influences what happens in the classrooms, what teachers communicate to their students and what students ultimately learn (p.144).

So, (Nisbet & Warren, 2000) however we think of beliefs, these impacts what happens in classroom and the decisions a teacher makes which then impacts what students learn.

In order to try and understand and answer my research question, I reflected on the debriefs with Mrs. W. to understand how she thought about planning, the impact of planning on her content and pedagogical content knowledge, and her instructional practice. The focus here is to understand Mrs. W.'s beliefs about her own understanding of mathematics she was teaching and how, through lesson planning, she shifted her beliefs and her instructional practice.

After the first lesson, Mrs. W. stated that she usually does the math in the upcoming lesson the night before she teaches the lesson to students. However, she noted that for this lesson, when she was doing the math and trying to get at what the students were supposed to know, it was scaring her. She mentioned that the lesson and the mathematics in the lesson didn't instill confidence but it helped her prepare for the struggle that students were going to have in her class. Paul Ernest (1991) states that one of a mathematics teacher's beliefs is in regard to how the teacher feels about themselves as a mathematician and their own confidence in understanding the mathematics. From our first debrief, Mrs. W. made it clear that she was not confident in her ability to do math and the math sometimes scares her. When she considered what shifts she made in her instructional practice as a result of the collaborative planning, she admitted that the only thing she felt was different was that she had a few more questions to ask students than she might have normally. Mrs. W. said that when she plans alone it is different than when she planned collaboratively and that if she planned alone, she would have stuck only with the questions in the book because of her lack of comfort with this lesson.

After the implementation of the second lesson, she admitted that she felt:

> "more prepared knowing that I really needed to listen to what the kids were talking about because I needed to tie this in to tomorrow's lesson because you (researcher) started showing me different triangles that I never, it never crossed my mind. All I saw was I need to make a right triangle, I need to make a right triangle…it impacted me because I know I

> needed to hear what they were saying so I could pull it in to tomorrow's lesson."

Jane Keiser (2012) argues that teachers that focus their instruction on procedural skills do not tend to listen to anything except the solution to the problem. She goes on to say that, as mathematics teachers, we need to take time to listen to students' thinking so that we can better understand their solutions and support their learning. I saw from Mrs. W. the desire to listen to students so that she could better support their learning within the lesson and make connection to an upcoming lesson.

After the third lesson, Mrs. W. mentioned during the debrief that she felt the lesson went well because doing the mathematics ahead of time helped her understand what the students may need support with for the lesson. She also continued to build on the previously planned lesson where she felt it was important for students to explain their thinking because *"there is so much you can learn by, just listening to the kids."* She noted that she did not settle for just an answer in this class because her goal was to be a "conversation facilitator" because she thought the students would learn better. This a notable shift she acknowledged in her instruction that was starting to align with how she felt as a mathematician and a teacher of mathematics, she was more confident in herself.

There was a pattern emerging with Mrs. W that showed as she was feeling more confident in the mathematics she was teaching, she wanted to open up her classroom to more discourse and listen to her students share their thinking. She wanted to move away from a solution-based classroom to one in which students ask and respond to questions that promote classroom discourse and sharing of ideas. This is aligned to Nisbet and

Warren's research (2000) that suggests teachers with a deeper content knowledge and SCK focus on more than just a solution in their mathematics classrooms.

When we sat down to debrief the final lesson together, Mrs. W. admitted that doing the mathematics helped her understand the mathematics that the students were doing and that helped her be more prepared for the kids. She stated that instead of just looking at the answer key to know what the solution is that students should be getting, she understood how they would get the answer and how they could even get the wrong answer. As she was talking through the lesson, she stopped to think about when she asked students to describe the relationship, and even though we worked through this together, she knew that this was going to be a challenge for students because it was a challenge for her in planning. But because she knew the math well enough from planning she was able to ask students probing and guiding questions that allowed them to be able to describe the relationship on their own.

Her own content knowledge and MKT had developed enough in planning, that even in a short period of time, we were seeing shifts in her instructional practice aligned to how research suggests teachers with deeper content and MKT move to a more facilitative classroom and away from solution oriented classrooms (Nisbet and Warren, 2000).

After Mrs. W. and I planned the last lesson together, she wrote me an unprompted note with her feelings of the overall experience of our time together. She had this to say:

> "My teaching has modified as a result of the collaborative planning. I was more prepared to help the students with their misconceptions, had better

> questions that were more thoughtful and related to the work that they were doing/showing (as opposed to generic how? why?), and I also had a little better understanding of the math. As I look at the lessons coming up this week, I will be doing similar planning (even though it will be by myself). I will be the crazy lady in her classroom at lunch working on the back board and talking out loud to myself in an effort to mimic the collaborative planning. I think that I will really be focusing on the misconceptions and questioning.
>
> Thank you for allowing me to be a part of this experience. I really feel like I am learning a lot. I know that all of this is going to help me to become a better teacher. Thank you!"

Through this journey, Mrs. W. and I had together, Mrs. W. acknowledged that the process of planning the mathematics lesson for deep understanding of the mathematical content and the learning goal had shifted her instructional focus. She was now admittedly more focused on understanding how students would approach the tasks in the lesson by considering misconceptions students may have and how she could support their thinking through questioning. This supports the research that argues how a teacher's beliefs about their own understanding of the mathematics changes how they think about teaching mathematics (Drageset, 2010 and Wilson & Cooney, 2002).

Analyzing patterns of questioning.

As I reflected on Mrs. W.'s patterns of questioning over the course of the six lessons, I noticed something that I thought was important in her patterns of questioning.

In the beginning of the research study, she was only situated with asking students funneling questions, questions in which she had a predetermined solution she wanted students to arrive at when she finished with her questions (Herbel-Eisenmann and Breyfogle, 2005). I would argue that this pattern of questioning came from a place of fear and avoidance. Because she did was admittedly afraid of the mathematics, she focused students learning and questions only on the content she knew.

In addition to the types of questions, her questions were situated between her and a single student in most situations. As she progressed, I saw more patterns of focusing questions (Leinwand et al, 2014 and Herbel-Eisenmann and Breyfogle, 2005), questions that validated student thinking and pushed students to make sense of and justify the mathematics. However, what I observed that was interesting was that she was able to be strategic with her patterns of questions based-on what she wanted students to accomplish in a given interaction. For example, when she was reviewing a prior concept, she had questions that where more funneling in nature because she wanted to review prior learning but when she was engaging students in a new concept or idea, she switched her patterns of questioning to be more aligned to focusing patterns of questioning that pushed students to make sense of the new learning they were engaged in during class. She had a strong command of the types and patterns of questions to accomplish the mathematical goal of the task and to promote classroom discourse (Smith and Stein, 2011). Also, towards the end of the research study together, Mrs. W. shifted her engagement of questions and questioning from a more teacher-to-student oriented pattern to a more teacher-to-multiple students (Leinwand et al, 2014). This shift is not something I really noticed until around the fifth lesson we planned together and it correlated with how Mrs. W. started to shift her thinking about her own ability to do the mathematics. In the debrief of the fifth lesson, Mrs. W. noted that she was particularly happy with her ability to push students to justify their thinking. She also admitted that the deep lesson planning and knowing the mathematics gave her confidence in the mathematics she was teaching. As Mrs. W. felt more comfortable with the mathematics and understood what students were going to be learning, she shifted her instructional practice to push students to justify their thinking which created a space for more focusing patterns of questioning where students had to justify their responses. This was a moment for me where I truly felt that lesson planning was having the intended impact on her instructional practice.

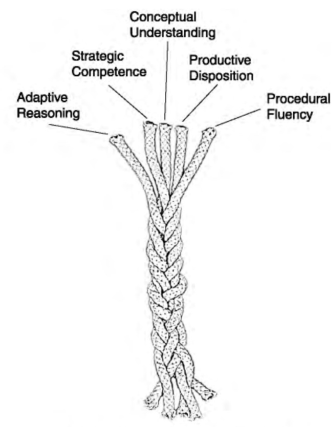
The Pedagogy of Knowing

As I watched the videos of the planning sessions and the classroom observations, I started to take note of when Mrs. W. felt like she had a solid understanding of the mathematical goal of the lesson and the mathematics that she would be teaching and how that translated into classroom practice. When Mrs. W. knew the mathematics of the lesson deeply and she understood what it was that she wanted students to take-away from the lesson, she had a different type of presence in the classroom. For example, when teaching the last lesson and students were tasked with describing the relationship, Mrs. W. knew which students she wanted to ask to explain their thinking and in which order to have then share. When students shared their thinking, she was prepared to have students consider their response and gave students an example to try and see if the students' solution and thinking was mathematically sound. She could respond to students and push their thinking in a way that involved the entire class and support the learning of all students. On the flip side of that was when I observed a video of the planning and did not quite feel like Mrs. W. was completely comfortable with the goal or the mathematics and the enacted implementation that occurred after planning. Two main patterns started to emerge from analyzing the planning and classroom implementation. First, as Mrs. W. was pushing students to think more critically and justify their thinking though more open-ended patterns of questioning she admitted, in the debrief, that she felt comfortable with her own understanding of the mathematics. And, second, Mrs. W. was more confident in responding to students' work as they were allowed more opportunities to make their thinking visible. Greenwood, Burroughs, Yopp, Higgs, and Sutton (2010) conducted a large-scale study to try and understand the relationship between teacher's MKT and instructional practice. While they did not find holistic statistical significance, they did suggest that there was a strong relationship between MKT and a classroom that was student centered. When Mrs. W. had a strong grasp of the mathematics, potential student misconceptions and the learning goal of the lesson, she allowed students more opportunities to present their thinking, respond to each other's thinking and ask questions that promoted discourse.

Here, I would like to define what I call The Pedagogy of Knowing. This idea relates to the instructional presence of teachers when they feel that they understand the content they are teaching at a deep level and are genuinely interested in the problem or task they are teaching. The instructional presence of the teacher is one in which the teacher has command of the classroom in such a way that the teacher is ready to ask questions, push students to think deeper, exudes a physical presence that shows confidence in both their content and pedagogical content knowledge, and prompts a teacher to want to put the ownership of learning back on the students. While I cannot define this broadly to other content areas from my research, I would argue that this is a term that can be aligned with mathematics education. However, I would also argue that, given further research in other content areas, this definition would hold true in other content areas as well.

The importance of this idea of Pedagogy of Knowing goes deeper than a teacher just having pedagogical content knowledge and content knowledge, it brings in other ideas from research as well. Kiplatrick, Swafford, and Bradford (2001) discuss the notion of strands of mathematical proficiency. Kilpatrick and colleagues are that these “five strands are interwoven and interdependent of in the development of proficiency in mathematics.” (p.116).

Figure 5.1. Strands of mathematical proficiency.

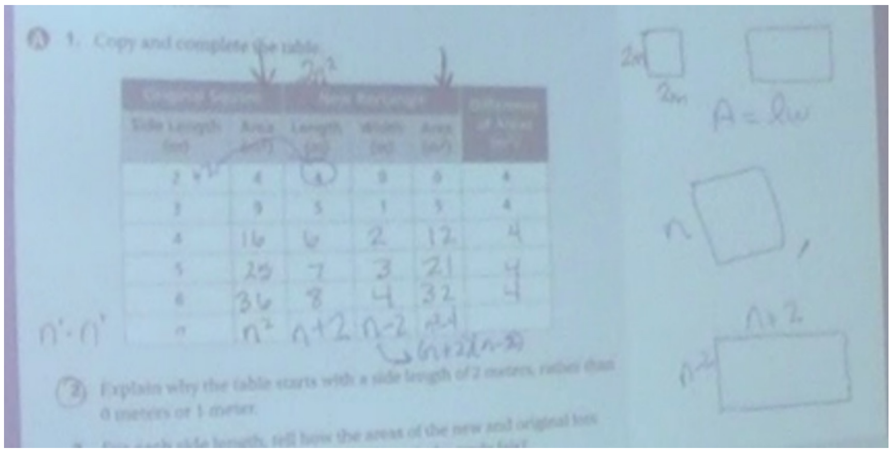


However, they are talking about a framework for learning for students, and not teachers, to acquire mathematical proficiency. Deborah Ball (2011) builds on this when discussing how teachers have to manage a classroom to help students engage in these five strands of mathematical proficiency. She argues that effective teachers need to make reasoned decisions about the use of instructional time by identifying the appropriate mathematical goal, establishing a safe and respectful learning environment, interpreting and responding to student errors, leading a class discussion, posing strategic questions and focusing on mathematical language; to name a few. Lipping Ma (1999) discusses why students in Asian countries tend to outperform their peers in the United States. She argues from her research that a teacher's disposition towards the mathematics they are teaching is a factor in how students perform. In China, for example, teachers are continually working to learn and deepen their understanding of the mathematics they are teaching. Ma describes teachers in China as actors performing in a play when they are teaching a lesson. She notes that a well-written play will not hinder an actor's performance but stimulate and inspire it. In contrast, teachers in the United States often see the mathematics they are teaching as trivial and do not take the time to understand the mathematics they are teaching. Thus, the teachers are only reading the script and providing stage directions, as Ma says, for students to follow.

As Mrs. W. was able to engage in the lessons, similar to how Ma (1999) saw the teachers in China engaging in the mathematics, and when she had a strong understanding of the mathematics she was teaching she implemented the lesson in a very different manner than I had previously observed (Ball, 2011). An example of this was from when Mrs. W. implemented the fourth lesson we planned together, Problem 2.1. In this lesson, as Mrs. W. and I planned together, I saw that she engaged in the mathematics of the lesson very differently than in previous lessons. She was excited to work on the mathematics and work through the task. She was asking questions to develop her own understanding further than just content knowledge by pushing both of us to consider how students may approach the problem and questions they may have along the way. During the planning of the lesson, there were visible signs that she was excited about the mathematics as we were able to make connections to several different equations and their equivalent forms. When Mrs. W. was implementing this lesson, her physical posture in the classroom showed one of confidence, as if she were performing, in how she asked students questions, leaned in to hear and respond to their thinking while they were working in their small groups, and responded to students' work when they shared their thinking. She knew the mathematics and was excited about the task the students were working on in class.

In addition to her own excitement and understanding of the mathematics, Mrs. W. was able to push student discourse and thinking though her questioning. The students were making connections to the equation $n^2 – 4$ and (n+2)(n-2) from the table.

Figure 5.2. Classwork on part A from Problem 2.1.



Mrs. W. was wanting students to explain how they came up with the two different equations:

Mrs. W.: (Student's name), how did you get the equation (n+2)(n-2)?

Student A: To find the area of a rectangle, you times the length and width. So, if I was to draw a rectangle, I would label the length n+2 and I would label the width n-2. So, to find the area, I would just times those two.

Mrs. W: Class, does that make sense to you? Yes or no?

Class: Yes

Mrs. W.: Okay, so how did you (Student B) get n^2-4?

Student B: If we take the area of the original square and compare it to the area of the new rectangle, for every number, the difference between the two is 4. And, so, that is where I got n^2 *-4.*

Mrs. W.: Okay, so help me with this one more time, say what you said again.

Student B: Okay, like on the first one the area of the square is 4 and the image of it, the area of that one is 0. And the difference between those two is 4. And the one where the length of the side of the square is 3, the area is 9 but the area of the image is 5 and, so, the difference is 4. And with 16 and 12, and so n^2*-4 would add the difference from the square.*

Mrs. W.: Does that make sense to you guys (the class), what he just did?

Class: Yes (except for Student C).

Student C: No.

Mrs. W.: Okay, who said no? Why did you say no?

Student C: I don't understand how the first equation, (n-2)(n+2), or whatever, can be the same as n^2*-4.*

Mrs. W.: That is a great question, so if (Student A) is right and (Student B) is right, then what does that tell us about these two expressions?

Class: (Starts to randomly respond out loud to the question. Several students respond that the expressions are the same.)

Mrs. W.: They are the same? But that is not very easy for us to see, is it?

I am going to pause here because there are a few things happening with this pattern of questioning. First, Mrs. W. is playing like she does not get it. This is a strategy the teachers use to have students revise their thinking (Daro, n.d.). This implies to me that she knows the content well enough to push students to clarify their explanations so that students, as a class, can move to a deeper understanding of the mathematics. Next, we see Mrs. W. focusing student ideas and not validating solutions to have the class react to each other's responses. This type of pattern of questioning is what Herbel-Eisenmann and Breyfogle (2005) discuss when thinking about purpose and patterns of questioning to promote discourse. And, finally, we see that Mrs. W. has the content knowledge for this task because as students are sharing their thinking, she is able to make connections and sequence their work appropriately (Smith and Stein, 2011).

The idea of a pedagogical practice that comes from a deep understanding of the content, knowing the learning goal of the lesson, and anticipating how students may approach solving a given task really starts to paint a picture of instruction when a teacher knows what they are going to teach.

I would argue that The Pedagogy of Knowing is really about moving teachers away from fear and avoidance to confidence and engaging. This holistic practice starts with lesson planning that sets teachers up to embody this idea. Mrs. W. is a case that clearly started the research study "afraid of the math" and moved to confident and wanting to engage students in "learning" mathematics.

Implications of Research

After reflecting on the research study and data analysis, there are several implications from this research that have arisen. First, this study reinforces and supports the notion that teachers learn through the act of teaching (Weber and Rhoads, 2011) and teacher education programs need to shift their pedagogical approach of teaching teachers. Ball and colleagues argue that subject matter courses in teacher preparation programs tend to be scholarly but are often irrelevant for prospective teachers because they are remote from classroom teaching. Pre-service teachers are gaining only content knowledge that is disconnected from useful application, lacks connections to students' thinking and is not useful in helping pre-service teachers consider misconceptions that students may have with the given content or connections to other domain sin mathematics. Pre-service teachers do not understand the extent to which formulation of MKT for teaching is culturally specific (Ball et al, 2011). Second, this research study supports the idea that deep lesson planning develops teachers' MKT and their confidence in being able to implement mathematics lessons and this should be a priority of focus for teacher education programs and for schools and school districts. Lastly, as the teacher in this study developed her MKT and confidence, her beliefs shifted and she was truly wanting students to have the same experience she had in planning that allowed her to deepen her own content knowledge. This shift in beliefs started to present itself in the way she allowed students to engage in the mathematics and in her patterns of questioning with students.

While there are critical ideas raised from this research, this study also opens the door for future work needed to better understand the connections between a teacher's content knowledge, pedagogical content knowledge and instructional practice. This is only a case study of a single teacher and is hard to make sweeping generalizations from a single case. While there is a body of research aligned to understanding how knowledge affects instructional practice on a small scale, very few large-scale studies have been conducted in this area. As researchers, we need to continue to evolve our work in this area as the relationship between teacher knowledge and instructional practice has important implications for policy and teacher education (Hill and colleagues, 2008).

Schools and school districts need to consider how a teacher's time in a school is distributed to provide rich opportunities for teachers to collaboratively plan lessons and engage in deeper learning of the mathematics and how to teach mathematics in an effective manner so students have a deeper understanding of concepts and develop procedural fluency. Teachers will need to learn how to set up tasks that promote discussion for student learning (Smith and Stein, 2011). This starts by having teachers engage in lesson planning that promotes learning the mathematics that is necessary to facilitate meaningful discourse, ask questions, and understand how students are considering the solutions and solution paths to various tasks (Smith and Stein, 2011).

Research strongly suggests that teachers who develop MKT produce better students that have better outcomes than teachers with less MKT (Hill, Rowan and Ball, 2005). Schools and districts are constantly working to improve outcomes for students and close achievement gaps. I would argue that to do this, schools and districts need to shift their resources to supporting effective models of lesson planning through professional learning opportunities. By prioritizing the work of schools and districts around developing MKT in mathematics teachers, they will be ensuring that students have teachers that are better prepared to implement lessons and close opportunity gaps that have historically been the root cause of achievement gaps (Flores, 2007).

Limitations

This research is not necessarily generalizable to the broader field of teacher education and the importance MKT and lesson planning has on shifting outcomes. However, this research study starts to ask questions that can generate further needed research to expand on the ideas from this study. Mrs. W. started out the research study discussing her fear of mathematics, and while this was a single case, it does add to the body of research about teacher's fear of content and may raise questions about the connections of fear of content and avoidance in classroom instruction. This research study is really designed as a method for starting to understand inter-connectedness of lesson planning, developing and the shifts in teacher's beliefs and practice. From here, there are many opportunities for further research studies related to these themes.

Missed Opportunities

In my work with Mrs. W., there is a piece of the planning that I regret not attending to in more detail and that is helping Mrs. W. understand how the mathematics of each lesson we were planning connected to prior learning and would be useful in upcoming learning for students. The idea of horizontal content knowledge is important in the broader umbrella of development of MKT. Teachers that have this specialized knowledge are better prepared to set the foundation and better prepare students for how the math they are currently learning will be built on for future learning (Ball, Thames and Phelps, 2008).

Conclusion

Shifting teacher beliefs is important because you are working to shift teachers' understandings about the world that are thought to be true (Phillip, 2007). These beliefs and understandings influence decisions that teachers make and serve as indicators of their decisions (Drageset, 2010). Beliefs and knowledge influence practice and as teachers shift their beliefs through gaining MKT they start to emphasize reasoning in students as they learn mathematics (Drageset, 2010). My research suggests that lesson planning is a vehicle that can shift beliefs and develop MKT in teachers to improve outcomes for students. While systemic change is challenging in educational settings, such as schools, districts and teacher preparation programs, they are necessary to improve outcomes for students in learning mathematics.

In addition to shifting beliefs and developing pedagogical content knowledge, schools and district need to invest in teachers more to allow more opportunities and time for collaborative planning. As I discussed the notion of a pedagogical practice that comes from a teacher's own understanding of the content, learning goals, and possible student solution paths leads to an instructional practice that is more aligned to what research suggests is best-practice in mathematics education. Schools and districts need to understand the work of the teacher doesn't only happen when students are in the classroom but also happens before students even walk through the door.

References

Askew, M. (n.d.). Mathematical discipline knowledge requirements for prospective primary teachers, and the structure and teaching approaches of programs designed to develop that knowledge. In K. Krainer, & T. Wood, *The International Handbook of Mathematics Teacher Education* (Vol. 1, pp. 13-36). Rotterdam, The Netherlands: Sense Publishers.

Ball, D. (2002). Connecting Mathematics Education Research and Classroom Practice. *Mathematics and Education Reform Newsletter*, 1-5.

Ball, D. (2011, October). *Supporting High Quality Mathematics Instruction.* Retrieved from Mathematics Teaching and Learning to Teach: http://www-personal.umich.edu/~dball/presentations/index.html

Ball, D., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*.

Banilower, E., Boyd, S., Pasley, J., & Weiss, I. (2006). *Lessons from a Decade of Mathematics and Science Reform: A Capstone Report for the Local Systemic Change through Teacher Enhancement Initiative.* Chapel Hill: Horizon Research, Inc.

Boerst, T., Shaugnessy, M., Ball, D., & Farmer, S. (2015). Attending to Teacher Preparation Outcomes from the Beginning: Learning from Baseline and Mid-Program Assessments. *Association of Mathematics Teacher Educators.* Orlando.

Brown, C., & Borko, H. (1992). Becoming a Mathematics Teacher. In D. Grouws, *Handbook of Research on Mathematics Teaching and Learning* (pp. 209-239). New York: Macmillan.

Castro, A. (2006). Planning for Mathematics Instruction: A Study of the Teacher Guide as a Resource. *Dissertation*.

Charalambous, C., & Hill, H. (2012). Teacher knowledge, curriculum materials, and qulaity of instruction: Unpacking a complex relationship. *Journal of Curriculum Studies*.

Common Core State Standards Initiative. (2015, July 23). *Common Core State Standards Initiative.* Retrieved from Common Core State Standards Initiative: http://www.corestandards.org/Math/

Cresewell, J. (2007). *Qualitative Inquiry & Researh Design: Choosing Among Five Approaches, 2nd Edition.* Thousand Oaks: Sage Publishing.

Daro, P. (n.d.). *The Deck Behind the 5x8 Card*. Retrieved from SERP Media: http://math.serpmedia.org/5x8card/deck/revise/

Doyle, W., & Carter, K. (1984). Academic Tasks in Classrooms. *Curriculum Inquiry*.

Drageset, O. (2010). The Interplay Between the Beliefs and the Knowledge of Mathematics Teachers. *Mathematics Teacher Education and Development, 12*(1), 30-49.

Ernest, P. (1989). The Impact of Beliefs on the Teaching of Mathematics. In P. Ernest, *Mathematics Teaching: The State of the Art* (pp. 249-254). London: Falmer Press.

Ernest, P. (1989). The knowledge, beliefs, and attitudes of the mathematics teacher: A model. *Journel of Education for Teaching, 15*, 13-33.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: The Falmer Press.

Flores, A. (2007). Examing Disparities in Mathematics Education: Achievement Gap or Opportunity Gap? *The High School Journal*.

Gagne, R. (1985). *The Conditions of Learning (4th.).* New York: Holt, Rinehart & Winston.

Gagne, R., & Briggs, L. (1979). *Principles of Instructional Design.* New York: Holt.

Gales, M. J., & Yan, W. (2001). *Relationship between Constructivist Teacher Beliefs and Instructional Practices to Students' Mathematical Achievement: Evidence from TIMMS.* College Park: ERIC.

Gay, L., Mills, G. E., & Airasian, P. (2009). *Educational Research: Competenceis for Analysis and Application.* New Jersey: Pearson Publication, Inc.

Gencturk, Y. (2012). Teachers' Mathematical Knowledge for Teaching, Instructional Practices and Student Outcomes. Ann Arbor: ProQuest LLC.

Handal, B. (2003). Teachers' Mathematical Beliefs: A Review. *Mathematics Educator*.

Herbel-Eisenmann, B., & Breyfogle, L. (2005). Questioning Our Patterns of Questioning. *Mathematics Teaching in the Middle School*.

Hiebert, J., & Stigler, J. (2009). Closing the Teaching Gap. *Kappan*.

Hiebert, J., Morris, A., Berk, D., & Jansen, A. (2007). Preparing Teachers to Learn from Teaching. *Journal of Teacher Education*.

Hill, H. (2007). Mathematical Knowledge of Middle School Teachers: Implications for No Child Left Behind Initiative. *Educational Evaluation and Policy Analysis*.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal*.

Ingersoll, R. (2011). Do We Produce Enough Mathematics and Science Teachers? *Kappan*.

Keiser, J. M. (2012). Student's Can Take Us Off Guard. *Mathematics Teaching in the Middle School*.

Khisty, L., & Chval, K. (2002). Pedagogic discourse and equity in mathematics: When teachers' talk matters. *Mathematics Education Research Journal, 14*(3), 154-168.

Kleickman, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., & Baumert, J. (2013). Teachers' Content Knowledge and Pedagogical Content Knowledge: The Role of Structural Differences in Teacher Education. *Journal of Teacher Education*.

Krussel, L., Edwards, B., & Springer, G. (2004). The Teacher’s Discourse Moves: A Framework for Analyzing Discourse in Mathematics Classrooms. *School Science and Mathematics*.

Lappan, G., Phillips, E. D., Fey, J. T., & Friel, S. N. (2014). *Connected Mathematics 3.* Boston: Pearson, Prentice Hall.

Lappan, G., Phillips, E., Fey, J., & Friel, S. (2014). *Connected Mathematics Project 3.* London: Pearson Education.

Leinwand, S., Brahier, D., & Huinker, D. (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston: The National Council of Teachers of Mathematics, Inc.

Lewis, C., Perry, R., & Hurd, J. (2004). A Deeper Look at Lesson Study. *Educational Leadership*, 18-22.

Liljedahl, P. (2008). Teachers' Beliefs as Teachers' Knowledge. *L'Enseignement Mathematique International Journal, 60*.

Ma, L. (1999). *Knowing and Teaching elementary Mathematics.* London: Lawrence Erlbaum Assicuates, Publishers.

Mills, G. (2007). *Action Research: A guide for the Teacher Researcher, 3rd Edition.* New Jersey: Pearson Education, Inc.

Monson, D. S. (2011). The Relationship Between Beliefs and Practices of Mathematics Teachers Who Use a Standards-Based Curriculum. Ann Arbor: Dissertation Publishing.

Morris, A., Hiebert, J., & Spitzer, S. (2009). Mathematical Knowledge for Teaching in Planning and Evaluating Instruction: What Can Preservice Teacher Learn? *Journal for Research in Mathematics Education*.

Morrone, A., Karkness, S., D'Ambrosio, B., & Caulfield, R. (2004). Patterns of Instructional Discourse that Promote the Perception of Mastery Goals in a Social Constructivist Mathematics Course. *Educational Studies in Mathematics*.

Mosvold, R., & Fauskanger, J. (2014). Teachers' Beliefs about Mathematical Horizon Content Knowledge. *International Journal for Mathematics Teaching and Learning, 9*(3), 311-327.

Natioanl Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academy Press.

Nisbet, S., & Warren, E. (2000). Primary school teachers' beliefs relating to mathematics, teaching and assessing mathematics and factors that influence these beliefs. *Mathematics Teacher Education and Development, 2*, 34-47.

Otten, S., & Soria, V. (2014). Relationships between students' learning and their participation during enactment of middle school algebra tasks. *Mathematics Education*.

Pajares, F. (1992). Teachers' Beliefs and Educational Research: Cleaning Up a Messy Construct. *Review of Educational Research*.

Philip, R. (2007). Mathematics Teachers' Beliefs and Affect. In J. Frank Lester, *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 275-315). Reston: NCTM.

Remillard, J. (2005). Examining Key Concepts in Researchon Teachers' Use of Mathematics Curricula. *Review of Educational Research*, 211-246.

Riegle-Crumb, C., & Humphries, M. (2012). Exploring Bias in Math Teachers' Perceptions of Students' Ability by Gender Race/Ethnicity. *Gender and Society*.

Romagnano, L. (1994). *Wrestling with Change: The Dilemnas of Teaching Real Mathematics.* Portsmouth: Reed Publishing Inc. .

Sam, L. C., & Ernest , P. (2000). A Survey of Publi Images of Mathematics. *Research in Mathematics Education*.

Seidle, T., Rimmele, R., & ManfredPrenzel. (2005). Clarity and Copherence of Lesson Goals as a Scaffold for Student Learning. *Learning and Instruction, 15*, 539-556.

Shulman, L. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, 4-14.

Simpson, A., Mokalled, S., Ellenburg, L., & Che, M. (2015). A Tool for Rethinking Questioning. *Mathematics Teaching in the Middle School*.

Smith, M., & Stein, M. (2011). *Five Practives for Orchestrating Productive Mathematics Discussions.* Reston: The National Council of Teachers of Mathematics, Inc.

Stein, M., & Kim, G. (2008). The role of mathematics curriculum materials in large-scale urban reform: An analysis of demands and opportunities for teacher learning. In J. Remillard, *Teachers' use of mathematics curriculum materials: Research perspectives on relationships between teachers and curriculum* (pp. 37-55). New York: Rutledge.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development.* New York: Teachers College Press.

Steunberg, R., Empson, S., & Carpenter, T. (2004). Inquiry in to Children's Mathematical Thinking as a Means to Teacher Change. *Journal of Mathematics Teacher Education, 7*(3), 237-67.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Stinson, D. (2006). African American Male Adolescent, Schooling (and Mathematics): Deficiency, Rejection and Achievement. *Review of Educational Research*.

Swan, M. (2006). Designing and using research intruments to describe the beliefs and practices of mathematics teachers. *Research in Education*.

The role of mathematics curriculum materials in large-scale urban reform: An analysis of demands and opportunities for teacher learning. (2008). In J. Remillard, *Teachers Use of Mathematics.*

Unknown. (2014, July 1). *Procedural Fluency - NCTM Position.* Retrieved from National Council of Teachers of Mathematics: http://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/Procedural%20Fluency.pdf

Wilhelm, A. G. (2014). Mathematics Teachers' Enactment of Cognitively Demanding Tasks: Investigating Links to Teachers' Knowledge and Conceptions. *Journal for Research in Mathematics Education*.

Wilson, M., & Cooney, T. (2002). Mathematics teacher change and development. . In G. &. Leder, *Beliefs: A Hidden Variabel in Mathematics Education* (pp. 127-148). Doordrecth, Netherlands: Kluwer Academic Publishers.