Polarized Bow Shock Nebulae Reveal Features of the Winds and Environments of Massive Stars

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Polarized bow shock nebulae reveal features of the winds and environments of massive stars

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Abstract

Massive stars strongly affect their surroundings through their energetic stellar winds during their lifetime and through their energetic deaths as supernovae. When a stellar wind interacts with the local interstellar medium (ISM), if the relative velocity between wind and ISM is supersonic, then a stellar wind bow shock is formed. Bow shocks and related density enhancements produced by the winds of massive stars moving through the interstellar medium provide important information regarding the motions of the stars, the properties of their stellar winds, and the characteristics of the local medium. Since bow shock nebulae are aspherical structures, light scattering within them produces a net polarization signal even if the region is spatially unresolved. Scattering opacity arising from free electrons and dust leads to a predictable distribution of polarized intensity across the bow shock structure. That polarization encodes information about the shape, composition, opacity, density, and ionization state of the material within the structure.

In my dissertation research, I use a Monte Carlo radiative transfer code that I optimized to simulate the polarization signatures produced by both resolved and unresolved stellar wind bow shocks (SWBS) illuminated by a central star and by emission from the bow shock. I derive bow shock shapes and densities from published analytical calculations and smooth particle hydrodynamic (SPH) models. In the case of the analytical SWBS and electron scattering, I find that higher optical depths produce higher polarization and position angle rotations at specific viewing angles compared to theoretical predictions for low optical depths. This is due to the
geometrical properties of the bow shock combined with multiple scattering effects. I also find that the source of illumination plays an important role in determining the distribution of polarization for resolved bow shocks. In the case of dust scattering, the polarization signature is strongly affected by wavelength, dust grain properties, dust temperature, and viewing angle. The behavior of the polarization as a function of wavelength in these cases can distinguish among different dust models. In the case of SPH density structures, I investigate how the polarization changes as a function of the dust grain size and composition present in the SWBS. I present preliminary results of this implementation. In each case, I discuss the observational implications of these models for the stellar winds and interstellar environments of these influential objects, and predict observable signatures that can help constrain quantities of particular interest.
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Chapter 1

Introduction

1.1 Background

Stars are an important component of galaxies and the universe. They are powered by the energy produced by nuclear fusion of lighter elements to heavier elements. The evolution of the star depends on the mass of the star. Both massive stars (greater than 8 solar masses, or 8 $M_\odot$) and lower mass stars (less than 8 $M_\odot$) burn hydrogen to produce energy in the main sequence phase. When they run out of hydrogen, then the helium fusion commences in the red giant phase. The evolution diverges from this phase depending on the stellar mass. If the stellar mass is less than 8 $M_\odot$, then the star cannot fuse any higher elements and ends its life as a white dwarf by expelling its outer layers in the form of a planetary nebula. If the star is more massive than 8 $M_\odot$ then its core temperature can get high enough to undergo non-degenerate carbon ignition. Massive stars fuse elements in the core until an iron core is formed. Since iron fusion requires energy instead of producing energy, hydrostatic equilibrium cannot be maintained and a massive star ends its life as a supernova explosion. This is a very energetic event which ejects the envelope of the
star, leaving a degenerate core (neutron star or black hole). This phenomenon is called a Core Collapse Supernova (CCSN).

Throughout their evolution, massive stars also lose mass in different forms. The mass-loss rate is higher for higher-mass stars. Mass loss in the form of a stellar wind impacts the composition and evolution of the star and the surrounding ambient medium. Thus massive stars play an important role in enriching the interstellar medium (ISM) throughout their lifetimes through both stellar winds and supernova explosions. Hence understanding the impacts massive stars have on the local ISM is important for many different aspects of astronomy. However, it is difficult to study the evolution of massive stars as their frequency is much lower than that of lower-mass stars and their lifetime is much shorter.

How does the interaction of a stellar wind with the ISM change the ISM dust grain properties? What is the mass loss rate of the stellar wind and how does it vary with spectral type, evolutionary phase, and other stellar properties? How does a star’s mass loss affect its ultimate fate (type of CCSN and remnant)? These are some of the open questions in the evolution of massive stars and their impact on the local ISM. One way to approach these questions is through the study of stellar wind bow shocks, which are interaction regions between the stellar wind and the ISM. These bow shocks have been observed around many stars at various evolutionary phases in different parts of the sky. As the bow shock structure depends both on the stellar wind and the local ISM, studying these phenomena can give us answers to some of the open-ended questions mentioned above.

1.2 Motivation

Mass loss from a massive star impacts the evolution of the star as well as the structure of the surrounding ambient medium. Massive stars lose mass during their lifetime via stellar winds. The mass-loss rates of these stars vary considerably as
they evolve from main sequence stars to red supergiants (RSGs), blue supergiants (BSGs), Wolf-Rayet stars, or luminous blue variables (LBVs; Crowther 2001). The circumstellar material around massive stars preserves imprints of mass loss at different phases of their stellar evolution (Ueta et al. 2006). Analyzing this material can help determine the mass-loss rates of individual stars, observe the transitions between stellar evolutionary phases (Mohamed et al. 2012), and predict the types of supernova produced by the stars (Smartt 2009).

In addition, the stellar winds from massive stars are major contributors to the composition of the ISM, as well as its evolution and dynamics (Castor et al. 1975). By itself, the ISM is hard to observe; thus many of its properties, such as its density and composition, are not well understood. Since these properties are not uniform throughout the Galaxy, it is particularly important to derive information about the density, composition, and grain size of the ISM in specific locations distributed throughout the Milky Way (Ueta et al. 2008a).

Stellar winds impact the surrounding ISM differently at different stellar evolutionary phases. Because of this, stellar wind bow shocks, which are produced by the interaction between stellar winds and the ISM, provide an excellent laboratory to study the properties of both (Wilkin 1996). For example, Ueta et al. (2008a) used bow shock observations to determine the ISM density around α Ori, while Gvaramadze et al. (2012) used properties of the bow shock around ζ Oph to measure its mass-loss rate. Studying the polarization produced in stellar wind bow shocks can give us further information about the star’s mass loss and the dust properties of the local ISM through which it travels.

1.3 Stellar wind bow shocks

When massive stars are ejected from their star clusters, they move through the ISM supersonically and are known as runaway stars. In this situation, when the
ram pressures of the stellar wind and ISM are balanced, a bow shock is formed in front of the star. In this dissertation, I use the term “bow shock” to describe not only a true physical shock, but also a region of enhanced density arising from wind-ISM interactions and having the same geometrical shape as a bow shock. The first stellar wind bow shock was observed around the runaway O star ζ Oph (Gull and Sofia 1979). Since these bow shocks are formed by the interaction of stellar wind and ISM, studying them gives us information regarding both the stellar wind of the star and ISM. Bow shocks have the potential to provide information regarding the direction of the motion of the star, mass loss history of the star (Mackey et al. 2012; Gvaramadze et al. 2014), and the structure of the surrounding ISM (Ueta et al. 2008a). There have been many bow shock detections in recent years (Peri et al. 2012; Kobulnicky et al. 2016). They have been observed at visible wavelengths (Gull and Sofia 1979) and in the infrared IR, mid-IR and far IR (Ueta et al. 2006; van Buren and McCray 1988). Bow shocks have been found associated with many different classes of objects such as pulsars (Cordes et al. 1993), cataclysmic variables (van Buren 1993), and Algol binaries (Mayer et al. 2016).

In this study, I focus on simulating polarization signatures of stellar wind bow shocks around massive single stars. Observationally, polarization has been measured for IRS 8, an infrared source identified as a bow shock around an O5-O6 star near the galactic center; this object displays 3.3% polarization at 13° and 4.3% polarization at 19° (Rauch et al. 2013). Thus modeling these structures to extract polarization behavior can help us constrain various properties of the observed bow shocks.

In the next two chapters, I use an analytic bow shock structure in the simulations. I assume shape and density functions for the bow shock derived analytically by Wilkin (1996). Some of the assumptions made in Wilkin (1996) are:

- The stellar wind is uniform
- The flow is hypersonic, thus pressure forces are neglected
• Cooling is efficient

• The ISM is uniform. Note that for the case with ISM density gradient perpendicular to the stellar velocity, a nonaxisymmetric bow shock is produced, whose analytic derivation is given in Wilkin (2000).

The standoff radius $R_0$ [cm] is derived by balancing the ram pressures of the ISM and the stellar wind: $\rho_w V_w^2 = \rho_a V_*^2$. This yields

$$R_0 = \sqrt{\frac{\dot{m}_w V_w}{4\pi \rho_a V_*^2}} \quad (1.3.1)$$

In this expression, $\dot{m}_w$ [$M_\odot$/year] is the stellar wind mass-loss rate and $V_w$ [cm/s] is the stellar wind velocity. $V_*$ [cm/s] is the velocity with which the star travels in a uniform ISM density of $\rho_a$ [g/cm$^3$]. Using momentum conservation and force balance and integrating these equations numerically, Wilkin (1996) obtains the equation for the radius of the shock:

$$R(\theta) = R_0 \csc(\theta) \sqrt{3 \left(1 - \theta \cot \theta\right)} \quad (1.3.2)$$

Equation 1.3.2 describes the shape of bow shock by giving the variation in radius with polar angle $\theta$.

Wilkin (1996) also defined the mass surface density $\sigma$ [g/cm$^2$] of the idealized, infinitely thin bow shock shell:

$$\sigma = R_0 \rho_a \frac{[2\alpha (1 - \cos \theta) + \tilde{\omega}^2]^2}{2\tilde{\omega} \sqrt{(\theta - \sin \theta \cos \theta)^2 + (\tilde{\omega}^2 - \sin^2 \theta)^2}} \quad (1.3.3)$$

Here $\alpha$ represents the ratio $V_* / V_w$, and the symbol $\tilde{\omega}$ is defined as

$$\tilde{\omega}^2 = 3(1 - \theta \cot \theta) \quad (1.3.4)$$
Given these relations, if we know the distance and the angular size of an observed bow shock, we can obtain the local ISM density using equation 1.3.1.

1.4 Polarimetry

When the vibration of the electric field in an electromagnetic wave is aligned to a particular direction, then the light is said to be polarized. There are various processes that can produce polarized light via emission (e.g., cyclotron and synchrotron radiation) or via scattering. Polarimetric studies of astronomical sources can give us information about the observed object such as geometry, magnetic field strength and orientation, particle densities, etc. Polarization studies have contributed to the mapping of stellar magnetic fields (Schrijver and Zwaan 2000), the polarization modes of the cosmic microwave background (BICEP2 Collaboration et al. 2014), and the understanding of gamma radiation from gamma-ray bursts (Gill and Granot 2018).

For this dissertation work, I focus on the polarization of light generated by the scattering of radiation. When the scattering medium is asymmetric in some way (such as aspherical geometry, clumps in the scattering region, alignment of dust due to magnetic fields, etc.), then the resultant scattered light from the region is polarized. Thus, polarization studies can provide complementary information about the scattering medium to that obtained from photometric and spectroscopic studies. Various research groups have used polarization signals from scattering regions around various astronomical objects including single stars, binary stars, and supernovae to study the properties of these objects and their surroundings (e.g., Code and Whitney 1995; Hoffman et al. 2003).

1.4.1 Stokes Parameters

One way to describe the polarization state of radiation is with Stokes vectors (Stokes 1852). The four components of the Stokes vector are
\[ I = \langle E_x^2 \rangle + \langle E_y^2 \rangle \]
\[ Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle \]
\[ U = 2 \langle E_x E_y \cos \delta \rangle \]
\[ V = 2 \langle E_x E_y \sin \delta \rangle \]

where \( E_{x,y} \) are the electric field components of the electromagnetic radiation in the \( x \) and \( y \) directions, respectively, given by \( E_x(t) = E_x(0)e^{i\omega t - \phi_1} \) and \( E_y(t) = E_y(0)e^{i\omega t - \phi_2} \). The quantity \( \delta \) is the difference between phase angles: \( \delta = \phi_2 - \phi_1 \).

In Eq. 1.4.1, \( I \) is the total intensity, \( Q \) and \( U \) measure linear polarization, and \( V \) measures circular polarization. Figure 1.1 presents a visual representation of the \( Q \) and \( U \) Stokes vectors. Using these quantities we can calculate the degree of polarization using

\[ p(\%) = \frac{\sqrt{Q^2 + U^2}}{I} \times 100, \]  
\[ (1.4.2) \]

and the polarization angle or position angle using

\[ \Psi = \frac{1}{2} \arctan \left( \frac{U}{Q} \right), \]
\[ (1.4.3) \]

1.5 Methods

For this dissertation work, I created simulations of the polarization produced by bow shock nebulae using the Supernova Line Profile (\( SLIP \)) code (Hoffman 2007; Shrestha et al. 2018). \( SLIP \) uses the Monte Carlo radiative transfer (MCRT) method (e.g., Whitney 2011) to track virtual photons through a three-dimensional spherical polar grid as in Whitney and Wolff (2002). \( SLIP \) does not rely on the Sobolev
Figure 1.1: Representation of linear polarization using Stokes vectors.
approximation (Jeffery 1989), but instead performs full radiative transfer in high optical depth regions. For the axisymmetric simulations presented in Chapters 2 and 3, I define a grid with 100 radial cells and 101 cells in the polar ($\theta$) direction. For the simulations with azimuthal ($\phi$) dependence in Chapter 4, the grid is defined by 100 radial cells, 101 cells in $\theta$, and 201 cells in $\phi$.

At the center of this grid I place a finite spherical photon source, surrounded by a circumstellar scattering region composed of pure hydrogen or dust (depending on the type of scattering) in local thermodynamic equilibrium (LTE). I do not assume this circumstellar material (CSM) is heated by the central star. Instead I define its temperature $T$ [K] (which for simplicity I assume is constant throughout the region) as a user-specified input parameter. I assume the CSM is stationary with respect to the star, which is a good approximation for bow shock nebulae. The optical depth ($\tau_0$) of the CSM at a specific reference angle is a user-defined input parameter in the code which is used to calculate the density variation throughout the CSM.

In the code, virtual photons can be emitted from the stellar photosphere or the CSM. Photon emission from the stellar photosphere is spherically symmetric, however the emission from the CSM is proportional to the density of the CSM. Dust emission is calculated differently, as I discuss in detail in Chapter 3. Once the photon is emitted, the code tracks it as it travels through the CSM. In the CSM it can scatter, be absorbed, or escape. At each scattering event, the code updates the photon’s polarization state by performing a transformation of its Stokes parameters (Chandrasekhar 1960; Whitney 2011) which define the polarization state of the light. When the photons exit the CSM, they are binned by outgoing angle and the Stokes parameters are summed appropriately in each bin. Hence the code produces a three-dimensional model whose polarization characteristics can be viewed from any inclination angle. These steps are shown in the flowchart in Fig. 1.2. Within each output bin, I sum the Stokes vectors due to all $N$ photons in the bin.
and apply normalization factors in $\theta$ and $\phi$ to ensure that output fluxes have the correct units. I determine the $1\sigma$ uncertainty for each Stokes parameter in each bin by calculating the standard deviation of that parameter over all $N$ photons in the bin and normalizing it to $\sqrt{N}$ to account for the Poisson statistics of this counting experiment (Wood et al. 1996b; Whitney 2011). This uncertainty is a numerical (internal) uncertainty. Computational (systematic) uncertainties have been investigated for the SLIP code and found to be small (Huk 2017); they are not included in the results I present here.

The methods by which I calculate opacity, albedo, and Stokes vectors vary with the composition of the scattering medium. These calculations are discussed in detail in the subsequent chapters. I discuss the analytic bow shock density structure and the case of electron scattering in Chapter 2. In Chapter 3, I present the implementation of dust scattering within the code. In addition to dust in the bow shock, I also discuss the polarization behavior when dust is present between the bow shock and the star. In Chapter 4, I discuss the extension of the models from an analytical shape to a more realistic CSM structure determined by smooth particle hydrodynamics models (Mohamed et al. 2012).

1.6 Goals of the dissertation

Most of the existing computational models of stellar wind bow shocks have not taken polarization into account. The few models that do study polarization have been done for a specific bow shock using analytic calculation or radiative transfer methods (Neilson et al. 2013; Shahzamanian et al. 2016). The goal of this dissertation is to create a systematic simulation grid of polarization signatures for a bow shock geometry with a range of physical input parameters. This dissertation provides a general polarization study for stellar wind bow shocks which can be used
Figure 1.2: Flowchart depicting the basic operation of the SLIP code. “PDF” stands for probability distribution function.

along with polarization observations to gain information about the scattering region and thus constrain properties of the stellar wind and ISM.

In the first part of my dissertation, I simulated bow shock polarization when electron scattering is the dominant contributor to the polarization signal. The aim of the study was to model cases where shock is detected and the temperature is high enough for ionization to occur. Thus electron scattering will be the dominant scattering mechanism for polarization. I conducted a parameter study for an analytic bow shock which studied the impact of density, temperature, inclination angle, and source of emission in polarization signatures. This study was published in the Monthly Notices of the Royal Astronomical Society in 2018 (Shrestha et al. 2018).

In the second part of my dissertation, I investigated the polarization signatures arising from dust scattering in bow shock nebulae. The aim of this study was to simulate polarization SEDs for different types of dust models to gain information
about the dust size, dust composition, dust temperature and how polarization varies with wavelength.

In the third part, I implemented a density structure from a smooth particle hydrodynamic (SPH) model of the bow shock nebula around Betelgeuse (Mohamed et al. 2012) and present preliminary results. This detailed study of a particular object will help us extract the properties of the stellar wind and local ISM for this red supergiant.

1.7 Outline

The dissertation is organized as follows: Chapter 2 presents the method and results for electron scattering case; this chapter has been published in a peer-reviewed journal (Shrestha et al. 2018). Chapter 3 discusses the method and results for the dust scattering case. Chapter 4 provides the method and preliminary results for the SPH density structure. Chapter 5 provides the conclusion and implications of this research, and discusses future directions for this project. The appendix (Chapter A) details the calculation of a geometric parameter, \( b(\theta) \), used in Chapter 2.
Chapter 2

Polarization signatures of stellar wind bow shock nebulae: The case of electron scattering

2.1 Introduction

Mass loss from massive stars impacts their evolution (e.g., Langer 2012) as well as the evolution and dynamics of the surrounding interstellar medium (ISM; Castor et al. 1975). One of the most visible manifestations of stellar mass loss, a bow shock, forms when the stellar wind emanating from a star moving through the ISM reaches supersonic relative velocities (e.g., Wilkin 1996). The properties of such stellar wind bow shocks encode information about the mass-loss history of the star (e.g., Raga and Cantó 2008; Mackey et al. 2012; Gvaramadze et al. 2014) and the structure of the surrounding ISM (e.g., Toalá and Arthur 2011).

Most observed bow shocks are associated with massive runaway stars; however, they are also observed around a variety of stellar sources including asymptotic giant branch stars (e.g., Ueta et al. 2006), pulsars (e.g., Cordes et al. 1993), cataclysmic
variables (e.g., van Buren 1993), and Algols (e.g., Mayer et al. 2016). These bow shocks are typically detected at optical (e.g., Gull and Sofia 1979) and infrared (IR) wavelengths (e.g., van Buren and McCray 1988; Ueta et al. 2006; Ueta et al. 2008b), though a few have been detected at X-ray (e.g., López-Santiago et al. 2012), ultraviolet (e.g., Le Bertre et al. 2012), and radio (e.g., Benaglia et al. 2010) wavelengths. In recent years, several dedicated surveys have revealed large numbers of bow shock nebulae in the Milky Way (e.g., Peri et al. 2012, 2015; Kobulnicky et al. 2016), opening new avenues of research into stellar winds and ISM characteristics.

In this chapter, we probe the connections between polarimetric observations and the physics of stellar wind bow shocks. (Hereafter, we will use the term “bow shock” to describe not only a true physical shock, but also a region of enhanced density arising from wind-ISM interactions and having the same geometrical shape as a bow shock.) Polarization by scattering samples the opacity of a medium, and encodes information about the relative orientation of a scattering region in relation to illuminating sources and the observer. In the case of electron (Thomson) scattering, interaction of unpolarized incident radiation with a free electron produces scattered radiation that is 100% linearly polarized when the scattering angle is 90°, independent of wavelength; the angle of polarization is perpendicular to the plane defined by the incident and scattered rays (Rybicki and Lightman 1979). In the case of dust scattering, asymmetric dust grains produce scattered radiation whose linear polarization magnitude and position angle are wavelength-dependent, and which may also be circularly polarized (Henyey and Greenstein 1941; White 1979). Polarization has been detected in two bow shock sources near the Galactic centre, with magnitudes up to a few percent (Buchholz et al. 2011; Rauch et al. 2013). Such values are easily measured with current polarimetric instrumentation, suggesting that polarization may be a valuable technique with which to study the wealth of newly discovered bow shocks.
Although many researchers have developed computational models of stellar wind bow shocks (e.g., Gustafsson et al. 2010; Mohamed et al. 2013; Christie et al. 2016), polarization signatures have not generally been considered. However, two recent studies have modelled the polarization of specific objects with bow shocks. Neilson et al. (2013) analytically modelled the near-IR polarization from a bow shock around Betelgeuse. Shahzamanian et al. (2016) used a sophisticated 3-D Monte Carlo radiative transfer (MCRT) code to simulate the polarization behaviour of a dust-scattering bow shock and other possible circumstellar structures around the Dusty S-cluster Object (DSO), an unusual infrared-excess source near the Galactic centre.

This contribution is the first of two papers in which we use Monte Carlo numerical methods to explore the polarization signatures arising from generalised stellar wind bow shock structures. Our code (SLIP; Hoffman (2007)) is related to the one used by Shahzamanian et al. (2016), but our implementation is different, as discussed below in Section 2.2. The MCRT approach is easily adaptable to nonspherical geometries while allowing for consideration of optical depth effects (i.e., the influence of multiple scattering on the polarization of escaping light). Our goal in this chapter is to formulate the problem of predicting the polarization produced within an idealised bow shock structure and to investigate the effects of various input parameters on the resulting polarization behavior, assuming Thomson scattering only for simplicity. The next chapter (hereafter Paper II) will investigate the effects of dust opacity on observed polarization, a scenario with broader applications.

Our chapter is organized as follows. In Section 2.2, we discuss the SLIP code and the features of our models. In Section 2.3, we present analytic results for our idealized bow shock cases, valid strictly in the optically thin limit. Although limited in applicability, the analytic results provide context for interpreting the numerical results from SLIP. In this section we also discuss comparisons between the analytic
and numerical simulations. In Section 2.4, we present and interpret numerical results for the polarization produced in both resolved and unresolved cases, as functions of the temperature and optical depth of the scattering material in the bow shock. We discuss how our results may aid in interpretation of observed polarization signals in Section 2.5. Finally, we offer concluding remarks in Section 2.6.

2.2 Methods

We constructed our simulations using the Supernova Line Profile (SLIP) code (Hoffman 2007). SLIP uses the MCRT method (e.g., Whitney 2011) to track photons through a three-dimensional spherical polar grid as in Whitney and Wolff (2002). For the axisymmetric simulations presented here, we define a grid with 100 radial cells and 101 cells in the polar (θ) direction.

At the centre of this grid we place a finite spherical photon source, surrounded by a circumstellar scattering region composed of pure hydrogen in local thermodynamic equilibrium (LTE). We do not assume this circumstellar material (CSM) is heated by the central star. Instead we define its temperature $T$ (which for simplicity we assume is constant throughout the region) as a user-specified input parameter governing the ionisation fraction $x$ within the scattering region. Given a specified reference optical depth $\tau_0$, SLIP first calculates the number density of free electrons via the equation $n_+ = \tau_0 / 0.4 m_H \Delta R_0$, where $m_H$ is the proton mass and $\Delta R_0$ is the radial thickness of the scattering region at the reference location. These quantities are defined in greater detail later in this section. With this value of $n_+$ and the input temperature $T$, we then apply the Saha equation to calculate $n_0$, the number density of neutral atoms:

$$\frac{n_+}{n_0} = \frac{Z_+}{Z_0} \frac{2}{n_e h^2} \frac{1}{(2\pi m_e kT)^{3/2}} e^{-\chi_i/kT} \left(2.2.1\right)$$
In this equation, \( n_e \) represents the number density of free electrons, \( m_e \) the electron mass, and \( k \) the Boltzman constant. \( Z_+ \) and \( Z_0 \) represent the partition functions of the ion and neutral atom, respectively, and \( \chi \) is the ionisation potential. From the calculated \( n_0 \) value, we obtain the ionisation fraction \( x = n_+/n_{\text{tot}} \) and finally the opacity of the CSM, \( \kappa = 0.4x \). By doing this, we assume a constant ionisation fraction and opacity throughout the CSM, which simplifies the Monte Carlo calculations described below. The code does not take into account any expansion of the CSM, which is a reasonable approximation for the case of a roughly stationary stellar wind bow shock.

Following the basic MCRT prescription, SLIP emits virtual, initially unpolarized “photons” from the central star (or other photon source) and tracks them as they scatter within the CSM. The code determines a photon’s behaviour by generating weighted random numbers corresponding to known probability distributions that depend on the optical depth \( \tau \) and albedo \( a \) of the scattering region (Whitney 2011). A strength of our implementation is that in addition to the star (or “central source”), SLIP also allows photons to be emitted from within the CSM itself (which we refer to as the “distributed source”). In the distributed emission case, we allow photons to be emitted isotropically from the volume of the CSM. Because the CSM density is not constant (see the discussion of the bow shock implementation below), we use the rejection method to ensure that the number of emitted photons at a given location is proportional to the local CSM density. In the sections below, we investigate the differences between these two emission scenarios.

As photons interact with the scattering region, SLIP performs the numerical optical depth integration described in Code and Whitney (1995) and Whitney (2011). After each integration, a random number compared with the photon’s albedo determines whether it scatters or becomes absorbed; the photon’s Stokes parameters are updated after each scattering event by applying the standard Mueller matrix mul-
tiplication (Chandrasekhar 1946; Code and Whitney 1995; Whitney 2011). Once a photon exits the simulation (i.e., it “escapes”), its Stokes parameters are combined with those of all previously tracked photons in the appropriate output bin corresponding to the observer’s viewing angle. A single SLIP run produces results for all viewing angles \((i = 0^\circ − 180^\circ)\). Within each output bin, we sum the Stokes vectors due to all \(N\) photons in the bin and apply normalisation factors in \(\theta\) and \(\phi\) to ensure that output fluxes have the correct units. We determine the 1\(\sigma\) uncertainty for each Stokes parameter in each bin by calculating the standard deviation of that parameter over all \(N\) photons in the bin and normalising it to \(\sqrt{N}\) to account for the Poisson statistics of this counting experiment (Wood et al. 1996b; Whitney 2011).

For simplicity, in this paper we consider electron (Thomson) scattering only, both for the case of pure scattering (albedo \(a = 1\)) and for the case of scattering plus hydrogen absorption \((a < 1)\). Although SLIP has the capability to simulate polarized spectra, because electron scattering is a gray process, our results are monochromatic for the pure-scattering case. That is, these results are comparable to polarization observations at any wavelength. When we consider hydrogen absorption, we choose a representative optical wavelength of 6040 Å and discuss how absorption effects modify the pure-scattering results. At higher temperatures for which our calculated ionisation fraction is very close to 1, these electron-scattering scenarios simulate a fully ionised environment such as a region of shocked gas. This focus on electron scattering and single bow-shock structures distinguishes the simulations in this paper from those of Shahzamanian et al. (2016). In Paper II, we will present wavelength-dependent dust-scattering results from SLIP and compare them with the bow-shock contribution to the polarization of the DSO as calculated by Shahzamanian et al.

Rather than simulating a particular object (as in Neilson et al. 2013 and Shahzamanian et al. 2016), our goal here is to understand the polarization produced by
Figure 2.1: Cross-section of our model geometry, along with a depiction of the bow shock density as a function of angle (greyscale). The star is at the origin and moving in the direction of the arrow (+z). The central green solid line represents the central radius of the bow shock, which in our models we define with the Wilkin analytical solution (Eq. 2.2.3). Due to the difficulty of representing this equation graphically, in this figure we have used a graphical approximation of this function; however, the greyscale image is a discretisation of the actual Wilkin equation. The red and blue outer dashed lines represent our adopted inner and outer CSM radii, separated by a constant radial thickness $f$ as described in Section 2.2. The density decreases from the bow head toward the wings of the shock (Eq. 3.2.4); we adopt an exponential decline in density in the far wings of the shock (Eq. 3.2.5). The central source is shown exaggerated in size for reference. The angle $\theta$ is the polar angle measured from the +z axis in our model grid, while the angle $i$ is the inclination or viewing angle for a distant observer.
electron scattering within a generalised bow shock. Thus, to describe our scattering region, we adopt the Wilkin (1996) analytic model of an axisymmetric bow shock formed when a star drives a wind into the stationary ISM while also moving along a straight line. This formulation assumes a spherically symmetric stellar wind and a locally uniform ISM. The resulting bow shock structure and properties depend on the properties of the stellar wind, the speed of the star through the ISM, and the local ISM density. The solution provides for the shape, mass surface density, and velocity flow in an infinitesimally thin axisymmetric bow shock. The essential properties of this solution are the standoff radius of the bow head, the opening angle of the bow shock, and a characteristic surface density for the structure.

The standoff radius \( R_0 \) is defined as the location along the star’s trajectory at which the ram pressures of the ISM and stellar wind are equal, i.e., \( \rho_w V_w^2 = \rho_I V_*^2 \). Here \( \rho_w \) represents the density of the stellar wind, \( V_w \) the stellar wind velocity, \( V_* \) the stellar velocity, and \( \rho_I \) the ISM density. With the stellar mass-loss rate represented by \( \dot{m}_w \), this condition yields

\[
R_0 = \sqrt{\frac{\dot{m}_w V_w}{4\pi \rho_I V_*^2}}
\]

(Wilkin 1996). Using momentum conservation and force balance, the bow shock radius as a function of polar angle is then given by

\[
R(\theta) = \sqrt{3} R_0 \csc \theta \sqrt{1 - \theta \cot \theta}
\]

We use this equation to define the central radius of our model bow shock structure (Fig. 2.1). As described in § 2.4, we choose \( R_0 \) to give a convenient scale to our simulations. Note that at \( \theta = \pi/2 \), the extent of the bow shock is \( \sqrt{3} R_0 \).
Wilkin (1996) also determined the mass surface density $\sigma$ of the idealized, infinitesimally thin bow shock shell as a function of polar angle using conservation of momentum:

$$\sigma(\theta) = \frac{1}{2} R_0 \rho_I \frac{[2\alpha(1 - \cos \theta) + \tilde{w}^2]^2}{\tilde{w} \sqrt{(\theta - \sin \theta \cos \theta)^2 + (\tilde{w}^2 - \sin^2 \theta)^2}}.$$  \hspace{1cm} (2.2.4)

Here $\tilde{w}$ is a convenient parametrization defined by $\tilde{w}^2 = 3(1 - \theta \cot \theta)$. In the wings of the bow shock, $\tilde{w} \gg 1$, giving $\sigma \propto \tilde{w}$. The symbol $\alpha$ parametrizes the ratio of the translational speed of the star to its stellar wind velocity ($\alpha = V_\ast/V_w$); in principle, the Wilkin (1996) model is valid only for $0 < \alpha < 1$. When $\alpha = 0$, the stellar wind forms a spherical bubble and the standoff radius is undefined, whereas $\alpha > 1$ means the star is travelling faster than its wind. For hot, massive stars with radiation-driven winds (Lamers and Cassinelli 1999), the wind velocity is much faster than that of the star, so that $\alpha \ll 1$. On the other hand, for cool stars, the wind velocity can be slow relative to that of the star. For instance, the value of $\alpha$ for the O star $\zeta$ Pup is 0.1 (Puls et al. 1996), while for Betelgeuse $\alpha$ is close to unity (Mackey et al. 2012). In our models, we assume $\alpha = 0.1$ to represent the hot-star case.

Within SLIP, it is not possible to encode an infinitesimally thin shell geometry with a divergent surface density. Instead, we construct a finite scattering region that reproduces the mass surface density function from Equation 2.2.4. As noted above, we define the shock’s mid-region with the Wilkin shape (Eq. 2.2.3). Then we calculate the volume density necessary to match the Wilkin mass surface density (Eq. 2.2.4) via $\rho(\theta) = \sigma(\theta) b(\theta)/\Delta R(\theta)$, where $\Delta R(\theta)$ is the radial thickness of the finite bow shock region. Here $b(\theta)$ is a geometrical correction factor arising from the $\theta$ dependence of the bow shock’s radius; we discuss this factor in detail in Appendix A.

Parametrising the CSM thickness with the fractional quantity $f$ (where $f$ is constant over the shape and $0 < f < 1$), we calculate $\Delta R(\theta)$ as follows:
\[ \Delta R(\theta) = R_{\text{out}}(\theta) - R_{\text{in}}(\theta) \equiv f R(\theta). \]  

(2.2.5)

In this equation, \( R(\theta) \) is the radius of the bow shock at the interface of the ISM and stellar wind, given by Eq. 2.2.3, \( R_{\text{in}}(\theta) \) is the inner radius of the finite structure, and \( R_{\text{out}}(\theta) \) is the outer radius. Approximations to these three functions are depicted as coloured lines in Fig. 2.1, while the actual discretised density is shown in greyscale. For a given value of \( \theta \), \( R_{\text{in}} \) and \( R_{\text{out}} \) are equidistant from \( R_0 \).

We checked how changing the radial thickness \( \Delta R(\theta) \) affects the simulated polarization signatures in the case of pure scattering \( (a = 1) \). For values ranging from \( f = 0.1 \) to \( f = 0.5 \) (representing physically thin shells), we found insignificant variation in the polarization behaviour at any viewing angle. Thus, in our simulations, we assume \( f = 0.25 \), which ensures the thickness of the shell is at least one grid cell within the code structure.

With the definitions above, the volume density within our scattering region is given by

\[
\rho(\theta) = \frac{R_0 \rho_I(\theta)}{2\Delta R(\theta)} \left\{ \frac{[2\alpha(1 - \cos \theta) + \bar{\omega}^2]^2}{\bar{\omega}^2(\theta - \sin \theta \cos \theta)^2 + (\bar{\omega}^2 - \sin^2 \theta)^2} \right\}. \quad (2.2.6)
\]

In the models presented here, we vary the density of the CSM by using as an input parameter the optical depth at a convenient arbitrary reference angle, \( \theta_0 = 1.76 \text{ rad} = 95.4^\circ \). We refer to this reference optical depth as \( \tau_0 \) and scale \( \rho(\theta_0) \) to match it (effectively choosing \( \rho_I \) to give the desired \( \tau_0 \)). We then use Eq. 2.2.6 to determine the density for other values of \( \theta \). This results in a CSM density that is nearly, but not exactly, constant with \( \theta \) (Fig. 2.2). We then calculate \( \tau(\theta) \) based on the density and thickness of the CSM. The variation of density and optical depth as a function of polar angle can be seen in Fig. 2.2. The increase in optical depth with \( \theta \) is due to the increasing behaviour of both \( \sigma(\theta) \) (Eq. 2.2.4; (see discussion
in Wilkin 1996) and $b(\theta)$ (Appendix A). To maintain a finite simulation size, we truncate the bow shock for large values of $\theta$ as described in Section 2.4 below.

In the geometry of Fig. 2.1, $+Q$ Stokes vectors correspond to equatorial scattering, vertical polarization vectors (i.e., in the $\pm z$ direction), and polarization position angles near $\Psi = 0^\circ$. Negative or $-Q$ Stokes vectors correspond to polar scattering, horizontal polarization vectors (i.e., in the plane orthogonal to $\pm z$), and position angles near $\Psi = 90^\circ$. Stokes $\pm U$ denotes diagonal polarization vectors rotated $45^\circ$ from the $\pm Q$ vectors. (In our axisymmetric models, $U$ averages to zero for unresolved cases.) Because we consider only electron scattering, a symmetric process, our models produce no Stokes $V$ (circular) polarization. Thus, the fractional polarization $p$ (usually expressed as a percentage) is defined as

$$
p(\%) = \frac{\sqrt{Q^2 + U^2}}{I} \times 100. \tag{2.2.7}
$$

### 2.3 Results from analytical model

Before embarking on a parameter study using the MCRT methods of the SLIP code, we first consider semi-analytic results for scattering within a bow shock in the optically thin limit. Because the stellar wind bow shock of Wilkin (1996) is explicitly axisymmetric, the methods of Brown and McLean (1977) can be used to determine its expected polarization as a function of viewing angle in the spatially unresolved case.

Brown and McLean (1977) derived a simple expression for the linear polarization from an axisymmetric and optically thin scattering region illuminated by a central point source. Considering scattered light only, the fractional polarization can be expressed as

$$
p(\%) = \frac{\sqrt{Q^2 + U^2}}{I} \times 100. \tag{2.2.7}
$$
Figure 2.2: Variation in mass density ($\rho$ [g cm$^{-3}$]; black points, right-hand axis) and local normalised optical depth ($\tau/\tau_0$; red points; left-hand axis) as a function of polar angle $\theta$. For each model, we specify the optical depth $\tau_0$ at the reference angle $\theta_0$ (dashed lines; § 2.2). The discrete nature of the optical depth is due to the distribution of the analytical bow shock shape across model grid cells. The behavior of the optical depth shows that the average number of scattering events per photon increases slowly with $\theta$ up to the cutoff angle (§ 2.4) and decreases rapidly thereafter.
\[ p = \frac{\sin^2 i}{h(\gamma) + \sin^2 i}, \]  
(2.3.1)

where \( i \) is the viewing angle measured from the z-axis as shown in Fig. 2.1, \( \gamma \) is a “shape factor” to be discussed below, and \( h(\gamma) = 2(1 + \gamma)/(1 - 3\gamma) \). Brown & McLean use the symbol \( \alpha \) in the expression for \( p \) (their Eqn. 17), but we choose to define \( h(\gamma) \equiv 2\alpha \) because we have already introduced a different \( \alpha \) in the context of the bow shock geometry.

The shape factor \( \gamma \) is given by

\[ \gamma = \frac{\int_{r=0}^{\infty} \int_{\mu=-1}^{1} n(r, \mu) \mu^2 dr d\mu}{\int_{r=0}^{\infty} \int_{\mu=-1}^{1} n(r, \mu) dr d\mu}, \]  
(2.3.2)

where \( \mu = \cos \theta \) (with \( \theta \) representing the polar angle measured from the z-axis; Fig. 2.1) and \( n(r, \mu) \) is the number density of the scattering region (Brown and McLean 1977). Values of \( \gamma \) range from 0 to 1, with \( \gamma = 1/3 \) representing a spherical envelope, \( \gamma = 0 \) a planar disk, and \( \gamma = 1 \) a bipolar jet. These geometries produce maximum polarization values (at viewing angles of 90°) of 0%, 33%, and 100% respectively. In the specific case of the Wilkin model, we have

\[ n(r, \mu) = \frac{\sigma(\mu)}{\Delta R(\mu)}. \]  
(2.3.3)

When we substitute our expressions for \( \sigma \) from Eq. (2.2.4) and \( \Delta R \) from Eq. (2.2.5) into Eq. 2.3.3, and then put the resulting expression for \( n(r, \mu) \) into Eq. 2.3.2, we determine the shape factor \( \gamma \) for our modified Wilkin bow shock. Because the bow shock is not a closed shape, we take the angular integrals from \( \theta = 0° \) to \( \theta = 131° \) only. The resulting \( \gamma \) factor depends only on \( f \), the fractional thickness of the shell, and \( \alpha \), the velocity ratio (both defined in Section 2.2). Numerical evaluation of the integrals in Eq. 2.3.2 for \( f = 0.25 \) and values of \( \alpha \) between 0.1 and 10 yields \( \gamma \approx 0.241 - 0.295 \). Corresponding values of \( h(\gamma) \) range from 8.96 to 22.52.
Given these generally large values of \( h(\gamma) \), we expect that for low scattering optical depths, the polarization should scale with viewing inclination as \( p \propto \sin^2 i \), which is symmetric about \( i = 90^\circ \). For representative values of \( \alpha = 0.1 \) and \( h(\gamma) = 8.96 \), we conclude that the theoretical electron-scattering polarization for an unresolved bow shock structure is

\[
p(\%) = 11.16 \sin^2 i. \tag{2.3.4}
\]

We constructed a set of SLIP models with \( f = 0.25 \), \( \alpha = 0.1 \), and \( a = 1 \), with photons arising from the central source only, to compare with these analytical results (Fig. 2.3). We considered reference optical depths of \( \tau_0 \leq 0.07 \) only in order to ensure that the average number of scatters per photon was very close to 1. Our simulations show a viewing angle dependence and symmetric behaviour about 90° in agreement with the prediction of Eq. 2.3.4, which serves to verify that our numerical approach is valid. The values arising from the simulation are generally consistent with the analytic model for these optical depths, with small differences attributable to our discretisation of the Wilkin function for the SLIP models. The symmetry about 90° begins to break down slightly as \( \tau_0 \) increases, which is expected given the variation in actual optical depth with viewing angle (Fig. 2.2).

### 2.4 Model predictions from SLIP

In order to perform numerical calculations of the polarization created in a Wilkin bow shock, we must take into account the fact that our simulations involve a grid of finite size, whose maximum extent we set at \( R_{\text{max}} = 6.68 \) AU. Our approach is to modify the density description in the Wilkin (1996) model to accommodate our finite grid. We use the density of the bow shock as prescribed by Eq. 2.2.6, up to a certain cutoff angle \( \theta_c \). For \( \theta > \theta_c \), we assume the bow shock density declines exponentially
Figure 2.3: Fractional polarization (with respect to scattered light only) as a function of optical depth at the standoff radius ($\tau_0$) for SLIP models of an optically thin, unresolved bow shock viewed at $i = 90^\circ$ (gold), $i = 75^\circ$ and $105^\circ$ (red), and $i = 45^\circ$ and $135^\circ$ (blue). Horizontal lines represent the analytical prediction (symmetric about $i = 90^\circ$) for each angle (Eq. 2.3.4). Our numerical simulations reproduce the theoretical predictions well, with some expected deviation from symmetry at larger optical depths. Error bars representing 1σ uncertainties in each model bin (§ 2.2) are smaller than the plotted symbols.
rather than being sharply truncated by the outer limit of our simulation (which we found resulted in spurious polarization at the edges). This modified density in the wings of the bow shock is given functionally by

$$\rho(\theta > \theta_c) = \rho(\theta_c, \infty) \exp\left[-\frac{(\theta - \theta_c)}{\delta \theta_0}\right],$$

(2.4.1)

where $\delta \theta_0$ is a constant angle governing the steepness of the density decline.

This modification of the Wilkin density structure does not affect the accuracy of our results, for two reasons. First, an infinitesimally physical thin shell is not physically realistic, especially at large distances from the bow head, as the shell must spatially “thicken” with distance by virtue of gas pressure gradients and Kelvin-Helmholtz instabilities (Mohamed et al. 2012; Mackey et al. 2014). Second, the geometry for a thin shell ensures that with increasing distance from the star, the solid angle subtended by a shell ring (i.e., a ring about the symmetry axis) decreases with distance. As a consequence, from the perspective of scattering stellar photons, the large-scale wings of the bow shock offer a diminishing cross-section for intercepting and scattering starlight. This also means that the increasing size of the grid cells at larger radii does not significantly affect our results.

We investigated the impact of the cutoff angle $\theta_c$ and the steepness $\delta \theta_0$ of the exponential decay function on polarization by varying both parameters in our simulations. We emphasize that in these and all our subsequent models we measure fractional polarization with respect to the total light, rather than scattered light only (as in Eqns. 2.3.1 and 2.3.4).

In testing the effects of $\theta_c$ and $\delta \theta_0$, we used the central photon source with reference optical depth of $\tau_0 = 0.5$ and a CSM temperature $T$ of 10,000 K. For an unresolved bow shock, we found that as the cutoff angle increases, the peak polarization value and the variation of polarization with viewing angle $i$ is nearly unchanged. We thus chose a convenient value of $\theta_c = 2.1$ rad ($122^\circ$) as the cutoff.
angle for all the other models presented in this paper. This choice for $\theta_c$ ensures that the entire CSM structure is included within our simulation grid. All the values we tested for $\delta \theta_0$ resulted in similar polarization values and behaviour. We chose $\delta \theta_0 = 0.3 \text{ rad (17}^\circ\text{)}$ for all the models shown hereafter.

We also tested the behaviour of the polarization in our simulations as a function of $\alpha$, the velocity ratio defined in Section 2.2. Fixing the albedo of the scattering region at $a = 1$, emitting photons from the central source, and using the same values of $\tau_0$ and $T$ as in our previous test cases, we found that as $\alpha$ increases, the polarization value increases as well. From Equation 2.2.6, we see that with a given thickness function $\Delta R(\theta)$, the volume density $\rho$ increases with $\alpha$ for angles greater than $\theta = 0$. Thus, increasing the value of $\alpha$ should have a similar effect to increasing the optical depth $\tau_0$ for $a = 1$, which does indeed increase polarization overall (Section 2.4.1). For the simulations presented below, we set $\alpha = 0.1$ as discussed in Section 2.2.

Finally, we studied how changing the standoff radius $R_0$ of the bow shock changes the polarization behaviour. When the albedo $a$ is fixed at 1 (the pure scattering case), changing $R_0$ does not affect the polarization.

However, when the albedo is not explicitly fixed (the case of scattering with absorption), changing the standoff radius changes the albedo and thus the polarization. This is because $R_0$ is used to calculate the physical thickness $\Delta R(\theta)$ of the bow shock (Eqs. 2.2.3 and 2.2.5), which in turn affects its opacity. When $a$ is not fixed, it is calculated using the opacity of the region (§ 2.4.2): a larger value of $R_0$ corresponds to a lower density for a given $\tau_0$, which leads to a larger opacity and a lower albedo.

We chose $R_0 = 1.4 \text{ AU}$ for all our models, because for variable $a$ this $R_0$ value produces polarization behaviour as a function of viewing angle similar to the analytical results in the optically thin case (Section 2.3). (For comparison, the radius of
our central source is $1R_\odot \approx 0.005$ AU; this value has no physical significance other than to make the central star effectively a point source.) With $R_0 = 1.4$ AU and $R_{\text{max}} = 6.68$ AU, the density within the bow shock goes to zero between $\theta = 134^\circ$ and $\theta = 140^\circ$ (where the bow shock radii intersect the boundary of the simulation).

To create our numerical simulations, we used the University of Denver’s high-performance computing cluster (HPC), which consists of 180 Intel Xeon processors running at 2.44 GHz. Each of our model runs used 16 CPUs with $10^8$ photons per CPU. This yielded polarization uncertainties on the order of $\sigma_p(\%) \sim 0.01$. Completing each run took $\sim 60 - 70$ minutes, with slightly longer times for larger values of $\tau_0$. Our simulations can be broadly divided into models assuming pure Thomson scattering with no absorption ($a = 1$) and those including some absorption (variable $a$). In each case, we studied the effect of various parameters on the polarization behaviour for both resolved and unresolved cases. In the resolved cases, we preserve spatial information from our simulations, while in the unresolved cases, we combine all photons from a given viewing angle into a single set of polarization values. We present our results below.

2.4.1 Pure Thomson Scattering

To simulate the case of pure Thomson scattering, we fixed the albedo of the bow shock environment at 1. In this case, all emitted photons scatter in the bow shock and ultimately escape. We explored the dependence of polarization on CSM temperature, standoff radius, and optical depth for both central and distributed photon sources. We found that for a given source, only the optical depth affects the simulated polarization; varying the CSM temperature and standoff radius produced no change in either polarization magnitude or behaviour as a function of viewing angle.
In the rest of this section, we present the detailed behaviour of polarization as a function of optical depth, for both resolved and unresolved scenarios. We investigated three representative optical depths: \( \tau_0 = 0.1, 0.5, \) and 2.0. In all the cases shown here, \( T = 10,000 \) K, \( \theta_c = 122^\circ, \) \( \delta \theta_0 = 17^\circ, \) and \( \alpha = 0.1. \) In all these simulations, we found polarization position angles very close to \( \Psi = 0^\circ, \) so we have not displayed the position angle results.

**Optical depth dependence – resolved bow shock**

In Fig. 2.4, we display the intensity, percent polarization, and polarized intensity images for a resolved bow shock with three different optical depths at two representative inclination angles symmetric around the \( z = 0 \) plane, 55\(^\circ\) and 125\(^\circ\). (Polarized intensity is calculated by multiplying \( \%p \) by intensity; in these maps it represents the polarized light arising from the system.) In the central-source cases (left column), the intensity maps show only a small dot at the location of the star due to our choice of a linear intensity scale that shows the distributed-source behavior well. The scattered light from the bow shock contributes intensity too faint to be seen on this scale.

The central-source polarization maps are similar for the two symmetric inclination angles; they show a generally elliptical polarization pattern, which is created by the combination of all 90\(^\circ\) scattering paths, as shown schematically in Fig. 2.5. For a given inclination angle, the overall polarization magnitude decreases with increasing optical depth, which is generally expected given that multiple scatters typically randomise the polarization of an ensemble of photons. For a given optical depth, the polarization near the bow head is smaller for the larger inclination angle. Figure 2.5 shows that the path length for photons scattering at 90\(^\circ\) near the bow head at the lower inclination angle (panel \( b, \) paths 1 and 2) is much smaller than in the case of the higher inclination angle (panel \( c, \) paths 1 and 2). Because of this,
multiple scattering is more important for higher inclinations and optical depths. In this case, because the outgoing photons scatter in the same plane, the dominant effect of multiple scattering is to remove polarized photons from the beam rather than randomising their position angles. This effect can be seen in the decrease of polarized intensity with inclination angle in the lower panels (Fig. 2.4).

The central-source polarized intensity maps show that the majority of scattered photons reach our line of sight from locations near the bow head; the scattering material is very tenuous in the outer regions, so very few photons scatter there (but those that do become highly polarized in the process). We note that although the resolved maps look similar in polarization between the two angles, they are quite distinct in polarized intensity, particularly at higher optical depths. This suggests that polarized intensity maps may provide an observational tool for constraining bow shock inclinations.

In the distributed-source case (photons arising only from the CSM; right side), Fig. 2.4 shows that the total intensity is concentrated near the bow head because the CSM density is higher in that region and thus more photons are emitted from there. In this case, photons are emitted with an isotropic distribution of initial directions from within the volume of the CSM. Thus, photons scatter more times on average than in the central-source model with the same input parameters. This increased scattering, combined with cancellation from neighbouring photon origins and the contribution from “surface” photons (those arising from the outer edge of the bow shock) that reach the observer directly, causes a significant decrease in the polarization arising from any given location in the CSM, compared with the central-source case (middle panels of Fig. 2.4). The polarization is highest at the edge of the CSM because of a scattering asymmetry. In most parts of the CSM, polarization angles are highly randomised, so photons that reach the viewer can have any polarization angle. However, limb photons cannot scatter in all directions and thus tend to have
a preferred polarization angle. The difference in polarization morphology between central-source and distributed-source models suggests that observational polarization maps (such as those of Rauch et al. 2013) can be useful for constraining the photon origin and thus the relative brightnesses of the star and the CSM.

By contrast, the distributed-source polarized intensity maps look very similar to those produced by the central-source models and show similar variations with inclination and optical depth. Thus, observed polarized intensity maps would not be able to distinguish reliably between photons emitted from the central star and photons emitted from the bow shock.

**Optical depth dependence – unresolved bow shock**

In Fig. 2.6, we display the polarization variation as a function of viewing angle for the unresolved case, considering four different values of the reference optical depth $\tau_0$. For both central and distributed emission cases, all models show a primary peak in percent polarization at an inclination angle of $90^\circ$, as well as a secondary peak at angles greater than $130^\circ$ whose exact location depends on $\tau_0$.

In Fig. 2.6, the maximum polarization occurs at an inclination angle of $90^\circ$ for all optical depths and both photon sources. This can be understood in terms of the analytical models of Brown and McLean (1977), who showed that for the optically thin case, the polarization produced by scattering in an axisymmetric envelope is proportional to $\sin^2 i$.

For higher $\tau_0$ values, however, our models depart from the theoretical $\sin^2 i$ dependence of the polarization, particularly at higher viewing angles. As the optical depth increases, the secondary peak becomes enhanced with respect to the primary peak, and even exceeds it at larger optical depths than we display here. (We tested a range of $\tau_0$ values to establish this behaviour, but only display a few in Fig. 2.6 for clarity.) We hypothesize that this effect is due to multiple scattering becoming
Figure 2.4: Intensity, polarization, and polarized intensity maps for resolved bow shocks illuminated by a central source (left) and the distributed source (right; photons arise from within the CSM as described in § 2.2). In the central-source intensity maps, arrows indicate the location of the star. We show two inclination angles symmetric about 90°. Optical depth increases from left to right in each row. Intensities are in arbitrary units.
Figure 2.5: Sketch showing the 90° scattering paths for central-source photons at four different viewing angles $i$. The numbered arrows indicate the limiting paths that produce negative $q$ polarization as seen by an observer in the $i$ direction (polar scattering). In each panel, there will also be 90° scattering paths for photons initially directed out of the page, defining the width of the scattering ellipses; these paths, which produce positive $q$ polarization (equatorial scattering), are not shown in the sketch. Dashed lines indicate the direction to the observer; short dotted segments mark the location of the density falloff in the wings of the bow shock (Section 2.4). Small coloured images for each inclination angle depict the distribution of $q$ polarization as seen by the observer, for $\tau_0 = 0.1$ (left) and $\tau_0 = 2.0$ (right). The colours range from $-100\%$ (darkest blue) to $+100\%$ (darkest red).
more common at higher optical depths. In order to understand the effect of multiple scattering on the polarization behaviour, we created central-source and distributed-source simulations for $\tau_0 = 0.5$ and $\tau_0 = 2.0$ in which we disaggregated the results by number of scatters; we display the results in Fig. 2.7. Indeed, we see from this figure that the singly scattered photons is consistent with the theoretical $\sin^2 i$ dependence (with a slight “shoulder” at low $\tau_0$ due to the onset of the density falloff; Eq. 2.4.1. Other slight departures from the idealised function are due to the discretisation effects discussed in § 2.3). The multiply scattered photons diverge from this behaviour more strongly as $\tau_0$ increases, particularly at larger viewing angles where the path length through the CSM is longer (Fig. 2.5).

We also see that the overall width of the polarization curve decreases for larger numbers of scatters (Fig. 2.7), particularly at higher optical depths. We attribute this to the increasing contribution from scattering paths producing negative $q$ (“polar scattering”) polarization in these cases. (Stokes $u$ is zero on average for these axisymmetric models, so $q$ is the dominant contributor to the total polarization $p$.) In the central-source case, the scattering paths producing positive $q$ polariza-
tion ("equatorial scattering") have a constant average initial (pre-scattering) path length through the CSM independent of viewing angle; thus the \( \pm q \) polarization varies as \( \sin^2 i \) due to projection effects. (These positive-\( q \) paths are not shown in Fig. 2.5: they initially run from the central source directly out of the page, then scatter toward the observer in the direction indicated by the arrows. They create the red regions in the inset \( q \) maps.) By contrast, the negative-\( q \) paths shown in Fig. 2.5 have path lengths through the CSM that vary with inclination angle, and these are longer than the \( \pm q \) paths for most angles. This means that increasing optical depth results in a higher magnitude of \(-q\) polarization, as shown explicitly in Fig. 2.8. With no absorption, more photons scatter into other lines of sight, while the few that escape toward the observer have scattered multiple times in the same plane and are thus more highly polarized (as discussed in Wood et al. 1996b). On the other hand, higher optical depths and more scatters produce more negative \( q \) polarization and smaller values of \( p \) in Fig. 2.7. For the viewing angles with negative \( q \) values, the polarization position angle \( \Psi \) flips from 0° to 90°.

We therefore conclude that the secondary peak near \( i = 130° \) in the unresolved, central-source models with higher optical depths (Fig. 2.6) is caused by a strong increase in \(-q\) polarization when multiple scattering becomes important. Most of our models also show a polarization peak near 150° due to the fact that at this angle, the line of sight no longer intersects the near side of the CSM because of our simulation boundary (§ 2.4). In this case, the path lengths that pass through the near side of the CSM are very long, so almost no photons escape there; the resulting polarization is primarily due to photons that are singly scattered from the interior far wall of the CSM (path 3 in Fig. 2.5, panel \( d \)).

In the distributed case, the polarization predominantly arises from the limb of the bow shock and from the wings farthest from the bow head (Fig. 2.4). Photons from the limb tend to produce \(+q\) polarization (in addition to some \( u \), which can-
cells out in the unresolved case) because they are most likely to reach the observer by singly scattering near the edge of the CSM, producing the familiar tangential polarization pattern. Photons arising from the plane facing the observer produce zero net polarization because they are equally likely to escape after scattering in any direction, and thus cancellation is high. In the wings, however, this symmetry breaks due to the density falloff; in this case photons are most likely to escape after singly scattering in the regions farthest from the bow head, producing negative $q$ values.

For the unresolved distributed models (Fig. 2.6), the polarization as a function of viewing angle behaves very similarly to the case of the central-source models, as expected because the bow-shock geometry of the CSM is the same between the two cases (Brown and McLean 1977). We see the same $\sin^2 \theta$ behaviour, modified by increasing contributions from $-q$ polarization at higher viewing angles (Fig. 2.8) as we see more contribution from the far side of the bow shock. The secondary peak in the distributed case occurs at larger viewing angles than in the central-source case because the CSM density falloff translates into fewer photons emitted from those angles.

Interestingly, although the central-source and distributed models show very similar polarization behaviour as a function of optical depth (Fig. 2.6), they behave quite differently as a function of number of scatters for a given optical depth (Fig. 2.7). In the distributed models, multiple scattering increases the polarization over single scattering at intermediate viewing angles. We attribute this effect to the fact that polarization in the distributed cases arises primarily from the limb, where column densities are high. Although this polarization is likely dominated by singly scattered photons originating near the outer surface, a few multiply scattered photons reaching us through the dense material at the limb can create large polarization percentages due to scattering in the same plane (Wood et al. 1996b). For higher
optical depths and more scatterings, however, the two emission cases become quite similar, as expected once the photon source becomes “forgotten.”

In Fig. 2.9, we compare the variation of polarization with optical depth for three different inclination angles and the two photon sources. As expected based on previous results, the central-source and distributed cases show similar behaviour. For the lower viewing angles, we see the “peaking” effect described by Wood et al. (1996b), in which polar scattering begins to dominate over equatorial scattering for higher optical depths. At $i = 45^\circ$, the polarization magnitude is relatively low for all $\tau_0$ values due to large contributions from $-q$ scattering paths (Fig. 2.5). At $i = 90^\circ$, the location of the first polarization peak in all our models, the polarization
Figure 2.8: Percent Stokes $q$ polarization as a function of inclination angle for four different values of optical depth $\tau_0$, for photons arising from the central source (left) and from the CSM (distributed source; right). Black points and lines represent optically thin cases, while red points and lines represent higher optical depths. Red dotted lines represent the theoretical $\sin^2(i)$ function normalised to the peak of the $\tau_0 = 2.0$ curves. Error bars representing $1\sigma$ uncertainties in each model bin (§ 2.2) are smaller than the plotted symbols. Positive values of $q$ correspond to polarization position angles of $\Psi = 0^\circ$, while negative values correspond to $\Psi = 90^\circ$.

Figure 2.9: Polarization as a function of optical depth $\tau_0$ at three different inclination angles (labelled in degrees), for photons arising from the central source (left) and from the CSM (distributed source; right). Error bars representing $1\sigma$ uncertainties in each model bin (§ 2.2) are smaller than the plotted symbols.
is a maximum for all optical depths due to the loss of paths 3 and 4 combined with a very short path length through the CSM at the bow head for paths 1 and 2 (which allows more photons to escape without scattering). At $i = 130^\circ$, the location of the second polarization peak for the central-source case, the behaviour is quite different: our models show a dramatic increase in polarization magnitude as a function of optical depth for $\tau_0 > 1$, with central-source models increasing more steeply than distributed models. At this inclination angle, the path lengths for scattering producing $-q$ polarization are at their longest (Fig. 2.5c); increasing optical depth increases the number of scatterings photons undergo in the same plane, while filtering out photons with lower polarization; this increases the $-q$ contribution as discussed above. Hence, the polarization increases with increasing optical depth, and the effect is more pronounced for the central-source models because the path lengths through the CSM are longer in these cases.

Our results can be used along with observational data to constrain the inclination angle and optical depth of a given bow shock nebula, assuming electron scattering is the primary polarizing mechanism. An unresolved bow shock would be observed at a single value of $i$ and $\tau_0$. Once corrected for interstellar polarization (and for orientation on the sky in the case of $q$, e.g. via proper motion measurements), observed values of $p$ and $q$ for such an object would yield horizontal lines in Figs. 2.6, 2.8, and 2.9. These lines would nearly always intersect the model curves in at least two places for Figs. 2.6 and 2.8, but this would place limits on the possible values of the inclination angle, especially in cases where the optical depth can be estimated from other measurements. Also, if the observed Stokes $q$ parameter were negative, we could say based on Fig. 2.8 that the bow shock was optically thick and viewed at an inclination angle greater than $90^\circ$. With an observed value of $p$, using Fig. 2.9 we could constrain the inclination angle if we had spectral information that probed the
CSM optical depth, or constrain the optical depth if we had radial and transverse velocity information that limited possible inclination angles.

2.4.2 Thomson Scattering with Absorption

In this section, we investigate cases in which the albedo \( a \) of the CSM is not unity (that is, at each interaction, photons have a chance of being absorbed rather than scattering). The \textit{SLIP} code can assign a user-specified albedo to the scattering material, but it also has the capability to calculate a self-consistent albedo using the input temperature and optical depth. In our simulations, the CSM is composed of pure hydrogen, both ionized and neutral. Thus, in the case of variable albedo, we assume photons may be absorbed by hydrogen atoms via both bound-free and free-free processes. The resulting absorption opacity is a function of photon wavelength. Although \textit{SLIP} can consider any range of wavelengths, for simplicity we assume a single optical wavelength of 6040 Å; this represents an intermediate value in the hydrogen opacity curve and avoids absorption edges. With this wavelength, the combinations of temperature and optical depth we consider give rise to albedo values that span the possible range from 0 to 1 (Table 2.1).

When we allow the albedo to vary, we first calculate the hydrogen absorption opacity \( \kappa_H \) for 6040 Å via Eq. 2 in Wood et al. (1996a). Using the ionisation fraction \( x \) found as above in § 2.2, we then set the albedo to be the ratio of scattering to total opacity: \( a = 0.4x/(0.4x + \kappa_H) \). Because we assume \( x \) to be constant throughout the CSM for computational simplicity, \( a \) is constant also. Table 2.1 presents the calculated albedo values for different temperatures and optical depths for our assumed wavelength of 6040 Å. For a given optical depth, the albedo increases with CSM temperature. In the subsections below, we discuss our model predictions of the polarization behaviour as a function of optical depth and temperature when
Table 2.1: Albedo values calculated by SLIP when $a$ is not constrained to be 1, for an assumed wavelength of 6040 Å and different CSM temperatures and reference optical depths (§ 2.4.2).

<table>
<thead>
<tr>
<th>$\tau_0$</th>
<th>5000 K</th>
<th>8000 K</th>
<th>10,000 K</th>
<th>20,000 K</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.468</td>
<td>0.862</td>
<td>0.927</td>
<td>0.985</td>
</tr>
<tr>
<td>2.0</td>
<td>0.180</td>
<td>0.609</td>
<td>0.761</td>
<td>0.942</td>
</tr>
</tbody>
</table>

the albedo is allowed to vary. As in the pure-scattering case, position angles ($\Psi$) for these models are generally near $\Psi = 0^\circ$.

**Temperature dependence – resolved bow shock**

As the CSM temperature increases, the albedo increases for a constant input optical depth, as shown in Table 2.1. This causes our results to deviate from the pure Thomson-scattering results (§ 2.4.1), especially at lower temperatures.

Fig. 2.10 shows maps of intensity, percent polarization and polarized intensity for two different viewing angles and three different temperatures for $\tau_0 = 0.5$. In the central-source case (*left side*), the scattered intensity is too faint to be seen on this linear scale, as discussed above in § 2.4.1. In this case we also see little change in polarization as the temperature increases (corresponding to increasing albedo; Table 2.1). This is because the overall number of photon interactions is small at this low optical depth. As in the pure scattering case, the polarization near the bow head is lower for the higher viewing angle. In this case, photons are removed from the beam by absorption in addition to scattering, but the result is the same. Polarized intensity is concentrated near the bow head as in the pure scattering case; it increases with increasing temperature as the photons undergo more scattering events relative to absorption events, which increases their likelihood of escaping.

In the distributed case (*right side*), there is little variation in polarization with respect to either temperature or viewing angle, again due to the low number of
Figure 2.10: Intensity, polarization, and polarized intensity maps for resolved bow shocks illuminated by a central source (left) and the distributed source (right) for the case of CSM albedo $a < 1$ (§ 2.4.2) and an optical depth of $\tau_0 = 0.5$. We show two inclination angles symmetric about $90^\circ$. CSM temperature increases from left to right in each row. Intensities are in arbitrary units. In the central-source intensity maps, arrows indicate the location of the star.
interactions. The polarized intensity maps show a very similar behaviour to those of the central-source case, with more polarized intensity at higher temperatures.

When absorption is present, the relation between the polarization and polarized intensity maps for central-source and distributed cases is quite similar to that discussed above for the pure-scattering scenario (§ 2.4.1). As we noted there, the difference in polarization maps suggests a possible observational diagnostic for the CSM:star brightness ratio. By contrast, if we compare the maps including absorption to the corresponding pure-scattering maps in Fig. 2.4 (middle column), we see very little difference, suggesting that polarization observations may not be able to constrain the albedo of the scattering material in cases of low optical depth.

In Fig. 2.11, we present the intensity, polarization, and polarized intensity maps for the case of variable albedo and an optical depth of $\tau_0 = 2.0$. These maps were created using models with the same number of input photons as Figs. 2.4 and 2.10, but look grainy because so many of the emitted photons become absorbed in the case of higher optical depth. Because of the relationship between albedo and temperature (Table 2.1), absorption effects are strongest for $T = 5000$ K (left column of each set).

In the central-source case (left side), we once again find a very small intensity contribution from scattered light (§ 2.4.1). At lower temperatures, the polarization maps show a “dark belt” at mid-latitudes that is not present at higher temperatures. This belt delineates the region of highest optical depth in the CSM, with $\theta$ values slightly less than the cutoff angle (Fig. 2.2; Fig. 2.5). In this region, photons that would normally reach the observer via multiple scattering are instead being absorbed. As the temperature increases, photons are again more likely to scatter at each interaction, so the dark belt disappears. At the higher viewing angle, the polarization is highest in the lower portion of the image. This can be attributed to the increased importance of photons backscattering from the CSM interior (Fig. 2.5, cases c and d), combined with a lower density in the CSM facing the observer. Like
the polarization, the polarized intensity is concentrated towards the lower portion of the image for the higher inclination angle, whereas for the lower angle the polarized intensity is highest near the bow head. These differences are explained by the longer line of sight for higher angles (described in § 2.4.1), which greatly increases the probability of absorption. Polarized intensity increases with temperature, as expected due to the decreasing importance of absorption at higher temperatures.

In the distributed case (right side), the intensity images for the first time show a significant contribution from the interior of the bow shock at higher inclination angles, as emission from the front side is suppressed by absorption. The polarization is more widely distributed across the shape for lower temperatures, but becomes more concentrated near the edges (similar to the cases of pure scattering and absorption at low optical depth) as temperature increases. At lower temperatures, most of the scattered photons become absorbed and very few escape, making cancellation effects less efficient and allowing a polarization signal to arise from regions other than the edges. At higher temperatures, more scatters increase cancellation and we approach previously considered cases. The polarized intensity maps behave similarly for the distributed case as for the central-source case.

Taken together, Figs. 2.4, 2.10, and 2.11 suggest that observational constraints on the temperature of the bow shock (in cases where electron scattering dominates) may be possible, but only in cases of higher density/optical depth. For less dense shock structures, the resolved polarization and polarized intensity maps appear similar whether or not absorption is included. However, at higher densities, new features appear when absorption is important, such as the dark belt in polarization and the interior of the shock cone in intensity and polarized intensity. These features could serve as temperature and density indicators in actual observations.
Figure 2.11: As in Fig. 2.10, but for $\tau_0 = 2.0$. “Ringing” patterns are not physical, but rather due to the discrete model grid (Fig. 2.2).
Figure 2.12: Polarization as a function of inclination angle for an unresolved bow shock with different CSM temperatures, for the case of CSM albedo $a < 1$ (§ 2.4.2). Photons arise from the central source (left) or from the CSM (distributed-source; right). Low optical depths are shown in the top row and higher optical depths in the bottom row. Error bars representing 1σ uncertainties in each model bin (§ 2.2) are smaller than the plotted symbols.
Temperature dependence – unresolved bow shock

In Fig. 2.12, we display the polarization variation as a function of viewing angle for models with absorption in the unresolved case, varying both optical depth (rows) and temperature (columns). In the lower optical depth regime (top row), the increase in albedo with temperature (Table 2.1) causes the degree of polarization to increase at most viewing angles for both central and distributed photon sources. When the albedo is low, photons tend to be absorbed rather than scattered, which lowers the overall degree of polarization (as seen in Wood et al. 1996a). As the albedo increases, photons that have been scattered and thus polarized are more likely to escape the bow shock. Hence we see an increase in polarization for higher temperatures.

At high optical depths, the albedo is generally small, and increases with increasing temperatures (Table 2.1). Thus at lower temperatures, only small numbers of photons can escape from the bow shock, and those that escape tend to be highly polarized. As the temperature increases, more photons can escape without scattering; this decreases the overall fractional polarization value. We see these effects in the case of the optically thick CSM illuminated by a central source (Fig. 2.12, lower left panel), where polarization values are very high (up to 45%) and the peak near 90° is suppressed for all temperatures. There is a prominent second peak near $i = 130^\circ$; as the temperature increases, the degree of polarization decreases at this higher viewing angle. We attribute the suppression of the 90° peak to the combination of higher optical depths and lower albedos, which together increase the chance for a photon to be absorbed. Inspection of the flux characteristics of these models shows that most of the photons escape in the wings of the bow shock, where the optical depth is lower due to our cutoff angle. Thus the secondary peak we discussed in the pure-scattering case (§ 2.4.1) dominates the polarization in these models.
The secondary peak is also prominent in the optically thick, distributed-source cases (lower right), although the polarization values are smaller than for the central-source models because more photons escape directly from near the surface of the CSM. The 90° peak is still present for most temperatures. At $T = 5000$ K, however, only the secondary peak contributes, while the 90° peak is completely suppressed by absorption (Fig. 2.11, right-hand side). The polarization is almost entirely due to photons arising and scattering near the interior surface of the CSM. Because very little polarized intensity arises from the outer surface, in this extreme scenario the secondary peak shifts to a viewing angle of $\approx 110°$, at which the interior first begins to be visible. In the high-density cases for both photon sources, the models with the highest temperatures approach the behaviour of the pure scattering case as $a \to 1$.

**Optical depth dependence – resolved bow shock**

Using Figs. 2.10 and 2.11, we can also assess our resolved results as a function of optical depth. The intensity maps vary significantly with optical depth in the case of the distributed source. At the higher inclination angle, the intensity is concentrated near the bow head for $\tau_0 = 0.5$, whereas for $\tau_0 = 2.0$ the intensity arises primarily from the wings and interior of the bow shock structure.

For all temperatures, the degree of polarization decreases with increasing optical depth. We attribute this behaviour to the decrease in albedo with $\tau_0$ shown in Table 2.1. For the central source at the lower temperature of 5000 K, the “dark belt” effect occurs for higher optical depths only, due to a lower albedo combined with increased photon interactions. For the distributed source, the polarization is primarily concentrated near the edges as in the pure scattering case. However, in the lower-temperature case viewed from $i = 125°$, some polarization arises from the upper portion of the bow shock for $\tau_0 = 2.0$, which is not seen at $\tau_0 = 0.5$. This occurs because when absorption is frequent, cancellation of Stokes vectors cannot
happen for $\tau_0 = 2.0$ as efficiently as in the case of $\tau_0 = 0.5$, so some net polarization remains.

In polarized intensity, the two optical depths produce very different maps. For the central-source case, at $\tau_0 = 0.5$ the polarized intensity is concentrated near the bow head for both viewing angles, while for $\tau_0 = 2.0$ at the higher viewing angle, the polarized intensity is concentrated towards the lower portion. This is because when the density near the bow head is high and $a < 1$, photons have a better chance of being absorbed in those regions. In the lower portion of the map, for $\theta$ values greater than the cutoff angle, the density is much lower; thus most of the photons that are polarized can escape the bow shock. These photons arise primarily from the interior of the shock cone, which is visible at the higher angle. We see a similar effect in the distributed-source case.

Because of these optical depth variations, observed polarized intensity maps can potentially constrain the optical depth of the bow shock material as well as the structure’s inclination angle. Comparison of observed maps with these predictions can also help identify the source of illumination and thus relative brightnesses of star and CSM, as discussed above (§ 2.4.1).

**Optical depth dependence – unresolved bow shock**

We can isolate optical depth-dependent behaviour for unresolved cases by comparing top to bottom panels in Fig. 2.12. For a constant temperature, the location of the polarization peak is different for the two optical depths. In the optically thin case, the peak is near 90° (as predicted by analytic models, e.g. Brown and McLean 1977) for both the central and distributed cases. In the optically thick case, the peak shifts to higher inclination angles for both photon sources. For a constant temperature, increasing optical depth leads to decreasing albedo. Thus, when $\tau_0$ is high, very few photons can escape from the denser central regions of the bow shock.
Instead they escape from higher viewing angles, giving rise to the secondary peaks for higher optical depth.

In the central-source case, the model with $T = 5000$ K and high optical depth produces the highest polarization in any of our models, because it has the lowest albedo. As discussed in Section 2.4.2, this scenario results in a low number of escaping photons (mainly those scattering from the interior surface) and thus high polarization magnitudes. At $90^\circ$, instead of a polarization peak, this extreme case shows a small “notch” that we attribute to the prominence of the “dark belt” discussed in §2.4.2: at edge-on inclinations, this belt will dominate the polarization signal, with very few photons escaping from either the bow head or the interior.

In the distributed-source case, the models evolve from single-peaked to a double-peaked shapes as $\tau_0$ increases. At higher optical depths, the $90^\circ$ peak is suppressed and the secondary peak begins to dominate, due to the fact that scattered photons can more easily escape at higher inclinations once absorption is present. At the lowest temperature, for which the albedo is close to 0, the $90^\circ$ peak completely disappears and the polarization is due entirely to photons arising and scattering near the interior surface of the CSM (§2.4.2).

### 2.5 Observational implications

We close by discussing potential observational implications of the electron scattering results presented here (subject to the model limitations discussed below in §2.6). These are useful as limiting cases and to lay the groundwork for future models that will include both electrons and dust as polarizing mechanisms.

In the case of a resolved bow shock, detailed polarization maps are rare in the literature, so it is not currently possible to compare our image predictions with actual observations. (The observations by Rauch et al. 2013 provide a notable exception, but these authors observed a known dusty source and obtained only 9 polarization
measurements across the bow shock.) Our results show that in future observational
efforts, both polarization and polarized intensity maps may provide useful diagnost-
tics. Polarization maps are relatively insensitive to viewing angle except in the case
where absorption is significant (Fig. 2.11). However, because the differences be-
tween central- and distributed-source models are greatest in polarization (Figs. 2.4,
2.10, and 2.11), these maps may provide information about the relative brightnesses
of source and bow shock. This could lead to more realistic models for individual
stars that consider both central and distributed photon sources (§ 2.6). Polarization
maps can also reveal information about the temperature of the bow shock when
absorption is important. In particular, an observed “dark belt” (Fig. 2.11) would
indicate a relatively low CSM temperature and high density. Polarized intensity
maps can distinguish between two symmetric viewing angles in the case of higher
optical depths (Figs. 2.4 and 2.11). Although we have not presented them here,
SLIP can also produce position angle maps for comparison with observations. The
position angles in our models are consistently $\approx 0^\circ$ for most viewing angles, but flip
to near $90^\circ$ at high inclinations and optical depths when $q$ is negative.

For unresolved bow shocks (or cases in which a bow shock is predicted to exist,
e.g. Neilson et al. 2014), we measure a single polarization value corresponding to
a single viewing angle. This corresponds to a horizontal line in figures such as
Figs. 2.6, 2.8, 2.9, and 2.12. If interstellar polarization can reliably be removed,
this could place constraints on the viewing angle if optical depth can be estimated
(Figs. 2.6 and 2.8), or vice versa (Fig. 2.9). A measurement of a negative value
of Stokes $q$ (accounting for the orientation of the bow shock on the sky, e.g. using
the proper motion of the star) would provide a particularly strong viewing angle
constraint (Fig. 2.8). Finally, a polarization measurement compared with the curves
in Fig. 2.12 could provide constraints on the CSM temperature, particularly at low
optical depths or for centrally-illuminated shocks.
2.6 Conclusions and future work

We investigated the polarization arising from electron scattering within an idealised stellar wind bow shock, for cases of illumination by a central star and self-illumination by the shock region. We studied how different parameters impacted the polarization behaviour for both pure scattering and scattering with absorption cases. As expected, polarization is highly dependent on viewing angle for all models. Multiple scattering significantly modifies the behaviour of the polarization with respect to analytical predictions assuming single scattering. For very low optical depths, our simulations reproduce the analytical $\sin^2 i$ dependence of Brown and McLean (1977), but many of our models show a secondary peak at higher inclination angles attributable to increased $-q$ polarization caused by multiple scattering.

In the case of pure scattering (albedo $a = 1$), we find that the optical depth of the bow shock significantly affects the resulting polarization behaviour, while its temperature does not. In addition, while changing the photon source (light arising from the central star vs. from within the bow shock) does not drastically modify the polarization curves for the unresolved case, it does change the appearance of the polarization and polarized intensity maps for resolved bow shocks. We have presented the central- and distributed-source cases separately here for clarity, but typically both should contribute simultaneously to the observed polarization. \textit{SLIP} has the capability to combine the two cases by specifying the relative brightnesses of the star and CSM; we will investigate these cases in the future when modeling particular bow shocks.

When the albedo is not fixed at 1, but instead calculated using input parameters, we find that the polarization depends both on temperature and optical depth. In this case, absorption effects cause dramatic departures from $\sin^2 i$ behaviour, particularly for higher optical depths and lower temperatures. These effects also produce resolved polarization maps that differ from those of the pure-scattering and low optical depth
cases. We have chosen a representative optical wavelength of 6040 Å to represent these cases, but this can be changed to correspond to specific observed scenarios.

We made several simplifying assumptions in creating these models, which should be kept in mind when interpreting the results. First, we chose a specific value of $\alpha = V_*/V_w = 0.1$ to correspond to winds from hot stars (§ 2.4). For cooler stars, $\alpha$ will be larger, and this will increase the density of the bow shock via Eq. 2.2.6 (see also Fig. 4 of Wilkin 1996). Thus, we expect that the results for cooler stars will be similar to those of the high optical-depth cases we discuss here.

We also chose a specific standoff radius $R_0$ (§ 2.4) for consistency in the models presented here. In the pure-scattering case, polarization behaviour does not depend on $R_0$, but for the more realistic case of variable albedo, the polarization may differ from the results presented here. This is due to the way we defined the thickness and density of the Wilkin (1996) bow shock, as discussed in § 2.4. A study investigating the use of polarization as a diagnostic of the stellar mass-loss rate or ISM density would need to assume or measure a value for $R_0$ in order to generate models with the appropriate CSM opacity and albedo. Such a study could be undertaken with SLIP, but is beyond the scope of this chapter because of the wide range of possible $R_0$ values. In the next chapter, we plan to compare SLIP models with polarization measurements of bow-shock sources with measured $R_0$ values, and will adjust the models accordingly.

We have not investigated the effect of ionised stellar wind material filling the interior of the bow shock, but we expect this would decrease the overall polarization magnitude without significantly affecting its behaviour as a function of viewing angle (particularly in the case of photons arising from the central source). We will explore the polarization contributions of interior scattering material in Chapter 3.

We also note that the bow shock solution presented by Wilkin (1996) is an idealisation that assumes a stable and highly evolved bow shock, as shown by hy-
drodynamic models (Mohamed et al. 2012). Resolved polarization or polarized intensity maps that show bow shock shapes similar to those in our models would thus provide information about the age of the observed bow shock, which in turn can reveal the evolutionary state of the star, as discussed in Mohamed et al. (2012). Younger bow shocks or bow shocks with instabilities due to a high-density region of the ISM (Meyer et al. 2014) or a star moving with a high space velocity (Meyer et al. 2015) will show different morphologies than the idealised shape considered here. We expect these cases will display broadly similar polarization features, but detailed studies will require additional modeling. We plan to investigate clumpy shock structures in a future contribution.

We recognize that dust scattering is an important contributor to the observed polarization of actual bow shocks that we have not treated here. In fact, most observations of stellar wind bow shocks have been obtained using IR data (e.g., Kobulnicky et al. 2016; Ueta et al. 2006; Ueta et al. 2008b; Peri et al. 2012). The SLIP code can treat dust scattering, and we will investigate its behaviour in Paper II. We will discuss the variation in polarization behaviour at different wavelengths as well as for different dust grain models.
Chapter 3

Polarization signatures of stellar wind bow shock nebulae: The case of dust scattering

3.1 Introduction

A stellar wind bow shock nebula is formed by the interaction of a stellar wind and the ambient medium when the relative velocity is supersonic (e.g., Wilkin 1996). In this chapter we are interested in bow shock nebulae around evolved massive stars. These stars lose mass throughout their lifetimes in different forms which impact their evolution (e.g., Langer 2012) as well as the local interstellar medium (ISM; Castor et al. 1975). Since stellar wind bow shock nebulae represent interaction regions between the stellar wind and the local ISM, they encode information about the dust properties of both (Ueta et al. 2008a).

In a previous study (Shrestha et al. 2018, hereafter Paper I), we carried out a computational investigation of the polarization arising from stellar wind bow shock nebulae when electron scattering is the only scattering mechanism. We considered
illumination by the star alone and by “distributed” emission from within the nebula itself. We found that the polarization thus produced is highly dependent on viewing angle, and that multiple scattering modifies the polarization significantly from the analytical predictions for single scattering. In cases involving significant multiple scattering, in addition to a polarization peak near a viewing angle of 90° predicted by single scattering models, our simulations also produced a second peak at a larger angle.

As an extension of the work presented in Chapter 2, we here expand the study to explore the effects of dust scattering on the polarization behavior of bow shock nebulae around evolved massive stars. Dust plays an important role in the density structure of bow shocks and bow shock nebulae around massive stars, and as a result many observational studies of these phenomena focus on the various infrared wavelengths (Kobulnicky et al. 2016; Ueta et al. 2006; Ueta et al. 2008a). Consequently, we cannot ignore the role of dust in scattering light and producing polarization in stellar wind bow shocks. The polarization observed in dusty bow shock nebulae has a magnitude of as high as a few percent (Buchholz et al. 2012; Rauch et al. 2013).

Several authors have previously modeled the polarimetric features arising from dust scattering in bow shock structures. Buchholz et al. (2012) used analytical calculations, which are applicable only for single scattering at very low optical depths. Shahzamanian et al. (2016) and Zajaček et al. (2017) used a sophisticated 3-D Monte Carlo radiative transfer (MCRT) code to simulate the polarization behaviour of a dust-scattering bow shock along with other possible circumstellar structures around the Dusty S-cluster Object (DSO) near the Galactic centre. These studies focused on a particular object and included scattering regions other than the bow shock nebula itself. Our aim is to build on these previous models and create numerical simulations for a generalized bow shock structure that include multiple scattering for consideration of higher optical depth regions.
This contribution is the second of two papers in which we use the MCRT method to simulate the polarimetric behavior of generalised stellar wind bow shock structures. We obtained our results using the \textit{SLIP} code ("Supernova LIne Polarization"; Hoffman 2007; Shrestha et al. 2018). This code is similar to the one used by Shahzamanian et al. (2016) and Zajaček et al. (2017), but our implementation is different, as discussed in Paper I. The ultimate goal of our study is to determine how polarization measurements may constrain the properties of the bow shock, which in turn provides constraints for the properties of the interstellar medium (ISM) and the stellar wind that produces the bow shock. Here we investigate the effects of various input parameters on the resulting polarization behavior, assuming dust is the only scattering mechanism. As in Paper I, we will use the term “bow shock” in a broad sense, describing not only a physical shock, but also the resulting nebula, or region of enhanced density, surrounding the shock and having the same shape.

This chapter is organized as follows. In Section 3.2 we discuss the implementation of dust scattering in the \textit{SLIP} code and provide details regarding the dust models adopted for use in our simulations. In Section 3.3, we present results from an analytical model to compare against the numerical model as well as comparing dust scattering results with electron scattering results. In Section 3.4 we present and interpret the predictions of simulations for a bow shock with and without dust emission, with different dust types, and at different wavelengths for both resolved and unresolved cases. In Section 3.5, we present observational implications and compare the results from our simulations with observational data. Finally, conclusions and future work are presented in Section 3.6.

3.2 Methods

The simulations for this paper were done with the Supernova LIne Profile (\textit{SLIP}) code (Hoffman 2007; Shrestha et al. 2018). \textit{SLIP} is a MCRT code Whitney (2011)
which tracks photon packets through a three-dimensional spherical polar grid as in Whitney and Wolff (2002). The results presented in this paper are for the azimuthally \( \phi \) symmetric case with 100 radial cells and 101 cells in the polar \( \theta \) direction.

A finite spherical photon source is placed at the center of the grid. The photon source is surrounded by a circumstellar material (CSM) composed of pure hydrogen in local thermodynamic equilibrium (LTE). The CSM is not assumed to be heated by the central star, instead we define its temperature \( T \), which governs the ionisation fraction \( x \) within the CSM and is assumed constant throughout. The reference optical depth \( \tau_0 \) is also defined by the user. Using the input \( T \) and \( \tau_0 \), \textit{SLIP} calculates the free electron density which in turn is used to calculate the opacity. More details about the calculation can be found in Chapter 2.

\textit{SLIP} emits virtual, initially unpolarized “photons” from the central star (or other photon source) and tracks them as they travel through the CSM. In the code, the photon’s behavior is determined by generating weighted random numbers corresponding to known probability distributions determined by the optical depth \( \tau \) and albedo \( a \) of the CSM (Whitney 2011). In addition to emission of photons from the central star, we can consider emission from the CSM as well. This capability of the code is one of the strengths of our implementation. The emission from the dusty CSM is described in detail in Section 3.2.1.

The emitted photons interact with the scattering region and \textit{SLIP} performs the numerical optical depth integration described in Code and Whitney (1995) and Whitney (2011). A random number is generated after each integration, which is compared with the albedo to determine whether the photon scatters or becomes absorbed. The photon’s Stokes parameters are updated after each scattering event by applying the standard Mueller matrix multiplication (Chandrasekhar 1960; Code and Whitney 1995; Whitney 2011). After a number of scattering events depend-
Table 3.1: Properties of the dust models implemented within SLIP for a wavelength of 2.2µm.

<table>
<thead>
<tr>
<th>Dust type</th>
<th>$g$</th>
<th>$\kappa$ g/cm$^2$</th>
<th>Composition</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRN</td>
<td>0.02</td>
<td>18.35</td>
<td>silicate, graphite</td>
<td>Mathis et al. (1977)</td>
</tr>
<tr>
<td>KHM</td>
<td>0.24</td>
<td>22.52</td>
<td>silicate, graphite</td>
<td>Kim et al. (1994)</td>
</tr>
<tr>
<td>R550</td>
<td>0.26</td>
<td>39.83</td>
<td>silicate, graphite</td>
<td>Clayton et al. (2003)</td>
</tr>
<tr>
<td>WW02</td>
<td>0.49</td>
<td>42.70</td>
<td>silicate, amorphous carbon</td>
<td>Cotera et al. (2001)</td>
</tr>
</tbody>
</table>

Depending on optical depth, a photon exits the simulation (i.e., it “escapes”). The Stokes parameters for all the photons in the appropriate output bin corresponding to the observer’s viewing angle are combined. A single SLIP simulation yields Stokes vectors at all the viewing angles ranging from $i = 0 – 180^\circ$. The summed Stokes vectors in a bin are normalised in $\theta$ and $\phi$ to ensure the output fluxes have correct units.

We calculate the uncertainty in the Stokes parameters in each bin by calculating the standard deviation of that parameter over all $N$ photons in the bin and normalising it to $\sqrt{N}$ to account for Poisson statistics (Wood et al. 1996b; Whitney 2011).

Within SLIP, we use tabular functions as described in Whitney (2011) to define the scattering properties for several different dust models. The data files available in the HOCHUNK3D code distribution (Whitney et al. 2013) contain the elements of the phase scattering matrix (Chandrasekhar 1960) and other optical properties for several common dust models as a function of wavelength. Different dust models and their properties, such as the dust scattering asymmetry $g$ and the opacity $\kappa$, are given in Table 3.1 and depicted graphically in Fig. 3.1.

MRN dust is based on a the standard interstellar grain model created by Mathis et al. (1977) using a fit to observed interstellar extinction in the wavelength range $0.11 – 1 \mu$m. The dust particles are spherical, uncoated, and composed of graphite and silicate, with a size distribution given by power law $n(a) \propto a^{-3.5}$, where $a$ ranges from $0.005 – 1 \mu$m. KMH dust represents interstellar grains for a wavelength range of $0.1 – 5 \mu$m (Kim et al. 1994). The composition and shape of the KMH dust grains
Figure 3.1: This plot shows dust properties of the dust models we have used in the simulations. Panel (a) shows how $g$ varies with wavelength for the different dust models. Panel (b) is for variation in opacity $\kappa$ with respect to wavelength. Panel (c) is for albedo $a$ with respect to wavelength.
is the same as in the MRN model; however, the MRN model has a sharp cutoff at 1 µm, while the KMH model decreases smoothly beginning at 0.2 µm. For both MRN and KMH models, the ratio of total to selective extinction is $R_V = 3.1$.

The WW02 dust model was obtained by extinction curve fitting to the disk of T Tauri star HH 30 (Cotera et al. 2001). The WW02 dust grains are larger than the ISM grains by a factor of approximately 2.1, and are composed of silicate and amorphous carbon. The R550 model comes from a maximum entropy method fit to HD 37022, a binary or multiple star in the Orion Nebula, with $R_V = 5.5$ in the wavelength range of $0.125 - 3$ µm, and assumes a graphite and silicate composition (Clayton et al. 2003).

We chose KMH and MRN dust models because they represent ISM dust. Hydrodynamic simulations have shown that ISM dust can be present in a bow shock (van Marle et al. 2015); thus it is important to understand the behaviour of polarization due to scattering by ISM dust. Another goal of the project was to understand the impact of dust size on polarization; we therefore chose WW02 and R550 which are larger dust grains compared to those of the ISM.

In this chapter the polarization due to dust scattering is done for a generalized bow shock rather than a particular object (as in Neilson et al. 2013, Shahzamanian et al. 2016, and Zajaček et al. 2017). We describe our CSM using an axisymmetric bow shock defined analytically by Wilkin (1996). This formulation assumes a spherically symmetric stellar wind and a locally uniform ISM. The analytic calculation provides the formulae for the shape, mass surface density, and velocity flow in an infinitesimally thin axisymmetric bow sock. The properties of this bow shock is determined by the stellar wind, the speed of the star through the ISM, and the local ISM density.

$$R_0 = \sqrt{\frac{\dot{m}_w V_w}{4\pi \rho_1 V_*^2}}$$  \hspace{1cm} (3.2.1)
Here $R_0$ is the standoff radius which is defined as the location along the star’s path at which the ram pressure of the ISM and stellar wind are equal (Wilkin 1996). In Eq. 3.2.1, $V_w$ represents the stellar wind velocity, $V_*$ the stellar velocity, $\rho_I$ the ISM density, and $\dot{m}_w$ the stellar mass-loss rate.

Wilkin (1996) calculated the bow shock radius as a function of polar angle using momentum conservation and force balance which is given by

$$R(\theta) = \sqrt{3} R_0 \csc \theta \sqrt{1 - \theta \cot \theta}.$$ (3.2.2)

We use this equation to calculate the shape of the generalised bow shock shape in the SLIP code. We choose $R_0$ to give a convenient scale to our simulations. Finally, mass surface density $\sigma$ as a function of polar angle is calculated using conservation of momentum and given by

$$\sigma(\theta) = \frac{1}{2} R_0 \rho_I \frac{[2\alpha(1 - \cos \theta) + \tilde{\omega}^2]^{1/2}}{\tilde{\omega} \sqrt{(\theta - \sin \theta \cos \theta)^2 + (\tilde{\omega}^2 - \sin^2 \theta)^2}}.$$ (3.2.3)

Here $\tilde{\omega}$ is a convenient parametrization defined by $\tilde{\omega}^2 = 3(1 - \theta \cot \theta)$. The symbol $\alpha$ parametrizes the ratio of the translational speed of the star to its stellar wind velocity ($\alpha = V_*/V_w$). In our models, we assume $\alpha = 0.1$ to represent the hot-star case, as for the O star $\zeta$ Pup (Puls et al. 1996).

In the code, we cannot simulate an infinitesimally thin bow shock. Thus we give a certain thickness to the bow shock and calculate the volume density using the thickness and mass surface density given by Eq. 3.2.3. The details of this implementation can be found in Paper I. Various thickness values were tested to check the impact of thickness on polarization signature; we found that polarization did not vary significantly with thickness in the physically thin regime. Thus we picked a thickness value which would make the thickness of bow shock at least one grid cell. The volume density is then given by
\[ \rho(\theta) = \frac{R_0 \rho_I b(\theta)}{2 \Delta R(\theta)} \left\{ \frac{[2\alpha (1 - \cos \theta) + \tilde{\omega}^2]^2}{\dot{\tilde{\omega}} \sqrt{(\theta - \sin \theta \cos \theta)^2 + (\tilde{\omega}^2 - \sin^2 \theta)^2}} \right\} \]  

Here \( b(\theta) \) is a geometrical term that accounts for the radius being \( \theta \) dependent; it is discussed in detail in Paper I and the appendix (Chapter A).

Since the simulation grid is of a finite size, we truncate the bow shock for large values of \( \theta \). Instead of abruptly making the density go to zero, we set a cutoff angle after which the density falls off exponentially:

\[ \rho(\theta > \theta_c) = \rho(\theta_c, \infty) \exp\left[-(\theta - \theta_c)/\delta\theta_0\right], \]  

where \( \delta\theta_0 \) is a constant angle governing the steepness of the density decline and \( \theta_c \) is the cutoff angle. After testing the effect of polarization with changing \( \delta\theta_0 \) and \( \theta_c \), we chose a convenient value of \( \theta_c = 2.1 \) rad (122°) and \( \delta\theta_0 = 0.3 \) rad (17°) for all the models shown hereafter.

For the case of dust, we need to account for the dust emission from the bow shock to understand the resultant polarization behavior. The calculation of dust emission and its implementation in SLIP is presented below.

### 3.2.1 Dust emission

Let \( j_\nu \) be the emissivity, and \( dV \) a differential element of volume. Then the luminosity for a certain frequency of dust emission is given by

\[ L_\nu = \int j_\nu \, dV. \]  

For isotropic emission, any coordinates can be used to obtain the same answer. The number of photons generated is invariant of the selection of coordinates when integrating over the entire volume. So we choose coordinates that are locally normal
and perpendicular. Let $dl$ be normal and $d\Sigma$ be a tangential area element. Then

$$L_\nu = \int j_\nu \, dl \, d\Sigma.$$  \hfill (3.2.7)

The emissivity is of the form $j_\nu = \kappa_\nu \, \rho \, B_\nu(T)$, where $\kappa_\nu$ is the dust opacity and $B_\nu(T)$ is the Planck function at temperature $T$. Now we have

$$L_\nu = \int \rho \, dl \int \kappa_\nu \, B_\nu(T) \, d\Sigma.$$  \hfill (3.2.8)

The first integral is just the mass surface density $\sigma$, which gives

$$L_\nu = \int \kappa_\nu \, B_\nu(T) \, \sigma \, d\Sigma.$$  \hfill (3.2.9)

If we adopt an assumption of isothermal grains, then the luminosity becomes

$$L_\nu = \kappa_\nu \, B_\nu(T) \int \sigma \, d\Sigma,$$  \hfill (3.2.10)

where $\sigma$ is the expression from the Wilkin (1996) model, which depends on $\theta$ (Eq. 3.2.3). The area element is

$$d\Sigma = 2\pi R^2(\theta) \, d\theta.$$  \hfill (3.2.11)

The integral part of Eq. 3.2.10 is the total mass $M$ in the bow shock at a particular viewing angle, which has contributions from both dust and gas. We define the dust-to-gas ratio as

$$\delta_d = \rho_d / \rho_{\text{gas}};$$  \hfill (3.2.12)

then the mass of dust in the bow shock is

$$M_d = \delta_d M.$$  \hfill (3.2.13)
Now Eq. 3.2.10 reduces to

\[ L_\nu(\theta) = \kappa_\nu B_\nu(T) M_d(\theta). \] (3.2.14)

Figure 3.2 represents the variation of the derived dust luminosity with viewing angle for a particular temperature of 1000 K, \( \kappa = 22.52 \) cm\(^2\)/g, and \( \lambda = 2.2 \) \( \mu \)m. This luminosity is implemented in SLIP by calculating the ratio of luminosity from dust to the luminosity of the central star. Using luminosity of Betelgeuse as the central source luminosity, i.e. \( L_\star^e = 1.1 \times 10^{38} \) erg/s/Hz (Smith et al. 2009), we calculated the ratio of the two \( L_\nu(\theta)/L_\nu^e \). This fraction is used to calculate the fractional number of photons that are emitted from the bow shock. These photons emitted from within the bow shock can be scattered or absorbed or escape similar to the photons emitted from the star.

### 3.3 Comparisons with Analytic Model and the Electron Scattering Case

#### 3.3.1 Comparisons with analytic model

Similar to the electron scattering case, we first considered semi-analytic results for scattering within a bow shock in the optically thin limit. Because a stellar wind bow shock is an asymmetric circumstellar phenomenon, we expect a bow shock to produce a net polarization even for an unresolved structure. We start by assuming a Wilkin (1996) analytic bow shock model defined by Eq. 3.2.2. This structure defines the bow shock as infinitesimally thin, which is the primary difference between the analytic and numerical calculation. Given the structure, we can apply the Brown and McLean (1977) analytic model for polarization due to dust scattering.
Figure 3.2: Dust luminosity as a function of $\theta$ as derived in § 3.2.1 for temperature of 1000 K, $\lambda = 2.2$ $\mu$m, and $\kappa = 22.52$ cm$^2$/g for the KMH dust model.
for an optically thin structure where we treat the dust scattering using the Henyey-Greenstein (H-G) functions (Whitney 2011) instead of dipole scattering.

Because the asymmetry is greatest for the side-on bow shock, it is reasonable to expect that the analytic model at this inclination will produce the greatest amount of detectable polarization. In Fig. 3.3, we compute the integrated polarization as a function of dust temperature for a wavelength of 2.2 µm using an analytic calculation. In this calculation we are treating dust emission as unpolarized light. Thus the emission from dust will dilute the polarization we get from scattering. In the low-temperature regime, the dust emission is small and does not dilute the polarization signal; we find a maximum polarization of about 1%. As the dust temperature increases there is more unpolarized emission from the bow shock, thus the polarization decreases. For $T_{\text{dust}} \approx 1,000$ K the polarization is about 0.3%.

We simulated an H-G run in SLIP using an optical depth of $\tau_0 = 0.1$, an opacity of 22.52 g/cm$^2$, a wavelength of 2.2 µm, and a $g$ value of 0.43 to closely match the parameters used in the analytic calculation. For the side-on view we found polarization of about 1% as shown in Fig. 3.4, which is close to the amount of polarization from analytic calculation. We also analytically calculated the dust emission as shown in Section 3.2.1. For comparison, we used the emission as unpolarized light and thus as a dilution factor. Fig. 3.4 has a similar shape to Fig. 3.3, but the decrease in polarization is sharper for the numerical method as compared to the analytic method. For $T_{\text{dust}} \approx 1,000$ K the polarization is about 0.1%.

### 3.3.2 Comparison with electron scattering case

We compared the results from the dust scattering case with electron scattering results. Here the albedo for the electron scattering case is chosen to be the same as the dust scattering case we are comparing with. For the electron scattering case, the geometrical parameter is the same as in the dust case described in Section 3.2.
Figure 3.3: Predicted polarization of an unresolved dusty bow shock from the analytic models as a function of dust temperature at 2.2 \( \mu \text{m} \).
Figure 3.4: Predicted polarization of an unresolved dusty bow shock from the SLIP code as a function of dust temperature at 2.2 µm. This plot is for an inclination angle of 90°.
The input CSM temperature is 10000 K and the albedo is fixed to 0.464 and 0.544 when comparing with dust of wavelengths 1.25 $\mu$m, and 0.55 $\mu$m, respectively. We chose these albedo values because they are the same for dust input files for the corresponding wavelengths.

Fig. 3.5 shows the variation in polarization with viewing angle for two different optical depths and two different wavelengths. We picked two representative wavelengths where the polarization behavior of dust was most similar to (for $\lambda = 1.25 \mu m$) and most different from (for $\lambda = 0.55 \mu m$) the electron scattering case. For all cases the amount of polarization is higher for electron scattering compared to the dust scattering case, which can be explained using comparison of the phase functions as shown in Fig. 3.6. Among all the phase functions, the electron case has higher probability of 90° scattering, which corresponds to overall higher polarization, compared to all dust types. For $\lambda = 1.25 \mu m$, the shape of different dust types and electron scattering looks similar. However, the amount of polarization varies. The MRN dust model is closest in shape of the phase function and amount of polarization compared to electron scattering at $\lambda = 1.25 \mu m$. WW02 has first peak at different angle than the electron scattering case for all the panels and the amount of polarization is lowest as well. For $\lambda = 0.55 \mu m$, the first peak seems to be shifted to higher angle for all the dust models compared to electron scattering and the amount of polarization is much lower for all the dust models as well. We see the phase function for all dust models is different compared to electron case at $\lambda = 0.55 \mu m$.

### 3.4 Model Predictions from *SLIP*

The geometrical and density setup of the bow shock is described in Section 3.2 and the setup in further detail is described in Paper I. The maximum extent of the grid is set at $R_{\text{max}} = 6.68$ AU, the cutoff angle is $\theta_c = 2.1$ rad ($122^\circ$), $\alpha$ is the ratio of speed of the star to its stellar wind is set to be 0.1, the stand off radius
Figure 3.5: Polarization as a function of inclination angle for different dust types for two different optical depths of $\tau_0 = 0.5$ (top) and $\tau_0 = 2.0$ (bottom) and for two different wavelengths of 1.25$\mu$m (right) and 0.55$\mu$m (left). Error bars are smaller than the plotted points.

Figure 3.6: Comparison of scattering phase functions for different dust types and electron scattering. For the dust cases, the phase function is wavelength dependent, thus the comparison for two wavelengths of $\lambda = 1.25\mu$m, and $\lambda = 0.55\mu$m. These phase functions are from the input dust files and Greenberg (1963).
$R_0 = 1.4$ AU, and exponential decay factor $\delta \theta_0 = 0.3$ rad ($17^\circ$) is set for all the results presented in this paper unless stated otherwise.

We used the University of Denver’s high-performance computing cluster (HPC) to create the simulations, which consists of 180 Intel Xeon processors running at 2.44 GHz. Each of our model runs used 16 CPUs with $10^8$ photons per CPU. This yielded polarization uncertainties on the order of $\sigma_p(\%) \sim 0.01$. Each run took $\sim 60 - 70$ minutes for completion, with slightly longer times for larger values of $\tau_0$.

Our simulations can be broadly divided into two categories. One is without dust emission and the other includes dust emission. For both cases, we studied the impact of dust size and composition on the polarization behaviour for resolved and unresolved cases. In the resolved cases, we preserve the spatial information from our simulations, whereas in the unresolved cases, we combine all the photons from a given viewing angle into a single set of polarization values. We also studied the impact of temperature and optical depth for both cases. For the simulations without dust emission, changing temperature does not change the polarization behaviour because the opacity and albedo are fixed from the input files for dust. We present the results for these different scenarios below.

### 3.4.1 No dust emission

We study the polarization behaviour of dust scattering for various dust sizes and composition with no dust emission from the bow shock. We present results for four dust types at four representative inclination angles. In all these simulations, we found polarization position angles very close to $\Psi = 0^\circ$, so we have not displayed the position angle results. We present results for both resolved and unresolved cases below.
Dust type dependence – resolved bow shock

In Fig. 3.7, we display the polarized intensity and percent polarization images for resolved bow shock for $\tau_0 = 0.5$ for four representative wavelengths. All the polarized intensity is coming from the bow head for all the dust types in four wavelengths and two different inclination angles. This result matches with the result in Chapter 2 for the low optical depth regime as well. The amount of polarized intensity is increasing with increasing wavelength for R550 and WW02 dust. However for KMH and MRN, the amount increases up to $H$ band and then it decreases for $K$ band. This behaviour is due to low albedo value of MRN and KMH at $K$ band. For R550 and WW02 the albedo value is almost constant but the $g$ parameter is decreasing slightly, thus slight increase in polarized intensity.

The polarization increases with increasing $\theta$ for all the dust models. At higher viewing angle, for most dust models the polarization is coming from lower part of the bow shock. This is similar to polarized intensity, however the polarized intensity decreases at $K$ band for MRN dust. The polarization magnitude does not do the same because albedo is decreasing thus the total intensity decreases and fractional polarization increases.

Fig. 3.8 displays polarized intensity and polarization maps for a resolved bow shock with $\tau_0 = 2.0$. The prominent difference between the high and low optical depth cases is the behaviour of the polarized intensity at the higher inclination angle. In this case, most of the polarized intensity is coming from the middle portion of bow shock. This distinction is similar to what is seen in Paper I. This suggests that polarized intensity maps may provide constraints in inclination angle of observed bow shock.
Figure 3.7: Predicted maps of a stellar wind bow shock from *SLIP* in polarized flux (left) and polarization magnitude (right) for an inclination angle of 125° and a reference optical depth of 0.5.
Figure 3.8: Predicted resolved maps of a stellar wind bow shock from \textit{SLIP} in polarized flux (left) and polarization magnitude (right) for an inclination angle of 125° and a reference optical depth of 2.0.
Dust type dependence – unresolved bow shock

In Fig. 3.9 we present the magnitude of polarization at different wavelengths for different types of dust in the central source case. Four different panels are for two different inclination angles for $\tau_0 = 0.5$ on the right side. For $i = 35^\circ$, the magnitude of polarization increases with wavelength for R550 and WW02 dust models. However, for MRN and KMH dust models the value increases up to wavelength of 1.25 $\mu$m and decreases for greater wavelengths. This behavior is similar to that of the Serkowski formula (Serkowski et al. 1975). A similar trend is seen for all the inclination angles. This trend can be explained by the behavior of $g$ and albedo with respect to wavelength for MRN and KMH dust types. We see that $g$ and albedo decrease with increasing wavelength; however, the albedo decreases slowly up to 1.25 $\mu$m and then the decrease is sharper. Thus at 1.25 $\mu$m there is an optimum point where $g$ is low and albedo is high enough to produce high polarization magnitude for both KMH and MRN models. Lower forward throwing means most of the scattered photons are polarized, corresponding to an increase in magnitude of polarization. The drop in $g$ is more drastic for MRN at 1.25 $\mu$m; thus we see a larger peak for MRN dust model in Fig. 3.9. The R550 and KMH models behave similarly for lower wavelengths in terms of magnitude and behavior. They diverge at higher wavelengths because the albedo value diverges for the two models at higher wavelengths as well. The WW02 dust model has the least amount of polarization for all the inclination angles at wavelengths less than 1.25 $\mu$m. This can be attributed to the high value of the forward throwing parameter $g$ and low albedo values in the low wavelength regime as shown in Fig. 3.1 as compared to other dust types.

The right panel of Fig. 3.9 shows results for for $\tau_0 = 2.0$. In this case multiple scattering plays an important role in the polarization behaviour; thus we see a second peak near 120$^\circ$ as shown in Fig. 3.5. The behaviour of polarization with respect to wavelength for $i = 90^\circ$ and $i = 120^\circ$ is different compared to the $\tau_0 = 0.5$
Figure 3.9: Polarization as a function of wavelength at two different inclination angles for four different dust types and reference optical depths of $\tau_0 = 0.5$ and $\tau_0 = 2.0$. The legend consists of different dust types. Error bars are smaller than the plotted points.
Figure 3.10: Polarization as a function of \( g \) value at two different inclination angles for four different dust types and reference optical depths of \( \tau_0 = 0.5 \) and \( \tau_0 = 2.0 \). The legend consists of different dust types. Error bars are smaller than the plotted points.

case. The behaviour seems to be switched for those two viewing angles. At 90°, the polarization increases with wavelength for WW02 because the albedo increases and \( g \) is almost constant. For MRN, KMH, and R550, the polarization increases up to 1.25 \( \mu \text{m} \) and then decreases as seen in the low optical depth case. R550 behaves differently for high optical depth compared to low optical depth. At low optical depth and \( i = 90° \), the polarization value increases with increasing wavelength, but for high optical depth the polarization values decrease after 1.25 \( \mu \text{m} \). This is the case because decreasing albedo has a higher impact on the polarization than does the decreasing \( g \) value in the high optical depth regime. For high optical depths there is a higher chance of photons interacting with dust, and if the albedo is lower then most of the photons are being absorbed at each interaction. Thus, the polarized light has a low chance of escaping. For 120° the density is lower due to the exponential falloff
in our model setup. Hence photons that have been multiply scattered can escape from this angle.

3.4.2 With dust emission

As discussed in Section 3.2.1, we incorporated photons emitted by the dust in the bow shock. This emission is dependent on the dust opacity, temperature, and wavelength. First we checked how changing temperature changes the polarization for different dust at various wavelengths. Then we checked how changing dust type changes the polarization results with dust emission.

Temperature dependence – resolved bow shock

Figure 3.11 shows intensity maps of resolved bow shocks for four different dust types at four different wavelengths. The top panel is for a reference optical depth of $\tau_0 = 0.5$, and the bottom panel is for $\tau_0 = 2.0$. The left panel is for a dust temperature of 750 K and the right panel is for $T = 1000$ K. For low optical depth and temperature of 750 K there is not much dust emission, while for 1000 K the amount of dust emission increases with increasing wavelength for all the dust models. For the higher optical depth as well, the emission increases with wavelength. Note that the value of intensity is different for different optical depths.

Figure 3.12 is same as Fig. 3.11, but shows polarization maps. The code is able to make polarized intensity maps as shown in Fig. 3.7, but we observed little variation with respect to temperature. Thus we present only polarization maps here. At shorter wavelengths ($V$ and $I$ bands), the polarization map is similar for both dust temperatures. However, at a wavelength of 1.65 $\mu$m, the polarization map is distinct between the two temperatures. This is because at the higher temperature, most of the photons are emitted from the bow shock, which thus acts as a distributed source (Chapter 2). Hence most of the polarization is coming from the edge of bow shock.
These resolved images can help constrain the dust temperature if we observe them in higher wavelength regime.

**Temperature dependence – unresolved bow shock**

We created numerical models for different dust types at various temperatures, considering both optically thin ($\tau_0 = 0.5$) case and optically thick ($\tau_0 = 2.0$) cases. For the optically thin case, we did not see much variation of polarization with temperature for any dust type. That is because in the optically thin case, multiple scattering is not the dominant factor in creating the polarization signature. Thus photons coming from the central star and from the bow shock itself will, on average, be scattered equally. This result agrees well with the result in Chapter 2 where we found that for an unresolved bow shock, the amount and behaviour of polarization at $\tau_0 = 0.5$ is similar for both central and distributed sources.

For the high optical depth ($\tau_0 = 2.0$) case, we see the behavior and amount of polarization changes with temperature when the temperature is greater than 500 K. At lower temperatures, the dust does not emit enough to change the polarization behaviour. Figure 3.13 shows the magnitude of polarization with respect to wavelength for two different dust types (KMH and WW02) at different inclination angles of 90° and 120°. We picked these two inclination angles because the two peaks in polarization are seen around those viewing angles. In top panel of Fig. 3.13, which represents $i = 90^\circ$, increasing temperature slightly increases the amount of polarization at longer wavelengths for both KMH and WW02 dust types. As wavelength increases, the fraction of photons coming from the bow shock increases. As temperature and wavelength increases, the fraction of photons from the bow shock increases as well. For the $i = 90^\circ$ case the polarization is mostly contributed by photons that have been scattered fewer of times (Chapter 2). Thus as more photons are emitted from the bow shock, they have higher chance of escaping after scattering.
Figure 3.11: Intensity maps for a resolved bow shock with temperatures 750 K (left) and 1000 K (right) for different dust models with reference optical depths of $\tau_0 = 0.5$ (top) and $\tau_0 = 2.0$ (bottom) for four different representative wavelengths. All these results are for $i = 125^\circ$. 
Figure 3.12: Polarization maps for a resolved bow shock with temperatures 750 K (left) and 1000 K (right) for different dust models with reference optical depth of $\tau_0 = 0.5$ (top) and $\tau_0 = 2.0$ (bottom) for four different representative wavelengths. All these results are for $i = 125^\circ$. 
Figure 3.13: Polarization as a function of wavelength at two different inclination angles for two different dust types and a reference optical depth of $\tau_0 = 2.0$. The legend consists of different temperatures of the model in Kelvin. “No-emission” stands for the case where all the photons are coming from the central star. Error bars are smaller than the plotted points.

Hence the amount of polarization increases with increasing temperature. However, for $i = 120^\circ$, multiple scattering contributes to the polarization peak as shown in Chapter 2. As the temperature increases, there are more photons from the bow shock which have average number of scattering less than the central-source photons, thus the polarization value at $i = 120^\circ$ decreases with increasing temperature.

**Dust type dependence – resolved bow shock**

In Fig. 3.12, we present polarization maps for four different dust types at four different wavelengths. For higher dust temperature we see very little difference among different dust types. However, for dust temperature of 750 K we see variations in the polarization maps with different dust types in the $H$ band. Thus polarization
maps can give us information about the dust properties. For the same dust type, as wavelength increases the polarization starts to concentrate near the edges because more photons are emitted from the bow shock.

**Dust type dependence – unresolved bow shock**

To simulate the polarization behaviour of different dust types with dust emission, we set a temperature for one set of models. Various dust temperatures of 500 K, 750 K, and 1000 K were simulated. We found that $T = 500$ K resulted in polarization similar to the case of no emission as shown in Section 3.4.1. The simulations were done for two reference optical depths of $\tau_0 = 0.5$ and $\tau_0 = 2.0$; we found that for $\tau_0 = 0.5$ the polarization behaviour is similar to that of the no-emission case. This result agrees well with the results of Chapter 2, in which we saw that for an unresolved bow shock the amount and behaviour of polarization at $\tau_0 = 0.5$ is similar for the central and distributed sources.

In Fig 3.14, we display the polarization variation as a function of wavelength for the unresolved case, considering two different viewing angles and two different temperatures for four different dust types. For all the cases, we see that at smaller wavelengths the polarization behaviour is similar for all the dust types. As wavelength increases the behaviour starts to diverge, depending on the dust type, at $i = 90^\circ$. At wavelengths up to 0.85 $\mu$m, the behavior is similar for no emission and emission cases. At higher wavelengths the behavior diverges for R550 and KMH from no-emission case. There is an increase in polarization at 2.2 $\mu$m which is not seen in no-emission case for temperature of 750 K. At 1000 K, the amount of polarization decreases for KMH model after 1.25 $\mu$m.

The behaviour at $i = 120^\circ$ is different compared to no-emission case. In this case, for all the dust models the polarization value increases up to 1.25 $\mu$m and then the value decreases for both temperature regimes. The amount of polarization is
lower for 1000 K. The polarization signal at $i = 120^\circ$ is due to multiple scattering and as temperature increases, most of the photons are emitted from the bow shock. This emission decreases the average number of scatters for photons that escape the bow shock, hence there is less polarization signal at $i = 120^\circ$. These results can be used along with observational data to constrain the dust temperature as well as inclination angle of the bow shock.

### 3.4.3 Dust between the star and bow shock

In a more physical scenario, there will be dust and gas between the bow shock and the star. As a first approximation I include dust with a density that decays as $r^{-2}$ from the star to the bow shock as shown in Fig. 3.15. The density is slightly higher in the bow shock structure as seen by a jump in density value in Fig. 3.15.
This density function is similar to that obtained from smooth particle hydrodynamic (SPH) model of Betelgeuse by Mohamed et al. (2012), but our model does not include dust clumps and the density value is slightly different. These differences will be addressed in future simulations.

All of the results in this section are for KMH dust in the $K$ band. We studied different optical depths, with results shown in Fig. 3.16. We found that dust between star and bow shock suppresses the overall polarization value for all optical depth values. When there is dust inside the bow shock, the density increases. Thus multiple scattering becomes more important and we get a suppression in polarization peak near 90° which is also seen in Chapter 2. The second peak at higher viewing angle is suppressed even more compared to the 90° peak when there is dust between the star and bow shock.
Figure 3.16: Polarization with respect to viewing angle for different optical depths for the case of dust between the star and the bow shock. All of the models are for KMH dust at 2.2 µm.

Figure 3.17: Polarized intensity and polarization maps for resolved bow shocks illuminated by a central source, for the case of dust between the star and the bow shock. We show two inclination angles symmetric about 90°. Polarized intensity is in arbitrary units.
Fig. 3.17 displays polarized intensity and polarization maps for a resolved bow shock with dust inside, with optical depth increasing from left to right, and for two different inclination angles symmetric around 90°. The polarized intensity maps look different than the previous cases. When there is no dust inside, the polarized intensity is concentrated near the bow head for $i = 55^\circ$ for all the optical depth cases, whereas when dust is inside the bow shock, the polarized intensity is concentrated near the center. This is because the light can scatter from the dust inside the bow shock and give us polarized light. For $i = 125^\circ$ and no dust inside, the polarized intensity is concentrated near the bow head or mid-region depending on the optical depth. However, when there is dust inside we see polarized intensity arising from the central part of the bow shock. The polarization maps for the case of dust inside are similar to those obtained for the previous cases.

### 3.5 Observational implications

Here we discuss the potential observational implications of our dust-scattering model results. These are important for using observed polarization measurements to understand the dust properties in and around real bow shocks.

Resolved polarization observations of bow shocks are rare in the literature; however, Rauch et al. (2013) presented $K_S$-band polarimetric observations of a resolved stellar wind bow shock arising from the source IRS 8 in the Galactic centre. Because they obtained only 9 polarization measurements across the bow shock, we cannot compare our resolved maps with these observations. However, the polarized intensity and polarization maps of resolved bow shocks from our simulations can be used to constrain dust properties. Polarized intensity maps can constrain the inclination angle for higher optical depth cases. If we can observe polarized intensity at a few different wavelengths, we can also constrain the type of dust particles responsible for the polarized intensity maps.
In cases of unresolved bow shocks (or cases in which a bow shock is predicted to exist, e.g., Neilson et al. 2014), the observed polarization will be a single value corresponding to our viewing angle. This can be compared with results as shown in Fig. 3.5 and we can constrain the optical depth and inclination angle. If we can obtain polarization observations in a few different wavelength bands as in Figs. 3.9, 3.13, 3.14, we can constrain the dust properties such as dust size and composition as well as the temperature of the dust in the bow shock. Broadband polarization uncertainties obtained with RSS/SALT are of order 0.05% at $V \sim 10$ (Fullard et al. 2018), while $V$-band observations of HD 230561 ($V \sim 11$) we obtained using DUSTPol (Wolfe et al. 2015) had uncertainties of 0.01% (Lin et al. 2018). Thus with a variety of current instruments we can observe the difference in polarization behavior depending on dust types as seen in Figs. 3.9, 3.13, and 3.14.

Buchholz et al. (2011) presented unresolved polarization measurements of many other Galactic centre sources, some of which contain known bow shocks. These include IRS 1W, IRS 21, IRS 10W, and IRS 5. For these sources, the intrinsic polarization ranges from about 2% to 16% for IRS 21. These observations show a great diversity for polarization measurements of bow shocks in the Galactic centre.

Even with that diversity, we argue that our models are consistent with these polarization measurements. For the case of IRS 1W, the polarization peaks at the sides of the bow shock away from the bow head, and the directions of motion of the bow shock and gas suggest the bow shock is oriented side on. Our numerical simulations suggest that the polarization increases as a function of optical depth and thickness of the bow shock. For the case of IRS 21, the $\sim 16\%$ observed polarization is likely due to the bow shock being oriented partially away from the observer. This is similar to our simulations with orientations of $i \approx 130^\circ$ and very large optical depths. If this dust is cold, the dust emission is smaller relative to the scattering
of photons, and if, again, our model bow shocks are too thin, then we are roughly consistent with this bow shock.

We appear to underestimate the polarization for one of the Galactic centre bow shock observations. This can be contributed to our model being based on the Wilkin (1996) bow shock geometry which misses the scale of the turbulence in the wings of the bow shock that increases the total polarization.

3.6 Conclusions and future work

We investigated the polarization arising from dust scattering within an analytic stellar wind bow shock defined by Wilkin (1996) for cases of dust emission and no dust emission within the bow shock. We studied how various parameters affect the polarization behaviour for both resolved and unresolved bow shocks. We see that geometrically, polarization is highly dependent on the inclination angle as expected. In the case of lower optical depth, the polarization behaves similarly to that of the analytic case and reproduces the analytical sin^2 i dependence of Brown and McLean (1977). As the optical depth increases, the polarization behaviour diverges from that of the analytic shape and we see a second peak at a higher inclination angle. We see a strong dependence on dust types for polarization variation with wavelength.

In the case of no dust emission from the bow shock, we see that dust types and optical depth both play important roles in polarization behaviour. We do not see any dependence on temperature for this case.

In the case of dust emission from the bow shock, polarization depends on temperature, optical depth, and dust type. We see that at lower optical depths, the dust emission does not change the behaviour of polarization with wavelength compared to the case of no dust emission. In this case the peak near 90° behaves similarly to the case of no dust emission; however, at higher inclination angles where multiple
scattering is dominant, we see a difference in polarization behaviour compared to
the no-emission case.

We also investigated the polarization behaviour when dust fills the interior of
bow shock for one particular dust type (KMH). We see the amount of polarization
near 90° is suppressed for all optical depth values. We also find a broader second
peak at higher inclination angles for higher optical depth regimes.

When interpreting these results, we need to keep in mind the several simplifying
assumptions we made in these models. We chose a specific standoff radius \( R_0 \)
(Section 3.2) for all the models presented here. We tested the effect of changing
\( R_0 \) on polarization behaviour and found that it does not change the in the dust
scattering case. We also fixed the parameter \( \alpha = V_s/V_w = 0.1 \), which is used to
calculate the density of bow shock (Section 3.2). Thus changing \( \alpha \) will have a similar
effect to changing the optical depth of the bow shock.

It is important to note that the analytic bow shock shape and density from
Wilkin (1996) represents an idealised case that assumes a stable and highly evolved
bow shock, as shown by Mohamed et al. (2012) using their smooth particle hy-
drodynamic models. Thus resolved images from this simulation can be compared
with observations to study the evolutionary phase of the observed bow shock. In
other cases with instabilities (Meyer et al. 2014, 2015), the bow shock structure will
have different morphologies. We plan to investigate these different morphologies
pertaining to particular bow shocks in a future contribution.
Chapter 4

Polarization signatures of Betelgeuse bow shocks

4.1 Introduction

In Chapters 2 and 3, I presented theoretical bow shocks following the prescription of Wilkin (1996). This analytic shape corresponds to a stable and evolved bow shock around the star. Using this shape to study a general polarization behavior is a good first approximation that can be used along with observational data to gain information about the properties of bow shocks such as density, temperature, inclination angle, source of emission, dust size, and dust composition, as shown in previous chapters. However, there are limitations on the scenarios that can be treated using this analytic shape. As seen in Fig. 4.1, the shape of a bow shock varies greatly depending on its evolutionary state. Thus, simulating the polarization signal from the bow shock of a particular star requires a more realistic model. Hydrodynamical simulations of bow shocks for various objects have been done by several research groups including Meyer et al. (2014); Mohamed et al. (2013); Shahzamanian et al. (2016); and Zajaček et al. (2017). Thus a code capable of implementing these more
physically realistic density structures will be useful in studying the polarization properties of a bow shock around a particular star.

In this chapter, I discuss the implementation of a realistic bow shock structure for alpha Orionis (Betelgeuse, HD 39801) from a smooth particle hydrodynamic (SPH) model by Mohamed et al. (2012). SPH is a computational method to simulate particles which behave like fluid elements. More details about the SPH method and numerical setup can be found in Mohamed et al. (2012). Using the density from SPH models gives SLIP the capability to simulate the polarization behavior of stellar wind bow shocks at various evolutionary stages. This in turn can provide information about the evolutionary phase of the star. It can also predict the detailed polarization signals for a particular bow shock structure.

This chapter is divided in the following way. In section 4.2, I give a brief description of the SPH code that produced the density structure I used, and outline the modifications I made to the SLIP code to use this density structure. In section 4.3, I present preliminary results for the SPH bow shock. In section 4.4, I discuss
preliminary conclusions and also the future work that we can do with this new code capability.

4.2 Methods

The major difference within SLIP between the SPH density structure and the analytic model is in the way code assigns density values to different grid cells. Even though SLIP is a three-dimensional code, the analytic density structure from Wilkin (1996) reduces to two-dimensional as it is symmetric in $\phi$. However, the SPH density structure is truly three-dimensional.

I modified the input file in SLIP to include a flag that signifies whether we are doing SPH or analytic density. If the flag is true for SPH density, then the code runs differently to set up the SPH density structure. For the setup, SLIP reads in a file which contains density information at different $r$, $\theta$, and $\phi$ values. Once this file is read, density is assigned at the corresponding $r$, $\theta$, and $\phi$ grid values within the code. In this case I used 100 radial bins, 101 $\theta$ bins, and 201 $\phi$ bins. One major difference from the analytic model is that the density is already defined in the SPH input file. Thus I do not rescale the density with input optical depth.

Once the density is set up, the rest of the code functions similar to the case of dust scattering; details can be found in Chapter 3. The results presented in this chapter are for different dust types with the Betelgeuse density structure from Mohamed et al. (2012). Figure 4.2 depicts this SPH model, while the corresponding density within the SLIP code is shown in Fig. 4.3.

4.3 Results

In this section, I present preliminary results from SPH models of the Betelgeuse bow shock (Mohamed et al. 2012) implemented within SLIP. Note that these results
Figure 4.2: Density structure arising from SPH simulations of the Betelgeuse bow shock, from Mohamed et al. (2012).
Figure 4.3: Density structure within *SLIP* created using the input density file from the SPH model of the Betelgeuse bow shock by Mohamed et al. (2012).
do not include dust emission as was done in Chapter 3. Here the results are for three different dust types with a low reference optical depth of $\tau_0 = 0.5$ and a high reference optical depth of $\tau_0 = 2.0$. The standoff radius is $R_0 = 8.5 \times 10^{17}$ cm, as observed by Ueta et al. (2008a). In this case, there is material inside the bow shock and there are clumps in the density distribution.

Since these simulations were computationally heavy, I used the Stampede supercomputer at University of Texas at Austin. Each of the model runs used $3.2 \times 10^8$ photons. This yielded internal polarization uncertainties on the order of $\sigma_p(\%) \sim 0.01$. The runtime for a simulation was 2–3 hours depending on the optical depth. The results are divided in two subsections for low and high optical depth cases.

### 4.3.1 Low optical depth

Figure 4.4 shows preliminary results for representative viewing angles for a central source; the left panels are for $\tau_0 = 0.5$. At all inclination angles we see that the amount of polarization varies greatly among dust types at longer wavelengths. For KMH and MRN dust, the amount of polarization increases and then decreases at higher wavelength values. However, for R550 dust this trend varies depending on the viewing angle. In the first panel, the amount of polarization remains almost constant after a certain wavelength value, whereas in the last panel the polarization keep increasing at higher wavelength as well.

Figure 4.5 present results for the distributed case and left panels are for $\tau_0 = 0.5$. In this case, the shape of polarization behavior is different from the central source case in all the viewing angles. Among different dust types, the amount of polarization varies greatly at longer wavelengths. For R550 dust type, the last data point in first and last panel seem to jump to higher value which is different from the central source case. At longer wavelengths the amount of polarization increases with increasing dust size.
Figure 4.4: Polarization as a function of wavelength at different inclination and $\phi$ angles for three different dust types. All the photons arise from the central star. Error bars are smaller than the symbols.
Figure 4.5: Polarization as a function of wavelength at different inclination and $\phi$ angles for three different dust types. All the photons arise from the bow shock. Error bars are smaller than the symbols.
4.3.2 High optical depth

The right-hand panels of Fig. 4.4 display the results for a high reference optical depth of $\tau_0 = 2.0$ for different viewing angles and dust types, assuming a central photon source. Between the low and high optical depth regimes, we see a major difference in the behavior of the KMH dust model. In the first and the last panel, the amount of polarization uniformly increases with increasing wavelength. In the last panel, we there is not much difference between the behavior of the KMH and R550 dust. Thus if the density is high, the polarization behaves similarly for different dust sizes and compositions. The right panels of Fig. 4.5 show the distributed source case. In this case there is no smooth trend with wavelength for any of the dust types.

4.4 Conclusions and Future Work

In this chapter, I presented a preliminary implementation of a density structure which makes the code more physical and allows consideration of a particular stellar wind bow shock (in this case, around Betelgeuse). The implementation is in an early phase, and more testing is needed. So far, I have studied how the polarization varies with wavelength for different dust types at various representative viewing angles. The results show that the amount of polarization at longer wavelengths depends sensitively on dust type. For lower optical depths and central source illumination, we see a smooth curve similar to that produced by the analytic density structure. However, in the case of high optical depth and distributed illumination, the trend is quite different and the curves are not smooth. Thus, the polarization behavior with wavelength is different compared to analytic density results for high density and distributed cases.

So far I have investigated only one SPH density structure. After more tests, I will implement different density structure representing various evolutionary phases of a
bow shock. This will predict the polarization signatures we can expect at various stellar ages. Another future step will be to include dust emission from the bow shock, as was done in Chapter 3. After the addition of the dust emission, I will compare results from these simulations with polarization observations of Betelgeuse.
Chapter 5

Conclusions and Future Work

5.1 Conclusions

In this dissertation I investigated the properties of stellar wind bow shock nebulae using polarization of light. I studied the effects of different parameters such as density, temperature, and dust types on the polarization behavior. I created a simulation grid of polarization behavior which can be used along with observational data to constrain various properties of the stellar wind and the local ambient medium in a bow shock scenario.

In Chapter 2, I investigated the polarization produced by electron scattering in cases of pure scattering and scattering with absorption for various combinations of parameters. At very low optical depths, my numerical simulation produced \( \sin^2 i \) behavior, where \( i \) is the inclination angle, as predicted by the analytic calculations of Brown and McLean (1977). I found that multiple scattering causes the polarization signatures to diverge from the analytic prediction. Thus, the density of bow shock plays an important role in determining the polarization behavior. In addition to density, I found that the inclination angle has a high impact on polarization as expected. In the cases where absorption is present, I found that temperature plays
an important role in polarization behavior. I also studied the impact of the source of illumination on polarization features. For the unresolved cases, I found little distinction between central and distributed sources, whereas distinct differences were seen in resolved polarization maps. Thus these results can be used along with observational data to extract the properties of the star and the scattering medium such as inclination angle, density and temperature of the bow shock, source of illumination, and stellar mass-loss rate.

In Chapter 3, I investigated the impact of various parameters on the polarization signals for the case of pure dust scattering, with and without emission from the dust in the bow shock. When the dust does not emit, I found that polarization varies with wavelength in a different manner depending on the dust type (grain size, grain composition, etc) and optical depth. At longer wavelengths, the polarization behavior diverges greatly for different dust types compared to shorter wavelengths. These simulation results can be used with multiwavelength broadband polarization observations to extract the properties of dust in the bow shock. When dust emission is included in the simulation, I found that for higher optical depth, temperature plays an important role in polarization. Thus these numerical results can be used to constrain the dust temperature in an observed bow shock. These results can help us understand what happens to the dust when the stellar wind and ISM interact.

In Chapter 4, I presented a more physical approach for the density structure of the bow shock to simulate polarization behavior from dust scattering. This setup can be used to study the properties of a particular bow shock. I have shown that SLIP can be modified to implement the density structure from hydrodynamic models and predict the resulting polarization for different combinations of parameters.

In conclusion, this dissertation work created a simulation grid which can be used along with observational data to constrain various properties of the stellar wind bow shocks using polarization of light. These simulations are general enough
to be used for various polarization observations of bow shock candidates. Such studies can help us better understand the orientation of the bow shock, density of the scattering medium, temperature of the scattering medium, and dust properties in the interaction region. Ultimately, this information can lead to a better understanding of stellar winds, stellar evolution, and the nature of the ISM in the Milky Way.

5.2 Future work

In this project, I created a simulation grid of polarization signatures using analytic expressions for the geometry and density of a stellar wind bow shock. However, the analytic bow shock is an idealized case that corresponds to an evolved and stable bow shock. There have been many observations of bow shocks with instabilities, clumps and distorted shapes. I expect these cases will broadly produce similar polarization signatures, but detailed study of particular bow shocks with different morphologies requires additional modeling.

In Chapter 4, I discussed the implementation of Betelgeuse’s bow shock structure in the code and provided preliminary polarization results. The use of density structure from hydrodynamic models for a particular bow shock will be a good way to study the polarization features of these complicated morphologies. Making the code capable of handling various kinds of density structures and then perform MCRT to produce polarization signals will be a good next step.

So far I have used electron and dust as the scattering material in the bow shock region separately. I would like to update the code such that the scattering region is composed of mixture of dust and gas. In this case polarization by both electron and dust scattering will be possible.

In addition to improving the code, comparing the simulation results with observational data and extracting physical properties of the bow shock will be important to study the environments around these massive evolved stars which will end their
life as a supernovae. I have worked with an undergraduate student at DU, Austin Lin (BS 2018), to obtain polarization observations of a bow shock candidate HD 230561 (Kobulnicky et al. 2016). We used the Denver University Small Telescope Polarimeter (DUSTPol; Wolfe et al. 2015) at the University of Denver to observe this star. We observed 3.88% polarization for the target before subtracting the interstellar polarization. After estimating the interstellar polarization from other stars in the observed field, we obtained an intrinsic polarization signal of 1.05% in a broad visual band. We compared these results with the simulation results from SLIP to constrain the inclination angle and density of the bow shock. Details of this observations and results can be found in Lin et al. (2018); Lin (2018).

Finally, I plan to implement the bow shock structure into MCRT models of the wind collision regions of Wolf-Rayet binaries to simulate the polarization data Dr. Hoffman’s group has obtained from the RSS spectropolarimeter on the Southern African Large Telescope. The bow shock formed in a colliding wind system would have a different shape than the analytic model I used for my single-star simulation; however, the analytic bow shock structure would serve as a first approximation to model polarization signal from these colliding winds.


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1996a. The Effect of Multiple Scattering on the Polarization from Axisymmetric


Appendix A

Calculation of $b(\theta)$

The $b$ factor referred in Chapter 2 is given by

\[ b(\theta) = \sqrt{1 + \frac{1}{4} \left( \frac{\theta \csc \theta^2 - 3 \cot \theta + 2 \theta \cot \theta^2}{1 - \theta \cot \theta} \right)^2}; \quad (A.0.1) \]

we show its functional form in Fig. A.1. This factor arises from the arc length formula involved in the calculation of surface area. Its presence here is due to the fact that an area element of the bow shock is not generally oriented normal to a radial vector from the star, with respect to which we define the optical depth $\tau$. The bow shock is axisymmetric and therefore can be considered a surface of revolution about the $z$-axis. The surface area, $S$, is defined in terms of the curve described by the bow shock at a fixed azimuth. The path length of the curve from the bow head to some point downstream along the shock at position $(r, \theta)$ is represented by $l$. The surface area for that portion of the bow shock is then

\[ S = \int 2\pi r \sin \theta \, dl, \quad (A.0.2) \]

where $r$ is the radius from the star to the curve, and $dl$ is given by
\[ dl = \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta. \] (A.0.3)

After substituting the expression for \( dl \) into the \( S \) integral and factoring, we find the surface area becomes

\[ S = \int 2\pi r^2 \sin \theta \, d\theta \sqrt{1 + \left( \frac{d\ln r}{d\theta} \right)^2}. \] (A.0.4)

The term under the square root is what we call the \( b \) factor. Thus,

\[ b(\theta) = \sqrt{1 + \left( \frac{d\ln r}{d\theta} \right)^2}, \] (A.0.5)

where \( r \) is given by Equation 3.2.2 for the bow shock. Putting Equation 3.2.2 into Equation A and simplifying, we obtain Equation A.0.1. In the code, we implement this factor discretely by calculating \( b \) for each grid cell.
Figure A.1: Variation of the $b$ factor with $\theta$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figureA1.png}
\end{figure}