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T-de Vries Algebra

Nawal Alznad
University of Denver

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Abstract

The main point of this dissertation is to introduce the action on de Vries algebra by a topological monoid and we denoted the resulting category by **dVT**. In order to reach our goal, we started with introducing new proofs for some well known results in the category of flows. Then, we studied the Generalized Smirnov's Theorem for flows. After we studied the new category (**dVT**), we were able to provide a new way to construct the Čech-stone flow compactification of a given flow. Finally, we developed the co-free T-de Vries algebra for a special case.

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Nikolaos Galatos

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Nawal Alznad

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Advisor: Richard Ball

Author: Nawal Alznad
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Abstract

The main point of this dissertation is to introduce the action on de Vries algebra by a topological monoid and we denoted the resulting category by **dVT**. In order to reach our goal, we started with introducing new proofs for some well known results in the category of flows. Then, we studied the Generalized Smirnov's Theorem for flows. After we studied the new category (**dVT**), we were able to provide a new way to construct the Čech-Stone flow compactification of a given flow . Finally, we developed the co-free T-de Vries algebra out of a given de Vries algebra for a spacial case.

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Contents

Chapter 1

Introduction And Motivation

1.1 Organization of the thesis

In this chapter we motivate the topics covered in this dissertation and outline the tools used in the analysis. The remaining sections of this chapter are not meant to serve as a comprehensive introduction. Instead, the reader may refer to the more detailed introductions at the beginning of each chapter. The appendix lists definitions and known facts about various structures which appear in chapters 2,3, and 4. It is best consulted as a supplementary reference from within the main chapters. In order to assist the reader in dealing with the features of this thesis, a list of symbols (pages vi, vii) is provided.

This dissertation covers three different topics: topological dynamics, proximity flows, and T-de Vries algebras. Each topic may be read separately and requires some background.

1.1.1 What is new

Chapter 2 covers well known facts about topological dynamics with some new proofs that are aligned with our main work. It covers all the basics needed to understand the topic from the definitions of flow, flow maps, and subflows. In particular, T-scales, one of the tools on which we rely heavily, are defined in this chapter. In fact, Lemma (2.1.11) and Proposition (2.1.14) are the most important and we will use them frequently in Chapters 3 and 4.

Chapter 3 begins with an introduction to Smirnov's Theorem (Theorem 3.1.10), for which we include a new proof. In order to extend this theorem to dynamical systems we need to define the concept of a T-proximity on a flow, which is our second tool used in Chapter 4. Even though T-proximities and the Generalized Smirnov's Theorem (Theorem 3.1.15) have been studied by other authors ([8]), we indicate new proofs using T-scales (our first tool) that will make this approach more helpful in later chapters. Although we assume that all spaces in this chapter are compactifiable flows, we found it useful to end this chapter with the consideration of non-compactifiable flows and non-separated T-proximities.

Chapter 4 contains our main topic. We start with the material needed to cover classical de Vries duality (Theorem 4.1.5). Then we use all of the machinery developed in the previous chapters to prove (1) the duality between compact flows and T-de Vries algebras, (2) the continuity of the action of T on a compact space is equivalent of the smoothness of the action on the dual De Vries algebra, and (3) subflows are dual to round T-filters.

In Chapter 5 we give the structure of the co-free T-de Vries algebra over a naked de Vries algebra, but only for a compact topological monoid T. So we had to start the chapter with a careful development of the sum of two de Vries algebras. This is a very short chapter; however, this topic is very interesting.

The last chapter (6) covers some applications of T-de Vries algebras. We conclude the chapter with some questions that may be considered for future work.

1.2 Why we study T-de Vries algebra

In 1962, de Vries developed an algebraic approach to the category of compact spaces. The duality, which now bears his name (see [5]), has been exploited by several authors, notably Bezhanishvili and coauthors ([3], [4]). The ideas have been generalized to different categories, such as locally compact spaces and frames, and have a number of applications, such as Stone duality and the Gleason cover of a space.

The idea of de Vries algebra is to associate to every compact Hausdorff space X the complete boolean algebra $RO(X)$ together with the proximity relation $\prec$ defined by $a \prec b$ if $\operatorname{cl} a \subseteq b$; the resulting object is written $(RO(X), \prec)$. In order to extend this duality to compact flows, the proximity is required to satisfy an additional axiom, and we we call it a T-proximity. It develops that every compactifiable flow admits a compatible T-proximity.

The Generalized Smirnov Theorem for flows was first proved by Google and Megrelishvili in [8]. We found it important to include this topic with new proofs mainly to generalize De Vries's work.

As we mentioned above, Stone duality can be viewed as a particular case of de Vries duality. An obvious question, then, is to ask whether this will be the case for Stone flows and **dVT**. There is a work done by Ball, Geschke, and Hagler (see [11]) on the duality between Stone flows and T-Boolean algebras, and we preferred to point out this duality in the last chapter as an application.

Our generalization of de Vries algebra may open the door to new areas of research in the field of dynamical systems. For example, it may provide insight into the structure of the Gleason cover of a compact flow or even the Gleason cover of a compactifiable flow. This is a rather mysterious object about which rather little seems to be known. The dual notion, namely the injective **dVT**-envelope of a T-de Vries algebra, is likewise a natural and important topic about which little is known. Furthermore, the category of frames with actions and its duality with **dVT** will be another area of interest.

Chapter 2

Introduction To Topological Dynamics

This chapter is a very short introduction to topological dynamics. This is a topic with a vast literature, and what follows is only enough to provide context for the development that follows. The material itself is well known, with proofs available mostly in [2]; only those facts which may be less well known or are particularly relevant to the development are proven in detail.

We define flows and flow compactifications. We also explain the relation between T-scales and compatibility of a given flow. We end the chapter by defining a special kind of flow map, called a T-irreducible flow map, and discussing some conditions equivalent to the definition.

2.1 Flows

Throughout T denotes a topological monoid, i.e. a monoid T endowed with a topology making multiplication a continuous map $\mathsf{T} \times \mathsf{T} \longrightarrow \mathsf{T}$, whose members we call actions and denote by r, s, or t, sometimes with subscripts. We use $\mathcal{N}_t$ to denote the neighborhood filter (see A.2.6 for definition) of an element $t \in T$.

Definition 2.1.1. *[2] We say that the topology of T is based at 1 if $N_1 t \in \mathcal{N}_t$ for all $t \in T$ and $N_1 \in \mathcal{N}_1$. This is equivalent to saying that the neighborhood filter of each $t \in T$ is generated by the translates $N_1 t$ of the neighborhoods N_1 of 1.*

Definition 2.1.2. *We say that T acts on a space X if there is a monoid homomorphism $\phi_X \colon T \to \hom_{\mathbf{Sp}}(X, X)$, where **Sp** designates the category of spaces with continuous maps. That is,*

1. *$\phi_X(1)$ is the identity function on X, and*

2. *$\phi_X(ts) = \phi_X(t)\phi_X(s)$ for all $t, s \in T$.*

We write $\phi_X(t)(x)$ as simply tx. A flow is a triple (X, T, ϕ_X), where X is a Tychonoff space, T is a topological monoid, and $\phi_X \colon T \to \hom_{\mathbf{Sp}}(X, X)$ is an action of T on X which makes the evaluation map $(t, x) \to tx$ continuous. A subset $Y \subseteq X$ is T-invariant if $ty \in Y$ for all $y \in Y$ and $t \in T$, and a T-invariant subspace of a flow is called a subflow. A subset U of a flow X is T-stable if $t^{-1}U \subseteq U$ for all $t \in T$; note that U is T-stable iff $X \setminus U$ is T-invariant. And finally, a flow map $f \colon X \to Y$ is a continuous function between flows which commutes with the actions.

Example 2.1.3. *Let $X = [0, 1]$ with the usual topology, and let*

$$T \equiv \{\, t \colon X \to X : t \textit{ is continuous and } t(0) = 0 \textit{ and } t(1) = 1 \,\}$$

Give T *a topology that makes* X *a flow; for instance, we could choose the topology on* T *to be the compact open topology (see A.2.8 for definition). The only* T-*invariant subsets are* $\{0\}$, $\{1\}$, $\{0, 1\}$, *and* X, *and these are all subflows.*

Notation 2.1.4. *Throughout we denote the archimedean lattice ordered group of continuous real-valued functions* $f\colon X \to \mathbb{R}$ *by* $C(X)$. *The bounded part of* $C(X)$ *is designated*

$$C^*(X) \equiv \{\, f \in C(X) : \exists n \in \mathbb{N}\ (|f| \leqslant n)\,\}.$$

Definition 2.1.5. *Let* X *be a space acted upon by* T. *A function* $g \in C(X)$ *is said to be* T-*uniformly continuous if for all* $t \in T$ *and* $\varepsilon > 0$ *there is a neighborhood* $N_t \in \mathcal{N}_t$ *such that*

$$\forall s \in N_t,\ x \in X\ (|g(sx) - g(tx)| < \varepsilon).$$

If the topology on T *is based at* 1, *this condition need only be checked at* $t = 1$. *We denote the collection of all* T-*uniformly continuous functions on* X *by* $C^T(X)$.

The relevance of the notion of T-uniform continuity for our purposes is given by Lemma 2.1.6. This lemma is well known (see [2] Theorem 3.2 and proposition 5.12), but we include a proof because it is fundamental for our purposes.

Lemma 2.1.6. *Let* X *be a compact space acted upon by* T. *Then* X *is a flow, i.e., the evaluation map* $T \times X \to X$ *is continuous, iff every member of* $C(X)$ *is* T-*uniformly continuous, i.e., iff* $C^T(X) = C(X)$.

Proof. Suppose X is a flow. Given $g \in C(X)$, $t \in T$, and $\varepsilon > 0$, use the continuity of g to find for each $x \in X$ a neighborhood V_x of tx such that $|g(x') - g(tx)| < \varepsilon$ for all $x' \in V_x$. Then use the continuity of evaluation to find neighborhoods U_x of x and N_{t_x} of t such that $t'x' \in V_x$ for all $x' \in U_x$ and $t' \in N_{t_x}$.
Let $\{ U_{x_i} : 1 \leqslant i \leqslant n \}$ be a finite subcover of $\{ U_x : x \in X \}$, and put $N \equiv \bigcap_{1 \leqslant i \leqslant n} N_{tx_i}$. Then an arbitrary $x \in X$ lies in some U_{x_i} and so we have

$$|g(t'x) - g(tx)| \leqslant |g(t'x) - g(tx_i)| + |g(tx) - g(tx_i)| < 2\varepsilon$$

for all $t' \in N$.

On the other hand, suppose every member of $C(X)$ is T-uniformly continuous. Let V be a neighborhood of tx for given $x \in X$ and $t \in T$. Because $C(X)$ separates points from closed sets, we can find $g \in C(X)$, $0 \leqslant g \leqslant 1$, such that $g(tx) = 1$ and $g(x) = 0$ for all $x \notin V$. Because g is T-uniformly continuous, there is a neighborhood N of t such that $|g(tx') - g(t'x')| < \varepsilon$ for all $t' \in N$ and $x' \in X$. Now $g^{-1}(1/2, \infty)$ is a neighborhood of tx and t is continuous, so we can find a neighborhood U of x such that $g(tx') > 1/2$ for all $x' \in U$. Then for all $x' \in U$ and $t' \in N$, the facts that $g(tx') > 1/2$ and $|g(tx') - g(t'x')| < 1/2$ imply that $g(t'x') > 0$, i.e., $g(t'x') \in V$. This proves the continuity of evaluation. □

Lemma 2.1.7. *Let* X *be a space acted upon by* T. *Then* $C^T(X)$ *is a uniformly closed* ℓ*-subgroup of* $C(X)$ *which contains the constant functions and which is* T*-invariant in the sense that* $g \in C^T(X)$ *implies* $gt \in C^T(X)$ *for all* $t \in T$.

Proof. See Proposition 4.3 in [2]. □

Definition 2.1.8. *For subsets* U *and* V *of a flow* X*, we say that* U *is* T*-disjoint from* V *if for all* $t \in T$ *there is a neighborhood* N_t *of* t *such that for all* $s \in N_t$ *it is true that*

$$s^{-1}U \cap t^{-1}V = t^{-1}U \cap s^{-1}V = \emptyset.$$

We say that U *is* T*-contained in* V*, and write* $U \subseteq_T V$*, if* U *is* T*-disjoint from* $X \backslash V$*. Thus* $U \subseteq_T V$ *means that for each* $t \in T$ *there is a neighborhood* N_t *of* t *such that*

$$\bigcup_{N_t} r^{-1}U \subseteq t^{-1}V \ \textit{ and } t^{-1}U \subseteq \bigcap_{N_t} s^{-1}V.$$

If the topology on T is based at 1 then this definition simplifies considerably. In that case U is T-disjoint from V if there is a neighborhood N_1 of 1 such that $s^{-1}U \cap t^{-1}V = \emptyset$ for all $s, t \in N_1$, and U is T-contained in V if there is a neighborhood N_1 of 1 such that $s^{-1}U \subseteq t^{-1}V$ for all $s, t \in N_1$.

Definition 2.1.9. *Two subsets* A *and* B *are said to be* T*-completely separated if there is a* T*-uniformly continuous function* g *on* X *which is* 0 *on* A *and* 1 *on* B.

It is worth remarking that two sets contained, respectively, in two T-completely separated sets are T-completely separated, and that two sets are T-completely separated if and only if their closures are.

Definition 2.1.10. *A scale is a collection* $S = \{U_p\}$ *of open subsets of a given space* X*, indexed by the rational numbers* $\mathbb{Q}$*, such that* $\mathrm{cl}(U_q) \subseteq U_p$ *for all* $q < p$ *in* $\mathbb{Q}$.

Actually, what is important is only that the index set Q be countable, totally ordered, without endpoints, and dense, meaning that for any $q_1 < q_2$ in Q there should exist $q_3 \in Q$ such that $q_1 < q_3 < q_2$. After all, any such set is isomorphic to $\mathbb{Q}$. Therefore we shall call any family $\{U_q : q \in Q\}$ of open subsets a scale provided that $\mathrm{cl}(U_q) \subseteq U_p$ whenever $q < p$, and provided that the index set is a countable dense totally ordered set without endpoints. But if no mention is made of the index set then we assume it to be $\mathbb{Q}$.

Lemma 2.1.11 shows how scales code continuous real-valued functions. It also shows why it is often convenient to index scales with the countable dense totally ordered set $(0,1)_{\mathbb{Q}}$.

Lemma 2.1.11. *For a continuous function* $f \colon X \to \mathbb{R}$*,* $f \geqslant 0$*, the set*

$$S \equiv \left\{\, f^{-1}(-\infty, q) : q \in (0,1)_{\mathbb{Q}} \,\right\}$$

is a scale, and given a scale $S = \{\, U_q : q \in (0,1)_{\mathbb{Q}} \,\}$*, the function* $f \colon X \to \mathbb{R}$ *defined by the rule*

$$f(x) = \begin{cases} \bigwedge \{\, q : x \in U_q \,\} & \text{if } x \in U_q \text{ for some } q \in (0,1)_{\mathbb{Q}}, \\ 1 & \text{if } x \notin U_q \text{ for all } q \in (0,1)_{\mathbb{Q}} \end{cases}$$

is positive and continuous and has the feature that $U_p \subseteq f^{-1}(-\infty, q) \subseteq U_q$ *for all* $p < q$ *in* $(0,1)_{\mathbb{Q}}$.

Proof. For a continuous function $f \colon X \to \mathbb{R}$, $f \geqslant 0$, and for $q \in (0,1)_{\mathbb{Q}}$, put $U_q \equiv f^{-1}(-\infty, q)$. Since f is a continuous function, $\mathrm{cl}(U_q) \subseteq f^{-1}(-\infty, q]$, i.e., $f(x) \leqslant q$ for all $x \in \mathrm{cl}(U_q)$. It follows at once that $\mathrm{cl}(U_q) \subseteq U_p$ for $q < p$ in

$(0,1)_{\mathbb{Q}}$, which is to say that $S \equiv \{U_q\}$ is a scale.

Now assume that a scale $S \equiv \{ U_q : q \in (0,1)_{\mathbb{Q}} \}$ is given and that the function f is defined from it as above. It is clear that $0 \leqslant f(x) \leqslant 1$ for every $x \in X$. To prove that f is continuous let $x \in X$ and let (c,d) be an open interval of $\mathbb{R}$ containing $f(x)$. Find $p, i, j \in \mathbb{Q}$ such that $c < p < i < f(x) < j < d$ and let $U \equiv U_j \setminus \mathrm{cl}(U_p)$. We first claim that U is an open neighborhood of x. For $f(x) = \bigwedge\{ q : x \in U_q \} < j$ implies that $x \in U_q$ for some $q < j$, hence $x \in U_j$. And we cannot have $x \in \mathrm{cl}(U_p)$, for otherwise $x \in \mathrm{cl}(U_p) \subseteq U_i$ would imply $f(x) = \bigwedge\{ q : x \in U_q \} \leqslant i$, contrary to assumption. This proves the first claim. The last claim is that $f(U) \subseteq (c,d)$. This holds because for every $x' \in U$ we have

$$x' \in U_j \implies f(x') = \bigwedge\{ q : x' \in U_q \} \leqslant j < d,$$

and because $f(x') \geqslant p$, for otherwise $f(x') = \bigwedge\{ q : x' \in U_q \} < p$ would imply that $x' \in U_q$ for some $q < p$, hence $x' \in U_p \subseteq \mathrm{cl}(U_p)$, contrary to assumption. We conclude that f is a continuous function. □

An important observation is that if $f \colon X \to Y$ is a continuous function and $\{U_q\}$ is a scale in Y then $\{f^{-1}(U_q)\}$ is a scale in X.

Definition 2.1.12. *[2] A T-scale on a flow X is a scale $S \equiv \{U_p : p \in \mathbb{Q}\}$ such that U_q is T-contained in U_p for all $q < p$ in $\mathbb{Q}$.*

Lemma 2.1.13. *A continuous real-valued function f on a flow X is T-uniformly continuous iff the scale associated with it in Lemma 2.1.11 is a T-scale.*

Proof. Let f be a T-uniformly continuous real-valued function on a flow X. By Lemma 2.1.11,

$$S \equiv \left\{ f^{-1}(-\infty, q) : q \in (0,1)_{\mathbb{Q}} \right\} \equiv \{ U_q : q \in (0,1)_{\mathbb{Q}} \}$$

is a scale. Fix $t \in T$ and $q < p$, and let $\varepsilon \equiv p - q$. Because $f \in C^T(X)$, there exists a neighborhood N_t of t such that for all $x \in X$ and $r, s \in N_t$ we have

$$|f(rx) - f(sx)| < \varepsilon,$$

and therefore

$$\begin{aligned} x \in r^{-1}U_q \implies rx \in U_q = f^{-1}(-\infty, q) \implies f(rx) < q \\ \implies f(sx) < p \implies sx \in U_p \implies x \in s^{-1}U_p. \end{aligned}$$

Thus U_q is T-contained in U_p for all $q < p$ in $\mathbb{Q}$, and hence S is a T-scale.

Now assume that a T-scale $S \equiv \{ U_q : q \in (0,1)_{\mathbb{Q}} \}$ is given, and use Lemma 2.1.11 to define a continuous function f on X. Fix $t \in T$ and $\varepsilon \in (0,1)_{\mathbb{Q}}$. We claim that there is a neighborhood $N_t \in \mathcal{N}_t$ such that

$$s^{-1}U_q \subseteq t^{-1}U_{q+\varepsilon} \quad \text{and} \quad t^{-1}U_q \subseteq s^{-1}U_{q+\varepsilon}$$

for all $s \in N_t$ and all $q \in (0,1)_{\mathbb{Q}}$ such that $q + \varepsilon \in (0,1)_{\mathbb{Q}}$. Fix a positive integer $n \geqslant 2/\varepsilon$, and for each integer k, $0 \leqslant k \leqslant n-1$, find a neighborhood $N_k \in \mathcal{N}_t$

such that $s^{-1}U_{\frac{k}{n}} \subseteq t^{-1}U_{\frac{k+1}{n}}$ and $t^{-1}U_{\frac{k}{n}} \subseteq s^{-1}U_{\frac{k+1}{n}}$ for all $s \in N_k$. Let $N_t \equiv \bigcap_0^{n-1} N_k \in \mathcal{N}_t$. Then for $s \in N_t$ and $q \in (0,1)_{\mathbb{Q}}$ such that $q + \varepsilon \in (0,1)_{\mathbb{Q}}$, let k be the least integer such that $k/n \geqslant q$. We have

$$s^{-1}U_q \subseteq s^{-1}U_{\frac{k}{n}} \subseteq t^{-1}U_{\frac{k+1}{n}} \subseteq t^{-1}U_{q+\varepsilon},$$

and likewise $t^{-1}U_q \subseteq s^{-1}U_{q+\varepsilon}$. This proves the claim. The claim then shows that for any $s \in N_t$ and $x \in X$ we have

$$f(sx) = \bigwedge_{sx \in U_q} q = \bigwedge_{x \in s^{-1}U_q} q \geqslant \bigwedge_{x \in t^{-1}U_{q+\varepsilon}} q = \bigwedge_{tx \in U_{q+\varepsilon}} (q+\varepsilon) - \varepsilon = f(tx) - \varepsilon.$$

A parallel argument yields $f(tx) \geqslant f(sx) - \varepsilon$. This completes the proof that f is T-uniformly continuous. □

Proposition 2.1.14. *[2] Two subsets A and B of a flow X are T-completely separated iff there is a T-scale $S \equiv \{U_q\}$ such that $A \subseteq U_q$ and $B \cap U_q = \emptyset$ for all $q \in \mathbb{Q}$.*

Proof. Suppose that subsets A and B of a flow X are T-completely separated. Then there exists a T-uniformly continuous function $f \in C^T(X)$ such that $f(A) = 0$ and $f(B) = 1$. Let $V_q = f^{-1}(-\infty, q)$ for $q \in (0,1)_{\mathbb{Q}}$. Then by Lemma 2.1.13 $S \equiv \{V_p\}$ is a T-scale, and clearly $A \in V_q$ and $B \cap V_q = \emptyset$ for all $q \in (0,1)_{\mathbb{Q}}$. Now assume that there is a T-scale $S = \{U_q\}$ such that $A \subseteq U_q$ and $B \cap U_q = \emptyset$ for all $q \in (0,1)_{\mathbb{Q}}$. Define $g\colon X \to \mathbb{R}$ as in Lemma 2.1.11. By construction $g(A) = 0$ and $g(B) = 1$. By Lemma 2.1.13 g is a T-uniformly continuous function. □

Given a flow X, let $\beta_X \colon X \to \beta X \equiv Y$ be its Čech-Stone compactification.

- Each action $t \colon X \to X$ lifts to a unique action $t^\beta \colon \beta X \to \beta X$ such that $t^\beta \circ \beta_X = \beta_X \circ t$.
- Because β is a functor, this defines an action of T on βX, i.e., $(t_1 t_2)^\beta(y) = t_1^\beta(t_2^\beta(y))$ and $1^\beta(y) = y$ for all $y \in \beta X$. This action need not make βX a flow, i.e., evaluation need not be continuous.
- Each bounded member f of $C(X)$ extends to a unique member f^β of $C(\beta X)$. Define an equivalence relation $\sim_T$ on βX by declaring $y_1 \sim_T y_2$ if $f^\beta(y_1) = f^\beta(y_2)$ for all bounded functions $f \in C^T(X)$. For each $y \in \beta X$ let $[y] \equiv \{ y' : y' \sim_T y \}$ designate the equivalence class of y, let $Z \equiv \beta X/{\sim_T}$ designate the quotient space, and let $q \colon \beta X \to Z$ designate the quotient map. Because q is continuous and surjective, Z is a compact Hausdorff space and q is a closed map. Finally, abbreviate $q \circ \beta_X$ to β_X^T.

$$\begin{tikzcd}
\beta X \arrow[r, "q"] & Z \arrow[d, "f^T"] \\
X \arrow[u, "\beta_X"] \arrow[ur, "\beta_X^T"] \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

Proposition 2.1.15. *Assume the foregoing terminology. Then the following hold.*

1. *By construction, for each $f \in C^T(X)$ the function f^β factors through q, say $f^\beta = f^T \circ q$ for some $f^T \in C(Z)$. Since these functions separate the points of Z, Z is compact, and C^T is uniformly closed by Lemma 2.1.7,*

$$\{ f^T : f \in C^T(X) \} = C(Z).$$

2. T *acts on* Z *by the rule* $t[y] = [ty]$, $t \in T$, $y \in Y$, *and* β_X^T *commutes with these actions. With respect to this action,* Z *is a compact flow and* β_X^T *is a flow map.*

3. *The map* β_X^T *is universal among flow compactifications of* X. *That is, any flow compactification* $\alpha\colon X \to Y$ *factors uniquely through* β_X^T.

Proof. (2) The action of T on Z is well defined, for $y_1 \sim_T y_2$ and $t \in T$ imply $t^\beta y_1 \sim_T t^\beta y_2$. That is because for each bounded $f \in C^T(X)$ we have $ft \in C^T(X)$, hence $f^\beta(t^\beta(y_1)) = (ft)^\beta(y_1) = (ft)^\beta(y_2) = f^\beta t^\beta(y_2)$. To show that β_X^T commutes with these actions, observe that

$$\begin{aligned}\beta_X^T(tx) = q(\beta_X(tx)) = q(t^\beta(\beta_X(x))) = [t^\beta(\beta_X(x))] = t^\beta[\beta_X(x)]\\ = t^\beta(q(\beta_X(x))) = t^\beta\beta_X^T(x).\end{aligned}$$

A routine calculation can then be used to show that the functions of $C(Z)$ are T-uniformly continuous, with the result that Z is a flow by Lemma 3.1.7 and β_X^T is a flow compactification.

(3) See Theorem 5.18 of [2]. □

Notation 2.1.16. *For a flow* X *let* $\beta_X^T\colon X \to Z \equiv \beta^T X$ *denote the flow compactification of Proposition 2.1.15.*

Theorem 2.1.17. β^T *is a functor. That is, any flow map* f *lifts uniquely to a flow map* $\beta^T f$ *which makes this diagram commute.*

$$\begin{array}{ccc} \beta^T X & \xrightarrow{\beta^T f} & \beta^T Y \\ {\scriptstyle \beta^T_X}\uparrow & & \uparrow{\scriptstyle \beta^T_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Definition 2.1.18. *A flow* X *is* compactifiable *if it is (flow homeomorphic to) a subflow of a compact flow.*

Proposition 2.1.19. *A flow is compactifiable if and only if each of its points is* T*-completely separated from every closed set not containing it.*

Proof. Suppose that Y is a flow compactification of X. Then $C^T(Y) = C(Y)$, and therefore $f|_X \in C^T(X)$ for every $f \in C^T(Y)$. Hence $C^T(X)$ separates points from closed sets.

Assume that every point in X is T-completely separated from every closed set not containing it. This condition gives to the quotient map $q\colon \beta X \to Z$ of Proposition 2.1.15 the property that its composition $q \circ \beta_X = \beta^T_X$ is one-one, making X a subflow of the compact flow Z. □

Theorem 2.1.20. *Every flow* X *has a finest compactifiable quotient flow* $Y = X/{\sim}$*, where* $x \sim y$ *if and only if* $g(x) = g(y)$ *for all* $g \in C^T(X)$*, and* $\beta^T X = \beta^T Y$*.*

Proof. This is the case in Proposition 2.1.15 when $\beta_X(x_1) \sim_T \beta_X(x_2)$ for points $x_1 \neq x_2$ in X. □

2.2 C^T-embeddings

A subflow Y of X is said to be C^T-embedded in X if every function in $C^T(Y)$ can be extended to a function in $C^T(X)$.

Proposition 2.2.1. *[2] The only flow compactification in which a compactifiable flow* X *is* C^T*-embedded is* $\beta^T X$.

The next theorem is the Urysohn's Extension Theorem for flows. For its proof see [2].

Theorem 2.2.2. *A subflow* Y *is* C^T*-embedded in a flow* X *if and only if any two* T*-completely separated subsets of* Y *are* T*-completely separated in* X.

Corollary 2.2.3. *[2] In a compactifiable flow any compact subset is* T*-completely separated from any closed set disjoint from it, and any compact subflow is* C^T*-embedded.*

2.3 T-irreducible maps

Recall that in general topology, a continuous function $f\colon X \to Y$ is called perfect if it is closed and has compact fibers, i.e., $f^{-1}(y)$ is compact for all $y \in Y$. f is called irreducible if it is a perfect surjection which maps no proper closed subset of X onto Y.

Definition 2.3.1. *Let* X *and* Y *be flows and let* f *be a perfect flow map from* X *onto* Y*. Then* f *is called* T*-irreducible if, whenever* A *is a proper closed subflow of* X*,* $f(A) \neq Y$.

Lemma 2.3.2. *Let* $f\colon X \to Y$ *be* T*-irreducible. Then:*

1. *If* $g\colon Y \to Z$ *is* T*-irreducible, then* $g \circ f\colon X \to Z$ *is* T*-irreducible;*

2. *Any initial factor of* f *which is a closed surjection is* T*-irreducible. That is, if* $f = k \circ h$ *for a closed surjection* $h\colon X \to Z$ *and a continuous function* $k\colon Z \to Y$*, then* h *is* T*-irreducible.*

3. *If* $k\colon Z \to X$ *is a closed flow surjection such that* $h \equiv f \circ k$ *is* T*-irreducible then* k *is* T*-irreducible.*

Proof. (1) Since f and g are closed surjective, so is $g \circ f$. If A is a proper closed subflow of X, then $f(A)$ is a proper closed subflow of Y, so $g(f(A)) = (g \circ f)(A)$ is a proper closed subflow of Z. Thus $g \circ f$ is T-irreducible.

(2) If f factors as $k \circ h$ then k must be surjective. If the closed surjection h were not T-irreducible, there would be a proper closed subflow A of X such that $h(A) = Z$. Thus $f(A) = k(h(A)) = k(Z) = Y$, which is a contradiction.

(3) If k were not T-irreducible there would be a closed subflow $A \subsetneq X$ such that $k(A) = X$. But then $h(A) = f(k(A)) = f(X) = Y$, contrary to the assumption that h is T-irreducible. □

Theorem 2.3.3. *Let* $f\colon Y \to X$ *be a perfect flow surjection. Then the following are equivalent.*

1. f *is* T*-irreducible.*

2. *For every non-empty open subset* $U \subseteq Y$*, there exist a non-empty open subset* $V \subseteq X$ *and a finite subset* $T_0 \subseteq T$ *such that* $f^{-1}(V) \subseteq \bigcup_{T_0} t^{-1}U$*.*

3. *For every proper closed subset* $A \subseteq Y$ *there exists a finite subset* $T_0 \subseteq T$ *such that* $f(\bigcap_{T_0} t^{-1}A)$ *is proper in* X*.*

4. *For every non-empty regular open subset* $U \subseteq Y$ *there exists a nonempty regular open subset* $V \subseteq X$ *and a finite subset* $T_0 \subseteq T$ *such that* $f^{-1}(V) \subseteq \mathrm{int}_Y \, \mathrm{cl}_Y(\bigcup_{T_0} t^{-1}U)$.

5. *For every proper regular closed set* A *in* Y *there exists a finite subset* T_0 *of* T *such that* $\mathrm{cl}_X \, \mathrm{int}_X \, f(\bigcap_{T_0} t^{-1}A)$ *is a proper regular closed subset of* X.

Proof. (1) iff (2). Assume (1) and let U be a non empty open set U in Y. There must be at least one point $x \in X$ such that $\bigcup_T t^{-1}U \supseteq f^{-1}(x)$, for otherwise $Y \setminus \bigcup_T t^{-1}U$ would be a proper closed subflow of Y which f maps onto X. Fix such an x; by the compactness of $f^{-1}(x)$, there exists a finite subset $T_0 \subseteq T$ such that $\bigcup_{T_0} t^{-1}(U) \supseteq f^{-1}(x)$. Then $V \equiv X \setminus f(Y \setminus \bigcup_{T_0} t^{-1}U)$ is an open subset of X such that $f^{-1}(V) \subseteq \bigcup_{T_0} t^{-1}U$.

Assume (2) and let W be a proper closed subflow of Y. Then $U = Y \setminus W$ is a nonempty open stable subset of Y. By (2) there exist a nonempty open set V in X and a finite subset T_0 of T such that $f^{-1}(V) \subseteq \bigcup_{T_0} t^{-1}U$, which means that $f(W) \neq X$.

(2) iff (3). Let A be a proper closed subset of Y. Then by (2) there exist a nonempty open set V in X and a finite subset T_0 of T such that

$$\begin{aligned} f^{-1}(V) \subseteq \bigcup_{T_0} t^{-1}(Y \setminus A) \implies & \bigcap_{T_0} t^{-1}A \subseteq Y \setminus (f^{-1}(V)) \\ \implies & f\Big(\bigcap_{T_0} t^{-1}A\Big) \subseteq X \setminus V. \end{aligned}$$

On the other hand, assume (3) and consider a nonempty open subset U of Y. By (3) there exists a finite subset $T_0 \subseteq T$ such that $f(A)$ is proper in X, where $A \equiv \bigcap_{T_0} t^{-1}(Y \setminus U)$. Then $V \equiv X \setminus f(A)$ is a nonempty open subset of X such that $f^{-1}(V) \cap A = \emptyset$, i.e.,

$$f^{-1}(V) \subseteq Y \setminus A = Y \setminus \bigcap_{T_0} t^{-1}(Y \setminus U) = \bigcup_{T_0} t^{-1}U.$$

(2) iff (4). The implication from (2) to (4) follows from the fact that

$$\bigcup_{T_0} t^{-1}U \subseteq \mathrm{int}_Y\, \mathrm{cl}_Y \bigcup_{T_0} t^{-1}U.$$

The opposite implication follows from the fact that every open set in a regular space is the union of the regular open subsets contained in it.

(3) iff (5). The implication from (3) to (5) follows from the fact that

$$\mathrm{cl}_X\, \mathrm{int}_X\, f\Big(\bigcap_{T_0} t^{-1}A\Big) \subseteq f\Big(\bigcap_{T_0} t^{-1}A\Big).$$

The opposite implication follows from the fact that every closed set in a regular space is the intersection of the regular closed sets containing it. □

Lemma 2.3.4. *If* $f : Y \to X$ *is a* T*-irreducible surjection, then*

1. *for every non empty open subset* U *of* Y *there exists a finite subset* T_0 *of* T *such that* $\mathrm{int}_X[f(\bigcup_{T_0} t^{-1}U)] \neq \emptyset$, *and*

2. *for every dense* T*-invariant subset* S *of* X, $f^{-1}(S)$ *is a dense* T*-invariant set.*

Proof. To prove (1), let $U \neq \emptyset$ be an open subset of Y. Since f is T-irreducible, there exist a finite subset T_0 of T and a non empty open subset V of X such that

$$f^{-1}(V) \subseteq \bigcup_{T_0} t^{-1}U \implies V = f(f^{-1}(V)) \subseteq f\left(\bigcup_{T_0} t^{-1}U\right)$$
$$\implies \operatorname{int}_X\left(f\left(\bigcup_{T_0} t^{-1}U\right)\right) \neq \emptyset.$$

Now suppose S is a dense T-invariant subset of X. Because f is a closed function, we have

$$X = \mathrm{cl}_X\, S = \mathrm{cl}_X(f[f^{-1}(S)]) = f(\mathrm{cl}_Y[f^{-1}(S)]).$$

Note that $\mathrm{cl}_Y[f^{-1}(S)]$ is a closed subflow of Y which mapped onto X and since f is T-irreducible, $\mathrm{cl}_Y[f^{-1}(S)] = Y$. $\square$

Chapter 3

Smirnov's Theorem With Actions

In this chapter we give a brief explanation of T-proximities and the Generalized Smirnov Theorem. The latter associates a flow compactification to each compatible T-proximity on a compactifiable flow and vice-versa. In pursuit of a completion, we will include a self-contained construction of the Smirnov flow compactification of a given compactifiable flow.

3.1 T-Proximity

Let X be a set. A *proximity on* X is a binary relation $\prec$ on the power set of X which satisfies axioms (P1)-(P6) given below. When $A \prec B$ we say that A *is strongly contained in* B. It is often convenient to use the associated binary relation δ defined from $\prec$ as follows.

$$A\delta B \iff A \not\prec (X \setminus B), \text{ and } A \prec B \iff A\bar{\delta}(X \setminus B).$$

Here the notation $A\bar{\delta}B$ expresses the negation of $A\delta B$.

Definition 3.1.1. *[9] Let* X *be a set and* $\prec$ *a binary relation on the power set of* X*. We call* $\prec$ *a* proximity on X*, and the pair* $(X, \prec)$ *a* proximity space*, if* $\prec$ *satisfies the following axioms:*

(P1) $X \prec X$*;*

(P2) $A \prec B \implies A \subseteq B$*;*

(P3) $A \subseteq B \prec C \subseteq D \implies A \prec D$*;*

(P4) $A \prec B, C \implies A \prec B \cap C$*;*

(P5) $A \prec B \implies X \setminus B \prec X \setminus A$*;*

(P6) $A \prec B \implies \exists\, C \subseteq X (A \prec C \prec B)$.

If the proximity $\prec$ *satisfies additionally the following axiom (P7), it will be called* separated*, or a* Hausdorff proximity.

(P7) $x \neq y \implies \{x\} \prec X \setminus \{y\}$.

(P4) and (P5) together imply

(P4') $A, B \prec C \implies A \cup B \prec C$.

Definition 3.1.2. *[9] Let* $(X, \prec_1)$ *and* $(Y, \prec_2)$ *be two proximity spaces. A mapping* $f : X \to Y$ *is called a* proximity mapping *if*

$$A \prec_2 B \implies f^{-1}(A) \prec_1 f^{-1}(B).$$

Each proximity $\prec$ on X induces a topology by declaring a subset $U \subseteq X$ to be open if $U = \{x \in X : \{x\} \prec U\}$. This is called *the topology induced by the proximity* $\prec$, and is designated $\tau_{\prec}$. If $\prec$ is a separated proximity then $\tau_{\prec}$ is a completely regular topology on X. If X is endowed with both a proximity $\prec$ and a topology τ, the proximity is called *compatible* if $\tau_{\prec} = \tau$.

Lemma 3.1.3. *Let* $(X, \prec)$ *be a proximity space, then*

(i) $A \prec B$ *implies* $\mathrm{cl}_{\tau_{\prec}}(A) \prec B$.

(ii) $A \prec B$ *implies* $A \prec \mathrm{int}_{\tau_{\prec}}(B)$.

Proof. See [9] Lemma (3.2). □

Corollary 3.1.4. *Let* X *be a topological space and let* $\prec$ *be a compatible proximity on* X*. If* $A \prec B$ *then there exists a regular open set* a *such that* $A \prec a \prec B$.

Proof. If $A \prec B$, then by (P6) and the previous lemma we can find C such that $A \prec \mathrm{int}(\mathrm{cl}(C)) \prec B$. Then $a = \mathrm{int}(\mathrm{cl}(C))$ is the desired regular open set. □

Lemma 3.1.5. *A compact Hausdorff space* X *has a unique compatible separated proximity, given by*

$$A \delta B \iff \mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \emptyset, \text{ or equivalently,}$$

$$A \prec B \iff \mathrm{cl}\, A \subseteq \mathrm{int}\, B.$$

Proof. It is easy to see that, as defined above, $\prec$ satisfies (P1)–(P5) and (P7); we need only prove (P6). Assume that $A \prec B$, then $\mathrm{cl}\, A \cap \mathrm{cl}(X \setminus B) = \emptyset$. Since

X is a compact Hausdorff space, it is normal and hence there exist open sets C and D such that $\operatorname{cl} A \subset C$ and $\operatorname{cl}(X \setminus B) \subseteq D$ and $\operatorname{cl} C \cap \operatorname{cl} D = \emptyset$. This gives $\operatorname{cl} A \subseteq C = \operatorname{int} C$ and $\operatorname{cl} C \subseteq X \setminus \operatorname{cl} D \subseteq \operatorname{int} B$, i.e., $A \prec C \prec B$.

Evidently $\prec$ is a compatible proximity, that is, a subset $U \subseteq X$ has the feature that $U = \{x : \{x\} \prec U\}$ iff U is open in X. To see that $\prec$ is the only such proximity on X, consider an arbitrary proximity $\prec'$ on X such that $U = \{x : \{x\} \prec' U\}$ iff U is open in X. We claim that $\operatorname{int} A = \{x : \{x\} \prec' A\} \equiv U$ for all $A \subseteq X$. For certainly if $x \in \operatorname{int} A$ then $\{x\} \prec' \operatorname{int} A \subseteq A$, hence $x \in U$. Furthermore U is open, for each $x \in U$ is contained in a set V such that $\{x\} \prec' V \prec' A$, and clearly $V \subseteq U$ since each point $y \in V$ satisfies $\{y\} \prec' A$. That is, $\{x\} \prec' U$ for all $x \in U$, hence U is open. The claim follows.

We have shown that $A \prec' B$ implies $A \subseteq \operatorname{int} B$. Since $A \prec' B$ also implies $(X \setminus B) \prec' (X \setminus A)$, we have $A \prec' B$ implies $\operatorname{cl} A \subseteq \operatorname{int} B$, i.e., $A \prec' B$ implies $A \prec B$. To verify the converse of the latter implication, consider $A \prec B$, i.e., $\operatorname{cl} A \subseteq \operatorname{int} B$. By the claim we know that $\operatorname{int} B = \{x : \{x\} \prec' B\}$, so for each $x \in \operatorname{cl} A$ we have $\{x\} \prec' B$, hence $\{x\} \prec' V_x \prec' B$ for some subset $V_x \subseteq X$. By replacing V_x by $\operatorname{int} V_x = \{x' : \{x'\} \prec' V_x\}$, we may assume that each V_x is open in X. Because X is compact, $\{V_x : x \in \operatorname{cl} A\}$ has a finite subcover $\{V_{x_i} : 1 \leqslant i \leqslant n\}$. By (P4') we get

$$A \subseteq \bigcup_{1 \leqslant i \leqslant n} V_{x_i} \prec' B. \qquad \square$$

Smirnov's Theorem 3.1.10 is the fundamental result concerning proximites. It establishes a bijection between the compactifications of a Tychonoff space X

and the compatible separated proximities on X. We give an outline of the proof, following [3]. The fundamental notion is that of a *round filter* on a proximity space.

Definition 3.1.6. *Let $\prec$ be a compatible separated proximity on the Tychonoff space X. Call a filter $\mathcal{F}$ on X* round *if for all $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $B \prec A$. A maximal among proper round filter is called an* end.

Let $\prec$ be a compatible separated proximity on the Tychonoff space X.

Lemma 3.1.7.

1. *Every proper round filter is contained in a maximal proper round filter.*

2. *A family $\mathcal{F}$ of subsets of X is an end if and only if it has these two properties.*

   - *For all $A, B \in \mathcal{F}$ there exist $C \in \mathcal{F}$ such that $C \prec A \cap B$.*
   - *For all $A \prec B$, either $X \setminus A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

3. *For a given ultrafilter $\mathcal{F}$ on X, the* round part of $\mathcal{F}$*,*

$$\mathcal{F}_{\prec} \equiv \{ A \in \mathcal{F} : \exists B \in \mathcal{F} \ (B \prec A) \},$$

   is an end.

4. *Every end is the round part of any ultrafilter containing it.*

5. *For each $x \in X$, $\mathcal{F}_x \equiv \{ A : \{x\} \prec A \}$ is an end.*

Proof. (1) For a given round filter $\mathcal{F}$ set $\mathcal{M}$ to be the set of all round filters that contains $\mathcal{F}$. Note that $\mathcal{M}$ is partially ordered by set inclusion and every chain in $\mathcal{M}$ has an upper bound in $\mathcal{M}$. Thus by Zorn's Lemma $\mathcal{M}$ contains a maximal element η which is an end containing $\mathcal{F}$.

(2) Assume that $\mathcal{F}$ is and end. For $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$ because $\mathcal{F}$ is a filter, and there exists $C \in \mathcal{F}$ such that $C \prec A \cap B$ because $\mathcal{F}$ is round. On the other hand, if $A \prec B$ then either $A \cap C \neq \emptyset$ for every $C \in \mathcal{F}$ or there exist $E \in \mathcal{F}$ such that $A \cap E = \emptyset$. In the first case $\eta = \{D : \exists C \in \mathcal{F}\ (A \cap C \prec D\}$ is a round filter containing $\mathcal{F}$ and B. By the maximality of $\mathcal{F}$ we have $\eta = \mathcal{F}$ and $B \in \mathcal{F}$. In the second case we have $X \setminus A \in \mathcal{F}$ because $E \subseteq X \setminus A$.

Suppose the family $\mathcal{F}$ satisfies the given conditions. Then one easily sees that $\mathcal{F}$ is a round filter by the first condition. By way of contradiction, assume that $\mathcal{F}$ is contained in a round filter η such that $B \in (\eta \setminus \mathcal{F})$ for some B. Since η is round there exists $A \in \eta$ with the property that $A \prec B$. By hypothesis we have $X \setminus A \in \mathcal{F} \subseteq \eta$, which leads to the contradiction $\emptyset = A \cap (X \setminus A) \in \eta$.

(3) If $A, C \in \mathcal{F}_{\prec}$ then there exist $B, D \in \mathcal{F}$ such that $B \prec A$ and $D \prec C$. Then $B \cap D \prec A, C$ by axiom (P3), hence $B \cap D \prec A \cap C$ by (P4), proving $\mathcal{F}_{\prec}$ to be a filter. To prove that $\mathcal{F}_{\prec}$ is round, consider $A \in \mathcal{F}_{\prec}$, say $B \prec A$ for $B \in \mathcal{F}$. Then by (P6) there exists some C for which $B \prec C \prec A$, and B witnesses the membership of C in $\mathcal{F}_{\prec}$. To prove that $\mathcal{F}_{\prec}$ has the second property listed under (2), consider $A \prec B$. Use axiom (P6) again to find C such that $A \prec C \prec B$. Because $\mathcal{F}$ is an ultrafilter, either $C \in \mathcal{F}$ or $X \setminus C \in \mathcal{F}$. Thus either $B \in \mathcal{F}_{\prec}$ or $X \setminus A \in \mathcal{F}_{\prec}$.

(4) Assume that η is an end and let $\mathcal{F}$ be an ultrafilter containing η. Then it is clear that η is contained in $\mathcal{F}_{\prec}$. Now if $A \in \mathcal{F}_{\prec}$ then there exist $B \in \mathcal{F}$ such that $B \prec A$. By (2) either $X \setminus B \in \eta$ or $A \in \eta$, and the former possibility is incompatible with the fact that $B \in \mathcal{F}$. We conclude that $\eta = \mathcal{F}_{\prec}$.

(5) It is well known that $\mathcal{F}_x$ is a filter. For every $A \in \mathcal{F}_x$, there exist a regular open set a such that $\{x\} \prec a \prec A$ and $a \in \mathcal{F}_x$. To show it is maximal assume $A \prec B$ and by (P6) find C such that $A \prec C \prec B$. Use the fact that either $x \in C \prec B$ or $x \in X \setminus C \prec X \setminus A$ to get that either $B \in \mathcal{F}_x$ or $X \setminus A \in \mathcal{F}_x$. Therefore $\mathcal{F}_x$ is an end. $\square$

For a given proximity space $(X, \prec)$, let Y be the set of ends of X, topologized by using sets of the form

$$O(U) \equiv \{\mathcal{F} : U \in \mathcal{F}\}, \qquad \text{open } U \subseteq X,$$

as a base for open sets.

Lemma 3.1.8. *For a family* $\{U_i : i \in I\}$ *of open subsets of* X,

$$\bigcup_I O(U_i) = Y \iff \forall i \in I\ \exists \textit{ open } V_i \prec U_i \left(\bigcup_I V_i = X\right).$$

Proof. Suppose $\bigcup_I V_i = X$ for some choice of open subsets $V_i \prec U_i$, and consider an arbitrary $\eta \in Y$. Let $\mathcal{F}$ be an ultrafilter on X containing η, so that $\eta = \mathcal{F}_{\prec}$ by Lemma 3.1.7. Then $\mathcal{F}$ must contain at least one of the V_i's, hence η must contain at least one of the U_i's, i.e., η must lie in one of the $O(U_i)$'s.

On the other hand, suppose $\bigcup_I V_i \neq X$ for any choice of open subsets $V_i \prec U_i$. Considering that the family $\{ X \smallsetminus V : \exists i \in I \text{ (open } V \prec U_i) \}$ has the finite intersection property, it must be contained in an ultrafilter $\mathcal{F}$ whose round part $\eta = \mathcal{F}_\prec$ is an end. But clearly $\eta \notin O(U_i)$ for any $i \in I$, for otherwise the roundness of η would require $V \in \eta$ for some $V \prec U_i$, contrary to assumption. This shows that $\bigcup_I O(U_i) \neq Y$. □

Lemma 3.1.9. Y *is a compact Hausdorff space, the map* $\alpha\colon X \to Y = (x \mapsto \mathcal{F}_x)$ *is continuous and one-one, and* $\alpha(X) = \{ \mathcal{F}_x : x \in X \}$ *is dense in* Y. *In short,* $\alpha\colon X \to Y$ *is a compactification of* X.

Proof. We are assuming that X is a Tychonoff space and that the proximity is compatible. To prove Y is a Hausdorff space choose distinct elements $\eta, \mathcal{F} \in Y$ and assume that $A \in \eta \smallsetminus \mathcal{F}$. Because η is round we can find $B \in \eta$ such that $B \prec A$, and by Lemma 3.1.3 and Corollary 3.1.4, we can then find a regular open set a such that $B \prec a \prec \mathrm{cl}(a) \prec A$. Since $\mathcal{F}$ is an end we have $X \smallsetminus \mathrm{cl}(a) \in \mathcal{F}$. Therefore $O(a)$ and $O(X \smallsetminus \mathrm{cl}\, a)$ are disjoint open sets containing η and $\mathcal{F}$, respectively.

To show that Y is compact, consider an open cover $\mathcal{C}$ of Y; without loss of generality we may assume that $\mathcal{C} = \{O(U_i)\}$ for some family $\{ U_i : i \in I \}$ of open subsets of X. If $\mathcal{C}$ has no finite subcover then $\{ X \smallsetminus V : \exists i \in I \ (V \prec U_i) \}$ is the basis of a proper filter by (the proof of) Lemma 3.1.8, and this filter is contained in at least one end $\mathcal{F}$.

But this is a contradiction, since any such end $\mathcal{F}$ could not contain any set $V \prec U_i$ for any $i \in I$, and therefore could not contain any U_i by virtue of its roundness.

In order to see that α is a one to one map let x_1, x_2 be distinct points in X. Then x_1 lies in the open set $X \setminus \{x_2\}$, and since the proximity is compatible, $\{x_1\} \prec X \setminus \{x_2\}$. By axiom (P6) and Lemma 3.1.3 there is a regular open subset $a \subseteq X$ such that $\{x_1\} \prec a \prec X \setminus \{x_2\}$. Since $\mathcal{F}_{x_1}$ and $\mathcal{F}_{x_2}$ are ends, $a \in \mathcal{F}_{x_1}$ but $a \notin \mathcal{F}_{x_2}$.

To show that α is continuous, let $O(a)$ be a nonempty basic open subset of Y. Then for every $x \in \alpha^{-1}(O(a))$ there exist $c \in \eta_x$ such that $\{x\} \prec c \prec a$. Since every end which contains c must contain a, it follows that $\alpha(p) \in O(a)$ for every $p \in O(c)$. Thus $x \in c \subseteq \alpha^{-1}(O(a))$. $\square$

Theorem 3.1.10 (Smirnov). *Let X be a Tychonoff space.*

1. *For every compactification $\alpha\colon X \to Y$, the proximity defined by the rule*

$$A \delta_Y B \iff \operatorname{cl}_Y \alpha(A) \cap \operatorname{cl}_Y \alpha(B) \neq \emptyset,$$

   is a compatible separated proximity on X.

2. *Conversely, for every compatible separated proximity $\prec$ on X there is a compactification $\alpha\colon X \to Y$ such that $\prec$ coincides with $\prec_Y$. The compactification is unique up to isomorphism with respect to this property.*

Proof. (1) It is clear that the corresponding relation

$$A \prec_Y B \iff A\overline{\delta_Y}(X \smallsetminus B) \iff \mathrm{cl}_Y\, \alpha(A) \cap \mathrm{cl}_Y\, \alpha(X \smallsetminus B) = \emptyset$$

satisfies axioms (P1)–(P5) and (P7); we need only verify (P6). For that purpose suppose that $A \prec_Y B$. We claim that $\alpha(A) \prec \alpha(B)$, where $\prec$ designates the unique compatible proximity on Y given in Lemma 3.1.5. For $A \prec_Y B$ means that $\mathrm{cl}_Y\, \alpha(A) \cap \mathrm{cl}_Y\, \alpha(X \smallsetminus B) = \emptyset$, from which follows $\mathrm{cl}_Y\, \alpha(A) \subseteq \mathrm{int}_Y(Y \smallsetminus \alpha(X \smallsetminus B))$. From the claim and the fact that $\prec$ satisfies axiom (P6), we can deduce the existence of a subset $C \subseteq Y$ such that $\alpha(A) \prec C \prec \alpha(B)$.

Finally, we claim that $A \prec_Y \alpha^{-1}(C) \prec_Y B$. Note that $\alpha(X \smallsetminus \alpha^{-1}(C)) \subseteq Y \smallsetminus C$, and $\alpha(X \smallsetminus B) \subseteq Y \smallsetminus \alpha(B)$. Thus $\mathrm{cl}_Y\, \alpha(A) \cap \mathrm{cl}_Y\, \alpha(X \smallsetminus \alpha^{-1}(C)) = \emptyset$ and $\mathrm{cl}_Y(C) \cap \mathrm{cl}_Y\, \alpha(X \smallsetminus B) = \emptyset$. Now $\mathrm{cl}_Y\, \alpha\alpha^{-1}(C) \cap \mathrm{cl}_Y\, \alpha(X \smallsetminus B) = \emptyset$ because $\alpha\alpha^{-1}(C) \subseteq C$, hence $A \prec_Y \alpha^{-1}(C) \prec_Y B$.

(2) It remains to show that for all subsets A, B of X,

$$A\bar{\delta}B \text{ iff } \mathrm{cl}_Y\, \alpha(A) \cap \mathrm{cl}_Y\, \alpha(B) = \emptyset$$

By Lemma 3.1.9 the space X is embedded in Y, the compact space of ends. Assume that for subsets A, B of X we have $\mathrm{cl}_Y\, \alpha(A) \cap \mathrm{cl}_Y\, \alpha(B) = \emptyset$. Then $\mathrm{cl}_Y\, \alpha(A) \subseteq Y \smallsetminus \mathrm{cl}_Y\, \alpha(B)$ and for every end $\mathcal{F}$ in $\mathrm{cl}_Y\, \alpha(A)$ there exist an open subset $b_{\mathcal{F}}$ of X such that $\mathcal{F} \in O(b_{\mathcal{F}}) \subseteq Y \smallsetminus \mathrm{cl}_Y\, \alpha(B)$. Since $b_{\mathcal{F}} \in \mathcal{F}$, there exist $a_{\mathcal{F}} \in \mathcal{F}$ such that $a_{\mathcal{F}} \prec b_{\mathcal{F}}$. The sets $\{O(a_{\mathcal{F}}) : \mathcal{F} \in \mathrm{cl}_Y\, \alpha(A)\}$ constitute an open cover of $\mathrm{cl}_Y\,\alpha(A)$, and by compactness it has a finite subcover, say $\{O(a_i) : i \in I\}$ for some finite index set I. It follows that $\bigcup_I a_i \prec \bigcup_I b_i$.

We claim that $A \prec X \setminus B$, which would imply that $A\bar{\delta}B$. For if $x \in A$ then $\mathcal{F}_x \in O(a_i)$ for some i which clearly implies that $x \in a_i$ and hence $A \subseteq \bigcup_I a_i$. If $x \in \bigcup_I b_i$ then $\mathcal{F}_x \in \bigcup_I O(b_i) \subseteq Y \setminus \mathrm{cl}_Y(B)$ which implies that $x \in X \setminus B$.

On the other hand, if for subsets $A, B \subseteq X$ we have $A \prec X \setminus B$ then by Corollary 3.1.4 there exist regular open sets a, b such that $A \prec a \prec b \prec X \setminus B$. It follows that $\alpha(A) \subseteq O(a)$ and $\alpha(B) \subseteq O(X \setminus \mathrm{cl}_X\, b)$. Since $O(a) \cap O(X \setminus \mathrm{cl}_X\, b) = \emptyset$, we get $\mathrm{cl}_Y(\alpha(A)) \cap \mathrm{cl}_Y(\alpha(B)) = \emptyset$. $\square$

We now enrich Smirnov's Theorem by the addition of actions. This is one of the main topics of the thesis.

Definition 3.1.11. *A proximity $\prec$ on a flow X is called a* T-proximity *provided that whenever $A \prec B$ and $t \in T$ there exists a neighborhood N_t of t such that*

$$\bigcup_{N_t} r^{-1}A \prec \bigcap_{N_t} s^{-1}B.$$

Lemma 3.1.12. *Let X be a flow and $\prec$ be a proximity on X. The following are equivalent:*

1. *$\prec$ is a T-proximity on X.*
2. *The following two conditions are satisfied.*

   (i) Every action $t\colon X \to X$ is a proximity mapping.

(ii) *For subsets* $A, B \subset X$ *with* $A \prec B$*,* A *is* T*-contained in* B*. That means that for every* $t \in T$ *there exists a neighborhood* N_t *of* t *such that* $r^{-1}A \subseteq s^{-1}B$ *for all* $r, s \in N_t$.

Proof. Assume that (1) holds. To prove (2)(i) consider $A \prec B$ and $t \in T$. Then by (1) there exist a neighborhood N_t of t such that $\bigcup_{N_t} r^{-1}A \prec \bigcap_{N_t} s^{-1}B$. In particular, $t^{-1}A \prec t^{-1}B$, which means that $t\colon X \to X$ is a proximity mapping. To verify (2)(ii) again consider $A \prec B$. Then for every $t \in T$ there exists a neighborhood N_t of t such that for all $r, s \in N_t$ we have

$$r^{-1}A \subseteq \bigcup_{N_t} r^{-1}A \prec \bigcap_{N_t} s^{-1}B \subseteq s^{-1}B.$$

On the other hand, suppose (2) holds and consider $A \prec B$ and $t \in T$. Then by (P6) there exist C, D such that $A \prec C \prec D \prec B$, and by (2) there exist neighborhoods N_t' and N_t'' of t such that $\bigcup_{N_t'} r^{-1}A \subseteq \bigcap_{N_t'} s^{-1}C$, $t^{-1}C \prec t^{-1}D$ for every $t \in T$, and $\bigcup_{N_t''} r^{-1}D \subseteq \bigcap_{N_t''} s^{-1}B$. Letting $N_t = N_t' \cap N_t''$, we have for all $r, s \in N_t$ that

$$\bigcup_{N_t} r^{-1}A \subseteq \bigcap_{N_t} s^{-1}C \subseteq s^{-1}C \prec s^{-1}D \subseteq \bigcup_{N_t} r^{-1}D \subseteq \bigcap_{N_t} s^{-1}B.$$

By (P3) we have $\bigcup_{N_t} r^{-1}A \prec \bigcap_{N_t} s^{-1}B$ for every $r, s \in N_t$, which proves (1). $\square$

3.1.1 Smirnov's Theorem with actions

Lemma 3.1.13. *The canonical proximity $\prec$ on a compact flow is a* T*-proximity.*

Proof. Let X be a compact Hausdorff flow, with its unique proximity $\prec$ defined by

$$A \prec B \text{ whenever } \mathrm{cl}_X(A) \cap \mathrm{cl}_X(X \setminus B) = \emptyset, \qquad A, B \subseteq X.$$

If $A \prec B$ then there exists a function $f \in C^*(X) = C^T(X)$ such that $f(\mathrm{cl}_X(A)) = 0$ and $f(\mathrm{cl}_X(X \setminus B)) = 1$. To prove $\prec$ is a T-proximity, fix $t \in T$. Since f is a T-uniformly continuous function, for every $0 < \varepsilon < 1$ there exists a neighborhood N_t of t such that

$$|f(rx) - f(sx)| < \varepsilon \text{ for } x \in X \text{ and } r, s \in N_t.$$

If $x \in r^{-1}(A)$ then $f(rx) = 0$ and therefore $|f(sx)| < \varepsilon$. Thus

$$\begin{aligned} sx \notin f^{-1}(1) \supseteq \mathrm{cl}_X(X \setminus B) &\implies sx \in X \setminus \mathrm{cl}_X(X \setminus B) \\ &\implies sx \in B \\ &\implies x \in s^{-1}(B) \end{aligned}$$

Hence, for every $t \in T$ there exist a neighborhood N_t of t such that $r^{-1}A \prec s^{-1}B$ for every $r, s \in N_t$, so $\prec$ is a T-proximity. $\square$

Theorem 3.1.14. *[9] Let (X_i, δ_i), $i = 1, 2$, be proximity spaces, and let $\alpha_i \colon X_i \to Y_i$ be the corresponding Smirnov compactifications (Theorem 3.1.10). Then for ev-*

ery proximity mapping $f\colon X_1 \to X_2$ *there is a unique continuous function* $f'\colon Y_1 \to Y_2$ *such that* $f' \circ \alpha_1 = \alpha_2 \circ f$.

Theorem 3.1.15 (Smirnov's Theorem with actions)**.** *Let* X *be a compactifiable flow.*

1. *Let* $\alpha\colon X \to Y$ *be a flow compactification and let us identify* X *with its image under* α *to simplify notation. Then the associated proximity* $\prec$ *from Smirnov's Theorem 3.1.10, namely*

$$A \prec B \iff \mathrm{cl}_Y A \cap \mathrm{cl}_Y(X \setminus B) = \emptyset, \qquad A, B \subseteq X,$$

   is a compatible separated T*-proximity on* X*.*

2. *Let* $\prec$ *be a compatible separated* T*-proximity on* X*, and let* $\alpha\colon X \to Y$ *be the compactification associated with* $\prec$ *by Smirnov's Theorem 3.1.10. Then the actions on* X *lift to* Y *so as to make* α *a flow compactification of* X*.*

3. *These two processes are inverses of one another.*

Thus the flow compactifications of X *are in bijective correspondence with the compatible separating* T*-proximities on* X*.*

It should be noted that the classical Smirnov Theorem 3.1.10 is the special case of Theorem 3.1.15 corresponding to the trivial action $T = 1$.

Proof. (1) Suppose subsets $A, B \subseteq X$ satisfy $A \prec B$. Then since $C^T(Y)$ separates disjoint closed subsets of Y, the fact that $\mathrm{cl}_Y A \cap \mathrm{cl}_Y(X \setminus B) = \emptyset$ implies the existence of $f \in C^T(Y)$ such that $f(\mathrm{cl}_Y A) = 0$ and $f(\mathrm{cl}_Y(X \setminus B)) = 1$. An argument along the lines of the proof of Lemma 3.1.13 can then be used to show that $\prec$ is a T-proximity.

(2) Lemma 3.1.12 asserts that each action $t \in T$ is a proximity map, and it extends to an action on Y by Theorem 3.1.14. We need only to show that the induced action is continuous, and this will follow from Lemma 3.1.7 if we can show that each $g \in C(Y)$ is T-uniformly continuous. But that follows directly from the fact that $g\alpha \in C^T(X)$. In more detail, for given $t \in T$ and $\varepsilon > 0$ find a neighborhood N_t of t such that $|g\alpha t(x) - g\alpha s(x)| < \varepsilon$ for all $s \in N_t$ and $x \in X$. This yields $|gt'\alpha(x) - gs'\alpha(x)| < \varepsilon$ for all $s \in N_t$ and $x \in X$, where s' and t' are the respective extensions of s and t to Y. Since $\alpha(X)$ is dense in Y, it follows that $|gt(y) - gs(y)| < \varepsilon$ for all $y \in Y$, i.e., g is T-uniformly continuous. $\square$

Theorem 3.1.16. *Let X be a compactifiable flow, and for A and B subsets of X define*

$$A \prec B \iff A \textit{ and } X \setminus B \textit{ are } T\textit{-completely separated.}$$

Equivalently by Proposition 2.1.14, $A \prec B$ iff there is a T-scale $\{U_q : q \in \mathbb{Q}\}$ such that $A \subseteq U_q \subseteq B$ for all $q \in \mathbb{Q}$. Then $\prec$ is the largest compatible separated T-proximity which can be defined on X.

Proof. We would like to prove that $\prec$ is a compatible separated T-proximity. Let $A, B, C \subseteq X$.

(P1) It is clear that $X \prec X$.

(P2) $A \prec B$ implies there is a T-scale $\{U_q : q \in \mathbb{Q}\}$ such that $A \subseteq U_q \subseteq B$ for all q. This clearly implies that $A \subseteq B$.

(P3) $A \subseteq B \prec C \subseteq D$ implies there is a T-scale $\{U_q : q \in \mathbb{Q}\}$ such that $B \subseteq U_q \subseteq C$ for all q. But then $A \subseteq U_q \subseteq D$ for all q, hence $A \prec D$.

(P4) $A \prec B, C$ implies that A and $X \setminus B$ are T-completely separated, and that A and $X \setminus C$ are T-completely separated. There are $f, g \in C^T(X)$, $0 \leqslant f, g \leqslant 1$, such that $f(A) = 0$ and $f(X \setminus B) = 1$ and $g(A) = 0$ and $g(X \setminus C) = 1$. Then $h = (f + g) \wedge 1 \in C^T(X)$, and $h(A) = 0$ and $h(X \setminus (B \cap C)) = 1$, yielding $A \prec B \cap C$.

(P5) $A \prec B$ implies that there exists $f \in C^T(X)$ such that $f(A) = 0$ and $f(X \setminus B) = 1$. Then $g \equiv 1 - f \in C^T(X)$ satisfies $g(X \setminus B) = 0$ and $g(X \setminus A) = 1$, hence $X \setminus B \prec X \setminus A$.

(P6) $A \prec B$ implies that there is a T-scale $\{U_q : q \in \mathbb{Q}\}$ such that $A \subseteq U_q \subseteq B$ for all q. Then for $C \equiv U_0$ we have $A \prec C$ by virtue of the T-scale $\{U_q : 0 > q \in \mathbb{Q}\}$ and $C \prec B$ by virtue of the scale $\{U_q : 0 < q \in \mathbb{Q}\}$.

(P7) If $x \neq y$ then since X is a compactifiable flow and $x \notin \{y\}$ there is some $g \in C^T(X)$ such that $g(x) = 0$ and $g(y) = 1$. So $\{x\} \prec X \setminus \{y\}$.

(Comp) In order to prove that $\prec$ is a compatible proximity, assume that U is an open subset of X and that $x \in U$. Since X is compactifiable, there exists a function $f \in C^T(X)$ such that $f(x) = 0$ and $f(X \setminus U) = 1$; i.e $\{x\} \prec U$ and hence $U \in \tau_\prec$. On the other hand, if $U \in \tau_\prec$ and $x \in U$, then $\{x\} \prec U$.

Hence there exist $f \in C^T(X)$ such that $f(x) = 0$ and $f(X \setminus U) = 1$. Thus $x \in f^{-1}([0, 1/4)) \subset U$ which means that $U \in \tau$.

(T) To show that $\prec$ is a T-proximity, consider $A \prec B$, say $\{U_q : q \in \mathbb{Q}\}$ is a T-scale such that $A \subseteq U_q \subseteq B$ for all q. Note that for any $p < q$ in $\mathbb{Q}$, $U_p \prec U_q$ because $\{U_r : p < r < q\}$ is a T-scale such that $U_p \subseteq U_r \subseteq U_q$ for all $p < r < q$. Now suppose that $t \in T$ is given. Since $U_0 \subseteq_T U_1$, we can find a neighborhood N_1 of t such that $\bigcup_{N_1} r^{-1}U_0 \subseteq \bigcap_{N_1} s^{-1}U_1$, and likewise another neighborhood N_3 of t such that $\bigcup_{N_3} r^{-1}U_2 \subseteq \bigcap_{N_3} s^{-1}U_3$. Letting $N_t \equiv N_1 \cap N_3$ gives

$$\begin{aligned} \bigcup_{N_t} r^{-1}A \subseteq \bigcup_{N_t} r^{-1}U_0 \subseteq \bigcap_{N_t} s^{-1}U_1 \subseteq t^{-1}U_1 & \\ \prec t^{-1}U_2 \subseteq \bigcup_{N_t} r^{-1}U_2 \subseteq \bigcap_{N_t} s^{-1}U_3 \subseteq \bigcap_{N_t} s^{-1}B. & \end{aligned}$$

The fact that $\prec$ is the largest T-proximity on X follows directly from Lemma 3.1.12. □

Corollary 3.1.17. *Let X be a compact Hausdorff flow. Then with respect to the canonical proximity on X,*

$$A \bar{\delta} B \iff \operatorname{cl} A \textit{ is } T\textit{-disjoint from } \operatorname{cl} B \textit{ in } X.$$

3.1.2 Non-separated T-proximity

In case the flow X is not compactifiable, then we can define an equivalence relation $\sim$ on X such that the quotient flow $Y \equiv X/\sim$ is compactifiable (see Theorem 2.2.3). The relation $\sim$ is defined by

$$x \sim y \iff g(x) = g(y) \text{ for all } g \in C^T(X),$$

and is respected by the actions, i.e., $x_1 \sim x_2$ implies $tx_1 \sim tx_2$ for all $x_i \in X$ and $t \in T$, thereby providing an action of T on Y. Furthermore, Y is a flow, i.e., the evaluation map $T \times Y \to Y$ is continuous, and the projection map $q \colon X \to Y$ is a flow surjection. (See [2] for a full development.) In fact, Y is Hausdorff and compactifiable, and is the finest among the quotient flows of X with these properties.

The point is that the proximity relation of Theorem 3.1.16 can be defined on a flow X even if X is not compactifiable. In that case the $\prec$ relation satisfies axioms (P1)–(P6), as shown by the proof of Theorem 3.1.16, though it cannot satisfy (P7). Nevertheless the finest compactifiable quotient flow Y of X does carry a coarsest compatible separated proximity inherited from $\prec$.

Theorem 3.1.18. *Any flow has a finest quotient flow which admits a compatible separated proximity.*

Chapter 4

De Vries Algebras With Actions

A proximity is a relation on the family of all subsets of a given space which satisfies axioms (P1)–(P7). It turns out that one may restrict this relation to the (complete Boolean algebra of) regular open subsets of the space without loss of information. The objects of study then become complete Boolean algebras equipped with a relation satisfying the axioms which appropriately generalize (P1)–(P7), so called de Vries algebras.

This chapter begins by defining these algebras and sketching a proof of the duality between de Vries algebras and compact Hausdorff spaces. Our purpose is to enrich the category of de Vries algebras by adding actions, thus obtaining the category of de Vries algebras with actions, or T-de Vries algebras. We then prove the duality between the category of compact flows and the category of T-de Vries algebras. Our main references are [5] by de Vries and [4] by Bezhanishvili.

4.1 Classical de Vries duality

Definition 4.1.1. *[4] A* de Vries algebra *is a pair* $(B, \prec)$*, where* B *is a complete Boolean algebra and* $\prec$ *is a binary relation on* B *satisfying the following axioms:*

(DV1) $1 \prec 1$*;*

(DV2) $a \prec b$ *implies* $a \leqslant b$*;*

(DV3) $a \leqslant b \prec c \leqslant d$ *implies* $a \prec d$*;*

(DV4) $a \prec b, c$ *implies* $a \prec b \wedge c$*;*

(DV5) $a \prec b$ *implies* $\neg b \prec \neg a$*;*

(DV6) $a \prec b$ *implies there exists* $c \in B$ *such that* $a \prec c \prec b$*;*

The algebra is called separating *if it also satisfies the following axiom.*

(DV7) $a \neq 0$ *implies there exists* $b \neq 0$ *such that* $b \prec a$.

An important and useful consequence of this definition is the following.

(DV8) For every $a \in B$ *we have* $a = \bigvee_{b \prec a} b$.

Remark 4.1.2. *The relation* $\prec$ *on a de Vries algebra* B *is a generalization of a proximity relation on a space. In fact, if* $(X, \prec)$ *is a proximity space then the restriction of the strong containment relation* $\prec$ *to* $\mathrm{RO}(X)$ *makes* $(\mathrm{RO}(X), \prec)$ *a de Vries algebra. For that reason we shall refer to the relation* $\prec$ *of a de Vries algebra* $(B, \prec)$ *as a proximity, trusting in the reader's ability to resolve any ambiguity that might arise.*

Before defining the action on a given de Vries algebra, let us recall the definition of a de Vries morphism, and of the composition of two de Vries morphisms.

Definition 4.1.3 ([5]). *Let* $(A, \prec_A)$ *and* $(B, \prec_B)$ *be de Vries algebras. We say that* $f \colon A \to B$ *is a* de Vries morphism *if the following conditions are satisfied:*

(M1) $f(0) = 0$*;*

(M2) $f(a \wedge b) = f(a) \wedge f(b)$*;*

(M3) $a \prec b$ *implies* $\neg f(\neg a) \prec f(b)$*;*

(M4) $f(a) = \bigvee_{b \prec a} f(b)$.

For a morphism $h \colon A \to B$ that satisfies (M1)-(M3), the associated morphism

$$f^*(a) = \bigvee_{b \prec a} f(a)$$

is a de Vries morphism. Since the composition of two de Vries morphisms f and g satisfies (M1)-(M3) but may not satisfy (M4), de Vries defined the composition of f and g by

$$f * g \equiv (f \circ g)^*.$$

Proposition 4.1.4 ([5]). *Let* $(A, \prec_A)$ *and* $(B, \prec_B)$ *be de Vries algebras and let* $h \colon A \to B$ *be a de Vries morphism. For all* $a, b \in A$*,*

1. $h(1) = 1$*;*

2. $h(a) \leqslant \neg h(\neg a)$*, i.e.* $h(\neg a) \leqslant h(a)$*;*

3. $a \prec b$ *implies* $h(a) \prec h(b)$*;*

4. $a \leqslant b$ *implies* $h(a) \leqslant h(b)$.

5. *If* $a \prec c$ *and* $b \prec d$*, then* $h(a \vee b) \prec h(c) \vee h(d)$.

Proof. 1. We know that $0 \prec 0$ by (DV5), and by (M3) and (M1) we get $\neg h(1) \prec 0$. Then $1 \prec h(1)$ by (DV5), hence $1 \leqslant h(1)$ which implies that $1 = h(1)$.

2. $0 = h(0) = h(a \wedge \neg a)$. By (M2) we have $0 = h(a) \wedge h(\neg a)$, which implies $h(a) \leqslant \neg h(\neg a)$.

3. If $a \prec b$ then by (M3) and by (2) above we have $h(a) \leqslant \neg h(\neg a) \prec h(b)$, hence $h(a) \prec h(b)$.

4. If $a \leqslant b$ then $h(a) = h(a \wedge b) = h(a) \wedge h(b)$, i.e., $h(a) \leqslant h(b)$.

5. Suppose that $a \prec c$ and $b \prec d$, then $h(a \vee b) \leqslant \neg h(\neg a \wedge \neg b)$. By (M2) we have $h(a \vee b) \leqslant \neg h(\neg a) \vee \neg h(\neg b)$. Thus $h(a \vee b) \prec h(c) \vee h(d)$.

$\square$

We sketch proofs of the main ideas behind de Vries's Theorem in Theorem 4.1.5. Theorem 4.1.6 is the full form of this famous result.

Theorem 4.1.5 (de Vries). *1. For a given compact Hausdorff space* X*, let* $B \equiv \mathrm{RO}(X)$ *and define* $U \prec V \iff \mathrm{cl}_X U \subseteq V, \qquad U, V \in \mathrm{RO}(X)$.

Then $(B, \prec)$ *is a de Vries algebra.*

2. *For every de Vries algebra* $(B, \prec)$ *there is a unique compact Hausdorff space* X *such that* B *is isomorphic to* $\mathrm{RO}(X)$*, and when* B *is identified with its image under this isormorphism,*

$$a \prec b \iff \mathrm{cl}_X\, a \subseteq b, \qquad a, b \in B.$$

3. *Any continuous function* $f\colon X \to Y$ *between compact Hausdorff spaces* X *and* Y *produces a de Vries homomorphism*

$$\tilde{f}\colon \mathrm{RO}(Y) \to \mathrm{RO}(X) = (U \mapsto \mathrm{int}_X\, \mathrm{cl}_X\, f^{-1}(U)).$$

Proof. (1) It is routine to check that $\prec$ satisfies (DV1)–(DV5). To prove (DV6) assume $U \prec V$, i.e., $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(X \smallsetminus V) = \emptyset$. Because X is a normal space, there are disjoint open subsets U', V' such that

$$U \subset \mathrm{cl}_X\, U \subseteq U' \subseteq X \setminus V' \subseteq V.$$

Let $W \equiv \mathrm{int}_X(X \smallsetminus V')$, then $W \in B$ and $U \prec W \prec V$. To prove (DV7), consider a regular open set $U \neq \emptyset$, say $x \in U$. Since X is a regular space, there is an open set U' such that $x \in U' \subseteq \mathrm{cl}_X\, U' \subseteq U$. Then $V \equiv \mathrm{int}_X\, U' \in B$ and $\emptyset \neq V \prec U$.

(2) Given a de Vries algebra $(B, \prec)$, let X be the Stone space of B, taken here to be the space of ultrafilters on B, and identify B with $\mathrm{RO}(X)$. It is always the case that B is isomorphic to the clopen algebra of X, but because B is complete X

is extremally disconnected, i.e., the closure of each open set is open. That means that a regular open subset, i.e., a subset which is the interior of its closure, is clopen. Thus $\mathrm{clop}(X)$ coincides with $\mathrm{RO}(X)$, and we may identify B with $\mathrm{RO}(X)$.

Following the development in the proof of Smirnov's Theorem 3.1.10, call a filter η on B *round* if for all $b \in \eta$ there exists $a \in \eta$ such that $a \prec b$. For each ultrafilter $\mathcal{F} \in X$ define the *round part of* $\mathcal{F}$ to be

$$\mathcal{F}_{\prec} \equiv \{\, b \in \mathcal{F} : \exists a \in \mathcal{F}\ (a \prec b)\,\}.$$

The same arguments used in the proof of Smirnov's Theorem 3.1.10 show that every round filter is contained in a maximal round filter which we call an end, that the round part of any ultrafilter is a maximal round filter, that any maximal round filter is the round part of any ultrafilter containing it, and that a maximal round filter is characterized by the properties

$$a, b \in \eta \implies \exists c \in \eta\ (c \prec a \wedge b) \quad \text{and} \quad a \prec b \implies \neg a \in \eta \text{ or } b \in \eta.$$

Let $Y \equiv \mathcal{E}nd(B, \prec)$ denote the set of all ends on B, and let $q \colon X \to Y = (\mathcal{F} \mapsto \mathcal{F}_{\prec})$ be the map which takes each ultrafilter to its round part.

In order to show that Y is a compact Hausdorff space we only have to prove that q is a closed quotient map. Fix $b \in B$ and let $A = \{\, \mathcal{F} \in X : b \in \mathcal{F} \,\}$ be a basic closed subset of X.

Assume for the sake of argument that $C \equiv q^{-1}(q(A))$ is not a closed subset of X, i.e., assume that there exists $\mathcal{F}' \in \mathrm{cl}_X C \setminus C$. Therefore $q(\mathcal{F}) \neq q(\mathcal{F}')$ for every $\mathcal{F} \in A$. Thus for every $\mathcal{F} \in A$ there exist $c_{\mathcal{F}} \in \mathcal{F}_{\prec} \setminus \mathcal{F}'_{\prec}$. So there exist $a_{\mathcal{F}} \in \mathcal{F}$ such that $a_{\mathcal{F}} \prec c_{\mathcal{F}}$ and $\neg a_{\mathcal{F}} \in \mathcal{F}'_{\prec}$. Let $O_{\mathcal{F}} \equiv \{x \in X : a_{\mathcal{F}} \in x\}$ and $U_{\mathcal{F}} \equiv \{x \in X : \neg a_{\mathcal{F}} \in x\}$. Then $\{O_{\mathcal{F}} : \mathcal{F} \in A\}$ is a cover of A, and it has a finite subcover $\{O_{\mathcal{F}_i} : i = 1, \ldots, n\}$. Let $U \equiv U_{\mathcal{F}_1} \cap \cdots \cap U_{\mathcal{F}_n}$, a neighborhood of $\mathcal{F}'$ which must meet C because $\mathcal{F}' \in \mathrm{cl}_X C$, say $\eta \in U \cap C$. That means that $\eta_{\prec} = q(\eta) = q(\mathcal{F}) = \mathcal{F}_{\prec}$ for some $\mathcal{F} \in A$. But $\mathcal{F} \in O_{\mathcal{F}_j}$ for some j because the $O_{\mathcal{F}_i}$'s cover A, and $\mathcal{F} \in U_{\mathcal{F}_j}$ by construction, a clear contradiction. We conclude that C is a closed subset of X.

Now we would like to show that the quotient map $q \colon X \to Y$ is irreducible, hence provides a bijection $\mathrm{RO}(X) \to \mathrm{RO}(Y)$. Let A be a proper closed subset of X, and consider $w \in X \setminus A$. We find $b, c, d \in B$ such that $0 < c \prec d \prec b$ and $w \in U \subseteq X \setminus A$, where $U \equiv \{u \in X : d \in u\}$. Thus we have $\neg c \in v_{\prec}$ for every $v \in A$, and since $b \in w_{\prec}$ we have $w_{\prec} \neq v_{\prec}$ for every $v \in A$. So $q(A)$ is a proper subset of Y, i.e. q is an irreducible quotient map. By Theorem 6.5(d) of [10] we have a bijection $\mathrm{RO}(X) \to \mathrm{RO}(Y)$.

We must show that, under the isomorphism $p \colon B \to \mathrm{RO}(Y)$, $a \prec b$ in B iff $\mathrm{cl}_Y\, p(a) \subseteq p(b)$. But this is not difficult, for the relation

$$U \prec' V \iff p^{-1}(U) \prec p^{-1}(V), \qquad U, V \in \mathrm{RO}(Y)$$

obviously makes $(\mathrm{RO}(Y), \prec')$ into a de Vries algebra, and therefore the $\prec'$ relation, when extended to the power set $P(X)$ of X by the convention that

$$A \prec' B \iff \exists U, V \in \mathrm{RO}(X)\ (A \subseteq U \prec' V \subseteq B), \qquad A, B \in \wp(X),$$

makes X into a compact separated proximity space by Remark 4.1.2. The desired conclusion follows from Lemma 3.1.5.

(3) We show that $\tilde{f}$ satisfies (M3) and (M4). For (M3) assume that $U \prec V$ for $U, V \in \mathrm{RO}(Y)$, i.e., $\mathrm{cl}_Y U \subseteq V$. Since

$$\begin{aligned}\neg\tilde{f}(\neg U) &= X \setminus \mathrm{cl}_X \tilde{f}(Y \setminus \mathrm{cl}_Y U) = X \setminus \mathrm{cl}_X \mathrm{int}_X \mathrm{cl}_X f^{-1}(Y \setminus \mathrm{cl}_Y U)\\ &= X \setminus \mathrm{cl}_X f^{-1}(Y \setminus \mathrm{cl}_Y U) \subseteq f^{-1}(\mathrm{cl}_Y U),\end{aligned}$$

it follows that $\mathrm{cl}_X \neg\tilde{f}(\neg U) \subseteq f^{-1}(\mathrm{cl}_Y U) \subseteq f^{-1}(V) \subseteq \mathrm{int}_X \mathrm{cl}_X f^{-1}(V) = \tilde{f}(V)$.

(M4) It is clear that we have $\bigvee_{U \prec V} \tilde{f}(U) \subseteq \tilde{f}(V)$ for all $V \in \mathrm{RO}(Y)$. For every $x \in f^{-1}(V)$ there exist $U \prec V$ such that $f(x) \in U \prec V$, hence $x \in f^{-1}(U) \subseteq f^{-1}(V)$. The point is that $\bigcup_{U \prec V} f^{-1}(U) = f^{-1}(V)$, so that

$$\bigvee_{U \prec V} \tilde{f}(U) = \mathrm{int}_X \mathrm{cl}_X \bigcup_{U \prec V} f^{-1}(U) = \mathrm{int}_X \mathrm{cl}_X f^{-1}(V) = \tilde{f}(V). \qquad \square$$

For every de Vries algebra $(B, \prec)$ the set $\mathrm{End}(B, \prec)$ of all ends on $(B, \prec)$ with the topology whose basic open sets of the form $O(a) = \{\mathcal{F} : a \in \mathcal{F}\}$ is a compact Hausdorff space.

Theorem 4.1.6. *The correspondences of Theorem 4.1.5 can be elaborated into a full categorical equivalence between the categories of compact Hausdorff spaces and that of de Vries algebras.*

For a given complete Boolean algebra B*, the separated de Vries proximities on* B *(Definition 4.1.1) are in bijective correspondence with the isomorphism classes of the compact Hausdorff spaces* X *for which* $\mathrm{RO}(X)$ *is isomorphic to* B*.*

An object P in a category **C** is said to be *projective* if every morphism $P \to B$ factors through every epimorphism $A \to B$. In the category **K** of compact Hausdorff spaces and continuous maps, the projective objects are the extremally disconnected spaces, i.e., those spaces in which the closure of each open set is open. These are the Stone spaces of the complete boolean algebras. Furthermore, a famous theorem of Gleason asserts the existence of a unique projective cover $q\colon P \to X$ for any compact Hausdorff space X, i.e., an irreducible surjection q from a projective object onto X. The Gleason cover is often referred to as the *absolute* of X, and two compact Hausdorff spaces are said to be *co-absolute* if their absolutes are homeomorphic.

Let B be a complete Boolean algebra with Stone space X. Among the various proximities on B which make it into a de Vries algebra, there is always a finest, namely the $\leqslant$ relation, and the de Vries dual compact Hausdorff space for this relation is X. In fact, for any other proximity $\prec'$ with corresponding compact Hausdorff space Y, the quotient map $q\colon X \to Y$ constructed in the proof of Theorem 4.1.5 is the Gleason cover of Y. An extensive development of the Gleason cover by means of de Vries algebras can be found in the excellent article [4].

Proposition 4.1.7. *Let* B *be a complete Boolean algebra with Stone space* X*. Then the poset of isomorphism types of those compact Hausdorff spaces having absolute* X *is isomorphic to the family of separating proximities on* B*.*

4.2 de Vries duality with actions

Definition 4.2.1. *Let* $(B, \prec)$ *be a de Vries algebra and let* T *be a topological monoid whose elements are designated by lower case letters* $t, r, s \ldots$ *An* action of T on B *is a monoid antimorphism.*

$$\theta\colon T \to \hom_{\mathbf{dv}}(B, B)$$

In detail,

- $\theta(1)$ *is the identity de Vries morphism on* B*, and*
- $\theta(rs) = \theta(s) * \theta(r)$.

Our convention is that T acts on B *on the right,* which means that we write $(a)\theta(t)$ for the result of applying the action $\theta(t)$ to the input $a \in B$. When we suppress mention of θ, which we shall do whenever possible, we write $(a)\theta(t)$ as at. Under that convention we have

- $a1 = a$ for all $a \in B$, and
- $a(rs) = (ar)s$ for all $a \in B$ and $r, s \in T$.

4.2.1 Smooth actions

Definition 4.2.2. *An action of* T *on a de Vries algebra* $(B, \prec)$ *is said to be* smooth *if for all* $a \prec b$ *in* B *and all* $t \in T$ *there exists a neighborhood* N_t *of* t *such that* $ar \prec cs$ *for all* $r, s \in N_t$.

Let X be a compact Hausdorff space with dual de Vries algebra B. Then every action of T on X induces an action of T on B, and vice-versa.

Theorem 4.2.3. *An action of* T *on a de Vries algebra* $(B, \prec)$ *is smooth iff the corresponding action makes the dual compact Hausdorff space into a flow.*

Proof. Take $B = \mathrm{RO}(X)$ for some compact Hausdorff space acted upon by T, and let $\prec$ be its unique compatible proximity, i.e., $a \prec b$ in B means $\mathrm{cl}\, a \subseteq b$. Then if X is a flow we know that $C(X) = C^T(X)$, so whenever $a \prec b$ in B there is an $f \in C^T(X)$ such that $f(a) = 0$ and $f(X \setminus b) = 1$. As in Theorem 3.1.16, this provides a T-scale $\{U_q\}$ such that $a \subseteq U_q \subseteq b$ for all $q \in \mathbb{Q}$. Then for $t \in T$ we have a neighborhood N_t of t such that for all $r, s \in N_t$ it is the case that $r^{-1}U_1 \subseteq s^{-1}U_2$, hence

$$r^{-1}a \subseteq r^{-1}U_0 \subseteq r^{-1}U_1 \subseteq s^{-1}U_2 \subseteq s^{-1}b.$$

Since $\{r^{-1}U_q\}$ is a scale, $\mathrm{cl}\, r^{-1}U_0 \subseteq r^{-1}U_1$, so that

$$\mathrm{cl}\, r^{-1}a \subseteq \mathrm{cl}\, r^{-1}U_0 \subseteq r^{-1}U_1 \subseteq s^{-1}b \subseteq \mathrm{int}\,\mathrm{cl}\, s^{-1}b.$$

In the language of de Vries algebras, this translates into the assertion that $ar \prec bs$ for all $r, s \in N_t$, which is to say that the action of T on B is smooth.

Now suppose T acts smoothly on the de Vries algebra $(B, \prec)$, let X be the compact Hausdorff space of ends of B, and identify B with $\mathrm{RO}(X)$. Consider an arbitrary continuous real-valued function $f\colon X \to \mathbb{R}$, $f \geqslant 0$, and let $\{U_q\}$ be the scale corresponding to f as in Lemma 2.1.11, i.e., $U_q = f^{-1}(-\infty, q)$ for all $q \in \mathbb{Q}$. Then $U'_q \equiv \operatorname{int}\operatorname{cl} U_q \in B$ for all $q \in \mathbb{Q}$, and $U'_p \prec U'_q$ for all $p < q$. The key point is that the smoothness of the action insures that U'_p is T-contained in U'_q for $p < q$, so that $\{U_p\}$ is a T-scale, and f is T-uniformly continuous by Lemma 2.1.13 and Lemma 2.1.6. With the aid of Lemma 3.1.7, we conclude that X is a flow. □

Definition 4.2.4. *A* T-de Vries algebra *is a de Vries algebra* $(B, \prec)$ *equipped with a smooth* T *action, and its proximity is referred to as a* T-proximity.

Definition 4.2.5. *Define* $\Phi : \mathbf{TK} \longrightarrow \mathbf{dVT}$ *by* $\Phi(X) = (\mathrm{RO}(X), \prec)$*, where* $\prec$ *is the unique proximity on* X *defined by* $b \prec a$ *if and only if* $\operatorname{cl} b \subseteq a$*. For a flow map* $f\colon X \to Y$*, define* $\Phi(f)\colon \mathrm{RO}(Y) \to \mathrm{RO}(X)$ *by* $\Phi(f)(a) = \operatorname{int}\operatorname{cl} f^{-1}(a)$.

Remark 4.2.6. *Note that the contravariant functor* Φ *is well defined because* $(\mathrm{RO}(X), \prec)$ *is a* T*-de Vries algebra by Theorem 4.2.3.*

Lemma 4.2.7. *Let* $f\colon X \to Y$ *be a flow map. Then* $\Phi(f)$ *is a* T*-de Vries morphism from* $\Phi(Y)$ *to* $\Phi(X)$.

Proof. It follows from de Vries duality that $\Phi(f)$ and $\phi(t)$ are de Vries morphisms. We only have to show that f commutes with the actions. Since f is a flow map, then $f(t(x)) = t(f(x))$ for every $t \in T$ and every $x \in X$.

Now, we would like to show that $\Phi(f \circ t) = \Phi(t) * \Phi(f)$. Let $a, b \in RO(Y)$ be such that $b \prec a$, so that $b \subseteq \operatorname{cl} b \subseteq a$ and it follows that $\operatorname{int}\operatorname{cl} f^{-1}(b) \subseteq f^{-1}(a)$. Thus $\operatorname{int}\operatorname{cl} t^{-1}(\operatorname{int}\operatorname{cl} f^{-1}(b)) \subseteq \operatorname{int}\operatorname{cl} t^{-1}(f^{-1}(a))$. Therefore, for $a \prec b$ we have,

$$
\begin{aligned}
(\Phi(t) \circ \Phi(f))(b) &= \operatorname{int}\operatorname{cl} t^{-1}[\operatorname{int}\operatorname{cl} f^{-1}(b)] \\
&\subseteq \operatorname{int}\operatorname{cl} t^{-1}[f^{-1}(b)] \\
&= \Phi(f \circ t)(a)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\Phi(t) * \Phi(f))(a) &= \bigvee_{b \prec a} (\Phi(t) \circ \Phi(f))(b) \\
&\subseteq \Phi(f \circ t)(a)
\end{aligned}
$$

For the reverse inclusion, let $a \in RO(Y)$. First we will show that

$$t^{-1}[f^{-1}(a)] = \textstyle\bigcup_{c \prec a} t^{-1}[f^{-1}(c)].$$

It is clear that $\bigcup_{c \prec a} t^{-1}[f^{-1}(c)] \subseteq t^{-1}[f^{-1}(a)]$. To prove the other inclusion let $x \in t^{-1}[f^{-1}(a)]$. Then $f(tx) \in a$, and because Y is Tychonoff we can find $c \in RO(Y)$ such that $f(tx) \in c \subseteq \operatorname{cl} c \subseteq a$ and therefore $x \in \bigcup_{c \prec a} t^{-1}[f^{-1}(c)]$.

Since a is open and since f and t are continuous functions we get

$$t^{-1}[f^{-1}(c)] \subseteq \operatorname{int}\operatorname{cl} t^{-1}[\operatorname{int}\operatorname{cl} f^{-1}(c)].$$

Hence

$$\begin{aligned}
\operatorname{int}\operatorname{cl} t^{-1}[f^{-1}(a)] &= \operatorname{int}\operatorname{cl} \bigcup_{c \prec a} t^{-1}[f^{-1}(c)] \\
&\subseteq \operatorname{int}\operatorname{cl} \bigcup_{c \prec a} \operatorname{int}\operatorname{cl} t^{-1}[\operatorname{int}\operatorname{cl} f^{-1}(c)] \\
&= \bigvee_{c \prec a} \operatorname{int}\operatorname{cl} t^{-1}[\operatorname{int}\operatorname{cl} f^{-1}(c)] \\
&= \bigvee_{c \prec a} (\Phi(t) \circ \Phi(f))(c).
\end{aligned}$$

So we get $\Phi(f \circ t)(a) \subseteq \Phi(t) * \phi(f)$ and therefore $\Phi(f \circ t)(a) = (\Phi(t) * \Phi(f))(a)$. A similar proof show that $\Phi(t \circ f)(a) = (\Phi(f) * \Phi(t))(a)$ and because $f \circ t = t \circ f$ it follows that $\Phi(t) * \Phi(f) = \Phi(f) * \Phi(t)$. $\square$

Theorem 4.2.8. *Φ is a contravariant functor from* **TK** *to* **dVT**.

Proof. Let X be a compact flow. Note that 1_X is the identity function and for any T-de Vries algebra B, $1_B = id_B$. Let $a \in RO(X)$, then

$$\begin{aligned}
\Phi(1_X)(a) &= \operatorname{int}\operatorname{cl} a \\
&= a \\
&= 1_{\Phi(X)}(a).
\end{aligned}$$

For every $t \in T$, it is clear that

$$\begin{aligned}\Phi(t) =&\Phi(t \circ 1_X)(a)\\ &= \operatorname{int} \operatorname{cl} t^{-1}(a)\\ &= at\\ &= (\Phi(t) * 1_{\Phi}(X))(a).\end{aligned}$$

Now let X, Y and Z be compact flows, and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are flow maps. Because the composition of flow maps is a flow map then by Lemma 4.2.7 we have $\Phi(g \circ f)$, $\Phi(f)$ and $\Phi(g)$ are T-de Vries morphisms. Following similar steps of the proof of Lemma 4.2.7 we get $\Phi(g \circ f) = \phi(f) * \Phi(g)$.

$\square$

Definition 4.2.9. *Define* $\Psi : \mathbf{dVT} \longrightarrow \mathbf{TK}$ *by*

$$\Psi(B) = \mathrm{End}(B)$$

For every T*-de Vries morphism* $f : A \longrightarrow B$ *define* $\Psi(f) : \mathrm{End}(B) \longrightarrow \mathrm{End}(A)$ *by*

$$\Psi(f)(\mathcal{F}) = \{a \in A : \exists b \in A \ b \prec a \ (f(b) \in \mathcal{F})\}.$$

The contravariant functor Ψ is well-defined because for every T-de Vries algebra the space of ends is a compact flow by Theorem 4.2.3. Also, it follows from de Vries duality that $\Psi(f)$ is a continuous function and $\Psi(f)(\mathcal{F}) \in \mathrm{End}(A)$.

Theorem 4.2.10. *Let* $f : A \longrightarrow B$ *be a* T*-de Vries morphism, then* $\Psi(f) : \mathrm{End}(B) \longrightarrow \mathrm{End}(A)$ *is a flow map.*

Proof. We only have to show that $\Psi(f)$ commutes with the actions, i.e, for every $t \in T$ we have $\Psi(f) \circ \Psi(t) = \Psi(t) \circ \Psi(f)$. Since f is a T-de Vries algebra we have $f * t = t * f$. So we only have to show that $\Psi(f * t) = \Psi(t) \circ \Psi(f)$.

Let $\mathcal{F} \in \mathrm{End}(B)$, then

$$\begin{aligned}(\Psi(t) \circ \Psi(f))(\mathcal{F}) &= \{a \in A : \exists c \in A \ (c \prec a \ \text{ and } \ ct \in \eta)\} \\ &= \{a \in A : \exists c \in A \ (c \prec a \ \text{ and } \exists d \in A \ [d \prec ct \text{ and } f(d) \in \mathcal{F}])\}.\end{aligned}$$

where $\eta = \{m \in A : d \in A \ (d \prec m \ \text{ and } f(d) \in \mathcal{F})\} = \Psi(f)(\mathcal{F})$.

If $a \in \Psi(t) \circ \Psi(f)$, then there exists $c \prec a$ and $d \in A$ such that $d \prec ct$ and $f(d) \in \mathcal{F}$. By (DV6) there exist c' such that $c \prec c' \prec a$. Because t and f are de Vries morphisms, we get $d \prec ct \prec c't$ and $f(d) \prec (f \circ t)(c) \prec (f \circ t)(c')$. It follows that $f(d) \prec \bigvee_{m \prec c'} (f \circ t)(m) = (f * t)(c')$ and therefore $(f * t)(c') \in \mathcal{F}$ which implies that $a \in \Psi(f * t)(\mathcal{F})$.

Because $\Psi(f*t)(\mathcal{F})$ and $(\Psi(t) \circ \Psi(f))(\mathcal{F})$ are ends on A and $(\Psi(t) \circ \Psi(f))(\mathcal{F}) \subseteq \Psi(f * t)(\mathcal{F})$, we have $\Psi(f * t)(\mathcal{F}) = (\Psi(t) \circ \Psi(f))(\mathcal{F})$. A similar proof shows that $\Psi(t * f) = \Psi(f) \circ \Psi(t)$ and because f is a T-de Vries morphism we get $\Psi(t) \circ \Psi(f) = \Psi(f) \circ \Psi(t)$. $\square$

Theorem 4.2.11. *Ψ is a contravariant functor from* **dVT** *to* **TK**.

Proof. Let 1_A be the idenity de Vries morphism, then

$$\begin{aligned}\Psi(1_B)(\mathcal{F}) &= \{a \in A : \exists c \in A \ (c \prec a \text{ and } c \in \mathcal{F})\} \\ &= \mathcal{F} \\ &= 1_{\mathrm{End}(A)}(\mathcal{F}) \\ &= 1_{\Psi(A)}(\mathcal{F}).\end{aligned}$$

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be a T-de Vries morphisms, then by de Vries duality $\Psi(g * f), \Psi(g)$ and $\Psi(f)$ are continuous functions and moreover, $\Psi(g * f) = \Psi(f) \circ \Psi(g)$. By Theorem 4.2.10 we have $\Psi(g * f), \Psi(g)$ and $\Psi(f)$ are flow maps. □

Remark 4.2.12. *From de Vries duality we have*

- *For any compact space X we have $(\Psi \circ \Phi)(X)$ is homeomorphic to X, and therefore, every compact flow X is homeomorphic to $(\Psi \circ \Phi)(X)$.*

- *For every de Vries algebra B we have $(\Phi \circ \Psi)(B)$ is isomorphic to B which implies that for every T-de Vries algebra B we have $(\Phi \circ \Psi)(B)$ is isomorphic to B.*

- *For every de Vries algebra B, the map $B \longrightarrow RO(\mathrm{End}(B))$ which maps b to $O(b)$ is a bijection.*

- *For every de Vries morphism* $f : A \longrightarrow B$ *and* $a \in A$ *we have* $(\Phi \circ \Psi)(f)(O(a)) = O(f(a))$. *Therefore, if* A *is a* T*-de Vries algebra then for every* $t \in T$ *we have* $(\Phi \circ \Psi)(t)(O(a)) = O(t(a))$. *We shall denote this by* $O(a)t = O(at)$.

- *For every continuous function* $f : X \longrightarrow Y$ *and* $x \in X$ *we have* $(\Psi \circ \Phi)(f)(\mathcal{F}_x) = \mathcal{F}_{f(x)} \cap RO(Y)$. *If* f *is a flow map then for every* $t \in T$ *we have* $t\mathcal{F}_x = \mathcal{F}_{tx}$*, which implies that*

$$\begin{aligned}
(\Psi \circ \Phi)(f)(t\mathcal{F}_x) &= \mathcal{F}_{f(tx)} \cap RO(Y) \\
&= \mathcal{F}_{tf(x)} \cap RO(Y) \\
&= t[\mathcal{F}_{f(x)} \cap RO(X)] \\
&= t[(\Psi \circ \Phi)(f)(\mathcal{F}_x)].
\end{aligned}$$

Theorem 4.2.13. *For a* T*-de Vries algebra* $(B, \prec)$ *define* $\xi_B : (RO(End(B)), \prec) \longrightarrow B$ *by* $\xi_B(O(b)) = b$. *Then* $\xi \colon \Phi \circ \Psi \longrightarrow 1_{dVT}$ *is a natural isomorphism.*

Proof. By remark 4.2.12 we have that ξ_B is a bijection; it remains only to show that ξ_B commutes with the actions, i.e., for every $t \in T$ we have $\xi_B t = t\xi_B$. Let $t \in T$, then

$$\begin{aligned}
\xi_B(O(b)t) &= \xi_B(O(bt)) \\
&= bt \\
&= (\xi_B O((b)))t.
\end{aligned}$$

Assuming that $f\colon A \longrightarrow B$ a T-de Vries morphism, then we have

$$\begin{tikzcd}
(\Phi \circ \Psi)(A) \arrow[r, "\xi_A"] \arrow[d, "(\Phi \circ \Psi)(f)"'] & A \arrow[d, "f"] \\
(\Phi \circ \Psi)(B) \arrow[r, "\xi_B"'] & B
\end{tikzcd}$$

we want to show this diagram commutes. First

$$(1_{dVT}(f) \circ \xi_A)(O(a)) = 1_{dVT}(f)(a) = f(a).$$

Also, by remark 4.2.12 we have

$$(\xi_B \circ (\Phi \circ \Psi)(f))(O(a)) = \xi_B(O(f(a))) = f(a).$$

Because ξ_B is an isomorphism for all T-de Vries algebra B it follows that ξ is a natural isomorphism. For every $t \in T$ we have

$$\begin{aligned}
(1_{dVT}(f) \circ \xi_A)(O(a)) &= 1_{dVT}(f)(at) \\
&= f(at) \\
&= f * t(a) \\
&= t * f(a) \\
&= t * (1_{dVT}(f) \circ \xi_A)(O(a)).
\end{aligned}$$

$\square$

Theorem 4.2.14. *For a compact flow, define* $\zeta_X : X \longrightarrow (\mathrm{End}(RO(X), \prec))$ *by* $\zeta_X(x) = \mathcal{F} \cap RO(X)$. *Then* $\zeta : 1_{TK} \longrightarrow (\Psi \circ \Phi)$ *is a natural isomorphism.*

Proof. For compact flows X, Y and flow map $f : X \longrightarrow Y$ we have the following diagram.

$$\begin{tikzcd} X \arrow[r, "\zeta_X"] \arrow[d, "f"'] & (\Psi \circ \Phi)(X) \arrow[d, "(\Psi \circ \Phi)(f)"] \\ Y \arrow[r, "\zeta_Y"'] & (\Psi \circ \Phi)(Y) \end{tikzcd}$$

we shall show that the diagram commutes. Note that

$$(\zeta \circ f)(x) = \zeta(f(x)) = \mathcal{F}_{f(x)} \cap RO(X).$$

Also, by remark 4.2.12 we have

$$((\Psi \circ \Phi)(f) \circ \zeta_X)(x) = (\Psi \circ \Phi)(f)(\mathcal{F}_x) = \mathcal{F}_x \cap RO(Y).$$

So the diagram commutes and hence ζ is a natural transformation. By remark 4.2.12 ζ_X is a homeomorphism for all compact flow X. Therefore it is a natural isomorphism.

We only have to show that the morphisms commutes with the action. Let $t \in T$, then

$$\begin{aligned}(\zeta \circ f)(tx) &= \zeta(f(tx)) \\ &= \mathcal{F}_{f(tx)} \cap RO(X) \\ &= t[\mathcal{F}_{f(x)} \cap RO(X)] \\ &= t[(\zeta \circ f)(x)].\end{aligned}$$

□

Theorem 4.2.15. *The categories* **TK** *and* **dVT** *are dually equivalent.*

Proof. By Theorem 4.2.8, Φ is a contravariant functor from **TK** to **dVT** and by Theorem 4.2.11, ψ is a contravariant functor from **dVT** to **TK**. By Theorem 4.2.13 and Theorem 4.2.14, $\xi : \Phi \circ \Psi \longrightarrow 1_{dVT}$ and $\zeta : 1_{TK} \longrightarrow \Psi \circ \Phi$ are natural isomorphisms. Therefore, **dVT** and **Tk** are dually equivalent. □

Theorem 4.2.16 (de Vries duality with actions). *A compact Hausdorff flow* X *induces an action making* $RO(X)$ *a separating* T*-de Vries algebra, and for every separating* T*-de Vries algebra* $(B, \prec)$ *there is a unique compact Hausdorff flow* X *for which* B *is isomorphic to* $RO(X)$ *and, when* B *is identified with its image under this isomorphism,* $a \prec b$ *iff* $\mathrm{cl}_X\, a \subseteq b$.

We point out that Theorem 4.2.16 includes classical de Vries duality as the special case in which the action of T is trivial, i.e., $T = 1$.

4.2.2 round T-filters and subflows

A round T-filter is a round filter that is closed under the actions. A round filter $\mathcal{F}$ is said to be *fixed* if $t\mathcal{F} = \mathcal{F}$ for all $t \in T$. The dual of round T-filters are called round T-ideals. It is well know that round filters are in bijective correspondence with round ideals by means of the complementation map.

Theorem 4.2.17. *If $(B, \prec)$ is a T-de Vries algebra, then there is a one-to one correspondence between round T-filters of B and subflows of $\mathrm{End}(B)$.*

Proof. For a round T-filter $\mathcal{F}$ of B, let us define:

$$Y_{\mathcal{F}} = \cap\{O(d) : d \in \mathcal{F}\}.$$

Let $\mathfrak{I} \in Y_{\mathcal{F}}$. Following the proof of (I.3.12) of [5] the set $Y_{\mathcal{F}}$ is a closed subset of the space $\mathrm{End}(B)$ because for every $a \in \mathcal{F}$ there exist $b \in \mathcal{F}$ such that $b \prec a$ and hence $\overline{O(b)} \subseteq O(a)$. Moreover, for every $t \in T$ we have $bt \in \mathcal{F}$ which implies that

$$\begin{aligned} bt \in \mathfrak{I} &\Rightarrow a \in t\mathfrak{I}\ \forall a \in \mathcal{F} \\ &\Rightarrow t\mathfrak{I} \in O(a)\ \forall a \in \mathcal{F} \\ &\Rightarrow t\mathfrak{I} \in Y_{\mathcal{F}}. \end{aligned}$$

Therefore, $Y_{\mathcal{F}}$ is a subflow and if $\mathcal{F} \subseteq \mathfrak{I}$ then $Y_{\mathfrak{I}} \subseteq Y_{\mathcal{F}}$.

Now suppose Y is a subflow of $\mathrm{End}(B)$, define

$$\mathcal{F}_Y = \{a \in B : Y \subseteq O(a)\}.$$

To show $\mathcal{F}_Y$ is a round filter

- let $a \in \mathcal{F}_Y$ and $a \leqslant b$, we have $Y \subseteq O(a) \subseteq O(b)$ which implies that $b \in \mathcal{F}_Y$;

- let $a, b \in \mathcal{F}_Y$, then $Y \subseteq O(a) \cap O(b) = O(a \wedge b)$. Thus for every $\mathfrak{I} \in Y$ we have $a \wedge b \in \mathfrak{I}$ so there exist $c_{\mathfrak{I}} \in \mathfrak{I}$ such that $c_{\mathfrak{I}} \prec a \wedge b$. Then $\{O(c_{\mathfrak{I}})\}_{\mathfrak{I} \in Y}$ is an open cover of Y and it has a finite subcover. Therefore we could fined $c \prec a \wedge b$ and $Y \subseteq O(c)$.

Since Y is closed under the actions, then $tY \subseteq Y$. Let $a \in \mathcal{F}_Y$

$$\begin{aligned}
a &\in \mathcal{F}_Y \Rightarrow tY \subseteq Y \subseteq O(a) \\
&\Rightarrow a \in t\mathfrak{I} \text{ for every } \mathfrak{I} \in Y \text{ and for every } t \in T \\
&\Rightarrow \exists b_{\mathfrak{I}} \in B \text{ such that } b_{\mathfrak{I}} \prec a \text{ and } b_{\mathfrak{I}} t \in \mathfrak{I} \\
&\Rightarrow at \in \mathfrak{I} \text{ for every } \mathfrak{I} \in Y \\
&\Rightarrow Y \subseteq O(at) \\
&\Rightarrow at \in \mathcal{F}_Y.
\end{aligned}$$

Therefore $\mathcal{F}_Y$ is a round T-filter. $\square$

Lemma 4.2.18. *Let* X *be a compact Hausdorff space with regular open algebra* B. *Then the maps*

$$\begin{aligned} U &\longrightarrow I_U \equiv \{ a \in B : \operatorname{cl} a \subseteq U \} \\ \bigcup I &\longleftarrow I \end{aligned}$$

are inverse bijections between the family $\mathcal{O}X$ *of open subsets of* X *and the family of round ideals of* B.

Proof. For a given $U \in \mathcal{O}X$, it is straightforward to check that I_U is an ideal of B. To verify that it is round, consider $a \in I_u$. Then $X \smallsetminus U$ and $\operatorname{cl} a$ are disjoint closed subsets of the normal space X, hence there is an open set V such that $\operatorname{cl} a \subseteq V \subseteq \operatorname{cl} V \subseteq U$, yielding $a \prec b \in I_U$ for $b \equiv \operatorname{int}\operatorname{cl} V$. And it is clear that $\bigcup I_U = U$, for obviously $\bigcup I_U \subseteq U$, and $U \subseteq \bigcup I_U$ by virtue of the regularity of X.

Given a round ideal I of B, let $U \equiv \bigcup I$. We claim that $I_U = I$. For $I \subseteq I_U$ because for every $a \in I$ there exists some $b \in I$ such that $\operatorname{cl} a \subseteq b \subseteq \bigcup I = U$. To show that $I_U \subseteq I$, consider $a \in B$ such that $\operatorname{cl} a \subseteq U$. Since $\operatorname{cl} a$ is compact, $\operatorname{cl} a \subseteq \bigcup I_0$ for a finite subset $I_0 \subseteq I$. Then clearly $a \subseteq \operatorname{int}\operatorname{cl} \bigcup I_0$, which is to say that $a \leqslant \bigvee I_0$ in B, hence $a \in I$. □

We remind the reader that, in a de Vries algebra $(B, \prec)$, any ideal I on B contains a largest round ideal

$$I_{\prec} \equiv \{ a \in I : \exists b \in I \, (a \prec b) \},$$

an ideal which we refer to as the *round part of* I. In case B arises from a compact Hausdorff space X as in Lemma 4.2.18, $\bigcup I_{\prec} = \bigcup I$. In connection with that lemma, note that for each open subset $U \subseteq X$ there are typically many ideals $I \subseteq B$ such that $\bigcup I = U$, but there is exactly one round ideal with this feature.

Let X be a compact flow. Recall that a subset $Y \subseteq X$ is T-invariant if $ty \in Y$ for all $y \in Y$ and T-stable if $t^{-1}Y \subseteq Y$ for all $t \in T$.

Lemma 4.2.19. *Let X be a compact flow and let $(B, \prec)$ be its dual T-de Vries algebra. Then the bijections of Lemma 4.2.18 restrict to inverse bijections between the families of T-stable open subsets of X and round T-ideals of B. Consequently, the maps*

$$\begin{aligned} Y &\longrightarrow \{ a \in B : \operatorname{cl} a \cap Y = \emptyset \} \\ X \setminus \bigcup I &\longleftarrow I \end{aligned}$$

are inverse bijections between the family of closed subflows of X and the family of round T-ideals of B.

Proof. A subset $Y \subseteq X$ is a closed subflow iff its complement $U = X \setminus Y$ is open and T-stable. $\square$

Let X be a compact Hausdorff space with de Vries dual $(B, \prec_B)$, and let Y be a closed subspace of X with de Vries dual $(A, \prec_A)$. Let $U \equiv X \setminus Y$ and $I \equiv I_U$ as in

Lemma 4.2.18. Then the insertion $Y \to X$ gives rise to a de Vries homomorphism

$$\mathfrak{m}\colon B \to A = (b \mapsto \operatorname{int}_Y \operatorname{cl}_Y(b \cap Y)), \qquad b \in B.$$

Lemma 4.2.20. *Assume the foregoing terminology. Then for elements* $b_i \in B$*,*

$$\mathfrak{m}(b_1) \leqslant \mathfrak{m}(b_2) \iff \forall b \in B \ (b \prec b_1 \wedge \neg b_2 \implies b \in I).$$

Consequently, for $b_i \in B$*,*

$$\mathfrak{m}(b_1) = \mathfrak{m}(b_2) \iff \forall b \in B \ (b \prec b_1 \wedge \neg b_2 \text{ or } b \prec b_2 \wedge \neg b_1 \implies b \in I).$$

Proof. In any boolean algebra, $b_1 \nleqslant b_2$ iff there exists $0 < a \leqslant b_1 \wedge \neg b_2$. Applying this principle to elements $\mathfrak{m}(b_i)$, we get

$$\mathfrak{m}(b_1) \nleqslant \mathfrak{m}(b_2) \iff \exists 0 < a \leqslant \mathfrak{m}(b_1) \wedge \neg \mathfrak{m}(b_2).$$

But $\mathfrak{m}$ is surjective and preserves meets, so we may take a to be of the form $\mathfrak{m}(b)$ for some $b \leqslant b_1$ such that $\mathfrak{m}(b) > 0$ and $\mathfrak{m}(b) \wedge \mathfrak{m}(b_2) = \mathfrak{m}(b \wedge b_2) = 0$. But if $\mathfrak{m}(b) = \operatorname{int}_Y \operatorname{cl}_Y(b \cap Y) > 0$ then $b \nsubseteq U$, from which it follows that $b' \notin I$ for some $0 < b' \leqslant b$. Here we have used the fact that $b = \bigcup\{\, b' : \operatorname{cl} b' \subseteq b \,\}$ by virtue of the regularity of X. $\square$

4.3 Proximity topology

This section is a digression from the main development. In it we observe that the proximity $\prec$ on a de Vries algebra $(B, \prec)$ is associated with a particular topology on B, here termed a *proximity topology*. This is Theorem 4.3.2. However, the proximity topology is not a Boolean topology, meaning that it does not render the Boolean operations continuous, as we show in Example 4.3.3. In fact, as of this writing we are unaware of any further implications of these observations.

Definition 4.3.1. *A* proximity topology *on a complete Boolean algebra* B *is a topology having a basis for open sets of the form*

$$\langle a, b\rangle \equiv \mathrm{int}[a, b], \qquad a \leqslant b,$$

with the following features:

(PT1) 1 *and* 0 *are isolated.*

(PT2) $\neg\langle a, b\rangle = \langle\neg b, \neg a\rangle$*;*

(PT3) *If* $\langle a, b\rangle \neq \emptyset$ *and* $\langle a, c\rangle \neq \emptyset$ *then* $\langle a, b \wedge c\rangle \neq \emptyset$*;*

(PT4) $\bigvee\langle\perp, a\rangle = a$ *for all* $a \in B$*;*

(PT5) *If* $\langle a, b\rangle \neq \emptyset$ *then there exists* $a \leqslant c \leqslant b$ *for which* $\langle a, c\rangle \neq \emptyset$ *and* $\langle c, b\rangle \neq \emptyset$.

Theorem 4.3.2. *Given a de Vries algebra* $(B, \prec)$*, the subsets of the form*

$$\langle a, b \rangle \equiv \{c : a \prec c \prec b\}, \qquad a \leqslant b,$$

form a basis for a proximity topology on B*. Conversely, given a proximity topology on a complete Boolean algebra* B*, the relation* $\prec$ *on* B *defined by the rule*

$$a \prec b \iff a \leqslant b \textit{ and } \operatorname{int}[a, b] \neq \emptyset.$$

is a proximity on B *which generates the given topology as above. These two processes are mutually inverse, so that the proximities on* B *are in bijective correspondence with the proximity topologies on* B*.*

Proof. For a given de Vries algebra $(B, \prec)$, we would like to show that the family of subsets of B defined as above forms a base for a proximity topology on B. Note that the family is certainly closed under intersection, for

$$\begin{aligned} \langle a, b \rangle \cap \langle c, d \rangle &= \operatorname{int}[a, b] \cap \operatorname{int}[c, d] = \operatorname{int}([a, b] \cap [c, d]) \\ &= \operatorname{int}[a \vee c, b \wedge d] = \langle a \vee c, b \wedge d \rangle. \end{aligned}$$

Furthermore, the family satisfies (PT1) because $(B, \prec)$ satisfies (DV1) of Definition 4.1.1; (PT2) follows from (DV4) in similar fashion, as does (PT4) from (DV8). Next observe that $\langle a, b \rangle \neq \emptyset$ iff $a \prec b$, hence (PT3) follows from (DV4) and (PT5) follows from (DV6).

Now suppose we are given a basis

$$\langle a, b\rangle \equiv \mathrm{int}[a, b], \qquad a \leqslant b,$$

for a proximity topology on a complete Boolean algebra B, and define the relation $\prec$ as above. Then $(B, \prec)$ satisfies (DV1) of Definition 4.1.1 because the topology satisfies (PT1), (DV2) and (DV3) by construction, (DV4) by (PT3), (DV5) by (PT2), (DV6) by (PT5), and (DV7) by (PT4). Finally, only a little reflection is required to conclude that these two processes are inverses of one another. $\square$

We note here that the proximity topology is not a Boolean topology, which is to say that the Boolean operations on B are not necessarily continuous in that topology.

Example 4.3.3. *Take* B *to be* $\mathrm{RO}[0, 1]$*, the regular open algebra on the unit interval, and let* $a_1 \equiv [0, 1/2)$ *and* $a_2 \equiv (1/2, 1]$*, elements which join to the top* $[0, 1]$ *in* B*. But the top element is isolated in the proximity topology, so that if the join operation were continuous then we could find regular open subsets* $b_i \prec a_i \prec c_i$ *such that* $a_1' \vee a_2' = [0, 1]$ *for all* $b_i \prec a_i' \prec c_i$*. Clearly no such elements* b_i *and* c_i *exist.*

Chapter 5

Coproduct in dVT

5.1 Coproduct of Boolean algebras

Theorem 11.2 of [12] states that every family of Boolean algebras has, up to isomorphism, a unique free product which is in fact the boolean algebra dual to the product space of the dual spaces of boolean algebras in the given family.

Assume that A and B are complete Boolean algebras, let $S_A = \mathtt{ult}(A)$ and $S_B = \mathtt{ult}(B)$ be their Stone spaces, and identify A and B with the algebras $\mathtt{Clop}(S_A)$ and $\mathtt{Clop}(S_B)$ of clopen subsets of S_A and S_B, respectively. By elementary Stone duality, the sum $A \oplus B$ in the category of boolean algebras is the clopen algebra of the product $S_A \times S_B$ in the category of compact Hausdorff spaces. The canonical insertions are given by the formulas $A \to A \oplus B = (a \mapsto a \times S_B)$ and $B \to A \oplus B = (b \mapsto S_A \times b)$. The rectangle determined by elements $a \in A$ and $b \in B$ is the set $(a \times S_B) \cap (S_A \times b)$, a set we denote by $a \wedge b$.

Two important features of rectangles are

$$a_1 \wedge b_1 \leqslant a_2 \wedge b_2 \iff a_1 \leqslant a_2 \text{ and } b_1 \leqslant b_2, \quad \text{and}$$

$$(a_1 \wedge b_1) \wedge (a_2 \wedge b_2) = 0 \iff a_1 \wedge a_2 = 0 \text{ or } b_1 \wedge b_2 = 0.$$

An arbitrary element of the sum is a finite union of rectangles, written $\bigvee_I (a_i \wedge b_i)$ for a finite index set I.

5.2 Coproduct of de Vries algebras

Let $(A, \prec_A)$ and $(B, \prec_B)$ be abstract de Vries algebras, with compact Hausdorff spaces X_A and X_B of ends. Our objective is the coproduct of $(A, \prec_A)$ and $(B, \prec_B)$ in the category **dV** of de Vries algebras. This, of course, is the de Vries dual $(\mathrm{RO}(X_A \times X_B), \prec)$ of the product $X_A \times X_B$ in the category of compact Hausdorff spaces. What we offer in this section is an alternative construction of the coproduct, one which sheds some light on its internal structure.

We start with the sum $A \oplus B$ in the category of Boolean algebras, with Stone spaces S_A and S_B and with canonical insertions $A \to A \oplus B$ and $B \to A \oplus B$ as in Subsection 5.1. We work in the complete boolean algebra $\mathrm{RO}(S_A \times S_B)$, which we denote by $A \boxplus B$. Since the compact space $S_A \times S_B$ has a base of rectangles, each element $u \in A \boxplus B$ can be expressed in the form $u = \bigvee_I (a_i \wedge b_i)$ for a possibly infinite index set I.

The elements which can be expressed in this form for a finite index set I are the clopen sets, and they make up the boolean subalgebra $A \oplus B$ of $A \boxplus B$. The expression $u = \bigvee_I (a_i \wedge b_i)$ is not unique, i.e., it is possible that $\bigvee_I (a_i \wedge b_i) = \bigvee_J (c_j \wedge d_j)$ for $I \neq J$. However, comparison of elements of $A \boxplus B$ is facilitated by recalling that, in any boolean algebra, $a \not\leqslant b$ iff there exist $0 < c \leqslant b$ such that $a \wedge c = 0$; we say that c witnesses the fact that $a \not\leqslant b$. This is the idea behind the proof of Lemma 5.2.1.

Lemma 5.2.1. *For elements* $u = a \wedge b$ *and* $v = \bigvee_J (a_j \wedge b_j)$ *of* $A \boxplus B$, $u \leqslant v$ *iff for all subsets* $J_0 \subseteq J$ *we have* $a \leqslant \bigvee_{J_0} a_j$ *or* $b \leqslant \bigvee_{J \smallsetminus J_0} b_j$. *Consequently, for elements* $u = \bigvee_I (a_i \wedge b_i)$ *and* $v = \bigvee_J (a_j \wedge b_j)$ *of* $A \boxplus B$, $u \leqslant v$ *iff for all* $i \in I$ *and all subsets* $J_0 \subseteq J$ *we have* $a_i \leqslant \bigvee_{J_0} a_j$ *or* $b_i \leqslant \bigvee_{J \smallsetminus J_0} b_j$.

Proof. If $a \wedge b \not\leqslant \bigvee_I (a_i \wedge b_i)$ then there is a witnessing rectangle $0 < a' \wedge b' \leqslant a \wedge b$ disjoint from $\bigvee_I (a_i \wedge b_i)$. But

$$0 = (a' \wedge b') \wedge \bigvee_I (a_i \wedge b_i) = \bigvee_I ((a' \wedge b') \wedge (a_i \wedge b_i))$$

implies that $a' \wedge a_i = 0$ or $b' \wedge b_i = 0$ for all $i \in I$. If we put $I_0 \equiv \{i : a' \wedge a_i = 0\}$ then we get that $a \not\leqslant \bigvee_{I_0} a_i$ as witnessed by a' and $b \not\leqslant \bigvee_{I \smallsetminus I_0} b_i$ as witnessed by b'.

Conversely, if there is a subset $I_0 \subseteq I$ such that

$$0 < a' \equiv a \wedge \left(\neg \bigvee_{I_0} a_i \right) \quad \text{and} \quad 0 < b' \equiv b \wedge \left(\neg \bigvee_{I \smallsetminus I_0} b_i \right)$$

then $a' \wedge b'$ is a nonempty open subset of $a \wedge b$ disjoint from $\bigvee_I (a_i \wedge b_i)$. $\square$

$A \boxplus B$ is the boolean completion of $A \oplus B$, i.e., $A \boxplus B$ is the unique complete boolean algebra into which $A \oplus B$ embeds as a dense subalgebra. But $A \boxplus B$ also has the status of a de Vries algebra, as we shall now explain.

We may regard the Stone space S_A of A as the de Vries dual, i.e., the space of ends of the de Vries algebra $(A, \leqslant)$, and likewise S_B may be regarded as the space of ends of $(B, \leqslant)$. Since $S_A \times S_B$ is the product in the category of compact Hausdorff spaces, its de Vries dual $(A \boxplus B, \prec_0)$ is the coproduct of $(A, \leqslant)$ and $(B, \leqslant)$ in the category of de Vries algebras. Here $\prec_0$ is the canonical proximity on the compact space $S_A \times S_B$, i.e., $u \prec_0 v$ iff $\operatorname{cl} u \leqslant v$. Note that if $u \in A \oplus B$ then u is clopen, hence $u \prec_0 u$.

Lemma 5.2.2 ([4], Section 4). *Assume the foregoing terminology. Then for all* $u, v \in A \boxplus B$*,* $u \prec_0 v$ *iff there exists some* $w \in A \oplus B$ *such that* $u \leqslant w \leqslant v$*.*

Proof. Certainly, if $\operatorname{cl} u \leqslant v = \bigvee_I (a_i \wedge b_i)$ then since $\operatorname{cl} u$ is compact, $\operatorname{cl} u \leqslant \bigvee_{I_0} (a_i \wedge b_i) \equiv w$ for some finite subset $I_0 \subseteq I$, and $w \in A \oplus B$. On the other hand, if $u \leqslant w \leqslant v$ for some $w \in A \oplus B$ then we have $u \leqslant w \prec_0 w \leqslant v$, hence $u \prec_0 v$. □

Thus $(A \boxplus B, \prec_0)$ is a zero-dimensional de Vries algebra. Now we shall introduce a new proximity $\prec$ on $A \boxplus B$, and we shall show that the space of ends of $(A \boxplus B, \prec)$ is $X_A \times X_B$. Finally we shall show that the identical insertion $(A \boxplus B, \prec) \to (A \boxplus B, \prec_0)$ is a de Vries homomorphism, and this will provide a surjection $S_A \times S_B \to X_A \times X_B$.

Definition 5.2.3. *For elements* $u = \bigvee_I (a_i \wedge b_i)$ *and* $v = \bigvee_J (a_j \wedge b_j)$ *of* $A \oplus B$*, declare* $u \prec v$ *to mean that for all* $i \in I$ *and* $J_0 \subseteq J$*,*

$$a_i \prec_A \bigvee_{J_0} a_j \quad \textit{or} \quad b_i \prec_B \bigvee_{J \smallsetminus J_0} b_j.$$

For elements $u, v \in A \boxplus B$*, declare* $u \prec v$ *to mean that there exist elements* $u', v' \in A \oplus B$ *such that* $u \leqslant u' \prec v' \leqslant v$*.*

Lemma 5.2.4. $(A \boxplus B, \prec)$ *is a de Vries algebra.*

Proof. It is clear that $\prec$ satisfies (dV1), and it satisfies (dV2) because $u' \prec v'$ implies $u' \leqslant v'$ for elements $u', v' \in A \oplus B$ by Lemma 5.2.1. It is also clear that $\prec$ satisfies (dV3). To verify the remaining de Vries axioms, consider elements $u_j = \bigvee_{I_j} (a_i^j \wedge b_i^j)$ for index sets I_j.

(dV4) It suffices to show that for elements $u_i \in A \oplus B$, $u_1 \prec u_2, u_3$ implies $u_1 \prec u_2 \wedge u_3$. Observe first that

$$\begin{aligned} u_2 \wedge u_3 &= \bigvee_{i \in I_2} (a_i^2 \wedge b_i^2) \wedge \bigvee_{j \in I_3} (a_j^3 \wedge b_j^3) \\ &= \bigvee_{(i,j) \in I_2 \times I_3} ((a_i^2 \wedge a_j^3) \wedge (b_i^2 \wedge b_j^3)). \end{aligned}$$

Fix $i_0 \in I_1$ and $J \subseteq I_2 \times I_3$. For each $k \in I_2$ let $J_k \equiv \{ j \in I_3 : (k, j) \in J \}$. Because $u_1 \prec u_3$, for each $k \in I_2$ either $a_{i_0}^1 \prec_A \bigvee_{i \in J_k} a_i^3$ or $b_{i_0}^1 \prec_B \bigvee_{i \notin J_k} b_i^3$. Let

$$K \equiv \left\{ k \in I_2 : a_{i_0}^1 \prec_A \bigvee_{i \in J_k} a_i^3 \right\}.$$

Because $u_1 \prec u_2$, either $a^1_{i_0} \prec_A \bigvee_{k\in K} a^2_k$ or $b^1_{i_0} \prec_B \bigvee_{k\notin K} b^2_k$. In the first case when $a^1_{i_0} \prec_A \bigvee_{k\in K} a^2_k$, we set $a \equiv \bigwedge_K \bigvee_{i\in J_k} a^3_i$. Since $a^1_{i_0} \prec_A a$ we get

$$\begin{aligned}\bigvee_{(i,j)\in J}(a^2_i \wedge a^3_j) &= \bigvee_{k\in I_2}\bigvee_{j\in J_k}(a^2_k \wedge a^3_j) \geqslant \bigvee_{k\in K}\left(a^2_k \wedge \bigvee_{j\in J_k} a^3_j\right) \\ &\geqslant \bigvee_{k\in K}(a^2_k \wedge a) = \bigvee_{k\in K} a^2_k \wedge a \;{}_A{\succ} a^1_{i_0}.\end{aligned}$$

In the second case when $b^1_{i_0} \prec_B \bigvee_{k\notin K} b^2_k$, we have for each $k \notin K$ that $b^1_{i_0} \prec_B \bigvee_{i\notin J_k} b^3_i$, so we put $b \equiv \bigwedge_{k\notin K} \bigvee_{i\notin J_k} b^3_i$. Since $b^1_{i_0} \prec b$ we get

$$\begin{aligned}\bigvee_{(i,j)\notin J}(b^2_i \wedge b^3_j) &= \bigvee_{k\in I_2}\bigvee_{j\notin J_k}(b^2_k \wedge b^3_j) \geqslant \bigvee_{k\notin K}\left(b^2_k \wedge \bigvee_{j\notin J_k} b^3_j\right) \\ &\geqslant \bigvee_{k\notin K}(b^2_k \wedge b) = \bigvee_{k\notin K} b^2_k \wedge b \succ b^1_{i_0}.\end{aligned}$$

(dV5) It is sufficient to show that $u_1 \prec U_2$ implies $\neg u_2 \prec \neg u_1$ for elements $u_i \in A \oplus B$. So assume $u_1 \prec u_2$, and observe that

$$\neg u_1 = \neg \bigvee_{I_1}(a^1_i \wedge b^1_i) = \bigwedge_{I_1}(\neg a^1_i \vee \neg b^1_i) = \bigvee_{K\subseteq I_1}\left(\bigwedge_{k\in K} \neg a^1_k \wedge \bigwedge_{k\notin K} \neg b^1_k\right).$$

Thus in order to show that $\neg u_2 \prec \neg u_1$ we must show that for any subset $J \subseteq I_2$ and for any family $\mathcal{K}$ of subsets of I_1 we have

$$\bigwedge_{i\in J} \neg a^2_i \prec_A \bigvee_{K\in\mathcal{K}} \bigwedge_{k\in K} \neg a^1_k \quad \text{or} \quad \bigwedge_{i\notin J} \neg b^2_i \prec_B \bigvee_{K\notin\mathcal{K}} \bigwedge_{k\notin K} \neg b^1_k,$$

which is to say that

$$\bigwedge_{K\in\mathcal{K}}\bigvee_{k\in K} a_k^1 \prec_A \bigvee_{i\in J} a_i^2 \quad\text{or}\quad \bigwedge_{K\notin\mathcal{K}}\bigvee_{k\notin K} b_k^1 \prec_B \bigvee_{i\notin J} b_i^2. \tag{$*$}$$

For that purpose fix a subset $J \subseteq I_2$ and family $\mathcal{K}$ of subsets of I_1, and put

$$K \equiv \left\{ k \in I_1 : a_k^1 \prec_A \bigvee_{i\in J} a_i^2 \right\}.$$

Since $\bigvee_{k\in K} a_k^1 \prec_A \bigvee_{i\in J} a_i^2$ by the dual of axiom (dV4) in A, if $K \in \mathcal{K}$ then the first alternative in $(*)$ holds. On the other hand, the assumption that $u_1 \prec u_2$ implies that $b_k^1 \prec_B \bigvee_{i\notin J} b_i^2$ for all $k \notin K$, hence $\bigvee_{k\notin K} b_k^1 \prec_B \bigvee_{i\notin J} b_i^2$. Therefore if $K \notin \mathcal{K}$ then the second alternative in $(*)$ holds.

(dV6) It is enough to show that whenever elements $u_i \in A\oplus B$ satisfy $u_1 \prec u_2$ there exists $v \in A \oplus B$ such that $u_1 \prec v \prec u_2$.

We first claim that for $a_i \in A$ and $b_i \in B$, $a_1 \wedge b_1 \prec a_2 \wedge b_2$ if $a_1 \prec_A a_2$ and $b_1 \prec_B b_2$. This is true because, for the purpose of applying Definition 5.2.3, we view the expression $a_2 \wedge b_2$ as of the form $\bigvee_{\{2\}}(a_i \wedge b_i)$. Then a subset $J \subseteq \{2\}$ is either $\{2\}$ or $\emptyset$, and if $J = \{2\}$ then $a_1 \prec_A \bigvee_{j\in J} a_j = a_2$ and if $J = \emptyset$ then $b_1 \prec \bigvee_{j\notin J} b_j = b_2$.

We next claim that for any $a \in A$ and $b \in B$ such that $a \wedge b \prec u = \bigvee_I (a_i \wedge b_i) \in A\oplus B$ there exist elements $a' \in A$ and $b' \in B$ such that $a\wedge b \prec a'\wedge b' \prec u$. For the assumption that $a \wedge b \prec u$ means that for any $J \subseteq I$ either $a \prec_A \bigvee_{j\in J} a_j$ or $b \prec_B \bigvee_{j\notin J} b_j$. By appealing to axiom (dV6) in A and B separately, we may choose elements $a' \in A$ and $b' \in B$ such that

$$a \prec_A a' \prec_A \bigwedge \left\{ \bigvee_{j\in J} a_j^2 : a \prec_A \bigvee_{j \in J} a_j^2 \right\} \qquad \text{and}$$
$$b \prec_B b' \prec_B \bigwedge \left\{ \bigvee_{j\in J} b_j^2 : b \prec_B \bigvee_{j \in J} b_j^2 \right\}.$$

Then by construction we have arranged for $a \wedge b \prec a' \wedge b' \prec u$.

Now assume that $\bigvee_{I_1}(a_i^1 \wedge b_i^2) = u_1 \prec u_2 = \bigvee_{I_2}(a_i^2 \wedge b_i^2)$ for finite index sets I_1 and I_2. The proof of axiom (dV6) is completed by using the second claim to choose, for each $i \in I_1$, elements $a_i' \in A$ and $b_i' \in B$ such that $a_i^1 \wedge b_i^1 \prec a_i' \wedge b_i' \prec u_2$, and then checking that $u_1 \prec \bigvee_{I_1}(a_i' \wedge b_i') \prec u_2$.

(dV7) If $0 < u \in A \boxplus B$ then there exists $0 < v \in A \oplus B$ with $v \leqslant u$ because $A \oplus B$ is dense in $A \boxplus B$, say $v = \bigvee_I (a_i \wedge b_i)$ for finite index set I. Moreover, there exist an index $i \in I$ such that $a_i \wedge b_i > 0$, which is to say $a_i > 0$ in A and $b_i > 0$ in B. By (dV7) in A and B there exist $0 < c \in A$ and $0 < d \in B$ such that $c \prec_A a_i$ and $d \prec_B b_i$. Therefore $c \wedge d \prec a_i \wedge b_i \leqslant u$, which implies that $0 < c \wedge d \prec u$. □

Lemma 5.2.5. *The compact Hausdorff space* Z *of ends of the de Vries algebra* $A \boxplus B$ *is (homeomorphic to)* $X_A \times X_B$*, where* X_A *and* X_B *represent the spaces of ends of* A *and* B*, respectively.*

Proof. We shall establish a bijection between $X_A \times X_B$ and Z. If $x \in X_A$ and $y \in X_B$ then it is straightforward to check that the filter $z(x, y)$ generated by $\{a \wedge b : a \in x,\ b \in y\}$ is round. To show that $z(x, y)$ is maximal among round filters, it is sufficient to show that if $a \wedge b \prec \bigvee_I (a_i \wedge b_i)$ in $A \oplus B$ then either $\neg(a \wedge b) \in z(x, y)$ or $\bigvee_I (a_i \wedge b_i) \in z(x, y)$. This means that for all $I' \subseteq I$, either $a \prec_A \bigvee_{i \in I'} a_i$ or $b \prec_B \bigvee_{i \notin I'} b_i$, so that, by the maximality of x and y, either $\neg a \in x$ or $\bigvee_{i \in I'} a_i \in x$ or $\neg b \in y$ or $\bigvee_{i \notin I'} b_i \in y$. If $\neg a \in x$ then $\neg(a \wedge b) = \neg a \vee \neg b \geqslant \neg a \wedge 1_B \in z(x, y)$, and similarly if $\neg b \in y$ then $\neg(a \wedge b) \in z(x, y)$. In the only case remaining we have that for all $I' \subseteq I$ either $\bigvee_{i \in I'} a_i \in x$ or $\bigvee_{i \notin I'} b_i \in y$. In view of the fact that

$$\bigvee_I (a_i \wedge b_i) = \bigwedge_{I' \subseteq I} \Big(\bigvee_{i \in I'} a_i \vee \bigvee_{i \notin I'} b_i \Big),$$

we can conclude that $\bigvee_I (a_i \wedge b_i) \in z(x, y)$.

Consider an arbitrary $z \in Z$ and put $x_z \equiv \{a : a \wedge 1_B \in z\}$. We claim that x_z is an end of A. Surely x_z is a filter on A; to show that it is round, consider an arbitrary $a \in x_z$. Since z is round there exists $u \in A \boxplus B$ such that $a \succ u \in z$; without loss of generality we may assume $u \in A \oplus B$, say $u = \bigvee_I (a_i \wedge b_i$ for finite index set I. Since $a_i \wedge b_i \prec a \wedge 1_B$ for each $i \in I$, it follows that $a_i \prec_A a$ for each i, hence $\bigvee_I a_i \prec_A a$ and $a \wedge 1_B \succ \bigvee_I a_i \wedge 1_B \in z$ and $a \succ_A \bigvee_I a_i \in x_z$. To show that x_z is maximal among round filters on A, consider $a_1 \prec_A a_2$. Then $a_1 \wedge 1_B \prec a_2 \wedge 1_B$, hence either $\neg(a_1 \wedge 1_B) \in z$ or $a_2 \wedge 1_B \in z$, which yields $\neg a_1 \in x_z$ or $a_2 \in x_z$. This proves the claim, and a parallel argument shows that $\{b : 1_A \wedge b \in z\} \equiv y_z \in X_B$.

Finally, since for any $z \in Z$ it is clear that $z(x_z, y_z) \subseteq z$, it follows from the maximality of $z(x_z, y_z)$ that $z(x_z, y_z) = z$. And since for any $x \in X_A$ and $y \in X_B$

it is clear that $x \subseteq x_{z(x,y)}$ and $y \subseteq y_{z(x,y)}$, it likewise follows from maximality that $x = x_{z(x,y)}$ and $y = y_{z(x,y)}$. Having established the bijection, the proof is completed by a routine argument showing its continuity in both directions. $\square$

Lemma 5.2.6. *The insertion map* $(A \boxplus B, \prec) \to (A \boxplus B, \prec_0)$ *is a de Vries morphism.*

Proof. It is clear that it is a boolean isomorphism; we only have to prove that if $u \prec v$ then $u \prec_0 v$. But this is immediate from Lemma 5.2.2 and Definition 5.2.3. $\square$

Lemma 5.2.7. $(A \boxplus B, \prec)$*, with the standard insertion maps, functions as the categorical sum in* **dV**.

Proof. Apply de Vries duality to the fact that Z serves as the product of X_A and X_B in the category of compact Hausdorff spaces. $\square$

5.3 Co-free T-de Vries algebras

In this section we show that an abstract de Vries algebra can be endowed with actions as freely as possible. Specifically, we develop the cofree T-de Vries algebra over a naked de Vires algebra. In detail, for a given de Vries algebra A we find a T-de Vries algebra D and de Vries homomorphism $g\colon D \to A$ with the following universal property. For every T-de Vries algebra C and de Vries homomorphism $f\colon C \to A$ there is a unique T-de Vries homomorphism $h\colon C \to D$ such that $g \circ h = f$.

$$\begin{tikzcd} A & C \arrow[l, "f"'] \arrow[dl, "h"] \\ D \arrow[u, "g"] & \end{tikzcd}$$

In order to carry out the construction of D we make the simplifying assumption that T is compact. That implies that T is a compact flow under the action $T \times T \to T = ((t, s) \mapsto ts)$ by left multiplication. Consequently, the action

$$RO(T) \times T \to T = ((b, t) \mapsto bt \equiv \operatorname{int} \operatorname{cl} t^{-1}b = \operatorname{int} \operatorname{cl}\{ s : ts \in b \})$$

is a smooth action on the de Vries algebra $(RO(T), \prec)$, where $a \prec b$ in $RO(T)$ if $\operatorname{cl} a \subseteq b$.

Now suppose we are given a given naked de Vries algebra $(A, \prec_A)$ with dual compact Hausdorff space X_A of ends. Form the sum $D \equiv A \boxplus RO(T)$ as in the previous section. The de Vries dual of D is the product $T \times X_A$, which is also a compact flow under left multiplication according to the rule

$$t(s, x) \equiv (ts, x), \qquad s, t \in T, \ x \in X.$$

Lemma 5.3.1. *The action of* T *on* $T \times X_A$ *by left multiplication gives rise to the dual action of* T *on* D *given by the rule*

$$ut = \bigvee \{ a \wedge bt : a \in A, \ b \in RO(T), \ a \wedge b \prec u \}, \qquad u \in D, \ t \in T.$$

This action is smooth.

Proof. Each element $u \in D$ is the join of rectangles $a \wedge b \prec u$, where a and b are regular open subsets of X_A and T, respectively. And

$$
\begin{aligned}
(a \wedge b)t &= \operatorname{int}\operatorname{cl}\{\,(s,x) : t(s,x) \in a \wedge b\,\} \\
&= \operatorname{int}\operatorname{cl}\{\,(s,x) : ts \in b \text{ and } x \in a\,\} = a \wedge bt.
\end{aligned}
$$

The smoothness of the action is explained by Theorem 4.2.3. □

We remark that a direct proof that the formula displayed in Lemma 5.3.1 gives a smooth action of T on D, one that does not go through the dual, appears to be subtle.

It is an important fact that every compact Hausdorff space can be freely embedded in a compact flow.

Proposition 5.3.2 ([1], 6.1)**.** *Let X be a compact Hausdorff space. Then the map $k \colon X \to T \times X = (x \mapsto (1,x))$ is the free compact flow over X. That is, for any compact flow Y and any continuous function m there is a unique flow map l such that $l \circ k = m$.*

$$
\begin{tikzcd}
T \times X \arrow[r, "l"] & Y \\
X \arrow[u, "k"] \arrow[ur, "m"'] &
\end{tikzcd}
$$

The map l satisfies $l((t,x)) = tm(x)$ for all $x \in X$ and $t \in T$.

Proof. Note that $l(r(t,x)) = l(rt,x) = rtm(x) = rl(t,x)$ which implies that l is a flow map. □

Proposition 5.3.3. *There is a unique cofree* T*-de Vries algebra over any naked de Vries algebra. That is, for any given de Vries algebra* A *there is a unique* T*-de Vries algebra* D *and de Vries homomorphism* g *with the following universal property. For any* T*-de Vries algebra* C *and de Vries homomorphism* f *there is a unique* T*-de Vries homomorphism* h *such that* $g \circ h = f$.

$$\begin{tikzcd} D \arrow[d, "g"'] & C \arrow[l, "h"'] \arrow[dl, "f"] \\ A & \end{tikzcd}$$

The map g *satisfies* $g(u) = \bigvee\{\, a \in A : a \wedge b \prec u$ *and* $1 \in b \,\}$ *for all* $u \in D$. *The map* h *satisfies* $h(c) = \bigvee\{\, a \wedge b : \forall t \in b\ (a \leqslant f(ct)\,\}$ *for all* $c \in C$.

Proof. This is the dual of Proposition 5.3.2, with X there taken to be X_A here, and the dual of $T \times X$ there taken to be $D = A \boxplus \mathrm{RO}(T)$ here. To verify the formula for g, first note that for any $u \in D$ we have

$$g(u) = \operatorname{int}\operatorname{cl} k^{-1}(u) = \operatorname{int}\operatorname{cl}\{\, x \in X_A : k(x) = (1, x) \in u \,\}.$$

where k has the meaning in Proposition 5.3.2. But a point $(1, x)$ lies in u iff there is a basic rectangle of the form $a \wedge b$ for $a \in \mathrm{RO}(X_A) = A$ and $b \in \mathrm{RO}(T)$ such that $(1, x) \in a \wedge b \prec u$.

To verify the formula for h, first observe that for any $c \in C$,

$$\begin{aligned} h(c) &= \operatorname{int}\operatorname{cl} l^{-1}c = \operatorname{int}\operatorname{cl}\{\,(t,x) : l(t,x) \in c\,\} = \operatorname{int}\operatorname{cl}\{\,(t,x) : tm(x) \in c\,\} \\ &= \operatorname{int}\operatorname{cl}\{\,(t,x) : m(x) \in ct\,\} = \operatorname{int}\operatorname{cl}\left\{\,(t,x) : x \in m^{-1}(ct)\,\right\} \end{aligned}$$

where l has the meaning in Proposition 5.3.2. But a point $(t, x) \in T \times X_A$ has the feature that $m(x) \in ct$ iff there is a basic rectangle of the form $a \wedge b$ for $a \in RO(X_A) = A$ and $b \in RO(T)$ such that $(t, x) \in a \wedge b$ and $m(x') \in ct'$ for all $x' \in a$ and $t' \in b$. The last condition is equivalent to $a \leqslant f(ct)$. $\square$

Chapter 6

Applications

6.1 Boolean flows

Recall that a boolean space is a space which is compact Hausdorff and zero-dimensional, i.e., compact Hausdorff with a clopen basis. Thus a space is boolean iff it is the Stone space of its clopen algebra; let us designate the category of such spaces by **bK**. Bezhanishvili conducted a penetrating analysis of de Vries duality confined to this subcategory of **K** in [4]. In that paper we find the following strengthening of de Vries axiom (DV6).

Definition 6.1.1. *We call a de Vries algebra* $(B, \prec)$ zero-dimensional *if axioms (DV6) is strengthened by the following axiom:*

(SDV6) $a \prec b$ *implies there exists* $c \in B$ *such that* $c \prec c$ *and* $a \prec c \prec b$.

Note that if X is a boolean space then it is the Stone space of its clopen algebra $\mathrm{clop}(X)$, whereas the de Vries dual of X is $\mathrm{RO}(X)$. These two algebras coincide iff $\mathrm{clop}(X)$ is complete, i.e., iff X is extremally disconnected.

In [11], the authors study injective and projective objects in the category of boolean flows; let us denote this category by **TbK**. They show that every left action on a boolean space gives rise to a right action on its clopen algebra, and vice-versa; moreover, the continuity of the left action (meaning the continuity of the evaluation map) is equivalent to the continuity of the right action (with respect to the discrete topology). The resulting duality is between **TbK** and the category of boolean algebras with actions, designated **baT**. The paper conducts an extensive analysis of injective objects in **baT**, and in particular shows that this category has enough injectives, and a more modest analysis of projective objects in **baT**. Passing to the dual, we get that every object of **TbK** has a projective cover, but that only a few objects have injective envelopes.

The two aforementioned investigations differ with one another in two important respects. First and foremost, [11] treats categories with actions and [4] does not. Second, the two use different boolean algebras in their dualities with boolean spaces X: $\mathrm{clop}(X)$ in the case of [11] and $\mathrm{RO}(X)$ in the case of [4]. Nevertheless, both articles are closely related to the topics of this thesis. We list here a few remarks regarding points of contact.

Remark 6.1.2.

- *In regard to Definition 6.1.1, note that if* $c \prec c$ *and the action on the algebra is smooth then for every* $t \in T$ *there exist a neighborhood* N_t *of* t *such that* $cr \prec cs$ *for every* $r, s \in N_t$. *This implies, in fact, that* $cr = ct$ *for every* $r \in N_t$*, which agrees with the result in [11] (Lemma 1.2.1) that* $N_t = \{r : cr = ct\}$ *is an open subset of* T.

- *Moreover, in the proximity topology of Section 4.3.1 we can see that for every* $c \prec c$ *we have* $\langle c, c \rangle = \{c\}$.

- *Thus if a flow* X *is extremally disconnected, so that* $\mathrm{clop}(X) = \mathrm{RO}(X)$*, then the proximity topology of Section 4.3.1 is discrete. It is this topology which makes the action of* T *on* $\mathrm{clop}(X)$ *continuous, and this is in agreement with [11] .*

6.2 Gleason Cover of a compact flow

The Gleason cover of a compact flow was first introduced by Richard Ball and James Hagler in [1]. They prove the existence and uniqueness of projective covers in the category of flows with perfect flow maps. In the classical case (no action), Guram Bezhanishvili gave a simple construction of the Gleason cover of a compact space by means of de Vries duality ([4]). Even though we were unable to extend Guram's work to include actions, here we can at least translate the Ball-Hagler result into terms of T-de Vries algebras. The key concept is that of a T-essential de Vries homomorphism.

The context for the following remarks are the categories **TK** of compact flows and **dVT** of T-de Vries algebras. Straighforward arguments can be used to show that, in both **TK** and **dVT**, the monomorphisms are the injective morphisms and the epimorphisms are the surjective morphisms. Of course, the monomorphisms in one category are dual to the epimorphisms in the other.

In a category **C**, a morphism $f\colon A \to B$ is called an irreducible surjection if it is an epimorphism such that, for any morphism $g\colon C \to A$, if $f \circ g$ is an epimorphism then so is g. The irreducible surjections of TK are characterized in Theorem 2.3.3, where they were given the name T-irreducible surjections. The dual notion is that of an essential extension; a morphism $f\colon A \to B$ of **C** is called an essential extension if it is a monomorphism such that, for any morphism $g\colon B \to C$, if $g \circ f$ is a monomorphism then so is g. We shall use the term T-essential extension for this type of monomorphism.

In a category **C**, a *maximal essential extension* is an essential extension $f\colon A \to B$ of which every other essential extension is a factor, i.e., for any essential extension $g\colon A \to C$ there is a morphism $h\colon C \to B$ such that $h \circ g = f$. The dual notion is that of a maximal irreducible preimage. It is known that a projective cover provides a maximal irreducible preimage.

Lemma 6.2.1. *The following are equivalent for a flow map* $f\colon Y \to X$ *in* TK *with dual* $g\colon A \to B$ *in* dVT.

1. f *is* T*-irreducible, i.e., irreducible in* **TK**.
2. f *maps no proper closed subflow of* Y *onto* X.
3. g *is a* T*-essential extension.*
4. *For all* $0 < b \in B$ *there exist* $0 < a \in A$ *and finite* $T_0 \subseteq T$ *such that* $g(a) \prec_B \bigvee_{T_0} bt$.

Proof. 2 is just a reformulation of 1 in light of the fact that the epimorphisms in **TK** are the surjections. The equivalence of 1 with 3 is a consequence of the duality between **TK** and **dVT**. The equivalence of 2 with 4 here is the equivalence of 1 with 4 in Theorem 2.3.3. □

An object I in a category **C** is called an *injective* if for any morphism $f\colon A \to I$ and monomorphism $g\colon A \to B$ there exists a morphism $h\colon B \to I$ such that $h \circ g = f$.

$$\begin{tikzcd} B \arrow[dr, "h"] & \\ A \arrow[u, "g"] \arrow[r, "f"'] & I \end{tikzcd}$$

An *injective envelope* of an object C of **C** is an essential extension $e\colon A \to I$ such that I is an injective in **C**. When they exist, injective envelopes are unique up to isomorphism over A. In many categories, including **dVT**, the injective envelopes are precisely the maximal essential extensions.

Theorem 6.2.2. *A* T*-de Vries algebra* A *has a maximal* T*-essential extension* $h\colon A \to E$*. This is the injective envelope of* A *in* **dVT***, and is unique up to isomorphism.*

Proof. What Ball and Hagler prove in [1] is that every object in **TK** has a maximal T-irreducible preimage. This theorem is simply the statement of the dual in **dVT**. □

6.3 Problems

Problem 6.3.1. *Assume we are given a complete boolean algebra* B *and a proximity* $\prec$ *relation that satisfies de Vries axioms (dV1)-(dV6), but not (dV7). Can one find a largest de Vries algebra subalgebra* $(C, \prec')$*?*

Problem 6.3.2. *Assume that the action of* T *on* $(B, \prec)$ *is not smooth. What is the largest* T*-de Vries algebra subalgebra* $(C, \prec')$ *on which the action is smooth?*

Problem 6.3.3. *What is the co-free* T*-de Vries algebra over a given de Vries* $(B, \prec)$ *algebra if we omit the compact assumption in chapter 5?*

Problem 6.3.4. *If* $(B, \prec)$ *is a* T*-de Vries algebra and* X *is its* T*-de Vries dual. What is the topology that can be defined on* B *such that continuity of the action on* B *is equivalent to continuity of the action on* X.

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Appendix A

This Appendix includes only definitions and well know facts without proves that we have used in one or more of the previous chapters and have not defined them there.

A.1 Set theory

A relation E on a set X, i.e., a subset of $X \times X$, is called an equivalence relation on X when it has the following properties:

- For every $x \in X$, xEx.
- If xEy, then yEx.
- If xEy and yEz, then xEz.

A partial order is a binary relation $\leqslant$ on a set A satisfying the following axioms:

- (relexivity) $a \leqslant a$ for all $a \in A$.
- (antisymmetry) If $a \leqslant b$ and $b \leqslant a$, then $a = b$.

- (transitivity) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant a$.

A set with a partial order is called a partially ordered set.

Lemma A.1.1. *(Zorn's Lemma) Let* $\underline{A} = (A, \leqslant)$ *be a partial order, and suppose that every subset* B *of* A *which is ordered by* $\leqslant$ *has an upper bound. Then* $\underline{A}$ *has a maximal member.*

A.2 General Topology

A topological space is a pair (X, τ) consisting of a set and a family τ of subsets of X satisfying the following conditions:

- $\emptyset \in \tau$ and $X \in \tau$.
- If $u, v \in \tau$, then $u \cap v \in \tau$.
- If $\mathcal{A} \subset \tau$, then $\bigcup \mathcal{A} \in \tau$.

The subsets of X belonging to τ are called open, and a subset of X is closed if its complement is open. A family $\mathcal{B}$ of τ is called a base for a topological space (X, τ) if every non-empty open subset of X can be represented as the union of subfamily of $\mathcal{B}$. Any base has the following properties:

- For any $u, v \in \mathcal{B}$ and every point $x \in u \cap v$ there exists a $w \in \mathcal{B}$ such that $x \in w \subseteq u \cap v$.
- For every $x \in X$ there exist $u \in \mathcal{B}$ such that $x \in u$.

A family $\mathcal{P}$ of τ is called a subbase if the family of all finite intersections $u_1 \cap u_2 \cap ..... \cap u_k$ where $u_i \in \mathcal{P}$ for $i = 1, ..., k$ a a base for (X, τ).

The closure of a subset A of X is defined be the intersection of all closed subsets of X that contains A and is denoted by $cl(A)$. The interior of A is the union of all open sets contained in A and is denoted by $int(A)$.

For a set X one can define different topologies. If τ_1 and τ_2 are two topologies on X and $\tau_2 \subset \tau_1$, then we say that the topology τ_1 is finer than the topology τ_2, or that τ_2 is coarser than τ_1.

Let X and Y are two topological spaces, a map $f : X \longrightarrow Y$ is called continuous if $f^{-1}(u)$ is open in X for every open subset u of Y. A continuous map $f : X \longrightarrow Y$ is called closed (open) if for every closed (open) set $A \subseteq X$ the image $f(A)$ is closed (open) in Y. A continuous map $f : X \longrightarrow Y$ is called a homeomorphism if it is one-to-one, onto and the inverse mapping $f^{-1} : Y \longrightarrow X$ is continuous.

A subspace Y of a topological space is a subset of X with the induced topology where open sets in Y has the form $U \cap Y$ for some open set U in X.

A.2.1 Axioms of Separation

A topological space X is called a

- T_0-space if for every pair of distinct points $x, y \in X$ there exists an open set containing exactly one of these points.

- T_1-space if for every pair of distinct points $x, y \in X$ there exists an open set $u \subset X$ such that $x \in u$ and $y \notin u$.

- T_2-space or (Hausdorff) if for every pair of distinct points $x, y \in X$ there exists open sets u, v such that $x \in u$ and $y \in v$ and $u \cap v = \emptyset$.

- T_3-space, or regular space, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exist open sets u, v such that $x \in u, F \subset v$ and $u \cap v = \emptyset$.

- $T_{3\frac{1}{2}}$-space, or Tychonoff space, or a completely regular space, if X is T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous function $f : X \longrightarrow I$ such that $f(x) = 0$ and $f(x) = 1$ for $y \in F$.

- T_4-space, or normal space, if X is a T_1-space and for every pair of disjoint closed subsets $A, B \subset X$ there exist open sets u, v such that $A \subset u$, $B \subset v$ and $u \cap v = \emptyset$.

Proposition A.2.1. *A T_1-space is a regular space if and only if for every $x \in X$ and every neighbourhood v of x in a fixed subbase $\mathcal{P}$ there exists a neighbourhood u of x such that $\mathrm{cl}\,u \subset v$.*

Proposition A.2.2. *A T_1-space is a Tychonoff space if and only if for every $x \in X$ and every neighbourhood v of x in a fixed subbase $\mathcal{P}$ there exists a continuous function $f : X \longrightarrow I$ such that $f(x) = 0$ and $f(x) = 1$ for $y \in X \setminus v$.*

Theorem A.2.3. *(Uryson's Lemma)[6] For every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f : X \longrightarrow I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

Theorem A.2.4. *[Urysohn [7]]A subspace Y is C^*-embedded in a Tychonoff space X if and only if any two completely separated subsets in Y are completely separated in X.*

Let $X = \prod_{s \in S} X_s$ be the Cartesian product of the family $\{X_s\}_{s \in S}$ of topological spaces and the family of mapping $\{p_s\}_{s \in S}$, where p_s assigns to the point $x \in X$ its sth coordinate $x_s \in X_s$. The set X with the topology generated by the family of mapping $\{p_s\}_{s \in S}$ is called the Cartesian product of the spaces $\{X_s\}_{s \in S}$ and the topology itself is called the Tychonoff topology.

Proposition A.2.5. *The family of all sets $\{W_s\}_{s \in S}$, where W_s is open subset of X_s and $W_s \neq X_s$ only for finite many $s \in S$, is a base for the Cartesian product $\prod_{s \in S} X_s$.*

Suppose we are given a topological space X and an equivalence relation E on the set X. Denote the set of all equivalence classes of E by X/E and the mapping of X to X/E by q which maps a point x to the equivalence class $[x]$. It turn out that in the class of all topologies on X/E that makes q continuous there exists the finest one; this is the family of all sets U such that $q^-1(U)$ is open in X. This topology is called the quotient topology and X/E is called the quotient space.

A.2.2 Filters

Let $\mathcal{R}$ be a family of sets that together with A and B contains the intersection $A \cap B$. By a filter in $\mathcal{R}$ we mean a non-empty subfamily $\mathcal{F} \subset \mathcal{R}$ satisfying the following conditions:

(F1) $\emptyset \notin \mathcal{F}$.

(F2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

(F3) If $A \in \mathcal{F}$ and $A \subset B \in \mathcal{R}$,then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ in $\mathcal{R}$ is a maximal filter or an ultrafilter in $\mathcal{R}$, if for every filter $\mathcal{G}$ in $\mathcal{R}$ that contains $\mathcal{F}$ we have $\mathcal{G} = \mathcal{F}$.

A filter base in $\mathcal{R}$ is a non-empty family $\mathcal{G} \subset \mathcal{R}$ such that $\emptyset \notin \mathcal{G}$ and

(FB) If $A_1, A_2 \in \mathcal{G}$, then there exist an $A_3 \in \mathcal{G}$ such that $A_3 \subset A_1 \cap A_2$.

Remark A.2.6. *One can easily prove that the set $\mathcal{N}_x$ of all topological neighbourhoods of the point x form a filter. We call it the neighbourhood filter at point $x \in X$*

A point x is called a limit of a filter $\mathcal{F}$ if every neighbourhood of x belongs to $\mathcal{F}$; we then say that the filter $\mathcal{F}$ converges to x. A point x is called a cluster point of a filter $\mathcal{F}$ if x belongs to the closure of every member of $\mathcal{F}$. Clearly, x is a cluster point of a filter $\mathcal{F}$ if and only if every neighbourhood of x intersects all members of $\mathcal{F}$.

A.2.3 Compact spaces

A cover of a set X is a family $\{A_s\}_{s \in S}$ of subsets of X such that $\bigcup_{s \in S} A_s = X$. If X is a topological space , $\{A_s\}_{s \in S}$ is an open (closed) cover of X if all sets A_s are open (closed). A topological space X is called compact space if X is a Hausdorff space and every open cover of X has a finite subcover.

We say that a family $\mathcal{A} = \{A_s\}_{s\in S}$ of subsets of X has the finite intersection property if $\mathcal{A} \neq \emptyset$ and $A_{s_1} \cap A_{s_2} \cap ..... \cap A_{s_k} \neq \emptyset$ for every finite set $\{s_1, s_2, ..., s_k\} \subset S$.

Theorem A.2.7. *A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has nonempty intersection.*

The following are well know facts about compact spaces, for more information about compact spaces see [6].

- Every closed subspace of compact space is compact.
- Every compact space is normal.
- Every compact subspace of a Hausdorff space X is a closed subspace of X.
- every continuous mapping of a compact space to a Hausdorff space is closed.
- Every filter in X has a cluster point.

Let Y^X be the set of all continuous mapping of X to Y. For subsets $A \subset X$ and $B \subset Y$, let

$$M(A, B) = \{f \in Y^X : f(A) \subset B\}$$

Definition A.2.8. *The compact-open topology on Y^X is the topology generated by the base consisting of all $\bigcup_{i=1}^{k} M(C_i, U_i)$, where C_i is a compact subset of X and U_i is an open subset of Y for $i = 1, 2, ..., k$.*

A pair (Y, α), where Y is compact space and $\alpha : X \longrightarrow Y$ is a homeomorphic embedding of X in Y such that $\mathrm{cl}(\alpha(X)) = Y$, is called a compactification of the space X.

Theorem A.2.9. *[6] A topological space* X *has a compactification if and only if* X *is a Tychonoff space.*

Let $\mathcal{C}(X)$ be the family of all compactification of X. Define an order relation on $\mathcal{C}(X)$ by; $\alpha_1 X \leqslant \alpha_2 X$ if there exists a continuous function $f : \alpha_1 X \longrightarrow \alpha_2 X$ such that $f\alpha_1 = \alpha_2$. The largest element in the family $\mathcal{C}(X)$ is called the Čech-stone compactification and is denoted by βX.

Theorem A.2.10. *(compactification Theorem) Every (Tychonoff) space* X *has a compactification* βX*, with the following equivalent properties*

(I) (Stone) Every continuous function g *from* X *into any compact space* Y *has a continuous extension* $\overline{g}$ *from* βX *into* Y.

(II) (Stone-Čech) Every function $f \in C^*(X)$ *has a continuous extension to a function* $f^\beta \in C(\beta X)$.

A.3 Algebra

Definition A.3.1. *A Boolean algebra is a set* B *together with operations* $\neg : B \to B, \wedge : B \times B \to B$ *and* $\vee : B \times B \to B$, *and special elements* $0 \in B$ *and* $1 \in B$, *which satisfies the following properties for all* $a, b, c \in B :$

1. $a \wedge 1 = a \vee 0 = a;$

2. $a \wedge \neg a = 0$ *and* $a \vee \neg a = 1$;

3. $a \wedge a = a \vee a = a$;

4. $\neg (a \wedge b) = \neg a \vee \neg b$ *and* $\neg (a \vee b) = \neg a \wedge \neg b$;

5. $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$;

6. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *and* $a \vee (b \vee c) = (a \vee b) \vee c$;

7. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *and* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A complete Boolean algebra B is a Boolean algebra such that for every subset M of B we have $\bigvee M$ and $\bigwedge M$ are in B.

Definition A.3.2. *Let* B *and* C *be Boolean algebras. Then a homomorphism* $f : B \to C$ *is a map that preserves all the structure of Boolean algebras:*

$$\begin{aligned} f(0) &= 0 \text{ and } f(1) = 1 \\ f(a \vee b) &= f(a) \vee f(b) \\ f(a \wedge b) &= f(a) \wedge f(b), \text{and} \\ f(\neg a) &= \neg f(a). \end{aligned}$$

If f *is also a bijection, we say* f *is an isomorphism.*

Example A.3.3. *Let* X *be any set and* $P(X)$ *be the set of all subsets of* X*. Then* $P(X)$ *is a Boolean algebra with*

$$\begin{aligned} A \wedge B &= A \cap B \\ A \vee B &= A \cup B \textit{ and} \\ \neg A &= X \setminus A. \end{aligned}$$

Let B be a Boolean algebra and $C \subseteq B$ be a subset containing 0 and 1 and closed under the boolean operations. Then C is a boolean algebra, and we say C is a subalgebra of B.

Example A.3.4. *Let* X *be a topological space and* $\mathrm{Clop}(X)$ *be the set of clopen (both closed and open) subsets of* X*. Then* $\mathrm{Clop}(X)$ *is a subalgebra of* $P(X)$*: clopen sets are closed under (finite) unions and intersections and complements.*

Definition A.3.5. *Let* B *be a Boolean algebra and* $a, b \in B$*. Then we say* $a \leqslant b$ *if* $a \wedge b = a$.

Definition A.3.6. *Let* B *be a Boolean algebra. A subset* $I \subseteq B$ *is a ideal if:*

1. $0 \in I$*;*

2. *If* $a \in I$ *and* $b \leqslant a$*, then* $b \in I$*;*

3. *If* $a, b \in I$*, then* $a \vee b \in I$*;*

4. *If* $1 \notin B$*, we say* B *is a proper ideal.*

Definition A.3.7. *A filter in a Boolean algebra* B *is a subset* $\mathcal{F}$ *of* B *such that*

1. $1 \in \mathcal{F}$*;*

2. *If* $a \in \mathcal{F}$ *and* $a \leqslant b$*, then* $b \in \mathcal{F}$*;*

3. *If* $a, b \in \mathcal{F}$*, then* $a \wedge b \in \mathcal{F}$*.*

A.3.1 Stone duality

Definition A.3.8. *For a Boolean algebra* B*,*

$$X_B = \{\mathcal{F} : \mathcal{F} \quad \textit{an ultrafilter of } B\}$$

is the set of ultrafilters of B*. The map* $s : B \longrightarrow P(X_B)$ *defined by*

$$s(b) = \{\mathcal{F} \in X_B : b \in \mathcal{F}\}$$

is the stone map. Sometimes $\mathtt{ult}(B)$ *is used for* X_B*.*

X_B *with* $s(b)$ *as a base for the topology is called the stone space or Boolean space. This space is known to be zero-dimensional space.*

Theorem A.3.9. *(Stone's representation theorem) Every Boolean algebra is isomorphic to the clopen algebra of a Boolean space.*

Proposition A.3.10. *A Boolean algebra is complete iff it is isomorphic to the regular open algebra of some topological space*