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Discrepancy Inequalities in Graphs and Their Applications

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Discrepancy inequalities in graphs and their applications

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ABSTRACT

Spectral graph theory, which is the use of eigenvalues of matrices associated with graphs, is a modern technique that has expanded our understanding of graphs and their structure. A particularly useful tool in spectral graph theory is the Expander Mixing Lemma, also known as the discrepancy inequality, which bounds the edge distribution between two sets based on the spectral gap. More specifically, it states that a small spectral gap of a graph implies that the edge distribution is close to random. This dissertation uses this tool to study two problems in extremal graph theory, then produces similar discrepancy inequalities based not on the spectral gap of a graph, but rather a different tool with motivations in Riemannian geometry.

The first problem explored in this dissertation is motivated by parallel computing and other communication networks. Consider a connected graph G, with a pebble placed on each vertex of G. The routing number, $rt(G)$, of G is the minimum number of steps needed to route any permutation on the vertices of G, where a step consists of selecting a matching in the graph and swapping the pebbles on the endpoints of each edge. Alon, Chung, and Graham introduced this parameter, and (among other results) gave a bound based on the spectral gap for general graphs. The bound they obtain is polylogarithmic for graphs with a sufficiently strong spectral gap. In this dissertation, we use the Expander Mixing Lemma, the probablistic method, and other extremal tools to investigate when this upper bound can be improved to be constant depending on the gap and the vertex degrees.

The second problem examined in this dissertation has motivations in a question of Erdős and Pósa, who conjectured that every sufficiently dense graph on n vertices, where

n is divisible by 3, decomposes into triangles. While Corradi and Hajnal proved this result true for graphs with minimum degree at least $\frac{2}{3}n$, their result spawned a series of similar questions about the number of vertex-disjoint subgraphs of a certain class that a graph with some degree condition must contain. While this problem is well-studied for dense graphs, many results give significantly worse bounds for less dense graphs. Using spectral graph theory, we show that every graph with some weak density and spectral conditions contains $O(\sqrt{nd})$ vertex-disjoint cycles. Furthermore, even if we require these cycles to contain a certain number of chords, a graph satisfying these conditions will still contain $O(\sqrt{nd})$ such vertex-disjoint cycles. In both cases, we show this bound to be best possible.

Finally, we conclude by obtaining local version of a discrepancy inequality. An oversimplification of the Expander Mixing Lemma states that a graph with a strong spectral condition must have nice edge distribution. We seek to mimic that idea, but by using discrete curvature instead of a spectral condition. Discrete curvature, inspired by its counterpart in Riemannian geometry, measures the local volume growth at a vertex. Thus, given a vertex x, our result uses curvature to quantify the edge distribution between vertices that are a distance one from x and vertices that are a distance two from x. In doing this, we are able to study the number of 3-cycles and 4-cycles containing a particular edge.

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Chapter 1: Introduction

A graph is a structure consisting of a set of vertices and a set of edges, where each edge joins of a pair of vertices. Graphs can be used to model any construct consisting of a set of objects with pairwise relationships between those objects. The study of graph theory increased in prominence with the rise of communication networks. As these networks grew in size and quantity, questions about these networks emerged. Thus, mathematicians began to study structural properties of these networks such as the greatest distance between two objects, the connectedness of the network, and how quickly information could travel through the network. With the advent of computers and the internet, applications of graph theory exploded even further. Today, graph theory is used to study clustering on social networks, in data analysis, and in computer chip design, among many other applications.

An extremely well-studied area of graph theory is extremal graph theory, which broadly is the study of relationships between various graph parameters. In other terms, how does one graph parameter control another? Problems of this type are often phrased in the following way: for a graph with parameter X, what is the minimum or maximum value of parameter Y? One classical problem of this type is the Ramsey number of a graph. The Ramsey number $R(k, l)$ is the smallest integer n such that in any two-coloring of the edges in a complete graph on n vertices with colors red and blue, there is either a red copy of K_k or a blue copy of K_l. Another classical problem is given a graph, what sort of substructures exist or do not exist? For example, how many vertex-disjoint triangles must exist within a graph having some given property?

A fundamental result in extremal graph theory is Hall's theorem [23]. This theorem is a result about *bipartite graphs*, which are graphs for which the vertex set V can be partitioned into subsets $A, B \subseteq V$ where each edge has one endpoint in A and the other endpoint in B. A *matching* in a graph is a set of vertex-disjoint edges. Hall's theorem characterizes when a matching of maximum size exists in a bipartite graph.

Theorem 1. Given a bipartite graph G with parts A and B, where $|A| \leq |B|$, G contains a matching of size $|A|$ if and only if for every set $S \subseteq A$, there are at least $|S|$ vertices in B that are adjacent to vertices in S.

We highlight this result in particular as we will use a more complicated version of this theorem for hypergraphs in Chapter 2.

In the search to answer these extremal questions, tools have been adapted from other areas of mathematics. Random objects play a crucial role in computer science and statistical physics, so the use of probability in graph theory has exploded over the last 50 years. Pioneered by Paul Erdős, the probabilistic method proves the existence of a structure by showing that a random object in some appropriate probability space has the desired properties with positive probability. This technique has proven fruitful in various areas of graph theory. For example, the best known lower bound for $R(k, k)$, the diagonal version of the aforementioned Ramsey number, is proven using the probabilistic method [48]. The use of concentration inequalities, which measure the probability that a graph property is close to its expectation in some probability space, has proven to be critical in this method and will be used later in this dissertation.

A second tool inspired by other areas of mathematics is spectral graph theory. Spectral graph theory is the application of linear algebraic techniques to graph theory. Since results from spectral graph theory motivate all of our work and are central to our results, it will be given a more complete treatment in the next section. The study of spectral graph theory is motivated in part by its counterpart in Riemannian geometry. In fact, Cheeger's inequality on graphs, which we will state later in this chapter, is a direct parallel to a theorem on manifolds. This connection between the application of spectral techniques in graph theory and Riemannian geometry has inspired research that has established other connections between the two fields, including curvature, which is the focus of the final chapter of this dissertation.

Before doing so, we must introduce a few terms that are used throughout this dissertation. The *distance* $d(u, v)$ between vertices u and v is the length of the shortest path between u and v. The *diameter* of a graph G is the maximum of $d(u, v)$ over all pairs u, v of vertices. The *degree* $\deg(v)$ of a vertex v is the number of edges that v is incident to. The *volume* $\mathrm{Vol}(S)$ of a set of vertices $S \subseteq V$ is given by

$$\mathrm{Vol}(S) = \sum_{v \in S} \deg(v).$$

The volume, therefore, can be thought of as the edge-weighted size of a set S.

The structure of this dissertation is as follows. The next section of this introductory chapter details the foundation of multiple later results: spectral graph theory. In this section, we introduce the concept of spectral graph theory and give two seminal results. One of these results, the Expander Mixing Lemma, is the discrepancy inequality that gives this dissertation its title and is a key tool in Chapters 2 and 3.

Chapter 2 examines a problem inspired by the aforementioned communication networks. In parallel computing, computations are carried out simultaneously and information must be shared and transfered between processors. Thus, it is important to know efficient algorithms and maximum times for this transfer of information. Motivated by this concept, Alon, Chung, and Graham [3] introduced the notion of the routing number of a graph. Suppose that a pebble is placed on every vertex of a connected graph. The routing number of the graph is the maximum number of steps required to route the pebbles according to any given permutation, where a step consists of selecting a matching and exchanging the pebbles on the endpoints of each edge in the matching. In this inaugural paper, Alon, Chung, and Graham establish a bound based on the spectral gap that is polylogarithmic in the number of vertices. Our main result takes a similar approach, but improves this polylogarithmic bound to constant in many cases by using a combination of spectral and probabilistic arguments.

Chapter 3 studies an extremal problem that explores the existence of certain substructures based on the size and degree conditions of a graph. Many historical results regarding the existence of vertex-disjoint cycles in a graph determine a linear relationship between the number of disjoint cycles and the minimum degree of a graph. However, these results are weaker in the case of sparse graphs or break down for graphs with even a few vertices of small degree. Using the Expander Mixing Lemma, we prove that for a graph satisfying a weak spectral condition and an average degree condition, the number of disjoint cycles grows with both the average degree and the number of vertices. Thus, this result is more effective on sparse graphs and graphs with some low-degree vertices.

Chapter 4 concludes with a local version of a discrepancy inequality. Discrete curvature, a tool adapted from its continuous counterpart on manifolds, describes the behavior of a graph within the first two neighborhoods of any vertex. Consequently, for any vertex x, we use curvature to describe the edge distribution between vertices that are distance one from x and those vertices that are distance two from x. Furthermore, we are able to use a similar result focusing on one neighbor of x to give a lower bound for the number of 3- and 4-cycles containing a particular edge.

A common theme throughout this dissertation is the reliance on discrepancy inequalities to find structures in graphs. In Chapters 2 and 3, the discrepancy inequality used is the Expander Mixing Lemma, based on the spectral gap. In Chapter 4, we show that curvature can be used instead of the spectrum to find discrepancy inequalities. Due to the definition of curvature only involving vertices that are a distance two from some fixed vertex, the discrepancy inequalities derived from curvature are necessarily localized in nature. Using these curvature-based discrepancy inequalities we can identify local structures in a graph.

1.1 Spectral Graph Theory

Determining various graph parameters is often not achievable in polynomial time. Thus, finding a way to study these and many other properties while avoiding exhaustive methods has been a central focus of modern graph theory. One avenue for such exploration is by representing the graph as a matrix. When written as matrices, graphs can be studied using the vast array of tools developed in linear algebra. More specifically, the eigenvalues of these matrices can reveal valuable information about the underlying structure of a graph. The study of this relationship between graph structure and matrix eigenvalues is known as spectral graph theory.

The most common matrix associated with a graph is the adjacency matrix. Given a graph G, the adjacency matrix A is an $n \times n$ matrix where each row and column corresponds to a vertex in $V(G)$. Then the entry A_{uv} is 1 if vertices u and v are adjacent and is 0 otherwise. The adjacency matrix is symmetric, which implies that it has n real eigenvalues. Immediately from these eigenvalues, we can determine the number of edges in the graph and whether or not the graph is bipartite, for example. For many other graph properties, however, this matrix is best applied for regular graphs, as we will see later in this section.

Another matrix that can be associated with a graph is the Laplacian matrix. The Laplacian matrix, denoted by L, is an $n \times n$ matrix where the entry $L_{uv} = -1$ if vertices u and v are adjacent, $L_{uv} = \deg(v)$ if $u = v$, and $L_{uv} = 0$ otherwise. The Laplacian can be written as a simple transformation of the adjacency matrix, as $L = D - A$, where D is the diagonal degree matrix of the graph. As an example of the utility of the Laplacian, we present the Matrix Tree Theorem, which characterizes the number of spanning trees in a graph. A spanning tree of G is a connected subgraph of G that includes every vertex and contains no cycles.

Theorem 2. For a graph G, if L^* is a matrix obtained by deleting row s and column t of L, then the number of spanning trees in G is $(-1)^{s+t}\det(L^*)$.

This matrix L is important due to its relationship with the Laplace-Beltrami operator in Riemannian geometry. In this setting, the Laplace-Beltrami operator measures how a function acts locally on a manifold. Moving to the discrete setting, we can also gain valuable local information about a graph by considering the Laplacian as an operator. As such, we will later look at the operator $\Delta = -L$, which acts on a vector f by $\Delta(f)(x) = \sum_{y\sim x}(f(y) - f(x))$. The sign reversal here is due to conventions in Riemannian geometry. This operator is one of a broad class of similar Laplace operators that can be used to study the local behavior of a graph, as we will do later in this dissertation.

While there are numerous well-studied matrices that are derived from graphs, this dissertation will primarily use the normalized Laplacian, as defined in [12]. The normalized Laplacian $\mathcal{L}$ is an $n \times n$ matrix where again each row and column correspond to a vertex in $V(G)$, and is defined by

$$\mathcal{L}(u,v) = \begin{cases} 1, & \text{if } u = v \\ -\frac{1}{\sqrt{\deg(u)\deg(v)}}, & \text{if } u \sim v \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, the normalized Laplacian can be defined as the matrix $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, where once again D is the diagonal degree matrix of G and A is the adjacency matrix of G defined above. Like the adjacency matrix, the normalized Laplacian is a real symmetric matrix and therefore has n real eigenvalues. Furthermore, when written in non-decreasing order, these eigenvalues can be written as $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$. Like with the adjacency matrix, these eigenvalues can give us instant information about the structure of the underlying graph. For example, the multiplicity of 0 as an eigenvalue equals the number of connected components in the graph. Also, $\lambda_n = 2$ if and only if one of the components of G is bipartite. While the adjacency matrix can give us a vast array of information about regular graphs, the normalized Laplacian can do the same for irregular graphs, since normalization allows many results for regular graphs to be generalized to irregular graphs. Additionally, these two matrices give the same information for regular graphs, as if ρ is an eigenvalue of the adjacency matrix of a d-regular graph, then $1 - \frac{\rho}{d}$ is an eigenvalue of the normalized Laplacian. Because it applies equally well to regular and irregular graphs, we use the normalized Laplacian for all spectral analysis in this dissertation.

In addition to the above immediate structural conclusions that can be derived from the spectrum of the normalized Laplacian, deeper information about the structure of a graph can be obtained from these eigenvalues. In applications such as computer networks and the heat equation on graphs, a crucial component to many questions is identifying bottlenecks in such graphs. In other words, are there partitions of the vertices into two parts where there are few edges between the parts? To state this more precisely, we need to properly measure the weight of a set of vertices, taking into account their degrees.

For a set of vertices S, let

$$h_G(S) = \frac{|e(S, \overline{S})|}{\min\{\mathrm{Vol}(S), \mathrm{Vol}(\overline{S})\}},$$

where $\overline{S} = V \setminus S$, the set of vertices not in S and $e(S, \overline{S})$ is the number of edges with one endpoint in S and the other endpoint in $\overline{S}$. The Cheeger constant h_G of a graph, then, is

$$h_G = \min_S h_G(S).$$

Therefore, the Cheeger constant is a measure of the sparsest cut of a graph relative to the size of the vertex sets that it partitions. We can relate this Cheeger constant to the eigenvalues of the normalized Laplacian in the following way.

Theorem 3 (Cheeger's Inequality [2])**.** For a connected graph G,

$$\frac{h_G^2}{2} \leq \lambda_2 \leq 2h_G.$$

This theorem is extremely useful when trying to control the edge expansion of a graph. However, it does not help determine where these edges go. In other words, Cheeger's inequality only controls the edges leaving a set. If we want to use spectral information to control the edges between two sets, we will need information about the entire spectrum, not just λ_2.

For this reason, we define the spectral gap of $\mathcal{L}$ to be

$$\sigma = \max\{|1 - \lambda_2|, |1 - \lambda_n|\}.$$

The spectral gap, in a sense, measures the randomness of the edge distribution in a graph. To make this precise, we introduce the following fundamental result.

Theorem 4 (Expander Mixing Lemma, [12])**.** For any two sets of vertices $X, Y \subseteq V(G)$,

$$\left| e(X,Y) - \frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(G)} \right| \leq \sigma\sqrt{\text{Vol}(X)\text{Vol}(Y)}$$

and

$$\left| e(X,Y) - \frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(G)} \right| \leq \sigma\sqrt{\text{Vol}(\overline{X})\text{Vol}(\overline{Y})}.$$

The term $\frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(G)}$ is important here because it is the expectation of the number of edges between X and Y in the following sense. The configuration model is a random graph model that uniformly randomly generates a multigraph with a given degree sequence. Let G' be such a random graph with degree sequence matching that of G. Let X' and Y' be the sets in this random graph with degree sequences corresponding to those of X and Y

in G. Fix an edge with one endpoint in X'. Then the probability that the other endpoint lands in Y' is $\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}$. Thus, since there are $\mathrm{Vol}(X)$ endpoints of edges in X', we expect there to be $\frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}$ edges between X' and Y'. It is in this sense that we consider the Expander Mixing Lemma a discrepancy inequality: it bounds the difference between the edge distribution of a graph and the expectation of the edge distribution in a random graph.

Due to the importance of this theorem in our work, we show a proof below. Many proofs of this theorem can be found in the literature.

Proof. Recall that the eigenvalues of $\mathcal{L}$ can be written as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$. We will denote their respective orthonormal eigenvectors by $\phi_1, \phi_2, \ldots, \phi_n$. Then $\phi_1 = \frac{D^{1/2}\mathbb{1}}{\sqrt{\mathrm{Vol}(G)}}$, where $\mathbb{1}$ is the vector with every entry as 1.

Let $\mathbb{1}_X$ be the vector such that $\mathbb{1}_X(v) = 1$ if $v \in X$ and $\mathbb{1}_X(v) = 0$ otherwise. Define $\mathbb{1}_Y$ similarly. We have that

$$e(X,Y) = \mathbb{1}_X A \mathbb{1}_Y^T,$$

where A is the adjacency matrix. Let $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that

$$D^{1/2}\mathbb{1}_X^T = \sum_{i=1}^n a_i \phi_i^T$$

and

$$D^{1/2}\mathbb{1}_Y^T = \sum_{i=1}^n b_i \phi_i^T.$$

We can use the fact that ϕ_i are eigenvalues to determine that

$$\begin{aligned} a_1 &= \left\langle \phi_1, D^{1/2}\mathbb{1}_X \right\rangle \\ &= \sum_{v \in V} \phi_1(v) \cdot D^{1/2}\mathbb{1}_X(v) \\ &= \sum_{v \in X} \frac{\sqrt{\deg(v)}}{\sqrt{\mathrm{Vol}(G)}} \sqrt{\deg(v)} \end{aligned}$$

$$= \frac{1}{\sqrt{\mathrm{Vol}(G)}} \sum_{v \in X} \deg(v)$$
$$= \frac{\mathrm{Vol}(X)}{\sqrt{\mathrm{Vol}(G)}}.$$

Similarly, $b_1 = \frac{\mathrm{Vol}(Y)}{\sqrt{\mathrm{Vol}(G)}}$.

As a result,

$$\begin{aligned}
e(X,Y) &= \mathbb{1}_X A \mathbb{1}_Y \\
&= \mathbb{1}_X D^{1/2}(I - \mathcal{L}) D^{1/2} \mathbb{1}_Y \\
&= \left(\sum_{i=1}^{n} a_i \phi_i\right) \left(\sum_{i=1}^{n} (1-\lambda_i) \phi_i^T \phi_i\right) \left(\sum_{i=1}^{n} b_i \phi_i\right) \\
&= \sum_{i=1}^{n} a_i b_i (1-\lambda_i) \\
&= \sum_{i=2}^{n} a_i b_i (1-\lambda_i) + a_1 b_1 \\
&= \sum_{i=2}^{n} a_i b_i (1-\lambda_i) + \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}.
\end{aligned}$$

Using this, we have that

$$\begin{aligned}
\left| e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| &= \left| \sum_{i=2}^{n} a_i b_i (1-\lambda_i) \right| \\
&\leq \sigma \sum_{i=2}^{n} |a_i b_i| \\
&\leq \sigma \sqrt{\sum_{i=2}^{n} a_i^2 \sum_{i=2}^{n} b_i^2},
\end{aligned}$$

where the last inequality is an application of Cauchy-Schwarz.

To determine $\sum_{i=2}^{n} a_i^2$, first calculate that

$$\begin{aligned}\sum_{i=1}^{n} a_i^2 &= \|D^{1/2}\mathbb{1}_X\|^2 \\ &= \left(\sqrt{\sum_{v\in X}\left(\sqrt{\deg(v)}\right)^2}\right)^2 \\ &= \sum_{v\in X}\deg(v) \\ &= \mathrm{Vol}(X).\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{i=2}^{n} a_i^2 &= \mathrm{Vol}(X) - \frac{(\mathrm{Vol}(X))^2}{\mathrm{Vol}(G)} \\ &= \frac{\mathrm{Vol}(X)(\mathrm{Vol}(G) - \mathrm{Vol}(X)}{\mathrm{Vol}(G)} \\ &= \frac{\mathrm{Vol}(X)\mathrm{Vol}(\overline{X})}{\mathrm{Vol}(G)}.\end{aligned}$$

Similarly, we also get that

$$\sum_{i=2}^{n} b_i^2 = \frac{\mathrm{Vol}(Y)\mathrm{Vol}(\overline{Y})}{\mathrm{Vol}(G)}.$$

Thus, we have that

$$\left|e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}\right| \le \sigma\frac{\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(\overline{X})\mathrm{Vol}(Y)\mathrm{Vol}(\overline{Y})}}{\mathrm{Vol}(G)}.$$

Now, note that $\mathrm{Vol}(\overline{X}) \le \mathrm{Vol}(G)$ and $\mathrm{Vol}(\overline{Y}) \le \mathrm{Vol}(G)$. Hence, $\sqrt{\mathrm{Vol}(\overline{X})\mathrm{Vol}(\overline{Y})} \le \mathrm{Vol}(G)$ and in turn, $\frac{\sqrt{\mathrm{Vol}(\overline{X})\mathrm{Vol}(\overline{Y})}}{\mathrm{Vol}(G)} \le 1$. Therefore,

$$\left|e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}\right| \le \sigma\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(Y)}.$$

We can obtain the second version of the conclusion with a similar last step. □

Observe that the first eigenvalue, 0, and its corresponding eigenvector, $\frac{D^{1/2}\mathbb{1}}{\sqrt{\mathrm{Vol}(G)}}$, play a crucial role in this proof. This first eigenvalue and eigenvector are always known for the normalized Laplacian, regardless of the underlying graph. However, the first eigenvalue and eigenvector of the adjacency matrix are only this clear when the graph is regular. Thus, this theorem can be proven using the adjacency matrix, but only in the case of regular graphs. In the case of regular graphs, this theorem can be stated using either matrix, and has a slightly simpler combinatorial interpretation.

Corollary 5. Let G be a d-regular graph. For any two sets of vertices $X, Y \subseteq V(G)$,

$$\left| e(X,Y) - \frac{d}{n}|X||Y| \right| \leq \sigma d \sqrt{|X||Y|}.$$

In the regular case, the term $\frac{d}{n}|X||Y|$ can be interpreted as the expectation of the number of edges between X and Y in a random graph where for every pair of vertices, there is an edge between them with probability $\frac{d}{n}$. Again, it is this comparison of the edge distribution of the graph to the expectation of a random graph that highlights this as a discrepancy inequality. We return to this notion later in this dissertation, when we prove a local version of a discrepancy inequality.

By the Expander Mixing Lemma, therefore, we can view the spectrum of the normalized Laplacian as a measure of randomness. Another way that this relationship between randomness and the normalized Laplacian manifests itself is in random walks. The transition probability matrix of the usual random walk on a graph is $W = D^{-1}A$, which is similar to $\mathcal{L}$. The matrix W is central to the study of random walks, meaning that $\mathcal{L}$ and its matrix can be used to gain valuable information about the properties of these walks. For example, Alon, Chung, and Graham use the spectral gap of $\mathcal{L}$ to get an upper bound on the routing number for certain classes of regular graphs by obtaining information about the random walks on the graph through the spectral gap, then constructing paths from these random walks. Furthermore, this transition probability matrix is often used in the study of discrete curvature as a weighted version of the aforementioned Laplacian operator Δ.

Chapter 2: Routing Number

2.1 Introduction

Let $G = (V, E)$ be a connected simple graph with n vertices. Alon, Chung, and Graham introduced the notion of the *routing number* of G, behind which is the following simple process: imagine a pebble on each vertex of the graph labeled with the vertex it sits on, and let π be an arbitrary permutation in S_V. The goal, then, is to move the pebbles according to π; that is, to move the pebble labeled v to $\pi(v)$. In any given step, a (not necessarily maximal) matching is selected in the graph and the pebbles at the endpoints are interchanged. The routing number of G for the permutation π, denoted by $rt(G, \pi)$, is the minimum number of steps needed to route all of the pebbles to their desired vertex as determined by π. Finally, the routing number of the graph G is

$$rt(G) = \max_{\pi \in S_V} rt(G, \pi).$$

Classes of graphs for which the routing number is known include complete graphs, $K_{n,n}$, paths, cycles, and stars (see [3, 34]). In particular, $rt(K_n) = 2$, $rt(K_{n,n}) = 4$, and $rt(P_n) = n$, where P_n is an n-vertex path. Additionally, there are a number of other classes of graphs for which bounds on the routing number are known. In [3], Alon, Chung, and Graham gave preliminary bounds for trees, general complete bipartite graphs, Cartesian products, hypercubes, and grids. For any tree T, Zhang showed in [53] that $rt(T) \leq \frac{3n}{2} + O(\log n)$, confirming a conjecture made by Alon, Chung, and Graham. In [34], Li, Lu, and Yang improved the bounds on general complete bipartite graphs and hypercubes. Specifically, they showed that for the n-dimensional hypercube Q_n, $n + 1 \leq rt(Q_n) \leq$

$2n-2$ using a computer search to prove that $rt(Q_3) = 4$, then applying the bound of [3] for Cartesian products of graphs. Alon, Chung, and Graham conjectured that $rt(Q_n) \sim \alpha n$, and while the above bounds show that $\alpha \in [1, 2]$, they conjecture that the correct value of α is closer to 1 than to 2. However, finding more precise asymptotics for the routing number of the hypercube is still an open and interesting question.

To address different questions in its applications with regards to parallel computing, a number of variations of this problem have arisen in the literature (see e.g. [7, 8, 42, 43, 33, 46, 49, 51]).

The motivation for our main theorem is the following result of Alon, Chung, and Graham.

Theorem 6 ([3])**.** For a d-regular graph G, $rt(G) \leq O\left(\frac{1}{(1-\sigma)^2}\log^2 n\right)$, where σ is the spectral gap of the normalized Laplacian.

We briefly note that this result was originally stated in terms of the second eigenvalue of the adjacency matrix for so-called (n, d, λ)-graphs; that is d-regular graphs with second adjacency eigenvalue λ, for which $\sigma = \frac{\lambda}{d}$. We state the result in terms of σ, however, to give a clearer comparison to our own results that in some cases apply to irregular graphs, for which the normalized Laplacian is more appropriate.

Our main theorem improves the upper bound of this result of Alon, Chung, Graham in the case where σ is small. In particular, among other results, we prove the following.

Theorem 7. For all $k > 0$ and $C > 0$, there exists $N_{k,C} \in \mathbb{N}$ such that for any d-regular graph on $n \geq N_{k,C}$ vertices with degree $d \geq \exp\left(\frac{C\log n}{\log\log n}\right)$, and $\sigma = kd^{-1/2} < \frac{1}{3}$,

$$\log(rt(G)) = O\left(\frac{\log n}{\log d}\right).$$

This theorem improves the Alon, Chung and Graham result throughout its range on d. However, the improvement is clearest when d is polynomial in n, in which case it gives the following constant bound on the routing number and hence improves the Alon, Chung and Graham result by a factor of $\log^2(n)$.

Corollary 8. For all $k > 0$ and $\epsilon > 0$, there exist $N_{k,\epsilon} \in \mathbb{N}$ and $C_{k,\epsilon} \in \mathbb{N}$ such that for any d-regular graph G on $n \geq N_{k,\epsilon}$ vertices with degree $d = n^{\epsilon}$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq C_{k,\epsilon}$.

At their heart, the strategy of our proofs is similar to that of Alon, Chung, and Graham: we use the fact that permutations can be written as the product of two permutations of order two and build disjoint paths between vertices involved in a transposition through which pebbles can be routed. However, instead of using random walks to find paths, we will build paths between vertices more directly using information about the spectrum of the normalized Laplacian. To accomplish this, we will use a random partitioning of the transpositions to select a collection of transpositions to be routed simultaneously, and Hall's theorem for hypergraphs [1] to select disjoint paths. As a result, we will get an upper bound for the routing number dependent upon the length of the paths and the number of partite sets.

Before we prove our main result, we begin with an easier case that will demonstrate the basis of our proof idea.

2.2 Warm-Up: Extremely Dense Graphs

As a starting point, we ask what bounds can one get on the routing number by density alone? If the minimum degree is at least half of the vertices, then between any two vertices there is some overlap in the neighborhoods of the vertices. More specifically, if we take any transposition in the decomposition of a permutation, the neighborhoods of the two vertices in the transposition share a nonempty intersection, which allows us to route this transposition through a three-vertex path. The larger this minimum degree, the more of these transpositions we will be able to route simultaneously because this minimum overlap will be larger.

As noted in the introduction, any permutation in S_n can be split into a product of two permutations of order two; that is, two permutations where the cycle structure consists entirely of transpositions. This is implicit in the work of Alon, Chung, and Graham in showing that $rt(K_n) = 2$. Due to its importance in the proof technique that is commonly used to explore this problem, we state a proof of this fact here.

Proposition 9 ([3])**.** Every permutation can be written as a product of two permutations of order two.

Proof. To show this, it is sufficient to show that any cyclic permutation can be routed in two steps, as any permutation can be written as the product of disjoint cycles. Thus, for any $m \in \mathbb{N}$, let $\pi = (m, m-1, \ldots, 2, 1)$.

Consider the following two permutations of order two:

$$\pi_1 : (1, m+1-1), (2, m+1-2), \ldots, (i, m+1-i), \ldots$$

and

$$\pi_2 : (1, m-1), (2, m-2), \ldots, (j, m-j), \ldots$$

For each $i \neq 1$, the composition $\pi_2\pi_1$ sends i to $m+1-i$ and then to $m-(m+1-i) = i-1$. Also, for $i = 1$, the composition $\pi_2 \circ pi_1$ sends 1 to m via π_1 and then π_2 does not move that pebble. Therefore, $\pi = \pi_2\pi_1$, where π_1 and π_2 each have order two. □

Note that this proposition directly implies that $rt(K_n) = 2$. When proving upper bounds for the routing number of a graph, it is common to use this fact to rewrite the arbitrary permutation as a product of two permutations of order two.

Theorem 10. Let $\epsilon > 0$. For a graph G on $n \geq \frac{3}{\epsilon}$ vertices with minimum degree $\delta(G) = \left(\frac{1}{2} + \epsilon\right) n$,

$$rt(G) \leq \frac{3n}{\left\lfloor \frac{\epsilon n}{3} \right\rfloor}.$$

We note here that $rt(G) = \Omega\left(\frac{1}{\epsilon}\right)$ is best possible as ϵ goes to 0. Consider a graph with vertex set $V = A \cup B \cup C$, where A and B have size $\left(\frac{1}{2} - \epsilon\right) n$ and C has size $2\epsilon n$. Let every vertex in A be adjacent to each vertex in $A \cup C$, every vertex in B be adjacent to each vertex in $B \cup C$, and every vertex in C also be adjacent to each other vertex in C. Then the minimum degree of this graph is $\left(\frac{1}{2} + \epsilon\right) n$. However, if a permutation took each vertex in A and swapped it with a vertex in B, then at each step only $2\epsilon n$ vertices from A or B could be moved into C, which is necessary to route them to their target. Thus, $\Omega\left(\frac{1}{\epsilon}\right)$ steps are required to move all vertices in A and B through C and to their respective targets.

Proof. Let G be a graph with minimum degree $\delta(G) = \left(\frac{1}{2} + \epsilon\right) n$ for some $\epsilon > 0$ and let π be a permutation on the vertices. Then $\pi = \pi_2 \pi_1$ for some permutations $\pi_1, \pi_2 \in S_V$ of order 2. To route the vertices according to π_1, write π_1 as the product of disjoint transpositions and order the transpositions arbitrarily. Now, select the first $\left\lfloor \frac{\epsilon n}{3} \right\rfloor \geq 1$ of these transpositions. For each transposition (v, v'), v and v' have at least ϵn common neighbors, meaning that at least $\frac{\epsilon n}{3}$ of these common neighbors are not in any of the transpositions in this selection. Thus, for each of the selected transpositions (v, v'), we select a middle vertex x that is adjacent to both v and v', is not in any of the selected transpositions, and also has not been selected as the middle vertex for any other transposition in this selection. Hence, we can simultaneously route each of these $\left\lfloor \frac{\epsilon n}{3} \right\rfloor$ transpositions through their corresponding selected vertex x, returning the pebble initially on x back to x, in three steps because vxv' is a path on three vertices. Since π_1 has at most $\frac{n}{2}$ disjoint transpositions, we must repeat this process at most $\frac{n}{2} / \left\lfloor \frac{\epsilon n}{3} \right\rfloor$ times to route all of the vertices according to π_1. Consequently, we can route all vertices according to π_1 in at most $\frac{3n}{2} / \left\lfloor \frac{\epsilon n}{3} \right\rfloor$ steps. Similarly, all vertices can be routed according to π_2 in at most $\frac{3n}{2} / \left\lfloor \frac{\epsilon n}{3} \right\rfloor$ steps. Therefore,

$$rt(G) \leq \frac{3n}{\left\lfloor \frac{\epsilon n}{3} \right\rfloor}.$$

□

This is the best that one can obtain by minimum degree alone. Indeed once $\delta < \frac{n}{2}$, then the graph need not even be connected. Thus, such a naive approach is insufficient, in general, to ensure that the graph has constant routing number. In order to decrease this minimum degree, we will use techniques from spectral graph theory as introduced in Chapter 1.

2.3 Graphs with Linear Degree

In order to guarantee a constant routing number for graphs with minimum degree cn, where c is some constant less than $\frac{1}{2}$, we will need to take a slightly different approach. Instead of relying on the neighborhoods of two vertices to overlap, we will use the Expander Mixing Lemma to guarantee that there are many edges between the neighborhoods of any two vertices. Notice that in the following theorem, in order to compensate for reducing the minimum degree, we need to add a condition on σ. This is a theme throughout this paper: in order to weaken the degree condition, we will need to strengthen the condition on σ, therefore bringing more structure to the graph.

Theorem 11. Fix $0 < c < 1$. Let G be a graph with minimum degree $\delta(G) \geq cn$, with $\sigma < c^2$. Then

$$rt(G) \leq \frac{12}{c^2(c^2 - \sigma)}.$$

Proof. Let G be a graph with minimum degree $\delta(G) = cn$ for some $c > 0$ and with $\sigma < c^2$. Let π be a permutation of $V(G)$. Then $\pi = \pi_2\pi_1$ for some $\pi_1, \pi_2 \in S_V$ of order 2, meaning that each of π_1 and π_2 can be written as a product of disjoint transpositions. Let (v, v') be a transposition in π_1 or π_2. Let $N(v)$ be the neighborhood of v and let $N(v')$ be the neighborhood of v'. Since $|N(v)| \geq cn$ and $|N(v')| \geq cn$, we have that $\text{Vol}(N(v)) \geq (cn)^2$ and $\text{Vol}(N(v')) \geq (cn)^2$.

Let $f(x,y) = \frac{xy}{\mathrm{Vol}(G)} - \sigma\sqrt{xy}$. Then

$$
\begin{aligned}
f_x(x,y) &= \frac{y}{\mathrm{Vol}(G)} - \frac{\sigma\sqrt{y}}{2\sqrt{x}} \\
&= \sqrt{y}\left(\frac{\sqrt{y}}{\mathrm{Vol}(G)} - \frac{\sigma}{2\sqrt{x}}\right) \\
&= \sqrt{y}\left(\frac{2\sqrt{xy} - \sigma \mathrm{Vol}(G)}{2\sqrt{x}\mathrm{Vol}(G)}\right) \\
&\geq cn\left(\frac{2c^2n^2 - c^2n^2}{2cn^3}\right) \\
&> 0
\end{aligned}
$$

when $x \geq c^2n^2$ and $y \geq c^2n^2$, which is the case in our situation. Thus, by symmetry, f is increasing in both parameters. Hence, since $\mathrm{Vol}(N(v)) \geq (cn)^2$ and $\mathrm{Vol}(N(v')) \geq (cn)^2$,

$$
\begin{aligned}
e(N(v), N(v')) &\geq \frac{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))}{\mathrm{Vol}(G)} - \sigma\sqrt{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))} \\
&\geq \frac{(cn)^4}{\mathrm{Vol}(G)} - \sigma(cn)^2 \\
&= (cn)^2\left(\frac{(cn)^2}{\mathrm{Vol}(G)} - \sigma\right) \\
&\geq (cn)^2(c^2 - \sigma).
\end{aligned}
$$

Let

$$\epsilon = \frac{c^2(c^2 - \sigma)}{4}.$$

Order the transpositions of π_1 arbitrarily and take a collection of the first ϵn transpositions in π_1. Then there are $2\epsilon n$ vertices in this collection. We will say that an edge is used if one of its vertices is either in the collection or is incident to an edge that has already been assigned. Thus, for the first transposition (v_1, v_1') of the collection, there are at most $2\epsilon n^2 \leq \frac{1}{2}(cn)^2(c^2 - \sigma)$ edges between $N(v_1)$ and $N(v_1')$ used of the at least $(cn)^2(c^2 - \sigma)$ edges that must be present. Select one of the unused edges to pair with this transposition.

For the next transposition (v_2, v_2'), there are at most $(2\epsilon n + 2)n$ used edges between $N(v_2)$ and $N(v_2')$. Since $(2\epsilon n + 2)n < 4\epsilon n^2 \leq (cn)^2(c^2 - \sigma)$, an unused edge between $N(v_2)$ and $N(v_2')$ can be selected to pair with this transposition. Proceeding inductively, for each $i \leq \epsilon n$, there are at most $(2\epsilon n + 2i - 2)n < 4\epsilon n^2 \leq (cn)^2(c^2 - \sigma)$ used edges between $N(v_i)$ and $N(v_i')$. Thus, an unused edge can be selected to pair with the transposition (v_i, v_i'). Since the selected paths between each v_i and v_i' are disjoint, we can route each of the transpositions (v_i, v_i') simultaneously in three steps, leaving the two middle vertices in each path back in their original positions. To see that this can be done in three steps (even though $rt(P_4 = 4)$, consider the path v_1, v_2, v_3, v_4. In the first step, swap the pebbles on the edges v_1v_2 and v_3v_4. In the second step, swap the pebbles on the edge v_2v_3. In the final step, again swap the pebbles on the edges v_1v_2 and v_3v_4. Thus, each of these transpositions (v_i, v_i') can be routed simultaneously in three steps.

Since there are at most $\frac{n}{2}$ transpositions in π_1, the above process must be repeated at most $\frac{1}{2\epsilon}$ times to route all of the transpositions in π_1. Since each collection of ϵn transpositions routes in three steps, it will take at most $\frac{3}{2\epsilon}$ steps to route all of the vertices according to π_1. By performing the same process on π_2, it will also take at most $\frac{3}{2\epsilon}$ steps to route all of the vertices according to π_2. Therefore,

$$rt(G) \leq \frac{3}{\epsilon} = \frac{12}{c^2(c^2 - \sigma)}.$$

□

2.4 Graphs with Sublinear Degree

If we desire a constant routing number, our goal is to route a positive proportion of the transpositions simultaneously. Unless these transpositions are spread out, this will be impossible because there could be too much overlap in the neighborhoods of these transpositions that we are seeking to route. For example, if we attempted to route a collection of transpositions including a vertex and all of its neighbors simultaneously, we would not be able to. While in the previous proof, we ordered the transpositions arbitrarily, we will now need to select the collections of transpositions more carefully. In order to do this, we will require regularity of the graph in order to better control the iterated neighborhoods of a vertex.

2.4.1 Preliminaries. Instead of partitioning the transpositions arbitrarily, we will partition the transpositions randomly. To do this, we will use Talagrand's inequality, which allows us to quantify the likelihood that a random variable is close to its mean given certain conditions.

Theorem 12 ([41])**.** Let $c > 0$, $r \geq 0$, and d be given and let the non-negative measurable function g on the product space $\Omega = \prod_i \Omega_i$ satisfy the following two conditions, for each $x \in \Omega$: (a) changing any coordinate x_j changes the value of $g(x)$ by at most c; and (b) if $g(x) = s$ then there is a set of at most $rs + d$ coordinates that certify that $g(x) \geq s$. Let $X_1, \ldots, X_n$ be independent random variables, where X_i takes values in Ω_i; let $X = (X_1, \ldots, X_n)$ and let $g(X)$ have mean μ. Then for each $t \geq 0$,

$$\mathbb{P}(g(X) - \mu \geq t) \leq \exp\left(-\frac{t^2}{2c^2(r\mu + d + rt)}\right)$$

and

$$\mathbb{P}(g(X) - \mu \leq -t) \leq \exp\left(-\frac{t^2}{2c^2(r\mu + d + t/3c)}\right).$$

When we say that there exists a set of at most $rs + d$ coordinates that certify that $g(x) \geq s$, we mean that there exists $I \subseteq \{1, \ldots, n\}$ such that $|I| = rs + d$ and if $x' \in \Omega$ and $x'_i = x_i$ for each $i \in I$, then $g(x) \geq s$.

We use Talagrand's inequality in the lemma that follows to provide more structure to the interactions between the neighborhoods of each vertex and the partition of transpositions that is used to route a number of the transpositions simultaneously. Specifically, this lemma states that we can partition a collection of disjoint transpositions so that most of the vertices that have a path of length j from any fixed vertex are not in any single part of the partition.

Lemma 13. Fix $C > 0$. There exists $N_C \in \mathbb{N}$ such that for all $n \geq N_C$, if G is a d-regular graph on n vertices with $d \geq \exp\left(C\frac{\log n}{\log\log n}\right)$, $\mathcal{T}$ is a collection of disjoint transpositions of the vertices, and $c \geq \exp\left(-\frac{C\log n}{2\log\log n}\right)$, then there exists a partition $X_1, \ldots, X_{4/c}$ of $\mathcal{T}$ so that both of the following hold.

1. $|X_i| \leq \frac{nc}{4}$ for all $i \in \{1, \ldots, \frac{4}{c}\}$.

2. Let $N_j(v) = \{u \in V(G) : \text{there is a path of length } j \text{ from } u \text{ to } v\}$. For any $v \in V(G)$, $i \in \left\{1, \ldots, \frac{4}{c}\right\}$, and $j \in \{1, \ldots, n\}$, at most $c|N_j(v)|$ vertices in $N_j(v)$ are in transpositions of X_i.

In the regime we care about, c will be significantly larger than the minimum asserted here – in the (most important) case that the minimum degree is a polynomial in n, for instance, c is a constant not depending on n. Even when d is of the form $\exp\left(\frac{\log n}{\log\log n}\right)$, c will be poly-logarithmic in $1/\log n$.

Proof. Create a partition $X_1, \ldots, X_{4/c}$ of $\mathcal{T}$ by, for each transposition (τ_1, τ_2), placing it in a part from $X_1, \ldots, X_{4/c}$ uniformly at random. Then $\mathbb{E}(|X_i|) = |\mathcal{T}| \cdot \frac{c}{4} \leq \frac{nc}{8}$. Since $|X_i|$ has a binomial distribution with $p = \frac{c}{4}$, $\sigma(|X_i|) = \sqrt{\frac{n}{2} \cdot \frac{c}{4}\left(1 - \frac{c}{4}\right)} = \sqrt{\frac{nc(4-c)}{32}}$. Thus, by Hoeffding's inequality [24], if $|\mathcal{T}| = \frac{n}{2}$,

$$\begin{aligned}
\mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) &= \mathbb{P}\left(|X_i| \geq \left(\frac{c}{4} + \frac{c}{4}\right)\frac{n}{2}\right) \\
&\leq \exp\left(-2\left(\frac{c}{4}\right)^2 \cdot \frac{n}{2}\right) \\
&= \exp\left(-\frac{c^2 n}{16}\right).
\end{aligned}$$

If $|\mathcal{T}| < \frac{n}{2}$, then Hoeffding's inequality would give a smaller upper bound. Thus,

$$\mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) \leq \exp\left(-\frac{c^2 n}{16}\right)$$

for each $i \in \{1, \ldots, 4/c\}$.

Now, fix $j \in \{1, \ldots, n\}$. Define $h_j(v, X_i)$ to be the number of vertices in $N_j(v)$ that are also in transpositions of X_i. First, note that by changing the placement of a single transposition, $h_j(v, X_i)$ changes by at most 2. Second, h_j is 1-certifiable because if part X_i is selected for s transpositions containing a neighbor of v, $h_j(v, X_i) \geq s$. Third, note that $\mathbb{E}(h_j(v, X_i)) \leq \frac{c \cdot |N_j(v)|}{4}$. Hence, by Talagrand's inequality,

$$\begin{aligned}
\mathbb{P}\left(h_j(v, X_i) \leq c \cdot |N_j(v)|\right) &= \mathbb{P}\left(h_j(v, X_i) \geq \frac{3c|N_j(v)|}{4} + \frac{c|N_j(v)|}{4}\right) \\
&\leq \mathbb{P}\left(h_j(v, X_i) - \mathbb{E}(h_j(v, X_i) \geq \frac{3c|N_j(v)|}{4}\right) \\
&\leq \exp\left(-\frac{\left(\frac{3c|N_j(v)|}{4}\right)^2}{2(2)^2\left(\mathbb{E}(h_j(v, X_i)) + \frac{3c|N_j(v)|}{4}\right)}\right) \\
&\leq \exp\left(-\frac{9}{128}c|N_j(v)|\right).
\end{aligned}$$

Note that $|N_j(v)| \geq d - 1$ for any $v \in V(G)$ and any $j \in \{1, \ldots, n\}$. The probability that $|X_i| \geq \frac{nc}{4}$ or $h_j(v, X_i) \geq c|N_j(v)|$ for any $v \in V(G)$, any $i \in \{1, \ldots, \frac{4}{c}\}$, and any $j \in \{1, \ldots, n\}$ is at most

$$\begin{aligned}
&\sum_{i=1}^{4/c} \mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) + \sum_{\substack{v \in V(G) \\ i \in \{1, \ldots, 4/c\} \\ j \in \{1, \ldots, n\}}} \mathbb{P}(h_j(v, X_i) \geq c|N_j(v)|) \\
&\qquad = \sum_{i=1}^{4/c} \exp\left(\frac{c^2 n}{16}\right) + \sum_{\substack{v \in V(G) \\ i \in \{1, \ldots, 4/c\} \\ j \in \{1, \ldots, q\}}} \exp\left(-\frac{9}{128}c|N_j(v)|\right)
\end{aligned}$$

$$\leq \frac{4}{c}\exp\left(-\frac{c^2 n}{16}\right) + \frac{4n^2}{c}\exp\left(-\frac{9}{128}c(d-1)\right)$$
$$< 1$$

for sufficiently large n, where here we use the fact that our bounds on c and d to ensure that the exponent in the second exponential is tending to negative infinity. Therefore, there exists such a partition $X_1, \ldots, X_{4/c}$ of $\mathcal{T}$. □

Once we have this partition, our goal will be to build paths between the vertices of the transpositions. For each part of the partition, we want to find a collection of disjoint paths through which we will be able to route all of the transpositions simultaneously. In order to do that, we will use Hall's theorem for hypergraphs, stated below.

Theorem 14 ([1])**.** Let $\mathcal{A}$ be a family of n-uniform hypergraphs. A sufficient condition for the existence of a system of disjoint representatives of $\mathcal{A}$ is that for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$.

First, we use Lemma 13 to partition the disjoint collection of transpositions that comprise π_1 into parts (X_i) satisfying the conclusions of the lemma. Our goal is to route the transpositions of a given part X_i simultaneously. In this direction, we select a positive integer z sufficiently large to guarantee many paths. For a particular i and for each transposition $(v_j, v'_j) \in X_i$, build a hypergraph $\Gamma_{(v_j,v'_j)}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, \ldots, u_{z-2}\} \in E\left(\Gamma_{(v_j,v'_j)}\right)$ if and only if $v_j, u_1, \ldots, u_{z-2}, v'_j$ is a path from v_j to v'_j and none of $u_1, \ldots, u_{z-2}$ are in any transposition of X_i. This yields that $\Gamma_{(v_j,v'_j)}$ is a $(z-2)$-uniform hypergraph for each $(v_j, v'_j) \in X_i$.

Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j,v'_j)} : (v_j, v'_j) \in X_i\}$, because this would give us a collection of z-vertex paths through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than

$(z-2)(|\mathcal{B}|-1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection of $(z-2)(|T|-1)$ vertex-disjoint paths joining the vertices of a transposition in T.

In order to do this, we will need to understand the vertex expansion of a graph. Toward this end, we present the following corollary of the Expander Mixing Lemma.

Corollary 15. Let G be a graph and let X be a subset of the vertices of G. If $N(X)$ denotes the set of vertices adjacent to at least one vertex of X, then

$$\mathrm{Vol}(N(X)) \geq \min\left\{\frac{1}{2}\mathrm{Vol}(G), \frac{1}{4\sigma^2}\mathrm{Vol}(X)\right\}.$$

Proof. Let G be a graph and let X be a subset of the vertices of G. By the Expander Mixing Lemma,

$$\mathrm{Vol}(X) = e(X, N(X)) \leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}{\mathrm{Vol}(G)} + \sigma\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}.$$

Then

$$\frac{1}{2}\mathrm{Vol}(X) \leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}{\mathrm{Vol}(G)}$$

or

$$\frac{1}{2}\mathrm{Vol}(X) \leq \sigma\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}.$$

Therefore, $\mathrm{Vol}(N(X)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or $\mathrm{Vol}(N(X)) \geq \frac{1}{4\sigma^2}\mathrm{Vol}(X)$. Thus

$$\mathrm{Vol}(N(X)) \geq \min\left\{\frac{1}{2}\mathrm{Vol}(G), \frac{1}{4\sigma^2}\mathrm{Vol}(X)\right\}.$$

□

2.4.2 Proofs of Results. We begin with a theorem whose proof has a similar flavor to our main theorem in that it uses the random partition of transpositions and Hall's theorem for hypergraphs described above, but has a stronger degree condition, which in turn will give us a better bound. This degree condition also allows us to use paths of length four through which to route the transpositions of our permutation.

Theorem 16. Fix $0 < c < \frac{1}{6}$. Then there exists an $N_c \in \mathbb{N}$ so that the following holds: Let G be a d-regular graph on $n \geq N_c$ vertices and suppose $\sigma < \frac{d(1-6c)^2}{n}$. Then $rt(G) \leq \frac{32}{c}$.

The c here can technically depend in a mild way on n (as per the statement of Lemma 13), however it cannot be too small – the point is that if σ is too large, then we lose sufficient control on the (iterated) neighborhoods to apply our techniques. In general, σ being small yields the best results, and in general σ is of order at least $\frac{1}{\sqrt{d}}$. The requirement in *this* result is in terms of $\sigma = O(\frac{d}{n})$ – and this becomes problematic once $d = o(n^{2/3})$. Hence, this result is really interesting only for graphs with degree $d = n^{\epsilon}$ for some $\epsilon \geq \frac{2}{3}$.

Proof. Let G be a d-regular graph where $d = n^{\epsilon}$. Consider a permutation π of the vertices. Then $\pi = \pi_2\pi_1$ for some $\pi_1, \pi_2 \in S_V$ of order two. Thus, π_1 and π_2 can each be written as a product of disjoint transpositions.

Let $\mathcal{T} = \{(v, v') \in \pi_1\}$, the collection of all transpositions in π_1. By Lemma 13, there exists a partition $X_1, \ldots, X_{4/c}$ in which each part X_i has size at most $\frac{nc}{4}$ and no vertex v has more than cd of its neighbors in X_i. To route the transpositions of X_i simultaneously, we will show that we can find disjoint paths between τ_1 and τ_2 for each $(\tau_1, \tau_2) \in X_i$.

For a particular i and for each transposition $(v_j, v_j') \in X_i$, build a hypergraph $\Gamma_{(v_j,v_j')}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, u_2\} \in E\left(\Gamma_{(v_j,v_j')}\right)$ if and only if v_j, u_1, u_2, v_j' is a path from v_j to v_j' and neither u_1 nor u_2 are vertices in transpositions of X_i. This yields that $\Gamma_{(v_j,v_j')}$ is a 2-uniform hypergraph for each $(v_j, v_j') \in X_i$. While a 2-uniform hypergraph is, of course, simply a graph, we state $\Gamma_{(v_j,v_j')}$ as a hypergraph to more easily use Hall's theorem for hypergraphs. Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j,v_j')} : (v_j, v_j') \in X_i\}$, because this would give us a collection of disjoint paths of length four (including v_j and v_j', the vertices in the transposition) through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $2(|\mathcal{B}|-1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection with size $2(|T|-1)$ of vertex-disjoint paths, where each path joins the vertices of some transposition in T.

Let $T \subseteq X_i$, let $t = |T|$, and let $N(T) = \bigcup_{v \in (v,v') \in T} N(v)$. Fix a maximum matching in $\bigcup_{(v,v') \in T} \Gamma_{(v,v')}$. Hall's condition is satisfied for this T unless this matching has cardinality less than $2t$; we assume, by way of contradiction, that the matching has size less than $2t$. Then this matching saturates fewer than $4t$ vertices. For convenience when counting, we will say that a vertex u is used if u is in this maximum matching or if there exists u' such that $(u, u') \in X_i$. Recall that for each vertex v in a transposition of T, there are at most cd neighbors in X_i, meaning that $|N(T) \cap X_i| \leq 2tcd$. Furthermore, each of the $4t$ vertices in the matching is adjacent to at most cd vertices in transpositions of X_i. Hence, the total number of unused vertices in $N(T)$ must be at least $2td - 2tcd - 4tcd = 2td - 6tcd$. Consequently, the average number of unused neighbors per transposition of T is at least $2d - 6cd$. Hence, there exists some transposition $(v, v') \in T$ such that the total unused neighbors of v and v' is at least $2d - 6cd$.

Since v has at most d unused vertices in its neighborhood and the sum of unused neighbors of v and the unused neighbors of v' is at least $2d - 6cd$, v' has at least $d - 6cd$ unused neighbors. Similarly, v must also have at least $d - 6cd$ unused neighbors. Thus, if V is the set of unused neighbors of v and V' is the set of unused neighbors of v', $\text{Vol}(V) \geq d(d-6cd)$ and $\text{Vol}(V') \geq d(d-6cd)$. Hence, by the Expander Mixing Lemma,

$$\begin{aligned} e(V, V') &\geq \frac{\text{Vol}(V)\text{Vol}(V')}{\text{Vol}(G)} - \sigma\sqrt{\text{Vol}(V)\text{Vol}(V')} \\ &\geq \frac{d^2(d-6cd)^2}{nd} - \sigma d^2 \end{aligned}$$

$$= d^2 \left(\frac{d(1-6c)^2}{n} - \sigma \right).$$

Now, since $\sigma < \frac{d(1-6c)^2}{n}$, $e(V, V') > 0$, which implies that there is an edge between an unused neighbor of v and an unused neighbor of v'. Consequently, there exists an edge in $\Gamma_{(v,v')}$ that is not in the matching. Therefore, this contradicts the maximality of the matching.

As a result, there exists a matching of size at least $2t$, meaning that by Hall's theorem for hypergraphs we can select a collection of disjoint edges $\{\{u_1, u_2\} \in E(G)\}$ so that for each transposition (v, v') in X_i, there exists an edge $\{u_1, u_2\}$ in this collection such that v, u_1, u_2, v' is a path. This implies we can route all of the transpositions of X_i through these disjoint paths simultaneously in 4 steps, returning the pebbles on u_1 and u_2 to their prior positions.

Since there are $\frac{4}{c}$ cells in this partition of $\mathcal{T}$, it will take at most $\frac{16}{c}$ steps to route all transpositions of the permutation π_1. By subsequently repeating this process for the transpositions of π_2, it will take at most $\frac{32}{c}$ steps to route all of the vertices according to the permutation π. Therefore, $rt(G) \leq \frac{32}{c}$. □

Notice that in this proof, the paths that we built between v and v' for a transposition $(v, v') \in X_i$ only contained four vertices. By extending these paths, we can weaken the restriction on the degree of the graph. However, this gives us a weaker result on the routing number, as the paths through which the transpositions are routed will be longer.

Theorem 7. For all $k > 0$, $C > 0$, there exists $N_{k,C} \in \mathbb{N}$ such that for any regular graph G on $n \geq N_{k,C}$ vertices with degree $d \geq \exp\left(\frac{C \log n}{\log \log n}\right)$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq (8z^5 + 8z^2)(2k)^z$, where z is the least even integer such that

$$z \geq \frac{2 \log \left(\frac{n}{4k^2}\right)}{\log \left(\frac{d}{4k^2}\right)} + 2.$$

In the introduction, this result was stated as $\log(rt(G)) = O\left(\frac{\log n}{\log d}\right)$. Note that if $rt(G) \leq (8z^5 + 8z^2)(2k)^z$, then for some constant C,

$$\begin{aligned}\log(rt(G)) &= \log(8z^5 + 8z^2) + Cz \\ &= O(z) \\ &= O\left(\frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2\right) \\ &= O\left(\frac{\log n}{\log d}\right).\end{aligned}$$

Corollary 8. For all $k > 0$ and $\epsilon > 0$, there exist $N_{k,\epsilon} \in \mathbb{N}$ and $C_{k,\epsilon} \in \mathbb{N}$ such that for any regular graph G on $n \geq N_{k,\epsilon}$ vertices with degree $d = n^\epsilon$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq C_{k,\epsilon}$.

Since $\log(rt(G)) = O\left(\frac{\log n}{\log d}\right)$ by Theorem 7, $\log(rt(G)) = O\left(\frac{1}{\epsilon}\right)$ when $d = n^\epsilon$.

Proof of Theorem 7. Let G be a d-regular graph where $d > \exp\left(\frac{C\log n}{\log\log n}\right)$ and $\sigma = kd^{-1/2} < \frac{1}{3}$. Consider a permutation π of the vertices. Then $\pi = \pi_2\pi_1$ for some $\pi_1, \pi_2 \in S_V$ of order two. Thus, π_1 and π_2 can each be written as a product of disjoint transpositions. Let z be the least even integer such that

$$z \geq \frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2$$

and let

$$c = \frac{1}{\lceil (z-1)(1+z^3)(4k^2)^{z/2} \rceil}.$$

We note that c here is (at least) polylogarithmic in $\frac{1}{\log n}$ – this follows from the computation in the remark above and our assumption that $d \geq \exp\left(\frac{C\log n}{\log\log n}\right)$. In particular, it satisfies the necessary lower bound for c in Lemma 13.

Let $\mathcal{T} = \{(v, v') \in \pi_1\}$, the collection of transpositions in π_1. Then by Lemma 13, there exists a partition $X_1, \ldots, X_{4/c}$ in which each part X_i has size at most $\frac{nc}{4}$ and for each $j \in \{1, \ldots, z\}$, no vertex $x \in V(G)$ has more than cd^j vertices in its jth neighborhood that are also in transpositions of X_i for any i. Fix $i \in \{1, \ldots, 4/c\}$. For each transposition $(v_j, v_j') \in X_i$, build a hypergraph $\Gamma_{(v_j, v_j')}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, \ldots, u_{z-2}\} \in E\left(\Gamma_{(v_j, v_j')}\right)$ if and only if $v_j, u_1, \ldots, u_{z-2}, v_j'$ is a path from v_j to v_j' and u_k is not in a transposition of X_i for all $k \in \{1, \ldots, z-2\}$. This yields that $\Gamma_{(v_j, v_j')}$ is a $(z-2)$-uniform hypergraph for each $(v_j, v_j') \in X_i$.

Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j, v_j')} : v_j, v_j' \in X_i\}$, because this would give us a collection of disjoint z-vertex paths through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $(z-2)(|\mathcal{B}|-1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection of $(z-2)(|T|-1)$ vertex-disjoint paths, each of which join the vertices of a transposition in T.

Let $T \subseteq X_i$ and let $t = |T|$. Fix a maximum matching in $\bigcup_{(v,v') \in T} \Gamma_{(v,v')}$. Hall's condition is satisfied for this T unless this matching has size less than zt; we assume, by way of contradiction, that the matching has size less than zt. Give each vertex a distance j away from a vertex in any transposition of T a weight of d^{z-j}. To count the weight used by the paths in this matching, first note that there are fewer than $z^2 t$ vertices in the matching. For each vertex x in the matching and for each $j \in \{1, \ldots, z\}$, there are at most cd^j paths of length j connecting x to a vertex in a transposition of X_i. From each of these paths, x gets weight cd^{z-j}. Thus, even if all of these paths connected x to a vertex in a transposition in T, x would get weight at most $(cd^j)(cd^{z-j})$ from being in the jth neighborhood of vertices in transpositions of T. Thus, summing over all $j \in \{1, \ldots, z\}$, each vertex in the matching has weight at most zc^2d^z. Hence, the total weight used by vertices in the matching is at most $z^3tc^2d^z$. Therefore, there exists a transposition $(v, v') \in T$ that uses weight at most $z^3c^2d^z$.

For notational purposes, define $N(v)$ to be the neighborhood of v and define $N^*(v) \subseteq N(v)$ to be the set of all unused vertices in $N(v)$. Then, define $N_2(v)$ to be the neighborhood of $N^*(v)$ and define $N_2^*(v) \subseteq N_2(v)$ to be the set of all unused vertices in $N_2(v)$. Proceed inductively in this way, defining $N_m(v)$ to be the neighborhood of $N_{m-1}^*(v)$ and defining $N_m^*(v) \subseteq N_m(v)$ to be the set of all unused vertices of $N_m(v)$.

To prove that there exists a path of unused vertices joining the vertices of a transposition in T, thus contradicting the maximality of the matching, we will prove the following lemma.

Lemma 17. In this case,

$$\mathrm{Vol}(N_m^*(v)) \geq \min \left\{ \frac{d^{m+1}\left(1 - (c + z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G) \right\}$$

for all $m \leq \frac{z}{2}$.

We leave the inductive proof of this lemma until after the proof of the main theorem. The crux of this lemma is that it implies by regularity that $|N_{z/2-1}^*(v)| \geq \left(\frac{1}{2} - \frac{z^2c}{8}\right)n$ or

$$|N_{z/2-1}^*(v)| \geq \frac{d^{z/2-1}\left(1 - (c + z^3c^2)\sum_{i=1}^{z/2-1}(4k^2)^{i-1}\right)}{(4k^2)^{z/2-2}}.$$

In the latter case, since $c < \frac{1}{(z-1)(1+z^3)(4k^2)^{(z-1)/2}}$,

$$1 - (c + z^3c^2)\sum_{i=1}^{z/2-1}(4k^2)^{i-1} \geq 1 - (c + z^3c^2)\left(\frac{z}{2} - 1\right)(4k^2)^{z/2-1} \geq \frac{1}{2}.$$

Furthermore, since $z \geq \frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2$,

$$\frac{z-2}{2} \log\left(\frac{d}{4k^2}\right) \geq \log\left(\frac{n}{4k^2}\right),$$

meaning that

$$\left(\frac{d}{4k^2}\right)^{z/2-1} \geq \frac{n}{4k^2},$$

which finally implies that

$$\frac{d^{z/2-1}}{2(4k^2)^{z/2-2}} \geq \frac{n}{2}.$$

Thus,

$$\begin{aligned}
|N^*_{z/2-1}(v)| &\geq \frac{d^{z/2-1}\left(1-(c+z^3c^2)\sum_{i=1}^{z/2-1}(4k^2)^{i-1}\right)}{(4k^2)^{z/2-2}} \\
&\geq \frac{d^{z/2-1}}{2(4k^2)^{z/2-2}} \\
&\geq \frac{n}{2}.
\end{aligned}$$

Therefore, in either case $|N^*_{z/2-1}(v)| \geq \left(\frac{1}{2} - \frac{z^2c}{8}\right) n$. By an identical argument, the same is true for $N^*_{z/2-1}(v')$.

Note that this implies that

$$\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v)}\right) \leq \left(\frac{1}{2} - \frac{z^2c}{8}\right) \mathrm{Vol}(G) < \frac{1}{2}\mathrm{Vol}(G)$$

and

$$\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v')}\right) < \frac{1}{2}\mathrm{Vol}(G).$$

By the Expander Mixing Lemma, then,

$$\begin{aligned} e\left(N^*_{z/2-1}(v), N^*_{z/2-1}(v')\right) &\geq \frac{\operatorname{Vol}\left(N^*_{z/2-1}(v)\right)\operatorname{Vol}\left(N^*_{z/2-1}(v')\right)}{\operatorname{Vol}(G)} \\ &\quad - \sigma\sqrt{\operatorname{Vol}\left(\overline{N^*_{z/2-1}(v)}\right)\operatorname{Vol}\left(\overline{N^*_{z/2-1}(v')}\right)} \\ &\geq \frac{1}{4}\operatorname{Vol}(G) - \sigma\sqrt{\frac{1}{4}[\operatorname{Vol}(G)]^2} \\ &= \left(\frac{1}{4} - \frac{1}{2}\sigma\right)\operatorname{Vol}(G) \\ &> 0 \end{aligned}$$

because $\sigma < \frac{1}{3}$. This implies that there is an edge between $N^*_{z/2-1}(v)$ and $N^*_{z/2-1}(v')$. Since these two sets are constructed by building paths of unused vertices in each iterated neighborhood of v and v', respectively, this means that there exists a $(z-2)$-vertex path of unused vertices that can be extended to a path between v and v', which contradicts the maximality of the matching on T. Therefore, there exists a matching that saturates X_i.

Since there is a matching that saturates X_i, there exist disjoint z-vertex paths such that for each transposition $(v, v') \in X_i$, one of these paths connects v and v'. Because these paths are all disjoint, each transposition can be routed along these paths simultaneously, returning all pebbles not on v or v' to their prior location, in z steps. Since there are $\frac{4}{c}$ parts of the partition, the permutation π_1 can be routed in at most $\frac{4z}{c}$ steps. By repeating this process for π_2, we can route the permutation π on G in at most $\frac{8z}{c}$ steps. Therefore, by the arbitrary selection of π,

$$\begin{aligned} rt(G) &\leq \frac{8z}{c} \\ &= 8z\left\lceil (z-1)(1+z^3)(4k^2)^{z/2}\right\rceil \\ &\leq (8z^5 + 8z^2)(4k^2)^{z/2}. \end{aligned}$$

□

We now return to prove the lemma that we omitted from the main proof.

Lemma 17. For all $m \leq \frac{z}{2}$,

$$\mathrm{Vol}(N_m^*(v)) \geq \min\left\{ \frac{d^{m+1}\left(1-(c+z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G) \right\}.$$

Proof. We will prove this by induction. For $m = 1$, note that $|N_1(v)| = d$. By construction of X_i, there are at most cd vertices in $N(v)$ that are also in transpositions of X_i. Furthermore, the total used weight of the transposition (v, v') is at most $z^3c^2d^z$, meaning that there must be used weight at most $z^3c^2d^z$ in $N(v)$. However, each vertex in $N(v)$ that has positive weight must have weight at least d^{z-1}. Thus, there must be at most z^3c^2d vertices of $N(v)$ used by the paths already in the matching. Hence, there are at least $d(1-(c+z^3c^2))$ unused vertices in $N(v)$, which implies that $\mathrm{Vol}(N_1^*(v)) \geq d^2(1-(c+z^3c^2))$. This proves the base case.

Now suppose as an induction hypothesis that

$$\mathrm{Vol}(N_{m-1}^*(v)) \geq \min\left\{ \frac{d^{m}\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}}, \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G) \right\}.$$

We will prove the induction through the following series of three claims.

Claim 18. If $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$, then $\mathrm{Vol}(N_m^*(v)) \geq \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G)$.

Proof of Claim 1. Since each path contains z vertices and the maximum matching in question contains less than zt such paths, there are at most z^2t vertices in the matching. Thus, since $t = |T|$, where $T \subseteq X_i$ and $|X_i| \leq \frac{cn}{8}$, there are at most $\frac{z^2cn}{8}$ used vertices in

$|X_i|$. Hence, because $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ implies that $|N_m(v)| \geq \frac{1}{2}n$, we get that $|N_m^*(v)| \geq \left(\frac{1}{2} - \frac{z^2c}{8}\right)n$. Therefore,

$$\mathrm{Vol}(N_m^*(v)) \geq \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G).$$

□

Claim 19. If $\mathrm{Vol}(N_{m-1}^*(v)) \geq \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G)$, then $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$.

Proof of Claim 2. By Lemma 15, $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or $\mathrm{Vol}(N_m(v)) \geq \frac{\mathrm{Vol}(N_{m-1}^*(v))}{4\sigma^2}$. However, note that since $c = \frac{1}{\lceil(z-1)(1+z^3)(4k^2)^{z/2}\rceil} < \frac{4-8\sigma}{z^2}$ as $\sigma < \frac{1}{3}$,

$$\begin{aligned}\frac{\mathrm{Vol}(N_{m-1}^*(v))}{4\sigma^2} &\geq \frac{\frac{1}{2} - \frac{z^2c}{8}}{4\sigma^2}\mathrm{Vol}(G) \\ &> \frac{\frac{1}{2} - \frac{z^2}{8}\cdot\frac{4-8\sigma}{z^2}}{4\sigma^2}\mathrm{Vol}(G) \\ &= \frac{1}{4\sigma}\mathrm{Vol}(G) \\ &> \frac{3}{4}\mathrm{Vol}(G).\end{aligned}$$

As a result, $\mathrm{Vol}(N_m(v))) \geq \frac{1}{2}\mathrm{Vol}(G)$ in either case. □

Claim 20. If

$$\mathrm{Vol}(N_{m-1}^*(v)) \geq \frac{d^m\left(1 - (c + z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}},$$

then $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or

$$\mathrm{Vol}(N_{m-1}^*(v)) \geq \frac{d^{m+1}\left(1 - (c + z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}$$

Proof of Claim 3. By Lemma 15, $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or

$$\begin{aligned}
\mathrm{Vol}(N_m(v)) &\geq \frac{\mathrm{Vol}(N^*_{m-1}(v_1))}{4\sigma^2} \\
&\geq \frac{d^m \left(1 - (c + z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}4\sigma^2} \\
&= \frac{d^{m+1} \left(1 - (c + z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.
\end{aligned}$$

By the construction of X_i, there are at most cd^m vertices in $N_m(v)$ that are also in transpositions of X_i. Furthermore, there must be used weight at most $z^3c^2d^z$ in $N_m(v)$. However, each vertex in $N_m(v)$ that has positive weight must have weight at least d^{z-m}. Thus, there must be at most $z^3c^2d^m$ vertices of $N_m(v)$ used by paths already in the matching. Hence,

$$\begin{aligned}
|N^*_m(v_1)| &\geq \frac{d^m \left(1 - (c + z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}} - z^3cd^m - cd^m \\
&= \frac{d^m \left(1 - (c + z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.
\end{aligned}$$

Therefore,

$$\mathrm{Vol}(N^*_m(v)) \geq \frac{d^{m+1} \left(1 - (c + z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.$$

□

As a result of these three claims, we have shown that for all $m \leq \frac{z}{2}$,

$$\mathrm{Vol}(N_m^*(v)) \geq \min \left\{ \frac{d^{m+1}\left(1 - (c + z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G) \right\}.$$

□

2.5 Graph Blowups

At the heart of this problem is the following question: what classes of graphs have small routing number? Above we have shown that random-like graphs, as defined by the spectral gap, are one such class. Another way to show that a class has a small routing number is to show that the class is constructed in some nice way. For example, Alon, Chung, and Graham showed that the routing number of a hypercube, which has very nice structure due to its simple, iterative definition, is logarithmic in the number of vertices. The spectral gap of the hypercube is approximately $\frac{1}{\log n}$, so our result doesn't apply. The following result, however, gives an avenue through which an upper bound for the routing number of a hypercube can be found, and is the motivation for our remaining work.

Theorem 21 ([3])**.** For graphs G and G', $rt(G \times G') \leq 2rt(G) + rt(G')$.

The general outline of the proof of this theorem is as follows. Let $(\pi(p), \pi'(p))$ be the target of pebble p. First, route within copies of G so that each copy of G' contains pebbles with distinct values of $\pi'(p)$. This takes at most $rt(G)$ steps. Then, route within copies of G' to place each pebble in its target copy of G. This takes at most $rt(G')$ steps. Finally, route again within copies of G to place each pebble on its target vertex. This takes at most $rt(G)$ steps.

We outline the proof of the above theorem because it is similar to our approach, albeit for a different class of graphs. The t-blowup of a graph G replaces a vertex in G with an independent set of size t. Two vertices in the blowup are adjacent if and only if their corresponding vertices in G are adjacent. Thus, every edge in G is replaced by $K_{t,t}$. For this upper bound on the routing number of blowups, first route all of the pebbles to their target part of the blowup. Now that each pebble is in the same part as its final target, each pebble must be placed on the proper vertex within the part. To do this, we will use matchings on the underlying graph G to form complete bipartite subgraphs in H. In [3], Alon, Chung, and Graham show that $rt(K_{n,n}) = 4$. Thus, routing the pebbles to their target vertices through these complete bipartite subgraphs takes only four steps for each underlying matching in G. However, we may need as many as $rt(G)$ such matchings to ensure that the pebbles in each part of the blowup are routed in this final step, thus bringing the total number of steps required to at most $5rt(G)$.

Theorem 22. For the t-blowup H of a graph G, $rt(H) \leq 5rt(G)$.

Proof. Let π be a permutation on $[tn]$. Since H is the t-blowup of G, H is an n-partite graph in which each part has exactly t vertices. Furthermore, two vertices in H are adjacent iff their corresponding vertices in G are adjacent.

Label the vertices of G using $1, \dots, n$. For each pebble p on H, label p the same number as the vertex in G corresponding to the part of H containing the final destination of p according to π. Thus, for each $i \in \{1, \dots, n\}$, there are t pebbles labeled i. Now, we will show that we can find t disjoint copies of G within H so that each copy has exactly one pebble labeled each of $1, \dots, n$.

Create a bipartite graph G' with parts A and B each of size n. Label the vertices of each part $1, \dots, n$. For a pebble p in H, draw an edge between the vertex in A corresponding to the part of H in which p starts and the vertex in B corresponding to the part of H containing the final destination of p according to π. Since there are t pebbles in each part of H to begin with and π dictates that there will be t pebbles in each part of H at the conclusion of the routing, G' is a t-regular multigraph. Hence, there is a perfect matching in G'. This perfect matching corresponds to an induced copy of G in H in which there is exactly one pebble labeled each of $1, \ldots, n$. By induction on t, we can find t such disjoint copies of G within H.

For each of these t copies of G within H, route the pebbles to the vertex corresponding to their label. Since each of these copies are disjoint, this can be done simultaneously. Thus, we can route every pebble to a vertex in the same part as its final destination in $rt(G)$ steps.

Find the minimum number of matchings $M_1, \ldots, M_s$ of G such that every vertex in G is incident to an edge in some M_i. By considering the permutation $(1, 2, \ldots, n)$, it can be seen that $s \leq rt(G)$. For each $i \in \{1, \ldots, s\}$, take the complete bipartite components in H corresponding to the edges in M_i. For each complete bipartite component, perform the permutation that routes the remaining pebbles to their final destinations according to π. Note that this can be done within these bipartite components because every pebble is on a vertex within the same part as their final destination. For each bipartite component, this can be done in four steps because $rt(K_{t,t}) = 4$ by [3]. Since M_i is a matching in G, this process can be run simultaneously for all bipartite components in H corresponding to an edge in M_i. Thus, for each $i \in \{1, \ldots, s\}$, this process takes four steps, which yields a total of $4s$ steps. Since every vertex of G is incident to some edge in $M_1, \ldots, M_s$, this process is completed for every part of H. Hence, every pebble in H has been routed according to π in at most $rt(G) + 4s \leq 5rt(G)$ steps. Therefore, by the arbitrary selection of the permutation π, $rt(H) \leq 5rt(G)$. □

In this proof, we used the fact that the number of matchings required to cover the vertices is bounded above by $rt(G)$, therefore giving us an upper bound entirely in terms of $rt(G)$. However, an immediate improvement of the above result says that $rt(H) \leq rt(G) + 4s(G)$, where $s(G)$ is the minimum number of matchings such that each vertex in G is incident to an edge in every collection of $s(G)$ matchings.

Observe that $K_{t,t}$ is the t-blowup of a single edge. Since the routing number of an edge is 1, the best possible upper bound for a t-blowup H of a graph G is $rt(H) \leq 4rt(G)$. This example giving us a result that is close to sharp is dependent on the fact that for an edge, $rt(G) = s(G)$. However, the best possible upper bound for the $rt(H)$ entirely in terms of $rt(G)$ is only slightly better than the above theorem.

2.6 Acknowledgements

Portions of this chapter are based on ”Routing numbers of dense and expanding graphs”, which was joint work with Paul Horn and has appeared in Journal of Combinatorics [29].

Chapter 3: Disjoint Cycles

3.1 Introduction

A classical problem in extremal graph theory is determining the number of vertex-disjoint subgraphs of some type within a given graph. In this direction, there are two major categories of graphs for which this is studied: cycles and complete graphs. For cycles, Corradi and Hajnal proved the following result, validating a conjecture of Erdős and Pósa.

Theorem 23. For any $t \geq 1$, if G is a graph on $n \geq 3t$ vertices with $\delta(G) \geq 2t$, then G contains at least t disjoint cycles.

For $n = 3t$, this gives a decomposition into triangles, thus also answering the question for cliques of size 3. For cliques of larger size, Hajnal and Szemerédi gave the following result for complete graphs.

Theorem 24 ([22])**.** For integers $t, k \geq 1$, if G is a graph on $n = t(k+1)$ vertices with $\delta(G) \geq tk$, then there exist t disjoint copies of K_{k+1} in G.

Note that here, Hajnal and Szemerédi obtain a decomposition of G into copies of K_{k+1}. While the result of Hajnal and Corradi doesn't give a decomposition in most cases, it does apply when $n \gg \delta(G)$, thus telling us something about the structure of a graph without such strong degree conditions as required in the clique case.

Both of these results have been generalized in various ways. For example, Finkel proved a result similar to that of Corradi and Hajnal for chorded cycles.

Theorem 25 ([15])**.** For any $t \geq 1$, if G is a graph on $n \geq 4t$ vertices with $\delta(G) \geq 3t$, then G contains t disjoint chorded cycles.

Once again, Finkel's result holds for sufficiently large sparse graphs. For sparse graphs, we cannot guarantee the existence of any complete graphs with at least three vertices (consider $K_{n,n}$, for example). Instead, we must loosen the notion of a complete graph in the following sense. We can consider the graph K_{k+1} as a cycle of size $k+1$ with $f(k) = \frac{(k+1)(k-2)}{2}$ chords. Thus, in an effort to encapsulate the general notion of Hajnal and Szemerédi for sparse graphs, we will say that a multiply chorded cycle is a cycle of any size with $f(k)$ chords, as a relaxation of K_{k+1}. We call this value $f(k)$ because K_{k+1} is the complete graph with minimum degree k.

This generalization was introduced by Gould, Horn, and Magnant, who proved the following result to parallel the work of Hajnal and Szemerédi, while allowing for the possibility of sparse graphs.

Theorem 26 ([20])**.** There exist t_0 and k_0 such that if $t \geq t_0$ and $k \geq k_0$, then there exists $n_0(t,k)$ such that for every graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq tk$, G contains t disjoint cycles, each with $f(k)$ chords.

While we only introduce here those theorems that we will seek to improve on, this work is part of a broad class of problems that attempt to find vertex-disjoint copies of some subgraph under some minimum degree or related conditions. Especially for cycles, this problem has been well studied (see e.g. [4, 9, 10, 11, 18, 19, 30, 40, 44, 52]).

The results of Corradi and Hajnal on cycles and of Hajnal and Szemerédi on cliques both yield that if G is a graph on $n = 3t$ vertices with minimum degree $\delta(G) \geq \frac{2}{3}n$, then G can be decomposed into triangles. In [32], Krivelevich, Sudakov, and Szabó weaken the degree condition required to guarantee a decomposition into triangles by introducing a spectral condition. While this theorem is originally stated in terms of the adjacency matrix, we convert it here to the normalized Laplacian, as our results will apply to irregular graphs, for which the normalized Laplacian is more appropriate.

Theorem 27 ([32])**.** Let G be a d-regular graph on n vertices such that n is divisible by 3. If the spectral gap of the normalized Laplacian matrix is $\sigma = o\left(\frac{d^2}{n^2 \log n}\right)$, then G has a decomposition into triangles.

Even though there is no explicit minimum degree condition in this theorem, the spectral condition given can only be satisfied on graphs with degree $d \geq n^{4/5} \log^{2/5} n$. It is this approach of introducing a spectral condition in order to weaken the degree condition that motivates our work. Instead of seeking a full decomposition into triangles, however, our goal is to find a large collection of disjoint cycles, chorded cycles, or multiply chorded cycles. In many cases, particularly for sparse graphs, our result will give significant improvements on the bounds given by Corradi and Hajnal for cycles, Finkel for chorded cycles, and Gould, Horn, and Magnant for multiply chorded cycles.

In Section 2, we define the notion of an admissible class. A class is admissible if it satisfies a linear Turán condition and a structural condition. Cycles, chorded cycles, and $f(k)$-chorded cycles are all admissible classes, as shown in Section 3.

Theorem 28. Let $\mathcal{H}$ be an admissible class of graphs. Then there exists $d_0 = d_0(\mathcal{H}) > 0$ such that if G is a graph on n vertices with average degree $d \geq d_0$ and $\sigma < \frac{1}{10}$, then G contains at least $\Omega(\sqrt{nd})$ vertex-disjoint members of $\mathcal{H}$.

There are many results that guarantee disjoint cycles for every graph satisfying some degree condition, only some of which are listed above. Our framework improves many of these results in the following ways. While earlier results depend only on the degree of the graph, our results depend both on the number of vertices and the degree. Thus, for graph with low minimum degree, our results are asymptotically much better. Furthermore, our result depends on an average degree condition instead of a minimum degree condition or regularity. For this reason, our result will apply to graphs with some vertices of low degree, whereas the earlier results will not. We explore these comparisons more explicitly for specific classes of graphs later, but these general trends are true for each class of cycles that we consider.

The remainder of this chapter is as follows. In Section 3.2, we prove the main theorem as it appears above for a general class of graphs. In Section 3.3, we prove that cycles, chorded cycles, and $f(k)$-chorded cycles each independently satisfy the conditions required of $\mathcal{H}$, detail the interpretation of the main theorem in each context, and give brief comparisons to the previously known results stated in this introduction. Finally, in Section 3.4, we give an example showing that this $\Omega(\sqrt{nd})$ lower bound is best possible.

3.2 Proof of Main Theorem

Let $\mathcal{H}$ be a class of graphs. A minimal $\mathcal{H}$-system is a maximum collection of vertex-disjoint members of $\mathcal{H}$ that is also minimal with respect to the number of vertices in the system. A class $\mathcal{H}$ is defined to be admissible if the following two properties hold:

- For a collection of vertices X, there exists $\pi(\mathcal{H}) > 0$ such that if $e(X, X) \geq \pi(\mathcal{H})n$, then X contains a member of $\mathcal{H}$.

- There exist $c(\mathcal{H}) > c'(\mathcal{H}) > 0$ such that for any H_1 and H_2 in a minimal $\mathcal{H}$-system, if $e(H_1, H_2) \geq c(\mathcal{H})$, then there exists a single vertex in H_i, for $i \in \{1, 2\}$ of degree at least $e(H_1, H_2) - c'(\mathcal{H})$ to H_j, where $j \in \{1, 2\}$ and $j \neq i$. Furthermore, there are only $t - 1$ such vertices within the system with degree at least $e(H_1, H_2) - c'(\mathcal{H})$ to another member of the system, where t is the number of members of $\mathcal{H}$ in the system. We call these high-degree vertices.

While these conditions are dense when presented in full generality, we prove in Section 3 that the classes of cycles, chorded cycles, and $f(k)$-chorded cycles each satisfy these two conditions and we find the corresponding constants for each class.

Theorem 28. Let $\mathcal{H}$ be an admissible class of graphs. If G is a graph on n vertices with average degree $d \geq \max\{37, 4\pi(\mathcal{H})\}$ and $\sigma < \frac{1}{10}$, then G contains at least

$$\sqrt{\frac{2}{c(\mathcal{H})}\left[\frac{1}{4}\left(1-\frac{\sigma+\sqrt{\sigma^2+\frac{4\pi(\mathcal{H})}{d}}}{2}\right)\left(1-\frac{\sigma+\sqrt{\sigma^2+\frac{4\pi(\mathcal{H})}{d}}}{2}-2\sigma\right)-\frac{1}{d}\right]nd}$$

vertex-disjoint members of $\mathcal{H}$.

Proof. Let $H_1, \ldots, H_t$ be a maximum collection of vertex-disjoint members of $\mathcal{H}$ that is also minimal with respect to the number of vertices in $\mathcal{C} = \bigcup_{i=1}^{t} V(H_i)$. Let $X = G \setminus \mathcal{C}$. By the maximality of t, X contains no member of $\mathcal{H}$. By assumption, this implies that $e(X, X) < \pi(\mathcal{H})n$. Let $\alpha \in [0, 1]$ such that $\mathrm{Vol}(X) = \alpha\mathrm{Vol}(G)$. By the expander mixing lemma, then,

$$\begin{aligned} e(X, X) &\geq \frac{\mathrm{Vol}(X)^2}{\mathrm{Vol}(G)} - \sigma\mathrm{Vol}(X) \\ &= \frac{\alpha^2\mathrm{Vol}(G)^2}{\mathrm{Vol}(G)} - \sigma\alpha\mathrm{Vol}(G) \\ &= \alpha^2\mathrm{Vol}(G) - \sigma\alpha\mathrm{Vol}(G) \\ &= (\alpha^2 - \sigma\alpha)\mathrm{Vol}(G). \end{aligned}$$

Hence,

$$\alpha^2 - \sigma\alpha - \frac{e(X, X)}{\mathrm{Vol}(G)} \leq 0.$$

The roots of this quadratic are

$$\alpha = \frac{\sigma \pm \sqrt{\sigma^2 + \frac{4e(X,X)}{\mathrm{Vol}(G)}}}{2}.$$

Since $\frac{1}{2}\left(\sigma - \sqrt{\sigma^2 + \frac{4e(X,X)}{\mathrm{Vol}(G)}}\right) < 0$, we get that

$$\alpha \leq \frac{\sigma + \sqrt{\sigma^2 + \frac{4e(X,X)}{\mathrm{Vol}(G)}}}{2} \leq \frac{\sigma + \sqrt{\sigma^2 + \frac{4\pi(\mathcal{H})}{d}}}{2} \leq \sigma + \sqrt{\frac{\pi(\mathcal{H})}{d}} < \frac{3}{5},$$

because $\sigma < \frac{1}{10}$ and $d \geq 4\pi(\mathcal{H})$.

Let S be the set of high-degree vertices in $\mathcal{C}$. By assumption, there are at most $t-1$ such vertices. Let $\epsilon = \frac{1}{2}$. We make all of the following calculations using ϵ instead of $\frac{1}{2}$ to illustrate that a more careful selection of ϵ improves the constant, even though $\epsilon = \frac{1}{2}$ has been chosen for clarity. Suppose first that $\mathrm{Vol}(S) \geq \epsilon(1-\alpha)\mathrm{Vol}(G)$. Since $\sigma < \frac{1}{10}$ implies that $\sigma < \epsilon(1-\alpha)$, we get that $\frac{2\mathrm{Vol}(S)}{\mathrm{Vol}(G)} - \sigma \geq 0$, meaning that $\frac{\mathrm{Vol}(S)^2}{\mathrm{Vol}(G)} - \sigma\mathrm{Vol}(S)$ is an increasing function in $\mathrm{Vol}(S)$. Hence, by the Expander Mixing Lemma,

$$\begin{aligned}
|S|^2 &\geq e(S,S) \\
&\geq \frac{\mathrm{Vol}(S)^2}{\mathrm{Vol}(G)} - \sigma\mathrm{Vol}(S) \\
&\geq \epsilon^2(1-\alpha)^2\mathrm{Vol}(G) - \sigma\epsilon(1-\alpha)\mathrm{Vol}(G) \\
&= \left[\epsilon^2(1-\alpha)^2 - \sigma\epsilon(1-\alpha)\right] nd \\
&= \epsilon(1-\alpha)[\epsilon(1-\alpha) - \sigma]nd.
\end{aligned}$$

Thus,

$$t \geq |S| \geq \sqrt{\epsilon(1-\alpha)(\epsilon(1-\alpha)-\sigma)nd}.$$

Conversely, suppose that $\mathrm{Vol}(S) \leq \epsilon(1-\alpha)\mathrm{Vol}(G)$. Then $\mathrm{Vol}(\mathcal{C}\setminus S) \geq (1-\epsilon)(1-\alpha)\mathrm{Vol}(G)$. As before, $\sigma < \frac{1}{10}$ implies that $\frac{\mathrm{Vol}(\mathcal{C}\setminus S)}{\mathrm{Vol}(G)} - \sigma\mathrm{Vol}(\mathcal{C}\setminus S)$ is an increasing function in $\mathrm{Vol}(\mathcal{C}\setminus S)$. Hence, by the Expander Mixing Lemma,

$$e(\mathcal{C}\setminus S, \mathcal{C}\setminus S) \geq \frac{\mathrm{Vol}(\mathcal{C}\setminus S)}{\mathrm{Vol}(G)} - \sigma\mathrm{Vol}(\mathcal{C}\setminus S)$$

$$\geq (1-\epsilon)^2(1-\alpha)^2\text{Vol}(G) - \sigma(1-\epsilon)(1-\alpha)\text{Vol}(G)$$
$$= [(1-\epsilon)^2(1-\alpha)^2 - \sigma(1-\epsilon)(1-\alpha)]nd$$
$$= (1-\epsilon)(1-\alpha)((1-\epsilon)(1-\alpha) - \sigma)nd.$$

Furthermore, we have that any pair H_i and H_j either has $e(H_i, H_j) < c(\mathcal{H})$ or there is a vertex v in H_i, without loss of generality, with at least $e(H_i, H_j) - c'(\mathcal{H})$ edges to H_j. In the latter case, the pair H_i and H_j contributes at most $c'(\mathcal{H}) \leq c(\mathcal{H})$ edges to $e(\mathcal{C} \setminus S, \mathcal{C} \setminus S)$. Additionally, each cycle H_i contains $|V(H_i)|$ edges. Thus,

$$e(\mathcal{C} \setminus S, \mathcal{C} \setminus S) \leq c(\mathcal{H})\binom{t}{2} + |\mathcal{C}| \leq \frac{c(\mathcal{H})}{2}t^2 + n.$$

By combining this edge count with the inequality derived from the Expander Mixing Lemma, we get that

$$t \geq \sqrt{\frac{2}{c(\mathcal{H})}\left[(1-\epsilon)(1-\alpha)((1-\epsilon)(1-\alpha) - \sigma) - \frac{1}{d}\right]nd}.$$

Therefore, from the two bounds on t we get that

$$t \geq \min\left\{\sqrt{\epsilon(1-\alpha)(\epsilon(1-\alpha) - \sigma)nd}, \sqrt{\frac{2}{c(\mathcal{H})}\left[(1-\epsilon)(1-\alpha)((1-\epsilon)(1-\alpha) - \sigma) - \frac{1}{d}\right]nd}\right\}.$$

Note again here that while we chose $\epsilon = \frac{1}{2}$ earlier, we could optimize ϵ to give us a better minimum by making these two terms of the minimum equal. If we did optimize ϵ, it would change the bounds required on d and σ, but they would remain constant. However, for readability purposes, we chose $\epsilon = \frac{1}{2}$. As a result, we get that the second term of this minimum is less than the first for any σ, d, and n. Therefore,

$$t \geq \sqrt{\frac{2}{c(\mathcal{H})}\left[\frac{1}{4}\left(1-\frac{\sigma+\sqrt{\sigma^2+\frac{4\pi(\mathcal{H})}{d}}}{2}\right)\left(1-\frac{\sigma+\sqrt{\sigma^2+\frac{4\pi(\mathcal{H})}{d}}}{2}-2\sigma\right)-\frac{1}{d}\right]nd}.$$

Note that this value is decreasing in σ and increasing in d. Furthermore, for $\sigma = \frac{1}{10}$ and $d = 37$, the coefficient on nd is positive, which implies that this coefficient is positive for all $\sigma < \frac{1}{10}$ and $d > 37$. □

In the next section, we will prove that the class of cycles, the class of chorded cycles, and the class of $f(k)$-chorded cycles, for any fixed $k \geq 3$, are each an admissible class. We then interpret this result for these classes of graphs and compare our results to previous work.

3.3 Main Result on Classes of Graphs

3.3.1 Cycles. Note first that if a graph G has average degree $d \geq 2$, then G contains a cycle. Thus, in the terminology of our main theorem, we have that for the class $\mathcal{H}$ of cycles, $\pi(\mathcal{H}) = 2$.

To establish the existence of high-degree vertices within a minimal cycle system, define the graph $G(a, b, c, d)$ to be the disjoint union of two cycles C_1 and C_2 of order a and b respectively, with two vertices v_1 and v_2 at distance d lying on the cycle C_c specified. The vertex v_1 is adjacent to all vertices in the other cycle and the vertex v_2 is adjacent to a single vertex of the other cycle. No other edges exist between the cycles. To satisfy the second condition on $\mathcal{H}$ in our main theorem, we will use the following result of Gould, Hirohata, and Horn [17] to prove the subsequent lemma.

Lemma 29 ([17])**.** Suppose C_1 and C_2 are two cycles with $e(C_1, C_2)$ edges between them. If $e(C_1, C_2) \geq 10$, then either $C_1 \cup C_2 \subseteq G(|C_1|, |C_2|, c, d)$ for some parameters c and d, or there exist two shorter disjoint cycles.

As a result, we get that if $e(C_1, C_2) \geq 10$, then there exists one vertex that is incident to all but one of the edges between them. In the terminology of our main theorem, this lemma states that $c(\mathcal{H}) = 10$ and $c'(\mathcal{H}) = 1$. We now use the above lemma to show that there are at most $t-1$ of these high-degree vertices in a minimal cycle system containing t cycles.

Lemma 30. If $C_1, \ldots, C_t$ is a minimal cycle system, then there are at most $t-1$ vertices with at least 9 edges to some other cycle.

Proof. Let $C_1, \ldots, C_t$ be a minimal cycle system. By Lemma 29, for any $i, j \in \{1, \ldots, t\}$, if $e(C_i, C_j) \geq 10$, then $C_i \cup C_j \subseteq G(|C_1|, |C_2|, c, d)$ for some parameters c, d. Specifically, this implies that there exists $v_{i,j} \in V(C_i)$ (without loss of generality) such that $e(v_{i,j}, C_j) \geq e(C_i, C_j) - 1 \geq 9$. We will refer to each such vertex $v_{i,j}$ as a vertex of high degree.

Suppose by way of contradiction that there are at least t of these high-degree vertices. Create an auxiliary graph G' with vertex set $\{v_1, \ldots, v_t\}$ where v_i corresponds to the cycle C_i. For each vertex of high degree in C_i, choose a single cycle C_j to which this vertex has at least 9 edges and add the edge $v_i v_j$ to G'. Since there are at least t high-degree vertices, there exists a cycle in G'. From this cycle in G', we can construct a replacement set of cycles using fewer vertices in the following way.

Order the vertices of this cycle in G' as $v'_1, \ldots, v'_l$. These vertices correspond to cycles $C'_1, \ldots, C'_l$ in G. For each $i \in \{1, \ldots, l\}$, there exists a vertex in C'_i or C'_{i+1} (where for $i = l$, we have $i + 1 = 1$) with at least 9 edges to the other cycle. We will call this vertex $v_{i,i+1}$. Suppose without loss of generality that $v_{1,2} \in C'_1$ (if not, the algorithm described below will simply work in the opposite direction around the cycle in G'). We will temporarily call $v_{1,2}$ the primary vertex and $v_{2,3}$ the secondary vertex. The vertex $v_{1,2}$ must have 9 edges to C'_2.

If $v_{2,3} \in C'_3$, select two vertices $x_{1,2}, y_{1,2} \in C'_2$ that are adjacent to $v_{1,2}$ and that have a path through C'_2 that excludes at least 4 neighbors of $v_{2,3}$. This is possible as even if $x_{1,2}$ and $y_{1,2}$ are adjacent to $v_{2,3}$, there are still 7 other neighbors of $v_{2,3}$ in C'_2. In this case, $v_{1,2}$

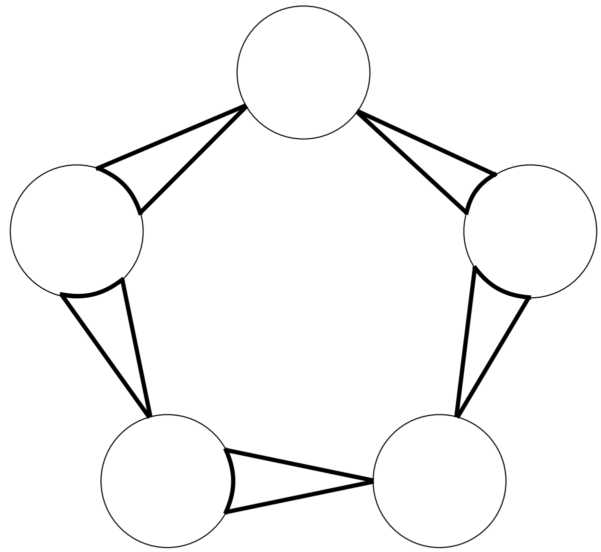

Figure 3.1

and this path including $x_{1,2}$ and $y_{1,2}$ excluding these 4 neighbors through C'_2 forms a new cycle. Then, let $x_{2,3}, y_{2,3} \in C'_2$ be the two excluded neighbors of $v_{2,3}$ that are closest when excluding the newly-formed cycle. Thus, $v_{2,3}$ and this shortest path including $x_{2,3}$ and $y_{2,3}$ through C'_2 also forms a new cycle, excluding two neighbors of $v_{2,3}$.

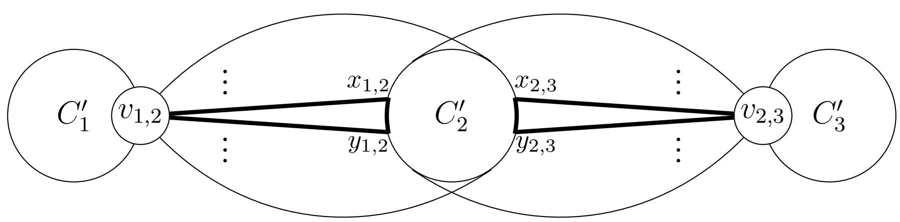

Figure 3.2

If $v_{2,3} \in C'_2$, select two vertices $x_{1,2}, y_{1,2} \in C'_2$ that are adjacent to $v_{1,2}$ and whose path through C'_2 that excludes $v_{2,3}$ is shortest. Then $v_{1,2}$ and this path between $x_{1,2}$ and $y_{1,2}$ that travels through C'_2 forms a new cycle that excludes all other neighbors of $v_{1,2}$ in C'_2.

In the latter case, proceed with the same decision process, considering $v_{2,3}$ as the primary vertex and $v_{3,4}$ as the secondary vertex. In the former case, proceed with the

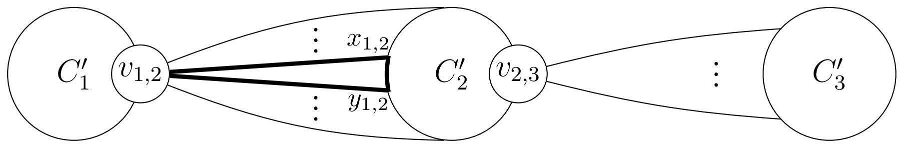

Figure 3.3

same decision process, considering $v_{3,4}$ as the primary vertex (since $v_{2,3}$ has already been included in a cycle) and $v_{4,5}$ as the secondary vertex. Continue this process until $v_{i,i+1}$ has been included in a new cycle for all $i \in \{1, \ldots, t\}$. Since there are l such vertices, we have created a replacement set of l cycles that is smaller than the set $C'_1, \ldots, C'_l$.

Therefore, by minimality, there must be at most $t - 1$ high-degree vertices. $\square$

Due to these results, the class $\mathcal{H}$ consisting of cycles satisfies the two conditions of our main theorem. Therefore, we get the following result.

Theorem 31. If G is a graph on n vertices with average degree $d \geq 37$ and $\sigma < \frac{1}{10}$, then G contains at least $.113\sqrt{nd}$ disjoint cycles.

To compare this to the previous work of Corradi and Hajnal on disjoint cycles, our result is much better for sparse graphs. First, note that our degree condition is on the average degree, which allows for low-degree vertices, whereas the result of Corradi and Hajnal requires a minimum degree condition. Even if we consider regular graphs to mitigate this difference, our result gives more disjoint cycles in most cases. For a regular graph with constant degree as n approaches infinity, the work of Corradi and Hajnal only guarantees a constant number of disjoint cycles, whereas our result will give $\Omega(\sqrt{n})$ disjoint cycles asymptotically. In fact, for the result of Corradi and Hajnal to guarantee at least $\sqrt{nd}$ disjoint cycles, a graph must have degree linear in n. While our result will never give a triangle decomposition as Corradi and Hajnal can under certain conditions, our result still will guarantee that the number of disjoint cycles is on the order of n with these conditions.

Similarly, the result of Krivelevich, Sudakov, and Szabó has the strength to guarantee a triangle decomposition that we cannot. However, our result has two distinct advantages over theirs. First, Krivelevich, Sudokov, and Szabó require regularity, which we avoided by using the normalized Laplacian matrix instead of the adjacency matrix. Second, since we are not seeking the strength of a triangle decomposition, our spectral condition is significantly weaker than theirs. In fact, the condition that $\sigma = o\left(\frac{d^2}{n^2 \log n}\right)$ guarantees that the degree must asymptotically be at least $d \geq n^{4/5} \log^{2/5} n$ [32]. Therefore, the class of graphs for which our result applies is significantly larger than the class of graphs to which Theorem 27 applies, which is natural since our result does not guarantee as many cycles.

3.3.2 Chorded Cycles. As stated in the introduction, Finkel proved in 2008 that if G is a graph on $n \geq 4t$ vertices and $\delta(G) \geq 3t$, then G contains t disjoint chorded cycles. Note, however, that if the average degree of G is at least $2r$, then G has a subgraph H of minimum degree at least $\delta(H) \geq r + 1$. Thus, Finkel's result gives the following corollary.

Corollary 32. If G is a graph with average degree $d \geq 4$, then G contains a chorded cycle.

This means that the class of chorded cycles satisfies the first condition of an admissible class with $\pi(\mathcal{H}) = 4$.

Lemma 33. If C_1 and C_2 are chorded cycles in a minimal cycle system with $e(C_1, C_2) \geq 31$, then there exists a vertex in one of these cycles with at least $e(C_1, C_2) - 9$ edges to the other cycle.

Before we prove the lemma, we need the following claim.

Claim 34. If P and Q are two paths with at least five edges between them, then there exists a chorded cycle in $P \cup Q$.

Proof. Let $P = p_1, \ldots, p_n$ and let $Q = q_1, \ldots, q_m$. Without loss of generality, assume that the vertex incident to the most amount of these chords is in P and name that vertex

p. If p is incident to at least three edges between paths, then this will yield a chorded cycle immediately. If p is incident to only one edge between paths, then all of the chords are pairwise disjoint. Then, by Erdős-Szekeres, there exists a collection $p_i < p_j < p_k$ of vertices in P with neighbors $q_{i'} < q_{j'} < q_{k'}$ or $q_{i'} > q_{j'} > q_{k'}$. In either case, the cycle $p_i, \ldots, p_k, q_{k'}, \ldots, q_{i'}$ has chord $p_j, q_{j'}$.

Finally, suppose that p is incident to exactly two edges between paths. Let q and q^* be the neighbors of p on Q with $q < q^*$. If there exists an edge $p'q'$ such that $p' < p$ and $q' < q$, then the cycle $p', \ldots, p, q^*, \ldots, q$ has chord pq. Similarly, if there exists an edge $p'q'$ such that $p' > p$ and $q' > q^*$, then the cycle $p, \ldots, p', q', \ldots, q$ has chord pq^*.

Otherwise, there must be two edges $p'q'$ and $p''q''$ with $p'' \geq p' > p$ or $p' \leq p'' < p$. Suppose the former without loss of generality. Then $q', q'' \leq q*$. If $q' \leq q$ or $q'' \leq q$, then the cycle $p, \ldots p'$, (or p'') q', (or q'') $\ldots, q*$ contains chord pq. Otherwise, if $q'' \leq q'$, then $p, \ldots, p'', q'', \ldots, q*$ contains chord $p'q'$. Similarly, if $q' \leq q''$, then $p, \ldots, p'', q'', \ldots, q$ contains chord $p'q'$. □

We now prove the existence of a high-degree vertex between two chorded cycles with sufficiently many edges between them.

Proof of Lemma 33. For the sake of clarity, orient the cycles C_1 and C_2 clockwise. For a vertex $v \in C_1$, we will denote the vertex prior to v in the cycle by v^- and we will denote the vertex subsequent to v in the cycle by v^+. For distinct vertices u_1 and u_2, the path $u_1C_1u_2$ will be the path starting at u_1, moving clockwise through C_1, and ending at u_2.

Let v be the vertex with the greatest number of edges to the opposite cycle. Without loss of generality, assume that v lies in C_1 and say that $e(v, C_2) = M$. By way of contradiction, assume that $8 \leq M < e(C_1, C_2) - 9$. Then there are at least 9 edges between C_1 and C_2 not incident to v. Let $x_1, x_2 \in C_2$ be neighbors of v such that there are at least three neighbors of v on each of the paths $x_1^+C_2x_2^-$ and $x_2^+C_2x_1^-$.

Consider the paths $x_1^+C_2x_2$ and $x_2^+C_2x_1$. Since there are at least 9 edges between C_1 and C_2 not incident to v, one of these paths, say $x_1^+C_2x_2$, has five edges to $C_1 \setminus \{v\}$. Thus, by the above claim, there is a chorded cycle between vertices on the path $x_1^+C_2x_2$ and $C_1 \setminus \{v\}$. Furthermore, there is a chorded cycle in $x_2^+C_2x_1^- \cup \{v\}$, as v has at least three neighbors on $x_2^+C_2x_1^-$. Therefore, these two cycles contradict the minimality of the initial cycle system.

Now, if $M < 8$, label the vertices of C_1 in order as $x_1, \ldots, x_n$ and label the vertices of C_2 in order as $y_1, \ldots, y_m$. Let x_i be the first vertex of C_1 such that $x_2C_1x_i$ has at least 9 edges to C_2. Since $M < 8$, we know that $x_2C_1x_i$ has at most 15 edges to C_2, as no one vertex in C_1 has more than 7 edges to C_2. Thus, since x_1 also has at most 7 edges to C_2 and $e(C_1, C_2) \geq 31$, $x_{i+1}C_1x_n$ has at least 9 edges to C_2. Similarly, let y_j be the first vertex on C_2 such that $y_1C_2y_j$ has at least 9 edges going to $C_1 \setminus \{x_1\}$. Then $y_1C_2y_j$ has at most 15 edges to $C_1 \setminus \{x_1\}$ and in turn, $y_{j+1}C_2y_m$ has at least 9 edges to $C_1 \setminus \{x_1\}$.

Suppose first that there are at most 4 edges between $x_2C_1x_i$ and $y_1C_2y_j$. This implies that there are at least 5 edges between $x_2C_1x_i$ and $y_{j+1}C_2y_m$ and there are at least 5 edges between $x_{i+1}C_1x_n$ and $y_1C_2y_j$. If there are at most 4 edges between $x_{i+1}C_1x_n$ and $y_{j+1}C_2y_m$, then there are at least 5 edges between $x_{i+1}C_1x_n$ and $y_1C_2y_j$ and there are at least 5 edges between $x_2C_1x_n$ and $y_{j+1}C_2y_m$. Finally, in the case that neither assumption holds, we have at least 5 edges between $x_2C_1x_i$ and $y_1C_2y_j$ and there are at least 5 edges between $x_{i+1}C_1x_n$ and $y_{j+1}C_2y_m$. In any case, there are two disjoint pairs of paths, one in C_1 and one in C_2, where within each pair, one path is in C_1 and one path is in C_2 and for each pair of paths, there are at least 5 edges between the paths. By Claim 34, there exist two chorded cycles that avoid x_1 by construction, which contradicts the minimality of the cycle system. □

Lemma 35. If $C_1, \ldots, C_t$ is a minimal chorded cycle system, then there are at most $t-1$ vertices with at least 22 edges another cycle.

Proof. Let $C_1, \ldots, C_t$ be a minimal chorded cycle system. We will refer to any vertex with at least 22 edges to another chorded cycle as a high-degree vertex. Suppose by way of contradiction that there are at least t of these high-degree vertices. Create an auxiliary graph G' with vertex set $\{v_1, \ldots, v_t\}$ where v_i corresponds to the cycle C_i. For each high-degree vertex in C_i, choose a single cycle C_j to which this vertex has at least 22 edges and add the edge v_iv_j to G'. Since there are at least t high-degree vertices by assumption, there exists a cycle in G'. From this cycle in G', we can construct a replacement set of chorded cycles using fewer vertices in the following way.

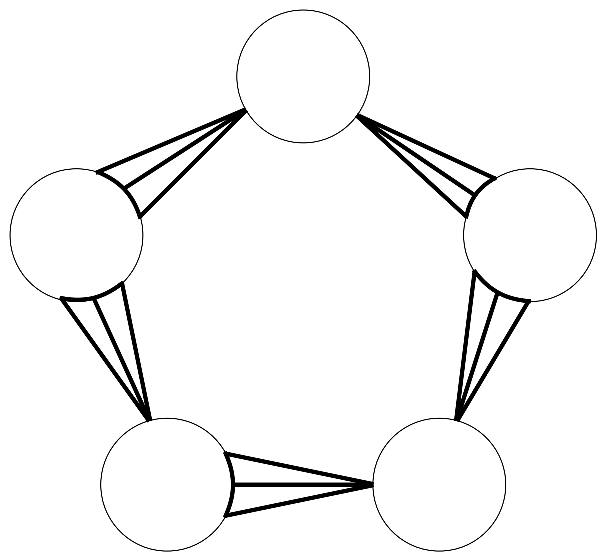

Figure 3.4

Order the vertices of this cycle in G' as $v'_1, \ldots, v'_l$. These vertices correspond to chorded cycles $C'_1, \ldots, C'_l$ in the minimal system of G. For each $i \in \{1, \ldots, l\}$, there exists a vertex in C'_i or C'_{i+1} with at least 22 edges to the other chorded cycle. We will call this vertex $v_{i,i+1}$ (where $i + 1 = 1$ when $i = l$). Suppose without loss of generality that $v_{1,2} \in C'_1$ (if not, the algorithm described below will simply work in the opposite direction around the cycle in G'). The vertex $v_{1,2}$ must have at least 22 edges to C'_2. We will again temporarily call $v_{1,2}$ the primary vertex and $v_{2,3}$ the secondary vertex.

If $v_{2,3} \in C'_3$, select three consecutive neighbors $x_{1,2}, y_{1,2}, z_{1,2} \in C'_2$ of $v_{1,2}$. There are at least 19 neighbors of $v_{2,3}$ on C'_2, exclusive of $x_{1,2}, y_{1,2}, z_{1,2}$. There are three paths, one between $x_{1,2}$ and $y_{1,2}$, one between $y_{1,2}$ and $z_{1,2}$, and one between $z_{1,2}$ and $x_{1,2}$, on which these other 19 vertices can fall. Thus, there exists one of these paths that must contain at least 7 neighbors of $v_{2,3}$. Say this path is the one between $x_{1,2}$ and $z_{1,2}$. Then there exists a cycle $v_{1,2}, x_{1,2}, C'_2, z_{1,2}$, where the path $x_{1,2}, C'_2, z_{1,2}$ contains $y_{1,2}$ and excludes 7 neighbors of $v_{2,3}$. This cycle has chord $v_{1,2}, y_{1,2}$. Furthermore, select three consecutive of the excluded neighbors of $v_{2,3}$. The path in C'_2 between these three neighbors forms a chorded cycle with $v_{2,3}$. This chorded cycle is disjoint from the new cycle containing $v_{1,2}$ and both cycles exclude at least 4 neighbors of $v_{2,3}$.

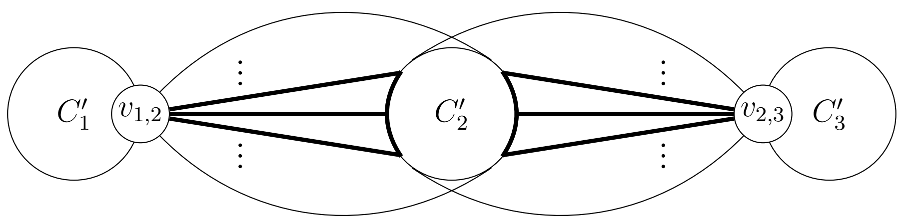

Figure 3.5

If $v_{2,3} \in C'_2$, select three consecutive neighbors $x_{1,2}, y_{1,2}, z_{1,2} \in C'_2$ of $v_{1,2}$ such that $v_{2,3}$ is on the path between $x_{1,2}$ and $z_{1,2}$ that excludes $y_{1,2}$. Then $v_{1,2}$ and the path between $x_{1,2}$ and $z_{1,2}$ that excludes $v_{2,3}$ forms a new cycle with chord $v_{1,2}, y_{1,2}$ that excludes all other neighbors of $v_{1,2}$ in C'_2.

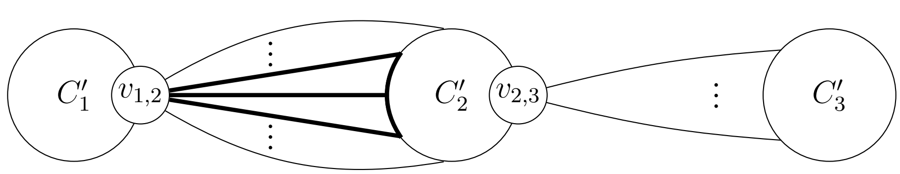

Figure 3.6

In the latter case, proceed with the same decision process, considering $v_{2,3}$ as the primary vertex and $v_{3,4}$ as the secondary vertex. In the former case, proceed with the same decision process, considering $v_{3,4}$ as the primary vertex (since $v_{2,3}$ has already been included in a new cycle) and $v_{4,5}$ as the secondary vertex. Continue this process until $v_{i,i+1}$ has been included in a new chorded cycle. Since there are l such vertices, we have created a replacement set of l chorded cycles that is smaller than the set $C'_1, \ldots, C'_l$.

Therefore, by minimality of the chorded cycle system, there must be at most $t-1$ high-degree vertices. $\square$

Therefore, the class of chorded cycles satisfies the second condition of an admissible class with $c(\mathcal{H}) = 31$ and $c'(\mathcal{H}) = 9$.

Theorem 36. If G is a graph on n vertices with average degree $d \geq 124$ and $\sigma < \frac{1}{10}$, then G contains at least

$$.0792\sqrt{nd}$$

disjoint chorded cycles.

The comparison between our result for chorded cycles and the previous work of Finkel is quite similar between the comparison between our result on cycles and the result of Corradi and Hajnal. Again, our work gives a much stronger result for sparse graphs, as the result of Finkel would only match the $\Omega(\sqrt{nd})$ disjoint chorded cycles that we guarantee when the minimum degree is linear in n. In this case, the constant that Finkel produces beats our constant, but our result still grows linearly with n.

3.3.3 Multiply Chorded Cycles. Fix an integer $k \geq 3$. Let $f(k) = \frac{(k+1)(k-2)}{2}$, the number of chords in K_{k+1}.

Theorem 37 ([20])**.** Let α denote the positive root of $g(x) = x(x-2) - (k+1)(k-2)$. Let $b = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil$ denote the largest integer strictly greater than α. If G has average degree at least $2b$, then G contains a $\frac{(k+1)(k-2)}{2}$-chorded cycle.

This theorem implies that the first condition of an admissible class holds for the class of $f(k)$-chorded cycles with $\pi(\mathcal{H}) = 2\left\lceil\sqrt{\frac{k(k-1)}{2}}\right\rceil$. Furthermore, the following lemma tells us that between any two $f(k)$-chorded cycles in a minimal system, if there are sufficiently many edges, then there exists a vertex in one of the cycles that is incident to most of the edges between these two cycles.

Lemma 38 ([20])**.** Suppose C_1 and C_2 are two $f(k)$-chorded cycles in a minimal cycle set with $e(C_1, C_2) \geq 28f(k) + 20$. Then there exists a single vertex in one of these cycles, say C_1, with at least $e(C_1, C_2) - (12f(k) + 3)$ edges to C_2. We call this a high-degree vertex.

In our terminology, this lemma states that $c(\mathcal{H}) = 28f(k)+20$ and $c'(\mathcal{H}) = 12f(k)+3$. Furthermore, this yields the following corollary, which states (within a stronger result) that there are at most $t-1$ high-degree vertices within any minimal $f(k)$-chorded cycle system.

Corollary 39 ([20])**.** If $C_1, \ldots, C_t$ is a minimal collection of cycles, each of which contains at least $f(k)$ chords, then there are at most $t-1$ high-degree vertices, as defined in Lemma 38.

This isn't the exact statement of the corollary given in [20], but is implicit in its proof. With this result, the class $\mathcal{H}$ consisting of $f(k)$-chorded cycles satisfies the two conditions of an admissible class, yielding the following result.

Theorem 40. For $k \geq 3$, if G is a graph on n vertices with average degree at least $d \geq \max\left\{37, 8\left\lceil\sqrt{\frac{k(k-1)}{2}}\right\rceil\right\}$ and $\sigma < \frac{1}{10}$, then there exists $K = K(k) > 0$ such that G contains $K\sqrt{nd}$ disjoint $f(k)$-chorded cycles in G.

For $k \geq 7$, $K(k) = \sqrt{\frac{1}{14f(k)+10}\left(.02679 - \frac{1}{8\left\lceil\sqrt{\frac{k(k-1)}{2}}\right\rceil}\right)}$.

For $k < 7$, $K(k) = .02575\sqrt{\frac{1}{14f(k)+10}}$.

Previously, Gould, Horn, and Magnant gave a result for $f(k)$-chorded cycles that, for sparse graphs, grows with the minimum degree rather than with n. Again, in order to match the $\sqrt{nd}\ f(k)$-chorded cycles that our result gives, Theorem 26 would require that the degree is linear in n. Thus, our result is better for sparse graphs. Furthermore, our result requires an average degree condition rather than the minimum degree dictated in Theorem 26, thus allowing for some low-degree vertices.

3.4 Sharpness Example

Let $G(m, l)$ be the graph with a clique of size m such that each vertex in K_m is adjacent to l leaves. Order the vertices in K_m as $v_1, \ldots, v_m$ and for any $i \in \{1, \ldots, l\}$, let $u_i^1, \ldots, u_i^l$ be the leaves adjacent to v_i. Note that if ϕ is an eigenvector of $D^{-1}A$ with eigenvalue λ, then for each $i \in \{1, \ldots, m\}$

$$d\lambda\phi(v_i) = (D^{-1}A)\phi(v_i) = \sum_{k \neq i} \frac{1}{l+m-1}\phi(v_k) + \sum_{j=1}^{l} \frac{1}{l+m-1}\phi(u_i^j) \tag{3.1}$$

and

$$\lambda\phi(u_i^j) = \phi(v_i). \tag{3.2}$$

For each $i \in \{1, \ldots, m-1\}$, let $\phi_i(v_i) = 1$ and let $\phi_i(v_{i+1}) = -1$. Then from the above two equations, we get that

$$\begin{aligned}
\lambda &= -\frac{1}{l+m-1} + \frac{l}{\lambda(l+m-1)} \\
0 &= \lambda^2 + \frac{1}{l+m-1}\lambda - \frac{l}{l+m-1} \\
\lambda &= \frac{-\frac{1}{l+m-1} \pm \sqrt{\frac{1}{(l+m-1)^2} + \frac{4l}{l+m-1}}}{2} \\
&= \frac{-1 \pm \sqrt{1 + 4l(l+m-1)}}{2(l+m-1)}.
\end{aligned}$$

Thus, if $\phi_i(u_k^j) = 0$ for all $k \neq i, i+1$ and $j \in \{1, \ldots, l\}$ while

$$\phi_i(u_i^j) = \frac{2(l+m-1)}{-1+\sqrt{1+4l(l+m-1)}} \text{ and } \phi_i(u_{i+1}^j) = \frac{2(l+m-1)}{1-\sqrt{1+4l(l+m-1)}}$$

for all $j \in \{1, \ldots, l\}$, then ϕ_i is an eigenvector of $D^{-1}A$ with eigenvalue

$$\lambda = \frac{-1+\sqrt{1+4l(l+m-1)}}{2(l+m-1)}.$$

Similarly, if $\phi_i(u_k^j) = 0$ for all $k \neq i, i+1$ and $j \in \{1, \ldots, l\}$ while

$$\phi_i(u_i^j) = \frac{2(l+m-1)}{-1-\sqrt{1+4l(l+m-1)}} \text{ and } \phi_i(u_{i+1}^j) = \frac{2(l+m-1)}{1+\sqrt{1+4l(l+m-1)}}$$

for all $j \in \{1, \ldots, l\}$, then ϕ_i is an eigenvector of $D^{-1}A$ with eigenvalue

$$\lambda = \frac{-1-\sqrt{1+4l(l+m-1)}}{2(l+m-1)}.$$

In either case, note that the vectors ϕ_i are linearly independent, giving us the eigenvalues $\lambda = \frac{-1+\sqrt{1+4l(l+m-1)}}{2(l+m-1)}$ and $\lambda = \frac{-1-\sqrt{1+4l(l+m-1)}}{2(l+m-1)}$ each with multiplicity $m-1$.

Now, for all $i \in \{1, \ldots, m\}$, let $\phi(v_i) = 1$. Then from equation (1), we get that

$$\lambda = \frac{m-1}{l+m-1} + \frac{l}{\lambda(l+m-1)},$$

which implies that

$$\lambda^2 - \frac{m-1}{l+m-1}\lambda - \frac{l}{l+m-1} = 0.$$

In turn, this yields that

$$\lambda = \frac{\frac{m-1}{l+m-1} \pm \sqrt{\left(\frac{m-1}{l+m-1}\right)^2 + \frac{4l}{l+m-1}}}{2}$$

$$= \frac{m-1 \pm \sqrt{(m-1)^2 + 4l(l+m-1)}}{2(l+m-1)}$$
$$= \frac{m-1 \pm (2l+m-1)}{2(l+m-1)},$$

which gives solutions
$$\lambda = -\frac{l}{(l+m-1)} \text{ and } \lambda = 1.$$

Thus, if $\phi_i(u_i^j) = -\frac{l+m-1}{l}$ for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, l\}$, then ϕ is an eigenvector of $D^{-1}A$ with eigenvalue $\lambda = -\frac{l}{l+m-1}$. Similarly, if $\phi_i(u_i^j) = l+m-1$ for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, l\}$, then ϕ is an eigenvector of $D^{-1}A$ with eigenvalue $\lambda = 1$.

Since each set of leaves yields an identical row in the matrix $D^{-1}A$, the rank of $D^{-1}A$ is at most $2m$. Thus, the set of $2m$ eigenvalues found above is the full collection of nonzero eigenvalues of $D^{-1}A$. Since $D^{-1}A$ and $D^{-1/2}AD^{-1/2}$ are similar matrices, these are also the full collection of nonzero eigenvalues of $D^{-1/2}AD^{-1/2}$. Hence, the eigenvalues of $\mathcal{L}$ are $1 + \frac{1-\sqrt{1+4l(l+m-1)}}{2(l+m-1)}$ with multiplicity $m-1$, $1 + \frac{1+\sqrt{1+4l(l+m-1)}}{2(l+m-1)}$ with multiplicity $m-1$, $1 + \frac{l}{l+m-1}$ with multiplicity 1, 0 with multiplicity 1, and 1 with multiplicity $n-2m$. Therefore,
$$\sigma = \max\left\{\frac{1+\sqrt{1+4l(l+m-1)}}{2(l+m-1)}, \frac{l}{l+m-1}.\right\}$$

Let $m = s$ and $l = \epsilon s$. Then the average degree d of $G(m,l)$ is $d = \frac{s(s-1+\epsilon s)+s(\epsilon s)}{s+s(\epsilon s)} = \frac{s^2+2\epsilon s^2-s}{\epsilon s^2+s}$, so as s approaches ∞, d approaches $\frac{1+2\epsilon}{\epsilon}$. Also, $G(m,l)$ has spectral gap

$$\sigma = \max\left\{\frac{1+\sqrt{1+4\epsilon s(s+\epsilon s-1)}}{2(s+\epsilon s-1)}, \frac{\epsilon s}{s+\epsilon s-1}\right\},$$

so σ approaches $\sqrt{\frac{\epsilon}{1+\epsilon}}$ as $s \to \infty$.

Thus, for $\epsilon = \frac{1}{100}$, $\sigma < \frac{1}{10}$ and d approaches 102, so $G(m, l)$ satisfies the conditions of our theorem. Furthermore, note that $G(m, l)$, for $m = s$ and $l = \epsilon s$ contains $\left\lfloor \frac{s}{3} \right\rfloor$ disjoint cycles. In this case, our theorem guarantees at least $\Omega(\sqrt{nd}) = \Omega(s)$ disjoint cycles. Hence, our result is asymptotically best possible for cycles. Similarly, this graph gives that our results for both chorded cycles and multiply chorded cycles are asymptotically best possible.

Chapter 4: Discrete Curvature

Let us recall the Expander Mixing Lemma, as stated in the introduction.

Lemma 41. For two vertex sets $X, Y \subseteq V(G)$, if $e(X,Y)$ is the number of edges with one endpoint in X and the other endpoint in Y, then

$$\left| e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| \leq \sigma \sqrt{\mathrm{Vol}(X)\mathrm{Vol}(Y)},$$

where σ is the spectral gap of the normalized Laplacian.

In this lemma, the number of edges between X and Y is compared to an expectation term in a random graph. As we get a stronger spectral condition, the realized number of edges approaches this expected number of edges. Thus, as our spectral condition gets stronger, our edges become more evenly distributed, when normalized for the volumes of the respective sets.

In general, using spectral information to obtain such structural results is fairly common. In this chapter, however, we will use discrete curvature, a relatively recent structural parameter on graphs, to build similar discrepancy inequalities. Due to the local nature of discrete curvature, our new discrepancy inequalities will bound the number of edges between subsets of vertex neighborhoods. While our results do not compare edge distribution in a graph to that of a random graph, the theme behind the discrepancy inequalities remains: as we place stronger conditions on our parameter, the edges of the graph become more evenly distributed.

4.1 Discrete Curvature Preliminaries

To understand the structure of graphs, a fruitful technique has been to take well-studied properties of manifolds and adapt them to the graph setting. Often, the graph analogs of these properties of manifolds are not clear due to the discrete nature of graphs. Therefore, much work has been done to find appropriate graph versions of significant results in Riemannian geometry. Many such properties of manifolds, however, have proven difficult to adapt to the discrete setting.

In manifolds, curvature is a measure of how much a Riemannian manifold deviates from a Euclidean ball at a particular point. Using curvature, one can obtain global information on manifolds, such as diameter and bottlenecks, from a local property. One way to do this is through the study of the heat equation. In this exploration, the Li-Yau inequality, which bounds a positive solution of the heat equation on a non-negatively curved manifold, was a breakthrough result. Using this result, it was shown that non-negatively curved manifolds satisfy the Harnack inequality, which can be used to prove other results on manifolds such as Gaussian bounds for heat dispersion, volume doubling, and the Poincaré inequality (see [21, 47]).

Defining curvature on graphs in such a way that these results hold in the discrete setting has proven quite challenging. Bakry and Émery realized that the key use of curvature in many proofs in Riemannian geometry was the Bochner identity, which immediately implies the following curvature-dimension inequality (also known as the CD-inequality) for all smooth functions f:

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \langle \nabla f, \nabla \Delta f \rangle + \frac{1}{n}(\Delta f)^2 + K|\nabla f|^2.$$

As a result, Bakry and Émery determined that this inequality could be used as an alternative definition of curvature in settings where a direct parallel to the definition of curvature on manifolds cannot be found [5].

Even using this definition, the lack of a chain rule for the Laplacian in the graph setting proved to be an obstacle to obtaining a discrete version of the Li-Yau inequality. In order to overcome this barrier, Bauer, et al. [6] gave the CDE-inequality and the CDE'-inequality, variations of Bakry and Émery's CD-inequality that baked in the chain rule. By doing this, they were able to prove a discrete version Li-Yau inequality, near-Gaussian bounds on heat dispersion, and polynomial bounds on volume growth. Using CDE', these results were later expanded in [26] to obtain Gaussian bounds for heat dispersion, volume doubling, and the Poincaré inequality, mirroring the results on manifolds. These different notions of curvature are actually closely related. It is clear that CDE' implies CDE and Münch proved that CDE' implies CD [45].

These results using CDE and CDE' are very analytic in nature, as they parallel similar results in Riemannian geometry. More generally, analytic arguments using curvature on graphs have become a very active area of research (see e.g. [16, 25, 26, 27, 28, 35, 36, 37, 38, 39]). However, we are interested in combinatorial applications of curvature. Therefore, the CD-inequality as defined by Bakry and Émery will be more appropriate as it is simpler. Therefore, we define their version of the curvature-dimension inequality below.

For our purposes, we will assume that the graph is locally finite; that is, $\deg(x) < \infty$ for all $x \in V(G)$. Given a measure $\mu : V \to \mathbb{R}$, the μ-Laplacian on G is the operator $\Delta : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} (f(y) - f(x)).$$

In our results, we use the measure $\mu(x) = 1$ for all $x \in V(G)$. In this case, $\Delta = -L$, where L is the standard graph Laplacian. Also of interest is when $\mu(x) = \deg(x)$ for all $x \in V(G)$, giving a weighted version of Δ that is analogous to the normalized graph Laplacian (with a sign change) introduced in Chapter 1 of this thesis. Both operators are used in various applications; since our goal is to obtain combinatorial results, we will use the unweighted version of Δ.

The gradient form $\Gamma = \Gamma^{\Delta}$ is defined by

$$\begin{aligned}\Gamma(f,g)(x) &= \frac{1}{2}\left(\Delta(f \cdot g) - f \cdot \Delta(g) - \Delta(f) \cdot g\right)(x) \\ &= \frac{1}{2\mu(x)} \sum_{y \sim x} (f(y) - f(x))(g(y) - g(x))\end{aligned}$$

for all $f, g \in \mathbb{R}^{|V|}$. When $f = g$, we simply write $\Gamma(f)$. The iterated gradient form $\Gamma_2 = \Gamma_2^{\Delta}$ is defined by

$$\Gamma_2(f,g) = \frac{1}{2}\left(\Delta\Gamma(f,g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g)\right)$$

for all $f, g \in \mathbb{R}^{|V|}$. Again, when $f = g$, we simply write $\Gamma_2(f)$. A graph G satisfies the CD-inequality $CD(\infty, K)$ at a vertex x if for any function $f : V \to \mathbb{R}$,

$$\Gamma_2(f)(x) \geq K\Gamma(f)(x).$$

Finally, the curvature of a graph G is defined to be the maximum value K for which $CD(\infty, K)$ holds for every vertex $x \in V(G)$.

This curvature parameter has been calculated for a number of classes of graphs over the past half-decade. The goal of curvature is to measure local volume growth, so classes of graphs that expand locally very quickly we expect to have large curvature. For example, it is known due to [31] that the curvature of the complete graph K_n is $1 + \frac{n}{2}$, the maximum possible curvature of a graph. This is expected, as the first neighborhood of a vertex is the entire graph and each vertex has an empty second neighborhood. The curvature of a d-regular tree, on the other hand, is $2 - d$ because the first neighborhood of any vertex has size d and the second neighborhood has size $d(d-1)$ [31]. Most graphs, of course, fall somewhere in between these two extremes.

In Riemannian geometry, there are numerous results about non-negatively curved manifolds. If we are to extend these notions to graphs, then a natural question is determining which classes of graphs have non-negative curvature. In [31], it is proven that the curvature of the hypercube Q_d is 2, regardless of the dimension. Furthermore, they show that all finite abelian Cayley graphs, such as the discrete torus, have non-negative curvature. A particularly interesting case is that of the complete bipartite graph $K_{s,t}$, as studied in [14]. When $s = t$, the curvature of $K_{s,t}$ is 2. As the parts become more unbalanced, the curvature decreases, and is in fact negative in many cases.

While the curvature of many classes of graphs have been previously discovered, we collectively have only scratched the surface in understanding it combinatorially. Our goal in this chapter, then, is not to calculate the curvature for graphs with given structure, but rather to understand what combinatorial structure must exist within a graph of given curvature.

4.2 Curvature Inequality

As defined above, curvature depends on the Laplacian Δ and the gradient Γ, each of which only depend on the first neighborhood of a vertex. Since the iterated gradient Γ_2 is a composition of those two operators, the curvature at a vertex x can be fully determined simply by analyzing the vertices of distance at most 2 from x. Therefore, we will translate $CD(\infty, K)$ into combinatorial terms, displaying the contribution of each edge type (that is, the edges within the first neighborhood of x, the edges between the first and second neighborhoods of x, etc.) to the curvature dimension inequality.

For notational purposes, let $N_1(x) = \{y : d(x,y) = 1\}$ be the neighborhood of x and let $N_2(x) = \{z : d(x,z) = 2\}$ be the second neighborhood of x. When x has been fixed, we will often write these simply as N_1 and N_2.

Theorem 42. A graph G satisfies $CD(\infty, K)$ at a vertex $x \in V(G)$ if for any function $f : V \to \mathbb{R}$,

$$\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] + \sum_{\substack{y,y'\in N_1\\ \{y,y'\}\in E(G)}}(f(y)-f(y'))^2 \geq \left(\frac{2K+\deg(x)-3}{2}\right)\Gamma(f)(x) - \frac{1}{2}\Delta(f(x))^2.$$

Proof. Expanding Γ_2 according to the combinatorial definitions of Δ and Γ yields

$$\begin{aligned}
\Gamma_2(f)(x) &= \frac{1}{2}\Delta\Gamma(f) - \Gamma(f,\Delta f)\\
&= \frac{1}{2}\sum_{y\sim x}[\Gamma(f)(y)-\Gamma(f)(x)] - \frac{1}{2}\sum_{y\sim x}(f(y)-f(x))(\Delta f(y)-\Delta f(x))\\
&= \frac{1}{4}\sum_{y_1\sim x}\left[\sum_{z\sim y_1}(f(z)-f(y_1))^2 - \sum_{y_2\sim x}(f(y_2)-f(x))^2\right]\\
&\qquad - \frac{1}{2}\sum_{y_1\sim x}(f(y_1)-f(x))\left[\sum_{z\sim y_1}(f(z)-f(y)) - \sum_{y_2\sim x}(f(y_2)-f(x))\right]\\
&= \frac{1}{4}\sum_{y\sim x}\sum_{z\sim y}(f(z)-f(y))^2 - \frac{1}{4}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_2)-f(x))^2\\
&\qquad - \frac{1}{2}\sum_{y_1\sim x}\sum_{z\sim y_1}(f(y_1)-f(x))(f(z)-f(y_1))\\
&\qquad + \frac{1}{2}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_1)-f(x))(f(y_2)-f(x))
\end{aligned}$$

$$= \frac{1}{4}\sum_{y\sim x}\sum_{z\sim y}(f(z)-f(y))^2 - \frac{\deg(x)}{4}\sum_{y\sim x}(f(y)-f(x))^2$$
$$- \frac{1}{2}\sum_{y_1\sim x}\sum_{z\sim y_1}(f(y_1)-f(x))(f(z)-f(y_1))$$
$$+ \frac{1}{2}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_1)-f(x))(f(y_2)-f(x)).$$

To this sum, the edges between N_1 and N_2 contribute

$$\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right].$$

To the sum, the edges between two vertices in N_1 contribute

$$\sum_{\substack{y,y'\in N_1\\ \{y,y'\}\in E(G)}}\left[\frac{1}{2}(f(y)-f(y'))^2\right.$$
$$\left. - \frac{1}{2}(f(y)-f(x))(f(y')-f(y)) - \frac{1}{2}(f(y')-f(x))(f(y)-f(y'))\right]$$
$$= \sum_{\substack{y,y'\in N_1(x)\\ \{y,y'\}\in E(G)}}(f(y)-f(y'))^2.$$

Finally, the edges between x and N_1 contribute

$$\sum_{y\sim x}\left[\frac{1}{4}(f(x)-f(y))^2 - \frac{\deg(x)}{4}(f(y)-f(x))^2\right.$$
$$\left. + \frac{1}{2}(f(y)-f(x))^2 + \frac{1}{2}(f(y)-f(x))\Delta f(x)\right]$$
$$= \frac{3-\deg(x)}{4}\sum_{y\sim x}(f(x)-f(y))^2 + \frac{1}{2}\Delta f(x)\sum_{y\sim x}(f(y)-f(x))$$
$$= \frac{3-\deg(x)}{2}\Gamma(f)(x) + \frac{1}{2}(\Delta f(x))^2.$$

Therefore, $CD(\infty, K)$ is equivalent to

$$\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}}\left[\frac{1}{4}(f(z)-f(y))^2-\frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right]$$

$$+\sum_{\substack{y,y'\in N_1\\ \{y,y'\}\in E(G)}}(f(y)-f(y'))^2$$

$$\geq\left(\frac{2K+\deg(x)-3}{2}\right)\Gamma(f)(x)-\frac{1}{2}\Delta(f(x))^2.$$

□

A similar combinatorial interpretation of the curvature dimension inequality was established by Klartag, et al. in [31]. While their version of the curvature dimension inequality is effective in many applications, ours proved more fruitful for the types of inequalities that we derive below.

4.3 Discrepancy Inequalities

The Expander Mixing Lemma bounds the difference between the term $\frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}$ and the edge distribution between two sets X and Y. This term is a measure of randomness in the following sense, as stated in the introduction. If G' is a random graph in the configuration model with degree sequence matching that of G and X' and Y' are the sets in this random graph with degree sequences corresponding to those of X and Y in G, then the expectation of the number of edges between X' and Y' is $\frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}$. Therefore, the term $\left|e(X,Y)-\frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}\right|$ measures the distance towards randomness of a graph G. It is this comparison that we will consider the defining feature of a discrepancy inequality.

Since curvature is a local property dependent upon the edges appearing in the first two neighborhoods of any given vertex, we will explore how curvature can measure the randomness of the edge distribution within these neighborhoods. In other words, how randomly are the edges distributed between the first and second neighborhoods of any vertex in a graph with high curvature? If we partition the first neighborhood into two sets of not necessarily equal size, we would expect in a random graph that the edges from the second neighborhood to the first neighborhood would be split proportionally according to the size of the two sets in N_1. In fact, this should be true not just for the total set of edges coming from the second neighborhood, but also for the edges between the first neighborhood and any particular vertex in the second neighborhood. We will also explicitly consider the case where the first neighborhood is partitioned into a set containing a single vertex and a set containing all other vertices, as this measure will help us quantify the number of 3- and 4- cycles appearing in the neighborhoods of any given vertex.

Just as the Expander Mixing Lemma measures the edge distribution of a graph based on its normalized Laplacian spectrum, in this section we seek to measure the local edge distribution of a graph based on its curvature.

4.3.1 Second neighborhood discrepancy. Our goal here is to use the combinatorial interpretation of curvature given above in order to quantify the distribution of edges between the first and second neighborhoods of a vertex. In the theorem that follows, we split the neighborhood of a vertex x into two (not necessarily equal) parts, X and X^C. In doing so, we show that a sufficiently large curvature condition implies that for most vertices z in N_2, the divide between the edges from z to X and the edges from z to X^C is proportional to the size of each set.

Theorem 43. Let G be a graph with curvature K. Let $x \in V(G)$ be any vertex and let $X \subseteq N_1(x)$ with $X^C = N_1(x) \setminus X$. If $|X| = \alpha|N_1(x)|$, then

$$\sum_{z \in N_2} \left(\frac{[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)]^2}{\deg_{N_1}(z)} \right) \leq \frac{3}{4}[\alpha^2 \cdot e(X^C, N_2) + (1-\alpha)^2 \cdot e(X, N_2)] + e(X, X^C) - \left(\frac{2K + \deg(x) - 3}{2} \right) \cdot \frac{1}{2}\alpha(1-\alpha)\deg(x).$$

The above theorem says that a graph with high curvature locally acts like a random graph in the following sense. For a fixed vertex x, split $N_1(x)$ into two sets, one of size $\alpha|N_1|$ (which we call X) and the other of size $(1-\alpha)|N_1|$ (which we call X^C). For any vertex $z \in V(G) \setminus N_1[x]$, if the edges from N_1 to z are distributed uniformly randomly, we expect that an α proportion of the edges will go to the set of size $\alpha|N_1|$ and a $1-\alpha$ proportion of the edges will go to the set of size $(1-\alpha)|N_1|$. Thus, $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)|$ should be small. Our theorem, therefore, states that for a graph with high curvature, most vertices have edge distribution according to the proportion of the split in $N_1(x)$. It is because this mimics the theme of the Expander Mixing Lemma, that a strong graph parameter implies a random-like edge distribution, that we call this theorem a discrepancy inequality.

Proof. Fix $x \in V(G)$ and define $f(x) = 0$. Let $X \subseteq N_1(x)$ such that $|X| = \alpha|N_1(x)|$ and let $X^C = N_1(x) \setminus X$. For each $y \in X$, let $f(y) = 1-\alpha$ and for each $y \in X^C$, let $f(y) = -\alpha$. Note that this implies that $\Delta f(x) = 0$ and

$$\begin{aligned}
\Gamma(f)(x) &= \frac{1}{2}[\alpha(1-\alpha)^2 + (1-\alpha)(\alpha)^2]\deg(x) \\
&= \frac{1}{2}(\alpha - 2\alpha^2 + \alpha^3 + \alpha^2 - \alpha^3)\deg(x) \\
&= \frac{1}{2}(\alpha - \alpha^2)\deg(x) \\
&= \frac{1}{2}\alpha(1-\alpha)\deg(x).
\end{aligned}$$

Our goal is, for each $z \in N_2(x)$, to select $f(z)$ in order to minimize the left side of the inequality derived from $CD(\infty, K)$. Thus, fix $z \in N_2(x)$. Then

$$\sum_{\substack{y \sim z \\ y \in N_1}} \left[\frac{1}{4}(f(z) - f(y))^2 - \frac{1}{2}(f(z) - f(y))(f(y) - f(x))\right]$$

$$= \deg_X(z)\left[\frac{1}{4}(f(z)-(1-\alpha))^2 - \frac{1}{2}(f(z)-(1-\alpha))(1-\alpha)\right]$$
$$+ \deg_{X^C}(z)\left[\frac{1}{4}(f(z)+\alpha)^2 + \frac{1}{2}(f(z)+\alpha)\alpha\right]$$
$$= \frac{1}{4}\deg_{N_1}(z) f(z)^2 + \left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right] f(z)$$
$$+ \frac{3}{4}[\alpha^2 \deg_{X^C}(z) + (1-\alpha)^2 \deg_X(z)].$$

Using calculus, we can see that this is minimized when

$$f(z) = -\frac{2\left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]}{\deg_{N_1}(z)}.$$

For this selection of $f(z)$,

$$\sum_{\substack{y\sim z\\ y\in N_1}} \left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right]$$
$$= -\frac{\left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]^2}{\deg_{N_1}(z)} + \frac{3}{4}\left[\alpha^2 \deg_{X^C}(z) + (1-\alpha)^2 \deg_X(z)\right].$$

Summing over $z \in N_2$, we get that

$$\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}} \left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right]$$
$$= \sum_{z\in N_2}\left(-\frac{\left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]^2}{\deg_{N_1}(z)} + \frac{3}{4}\left[\alpha^2 \deg_{X^C}(z) + (1-\alpha)^2 \deg_X(z)\right]\right)$$
$$= \sum_{z\in N_2}\left(-\frac{\left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]^2}{\deg_{N_1}(z)}\right)$$
$$+ \frac{3}{4}[\alpha^2 \cdot e(X^C, N_2) + (1-\alpha)^2 \cdot e(X, N_2)]$$

As a result, $CD(\infty, K)$ implies that

$$
\begin{aligned}
-\sum_{z\in N_2}\left(\frac{[\alpha\deg_{X^C}(z)-(1-\alpha)\deg_X(z)]^2}{\deg_{N_1}(z)}\right)& \\
+\frac{3}{4}[\alpha^2\cdot e(X^C,N_2)+(1-\alpha)^2\cdot e(X,N_2)]+e(X,X^C)&\geq \\
\left(\frac{2K+\deg(x)-3}{2}\right)\cdot\frac{1}{2}\alpha(1-\alpha)\deg(x).&
\end{aligned}
$$

Solving for this first term yields

$$
\begin{aligned}
\sum_{z\in N_2}\left(\frac{[\alpha\deg_{X^C}(z)-(1-\alpha)\deg_X(z)]^2}{\deg_{N_1}(z)}\right)\leq& \\
\frac{3}{4}[\alpha^2\cdot e(X^C,N_2)+(1-\alpha)^2\cdot e(X,N_2)]+e(X,X^C)& \\
-\left(\frac{2K+\deg(x)-3}{2}\right)\cdot\frac{1}{2}\alpha(1-\alpha)\deg(x).&
\end{aligned}
$$

□

Corollary 44. Let G be a graph with curvature K. Let $x\in V(G)$ be any vertex and let $X\subseteq N_1(x)$ with $X^C=N_1(x)\setminus X$. If $|X|=\alpha|N_1(x)|$, then

$$
\begin{aligned}
\frac{\left(\sum_{z\in N_2}|\alpha\deg_{X^C}(z)-(1-\alpha)\deg_X(z)|\right)^2}{e(N_1,N_2)}\leq& \\
\frac{3}{4}\left[\alpha^2 e(X^c,N_2)+(1-\alpha)^2 e(X,N_2)\right]+e(X,X^C)& \\
-\left(\frac{2K+\deg(x)-3}{2}\right)\cdot\frac{1}{2}\alpha(1-\alpha)\deg(x).&
\end{aligned}
$$

Proof. We will use Cauchy-Schwarz to simplify the sum

$$\sum_{z\in N_2}\left(\frac{[\alpha\deg_{X^C}(z)+(1-\alpha)\deg_X(z)]^2}{\deg_{N_1}(z)}\right).$$

For our purposes, we will use Cauchy-Schwarz in the form of

$$\sum a_i^2 \geq \frac{\left(\sum a_i b_i\right)^2}{\sum b_i^2},$$

with

$$a_i = \frac{\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)}{\sqrt{\deg_{N_1}(z)}} \text{ and } b_i = \sqrt{\deg_{N_1}(z)}.$$

Thus,

$$\sum_{z \in N_2} \left(\frac{\left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]^2}{\deg_{N_1}(z)} \right) \geq \frac{\left(\sum_{z \in N_2} \left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]\right)^2}{\sum_{z \in N_2} \deg_{N_1}(z)} = \frac{\left(\sum_{z \in N_2} \left[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)\right]\right)^2}{e(N_1, N_2)}.$$

As a result, $CD(\infty, K)$ implies that

$$-\frac{\left(\sum |\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)|\right)^2}{e(N_1, N_2)} + \frac{3}{4}\left[\alpha^2 e(X^c, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, X^C) \geq \left(\frac{2K + \deg(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\deg(x).$$

□

This corollary gives us a much cleaner version of the above theorem. However, in some situations, this corollary gives away too much in its use of Cauchy-Schwarz. For example, our theorem gives a sharp bound on curvature for the graph $\mathbb{Z}^d$, while the corollary only gives us an asymptotic upper bound of d on the curvature. We will more fully explore this example later.

While the above theorem and corollary are the most general versions, as they apply to all subsets X of the neighborhood of some vertex x, we state the following corollary that restricts X to exactly half of the first neighborhood of x for the sake of clarity.

Corollary 45. Let G be a graph with curvature K. For any vertex $x \in V(G)$ and any $X \subseteq N(x)$ with $|X| = \frac{1}{2}|N(x)|$, if $X^C = N(x) \setminus X$, then

$$\frac{\left(\sum_{z\in N_2} |\deg_{X^C}(z) - \deg_X(z)|\right)^2}{e(N_1, N_2)} \leq \frac{3}{4}e(N_1, N_2) + 4e(X, X^C) - \frac{2K + \deg(x) - 3}{4}\deg(x).$$

Proof. Taking $\alpha = \frac{1}{2}$ in the above theorem directly gives the corollary. □

Corollary 46. Let G be a triangle-free graph with curvature K. Let $x \in V(G)$ be any vertex and let $X \subseteq N_1(x)$ with $X^C = N_1(x) \setminus X$. If $|X| = \alpha|N_1(x)|$, then

$$\frac{\left(\sum_{z\in N_2} |\alpha\deg_{X^C}(z) - (1-\alpha)\deg_X(z)|\right)^2}{e(N_1, N_2)} \leq \frac{3}{4}\left[\alpha^2 e(X^c, N_2) + (1-\alpha)^2 e(X, N_2)\right] - \left(\frac{2K + \deg(x) - 3}{2}\right)\cdot\frac{1}{2}\alpha(1-\alpha)\deg(x).$$

Proof. If $e(X, X^C) > 0$, then there exist adjacent vertices $y, y' \in N_1$. This edge then creates a triangle with x. Since G is triangle-free, then, $e(X, X^C) = 0$. □

In [31], it is shown that if G is a triangle-free graph, then G has curvature $K \leq 2$. However, as we show when we apply this corollary to trees, our result can clearly differentiate between trees and graphs with curvature zero, so this triangle-free corollary is still quite useful.

Example 1: Regular Trees As an example of Corollary 45, suppose that G is a d-regular tree. In any tree, every vertex in N_2 must be adjacent to exactly one vertex in N_1, as any vertex $z \in N_2$ with $\deg_{N_1}(z)$ would form a 4-cycle with x and its neighbors in N_1. Thus, we have that

$$\sum_{z \in N_2} |\deg_{X^C}(z) - \deg_X(z)| = \sum_{z \in N_2} 1 = d(d-1).$$

Furthermore, every tree is triangle-free, which implies that $e(X, X^C) = 0$. Our theorem, in this case, states that

$$d(d-1) \leq \frac{3}{4}d(d-1) - \frac{2K+d-3}{4}d.$$

Solving for K here yields that $K \leq 2-d$, and as shown in [31], the curvature of a tree is exactly $K = 2-d$. Therefore, our theorem gives a sharp upper bound on the curvature of a d-regular tree.

Example 2: $\mathbb{Z}^d$ As an example of Theorem 43, consider the graph $\mathbb{Z}^d$. Let $x = (0, \ldots, 0)$. Then $N_1(x) = \{(y_1, \ldots, y_d) : \sum |y_i| = 1\}$ and $N_2(x) = \{(z_1, \ldots, z_d) : \sum |z_i| = 2\}$. Define $X \subseteq N_1(x)$ to be the points $(y_1, \ldots, y_d)$ where $y_i = 1$ for some $i \in \{1, \ldots, d\}$ and $y_j = 0$ for all $j \neq i$. Then $X^C \subseteq N_1(x)$ is the set of points $(y_1, \ldots, y_d)$ where $y_i = -1$ for some $i \in \{1, \ldots, d\}$ and $y_j = 0$ for all $j \neq i$. Note here that $\alpha = \frac{1}{2}$, as N_1 is split evenly according to this partition.

Let $z \in N_2(x)$. We will analyze how each possible z contributes to the sum on the left side.

- If z contains two 1s, then $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)| = 1$ and $\deg_{N_1}(z) = 2$. Thus, the contribution of z to the sum is $\frac{1}{2}$. There are $\binom{d}{2}$ such vertices.
- If z contains two -1s, then $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)| = 1$ and $\deg_{N_1}(z) = 2$. Thus, the contribution of z to the sum is $\frac{1}{2}$. There are again $\binom{d}{2}$ such vertices.

- If z contains a 2, then $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)| = \frac{1}{2}$ and $\deg_{N_1}(z) = 1$. Thus, the contribution of z to the sum is $\frac{1}{4}$. There are d such vertices.

- If z contains a -2, then $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)| = \frac{1}{2}$ and $\deg_{N_1}(z) = 1$. Thus, the contribution of z to the sum is $\frac{1}{4}$. There are d such vertices.

- If z contains a 1 and a -1, then $|\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)| = 0$ and $\deg_{N_1}(z) = 2$. Thus, the contribution of z to the sum is 0. There are $2\binom{d}{2}$ such vertices.

Thus, the sum over all z yields

$$\sum_{z \in N_2} \left(\frac{[\alpha \deg_{X^C}(z) - (1-\alpha)\deg_X(z)]}{\deg_{N_1}(z)} \right) = \frac{1}{2}\binom{d}{2} + \frac{1}{2}\binom{d}{2} + \frac{1}{4}d + \frac{1}{4}d + 0 \cdot 2\binom{d}{2} = \frac{1}{2}d^2.$$

On the right side, $\alpha = \frac{1}{2}$. Thus,

$$\frac{3}{4}[\alpha^2 \cdot e(X^C, N_2) + (1-\alpha)^2 e(X, N_2)] = \frac{3}{16} e(N_1, N_2).$$

To compute $e(N_1, N_2)$, every vertex in N_2 with two nonzero coordinates has two neighbors in N_1 and there are $4\binom{d}{2}$ of these vertices. Also, every vertex in N_2 with one nonzero coordinate has one neighbor in N_1 and there are $2d$ of these vertices. Thus, $e(N_1, N_2) = 8\binom{d}{2} + 2d = 4d^2 - 2d$. Since $\mathbb{Z}^d$ is triangle-free, we have that $e(X, X^C) = 0$. Finally, the curvature term yields $\frac{d(2K+2d-3)}{8}$.

Therefore, our discrepancy inequality yields that

$$\frac{1}{2}d^2 \leq \frac{3}{16}(4d^2 - 2d) - \frac{d(2K + 2d - 3)}{8}.$$

Solving for K yields that $K \leq 0$, which again is a tight upper bound as [31] gives that $K = 0$.

4.3.2 Discrepancy of the neighbors of a single vertex. In the theorem that follows, we isolate a single vertex in $N(x)$. In doing so, counting the number of edges from y to $N(x)$ becomes equivalent to counting the number of triangles that include the edge xy. Similarly, for any vertex $z \in N(y)$ such that $d(x,z) = 2$ (in other words, for any vertex $z \in N(y) \cap N_2(x)$), every edge from z to any vertex in $N(x)$ other than y forms a 4-cycle that includes the edge xy. Therefore, the following inequality will give us a way to count the number of short cycles that include any given edge.

Theorem 47. Let G be a graph with curvature K. Let $x \in V(G)$, let $y \in N(x)$, and let $A = \{y' \in N_1(x) : y' \neq y\}$. Then

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \frac{1}{\deg_A(z)+1} \leq \deg_{N_1}(y) + \frac{3}{4}\deg_{N_2}(y) - \frac{1}{4}\deg(x) - \frac{1}{2}K + \frac{5}{4}.$$

Since we saw in the previous section that edges from N_2 to N_1 are split nearly randomly in a graph with high curvature, if an edge exists from a particular vertex $y \in N_1$ to some vertex $z \in N_2$, there must be many edges from z to the rest of N_1.

Proof. Fix $x \in V(G)$ and define $f(x) = 0$. Then fix $y \in N_1(x)$ and define $f(y) = 1$. Let $A = \{y' \in N_1(x) : y' \neq y\}$ and for all $y' \in A$, define $f(y') = 0$. For all $z \in N_2(x)$ such that $z \not\sim y$, let $f(z) = 0$. Here we get that $\Delta f(x) = 1$ and $\Gamma(f)(x) = \frac{1}{2}$.

Then for each $z \in N_2(x)$ that is adjacent to y,

$$\begin{aligned}
&\sum_{y \sim z} \left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] \\
&= \frac{1}{4}(f(z)-1)^2 - \frac{1}{2}(f(z)-1) + \frac{1}{4}\deg_A(z)f(z)^2 \\
&= \frac{1}{4}(\deg_A(z)+1)f(z)^2 - f(z) + \frac{3}{4}.
\end{aligned}$$

This term is maximized when $f(z) = -\frac{2}{\deg_A(z)+1}$, which yields that

$$\sum_{y\sim z}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] = -\frac{1}{\deg_A(z)+1} + \frac{3}{4}.$$

Therefore, $CD(\infty, K)$ implies that

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} -\frac{1}{\deg_A(z)+1} + \frac{3}{4}\deg_{N_2}(y) + \deg_{N_1}(y) \geq \frac{2K+\deg(x)-3}{4} - \frac{1}{2}.$$

Rearranging this inequality yields the result of the theorem. □

4.4 Combinatorial Applications

Earlier, we explored the extremal question of how many disjoint cycles a graph with certain degree conditions can contain. In their initial work, Corradi and Hajnal found that a sufficiently large graph with minimum degree δ contains at least $\frac{\delta}{2}$ vertex-disjoint cycles. In this dissertation, we studied how adding a spectral condition can improve this lower bound for sparse graphs. In this same vein, can we produce lower bounds on the number of disjoint cycles in a sparse graph using a curvature condition? In order to do this, we will only be able to study cycles of size 3 and 4, as the curvature of a graph only gives us information within the first two neighborhoods of any given vertex. However, the local discrepancy inequalities above can be used to quantify the number of these short cycles that an edge must be contained in.

Proposition 48. If G is a graph with minimum degree $\delta(G) \geq 4$ and curvature $K \geq 0$, then for every pair of adjacent vertices $x, y \in V(G)$, there exists a 3-cycle or a 4-cycle containing the edge xy.

Proof. Fix two adjacent vertices $x, y \in V(G)$. By Theorem 47, we have that

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \frac{1}{\deg_A(z)+1} \leq -\frac{1}{4}\deg(x) + \deg_{N_1}(y) + \frac{3}{4}\deg_{N_2}(y) + \frac{5}{4}.$$

Suppose that there are no triangles involving edge xy. Then $\deg_{N_1}(y) = 0$ as every edge within N_1 forms a triangle with x. Furthermore, if there exists an edge between a vertex in $z \in N(y)$ with $d(x,z) = 2$ and $y' \in A$, then $xyzy'$ forms a four-cycle including the edge xy. Suppose by way of contradiction that $\deg_A(z) = 0$ for all $z \in N(y)$ such that $d(x,z) = 2$. Then

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \frac{1}{\deg_A(z)+1} = \deg_{N_2}(y) = d - 1.$$

This implies that the above inequality derived from Theorem 47 is equivalent to

$$\deg_{N_2}(y) \leq -\frac{1}{4}\deg(x) + \frac{3}{4}\deg_{N_2}(y) + \frac{5}{4}.$$

Therefore, we have that

$$\deg_{N_2}(y) + \deg(x) \leq 5,$$

which contradicts the assumption that $\delta(G) \geq 4$. $\square$

By only asking for one 4-cycle, we forced the $\deg_{N_2}(y)$ term on the left side to have a greater coefficient than that on the right side. In order to optimize the number of 4-cycles for a graph, these two coefficients should be equal.

Proposition 49. If G is a graph with minimum degree $\delta(G) \geq 6$ and curvature $K \geq 0$, then for every pair of adjacent vertices $x, y \in V(G)$, there exists at least $\frac{1}{4}(\deg(x) - 5)$ 3-cycles containing the edge xy or at least $\frac{1}{2}\deg(y)$ 4-cycles containing the edge xy.

Before proving this statement, we need the following lemma.

Lemma 50. For all nonnegative integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i \leq \frac{1}{2}n$,

$$\sum_{i=1}^{n} \frac{1}{a_i + 1} \geq \frac{3}{4}n.$$

Proof. Consider a sequence of nonnegative integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i \leq \frac{1}{2}n$ that minimizes $\sum_{i=1}^{n} \frac{1}{a_i+1}$. Then there exists $j \in \{1, \ldots, n\}$ such that $a_j = 0$. If there exists k with $a_k \geq 2$, then replacing a_j and a_k with ones would preserve or decrease $\sum_{i=1}^{n} a_i$. However, this replacement would also reduce $\sum_{i=1}^{n} \frac{1}{a_i+1}$ because $\frac{1}{2} + \frac{1}{2} < 1 + \frac{1}{a_k+1}$, which contradicts the minimality of the sequence. Thus, $a_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$.

Since $a_1, \ldots, a_n$ must be a $\{0, 1\}$-sequence, $\sum_{i=1}^{n} \frac{1}{a_i+1}$ decreases as the number of ones increases. Thus, the $\{0, 1\}$ sequence that minimizes this sum must contain $\lfloor \frac{1}{2}n \rfloor$ ones, the maximum allowable subject to the constraint $\sum_{i=1}^{n} a_i \leq \frac{1}{2}n$. For this minimizing sequence, with n even (n odd produces a slightly larger sum), we have

$$\sum_{i=1}^{n} \frac{1}{a_i + 1} = \frac{3}{4}n.$$

□

Proof of Proposition 49. Fix two adjacent vertices $x, y \in V(G)$. By Theorem 47, we have that

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \frac{1}{\deg_A(z) + 1} \leq -\frac{1}{4}\deg(x) + \deg_{N_1}(y) + \frac{3}{4}\deg_{N_2}(y) + \frac{5}{4}.$$

If there exists an edge between a vertex in $z \in N(y)$ with $d(x, z) = 2$ and $y' \in A$, then $xyzy'$ forms a four-cycle including the edge xy. Suppose that xy is contained in fewer than $\frac{1}{2}\deg(y)$ 4-cycles. Then

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \deg_A(z) \leq \frac{1}{2}\deg_{N_2}(y).$$

By the above lemma, this implies that

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} \frac{1}{\deg_A(z)+1} \geq \frac{3}{4}\deg_{N_2}(y).$$

Combining this inequality with the inequality derived from Theorem 47 yields

$$\frac{3}{4}\deg_{N_2}(y) \leq -\frac{1}{4}\deg(x) + \deg_{N_1}(y) + \frac{3}{4}\deg_{N_2}(y) + \frac{5}{4}.$$

This is equivalent to $\deg_{N_1}(y) \leq \frac{1}{4}(\deg(x) - 5)$. Since every edge within N_1 forms a triangle with x, this means that the edge xy is contained in at least $\frac{1}{4}(\deg(x) - 5)$ triangles.

□

In this calculation, it is important that $\deg_A(z)$ is an integer. In Lemma 50, if $a_1, \ldots, a_n$ were nonnegative real numbers, the lower bound on $\sum_{i=1}^{n} \frac{1}{a_i+1}$ would instead by $\frac{2}{3}n$ by Jensen's inequality. Furthermore, such a lower bound could not be improved, as taking $a_i = \frac{1}{2}$ for all i realizes this lower bound. However, due to our application on graphs, a_i must be an integer, giving us this coefficient of $\frac{3}{4}$. This lower bound is ideal, as the left side cancels out the $\frac{3}{4}\deg_{N_2}(y)$ on the right side. Furthermore, this lower bound is realized when $\deg_A(z) = 1$ for half of the neighbors of y with $d(x, z) = 2$ and $\deg_A(z) = 0$ for the other such neighbors. With this partition,

$$\sum_{\substack{z\in N(y)\\ d(x,z)}} \frac{1}{\deg_A(z)+1} = \frac{3}{4}\deg_{N_2}(y).$$

If this sum were any smaller, as it would be if more than half of the neighbors of y had an edge to A, then we would get a result comparing $\deg(x)$ and $\deg(y)$, which would in turn force a regularity-type condition on the graph to obtain a similar result.

This local result about the number of 4-cycles containing an edge can be transformed into global results about edge-disjoint and vertex-disjoint 4-cycles.

Corollary 51. Let G be a d-regular, triangle-free graph with $d \geq 6$ and curvature $K \geq 0$. Then G contains at least $\frac{n}{64}$ edge-disjoint 4-cycles.

Proof. By Proposition 49, every edge is in at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{d^2 n}{16}$ total 4-cycles in G. Note that any edge can appear in at most d^2 4-cycles, so each 4-cycle shares an edge with at most $4d^2$ 4-cycles. Therefore, the graph must contain at least $\frac{n}{64}$ edge-disjoint 4-cycles. □

In the above proof, we examined the number of 4-cycles in which a given edge can appear in order to get a lower bound on the number of edge-disjoint 4-cycles. Similarly, examining the number of 4-cycles in which a given vertex can appear produces a lower bound on the number of vertex-disjoint 4-cycles in a graph.

Corollary 52. Let G be a d-regular, triangle-free graph with $d \geq 6$ and curvature $K \geq 0$. Then G contains at least $\frac{n}{32d}$ vertex-disjoint 4-cycles.

Proof. By Proposition 49, every edge is in at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{d^2 n}{16}$ total 4-cycles in G. Fix a vertex x. Since a 4-cycle containing x must have two neighbors of x and a second vertex adjacent to each of these neighbors, x is contained in at most $\binom{d}{2}(d-1) \leq \frac{d^3}{2}$ 4-cycles. Thus, every 4-cycle shares a vertex with at most $2d^3$ other 4-cycles. Dividing the total number of 4-cycles in G by this maximum number of 4-cycles that share a vertex yields a total of at least $\frac{n}{32d}$ vertex-disjoint 4-cycles. □

Unfortunately, this result decreases in d, which does not mesh with our intuition. Ultimately, as the degree of a regular graph increases, so too should the number of disjoint cycles. For graphs with large degree, this bound is quite bad. For example, this bound gives only 1 disjoint cycle in a complete graph. For graphs with small degree, however, this bound is much closer to correct. For example, the hypercube Q_d can be decomposed into $\frac{n}{4}$ 4-cycles while the above bound gives at least $\Omega(\frac{n}{\log n})$ vertex-disjoint 4-cycles. We believe that the above technique can be improved to guarantee $\Omega(n)$ disjoint 4-cycles.

Even obtaining this many 4-cycles, however, shows the impact of this curvature term. For a fixed degree d, the Turán number for 4-cycles is $n^{3/2}$, meaning that there exists a regular graph with degree as large as $d = \sqrt{n}$ that contains no 4-cycles. By simply adding this curvature condition, however, we guarantee that a d-regular graph with $d = \sqrt{n}$ contains at least $\Omega(\sqrt{n})$ 4-cycles.

Additionally, this result gives us an improvement on the result by Corradi and Hajnal if d is small. In fact, these two results can be combined to give at least $\Omega(\sqrt{n})$ disjoint cycles on a sufficiently large graph.

Corollary 53. Let G be a d-regular, triangle-free graph on at least $\frac{3}{2}d$ vertices with $d \geq 6$ and curvature $K \geq 0$. Then G has at least $\Omega(\sqrt{n})$ disjoint cycles.

Proof. If $d \geq \Omega(\sqrt{n})$, then the above result of Corradi and Hajnal states that G has at least $\frac{d}{2} \geq \Omega(\sqrt{n})$ disjoint cycles. If $d \leq O(\sqrt{n})$, then our above result states that G contains at least $\frac{n}{32d} \geq \Omega(\sqrt{n})$ disjoint 4-cycles. □

Each of the above corollaries applies only to a quite restrictive class of graphs, as the maximum curvature of a triangle-free graph is 2 [31]. While there are graphs such as Q_d and $K_{t,t}$ that satisfy this condition, ideally these results would hold for a larger class of graphs. By eliminating the triangle-free restriction and using the full capacity of Proposition 49, we can obtain similar results on the number of 3- and 4-cycles (which we refer to as short cycles) with slightly worse constants.

Corollary 54. Let G be a d-regular graph with $d \geq 6$ and curvature $K \geq 0$. Then G contains at least $\frac{n(d-5)}{128(d+1)}$ edge-disjoint cycles of size 3 or 4.

Proof. By Proposition 49, every edge is in at least $\frac{1}{4}(d-5)$ 3-cycles or at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{nd(d-5)}{32}$ total short cycles. Any edge is in at most d 3-cycles and at most d^2 4-cycles. Thus, any 3-cycle shares an edge with at most $3d$ other 3-cycles and at most $3d^2$ 4-cycles. Similarly, any 4-cycle shares an edge with at most $4d$ 3-cycles and at most $4d^2$ 4-cycles. Dividing the total number of short cycles in G by the maximum number of short cycles that share an edge gives that G must contain at least $\frac{n(d-5)}{128(d+1)}$ edge-disjoint short cycles. □

Again, we can determine the maximum number of 3- and 4-cycles in which a vertex can appear to find a lower bound on the number of vertex-disjoint short cycles in a graph.

Corollary 55. Let G be a d-regular graph with $d \geq 6$ and curvature $K \geq 0$. Then G contains at least $\frac{n(d-5)}{64(d-1)^2}$ disjoint cycles of size 3 or 4.

Proof. By Proposition 49, every edge is in at least $\frac{1}{4}(d-5)$ 3-cycles or at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{nd(d-5)}{32}$ total cycles of size 3 or 4. Fix a vertex x. Since a triangle containing x must consist of two neighbors of x, x is contained in at most $\binom{d}{2}$ 3-cycles. Thus, every 3-cycle shares a vertex with at most $\frac{3d(d-1)}{2}$ other 3-cycles and at most $\frac{3d(d-1)^2}{2}$ 4-cycles. Furthermore, any 4-cycles containing x must consist of two neighbors of x and another vertex adjacent to each of these neighbors, so x is contained in at most $\binom{d}{2}(d-1)$ 4-cycles. Thus, every 4-cycle shares a vertex with at most $2d(d-1)$ 3-cycles and at most $2d(d-1)^2$ other 4-cycles. Dividing the total number of short cycles in G by the maximum number of short cycles that share a vertex yields that G must contain at least $\frac{n(d-5)}{64d(d-1)}$ vertex-disjoint short cycles. □

While many of these results are immediate applications of the discrepancy-type inequalities that we have derived, we believe that there is much more room for exploration using these tools. For example, we have made significant progress on results giving lower bounds for connectivity based on the curvature. Furthermore, other techniques that use local information to prove global results should be fertile territory for the application of curvature bounds. For example, a graph can be shown to be Hamiltonian by considering the size of particular neighborhoods. Could such conditions be altered to conditions on curvature that still guarantee Hamiltonicity? The examples above give a slight glimpse into what is possible using these local discrepancy inequalities, but there is still plenty of room for exploration.

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