University of Denver

Digital Commons @ DU

Electronic Theses and Dissertations | Graduate Studies

2021

Zn Orbifolds of Vertex Operator Algebras

Daniel Graybill
University of Denver

Follow this and additional works at: https://digitalcommons.du.edu/etd

Part of the Algebra Commons

Recommended Citation

Graybill, Daniel, "Zn Orbifolds of Vertex Operator Algebras" (2021). *Electronic Theses and Dissertations*. 1924.
https://digitalcommons.du.edu/etd/1924

This Dissertation is brought to you for free and open access by the Graduate Studies at Digital Commons @ DU. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of Digital Commons @ DU. For more information, please contact jennifer.cox@du.edu,dig-commons@du.edu.

$\mathbb{Z}_n$ Orbifolds of Vertex Operator Algebras

A Dissertation

Presented to

the Faculty of the College of Natural Sciences and Mathematics

University of Denver

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Daniel Graybill

June 2021

Advisor: Andrew Linshaw

Author: Daniel Graybill
Title: $\mathbb{Z}_n$ Orbifolds of Vertex Operator Algebras
Advisor: Andrew Linshaw
Degree Date: June 2021

ABSTRACT

Given a vertex algebra $\mathcal{V}$ and a group of automorphisms of $\mathcal{V}$, the invariant subalgebra $\mathcal{V}^G$ is called an orbifold of $\mathcal{V}$. This construction appeared first in physics and was also fundamental to the construction of the Moonshine module in the work of Borcherds. It is expected that nice properties of $\mathcal{V}$ such as C_2-cofiniteness and rationality will be inherited by $\mathcal{V}^G$ if G is a finite group. It is also expected that under reasonable hypotheses, if $\mathcal{V}$ is strongly finitely generated and G is reductive, $\mathcal{V}^G$ will also be strongly finitely generated. This is an analogue of Hilbert's theorem on the finite generation of classical invariant rings, and it is known in certain classes of vertex algebras such as free field algebras, affine vertex algebras, and $\mathcal{W}$-algebras. Unfortunately, the proof is nonconstructive and it is difficult to give explicit strong generating sets. Finding strong generators is very useful for understanding the representation theory of vertex algebras since strong generators give rise to generators for their Zhu algebras.

There are two main results in this thesis. First, the rank n Heisenberg algebra $\mathcal{H}(n)$ has full automorphism the orthogonal group. It is known to be strongly finitely generated, but no upper bound is known for how many generators are needed. We show that it is if type $\mathcal{W}(2, 4, \dots, 2N)$ for some N satisfying $n^2+3n \leq 2N \leq 2n^2+4n$. This means that it has a minimal strong generating set consisting of one field in each weight $2, 4, \dots, 2N$. Second, we consider the $\mathbb{Z}_n$ orbifold of the rank 2 Heisenberg algebra $\mathcal{H}(2)$. For $n = 2$ and $n = 3$, the structure of this orbifold was already understood, but it is considerably more difficult to find the minimal strong generating type for all n. We will show that for all $n \geq 3$, $\mathcal{H}(2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(2, 3, 4, 5, n^2, (n+2)^2)$.

This strong generating set immediately gives rise to generators for Zhu's associative algebra $A(\mathcal{H}(2)^{\mathbb{Z}_n})$ and Zhu's commutative algebra $R_{\mathcal{H}(2)^{\mathbb{Z}_n}}$. It is an interesting question how the orbifold functor interacts with the passage to Zhu's commutative algebra. In particular, if $\mathcal{V}$ is a vertex algebra and G is a finite group of automorphisms of $\mathcal{V}$, there is an induced action of G on $R_{\mathcal{V}}$, and we have a map $R_{\mathcal{V}^G} \to (R_{\mathcal{V}})^G$. In general, this map is not an isomorphism but it an interesting question whether it is an isomorphism at the level of reduced rings, that is, after quotienting by the nilradical. Using the strong generating set for $\mathcal{H}(2)^{\mathbb{Z}_3}$, we will prove that this is the case in this example.

More generally, if $R_{\mathcal{V}^G}$ and $(R_{\mathcal{V}})^G$ become isomorphic as reduced rings, they must have the same Krull dimension. For strongly finitely generated vertex algebras, this can be viewed as a generalization of the conjecture that taking G-invariants preserves C_2-cofiniteness, since C_2-cofiniteness of $\mathcal{V}$ means that $R_{\mathcal{V}}$ has Krull dimension zero.

ACKNOWLEDGEMENTS

I would like to thank all the math teachers I had along the way. Thank you to all the professors and staff of the University of Denver Mathematics Department. I would like to add special thanks to Dr. Jim Hagler. My profound thanks and appreciation to Dr. Andrew Linshaw for guiding me, sharing with me his incredible perspective, and most importantly his patience.

My most sincere thanks to my mom, dad, and my wife Dr. Chiharu Graybill.

Thank you for the aspiration.

TABLE OF CONTENTS

Chapter 1: Introduction

1.1 Introduction

At the end of the nineteenth century and into the beginning of the twentieth century physicists were laying the groundwork for what we now refer to as general relativity and quantum mechanics. In doing so they split the physical world into two perspectives.

Concurrent with these developments both physicists and mathematicians worked to build a robust set of mathematical tools and techniques for modeling these two theories, and by the middle of the twentieth century both theories had successfully explained previously unexplained observations and successfully predicted new ones. Physicists began to make theories and hypotheses that relied on combining findings from both theories. Subsequently much work has been done developing mathematical models for relativistic quantum mechanics.

Several mathematical formalisms of quantum mechanics have emerged. One framework that has met with particular success is quantum field theory which combines relativistic quantum mechanics with the mathematical techniques of classical electrodynamic field theory.

In order to narrow the search for viable quantum field theories physicists have found it fruitful to consider systems which are invariant under certain transformations of the underlying space. Invariance under conformal transformations leads to some particularly nice structure.

In two dimensions there are infinitely many conformal transformations while they are much more restricted in higher dimensions. This means that in two dimensions being invariant under these transformations is particularly constraining. The two dimensional, infinitesimal conformal transformations are given by the Witt algebra, but for quantum reasons we consider the unique central extension called the Virasoro algebra. Since it is the action of this algebra that is of interest the representation theory of the Virasoro algebra has been much studied.

Meanwhile in 1978 John McKay made his observation connecting the j-function with the representations of the monster group. This was furthered by John Conway and Simon Norton leading to the Moonshine Conjecture that there exists an infinite dimensional graded representation of the monster group with q-traces matching the Hauptmaduln.

In 1986 Richard Borcherds axiomatized the notion of a vertex algebra. This was followed by Igor Frenkel, James Lepowsky, and Arne Meurman constructing the moonshine module. In 1992 Borcherds completed the proof that the graded traces matched Conway and Norton's list of Hauptmoduln.

In order for a quantum field theory to be invariant under conformal transformations its space of states must be the direct sum of eigenspaces for the L_0 element of the Virasoro algebra. The eigenvalue for a homogeneous element is referred to as its *conformal weight*. Frenkel, Lepowsky, and Meurman captured this structure in the additional axioms defining a vertex operator algebra, or VOA [FLM]. In particular VOAs come equipped with an grading by weight which is compatible with the circle products and the translation operator.

For a vertex algebra $\mathcal{V}$ and a group G of automorphisms of $\mathcal{V}$, the invariant subalgebra $\mathcal{V}^G$ is called an orbifold. This construction first appeared in the physics literature [DVVV, DHVWI, DHVWII]. It was also essential in the construction of the Moonshine module $V^\natural$ appearing in [FLM], since $V^\natural$ is an extension of the $\mathbb{Z}/2\mathbb{Z}$-orbifold of the vertex algebra associated to the Leech lattice. The orbifold construction is one of the fundamental ways for constructing new vertex algebras from old ones, and it has been important in the longstanding problem of attempting to classify rational conformal field theories. For example, a problem which has received a lot of attention for the last thirty years is the classification of holomorphic VOAs, that is, rational VOAs $\mathcal{V}$ which admit only one irreducible module, namely $\mathcal{V}$ itself. If $\mathcal{V}$ is holomorphic, its central charge c must be a positive integer multiple of 8 [DM, Sch]. In the first two cases $c = 8$ and $c = 16$, the classification is easy and was achieved by Dong and Mason in [DM], but the case $c = 24$ is very nontrivial. In [Sch], Schellekens introduced his famous list of 71 possible characters for such VOAs and conjectured that there is a unique holomorphic VOA with $c = 24$ realizing each of these characters. All of these VOAs have now been constructed using orbifold theory in recent years, and their uniqueness has also been proven in all cases except for the Moonshine module $V^\natural$ [EMS1, EMS2, ELMS, LS].

Since it is the representation theory of these algebras which is of most interest in quantum mechanics and string theory, properties which constrain the representation theory are significantly studied. The properties of rationality and C_2-cofiniteness are chief among these. Here C_2-cofiniteness is a finiteness condition introduced by Zhu that that a commutative algebra $R_\mathcal{V}$ obtained from a VOA $\mathcal{V}$, is finite-dimensional as a vector space. This condition implies that there are finitely many irreducible $\mathbb{Z}$-graded $\mathcal{V}$-modules. Rationality is the property that all $\mathcal{V}$-modules are completely reducible. If a vertex operator algebra both C_2-cofinite and rational it is referred to as regular.

A longstanding conjecture has been that if $\mathcal{V}$ is C_2-cofinite and rational and G is a finite group then $\mathcal{V}^G$ inherits both these properties. This was proven by Carnahan and Miyamoto [CM] when G is cyclic, and therefore by iteration it holds when G is solvable as well. McRae has shown under mild hypotheses on $\mathcal{V}$, namely that it is of CFT type and self-contragredient, the C_2-cofiniteness of $\mathcal{V}^G$ would imply the rationality of $\mathcal{V}^G$ for any finite G.

Another important property of vertex algebras $\mathcal{V}$ is strong finite generation. This says that $\mathcal{V}$ is spanned by all normally ordered monomials in a finite set of fields and their derivatives. In particular, it has a finite generating set that closes under OPE with possible nonlinear terms. Strong generators are very useful because they give rise to generators for both Zhu's associative algebra $A(\mathcal{V})$ and Zhu's commutative algebra $R_{\mathcal{V}}$. A natural question to ask is the following: given a strongly finitely generated VOA $\mathcal{V}$ and a reductive group G of automorphisms of $\mathcal{V}$, is $\mathcal{V}^G$ also strongly finitely generated? This is an analogue of Hilbert's theorem that says that if G is any reductive group and V is a finite-dimensional G-module, the ring of invariant polynomial functions $\mathbb{C}[V]^G$ is finitely generated [HI, HII]. This problem been called the *vertex algebra Hilbert problem* in the recent survey by Lian and Linshaw [LL]. In fact, it does not hold for abelian vertex algebras and it fails even for the abelian vertex algebra with one generator with the action of $G = \mathbb{Z}_2$ acting by -1. It was conjectured in [LL] that it holds when $\mathcal{V}$ is simple, and so far it has been proven for a class of vertex algebras that includes all free field algebras, affine vertex algebras, and $\mathcal{W}$-algebras at generic levels [L1, L2, L3, L4, L5, CL2, CL3, CL4]. We mention in passing that the strong finite generation of orbifold has an application to the structure of cosets of affine vertex algebras inside larger structures, since these cosets often have a large level limit that is isomorphic to an orbifold of a free field algebras. For example, one can show using this idea that for any $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ where $\mathfrak{g}$ is a simple Lie superalgebra and f is an even nilpotent element in $\mathfrak{g}$, the coset of $\mathcal{W}^k(\mathfrak{g}, f)$ by an affine subalgebra is strongly finitely generated at generic level. The explicit description of the affine cosets of a large family of $\mathcal{W}$-algebras has recently led to the solution of the Gaiotto-Rapčák triality conjectures in [CL3, CL4]; this is the statement that the affine cosets of three different $\mathcal{W}$-algebras are isomorphic.

Unfortunately, there are only a few settings where then description of $\mathcal{V}^G$ can be given constructively so far. These include the case where $\mathcal{V}$ is of the standard free field algebras, namely the Heisenberg algebra $\mathcal{H}(n)$, the free fermion algebra $\mathcal{F}(n)$, the $\beta\gamma$-system $\mathcal{S}(n)$, and the symplectic fermion algebra $\mathcal{A}(n)$, and G is the full autormorphism group Aut $\mathcal{V}$, which is the orthogonal group $O(n)$ for $\mathcal{H}(n)$ and $\mathcal{F}(n)$, and the symplectic group $Sp(2m)$

for $\mathcal{S}(n)$ and $\mathcal{A}(n)$. In these cases, the description of $\mathcal{V}^G$ is ultimately based on Weyl's first and second fundamental theorems of invariant theory for the standard representations of $O(n)$ and $Sp(2n)$. These theorems give generators and relations for the ring of G-invariant polynomial functions on a sum of many copies of the standard representation of G. In both cases the generators are quadratic and correspond to pairings between different copies of the standard representation, and the relations are given by determinants in the case of $O(n)$, and Pfaffians in the case of $Sp(2n)$. Since the description of these classical invariant rings is so nice, it is possible to find explicit genertaing sets for the orbifolds $\mathcal{V}^{\text{Aut } \mathcal{V}}$. In the case where $\mathcal{V}$ is either $\mathcal{F}(n)$, $\mathcal{S}(n)$, or $\mathcal{A}(n)$, minimal strong generating sets for $\mathcal{V}^{\text{Aut } \mathcal{V}}$ are known, but in the case of $\mathcal{H}(n)^{O(n)}$ the minimal set is only known for $n \leq 6$.

Another setting where explicit minimal strong generating sets can be found in some generality is the case where $G = \mathbb{Z}_2$. One such result established by Masoumah Al-Ali asserts that the $\mathbb{Z}_2$ orbifold of the universal affine VOA associated to a simple, finite-dimensional Lie algebra $\mathfrak{g}$ at generic level k, $V^k(\mathfrak{g})^{\mathbb{Z}_2}$, with one minor exception is of type $\mathcal{W}(1^m, 2^{d+\binom{d}{2}}, 3^{\binom{d}{2}}, 4)$ [Al-A]. Here m is the number of positive roots of $\mathfrak{g}$ and $d = m + \text{rank}(\mathfrak{g})$. It is also worth pointing out that in this example the number of generators varies across the family while the weights in which they appear remains fixed. In her thesis Al-Ali also provided strong generating sets for $\mathbb{Z}_2$ orbifold of the rank n Heisenberg, and showed it is of type $\mathcal{W}(2,4)$, $\mathcal{W}(2^3, 3, 4^2)$, or $\mathcal{W}(2^{n+\binom{n}{2}}, 3^{\binom{n}{2}}, 4)$. However, it is very nontrivial to produce minimal strong generating sets for orbifolds in a more general setting.

The first main result in this thesis is an improvement of the description of $\mathcal{H}(n)^{O(n)}$. It is conjectured to be of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$ in [L3], and this conjecture was proven for $n \leq 6$ in [L4]. It was also shown in [L4] that $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \dots, 2N)$ for some $2N \geq n^2 + 3n$, but until recently there was no known bound on the size of N. In Chapter, we show that $\mathcal{H}(7)^{O(7)}$ is of type $\mathcal{W}(2, 4, \dots, 70)$, and showing that for all $n \geq 8$, $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \dots, 2N)$ for $2N \leq 2n^2 + 4n$. The main idea is that the associated graded algebra of $\mathcal{H}(n)^{O(n)}$ is isomorphic to the ring of $O(n)$-invariant polynomial functions on the sum of infinitely many copies of the standard representation $\mathbb{C}^n$ of $O(n)$. Recall that this ring is generated by quadratics polynomials and the relations are $(n+1) \times (n+1)$ determinants in the generators. In the vertex algebra $\mathcal{H}(n)^{O(n)}$, these determinants require quantum corrections, and if the coefficient of a certain term in the quantum correction is nonzero, this relation yields a decoupling relation that allows the generators in the appropriate weight, as well as all higher weights, to be be eliminated. This leaves behind a finite set that will strongly generate the algebra. These determinantal relations are specified by two lists of distinct non-negative integers, and the above quantum corrections are determined by a complicated recursive formula that was written down in [L4], but no closed formula was given. When these lists of integers only contain even integers, we will give a closed formula for the desired quantum correction coefficient and it will be apparent from our formula that it is always nonzero.

The second result in this thesis is the explicit minimal strong generating set for $\mathcal{H}(2)^{\mathbb{Z}_n}$ for all $n \geq 3$. This is an infinite family of examples that are much more challenging to analyze in a uniform way that the cases involving either the standard representation of classical groups, or $\mathbb{Z}_2$-actions. Our approach will be to first analyze the orbifold $\mathcal{H}(2)^{SO(2)}$, which is easily seen to be of type $\mathcal{W}(2,3,4,5)$, and then decompose $\mathcal{H}(2)^{\mathbb{Z}_n}$ as a module over $\mathcal{H}(2)^{SO(2)}$ for all n. It has a strong generating set that lies in the sum of just three irreducible $\mathcal{H}(2)^{SO(2)}$, and we will show that in addition to the generators of $\mathcal{H}(2)^{SO(2)}$, we require only two more fields in each of these two modules, which have weights n and $n+2$. In particular, $\mathcal{H}(2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(2,3,4,5,n^2,(n+2)^2)$.

This explicit generating set has several applications. First, it immediately gives generators for the Zhu algebra $A(\mathcal{H}(2)^{SO(2)})$, from which we can deduce some properties of its modules. By a deformation argument, we also get a description of the orbfold $V^k(\mathfrak{sl}_2)^{\mathbb{Z}_n}$. For a general abelian group G acting on the rank m Heisenberg algebra $\mathcal{H}(m)$, the structure of $\mathcal{H}(m)^G$ can also be given in terms of many copies of algebras of the form $\mathcal{H}(2)^{\mathbb{Z}_n}$ and $\mathcal{H}(1)^{\mathbb{Z}_2}$.

Finally, in the case $n = 3$, we will use our explicit generators of $\mathcal{H}(2)^{\mathbb{Z}_3}$ to describe the commutative algebra $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$. In fact, $\mathbb{Z}_3$ acts on $R_{\mathcal{H}(2)}$ and there is a homomorphism

$$R_{\mathcal{H}(2)^{\mathbb{Z}_3}} \to (R_{\mathcal{H}(2)})^{\mathbb{Z}_3}.$$

This map is not an isomorphism, but we prove that it becomes an isomorphism at the level of reduced rings. This means that the functor which assigns to a vertex algebra $\mathcal{V}$ the reduced ring of $R_{\mathcal{V}}$ actually commutes with the G-invariant functor in this example. It raises the very interesting question whether this occurs in more generality.

1.2 Outline

♣ Chapter 2

In chapter 2 we review the definition of vertex algebras and some basic facts. We then define several fundamental examples of VOAs that are needed in this work. We present the axioms first as reference. We explore the meaning of this definition from the perspective of quantum commuting quantum operator algebras, CQOAs. This perspective highlights the operator product expansion and will allow us to show the relation to differential graded algebras.

We present a construction of the rank 1 Heisenberg VOA, $\mathcal{H}$. We then discuss a few important properties such as its strong finite generation. The definition of the rank n Heisenberg VOA, $\mathcal{H}(n)$, is then given.

This is followed by a construction of the Virasoro VOA. This includes some motivation from conformal field theory and discussion of the conformal structure.

We then move on to defining the Universal affine VOAs. We can then define the large k limit of these algebras. Furthermore, it turns out that the limit algebras obtained from the universal affine VOAs are free field algebras and this will allow us to extend the main result to many more algebras.

We conclude the chapter with a discussion of rationality and C_2-cofiniteness. We give their definitions and discuss their relation to orbifold theory.

♣ Chapter 3

Here we define orbifolds of VOAs. As part of the presentation of cosets we present the parafermion VOA. We review an important theorem from Dong, Li, and Mason which yields a decomposition of a VOA as irreducible modules for an orbifold. We then present the parafermion VOA and discuss some recent results in VOA orbifold theory.

♣ Chapter 4

This chapter briefly explains Zhu's associative algebra, $A(\mathcal{V})$. This is an algebra functorially connected to a VOA $\mathcal{V}$. In his thesis [Z], Zhu proved that the irreducible representations of $A(\mathcal{V})$ are in one-to-one correspondence with the irreducible highest weight representations of $\mathcal{V}$. This implies that if $A(\mathcal{V})$ is abelian then all the irreducible highest weight representations of $\mathcal{V}$ are one dimensional. We will make significant use of this result as it applies to a particular subalgebra.

♣ Chapter 5

Here we present the proof of an improved bound for the strong generators needed for orbifolds of the rank n Heisenberg VOA under its full automorphism group $O(n)$. It was previously shown that these orbifolds are strongly finitely generated. However, no explicit bound on the number of generators needed is yielded by the proof. We present a proof that this bound can be limited to an explicit finite set.

♣ Chapter 6

This chapter presents the proof of the second main result that for $n \geq 3$ the $\mathbb{Z}_n$ orbifold $\mathcal{H}(2)^{\mathbb{Z}_n}$ of the rank 2 Heisenberg VOA is strongly finitely generated by elements in weights

2, 3, 4, and 5 together with two generators in weights n, and $n+2$. We conclude with some remarks about the Zhu associative algebra $A(\mathcal{H}(2)^{\mathbb{Z}_n})$.

♣ **Chapter 7**

We give a few corollaries to the main result here. The first being the decomposition of the Heisenberg orbifolds for arbitrary finite abelian group. We also show that since the large k limit of the universal affine VOA $V^k(\mathfrak{sl}_2)$ is the rank 3 Heisenberg algebra given an action by $\mathbb{Z}_n$ which fixes the Cartan generator we can deduce it's generating type for generic k.

♣ **Chapter 8** Here we discuss the relationship between orbifolds and Zhu's commutative algebra. We prove that in the case $\mathcal{V} = \mathcal{H}(2)$ and $G = \mathbb{Z}_3$, the reduced rings of $R_{\mathcal{V}^G}$ and $(\mathcal{R}_{\mathcal{V}})^G$ are isomorphic. This provides a non-trivial example of a much broader conjectured isomorphism which is also discussed.

♣ **Appendix**

The first section of the appendix contains the lengthy formulas for generating the needed elements to show that the set of strong generators derived in the main result close under OPE.

The second section of the appendix contains a lengthy relation needed for determining the nilradical of Zhu's commutative algebra for $\mathcal{H}(2)^{\mathbb{Z}_3}$.

Chapter 2: Vertex Algebras

2.1 Axioms

Throughout this thesis, our base field will be taken to be $\mathbb{C}$. A vertex algebra is a vector space V over $\mathbb{C}$ together with

1. There is a distinguished endormorphism $\partial : V \to V$ called the translation endomorphism.

2. V contains a distinguished element $|0\rangle$ called the vacuum vector.

3. There exists a linear map $Y : V \to End(V)[[z, z^{-1}]]$ where for any vector $a, b \in V$ we have $Y(a, z)b$ has finitely many non-zero coefficients in the negative powers of z.

4. This map is typically written

$$Y(a, z)b = \sum_{n \in \mathbb{Z}} (a \circ_n b) z^{-n-1}$$

These data follow the following axioms:

1. (Identity): $Y(1, z)v = vz^0 = v$.

2. (Translation): $[\partial, Y(a, z)]v = \partial Y(a, z)v - Y(a, z)\partial v = (\frac{d}{dz} Y(a, z))v$.

3. Locality: $(z-w)^n Y(a, z)Y(b, w) = (z-w)^n Y(b, w)Y(a, z)$ for all $n \geq N$ for some positive N depending on a and b.

2.2 Basic Facts

Let V be a vector space. We will assume V is a vector space over $\mathbb{C}$, but many results in the field can be generalized further.

Let:

$V[z]$ denote the polynomials with coefficients in V

$V((z))$ denote the Laurent series over V

$V[[z^{\pm 1}]]$ denotes the space of formal distributions over V

If V is an algebra then the product of two formal distributions in two distinct formal variables is a well defined element in the space of V-valued formal distributions in both variables.

That is for

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in V[[z^{\pm 1}]] \tag{2.2.1}$$

and

$$b(w) = \sum_{m \in \mathbb{Z}} b(m) w^{-m-1} \in V[[w^{\pm 1}]] \tag{2.2.2}$$

we have

$$a(z)b(w) = \sum_{n,m \in \mathbb{Z}} a(n)b(m) z^{-n-1} w^{-m-1} \in V[[z^{\pm 1}, w^{\pm 1}]] \tag{2.2.3}$$

However, the product of two formal distributions in the same formal variable is not well defined since the coefficients do not converge algebraically. Thankfully the product of two Laurent series in one variable is well defined.

From here we take the approach of defining vertex algebras as the equivalent concept of a quantum commuting quantum operator algebra, or CQOA.

Definition 2.2.1. $QO(V)$, the Quantum Operators of V, are the set of linear maps from V to the formal Laurent series $V((z))$

Each element of $QO(V)$ can be uniquely represented by $End(V)$-valued formal distribution,

$$a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in End(V)[[z^{\pm1}]] \tag{2.2.4}$$

such that for any $v \in V$ $a(n)v = 0$ for $n >> 0$. That is for $a(z) \in QO(V)$ and $v \in V$ $\exists N \in \mathbb{Z}$ such at for all $n \geq N$ $a(n)v = 0$.

The product is then in general not well defined. However, the space $End(V)$ can be embedded in $QO(V)$ by the map

$$a \to \sum_{n\in\mathbb{Z}} \delta_{n,-1}a(n)z^{-n-1} \tag{2.2.5}$$

Here $\delta_{n,-1}$ is the Kronecker delta function.

One known way of extending the product in $End(V)$ to the rest of $QO(V)$ is the Wick product, also referred to as normal ordering.

Definition 2.2.2. The Wick product of two series is defined as

$$: a(z)b(w) := \sum_{n<0} a(n)z^{-n-1}b(w) + b(w)\sum_{n\geq 0} a(n)z^{-n-1} \tag{2.2.6}$$

On $QO(V)$ there is a set of nonassociative bilinear operations $\circ_n$, indexed by $n \in \mathbb{Z}$.

Definition 2.2.3. For $a(w)$ and $b(w) \in QO(V)$ the nth circle product is defined by

$$a(w) \circ_n b(w) := Res_z a(z)b(w)(z-w)^n - Res_z b(w)a(z)(w-z)^n \tag{2.2.7}$$

Here we follow the ordering convention that $(z-w)^n$ for $n < 0$ is understood to be the Laurent series expansion for the inverse of $(z-w)^{-n}$ in the region where $|z| > |w|$. $(w-z)^n$ is understood to be the expansion in the region where $|w| > |z|$.

These are an extension of the Wick product which corresponds to the $n = -1$ circle product.

The circle products are connected to the derivative by the formula

$$na(z) \circ_{-n-1} b(z) = (\partial a(z)) \circ_{-n} b(z) \tag{2.2.8}$$

Or by induction for negative n

$$n!a(z) \circ_{-n-1} b(z) =: (\partial^n a(z))b(z) : \tag{2.2.9}$$

The nonnegative circle products are related by the Operator Product Expansion (OPE)

$$a(z)b(w) = \sum_{n\geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} + : a(z)b(w) : \tag{2.2.10}$$

It is often written

$$a(z)b(w) \sim \sum_{n\geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} \tag{2.2.11}$$

meaning they are the equivalent modulo the normally ordered term $: a(z)b(w) :$ which is regular at $z = w$.

The following identities are very useful. They measure the nonassociativity and noncommutativity of the Wick product, and the failure of the positive circle products to be derivations of the Wick product respectively. Let a, b, c be elements of a vertex algebra and $n > 0$. Then

$$: (: ab :)c : - : abc := \sum_{k\geq 0} \frac{1}{(k+1)!} \big(: (\partial^{k+1} a)(b \circ_k c) : + (-1)^{|a||b|} : (\partial^{k+1} b)(a \circ_k c) : \big) \tag{2.2.12}$$

$$: ab : - : ba := \sum_{k \geq 0} \frac{(-1)^k}{(k+1)!} \partial^{k+1}(a \circ_k b) \tag{2.2.13}$$

$$a \circ_n (: bc :) - : (a \circ_n b)c : - : b(a \circ_n c) := \sum_{k=1}^{n} \binom{n}{k} (a \circ_{n-k} b) \circ_{k-1} c \tag{2.2.14}$$

$$(: ab :) \circ_n c = \sum_{k \geq 0} \frac{1}{k!} : (\partial^k a)(b \circ_{n+k} c) : + \sum_{k \geq 0} b \circ_{n-k-1} (a \circ_k c) \tag{2.2.15}$$

Note that the space $QO(V)$ is a nonassociative algebra with the operations $\circ_n$. These operations satisfy $1 \circ_n a = \delta_{n,-1} a$ for all n, and $a \circ_n 1 = \delta_{n,-1} a$ for $n \geq -1$. Therefore 1 behaves as a unit with respect to $\circ_{-1}$. A linear subspace $\mathcal{A} \subset QO(V)$ containing 1 which is closed under the circle products will be called a *quantum operator algebra*, or a QOA. Note that any QOA $\mathcal{A}$ is closed under ∂ since $\partial a = a \circ_{-2} 1$. Many formal notions can be defined now in a natural way: for example, a homomorphism of QOAs is just a linear map that sends 1 to 1 and preserves all circle products, and a module over as QOA $\mathcal{A}$ is a vector space M equipped with a QOA homomorphism $\mathcal{A} \to QO(M)$.

Let $\mathcal{A}$ be a QOA and let $S = \{a_i | \; i \in I\}$ be a subset of $\mathcal{A}$. We say that S *generates* $\mathcal{A}$ if every element $a \in \mathcal{A}$ can be written as a linear combination of (nonassociative) words in the letters a_i and $\circ_n$, for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* $\mathcal{A}$ if every $a \in \mathcal{A}$ can be written as a linear combination of words in the letters a_i, $\circ_n$ for $n < 0$. An equivalent statement is that $\mathcal{A}$ is spanned by the set of normally ordered monomials $\{: \partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z) : | \; i_1, \dots, i_m \in I, \; k_1, \dots, k_m \geq 0\}$.

We say that two elements $a, b \in QO(V)$ are *local* if

$$(z-w)^N [a(z), b(w)] = 0,$$

for some integer $N \geq 0$. Here $[,]$ denotes the bracket $a(z)b(w) - b(w)a(z)$ which is a well-defined element of $End\, V[[z, w, z^{-1}, w^{-1}]]$. Note that condition implies that $a \circ_n b = 0$ for $n \geq N$, so the OPE of $a(z)b(w)$ becomes a finite sum. A *commutative quantum operator algebra* (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is well known to be equivalent to the notion of a vertex algebra, and we use these notions interchangeably for the rest of this thesis.

2.3 Heisenberg VOA

The Heisenberg algebra, also known as the rank 1 free boson algebra, is defined as a one-dimensional central extension over the Laurent polynomials $\mathbb{C}[t, t^{-1}]$

Regard $\mathbb{C}[t, t^{-1}]$ as an abelian Lie algebra. Then define a one-dimensional central extension $\mathfrak{h} \coloneqq \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ by the bracket formula:

$$[A(t), B(t)] = \text{Res}_t A'(t)B(t)c \tag{2.3.1}$$

The element c is referred to as the central charge.

We now build the Fock module as follows. We first introduce a grading by degree such that $deg(t^n) = n$ and $deg(c) = 0$. This gives us the decomposition $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+$ where $\mathfrak{h}_- = \oplus_{n<0}\mathfrak{h}_n$, $\mathfrak{h}_+ = \oplus_{n\geq 0}\mathfrak{h}_n$, and $\mathfrak{h}_n$ is the subspace of degree n. We then define a one dimensional action of $\mathfrak{h}_+$ where t^n $n \geq 0$ acts by zero, and c acts by the identity. This is followed by inducing a representation to the rest of $\mathfrak{h}$.

This yields a module which is isomorphic to $V = \mathbb{C}[(t_{-1}), (t_{-2}), (t_{-3}), \ldots]$ where $t^n \in \mathfrak{h}_-$ act by multiplication by (t_n). Then we consider a t^n $n \geq 0$ multiplying the left of a term with elements from $\mathfrak{h}_-$. Using the bracket relation we then move the positive t^n past the negative elements until it acts by zero on V. By induction we can see for $n \geq 0$ t^n acts by $n\frac{\partial}{\partial t^{-n}}$.

We now construct the generating field by letting $h(n)$ represent t^n thought of as an endomorphism of V so that

$$h(z) = Y(h, z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} \tag{2.3.2}$$

From this it is a standard exercise to calculate the OPE of $h(z)$ with it self as being

$$h(z)h(w) \sim \frac{1}{(z-w)^2}. \tag{2.3.3}$$

The rank n Heisenberg system is the tensor product of n rank one Heisenberg algebras. One can again choose a generating set $h_1, h_2, \cdots h_n$, which satisfy

$$h_i(z)h_j(w) \sim \delta_{i,j} \frac{1}{(z-w)^2}. \tag{2.3.4}$$

We will discuss several more properties and structure displayed by the Heisenberg algebras as they are introduced.

2.4 Virasoro VOA

The Virasoro vertex operator algebra plays an important central role in the theory. In two dimensional conformal field theory the conformal transformations are given by holomorphic functions. The infinitesimal conformal maps are given by two copies of the Witt algebra spanned by $\{L_n = -t^{n+1}\frac{\partial}{\partial t} : n \in \mathbb{Z}\}$. This is equivalently the algebra of continuous derivations, $Der\mathcal{K}$, of the loop algebra $\mathcal{K} = \mathbb{C}[t, t^{-1}]$. Then due to a quantum anomaly the symmetry algebra is given by two copies of the central extensions of the Witt algebras. The unique central extension of the Witt algebra is the Virasoro algebra Vir. It is spanned by the L_n and the element κ also referred to as the central charge. These satisfy the bracket relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\kappa}{12}(m^3 - m)\delta_{n+m,0} \tag{2.4.1}$$

The scaling factor $\frac{1}{12}$ is a convention from physics. We give Vir the $\mathbb{Z}$-grading where L_n is declared to have degree n, and κ has degree 0. Define $\text{Vir}_{\geq -1}$ be the span of the elements of degree $n \geq -1$, and for each $c \in \mathbb{C}$, let C^c be the 1-dimensional $\text{Vir}_{\geq -1}$-module on which L_n acts by zero for $n \geq -1$ and κ acts by $c \cdot \text{id}$. Let N^c be the induced modules $U(\text{Vir}) \otimes_{U(\text{Vir}_{\geq -1})} C^c$. We then have the field

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in End\ N^c[[z, z^{-1}]]. \tag{2.4.2}$$

A calculation using only the Lie bracket relations (2.4.1) then shows that $L(z)$ satisfies the OPE

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1} \tag{2.4.3}$$

Therefore $L(z)$ is local with itself, and hence it generates a vertex algebra known as the universal *Virasoro vertex algebra* of central charge c.

A quantum field theory is *conformal* if its underlying space of states is given by eigenspaces of the operator L_0. Together with theory of weight modules this leads us to a very important definition and much of the mathematical significance of the Virasoro vertex algebra.

A *Vertex Operator Algebra*, VOA, is a vertex algebra that comes equipped with a conformal vector ω, such that

$$\omega(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega(n) z^{-n-1} = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-1} \tag{2.4.4}$$

is the weight two Virasoro field satisfying the following properties:

- $\omega(1) = L_0$ acts semisimply on the underlying vector space V with integral eigenvalues which are bounded below.

- Under the weight grading given by the eigenvalues of L_0 each circle product is homogeneous with $u \circ_n v$ having weight $wt(u) + wt(v) - n - 1$ for weight homogeneous elements u and v.

- $\omega(0) = L_{-1} = T$ the translation operator.

It is this added structure assumed about the whole algebra which underlies much of the mathematical importance of the Virasoro VOA. For example, it is well known that the Heisenberg algebra $\mathcal{H}$ has a conformal vector $L(z) = \frac{1}{2} : h(z)h(z) :$ which has central charge 1. Also, it satisfies the OPE

$$L(z)h(w) \sim h(w)(z-w)^{-2} + \partial h(w)(z-w)^{-1}.$$

We say that $h(z)$ is primary of conformal weight 1; more generally, a field α in a vertex algebra $\mathcal{A}$ is primary of conformal weight d if

$$L(z)a(w) \sim d\alpha(w)(z-w)^{-2} + \partial\alpha(w)(z-w)^{-1}.$$

Note on notation: In order to avoid confusion with lengthy equations we adopt the convention of using $L(n) = L_{n-1}$ for the Virasoro field so as to be consistent with the n^{th} circle product. When confusion arises the symbol $\omega(n)$ is commonly used for $L(n)$. We use $L(n)$ since we have no further use for L_n in this thesis.

As a matter of notation, we say that a vertex operator algebra $\mathcal{V}$ is of type $\mathcal{W}(n_1^{d_1}, n_2^{d_2}, \dots, n_r^{d_r})$ if it has a minimal strong generating set consisting of d_i fields in weight n_i for $i = 1, \dots, r$.

2.5 Universal Affine Vertex Algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, with dual Coxeter number $h^\vee$. We equip $\mathfrak{g}$ with its standard bilinear form $(-|-)$ given by $(\zeta|\eta) = \frac{1}{2h^\vee}\langle \xi, \eta \rangle$, where

$\langle \zeta, \eta \rangle = \text{Tr}(\text{ad } \zeta)(\text{ad } \eta)$ denotes the Killing form. The *affine Kac-Moody Algebra* $\hat{\mathfrak{g}}$ is defined as the one-dimensional central extension of the loop algebra on $\mathfrak{g}$.

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\kappa \tag{2.5.1}$$

It is spanned by $\{\zeta \otimes t^n : \zeta \in \mathfrak{g}, n \in \mathbb{Z}\}$ and the central charge κ, and these generators satisfy the bracket relations

$$[\zeta \otimes t^n, \eta \otimes t^m] = [\zeta, \eta]_{\mathfrak{g}} \otimes t^{n+m} + n(\zeta|\eta)\delta_{n+m,0}\kappa, \quad [\zeta \otimes t^n, \kappa] = 0 \tag{2.5.2}$$

For simplicity, we write ζt^n instead of $\zeta \otimes t^n$. Note that $\hat{\mathfrak{g}}$ is equipped with a $\mathbb{Z}$-grading defined by deg $\zeta \otimes t^n = n$, and $deg(\kappa) = 0$. Let $\hat{\mathfrak{g}}_{\geq 0} \subset \hat{\mathfrak{g}}$ be the Lie subalgebra of elements of non-negative degree, and let C^k be the 1-dimensional $\hat{\mathfrak{g}}_{\geq 0}$-module on which ζt^n acts by zero for all $n \geq 0$, and κ acts by $k \cdot \text{id}$. Now consider the induced $\hat{\mathfrak{g}}$-module

$$N^k = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} C^k,$$

which is clearly graded by the non-positive integers For each $\zeta \in \mathfrak{g}$, let $X^\zeta(n)$ denote the linear operator on N^k representing ζt^n, and consider the fields

$$X^\zeta(z) = \sum_{n \in \mathbb{Z}} X^\zeta(n) z^{-n-1} \in \text{End } N^k[[z, z^{-1}]]. \tag{2.5.3}$$

It is straightforward to verify that $X^\zeta(z) \in \text{QO}(N^k)$ and that these fields satisfy OPE relations

$$X^\zeta(z) X^\eta(w) \sim k(\zeta|\eta)(z-w)^{-2} + X^{[\zeta,\eta]}(w)(z-w)^{-1}. \tag{2.5.4}$$

Therefore these fields generate a vertex algebra $V^k(\mathfrak{g})$ which is known as the universal affine vertex algebra associated to $\mathfrak{g}$ at level k.

It is well known that $V^k(\mathfrak{g})$ is a VOA as long as $k \neq -h^\vee$. It has a conformal vector given by the Sugawara construction. Given a basis $\{J^a\}$ for $\mathfrak{g}$, $\{J_a\}$ its dual basis with respect to $(-|-)$, the conformal vector is given by

$$L(z) = \frac{1}{2(k+h^\vee)} \sum_a : J^a J_a : \tag{2.5.5}$$

The level $k = -h^\vee$ is referred to as the *critical level*. At this level, there is no comformal structure, and the algebra has a large center known as the Feigin-Frenkel center which is isomorphic to the polynomial algebra generated by the center of $U(\mathfrak{g})$.

We should note at this point that for any simple $\mathfrak{g}$, we can rescale the generating fields X^ζ by $\frac{1}{\sqrt{k}}$ for $k \neq 0$. This yields an isomorphic vertex algebra with the OPE relations

$$X^\zeta(z)X^\eta(w) \sim (\zeta|\eta)(z-w)^{-2} + \frac{X^{[\zeta,\eta]}(w)}{\sqrt{k}}(z-w)^{-1} \tag{2.5.6}$$

Taking the limit $k \to \infty$ now has the effect of eliminating the first order pole. This is referred to as the *large k limit*, and it is isomorphic to the Heisenberg algebra $\mathcal{H}(d)$ where $d = \dim \mathfrak{g}$.

Later, we will consider the special case $V^k(\mathfrak{sl}_2)$ with $\{e, f, h\}$ the usual basis for $\mathfrak{sl}_2$. In this case, the large k limit is just the rank 3 Heisenberg algebra $\mathcal{H}(3)$ with the generators $X^h(z)$, $X^{e+f}(z)$, and $X^{e-f}(z)$.

2.6 Rationality and C_2-cofiniteness

Given a vertex algebra $\mathcal{V}$, define

$$C(\mathcal{V}) = \text{Span}\{a_{(-2)}b |\, a, b \in \mathcal{V}\}, \qquad R_{\mathcal{V}} = \mathcal{V}/C(\mathcal{V}). \tag{2.6.1}$$

It follows immediately from the formulas (2.2.12) and (2.2.13) which measure the noncommutativity and nonassociativity of the Wick product, that $R_{\mathcal{V}}$ is a commutative, associative algebra with product induced by the Wick product [Z]. Note also that $\mathcal{V}$ is graded by conformal weight, $R_{\mathcal{V}}$ always inherits this grading. If $\{\alpha_i|\, i \in I\}$ is a strong generating set for $\mathcal{V}$, the images of these fields in $R_{\mathcal{V}}$ will generate $R_{\mathcal{V}}$ as a ring. In particular, $R_{\mathcal{V}}$ is finitely generated if and only if $\mathcal{V}$ is strongly finitely generated.

A vertex algebra $\mathcal{V}$ is called C_2-cofinite if $R_{\mathcal{V}}$ is finite-dimensional as a vector space over $\mathbb{C}$. If $\mathcal{V}$ is strongly finitely generated, it is clearly C_2-cofinite if and only if every element of $R_{\mathcal{V}}$ is nilpotent, and in particular it has Krull dimension zero.

A vertex algebra $\mathcal{V}$ is called *rational* if every $\mathbb{Z}_{\geq 0}$-graded $\mathcal{V}$-module is completely reducible. It is expected that rational vertex algebras are C_2-cofinite, and this is an important open question in the subject. The C_2-cofiniteness condition was introduced by Zhu [Z], and has the important consequence that $\mathcal{V}$ has finitely many simple $\mathbb{Z}_{\geq 0}$-graded modules. It also plays it a fundamental notion in his proof of modular invariance of characters of rational vertex algebras which satisfy this condition. On the other hand, C_2-cofiniteness does not imply rationality; there are several known examples such as the triplet vertex algebras that are C_2-cofinite but are not rational [AM].

Chapter 3: Orbifolds and Cosets

3.1 Orbifolds

Definition 3.1.1. Given a vertex algebra $\mathcal{V}$ and a group of automorphisms G, the invariant subalgebra $\mathcal{V}^G \subset \mathcal{V}$ is always a new vertex algebra, and is called a an *orbifold* of $\mathcal{V}$.

This is sometimes referred to as a fixed point subalgebra. Also the term orbifold is sometimes used to refer to a different construction where one adjoins twisted modules and then takes the fixed points under certain automorphisms. We adopt the convention used in [FBZ] as defined above. We will always assume that $\mathcal{V}$ has conformal vector L which is invariant under G, so that V^G has the same conformal vector, and hence is conformally embedded in $\mathcal{V}$.

The orbifold construction was introduced originally in the physics literature; see for example [DVVV, DHVWI, DHVWII]. It was also essential in the construction of the Moonshine module $V^\natural$ given in [FLM], since $V^\natural$ is an extension of the $\mathbb{Z}/2\mathbb{Z}$-orbifold of the vertex algebra associated to the Leech lattice.

Recall that a vertex algebra $\mathcal{V}$ is called *rational* if every $\mathbb{Z}_{\geq 0}$-graded $\mathcal{V}$-module is completely reducible. It is a well-known conjecture that if $\mathcal{V}$ is C_2-cofinite and rational, then $\mathcal{V}^G$ inherits these properties for any finite group G of automorphisms of $\mathcal{V}$. This was proven in 2016 by Carnahan and Miyamoto when G is a cyclic group [CM], and hence it holds also when G is solvable. In a very recent paper [McR], McRae has shown that for any finite groups G, if $\mathcal{V}$ is rational and C_2-cofinite, the rationality of $\mathcal{V}^G$ would follow if the C_2-cofiniteness of $\mathcal{V}^G$ were known [McR].

Another property which can be inherited by orbifolds is strong finite generation. If $\mathcal{V}$ is strongly finitely generated and G is a reductive group, one can ask whether $\mathcal{V}^G$ is also strongly finitely generated. This is a natural analogue of Hilbert's theorem that says that if G is any reductive group and V is a finite-dimensional G-module, the ring of invariant polynomial functions $\mathbb{C}[V]^G$ is finitely generated [HI, HII]. This has been called the vertex algebra Hilbert problem; see [L2] as well as the recent survey by Lian and Linshaw [LL]. In fact, the answer to this question in general is no. As explained in [LL], it is instructive to compare the case of the the rank one Heisenberg algebra $\mathcal{H}(1)$ with its $\mathbb{Z}_2$-action, to the case of the degenerate rank one Heisenberg algebra with its $\mathbb{Z}_2$ action. It is a well known theorem of Dong and Nagatomo that $\mathcal{H}(1)^{\mathbb{Z}_2}$ is strongly generated by the Virasoro field together with one additional weight 4 primary field, but for the degenerate Heisenberg algebra, even though it is linearly isomorphic to $\mathcal{H}(1)^{\mathbb{Z}_2}$ it is not strongly finitely generated.

In a series of papers, Linshaw [L1, L2, L3, L4, L5], has given a positive answer to the vertex algebra Hilbert problem whenever V is a free field algebra and G is a general reductive group. Here a free field algebra is a simple vertex (super)algebra such that the nontrivial terms in the OPE relations among the generators only involving the vacuum. There are four standard families of free field algebras, namely, the Heisenberg algebra $\mathcal{H}(n)$, free fermion algebra $\mathcal{F}(n)$, $\beta\gamma$-system $\mathcal{S}(n)$, and symplectic fermion algebra $\mathcal{A}(n)$. These are characterized by whether the generators are even or odd, and whether their full automorphism group is a symplectic or orthogonal group. There are also similar algebras where the poles are of higher order, and we can take arbitrary tensor products of finitely many such algebras.

The proof of the vertex algebra Hilbert theorem for free field algebras has the following outline. One begins with the case where $\mathcal{V}$ is one of the above standard algebras $\mathcal{H}(n)$, $\mathcal{F}(m)$, $\mathcal{S}(r)$, $\mathcal{A}(s)$, and G is the full automorphism group, which is either $O(n)$ or $Sp(2n)$. Then $\text{gr}(\mathcal{V})$ is either a symmetric or exterior algebra on $\bigoplus_{i\geq 0} V_i$, where each V_i is the standard Aut $\mathcal{V}$-module. The generators for this kind of invariant rings are given by Weyl's first fundamental theorem of invariant theory (FFT) for the orthogonal and symplectic groups [W]. From this, we get infinite generating sets for $\text{gr}(\mathcal{V})^{\text{Aut}\,\mathcal{V}}$, and hence infinite strong generating sets for $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$. The relations among these generators are given by Weyl's second fundamental theorem of invariant theory (SFT), and using these relations we can eliminate all but a finite subset of the generators. It turns out that in all cases, $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$ is a quotient of the universal two-parameter even-spin $\mathcal{W}_{\infty}$-algebra constructed by Kanade and Linshaw [KL]. All quotients this algebra have abelian Zhu algebras. It follows that all irreducible ordinary $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$-modules have one-dimensional top component, and are parametrized by the variety Specm $A(\mathcal{V}^{\text{Aut}\,\mathcal{V}})$ whose coordinate ring is $A(\mathcal{V}^{\text{Aut}\,\mathcal{V}})$.

The next step is to consider an arbitrary reductive group $G \subseteq$ Aut $\mathcal{V}$. One can then decompose $\mathcal{V}^G$ as a module over $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$. Typically there will be infinitely many irreducible $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$-modules in this decomposition. However, Weyl's theorem on polarizations [W] implies that $\mathcal{V}^G$ has an infinite strong generating set that lies in the direct sum of finitely many of these modules. In particular, $\mathcal{V}^G$ is finitely generated as a module over $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$.

The final step is to show that all the irreducible $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$-modules appearing in $\mathcal{V}$ have the C_1-cofiniteness property; together with the strong finite generation of $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$ and the fact that $\mathcal{V}^G$ is finitely generated as a module over $\mathcal{V}^{\text{Aut}\,\mathcal{V}}$, the result follows.

The vertex algebra Hilbert problem has been solved affirmatively for a large class of vertex algebra that include all affine vertex (super)algebras $V^k(\mathfrak{g})$ and all $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra and $f \in \mathfrak{g}$ is an even nilpotent [CL3]. These vertex algebra have the property that their large level limits are isomorphic to free field algebras, and the result can be deduced from the Hilbert problem for free field algebras.

3.2 Cosets

The coset construction is another standard way to create new vertex algebras from old ones. Given a vertex algebra $\mathcal{V}$ and a vertex subalgebra $\mathcal{U} \subset \mathcal{V}$, the coset $\text{Com}(\mathcal{U}, \mathcal{V})$ is the set of elements of $\mathcal{V}$ which commute with all elements of $\mathcal{U}$, that is,

$$\text{Com}(\mathcal{U}, \mathcal{V}) = \{v \in \mathcal{V} |\, [u(z), v(w)] = 0, \text{ for all } u \in \mathcal{U}\}.$$

Equivalently, $v \in \text{Com}(\mathcal{U}, \mathcal{V})$ if an only if $u \circ_n v = 0$ for all $u \in \mathcal{U}$ and $n \geq 0$. This concept was first formalized by Frenkel and Zhu in [FZ]. Note that if $\mathcal{V}$ has conformal vector $L^{\mathcal{V}}$ and $\mathcal{U}$ has conformal vector $L^{\mathcal{U}}$, then $\text{Com}(\mathcal{U}, \mathcal{V})$ has conformal vector $L^{\mathcal{C}} = L^{\mathcal{V}} - L^{\mathcal{U}}$. In particular, $\mathcal{U} \otimes \text{Com}(\mathcal{U}, \mathcal{V})$ is conformally embedded in $\mathcal{V}$.

An important family of examples are the *parafermion algebras* investigated by Dong, Lam, and Yamada [DLY]. For any simple Lie algebra $\mathfrak{g}$ the parafermion algebra $N^k(\mathfrak{g})$ is defined to be the coset $\text{Com}(\mathcal{H}, V^k(\mathfrak{g}))$, where $\mathcal{H}$ is the Heisenberg subalgebra generated by the fields corresponding to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, $N^k(\mathfrak{sl}_2)$ was shown in [DLY] to be of type $\mathcal{W}(2, 3, 4, 5)$, and to be generated (but not strongly) by the weights 2 and 3 fields. The large k limit of $V^k(\mathfrak{sl}_2)$ is $\mathcal{H}(3)$, and it follows from the general result Theorem 6.10 of [CL2] that the large k limit of $N^k(\mathfrak{sl}_2)$ is just the orbifold $\mathcal{H}(2)^{SO(2)}$.

In general, it is difficult to understand the structure of coset vertex algebras, but there are powerful approaches in the case where $\mathcal{U}$ is an affine vertex algebra or a Heisenberg algebra. In fact, there is a general setting which includes the case where $\mathcal{V}$ is the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ associated to any simple Lie algebra $\mathfrak{g}$ and any even nilpotent element $f \in \mathfrak{g}$, such that the affine coset has large level limit isomorphic to a certain orbifold of a free field algebra. In these cases, the strong finite generation of such cosets is then a corollary of the Hilbert problem for free field algebras. The explicit description of the affine cosets of large family of $\mathcal{W}$-algebras has recently led to the solution of the Gaiotto-Rapcak triality conjectures in [CL3, CL4]; this is the statement that the affine cosets of three different $\mathcal{W}$-algebras are isomorphic.

Despite some recent success in determining the structure of orbifolds and cosets using invariant theory, there are still many deficiencies in this theory that require further study. One gap in the theory is that the unlike the case of $\mathcal{F}(n)$, $\mathcal{S}(n)$, and $\mathcal{A}(n)$, the minimal strong generating type of $\mathcal{H}(n)^{O(n)}$ is still an open problem for $n \geq 7$. While it is conjectured to be of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$ and it is known that this is a lower bound for the number of fields needed, there is no known upper bound. In Chapter 5, we will remedy this situation in by proving that $\mathcal{H}(7)^{O(7)}$ is of type $\mathcal{W}(2, 4, \dots, 70)$, and showing that for all $n \geq 8$, $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \dots, 2N)$ for $2N \leq 2n^2 + 4n$.

The other deficiency in the theory is that even though strong finite generation of $\mathcal{V}^G$ is known when $\mathcal{V}$ is a free field algebra and G is reductive, the proof is nonconstructive. In fact, aside from a few isolated exceptions, it is only possible to find explicit minimal strong generating sets when G is one of the classical groups and $\mathcal{V}$ is linearly isomorphic to the polynomial ring on many copies of the standard representation of V. Note that this includes the case where $G = \mathbb{Z}_2$ acting by -1, which we interpret as the orthogonal group $O(1)$ acting on its standard module. It is a very nontrivial task to give explicit minimal strong generating sets for orbifolds where the action of G is not of this kind. The main result in this thesis, namely, the explicit minimal strong generating set for $\mathcal{H}(2)^{\mathbb{Z}_n}$ for all n, is an infinite family of examples that are not covered by existing results and methods. Finally, we point out that an explicit strong generating set for a vertex algebra is very useful. It gives generators for both Zhu's associative algebra $A(\mathcal{V})$ and Zhu's commutative algebra $R_{\mathcal{V}}$. In the case of $\mathcal{H}(2)^{\mathbb{Z}_n}$, we will give some information about the Zhu algebra and how it can be used to deduce some features of the positive-energy modules over $\mathcal{H}(2)^{\mathbb{Z}_n}$.

Finally, in the case $n = 3$, we will use our explicit generators of $\mathcal{H}(2)^{\mathbb{Z}_3}$ to describe the commutative algebra $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$. In fact, $\mathbb{Z}_3$ acts on $R_{\mathcal{H}(2)}$ and there is a homomorphism

$$R_{\mathcal{H}(2)^{\mathbb{Z}_3}} \to (R_{\mathcal{H}(2)})^{\mathbb{Z}_3}.$$

This map is certainly not an isomorphism since it has a big kernel, but we prove that it becomes an isomorphism at the level of reduced rings. This means that the functor which assigns to a vertex algebra $\mathcal{V}$ the reduced ring of $R_{\mathcal{V}}$ actually commutes with the G-invariant functor in this example. It raises the very interesting question whether this occurs in more generality.

3.3 Dong, Li, Mason Theorem

In a 1996 paper [DLM], Chongying Dong, Haisheng Li, and Geoffrey Mason established the following important theorem.

Theorem 3.3.1. Let $\mathcal{V}$ be a simple vertex algebra let G be a compact group acting faithfully on $\mathcal{V}$ by vertex algebra automorphisms. Then $\mathcal{V}$ has a Schur-Weyl type decomposition as a $G \times \mathcal{V}^G$-module

$$V = \bigoplus_{\lambda \in I} W_\lambda \otimes V_\lambda \tag{3.3.1}$$

Here the set I indexes the finite-dimensional irreducible representations W_λ of G, and the multiplicity spaces V_λ are nonzero, inequivalent, irreducible $\mathcal{V}^G$-modules.

In this thesis, we will apply this theorem only in the case where G is abelian. In this case, all modules W_λ are therefore one-dimensional, so $W_\lambda \otimes V_\lambda \cong V_\lambda$. One case where we apply this result is when $\mathcal{V}$ is the rank 2 Heisenberg algebra $\mathcal{H}(2)$ and $G = SO(2)$. In this case, it is known that the Zhu algebra of $\mathcal{H}(2)^{SO(2)}$ is abelian, so all the $\mathcal{H}(2)^{SO(2)}$ modules have one-dimensional top component.

Chapter 4: Zhu's Associative Algebra

An important tool in the study of the representation theory of vertex operator algebras is the Zhu associative algebra first introduced by Zhu in [Z]. Given a vertex operator algebra $\mathcal{W}$ with a conformal weight grading $\mathcal{W} = \bigoplus_{n\geq 0} \mathcal{W}_n$, the Zhu functor surjectively maps $\mathcal{W}$ to an associative algebra $A(\mathcal{W})$. If $a(z)$ is homogeneous of weight m the product in $A(\mathcal{W})$ is given in terms of the circle products as

$$a(z) * b(w) = Res_z\Bigg(a(z)\frac{(z+1)^m}{z}b(w)\Bigg) \tag{4.0.1}$$

Or equivalently

$$a(z) * b(w) = \sum_{n=-1}^{m-1} \binom{m}{n+1} a \circ_n b \tag{4.0.2}$$

This product is extended linearly to a bilinear operation $\mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$. Let $O(\mathcal{W})$ be the subspace spanned by elements of the form

$$a(z) * b(w) = Res_z\Bigg(a(z)\frac{(z+1)^m}{z^2}b(w)\Bigg) \tag{4.0.3}$$

$A(\mathcal{W})$ is then defined to be the quotient $\mathcal{W}/O(\mathcal{W})$. There are many useful facts established in [Z]. First that the conformal element L is always in the center of $A(\mathcal{W})$. Modulo $O(\mathcal{W})$ we can eliminate the formal derivative using the formula

$$\partial a(z) \cong (-m)a(z) \text{ mod } O(\mathcal{W}) \tag{4.0.4}$$

Where m is the conformal weight of $a(z)$.

The commutator of two elements is given by

$$a(z) * b(w) - b(w) * a(z) \cong Res_z\bigg(a(z)(z+1)^{m-1}b(w)\bigg) \bmod O(\mathcal{W}) \qquad (4.0.5)$$

If a vertex algebra is strongly finitely generated then the images of a set of strong generators generate the Zhu associative algebra.

Much of the utility of the Zhu associative algebra comes from the one-to-one correspondence between irreducible representations of the Zhu associative algebra and irreducible positive energy representations of the VOA. That is an irreducible representation of the Zhu associative algebra has a natural action on the top weight space of the VOA which uniquely induces an irreducible positive energy representation of the VOA.

4.1 Examples of Zhu associative algebras

The above facts allow us to easily describe the Zhu associative algebra for the rank one Heisenberg algebra $A(\mathcal{H}(1))$. Since $\mathcal{H}(1)$ is strongly finitely generated by the single weight one field h. We can also inductively eliminate derivatives using 4.0.4. Therefore $A(\mathcal{H}(1))$ is abelian and isomorphic to $\mathbb{C}[x]$. This in turn implies that all the irreducible positive energy representations are one dimensional and are parametrized by a complex parameter λ.

The story is similar for the Virasoro VOA. It is again strongly finitely generated by the single weight 2 field $L(z)$. Hence it is clear that $A(Vir)$ is abelian and isomorphic to $\mathbb{C}[x]$. We further note that Zhu proved that this field is always in the center of any Zhu associative algebra.

The Zhu associative algebra for the parafermion algebra $N^k(\mathfrak{sl}_2)$ has also been previously studied [DLY]. Since $N^k(\mathfrak{sl}_2)$ is of type $\mathcal{W}(2,3,4,5)$, the corresponding fields generate $A(N^k(\mathfrak{sl}_2))$, and it was shown in [DLY] to be a commutative algebra. Since $\mathcal{H}(2)^{SO(2)}$ is the large k limit of $N^k(\mathfrak{sl}_2)$, it is easy to see that $A(\mathcal{H}(2)^{SO(2)})$ is abelian;

alternatively this can be checked by direct calculation using the generating fields and their OPEs. In particular, all irreducible $A(\mathcal{H}(2)^{SO(2)})$-modules are one-dimensional.

Chapter 5: Bound on Strong Generators for $\mathcal{H}(n)^{O(n)}$

Recall that there is a missing result in the invariant theory of free field algebras, namely the minimal strong generating type of the orbifold $\mathcal{H}(n)^{O(n)}$ of the rank n Heisenberg algebra under its full automorphism group $O(n)$. In [L3], it was conjectured that this algebra is of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$ and in [L4] it was shown that it is of type $\mathcal{W}(2, 4, \dots, 2N)$ for some N satisfying $2N \geq n^2 + 3n$, and it was shown that $2N = n^2 + 3n$ for $n \leq 6$. Unfortunately, the proof of strong finite generation given in [L4] is nonconstructive and it is not clear how to even find an upper bound for N based on this proof. We will extend the above result to the case $n = 7$, and we will also give a bound for N which is quadratic in n:

Theorem 5.0.1. For all $n \geq 8$, $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \dots, 2N)$ where $2N \leq 2n^2+4n$.

First, we review the setting and main results of [L3, L4]. Recall that $\mathcal{H}(n)$ has a good increasing filtration in the sense of Li, that is $\mathcal{H}(n) = \bigcup_{d\geq 0} \mathcal{H}(n)_{(d)}$ where $\mathcal{H}(n)_{(d)}$ is the span of normally ordered monomials in the generators $\alpha^1, \dots, \alpha^d$ and their derivatives, of total length at most d. The associated graded algebra $gr(\mathcal{H}(n))$ is just the differential polynomial algebra generated by $\partial^j \alpha^i$ for $j \geq 0$ and $i = 1, \dots, n$. In particular, $\mathcal{H}(n) \cong \text{gr}(\mathcal{H}(n))$ as vector spaces, and

$$\text{gr}(\mathcal{H}(n)) \cong \text{Sym} \bigoplus_{j\geq 0} V_j$$

as differential algebras. Here V_j is spanned by $\{x_{i,j}|\ i = 1, \dots, n\}$ where $x_{i,j}$ corresponds to $\partial^j \alpha^i$. In particular, $V_j \cong \mathbb{C}^n$ as $O(n)$-modules for all $j \geq 0$. We have a linear isomorphism $\mathcal{H}(n)^{O(n)} \cong \text{gr}(\mathcal{H}(n)^{O(n)})$, and isomorphisms of differential algebras

$$\text{gr}(\mathcal{H}(n)^{O(n)}) \cong (\text{gr}(\mathcal{H}(n)))^{O(n)} \cong (\text{Sym} \bigoplus_{j\geq 0} V_j)^{O(n)}. \tag{5.0.1}$$

Next, we recall Weyl's first and second fundamental theorems of invariant theory for the standard representation of $O(n)$ (Theorems 2.9A and 2.17A of [W]); this gives the generators and relations for the ring $(\text{Sym} \bigoplus_{j\geq 0} V_j)^{O(n)}$.

Theorem 5.0.2. For $j \geq 0$, let V_j be the copy of the standard $O(n)$-module $\mathbb{C}^n$ with orthonormal basis $\{x_{i,j}|\ i = 1, \dots, n\}$. Let $R = (\text{Sym} \bigoplus_{j\geq 0} V_j)^{O(n)}$ be the ring of invariants. Then R is generated by

$$q_{a,b} = \sum_{i=1}^{n} x_{i,a} x_{i,b}, \qquad 0 \leq a \leq b. \tag{5.0.2}$$

For $a > b$, define $q_{a,b} = q_{b,a}$, and let $\{Q_{a,b}|\ a, b \geq 0\}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$ and no other relations. The kernel of the map $\mathbb{C}[Q_{a,b}] \to (\text{Sym} \bigoplus_{j\geq 0} V_j)^{O(n)}$ sending $Q_{a,b} \mapsto q_{a,b}$ is generated by the $(n+1) \times (n+1)$ determinants

$$d_{I,J} = \begin{bmatrix} Q_{i_0,j_0} & \cdots & Q_{i_0,j_n} \\ \vdots & & \vdots \\ Q_{i_n,j_0} & \cdots & Q_{i_n,j_n} \end{bmatrix}. \tag{5.0.3}$$

Here $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$ are lists of integers satisfying

$$0 \leq i_0 < \cdots < i_n, \qquad 0 \leq j_0 < \cdots < j_n. \tag{5.0.4}$$

Corresponding to the generators $q_{a,b} \in R$ are the fields

$$\omega_{a,b} = \sum_{i=1}^{n} :\partial^a \alpha^i \partial^b \alpha^i: \in \mathcal{H}(n)^{O(n)}, \qquad 0 \leq a \leq b,$$

which are easily seen to strongly generate $\mathcal{H}(n)^{O(n)}$. Not all of these fields are necessary because of the differential algebraic relations $\partial \omega_{a,b} = \omega_{a+1,b} + \omega_{a,b+1}$. It is easy to check that a minimal generating set for R as a differential algebra is $\{\partial^k q_{0,2m} | \, k, m \geq 0\}$, so that the smaller set $\{\omega_{0,2m} | \, m \geq 0\}$ also strongly generates $\mathcal{H}(n)^{O(n)}$. Following [L3], we use the notation

$$j^{2m} = \omega_{0,2m}, \qquad m \geq 0.$$

As in [L3, L4], it is convenient to regard $\mathcal{H}(n)^{O(n)}$ as the quotient of a vertex algebra $\mathcal{V}_n$ under surjective homomorphism

$$\pi_n : \mathcal{V}_n \to \mathcal{H}(n)^{O(n)}.$$

Here $\mathcal{V}_n$ is the vertex algebra freely generated by fields J^{2m} for $m \geq 0$ satisfying the same OPE relations as the j^{2m}'s. Also, we can change variables to give an alternative strong generating set $\{\Omega_{a,b} | \, 0 \leq a \leq b\}$ for $\mathcal{V}_n$ such that $\pi_n(\Omega_{a,b}) = \omega_{a,b}$. This is apparent because the sets $\{\omega_{a,b} | \, 0 \leq a \leq b\}$ and $\{\partial^k j^{2m} | \, k, m \geq 0\}$ span the same vector space, and we define $\Omega_{a,b}$ as a linear combination of the fields $\partial^j J^{2m}$ using the same formulas. Note also that $\mathcal{V}_n$ has a good increasing filtration such that $(\mathcal{V}_n)_{(2k)}$ is spanned by iterated Wick products of the generators $\Omega_{a,b}$, of length at most k, and $(\mathcal{V}_n)_{(2k+1)} = (\mathcal{V}_n)_{(2k)}$.

Let $\mathcal{I}_n$ denote the kernel of the map π_n. Let $D^{2n+2}_{I,J} \in (\mathcal{V}_n)_{(2n+2)}$ be some *normal ordering* of $d_{I,J}$, that is, a normally ordered polynomial obtained by replacing each $Q_{a,b}$ with $\Omega_{a,b}$, and replacing ordinary products with iterated Wick products. Then $\pi_n(D^{2n+2}_{I,J})$ lies in the lower filtered component $(\mathcal{H}(n)^{O(n)})_{(2n)}$. As explained in [L3, L4], we can find a sequence of quantum corrections $D^{2k}_{I,J}$ for $k = 1, \dots, n$, which are homogeneous, normally ordered polynomials of degree k in the variables $\Omega_{a,b}$, such that

$$D_{I,J} = \sum_{k=1}^{n+1} D^{2k}_{I,J}$$

lies in $\mathcal{I}_n$. Since the corresponding elements $d_{I,J}$ generate the classical relations among the generators of $R = (\text{Sym} \bigoplus_{j\geq 0} V_j)^{O(n)}$, it is apparent that the elements $D_{I,J}$ generate $\mathcal{I}_n$.

In this decomposition, the term $D^2_{I,J}$ lies in the space A_r spanned by $\{\Omega_{a,b} | \ a + b = r\}$, for $r = |I| + |J| + 2n$. Assume now that r is an even integer $2m$. In this case, $A_{2m} = \partial^2(A_{2m-2}) \oplus \langle J^{2m} \rangle$, where $\langle J^{2m} \rangle$ denotes the linear span of J^{2m}, and we define $\text{pr}_{2m} : A_{2m} \to \langle J^{2m} \rangle$ to be the projection onto the second term. In [L3] the *remainder* $R_{I,J}$ was defined to be $\text{pr}_{2m}(D^2_{I,J})$. Following [L4], write

$$R_{I,J} = R_n(I, J) J^{2m},$$

so that $R_n(I, J)$ denotes the coefficient of J^{2m} in $\text{pr}_{2m}(D^2_{I,J})$. As shown by Lemma 4.4 of [L3], $R_n(I, J)$ is independent of all choices in how we decompose $D_{I,J}$.

Suppose that $R_n[I, J] \neq 0$ for some I, J. Since $\pi_n(D_{I,J}) = 0$, we obtain a decoupling relation

$$j^{2m} = P(j^0, j^2 \dots, j^{2m-2}), \qquad m = \frac{1}{2}(|I| + |J| + 2n),$$

where P is a normally ordered polynomial in $j^0, j^2, \dots, j^{2m-2}$ and their derivatives. By applying the operator $j^2\circ_1$ repeatedly, we can construct higher decoupling relations

$$j^{2r} = Q_{2r}(j^0, j^2, \dots, j^{2m-2})$$

for all $r > m$. The argument is the same as the proof of Theorem 4.7 of [L3]. This implies that $\mathcal{H}(n)^{O(n)}$ is strongly generated by $\{j^0, j^2, \dots, j^{2m-2}\}$. If m is the smallest integer such that $R_n(I,J) \neq 0$ for some I, J with $m = \frac{1}{2}(|I| + |J| + 2n)$, the strong generating set $\{j^0, j^2, \dots, j^{2m-2}\}$ for $\mathcal{H}(n)^{O(n)}$ is minimal. For $I = (0, 1, \dots, n) = J$, $D_{I,J}$ is the unique element of the ideal $\mathcal{I}_n$ (up to scalar multiples) of minimal weight $n^2 + 3n + 2$. Define $R_n = R_n(I,J)$ in this case. In [L3], it was conjectured that $R_n \neq 0$, and this conjecture implies that $\{j^0, j^2, \dots, j^{n^2+3n-2}\}$ is a minimal strong generating set for $\mathcal{H}(n)^{O(n)}$.

The key innovation of [L4] was to give a recursive formula for this remainder, which we now recall. First, in the case $n = 1$, so that $I = (i_0, i_i)$ and $J = (j_0, j_1)$, we have the following formula:

$$\begin{aligned} R_1[I,J] = &\frac{(-1)^{i_0+j_0}}{2+i_0+i_1} - \frac{(-1)^{i_0+j_1}}{2+i_0+i_1} + \frac{(-1)^{i_0+i_0}}{2+i_0+j_0} - \frac{(-1)^{j_0+j_1}}{2+i_1+j_0} - \\ &\frac{(-1)^{i_0+i_0}}{2+i_0+j_1} + \frac{(-1)^{j_0+j_1}}{2+i_1+j_1} - \frac{(-1)^{i_1+j_0}}{2+j_0+j_1} + \frac{(-1)^{i_1+j_1}}{2+j_0+j_1}. \end{aligned} \tag{5.0.5}$$

Then for $n \geq 2$, we have

$$R_n[I,J] = -\sum_{r=0}^{n} (-1)^r (-1)^{i_r} \left(\sum_k \frac{R_{n-1}[I_{r,k}, J']}{i_k + i_r + 2} + \sum_l \frac{R_{n-1}[I_r, J'_l]}{j_l + i_r + 2} \right) \tag{5.0.6}$$

$$-\sum_{r=0}^{n} (-1)^r (-1)^{j_0} \left(\sum_k \frac{R_{n-1}[I_{r,k}, J']}{i_k + j_0 + 2} + \sum_l \frac{R_{n-1}[I_r, J'_l]}{j_l + j_0 + 2} \right).$$

In this notation, $J' = (j_1, \dots, j_n)$ is the list of length n obtained from J by omitting j_0, and $I_r = (i_0, \dots, \widehat{i_r}, \dots, i_n)$ is the list of length n obtained from I by omitting i_r. For $k = 0, \dots, n$ and $k \neq r$, $I_{r,k}$ is the list of length n obtained from $I_r = (i_0, \dots, \widehat{i_r}, \dots, i_n)$ by replacing the entry i_k with $i_k + i_r + j_0 + 2$. Finally, for $l = 1, \dots, n$, J'_l is obtained from $J' = (j_1, \dots, j_n)$ by replacing j_l with $j_l + i_r + j_0 + 2$.

Using this recursion, the value of R_n for $1 \leq n \leq 6$ were given in [L4]. We were able to compute the next value which was previously unknown:

$$R_7 = \frac{-106186063}{1599733892409490479882240000000} \tag{5.0.7}$$

This implies that $\mathcal{H}(7)^{O(7)}$ is of type $\mathcal{W}(2, 4, 6, \dots, 70)$. But unfortunately, a closed formula for $R_n(I, J)$ is not known even though a similar recursion for Pfaffians that governs the $Sp(2n)$-orbifold of the rank n $\beta\gamma$-system was shown to have a nice closed formula [L5]. Although the recursion can be shown to have the property that it is nonzero asymptotically, no good bounds for the quantity N were given in [L4].

The main technical result in this section is that if we restrict to the case where all entries of I and J are even, then we can give a closed formula for $R_n(I, J)$ and from this formula it will be apparent that $R_n(I, J) \neq 0$ whenever I consists of distinct even entries, and J also consists of distinct even entries. Therefore when $I = (0, 2, 4, \dots, 2n+2) = J$, $R_n(I, J) \neq 0$. This will complete the proof of Theorem 5.0.1.

First, if we restrict to the case where all entries in the I and J blocks are even, the formulas (5.0.5) and (5.0.6) simply as follows.

$$\begin{aligned} R_1[I, J] &= \frac{1}{2 + i_0 + j_0} - \frac{1}{2 + i_1 + j_0} - \frac{1}{2 + i_0 + j_1} + \frac{1}{2 + i_1 + j_1} \\ &= \frac{(4 + i_0 + i_1 + j_0 + j_1)(i_0 - i_1)(j_0 - j_1)}{(2 + i_0 + j_0)(2 + i_1 + j_0)(2 + i_0 + j_1)(2 + i_1 + j_1)}. \end{aligned} \tag{5.0.8}$$

$$\begin{aligned} R_n[I, J] = \sum_{r=0}^{n} \Bigg(& \sum_{k \neq r} (-1)^{r+1} \frac{R_{n-1}[I_{r,k}, J']}{i_r + i_k + 2} + \sum_{l \neq 0} (-1)^{r+1} \frac{R_{n-1}[I_r, J'_l]}{i_r + j_l + 2} + \\ & \sum_{k \neq r} (-1)^{r+1} \frac{R_{n-1}[I_{r,k}, J']}{j_0 + i_k + 2} + \sum_{l \neq 0} (-1)^{r+1} \frac{R_{n-1}[I_r, J'_l]}{j_0 + j_l + 2} \Bigg). \end{aligned} \tag{5.0.9}$$

Given $I = [i_0, i_1, ...i_{n-1}]$ and $J = [j_0, j_1, ...j_{n-1}]$ define $I_x = [i_0, i_1, ...i_{n-1}, x]$ and $J_x = [j_0, j_1, ...j_{n-1}, x]$. For $n > 2$ define $g_n(R_n[I, J])$ as the terms in the restricted recursive formula for which i_r, i_k, j_l, or j_0 depend on x. Such terms are referred to as x-active. Note each term in $g_n(R_n[I_x, J_x])$ is a function of $R_{n-1}[I_{r,k}, J']$ or $R_{n-1}[I_r, J'_l]$, and elements from I and J. We can then for $n > 2$ define $f_n(R_n[I_x, J_x])$ as the terms for which i_r, i_k, j_l, or j_0 does not depend on x. Such terms are referred to as non-x-active. Then $R_n[I_x, J_x] = f_n(R_n[I_x, J_x]) + g_n(R_n[I_x, J_x])$. Define $h_n(R_n[I_n, J_n]) = \lim_{x\to\infty} R_n[I, J]$

Lemma 5.0.3. For all $n > 2$ and $I_x = [i_0, i_1, ...i_{n-1}, x]$ and $J_x = [j_0, j_1, ...j_{n-1}, x]$ with i_s and j_t even for all s and t, we have

$$h_n(R_n[I_x, J_x]) = \frac{n}{2n + \sum_{s=0}^{n-1}(i_s + j_s)} R_{n-1}[I, J] \tag{5.0.10}$$

Proof. For $n = 3$ this is a direct calculation, so we proceed by induction on n. We will simultaneously prove the formula

$$h_{n-1}(g_n(R_n[I_x, J_x])) = \frac{1}{2n + \sum_{s=0}^{n-1}(i_s + j_s)} R_{n-1}[I, J] \tag{5.0.11}$$

This too can be checked by direct calculation for $n = 3$, so we assume both formulas hold for $n-1$. Each term appearing in $f_n(R_n[I_x, J_x])$ depends on $R_{n-1}[K_x, L_x]$ for some K and L of length $n-1$ with entries depending on only those in I or J respectively. Therefore by our inductive hypothesis that the two formulas hold for $n-1$ we have that

$$h_{n-1}(f_n(R_n[I_x, J_x])) = \frac{n-1}{2n + \sum_{s=0}^{n-1}(i_s + j_s)} R_{n-1}[I, J] \tag{5.0.12}$$

It has been previously shown [L4] that

$$h_{n-2}(g_{n-1}(f_n(R_n[I_x, J_x]))) = h_{n-2}(f_{n-1}(g_n(R_n[I_x, J_x])))$$

and

$$g_{n-1}(g_n(R_n[I_x, J_x])) = 0$$

Therefore

$$h_{n-1}(g_n(R_n[I_x, J_x])) = h_{n-2}(f_{n-1}(g_n(R_n[I_x, J_x]))) + h_{n-2}(g_{n-1}(g_n(R_n[I_x, J_x]))) =$$
$$h_{n-2}(f_{n-1}(g_n(R_n[I_x, J_x]))) = h_{n-2}(g_{n-1}(f_n(R_n[I_x, J_x])))$$

Hence by the inductive hypothesis we have

$$h_{n-1}(g_n(R_n[I_x, J_x])) = h_{n-2}(g_{n-1}(f_n(R_n[I_x, J_x]))) = \frac{1}{2n + \sum_{s=0}^{n-1}(i_s + j_s)} R_{n-1}[I, J]$$

Now the claim follows from (5.0.11) and (5.0.12). □

Theorem 5.0.4. Given lists of non-negative even integers $I = [i_0, i_1, ..., i_n]$ and $J = [j_0, j_1, ..., j_n]$ the closed formula holds

$$R_n[I, J] = \frac{n!(2(n+1) + \sum_{a=0}^{n}(i_a + j_a))(\prod_{0 \le a < b \le n}(i_a - i_b)(j_a - j_b))}{\prod_{0 \le a \le n, 0 \le b \le n}(2 + i_a + j_b)} \tag{5.0.13}$$

Proof. We proceed by induction on n, assuming the formula holds for $n-1$. By plugging in our closed formula into the recursive formula once and expressing it as a rational function of the entries of I and J we see the denominator must be expressible as the product of terms of the form

$2 + i_a + i_b$, $2 + j_0 + j_b$, $4 + i_a + i_b + j_c$, $4 + i_a + j_b + j_c$, $2 + i_a + j_b$

It was shown in [L4] that the recursive formula inherits symmetries from the determinant. One of these is that if we swap two indices in either the I or J blocks this only changes the sign. This implies that terms of the form $4 + i_a + i_b + j_c$, $4 + i_a + j_b + j_c$, or $2 + j_0 + j_b$ can not appear. Additionally since the transpose preserves the determinant swapping the I and J blocks does not change the denominator. Together with noting that $2 + j_a + j_b$ has already been precluded we see $2 + i_a + i_b$ can not appear either.

Having now reduced our list to terms of the form $2 + i_a + j_b$ we can see that the symmetries imply that all such terms must appear. Hence the denominator is

$$\prod_{0 \le a \le n, 0 \le b \le n} (2 + i_a + j_b) \tag{5.0.14}$$

This shows that the symmetries above only lead to permutations of the factors in the denominator. Therefore the sign change from swapping two elements within a block must be apparent in the numerator. Hence

$$\prod_{0 \le a \le n-1, a < b < n} (i_a - i_b)(j_a - j_b) \tag{5.0.15}$$

divides the numerator.

This product has degree $n^2 + n$ and our denominator has degree $(n+1)^2$. The inductive hypothesis gives that each term appearing in 5.0.9 has total degree at least $-n$. So we must have one additional linear factor in the numerator.

Since we have already taken care of the sign changes this linear factor must be invariant under all the permutations discussed before. Hence it must have the form

$$a_n(\sum_{k=0}^{n} i_k + j_k) + c_n \tag{5.0.16}$$

To show $a_n = n!$ given $I = [i_0, i_1, ...i_{n-1}]$ and $J = [j_0, j_1, ...j_{n-1}]$ define $I_x = [i_0, i_1, ...i_{n-1}, x]$ and $J_x = [j_0, j_1, ...j_{n-1}, x]$ as in the lemma.

The highest power of x appearing in the numerator is x^{2n+1} and its coefficient is given by

$$2a_n \prod_{0 \le a \le n-2, a < b < n-1} (i_a - i_b)(j_a - j_b) \tag{5.0.17}$$

The highest power of x appearing in the denominator is also x^{2n+1} and its coefficient is given by

$$2 \prod_{0 \leq a \leq n-1, 0 \leq b \leq n-1} (2 + i_a + j_b) \tag{5.0.18}$$

Therefore defining $h_n(R_n(I_x, J_x))$ as in the lemma we get

$$h_n(R_n(I_x, J_x)) = \frac{a_n \prod_{0 \leq a \leq n-2, a<b<n-1} (i_a - i_b)(j_a - j_b)}{\prod_{0 \leq a \leq n-1, 0 \leq b \leq n-1} (2 + i_a + j_b)} \tag{5.0.19}$$

Then from our inductive hypothesis we have that $a_{n-1} = (n-1)!$, and

$$R_{n-1}[I, J] = \frac{(n-1)!(2(n) + \sum_{a=0}^{n-1}(i_a + j_a))(\prod_{0 \leq a<b<n}(i_a - i_b)(j_a - j_b))}{\prod_{0 \leq a<n, 0 \leq b<n}(2 + i_a + j_b)} \tag{5.0.20}$$

Putting these together with our lemma we see

$$\begin{aligned} h_n(R_n(I_x, J_x)) &= \frac{a_n \prod_{0 \leq a \leq n-2, a<b<n-1} (i_a - i_b)(j_a - j_b)}{\prod_{0 \leq a \leq n-1, 0 \leq b \leq n-1} (2 + i_a + j_b)} \\ &= \frac{n}{2(n) + \sum_{a=0}^{n-1}(i_a + j_a)} R_{n-1}(I, J) \end{aligned} \tag{5.0.21}$$

Therefore $a_n = n!$.

To show $c_n = 2(n+1)!$ we notice that by following the procedures to obtain I_r, $I_{r,k}$, J', and J'_l from I and J together with our inductive hypothesis shows that $2(n)+\sum_{a=0}^{n-1}(i_a+j_a)$ divides each term of $R_n[I, J]$.

□

Chapter 6: $\mathcal{H}(2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(2,3,4,5,n^2,(n+2)^2)$

In this chapter, we consider the $\mathbb{Z}_n$-orbifold of the rank 2 Heisenberg algebra $\mathcal{H}(2)$. First, we begin with the standard generating set $h_1(z), h_2(z)$ satisfying

$$h_i(z)h_j(z) \sim \delta_{i,j}(z-w)^{-2}, \qquad i,j = 1,2.$$

The automorphism group of $\mathcal{H}(2)$ is $O(2)$ acting on the Span(h_1, h_2) as the standard representation. Note that the subgroup $SO(2)$ acts on this space by rotation. Identifying $SO(2)$ with the circle group, we can identify $\mathbb{Z}_n$ as the subgroup consisting of the nth roots of 1. The main result in this chapter is the following.

Theorem 6.0.1. For $n \geq 3$, $\mathcal{H}(2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(2,3,4,5,n^2,(n+2)^2)$

In order to prove this theorem we utilize the decomposition of $\mathcal{H}(2)^{\mathbb{Z}_n}$ as modules over the subVOA $\mathcal{H}(2)^{SO(2)}$. As mentioned, this is the large k limit algebra of the parafermion VOA $N^k(\mathfrak{sl}_2)$. It is known to be strongly finitely generated of type $\mathcal{W}(2,3,4,5)$, and have an abelian Zhu associative algebra. We proceed by constructing $\mathcal{H}(2)^{SO(2)}$.

First, it will be helpful to diagonalize the action of $SO(2)$ and choose a different set of strong generators which are eigenvectors for each $e^{i\theta} \in SO(2)$. Consider the action of $SO(2)$ on the $\mathbb{C}^2$ with basis $\{x_1, x_2\}$ by rotation matrices of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{6.0.1}$$

Since $SO(2)$ is abelian we can simultaneously diagonalize this action as follows

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}. \tag{6.0.2}$$

This implies

$$a\cos(\theta) - b\sin(\theta) = \lambda a, \tag{6.0.3}$$

$$a\sin(\theta) + b\cos(\theta) = \lambda b. \tag{6.0.4}$$

So

$$\cos(\theta) - \frac{b}{a}\sin(\theta) = \frac{a}{b}\sin(\theta) + \cos(\theta). \tag{6.0.5}$$

Therefore

$$-a^2 = b^2. \tag{6.0.6}$$

Thsi gives us that $a = i$ and $b = 1$ or $a = -i$ and $b = 1$. And plugging these in to solve for λ, we see

$$\cos(\theta) + i\sin(\theta) = e^{i\theta} = \lambda_1, \tag{6.0.7}$$

or

$$\cos(\theta) - i\sin(\theta) = e^{-i\theta} = \lambda_2. \tag{6.0.8}$$

This yields a new basis $y_1 = ix_1 + x_2$ and $y_2 = -ix_1 + x_2$. In this basis, it is clear that any monomial with the same number of y_1 as y_2 is invariant under all rotations. Note also that for each $n \in \mathbb{N}$ we can embed the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ as generated by the rotation $\theta = \frac{2\pi}{n}$. It is then immediate that this rotation corresponds to the eigenvalue $\zeta_n = e^{\frac{2\pi i}{n}}$. It is also clear that terms of the form $y_1^{m_1 n}$, and $y_2^{m_2 n}$ are invariant under the action for all $m_1, m_2 \in \mathbb{Z}_{\geq 0}$.

We now note that from equations 2.2.12 and 2.2.13, commuting two factors in a Wick product in general lead to a correction term which has the same weight. However for the Heisenberg algebra, since each term appearing in the OPE's among the generators has lower degree we can assume commutativity up to lower degree corrections. Hence if we are careful to account for induction degree by degree we can assume a standard ordering for weight homogeneous monomials.

Using the above diagonalization of the action of $SO(2)$, we adopt a new strong generating set as follows.

$$B_1 = ih_1 + h_2, \qquad B_2 = -ih_1 + h_2 \tag{6.0.9}$$

with OPE's

$$B_1(z)B_1(w) \sim B_2(z)B_2(w) \sim 0 \tag{6.0.10}$$

$$B_1(z)B_2(w) \sim 2(z-w)^{-2} \tag{6.0.11}$$

Since we can recover the original basis it is clear that B_1 and B_2 strongly finitely generate $\mathcal{H}(2)$. From here on in this section, we shall work with this new generating set, and also for simplicity we shall omit the colons and let juxtaposition represent the Wick product with its usual right-associated convention.

Then the $SO(2)$ orbifold $\mathcal{H}(2)^{SO(2)}$ is generated by monomials with the same degree in the B_1's and B_2's. By induction on degree, we immediately obtain

Lemma 6.0.2. $\mathcal{H}(2)^{SO(2)}$ is strongly generated by the set $\{\partial^i B_1 \partial^j B_2 | \; i, j \geq 0\}$. In particular, it is spanned by all iterated Wick products of these fields.

We can further reduce our consideration to elements of the form $B_1 \partial^j B_2$ by taking derivatives and induction on i. Then at weight 6, we observe that

$$B_1\partial^4 B_2 = (B_1B_2)(\partial B_1\partial B_2) - (\partial B_1 B_2)(B_1\partial B_2) + \frac{1}{3}\partial B_1\partial^3 B_2 - \partial^2(B_1\partial^2 B_2) + \frac{1}{3}\partial^3(B_1\partial B_2)$$

Using the identities 2.2.12, 2.2.13, 2.2.14, and 2.2.15 we can calculate

$$(B_1\partial B_2)\circ_1 (B_1\partial^j B_2) = (2j+6)B_1\partial^{j+1}B_2 - (4)\partial(B_1\partial^j B_2)$$

This implies that if we can obtain an expression for $B_1\partial^j B_2$ that is a normally ordered polynomial in the terms $B_1\partial^k B_2$ where $k \in \{0,1,2,3\}$ we can lift it to an expression for $B_1\partial^{j+1}B_2$ which is a normally ordered polynomial in the terms $B_1\partial^k B_2$ where $k \in \{0,1,2,3,4\}$. We can them use the above formula to replace each of the $B_1\partial^4 B_2$. Thus by induction we can obtain an expression for all $B_1\partial^j B_2$ where $j \geq 4$. This yields the following theorem.

Theorem 6.0.3. The $SO(2)$ orbifold, $\mathcal{H}(2)^{SO(2)}$, is strongly finitely generated by $B_1\partial^k B_2$ where $k \in \{0,1,2,3\}$. Therefore $\mathcal{H}(2)^{SO(2)}$ is of type $\mathcal{W}(2,3,4,5)$.

Note that this also can be obtained from the theorem of [DLY] that $N^k(\mathfrak{sl}_2)$ is of type $\mathcal{W}(2,3,4,5)$ by passing to the large level limit.

It is then follows from Theorem 3.3.1 [DLM] that if G is a compact group then $\mathcal{H}(2)$ decomposes as

$$\mathcal{H}(2) \cong \bigoplus_{\nu\in I} L(\nu)\otimes M^{\nu}$$

where I indexes the finite-dimensional G-modules $L(\nu)$, and the M^{ν}'s are nonzero, inequivalent, irreducible $\mathcal{H}(2)^G$ modules. In the case $G = SO(2)$, the L_ν's are all one dimensional and are indexed by the integers, so our decomposition becomes

$$\mathcal{H}(2) \cong \bigoplus_{m\in\mathbb{Z}} M^m$$

Furthermore, since the Zhu algebra of $\mathcal{H}(2)^{SO(2)}$ is abelian, its irreducible positive energy modules all have one-dimensional top component and are cyclically generated by their lowest weight element.

With our choice of basis we can see that the modules can be indexed by the exponent of their eigenvalues under the action of suitable elements of $SO(2)$. This can be seen as the total degree of the B_1's and its derivatives minus the total degree of B_2's and its derivatives in a monomial term. This naturally separates the action of elements $\mathcal{H}(2)^{SO(2)}$ which have eigenvalue 1, and the remaining monomial term consisting of purely B_1's or B_2's. This is equivalent to reordering and reassociating monomials up to lower weight corrections into the product of paired terms of the form $\partial^i B_1 \partial^j B_2$ and one term of the form $\partial^{\alpha_1} B_\beta \partial^{\alpha_2} B_\beta ... \partial^{\alpha_m} B_\beta$, $\beta \in \{1, 2\}$.

Next, recall that $\mathbb{Z}_n$ acts on $\mathcal{H}(2)$ as the subgroup of $SO(2)$ generated by the matrices

$$\begin{bmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{bmatrix}$$

Then B_1 is the eigenvector corresponding to the eigenvalue $e^{\frac{2\pi i}{n}}$ and similarly B_2 corresponds to $e^{\frac{-2\pi i}{n}}$. As these are primitive n^{th} roots of unity it is immediate that monomials which can be expressed as the product of terms of the form $\partial^i B_1 \partial^j B_2$ and one term of the form $\partial^{\alpha_1} B_\beta, \partial^{\alpha_2} B_\beta, ..., \partial^{\alpha_m} B_\beta$, $\beta \in \{1, 2\}$ and m a multiple of n have an total eigenvalue of one under the action of the generator, and therefore all of $\mathbb{Z}_n$. The linear span of such elements is the $\mathbb{Z}_n$ orbifold, $\mathcal{H}(2)^{\mathbb{Z}_n}$.

It is then easily seen that as an $\mathcal{H}(2)^{SO(2)}$-module, the $\mathbb{Z}_n$ orbifold is obtained as the direct sum of all irreducible $\mathcal{H}(2)^{SO(2)}$-modules indexed by integer multiples of n in our decomposition of $\mathcal{H}(2)$, that is,

$$\mathcal{H}(2)^{\mathbb{Z}_n} \cong \bigoplus_{m\in\mathbb{Z}} M^{mn}$$

Additionally we have that the modules corresponding to $m > 1$ can be obtained from normally ordered products of elements in the module corresponding to $m = 1$. Similarly those in the modules corresponding to $m < -1$ can be obtained from those in the module corresponding to $m = -1$.

Therefore if we show that we can obtain all monomials of the form

$$\partial^{\alpha_1} B_\beta \partial^{\alpha_2} B_\beta ... \partial^{\alpha_n} B_\beta$$

from the action of $\mathcal{H}(2)^{SO(2)}$ on the element B_β^n for $\beta = 1$ and 2, by then taking normally ordered products with elements of $\mathcal{H}(2)^{SO(2)}$ we can generate all of the $m = -1, 0, 1$ modules and hence all of $\mathcal{H}(2)^{\mathbb{Z}_n}$ as a module over $\mathcal{H}(2)^{SO(2)}$.

We now consider the linear span $\mathcal{L}$ of all annihilation operators of the form

$$(B_1 \partial^k B_2)\circ_k, \qquad k \geq 0,$$

which are homogeneous of weight 1. Since the fields $(B_1\partial^k B_2)$ close linearly under OPE, is straightforward to check that $\mathcal{L}$ is a Lie algebra; see [L3] for a more detailed discussion.

Lemma 6.0.4. All terms of the form $\partial^{\alpha_1} B_1 \partial^{\alpha_2} B_1 ... \partial^{\alpha_n} B_1$ lie in the span of all fields obtained by applying finitely many elements of $\mathcal{L}$ to B_1^n.

Proof. Note first that $(B_1\partial^k B_2)\circ_k \in \mathcal{L}$ acts on $\partial^{\alpha_1} B_1 \partial^{\alpha_2} B_1 ... \partial^{\alpha_n} B_1$ as a derivation:

$$(B_1 \partial^k B_2) \circ_k (\partial^{\alpha_1} B_1 ... \partial^{\alpha_n} B_1) = \sum_{i=1}^{n} \frac{2(-1)^k(\alpha_i + k + 1)!}{(\alpha_i + 1)!} \partial^{\alpha_i + 1} B_1 C_i$$

Here C_i is obtained by omitting $\partial^{\alpha_i} B_1$ from $\partial^{\alpha_1} B_1 ... \partial^{\alpha_n} B_1$. Consider the $p \times p$ matrix with ij^{th} entry given by $\frac{2(-1)^k (j+i+1)!}{(j+1)!}$, where $p = 2N+1$ and N is the maximum of $\alpha_1, \dots, \alpha_n$. We observe that this matrix is row equivalent to one with entries given by $\frac{2(j+i+1)!}{(j+1)!}$. This row equivalent matrix has the property that the entries are strictly increasing in both i and j, and that the determinant of each 2×2 submatrix is non zero. Such matrices are said to be *totally increasing* and are invertible. As shown in Lemma 6.5 of [L3], this implies that there exists an element of $\mathcal{L}$ taking $\partial^{(\alpha_1)} B_1 ... \partial^{(\alpha_i)} B_1 ... \partial^{(\alpha_n)} B_1$ to $\partial^{(\alpha_1)} B_1 ... \partial^{(\alpha_i + 1)} B_1 ... \partial^{(\alpha_n)} B_1$ for each $i \in \{1...n\}$. Hence by induction on the weight we can obtain any term of the form $\partial^{\alpha_1} B_1 ... \partial^{\alpha_n} B_1$ can be obtained by applying a finite sequence of elements of $\mathcal{L}$ to B_1^n. □

By a symmetric argument we get that $\partial^{\alpha_1} B_2 \partial^{\alpha_2} B_2 ... \partial^{\alpha_n} B_2$ can be obtained by applying finitely many elements of $\mathcal{L}$ to B_2^n. Since $\mathcal{L}$ consists of modes of fields $B_1 \partial^k B_2 \in \mathcal{H}(2)^{SO(2)}$, it follows that $\mathcal{H}(2)^{\mathbb{Z}_n}$ is generated as an $\mathcal{H}(2)^{SO(2)}$-module by the vacuum vector 1 together with two elements B_1^n and B_2^n. Furthermore since $\mathcal{H}(2)^{SO(2)}$ is strongly generated by $B_1 B_2$ $B_1 \partial B_2$ $B_1 \partial^2 B_2$ $B_1 \partial^3 B_2$, any finite set which contains these seven elements and closes under OPE, must strongly generate $\mathcal{H}(2)^{\mathbb{Z}_n}$.

Now of course the preceding lemma is completely unnecessary since we already know that each $\mathcal{H}(2)^{SO(2)}$ module is cyclically generated by one element. Let this be B_β^{mn} and we can see that these are generated as products of B_β^n within $\mathcal{H}(2)^{\mathbb{Z}_n}$. It is included simply to provide an alternative explicit construction. It also demonstrates how we may access the larger VOA structure to translate our commutative perspective around the VOA.

Up to this point we have established a set monomials which linearly span $\mathcal{H}(2)^{\mathbb{Z}_n}$. We have also shown that within the orbifold VOA these are generated by the four strong generators for $\mathcal{H}(2)^{SO(2)}$ and two degree n elements. In order pass to strong generators we view the orbifold as an OPE algebra and show that a set of strong generators which contains the minimal set of VOA generators closes under OPE. That is we express each pairwise circle product among the set as a linear combination of Wick products of our strong generators and their derivatives. Hence by induction on length we establish a process for strongly generating this possibly larger OPE algebra. However if the new set of strong generators is contained in the VOA they must be the same as OPE algebras. This establishes the strong generation of the VOA.

In generating these elements it is immediately apparent that we don't need to include the element $\frac{1}{n}\partial(B_\beta^n)$. However we do need two elements to span the weight $n+2$ space. There is only one derivative available for each $\beta \in \{1,2\}$. Further there is no hope for a decoupling relation of weight less than $n+3$ since it must involve a generator of degree at least two and at least one derivative. As we shall see this turns out to be all we need.

The set containing B_1B_2, $B_1\partial B_2$, $B_1\partial^2 B_2$, $B_1\partial^3 B_2$, B_1^n, B_2^n, $\partial^2 B_1 B_1^{n-1}$, and $\partial^2 B_2 B_2^{n-1}$ is a minimal set which can be shown to close under OPE by direct calculation.

Before carrying out any calculations we should recall that it has already been shown that the generators for the $SO(2)$ orbifold close under OPE. We observe that the OPE of any two terms of the form $\partial^i B_\beta B_\beta^{n-1}$ is zero since both B_1 and B_2 commute with themselves or is an element of $\mathcal{H}(2)^{SO(2)}$ since there would be an equal number of $\partial^i B_1$ and $\partial^j B_2$ type factors. This shows that we need only check the mixed pairings between the quadratics and the degree n terms. It is easily seen that all the resulting elements are pure degree n terms with weight $\leq n+6$. It then suffices to show that we can generate all degree n terms of weight $\leq n+6$.

This was done by generating a matrix of coefficients as polynomials in n from the previous weights first derivatives and utilizing decoupling relations to gain the needed degrees of freedom. Inverting this matrix yields the coefficients as rational functions of n needed to solve for all elements of each weight in terms of those of the previous weight. It is immediate from the factorization of the denominators that these coefficients are well defined for each needed element.

Since the partition for a natural number at most 6 is at most length 6 and we give ourselves one more unaffected spot to avoid possible issues with zeros. These calculations stabilize for $n \geq 7$ since we can obtain all possible terms with coefficients depending only on n. It is a straightforward matter to directly check all the calculations for $2 \leq n \leq 6$. This simply requires choosing linearly independent subsets of decoupling relations and omitting derivatives which do not appear due to degree constraints. Below we present the coefficients for $n \geq 7$. We also only present the calculations for the B_1's. These show the existence of those for the B_2's by symmetry, but the exact formulas differ using this set of strong generators for $\mathcal{H}(2)^{SO(2)}$.

Weight $n+1$

$$\partial B_1 B_1^{n-1} = \frac{1}{n}\partial(B_1^n) \tag{6.0.12}$$

Weight $n+2$

$$\partial B_1 \partial B_1 B_1^{n-2} = \frac{1}{n-1}(\partial(\partial B_1 B_1^{n-1}) - \partial^2 B_1 B_1^{n-1}) \tag{6.0.13}$$

Weight $n+3$

$$\begin{aligned}\partial B_1 \partial B_1 \partial B_1 B_1^{n-3} = &\frac{\frac{2}{3}+\frac{n}{3}-n^2}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}\partial(\partial B_1 \partial B_1 B_1^{n-2})+\\ &\frac{-\frac{2}{3}+2n}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}\partial(\partial^2 B_1 B_1^{n-1})+\\ &\frac{2}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}(B_1 B_2 \partial B_1 B_1^{n-1} - \partial B_1 B_2 B_1^n)\end{aligned} \tag{6.0.14}$$

$$\begin{aligned}\partial^2 B_1 \partial B_1 B_1^{n-2} = &\frac{-\frac{2}{3}+\frac{7n}{3}-n^2}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}\partial(\partial^2 B_1 B_1^{n-1})+\\ &\frac{2-n}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}(B_1 B_2 \partial B_1 B_1^{n-1} - \partial B_1 B_2 B_1^n)\end{aligned} \tag{6.0.15}$$

$$\partial^3 B_1 B_1^{n-1} = \frac{-2+3n-n^2}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}\partial(\partial^2 B_1 B_1^{n-1})+ \frac{2-3n+n^2}{(\frac{-1}{3})(n+\frac{2}{3})(n-1)(n-2)}(B_1 B_2 \partial B_1 B_1^{n-1} - \partial B_1 B_2 B_1^n) \tag{6.0.16}$$

The formulas for all the needed elements in weights $n+4$ through $n+6$ are given in the Appendix.

Therefore we have show that any term appearing in an pairwise OPE between any of our strong generators can be expressed as normally ordered polynomials of these generators and their derivatives. Hence they close under OPE. Since these generators are contained in $\mathcal{H}(2)^{\mathbb{Z}_n}$ and contain a generating set they must strongly generate $\mathcal{H}(2)^{\mathbb{Z}_n}$. This proves the main result that $\mathcal{H}(2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(2,3,4,5,n^2,(n+2)^2)$.

6.1 Some remarks on the Zhu algebra of $\mathcal{H}(2)^{\mathbb{Z}_n}$

It is a general fact that if a vertex algebra $\mathcal{V}$ is strongly generated by a set $\{\alpha_i | \ i \in I\}$, then the images of these fields in the Zhu algebra $A(\mathcal{V})$ will generate $A(\mathcal{V})$. Therefore the Zhu algebra $A(\mathcal{H}(2)^{\mathbb{Z}_n})$ is generated by fields $w^2, w^3, w^4, w^5, c^n, d^n, c^{n+2}, d^{n+2}$ corresponding to the above strong generators in weights $2,3,4,5,n,n,n+2,n+2$.

The inclusion $\mathcal{H}(2)^{SO(2)} \hookrightarrow \mathcal{H}(2)^{\mathbb{Z}_2}$ induces a homomorphism of Zhu algebras

$$A(\mathcal{H}(2)^{SO(2)}) \to A(\mathcal{H}(2)^{\mathbb{Z}_n}),$$

which need not be injective, and w^2, w^3, w^4, w^5 lie in the image of this map. Since $\mathcal{H}(2)^{SO(2)}$ has an abelian Zhu algebra, these fields then generate an abelian subalgebra of $A(\mathcal{H}(2)^{\mathbb{Z}_n})$. In fact, these four elements are not algebraically independent and it is straightforward to find two independent relations among them. In fact, viewing $\mathcal{H}(2)^{SO(2)}$ as the large k limit of the parafermion algebra $N^k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V^k(\mathfrak{sl}_2))$, these relations are already known [DLY]. In the parafermion algebra, there exist two normally ordered relations in weight 8 that give rise to relations in the Zhu algebra, and taking the limit of these relations and applying the Zhu map gives the relations in $A(\mathcal{H}(2)^{SO(2)})$.

We observe that $A(\mathcal{H}(2)^{\mathbb{Z}_n})$ admits a triangular decomposition

$$A(\mathcal{H}(2)^{\mathbb{Z}_n}) \cong C \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} D,$$

where A is the commutative algebra generated by w^2, w^3, w^4, w^5. C is generated by c^n, c^{n+2}, and D is generated by d^n, d^{n+2}. Note that we have complicated relations expressing $c^i d^j$ as polynomials in w^2, w^3, w^4, w^5 for all $i, j = n, n+2$. Therefore we can construct modules for $A(\mathcal{H}(2)^{\mathbb{Z}_n})$ in the following way.

1. Begin with a one-dimensional module V_χ over $A \otimes_{\mathbb{C}} D$, where A acts by the character $\chi : A \to \mathbb{C}$ on V, and D acts by zero. Note that χ is required to preserve the algebraic relations satisfied by w^2, w^3, w^4, w^5.

2. We now consider the induced module

$$M_\chi = A(\mathcal{H}(2)^{\mathbb{Z}_n}) \otimes_{A \otimes D} V_\chi.$$

3. Using commutation relations in $A(\mathcal{H}(2)^{\mathbb{Z}_n})$, it is easy to see that M_χ is spanned by the elements $\{(c^n)^i (c^{n+2})^j v | \ i, j \geq 0\}$, where v is a basis vector for V_χ. We then consider the quotient L_χ of M_χ by its maximal proper $A(\mathcal{H}(2)^{\mathbb{Z}_n})$-submodule.

Note that M_χ is an analogue of Verma modules over finite-dimensional Lie algebras. It is an interesting question to classify for which characters χ the simple modules L_χ are finite dimensional. It is expected that there are further algebraic constraints on the χ which imply finite-dimensionality. The corresponding positive energy modules for $\mathcal{H}(2)^{\mathbb{Z}_n})$ should then exhaust the modules for which the graded components are finite-dimensional. In the case $n = 3$ we believe it is possible to carry out this classification because the full OPE algebra can be computed, and hence the explicit commutation relations in $A(\mathcal{H}(2)^{\mathbb{Z}_n})$ can be deduced.

Chapter 7: Some Corollaries

7.1 General Finite Abelian Group

Let $G \leq Aut(H(N))$ be a finite abelian subgroup of the automorphism group of $H(N)$ for suitable N. Then from the fundamental theorem of finite abelian groups we may assume G is the direct sum of finite cyclic groups. That is

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \tag{7.1.1}$$

for some integers $n_i \geq 2$, not necessarily unique.

Since G is abelian the action under all of G is simultaneously diagonalizable. We are in fact viewing G as embedded in $End(V)$ where V is the underlying complex vector space for $\mathcal{H}(N)$. Since this action is assumed to preserve the dimension and bilinear form on V we may assume without any loss of generality that G is embedded in $O(V)$ where it is represented by $N \times N$ complex matrices. From this perspective we can see that in order to preserve the bilinear form, eigenspaces must come in conjugate pairs. If the only eigenvalues acting on a space are ± 1, we can view this space as self-conjugate; hence this action can be represented on a single rank 1 Heisenberg tensor factor. If an element is in an eigenspace for which there is a different complex eigenvector than ± 1, we must be able to pair it with a conjugate space. Hence each such pairing forms a rank 2 Heisenberg tensor factor.

Then we can view $\mathcal{H}(N)$ and the tensor product of $\mathcal{H}(2)$'s and $\mathcal{H}(1)$'s, and we can choose a basis for our Heisenberg algebra such that the generator of the j^{th} summand of G acts on the i^{th} tensor factor by the eigenvalues of an $\zeta_{d_{j_i}}$ and its complex conjugate, with

$\zeta_{d_{j_i}}$ a primitive $d_{j_i}^{th}$ root of unity, where d_{j_i} divides n_i. We can also assume we have taken care of any $\mathcal{H}(1)$'s who's basis vector remains invariant under G. Such factors are strongly generated by their weight one generator, and are simply unaffected.

Since we wish to find invariants under the action from all of G not just individual elements we consider the homomorphic image of G into the automorphism group of each tensor factor. If the factor is rank one then the automorphism group is just $\mathbb{Z}_2$ and since we have removed unaffected factors, the action of G on each such factor is equivalent to an action by $\mathbb{Z}_2$. If the factor is rank two the automorphism group is $O(2)$. Since G is an abelian group its homomorphic image must live in a maximal abelian subgroup isomorphic to $SO(2)$. Since G is finite it must map to a finite group in $SO(2)$. All such groups are cyclic and hence isomorphic to $\mathbb{Z}_{m_i}$ for some m_i.

We can assume we have diagonalized the action within each factor. Let us label these generators B_{1_i}, and B_{2_i} if the factor is rank 2. Then the action of G on B_{1_i} and its derivatives is generated by a primitive ${m_i}^{th}$ root of unity, and its conjugate on B_{2_i} and its derivatives if rank 2.

We can then see that if the total eigenvalue for a monomial under the action of G is given by the product of these eigenvalues for each factor appearing. In other words if the product $\prod_i \zeta_{m_i}^{k_i} = 1$. Where k_i is the signed power of the number of B_{1_i}'s assigned to positive k and B_{2_i}'s assigned to negative values. Decorating with derivatives does not affect this.

This cuts out a sublattice from the lattice of monomials in the B's. For each tensor factor i we then need the corresponding L_i, $W_{3,i}$, $W_{4,i}$, $W_{5,i}$, unless $\zeta_{m_i} = -1$ when we need only the weight two and four generators. It is then clear that by factoring out such generators we can generate the other invariants by considering only $\prod_i \zeta_{m_i}^{k_i} = 1$ for which $k_i > 0$. These correspond to monomials in the B_{1_i}'s and their derivatives. By symmetry we get the conjugate set.

Then since we have proven that we can generate each degree k_i factor with fixed i within each of these monomials for which $k_i \geq 2$ from terms with no derivatives and with two derivatives. Putting this together with the fact that each of the tensor factors commute we can see that our lattice points correspond to needing strong generators constructed as all possible products choosing either $B_{1_i}^{k_i}$ or $\partial^2 B_{1_i} B_{1_i}^{k_i-1}$ for each k_i appearing.

If two tensor factors are being acted upon by a single summand of G we can obtain additional decoupling and derivative relations which can eliminate some needed generators. An example of this is $\mathcal{H}(n)^{\mathbb{Z}_2}$ worked out by Al-Ali. This in effect covers our case since by degree considerations this can only come about between factors for which $m_i = 2$. We can group these factors and simply apply Al-Ali's result.

7.2 Implications for $V^k(\mathfrak{g})^{\mathbb{Z}_n}$

As we saw in section 2.5 the large k limit of $V^k(\mathfrak{sl}_2)$ is the rank 3 Heisenberg VOA. Additionally it has been shown that a set of strong generators for the orbifold of the limit algebra can be pulled back to generic values of k [L4]. At some levels where the $V^k(\mathfrak{sl}_2)$ fails to be simple the strong generators pull back to generators for the unique simple quotient though additional relations may be introduced, and we need to check for closure under OPE. At some values of k the equations for coefficients fail to be defined. These are not considered generic points here. Some of these poles are non-essential and can be moved by rescaling. Others can not be removed.

Let us define our action of $\mathbb{Z}_n$ as acting on the generators X^{e+f} and X^{e-f} as they did on h_1 and h_2, and by identity on X^h. Then in the limit algebra we have an action of $\mathbb{Z}_n$ on the tensor product of two Heisenberg VOAs and a third unaffected Heisenberg. The orbifold $V^k(\mathfrak{sl}_2)^{\mathbb{Z}_n}$ is then given by the strong generators for the rank 2 part the same as before together with the single weight one generator for the unaffected Heisenberg.

Therefore for generic values of k $V^k(\mathfrak{sl}_2)^{\mathbb{Z}_n}$ is of type $\mathcal{W}(1, 2, 3, 4, 5, n^2, (n+2)^2)$.

In the case of the $\mathbb{Z}_3$-orbifold we calculate the OPE's for all pairwise combinations of X, Y, $X, \partial Y$, $X, \partial^2 Y$, $X, \partial^3 Y$, X^3 $X^2, \partial^2 X$ and expressing the poles as normally ordered products of these same elements and their derivatives we can verify closure as long as the coefficients are defined for that value of k.

If fact we needn't perform all these calculations. We can express some of the poles as circle products of those with lower weight. This allows us to avoid the rather large calculation of $X, \partial^3 Y$ with $X^2, \partial^2 X$

From these calculations we found that $k = \frac{-9}{2}, \frac{27}{10}, \frac{27}{22}, \frac{9}{244}(33-\sqrt{113}), \frac{9}{244}(33+\sqrt{113})$ were poles of the OPE coefficients.

Hence if k is not one of these points we have a finite set of strong generators for the $\mathbb{Z}_3$ orbifold if it is simple and at least the simple quotient if it is not.

Chapter 8: Orbifolds and Zhu's commutative algebra

In this chapter we discuss the relationship between Zhu's commutative algebra $R_{\mathcal{V}}$ of a vertex algebra, and orbifolds under finite groups. We recall an important open conjecture in the subject: given a rational C_2-cofinite vertex algebra $\mathcal{V}$ and a finite group G of automorphisms of $\mathcal{V}$, it is expected that $\mathcal{V}^G$ is also rational and C_2-cofinite. In the case where G is a cyclic group, the C_2-cofiniteness of $\mathcal{V}^G$ was proven by Miyamoto [M], and the rationality was proven by Miyamoto and Carnahan in [CM]. McRae has shown under some very natural hypotheses on $\mathcal{V}$, namely that it is of CFT type and self-contradredient, the C_2-cofiniteness of $\mathcal{V}^G$ would imply the rationality of $\mathcal{V}^G$ for any finite G. It is then apparent that a fundamental problem is to show that finite group orbifolds preserve C_2-cofiniteness.

A related question is whether for a vertex algebra $\mathcal{V}$ and a finite group G of automorphisms of $\mathcal{V}$, the rings $R_{\mathcal{V}}$ and $R_{\mathcal{V}^G}$ have the same Krull dimension. Note for a strongly finitely generated vertex algebra $\mathcal{A}$, C_2-cofinitenss of $\mathcal{A}$ is equivalent to $R_{\mathcal{A}}$ having Krull dimension zero. Therefore if $\mathcal{V}$ and $\mathcal{V}^G$ are both strongly finitely generated, the statement that $R_{\mathcal{V}}$ and $R_{\mathcal{V}^G}$ have the same Krull dimension is really a generalization of the statement that C_2-cofiniteness is preserved by taking G-invariants.

Note that if G is a finite group of automorphisms of $\mathcal{V}$, G will act on $R_{\mathcal{V}}$ by automorphisms as well. The inclusion $\mathcal{V}^G \hookrightarrow \mathcal{V}$ induces a homomorphism

$$R_{\mathcal{V}^G} \to \mathcal{R}_{\mathcal{V}}. \tag{8.0.1}$$

This map need not be injective, but its image certainly lies in the invariant subalgebra $(R_{\mathcal{V}})^G$. A much more precise statement, which would in particular imply that $R_{\mathcal{V}}$ and

$R_{\mathcal{V}^G}$ have the same Krull dimension, is that (8.0.1) induces an isomorphism at the level of reduced rings, that is, the induced map $R_{\mathcal{V}^G}/\mathcal{N} \to (R_{\mathcal{V}})^G/\mathcal{N}'$ is an isomorphism, where $\mathcal{N}$ and $\mathcal{N}'$ are the respective nilradicals.

We end this chapter by showing that this is indeed the case in the example $\mathcal{V} = \mathcal{H}(2)$ and $G = \mathbb{Z}_3$. This is a very nontrivial statement because the nilradical of $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$ is quite large, but $(R_{\mathcal{H}(2)})^{\mathbb{Z}_3}$ has trivial nilradical.

Recall the Heisenberg fields B_1, B_2 satisfying $B_1(z)B_1(w) \sim 0$, $B_2(z)B_2(w) \sim 0$, and $B_1(z)B_2(w) \sim 2(z-w)^{-2}$. For convenience, we choose a slightly different strong generating set for the subalgebra $\mathcal{H}(2)^{SO(2)}$ than we did earlier. Recall that this subalgebra is of type $\mathcal{W}(2,3,4,5)$ and we use the generators L, W^3, W^4, W^5. The Virasoro field is

$$L = B_1B_2,$$

which has central charge $c = 2$, and we define

$$W^3 = \frac{1}{2}(B_1\partial B_2 - (\partial B_1)B_2,$$

which is primary of weight 3. Then we take

$$\begin{aligned} W^4 &= (W^3)_{(1)}W^3 = \frac{1}{4}\big(5B_1(\partial^2 B_2) - 6(\partial B_1)(\partial B_2) + 5(\partial^2 B_1)B_2\big), \\ W^5 &= (W^3)_{(1)}W^4 = \frac{5}{8}\big(7B_1(\partial^3 B_2) - 9(\partial B_1)(\partial^2 B_2) + 9(\partial^2 B_1)(\partial B_2) - 7(\partial^3 B_1)B_2\big). \end{aligned} \tag{8.0.2}$$

Then we need the additional fields of weight 3:

$$C^3 = (B_1)^3, \qquad D^3 = (B_2)^3,$$

and finally the fields of weight 5:

$$C^5 = (\partial^2 B_1)(B_1)^2, \qquad D^5 = (\partial^2 B_2)(B_2)^2.$$

As shown earlier (and also in [MPS]), this is a minimal strong generating set for $\mathcal{H}(2)^{\mathbb{Z}_3}$. Therefore the corresponding elements of $R_{\mathcal{H}(2)^{\mathbb{Z}/3\mathbb{Z}}}$, which we also denote by

$$L, W^3, W^4, W^5, C^3, D^3, C^5, D^5,$$

are a minimal generating set for $R_{\mathcal{H}(2)^{\mathbb{Z}/3\mathbb{Z}}}$. In particular, this shows that

$$R_{\mathcal{H}(2)^{\mathbb{Z}/3\mathbb{Z}}} \cong \mathbb{C}[L, W^3, W^4, W^5, C^3, D^3, C^5, D^5]/I,$$

for some ideal I.

Next, we claim that W^5 is in fact nilpotent of order 2 in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$. To see this, we construct a normally ordered relation of weight 10 in $\mathcal{H}(2)^{\mathbb{Z}_3}$ such that the only term that does not vanish in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$ is $:W^5W^5:$. The relation is quite nontrivial and was found with the help of Thielemans' Mathematica package [T], and it appears in the Appendix. There is a similar relation of weight 12 which shows that W^4 is nilpotent of order 3; this relation is omitted but can be found similarly by computer.

Next, we have the following normally ordered relation in $\mathcal{H}(2)^{\mathbb{Z}_3}$:

$$\begin{aligned}0 = W_3W_3W_3 - \frac{8}{27}LW_3W_4 + \frac{1}{108}LLW_5 - \frac{1}{1296}W_4W_5 + \frac{5}{18}\partial^2 LLW_3 \\ - \frac{1}{4}\partial L\partial LW_3 + \frac{5}{54}\partial LL\partial W_3 - \frac{35}{432}LL\partial^2 W_3 + \frac{671}{6912}\partial W_3\partial W_4 \\ - \frac{845}{6912}W_3\partial^2 W_4 - \frac{299}{20736}\partial^2 LW_5 - \frac{131}{6912}\partial L\partial W_5 - \frac{97}{51840}L\partial^2 W_5 \\ + \frac{10387}{82944}\partial^4 LW_3 - \frac{5441}{82944}\partial^3 L\partial W_3 + \frac{869}{9216}\partial^2 L\partial^2 W_3\end{aligned}$$

$$-\frac{11221}{82944}\partial L\partial^3 W_3+\frac{7015}{41472}L\partial^4 W_3+\frac{247}{155520}\partial^4 W_5-\frac{10439}{1658880}\partial^6 W_3 \quad (8.0.3)$$

Taking the image in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$ shows that the following relation holds in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$:

$$(W^3)^3-\frac{8}{27}LW^3W^4+\frac{1}{108}L^2W^5-\frac{1}{1296}W^4W^5=0.$$

Since W^4 and W^5 lie in the nilradical, this proves that W^3 lies in the nilradical as well.

We also have the normally ordered relations

$$\begin{aligned}
&C^5C^5+\frac{2}{27}(\partial C^3)(\partial^3 C^3)-\frac{1}{27}(\partial^2 C^3)(\partial^2 C^3)\\
&\quad-\frac{2}{9}(\partial^2 C^3)C^5-\frac{2}{9}(\partial C^3)(\partial C^5)\\
&D^5D^5+\frac{2}{27}(\partial D^3)(\partial^3 D^3)-\frac{1}{27}(\partial^2 D^3)(\partial^2 D^3)\\
&\quad-\frac{2}{9}(\partial^2 D^3)D^5-\frac{2}{9}(\partial D^3)(\partial D^5),
\end{aligned} \quad (8.0.4)$$

which imply that C^5 and D^5 are both nilpotent of order 2 in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$.

So far, these calculations show that the reduced ring $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}/\mathcal{N}$ is generated by L, C^3, D^3. Finally, we have the normally ordered relation in $\mathcal{H}(2)^{\mathbb{Z}_3}$

$$\begin{aligned}
&C_3D_3+117W_3W_3-33LW_4+9\partial LW_3+9L\partial W_3-LLL+\frac{153}{4}\partial^2 LL\\
&\quad-\frac{135}{4}\partial L\partial L-\frac{3}{10}\partial W_5-\frac{43}{4}\partial^2 W_4+\frac{3}{8}\partial^3 W_3+\frac{97}{16}\partial^4 L=0.
\end{aligned} \quad (8.0.5)$$

This yields the relation

$$C^3D^3+117(W^3)^2-33LW^4-L^3=0$$

in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$, and hence we get the relation

$$C^3D^3 - L^3 = 0$$

in the reduced ring $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}/\mathcal{N}$. It is not difficult to check that there are no further relations, so that

$$R_{\mathcal{H}(2)^{\mathbb{Z}_3}}/\mathcal{N} \cong \mathbb{C}[L, C^3, D^3]/\langle C^3D^3 - L^3\rangle.$$

Finally, we claim that this is isomorphic to $(R_{\mathcal{H}(2)})^{\mathbb{Z}/3\mathbb{Z}}$ since $R_{\mathcal{H}(2)} \cong \mathbb{C}[B_1, B_2]$ and the generators of $(R_{\mathcal{H}(2)})^{\mathbb{Z}/3\mathbb{Z}}$ are clearly B_1B_2, $(B_1)^3$, and $(B_2)^3$. Note that $(R_{\mathcal{H}(2)})^{\mathbb{Z}/3\mathbb{Z}}$ is already a reduced ring. Therefore we have proven

Theorem 8.0.1. In the example $\mathcal{V} = \mathcal{H}(2)$ and $G = \mathbb{Z}_3$, the reduced rings of $(R_{\mathcal{H}(2)})^{\mathbb{Z}_3}$ and $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$ are isomorphic.

It is an interesting question whether this holds in a more general setting. We note that the strong finite generation set for $\mathcal{H}(2)^{\mathbb{Z}_3}$ makes it possible to answer this question by direct computation, but significant new ideas are needed to study it in a general setting.

BIBLIOGRAPHY

[AM] D. Adamovic and A. Milas, *On the triplet vertex algebra $\mathcal{W}(p)$*, Adv. Math. 217 (2008), no. 6, 2664-2699.

[Al-A] M. Al-Ali, *The $\mathbb{Z}_2$-orbifold of the universal affine vertex algebra*, J. Pure Appl. Algebra 223 (2019), no. 12, 5430-5443.

[AL] M. Al-Ali and A. Linshaw, *The $\mathbb{Z}_2$-orbifold of the $\mathcal{W}_3$-algebra*, Comm. Math. Phys. 353 (2017), no. 3, 1129-1150.

[B] R. Borcherds, *Vertex operator algebras, Kac-Moody algebras and the monster*, Proc. Nat. Acad. Sci. USA 83 (1986) 3068-3071.

[CM] S. Carnahan and M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras*, arXiv:1603.05645.

[CL1] T. Creutzig and A. Linshaw, *Orbifolds of symplectic fermion algebras*, Trans. Amer. Math. Soc. 369, No. 1 (2017), 467-494.

[CL2] T. Creutzig and A. Linshaw, *Cosets of affine vertex algebras inside larger structures*, J. Algebra. 517 (2019) 396-438.

[CL3] T. Creutzig and A. Linshaw, *Trialities of $\mathcal{W}$-algebras*, arXiv:2005.10234,

[CL4] T. Creutzig and A. Linshaw, *Trialities of orthosymplectic $\mathcal{W}$-algebras*, arXiv:2102.10224

[DVVV] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models*, Comm. Math. Phys. 123 (1989), 485-526.

[DHVWI] L. Dixon, J. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds*, Nucl. Phys. B 261 (1985) 678-686.

[DHVWII] L. Dixon, J. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds II*, Nucl. Phys. B 274, (1986), 285-314.

[DM] C. Dong and G. Mason, *Holomorphic vertex operator algebras of small central charge*, Pacific J. Math. 213 (2004), no. 2, 253-266.

[DN] C. Dong and K. Nagatomo, *Classification of irreducible modules for the vertex operator algebra* $M(1)_+$, J. Algebra 216 (1999) no. 1, 384-404.

[DLY] C.Dong, C. Lam, H. Yamada, $\mathcal{W}$*-algebras related to parafermion algebras*, J. Algebra. no. 322, (2009), 2366-2403.

[DLM] C. Dong, H. Li, and G. Mason, *Compact automorphism groups of vertex operator algebras*, Int. Math. Res. Not. 18 (1996), 913-921.

[EMS1] J. van Ekeren, S. Möller, and N. Scheithauer, *Construction and classification of holomorphic vertex operator algebras*, J. Reine Angew. Math. 759 (2020), 61-99.

[EMS2] J. van Ekeren, S. Möller, and N. Scheithauer, *Dimension formulae in genus zero and uniqueness of vertex operator algebras*, Int. Math. Res. Not. IMRN 2020, no. 7, 2145-2204.

[ELMS] J. van Ekeren, C. H. Lam, S. Möller, and H Shimakura, *Schellekens' list and the very strange formula*, Adv. Math. 380 (2021), 107567.

[FBZ] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys and Monographs, Vol. 88, American Math. Soc., 2nd edition, 2004.

[FF] B. Feigin and E. Frenkel, *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. B. 246(1-2): 75-81, 1990.

[FLM] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1988.

[FZ] I. B. Frenkel and Y. C. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J, Vol. 66, No. 1, (1992), 123-168.

[HI] D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. 36 (1890) no. 4, 473-534.

[HII] D. Hilbert, *Über die vollen Invariantensysteme*, Math. Ann. 42 (1893) no. 3, 313-373.

[GR] D. Gaiotto and M. Rapčák, *Vertex Algebras at the Corner*, J. High Energy Phys. 2019, no. 1, 160, front matter+85 pp.

[K] V. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10. American Math. Soc., 1998.

[KL] S. Kanade and A. Linshaw, *Universal two-parameter even spin $\mathcal{W}_\infty$-algebra*, Advances in Mathematics 355 (2019), 106774, 58pp.

[LS] C. H. Lam and H. Shimakura, *On orbifold constructions associated with the Leech lattice vertex operator algebra*, Math. Proc. Cambridge Philos. Soc. 168 (2020), no. 2, 261-285.

[LiI] H. Li, *Local systems of vertex operators, vertex superalgebras and modules*, J. Pure Appl. Algebra 109 (1996), no. 2, 143-195.

[LiII] H. Li, *Vertex algebras and vertex Poisson algebras*, Commun. Contemp. Math. 6 (2004) 61-110.

[LZ] B. Lian and G. Zuckerman, *Commutative quantum operator algebras*, J. Pure Appl. Algebra 100 (1995) no. 1-3, 117-139.

[L1] A. Linshaw, *Invariant theory and the $\mathcal{W}_{1+\infty}$ algebra with negative integral central charge*, J. Eur. Math. Soc. 13, no. 6 (2011) 1737-1768.

[L2] A. Linshaw, *A Hilbert theorem for vertex algebras*, Transform. Groups, Vol. 15, No. 2 (2010), 427-448.

[L3] A. Linshaw, *Invariant theory and the Heisenberg vertex algebra*, Int. Math. Res. Notices, 17 (2012), 4014-4050.

[L4] A. Linshaw, *Invariant subalgebras of affine vertex algebras*, Adv. Math. 234 (2013), 61-84.

[L5] A. Linshaw, *The structure of the Kac-Wang-Yan algebra*, Commun. Math. Phys. 345, No. 2, (2016), 545-585.

[LL] B. Lian and A. Linshaw, *Vertex algebras and commutative algebra*, to appear in Harvard CMSA , The Literature and History of Mathematical Science.

[L6] A. Linshaw, *Universal two-parameter $\mathcal{W}_\infty$-algebra and vertex algebras of type $\mathcal{W}(2,3,\dots,N)$*, Compos. Math. 157 (2021), no. 1, 12-82.

[McR] R. McRae, *Twisted modules and G-equivariantization in logarithmic conformal field theory*, published online in Comm. Math. Phys. doi.org/10.1007/s00220-020-03882-2.

[MPS] A. Milas, M. Penn, and H. Shao, *Permutation orbifolds of the Heisenberg vertex algebra $\mathcal{H}(3)$*, J. Math. Phys. 60 (2019), no. 2, 021703, 17 pp.

[M] M. Miyamoto, C_2-*cofiniteness of cyclic orbifold models*, Comm. Math. Phys. 335 (2015) 1279-1286.

[Sch] A. N. Schellekens, Meromorphic $c = 24$ conformal field theories, Comm. Math. Phys. 153 (1993), no. 1, 159-185.

[T] Thielemans, K. A Mathematica package for computing operator product expansions. Internat. J. Modern Phys. C 2 (1991), no. 3, 787-798.

[W] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1946.

[Z] Y. Zhu, *Modular invariants of characters of vertex operators*, J. Amer. Math. Soc. 9 (1996) 237-302.

APPENDIX

.1 Additional generation formulas

Formulas for generating elements needed to show closure under OPE.

Since the equations become quite lengthy we from here we present one general form equation with a lists of coefficients

Weight $n+4$

$$A_i = \Big(\frac{6}{(n-3)(n-2)(n-1)(4n+1)}\Big)\big((a_i)\partial B_1\partial B_1\partial B_1 B_1^{n-3} + (b_i)\partial^2 B_1\partial B_1 B_1^{n-2} + (c_i)\partial^3 B_1 B_1^{n-1} + (d_i)(B_1B_2\partial^2 B_1 B_1^{n-1} - \partial^2 B_1 B_2 B_1^{n}) + (e_i)(B_1B_2\partial B_1\partial B_1 B_1^{n-2} - \partial B_1 B_2 \partial B_1 B_1^{n-1})\big)$$

$$A_1 = \partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-4} \qquad A_2 = \partial^2 B_1\partial B_1\partial B_1 B_1^{n-3} \qquad A_3 = \partial^2 B_1\partial^2 B_1 B_1^{n-2}$$
$$A_4 = \partial^3 B_1\partial B_1 B_1^{n-2} \qquad A_5 = \partial^4 B_1 B_1^{n-1} \tag{.1.1}$$

$$a_1 = \frac{2n^3}{3} - \frac{11n^2}{6} + \frac{5n}{6} + \frac{1}{3}, \qquad b_1 = n^2 - 6n + 5, \qquad c_1 = -3n^2 + \frac{21n}{2} - \frac{11}{2},$$
$$d_1 = 5 - n, \qquad e_1 = -3n^2 + \frac{15n}{2} - \frac{9}{2} \tag{.1.2}$$

$$a_2 = 0, \qquad b_2 = -\frac{n^3}{3} + 3n^2 - \frac{23n}{3} + 5, \qquad c_2 = n^3 - \frac{13n^2}{2} + \frac{37n}{3} - \frac{11}{2},$$
$$d_2 = \frac{n^2}{3} - \frac{8n}{3} + 5, \qquad e_2 = n^3 - \frac{11n^2}{2} + 9n - \frac{9}{2}, \tag{.1.3}$$

$$a_3 = 0, \qquad b_3 = n^4 - \frac{13n^3}{2} + 14n^2 - \frac{23n}{2} + 3,$$
$$c_3 = -n^4 + \frac{41n^3}{6} - \frac{95n^2}{6} + \frac{43n}{3} - 4, \qquad d_3 = -\frac{n^3}{3} + \frac{7n^2}{3} - \frac{16n}{3} + 4, \tag{.1.4}$$
$$e_3 = -n^4 + \frac{13n^3}{2} - 14n^2 + \frac{23n}{2} - 3,$$

$$a_4 = 0, \qquad b_4 = -n^3 + 6n^2 - 11n + 6, \qquad c_4 = \frac{5n^3}{3} - \frac{19n^2}{2} + \frac{95n}{6} - 7,$$
$$d_4 = n^2 - 5n + 6, \qquad e_4 = n^3 - 6n^2 + 11n - 6 \tag{.1.5}$$

$$a_5 = 0, \qquad b_5 = n^4 - 7n^3 + 17n^2 - 17n + 6,$$
$$c_5 = -n^4 + \frac{22n^3}{3} - 19n^2 + \frac{62n}{3} - 8, \qquad d_5 = -n^3 + 6n^2 - 11n + 6, \tag{.1.6}$$
$$e_5 = -n^4 + 7n^3 - 17n^2 + 17n - 6$$

Weight $n + 5$

$$A_i = \frac{6}{(n-4)(n-3)(n-2)^2(n-1)(38n-11)}$$
$$\big(a_i\partial(\partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-4} + b_i\partial(\partial^2 B_1\partial B_1\partial B_1 B_1^{n-3}) + c_i\partial(\partial^2 B_1\partial^2 B_1 B_1^{n-2})$$
$$+ d_i\partial(\partial^3 B_1\partial B_1 B_1^{n-2}) + e_i\partial(\partial^4 B_1 B_1^{n-1}) + f_i((B_1B_2)(\partial^3 B_1 B_1^{n-1}) - (\partial^3 B_1 B_2)(B_1^n))$$
$$+ g_i((B_1B_2)(\partial^2 B_1\partial B_1 B_1^{n-2}) - (\partial^2 B_1 B_2)(\partial B_1 B_1^{n-1}))$$

$$A_1 = \partial B_1 \partial B_1 \partial B_1 \partial B_1 \partial B_1 B_1^{n-5}, \qquad A_2 = \partial^2 B_1 \partial B_1 \partial B_1 \partial B_1 B_1^{n-4},$$
$$A_3 = \partial^2 B_1 \partial^2 B_1 \partial B_1 B_1^{n-3}, \qquad A_4 = \partial^3 B_1 \partial B_1 \partial B_1 B_1^{n-3}, \qquad A_5 = \partial^3 B_1 \partial^2 B_1 B_1^{n-2},$$
$$A_6 = \partial^4 B_1 \partial B_1 B_1^{n-2}, \qquad A_7 = \partial^5 B_1 B_1^{n-1} \tag{.1.7}$$

$$\begin{aligned}
a_1 &= \frac{19n^5}{3} - \frac{105n^4}{2} + \frac{481n^3}{3} - \frac{439n^2}{2} + \frac{382n}{3} - 22, \\
b_1 &= -\frac{76n^4}{3} + 134n^3 - \frac{718n^2}{3} + 160n - \frac{88}{3}, \\
c_1 &= -\frac{148n^3}{3} + \frac{532n^2}{3} - \frac{560n}{3} + \frac{176}{3}, \\
d_1 &= \frac{76n^3}{3} - \frac{250n^2}{3} + \frac{218n}{3} - \frac{44}{3}, \\
e_1 &= 100n^3 - \frac{1258n^2}{3} + 512n - \frac{440}{3}, \\
f_1 &= \frac{20n^2}{3} - \frac{106n}{3} + 44, \\
g_1 &= 100n^3 - 344n^2 + 332n - 88
\end{aligned} \tag{.1.8}$$

$$\begin{aligned}
a_2 &= 0, \\
b_2 &= \frac{19n^5}{3} - \frac{353n^4}{6} + \frac{1163n^3}{6} - \frac{838n^2}{3} + \frac{502n}{3} - \frac{88}{3}, \\
c_2 &= \frac{37n^4}{3} - \frac{281n^3}{3} + 224n^2 - \frac{604n}{3} + \frac{176}{3}, \\
d_2 &= -\frac{19n^4}{3} + \frac{277n^3}{6} - \frac{203n^2}{2} + \frac{229n}{3} - \frac{44}{3}, \\
e_2 &= -25n^4 + \frac{1229n^3}{6} - \frac{1642n^2}{3} + \frac{1646n}{3} - \frac{440}{3}, \\
f_2 &= -\frac{5n^3}{3} + \frac{31n^2}{2} - \frac{139n}{3} + 44, \\
g_2 &= -25n^4 + 186n^3 - 427n^2 + 354n - 88
\end{aligned} \tag{.1.9}$$

$$
\begin{aligned}
a_3 &= 0,\\
b_3 &= 0,\\
c_3 &= -\frac{11n^5}{3} + \frac{233n^4}{6} - 150n^3 + \frac{1555n^2}{6} - \frac{589n}{3} + 52,\\
d_3 &= 0, \qquad (.1.10)\\
e_3 &= 10n^5 - 109n^4 + 437n^3 - 788n^2 + 612n - 144,\\
f_3 &= \frac{2n^4}{3} - \frac{23n^3}{3} + \frac{97n^2}{3} - \frac{178n}{3} + 40,\\
g_3 &= 10n^5 - 104n^4 + 390n^3 - 640n^2 + 440n - 96
\end{aligned}
$$

$$
\begin{aligned}
a_4 &= 0,\\
b_4 &= 0,\\
c_4 &= -5n^5 + 53n^4 - 205n^3 + 355n^2 - 270n + 72,\\
d_4 &= \frac{19n^5}{3} - \frac{391n^4}{6} + 240n^3 - \frac{2285n^2}{6} + \frac{731n}{3} - 44, \qquad (.1.11)\\
e_4 &= 5n^5 - \frac{371n^4}{6} + \frac{1727n^3}{6} - \frac{1844n^2}{3} + \frac{1706n}{3} - 152,\\
f_4 &= \frac{n^4}{3} - \frac{31n^3}{6} + \frac{169n^2}{6} - \frac{193n}{3} + 52,\\
g_4 &= 5n^5 - 53n^4 + 205n^3 - 355n^2 + 270n - 72,
\end{aligned}
$$

$$
\begin{aligned}
a_5 &= 0, \\
b_5 &= 0, \\
c_5 &= 5n^6 - 62n^5 + 299n^4 - 710n^3 + 860n^2 - 488n + 96, \\
d_5 &= 0, \\
e_5 &= -5n^6 + \frac{129n^5}{2} - \frac{655n^4}{2} + 831n^3 - 1094n^2 + 684n - 144, \\
f_5 &= -\frac{n^5}{3} + \frac{9n^4}{2} - \frac{143n^3}{6} + 62n^2 - \frac{238n}{3} + 40, \\
g_5 &= -5n^6 + 62n^5 - 299n^4 + 710n^3 - 860n^2 + 488n - 96,
\end{aligned}
\tag{.1.12}
$$

$$
\begin{aligned}
a_6 &= 0, \\
b_6 &= 0, \\
c_6 &= -n^5 + 12n^4 - 55n^3 + 120n^2 - 124n + 48, \\
d_6 &= 0, \\
e_6 &= \frac{22n^5}{3} - 84n^4 + \frac{1078n^3}{3} - 704n^2 + \frac{1816n}{3} - 160, \\
f_6 &= \frac{4n^4}{3} - \frac{44n^3}{3} + \frac{176n^2}{3} - \frac{304n}{3} + 64, \\
g_6 &= n^5 - 12n^4 + 55n^3 - 120n^2 + 124n - 48,
\end{aligned}
\tag{.1.13}
$$

$$
\begin{aligned}
a_7 &= 0, \\
b_7 &= 0, \\
c_7 &= n^6 - 13n^5 + 67n^4 - 175n^3 + 244n^2 - 172n + 48, \\
d_7 &= 0, \\
e_7 &= -n^6 + \frac{27n^5}{2} - 73n^4 + \frac{405n^3}{2} - 304n^2 + 234n - 72, \\
f_7 &= -\frac{4n^5}{3} + 16n^4 - \frac{220n^3}{3} + 160n^2 - \frac{496n}{3} + 64, \\
g_7 &= -n^6 + 13n^5 - 67n^4 + 175n^3 - 244n^2 + 172n - 48,
\end{aligned}
\tag{.1.14}
$$

Weight $n + 6$

$$
A_i = \frac{-90}{(-5+n)(-4+n)(-3+n)(-2+n)^2(-1+n)(-85-253n+180n^2)}
$$

$$
\big(a_i\partial(\partial B_1\partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-5} + b_i\partial(\partial^2 B_1\partial B_1\partial B_1\partial B_1 B_1^{n-4})
$$

$$
+ c_i\partial(\partial^2 B_1\partial^2 B_1\partial B_1 B_1^{n-3}) + d_i\partial(\partial^3 B_1\partial B_1\partial B_1 B_1^{n-3}) + e_i\partial(\partial^3 B_1\partial^2 B_1 B_1^{n-2})
$$

$$
+ f_i\partial(\partial^4 B_1\partial B_1 B_1^{n-2}) + g_i\partial(\partial^5 B_1 B_1^{n-1}) + h_i((B_1B_2)(\partial^4 B_1 B_1^{n-1}) - (\partial^4 B_1 B_2)(B_1^n))
$$

$$
+ l_i((B_1B_2)(\partial^3 B_1\partial B_1 B_1^{n-2}) - (\partial^3 B_1 B_2)(\partial B_1 B_1^{n-1}))+
$$

$$
m_i((B_1B_2)(\partial^2 B_1\partial^2 B_1 B_1^{n-2}) - (\partial^2 B_1 B_2)(\partial^2 B_1 B_1^{n-1}))+
$$

$$
q_i((B_1B_2)(\partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-4}) - (\partial B_1 B_2)(\partial B_1\partial B_1\partial B_1 B_1^{n-3}))\big)
$$

$$
\begin{aligned}
&A_1 = \partial B_1\partial B_1\partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-6}, \qquad A_2 = \partial^2 B_1\partial B_1\partial B_1\partial B_1\partial B_1 B_1^{n-5},\\
&A_3 = \partial^2 B_1\partial^2 B_1\partial B_1\partial B_1 B_1^{n-4}, \qquad A_4 = \partial^2 B_1\partial^2 B_1\partial^2 B_1 B_1^{n-3},\\
&A_5 = \partial^3 B_1\partial B_1\partial B_1\partial B_1 B_1^{n-4}, \qquad A_6 = \partial^3 B_1\partial^2 B_1\partial B_1 B_1^{n-3},\\
&A_7 = \partial^3 B_1\partial^3 B_1 B_1^{n-2},\\
&A_8 = \partial^4 B_1\partial B_1\partial B_1 B_1^{n-3},\\
&A_9 = \partial^4 B_1\partial^2 B_1 B_1^{n-2},\\
&A_{10} = \partial^5 B_1\partial B_1 B_1^{n-2},\\
&A_{11} = \partial^1 B_1 B_1^{n-1},
\end{aligned}
\tag{.1.15}
$$

$$
\begin{aligned}
a_1 &= -\frac{1}{90}(n-4)(n-3)(n-2)^2(n-1)\left(180n^2-253n-85\right),\\
b_1 &= \frac{1}{9}(n-3)(n-2)^2(n-1)\left(45n^2-149n-254\right),\\
c_1 &= -\frac{1}{3}(n-2)^2(n-1)\left(45n^2-149n-254\right),\\
d_1 &= \frac{1}{18}(n-2)^2(n-1)\left(90n^3-555n^2+466n-1889\right),\\
e_1 &= \frac{10}{9}(n-2)(n-1)(3n-1)\left(6n^2-53n+73\right),\\
f_1 &= \frac{1}{18}(n-2)(n-1)\left(90n^3+105n^2+110n-649\right),\\
g_1 &= -\frac{1}{18}(n-2)\left(540n^4-5616n^3+15195n^2-13504n+2417\right),\\
h_1 &= -\frac{1}{6}(n-2)\left(60n^3-587n^2+1762n-1719\right),\\
l_1 &= \frac{1}{3}(-5)(n-2)(n-1)(3n-1)\left(6n^2-53n+73\right),\\
m_1 &= \frac{1}{3}(n-2)(n-1)\left(45n^2-149n-254\right),\\
q_1 &= \frac{1}{2}(n-3)(n-2)^2(n-1)\left(10n^2+5n+47\right)
\end{aligned}
\tag{.1.16}
$$

$$
\begin{aligned}
a_2 &= 0, \\
b_2 &= -\frac{1}{45}(n-5)(n-3)(n-2)^2(n-1)\left(45n^2-149n-254\right), \\
c_2 &= \frac{1}{15}(n-5)(n-2)^2(n-1)\left(45n^2-149n-254\right), \\
d_2 &= -\frac{1}{90}(n-5)(n-2)^2(n-1)\left(90n^3-555n^2+466n-1889\right), \\
e_2 &= \frac{1}{9}(-2)(n-5)(n-2)(n-1)(3n-1)\left(6n^2-53n+73\right), \\
f_2 &= -\frac{1}{90}(n-5)(n-2)(n-1)\left(90n^3+105n^2+110n-649\right), \\
g_2 &= \frac{1}{90}(n-5)(n-2)\left(540n^4-5616n^3+15195n^2-13504n+2417\right), \\
h_2 &= \frac{1}{30}(n-5)(n-2)\left(60n^3-587n^2+1762n-1719\right), \\
l_2 &= \frac{1}{3}(n-5)(n-2)(n-1)(3n-1)\left(6n^2-53n+73\right), \\
m_2 &= -\frac{1}{15}(n-5)(n-2)(n-1)\left(45n^2-149n-254\right), \\
q_2 &= -\frac{1}{10}(n-5)(n-3)(n-2)^2(n-1)\left(10n^2+5n+47\right)
\end{aligned}
\tag{.1.17}
$$

$$
\begin{aligned}
a_3 &= 0, \\
b_3 &= -\frac{1}{90}(n-5)(n-4)(n-3)(n-2)^2(n-1)\left(30n^2+5n+97\right), \\
c_3 &= -\frac{1}{45}(n-5)(n-4)(n-2)^2(n-1)\left(45n^2-134n-188\right) \\
d_3 &= \frac{1}{270}(n-5)(n-4)(n-2)^2(n-1)(3n-14)\left(30n^2+5n+97\right) \\
e_3 &= \frac{2}{27}(n-5)(n-4)(n-2)(n-1)\left(18n^3-135n^2+205n-46\right) \\
f_3 &= \frac{1}{270}(n-5)(n-4)(n-2)(n-1)(3n-4)\left(30n^2+5n+97\right) \\
g_3 &= -\frac{1}{270}(n-5)(n-4)(n-2)\left(540n^4-4716n^3+11715n^2-9719n+1544\right) \\
h_3 &= -\frac{1}{90}(n-5)(n-4)(n-2)\left(60n^3-487n^2+1297n-1188\right) \\
l_3 &= -\frac{1}{9}(n-5)(n-4)(n-2)(n-1)\left(18n^3-135n^2+205n-46\right) \\
m_3 &= \frac{1}{45}(n-5)(n-4)(n-2)(n-1)\left(45n^2-134n-188\right) \\
q_3 &= \frac{1}{90}(n-5)(n-4)(n-3)(n-2)^2(n-1)\left(30n^2+5n+97\right)
\end{aligned}
\tag{.1.18}
$$

$$
\begin{aligned}
a_4 &= 0\\
b_4 &= \frac{1}{18}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)\left(6n^2-11n+1\right)\\
c_4 &= -\frac{1}{6}(n-5)(n-4)(n-3)(n-2)^2(n-1)\left(6n^2-11n+1\right)\\
d_4 &= -\frac{1}{54}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-14)\left(6n^2-11n+1\right)\\
e_4 &= \frac{1}{27}(-2)(n-5)(n-4)(n-3)(n-2)(n-1)(3n-5)\left(6n^2-11n+1\right)\\
f_4 &= -\frac{1}{54}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-4)\left(6n^2-11n+1\right)\\
g_4 &= \frac{1}{270}(n-5)(n-4)(n-3)(n-2)\left(6n^2-11n+1\right)\left(90n^2-261n+190\right)\\
h_4 &= \frac{1}{90}(n-5)(n-4)(n-3)(n-2)(3n-5)\left(20n^2-49n+48\right)\\
l_4 &= \frac{1}{9}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-5)\left(6n^2-11n+1\right)\\
m_4 &= -\frac{1}{45}(n-5)(n-4)(n-3)(n-2)(n-1)\left(45n^2-44n-50\right)\\
q_4 &= -\frac{1}{18}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)\left(6n^2-11n+1\right)
\end{aligned}
\tag{.1.19}
$$

$$
\begin{aligned}
a_5 &= 0,\\
b_5 &= -\frac{1}{15}(n-5)(n-4)(n-3)(n-2)^2(n-1)(5n+22),\\
c_5 &= \frac{1}{5}(n-5)(n-4)(n-2)^2(n-1)(5n+22),\\
d_5 &= -\frac{1}{90}(n-5)(n-4)(n-2)^2(n-1)\left(150n^2-245n+531\right),\\
e_5 &= \frac{1}{9}(-2)(n-5)(n-4)(n-2)(n-1)\left(30n^2-67n+27\right),\\
f_5 &= \frac{1}{90}(n-5)(n-4)(n-2)(n-1)\left(210n^2-161n-261\right),\\
g_5 &= \frac{1}{90}(n-5)(n-4)(n-2)\left(900n^3-3480n^2+3785n-873\right),\\
h_5 &= \frac{1}{30}(n-5)(n-4)(n-2)\left(100n^2-465n+531\right),\\
l_5 &= \frac{1}{3}(n-5)(n-4)(n-2)(n-1)\left(30n^2-67n+27\right),\\
m_5 &= -\frac{1}{5}(n-5)(n-4)(n-2)(n-1)(5n+22),\\
q_5 &= \frac{1}{15}(n-5)(n-4)(n-3)(n-2)^2(n-1)(5n+22),
\end{aligned}
\tag{.1.20}
$$

$$
\begin{aligned}
a_6 &= 0,\\
b_6 &= \frac{1}{45}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)(15n+23),\\
c_6 &= -\frac{1}{15}(n-5)(n-4)(n-3)(n-2)^2(n-1)(15n+23),\\
d_6 &= -\frac{1}{135}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-14)(15n+23),\\
e_6 &= \frac{1}{27}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-1)(24n-41),\\
f_6 &= -\frac{1}{135}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-4)(15n+23),\\
g_6 &= -\frac{1}{270}(n-5)(n-4)(n-3)(n-2)\left(1080n^3-3807n^2+3684n-677\right),\\
h_6 &= -\frac{1}{45}(n-5)(n-4)(n-3)(n-2)\left(60n^2-227n+237\right),\\
l_6 &= -\frac{1}{18}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-1)(24n-41),\\
m_6 &= \frac{1}{15}(n-5)(n-4)(n-3)(n-2)(n-1)(15n+23),\\
q_6 &= -\frac{1}{45}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)(15n+23),
\end{aligned}
\tag{.1.21}
$$

$$
\begin{aligned}
a_7 &= 0, \\
b_7 &= \frac{1}{15}(-2)(n-5)(n-4)(n-3)^2(n-2)^3(n-1)(5n+3), \\
c_7 &= \frac{2}{5}(n-5)(n-4)(n-3)(n-2)^3(n-1)(5n+3), \\
d_7 &= \frac{2}{45}(n-5)(n-4)(n-3)(n-2)^3(n-1)(3n-14)(5n+3), \\
e_7 &= -\frac{1}{90}(n-5)(n-4)(n-3)(n-2)^2(n-1)\left(300n^2-503n-15\right), \\
f_7 &= \frac{2}{45}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-4)(5n+3), \\
g_7 &= \frac{1}{90}(n-5)(n-4)(n-3)(n-2)^2\left(180n^3-687n^2+674n-99\right), \\
h_7 &= \frac{1}{30}(n-5)(n-4)(n-3)(n-2)^2\left(20n^2-69n+83\right), \\
l_7 &= \frac{1}{6}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-1)(4n-7), \\
m_7 &= \frac{1}{5}(-2)(n-5)(n-4)(n-3)(n-2)^2(n-1)(5n+3), \\
q_7 &= \frac{2}{15}(n-5)(n-4)(n-3)^2(n-2)^3(n-1)(5n+3),
\end{aligned}
\tag{.1.22}
$$

$$
\begin{aligned}
a_8 &= 0, \\
b_8 &= -\frac{1}{9}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)(3n-4), \\
c_8 &= \frac{1}{3}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-4), \\
d_8 &= \frac{1}{27}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-14)(3n-4), \\
e_8 &= \frac{4}{27}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-5)(3n-4), \\
f_8 &= -\frac{1}{270}(n-5)(n-4)(n-3)(n-2)(n-1)\left(450n^2-519n-415\right), \qquad (.1.23) \\
g_8 &= -\frac{1}{270}(n-5)(n-4)(n-3)(n-2)\left(540n^3-2826n^2+3987n-1265\right), \\
h_8 &= -\frac{1}{90}(n-5)(n-4)(n-3)(n-2)(3n-5)(20n-129), \\
l_8 &= \frac{1}{9}(-2)(n-5)(n-4)(n-3)(n-2)(n-1)(3n-5)(3n-4), \\
m_8 &= -\frac{1}{3}(n-5)(n-4)(n-3)(n-2)(n-1)(3n-4), \\
q_8 &= \frac{1}{9}(n-5)(n-4)(n-3)^2(n-2)^2(n-1)(3n-4),
\end{aligned}
$$

$$
\begin{aligned}
a_9 &= 0,\\
b_9 &= \frac{1}{9}(n-5)(n-4)(n-3)^2(n-2)^3(n-1)(3n-1),\\
c_9 &= -\frac{1}{3}(n-5)(n-4)(n-3)(n-2)^3(n-1)(3n-1),\\
d_9 &= -\frac{1}{27}(n-5)(n-4)(n-3)(n-2)^3(n-1)(3n-14)(3n-1),\\
e_9 &= \frac{1}{27}(-4)(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-5)(3n-1),\\
f_9 &= -\frac{1}{27}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-4)(3n-1),\\
g_9 &= \frac{1}{135}(n-5)(n-4)(n-3)(n-2)^2(3n-1)\left(90n^2-261n+190\right),\\
h_9 &= \frac{1}{90}(n-5)(n-4)(n-3)(n-2)^2\left(60n^2-247n+225\right),\\
l_9 &= \frac{2}{9}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-5)(3n-1),\\
m_9 &= \frac{1}{3}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-1),\\
q_9 &= -\frac{1}{9}(n-5)(n-4)(n-3)^2(n-2)^3(n-1)(3n-1),
\end{aligned}
\tag{.1.24}
$$

$$
\begin{aligned}
a_{10} &= 0,\\
b_{10} &= -\frac{1}{3}(n-5)(n-4)(n-3)^2(n-2)^3(n-1),\\
c_{10} &= (n-5)(n-4)(n-3)(n-2)^3(n-1),\\
d_{10} &= \frac{1}{9}(n-5)(n-4)(n-3)(n-2)^3(n-1)(3n-14),\\
e_{10} &= \frac{4}{9}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-5),\\
f_{10} &= \frac{1}{9}(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-4),\\
g_{10} &= -\frac{1}{18}(n-5)(n-4)(n-3)(n-2)^2\left(72n^2-155n+59\right),\\
h_{10} &= \frac{1}{3}(-2)(n-5)(n-4)(n-3)(n-2)^2(4n-7),\\
l_{10} &= \frac{1}{3}(-2)(n-5)(n-4)(n-3)(n-2)^2(n-1)(3n-5),\\
m_{10} &= -(n-5)(n-4)(n-3)(n-2)^2(n-1),\\
q_{10} &= \frac{1}{3}(n-5)(n-4)(n-3)^2(n-2)^3(n-1),
\end{aligned}
\tag{.1.25}
$$

$$
\begin{aligned}
a_{11} &= 0, \\
b_{11} &= \frac{1}{3}(n-5)(n-4)(n-3)^2(n-2)^3(n-1)^2, \\
c_{11} &= -(n-5)(n-4)(n-3)(n-2)^3(n-1)^2, \\
d_{11} &= -\frac{1}{9}(n-5)(n-4)(n-3)(n-2)^3(n-1)^2(3n-14), \\
e_{11} &= \frac{1}{9}(-4)(n-5)(n-4)(n-3)(n-2)^2(n-1)^2(3n-5), \\
f_{11} &= -\frac{1}{9}(n-5)(n-4)(n-3)(n-2)^2(n-1)^2(3n-4), \\
g_{11} &= \frac{1}{45}(n-5)(n-4)(n-3)(n-2)^2(n-1)\left(90n^2-261n+190\right), \\
h_{11} &= \frac{2}{3}(n-5)(n-4)(n-3)(n-2)^2(n-1)(4n-7), \\
l_{11} &= \frac{2}{3}(n-5)(n-4)(n-3)(n-2)^2(n-1)^2(3n-5), \\
m_{11} &= (n-5)(n-4)(n-3)(n-2)^2(n-1)^2, \\
q_{11} &= -\frac{1}{3}(n-5)(n-4)(n-3)^2(n-2)^3(n-1)^2,
\end{aligned}
\tag{.1.26}
$$

.2 Relation in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$

Here we give the normally ordered relation of weight 10 in $\mathcal{H}(2)^{\mathbb{Z}_3}$ which shows that the image of W^5 in $R_{\mathcal{H}(2)^{\mathbb{Z}_3}}$ is nilpotent of order 2. All products shown are normally ordered products.

$$
\begin{aligned}
0 = W_5W_5 &+ \frac{2678685876576455}{18149101955424}\partial^4W_3W_3 - \frac{2459792895829160 3}{4537275488856}\partial^3W_3\partial W_3 \\
&- \frac{70459390989449017}{18149101955424}\partial^2W_3\partial^2W_3 + \frac{550229856861384460603}{112052555472787776 0}\partial^5LW_3 \\
&+ \frac{988237374940010304 29}{74701703648525184}\partial^4L\partial W_3 + \frac{242947343720488465879}{112052555472787776}\partial^3L\partial^2W_3 \\
&+ \frac{2615174311696151}{4537275488856}\partial^2L\partial^3W_3 + \frac{133200897472088669 89}{27441442156601088}\partial L\partial^4W_3
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{5425880387358766690457}{747017036485251840} L\partial^5 W_3 + \frac{3250577620272965680997}{4482102218911511040} \partial^6 LL \\
&+ \frac{1530862961404178680087}{498011357656834560} \partial^5 L\partial L - \frac{7199212099288233017775}{4482102218911511040} \partial^4 L\partial^2 L \\
&- \frac{2340006490164784065523}{336157666418363328} \partial^3 L\partial^3 L - \frac{3383993617064571031}{231513544365264} \partial LLLW_3 \\
&- \frac{1975692656753484934 9}{2546648988017904} LLL\partial W_3 + \frac{810233150077363}{1649918359584} \partial^2 LLLL \\
&- \frac{5769292145513}{149992578144} \partial L\partial LLL - \frac{4837420}{729} \partial W_3 W_3 W_3 \\
&- \frac{59186849919762035}{17324142775632} \partial LW_3 W_4 + \frac{4659534277842055081}{2546648988017904} L\partial W_3 W_4 \\
&- \frac{1720627402630504009}{848882996005968} LW_3 \partial W_4 + \frac{573978570316214009}{389793212451720} \partial LLW_5 \\
&+ \frac{5658707416671436417}{12733244940089520} LL\partial W_5 + \frac{169096934724515}{17186649579} \partial^2 LW_3 W_3 \\
&+ \frac{117085729880576503}{824959179792} \partial L\partial W_3 W_3 + \frac{11924173973893315}{824959179792} L\partial^2 W_3 \partial W_3 \\
&+ \frac{17124387338608915}{824959179792} L\partial W_3 \partial W_3 - \frac{36368670441116605}{4949755078752} \partial^2 LLW_4 \\
&- \frac{108118564782505561}{4949755078752} \partial L\partial LW_4 - \frac{133561545160379497}{4949755078752} \partial LL\partial W_4 \\
&- \frac{566811815984507}{206239794948} LL\partial^2 W_4 + \frac{4507075693834468103}{926054177461056} \partial^3 LLW_3 \\
&+ \frac{349717658025915187}{69296571102528} \partial^2 L\partial LW_3 + \frac{2619911019397881283}{199737175530816} \partial^2 LL\partial W_3 \\
&+ \frac{2376590794970953277}{103944856653792} \partial L\partial L\partial W_3 + \frac{7662799231498399301}{636662247004476} \partial LL\partial^2 W_3 \\
&+ \frac{7165692706619468687}{1455227993153088} LL\partial^3 W_3 + \frac{9675799558089924195 5}{56026277736393888} \partial^4 LLL \\
&+ \frac{4476379133443605450521}{224105110945575552} \partial^3 L\partial LL - \frac{1347236295730456170 47}{747017036485251 84} \partial^2 L\partial^2 LL \\
&- \frac{529168293094776413}{24198802607232} \partial^2 L\partial L\partial L + \frac{2140178990477309}{1088946117325440} \partial^5 W_5 \\
&- \frac{116808330690631 43}{3920206022371584} \partial^6 W_4 + \frac{5559836130078844321}{274414421566010880} \partial^7 W_3 \\
&+ \frac{1070001793728119388149}{31374715532380577280} \partial^8 L + \frac{4443860755569613}{7424632618128} \partial^2 C_3 D_5 \\
&+ \frac{1263672463162999}{1856158154532} \partial C_3 \partial D_5 - \frac{857226698097659}{2474877539376} C_3 \partial^2 D_5 \\
&+ \frac{708191543347217}{1856158154532} \partial W_3 C_3 D_3 - \frac{968559207665923}{1856158154532} W_3 \partial C_3 D_3
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{264594040910083}{1856158154532} W_3 C_3 \partial D_3 + \frac{9275041328429}{154679846211} \partial^2 L C_3 D_3 \\
&+ \frac{197627596332868}{1392118615899} \partial L \partial C_3 D_3 + \frac{9550332016995761}{22273897854384} \partial L C_3 \partial D_3 \\
&- \frac{10795943032606898659}{7639946964053712} L \partial^2 C_3 D_3 - \frac{1788880550400697}{7424632618128} L \partial C_3 \partial D_3 \\
&+ \frac{3994672368727784483}{3819973482026856} L C_3 \partial^2 D_3 + \frac{929018319315929273}{2940154516778688} \partial^4 C_3 D_3 \\
&+ \frac{112793621764570}{567159436107} \partial^3 C_3 \partial D_3 - \frac{113079691834929103}{490025752796448} \partial^2 C_3 \partial^2 D_3 \\
&- \frac{10604820623892227}{122506438199112} \partial C_3 \partial^3 D_3 + \frac{1198900593926728 57}{2940154516778688} C_3 \partial^4 D_3 \quad (.2.1)
\end{aligned}
$$