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Banach Spaces on Topological Ramsey Structures

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Abstract

A Banach space $T_1(d, \theta)$ with a Tsirelson-type norm is constructed on the top of the topological Ramsey space T_1 defined by Dobrinen and Todorcevic [6]. Finite approximations of the isomorphic subtrees are utilised in constructing the norm. The subspace on each "branch" of the tree is shown to resemble the structure of an ℓ_∞^{n+1} -space where the dimension corresponds to the number of terminal nodes on that branch. The Banach space $T_1(d, \theta)$ is isomorphic to $(\sum_{n\in\mathbb{N}} \oplus \ell_\infty^{n+1})_p$, where $d \in \mathbb{N}$ with $d \geq 2$, $0 < \theta < 1$, $d\theta > 1$, and $d\theta = d^{1/p}$. Banach spaces with analogous norms are also constructed on extensions of the tree defined by Dobrinen and Todorcevic [6] and Trujillo [17]. They are shown to be isomorphic to $(\sum_{n\in\mathbb{N}} \oplus \ell_\infty^{n+1})_p$ as well.

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ABSTRACT

A Banach space $T_1(d, \theta)$ with a Tsirelson-type norm is constructed on the top of the topological Ramsey space T_1 defined by Dobrinen and Todorcevic [6]. Finite approximations of the isomorphic subtrees are utilised in constructing the norm. The subspace on each "branch" of the tree is shown to resemble the structure of an ℓ_∞^{n+1}-space where the dimension corresponds to the number of terminal nodes on that branch. The Banach space $T_1(d, \theta)$ is isomorphic to $\left(\sum_{n\in\mathbb{N}} \oplus\, \ell_\infty^{n+1}\right)_p$, where $d \in \mathbb{N}$ with $d \geq 2$, $0 < \theta < 1$, $d\theta > 1$, and $d\theta = d^{1/p}$. Banach spaces with analogous norms are also constructed on extensions of the tree defined by Dobrinen and Todorcevic [6] and Trujillo [17]. They are shown to be isomorphic to $\left(\sum_{n\in\mathbb{N}} \oplus\, \ell_\infty^{n+1}\right)_p$ as well.

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CHAPTER 1: INTRODUCTION

Let X be a linear space over the scalar field $\mathbb{R}$. A norm $\|\cdot\| : X \longrightarrow [0, \infty)$ on the linear space X is a function such that

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) for every $\lambda \in \mathbb{R}$ and for every $x \in X$, $\|\lambda x\| = |\lambda|\|x\|$.

(3) for every $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

A linear space with a norm on it is called a normed space.

For a normed space $\big(X, \|\cdot\|\big)$, setting, for every $x, y \in X$, $d(x, y) = \|x - y\|$ defines a metric on X. The topology of X induced by $\|\cdot\|$ is the metric topology given by this metric d. A Banach space is a normed space that is complete with respect to this metric, in the sense that every Cauchy sequence in the space X converges to a point in X.

Here are some of the classical examples of Banach spaces.

Example. The scalar field $\mathbb{R}$ is a Banach space with the norm $\|x\| = |x|$ for $x \in \mathbb{R}$.

Example. $c_0 = \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\longrightarrow\infty} x_n = 0\big\}$ with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\|_{c_0} = \sup\big\{|x_n| : n \in \mathbb{N}\big\}$.

Example. For $1 \leq p < \infty$, $\ell_p = \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\big\}$ with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\|_p = \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p}$.

Example. $\ell_\infty = \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < \infty\big\}$ with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\|_\infty = \sup\big\{|x_n| : n \in \mathbb{N}\big\}$.

There are normed spaces that are incomplete, though they must be infinite dimensional. A typical example is the space $c_{00} = \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}\ s.t.\ \forall n \geq N, x_n = 0\big\}$ with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\| = \sup\big\{|x_n| : n \in \mathbb{N}\big\}$. Indeed, take $x_n = \big(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}, 0, 0, \dots\big) \in \mathbb{R}^{\mathbb{N}}$ for each $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in c_{00}, and yet it does not converge in c_{00}.

While there are both complete and incomplete normed spaces, every normed space X can be isometrically imbedded as a dense subspace into a complete space by taking quotient of the set of all Cauchy sequences in X with respect to the equivalence relation that two Cauchy sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are equivalent if $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$. This complete space is called the completion of the normed space. Therefore every normed space has a completion, and if a normed space is complete, then it is isomorphic to its completion.

In 1974 Tsirelson [18] constructed a Banach space that did not contain a copy of c_0 or any of ℓ_p, $1 \leq p < \infty$, answering a long standing question. This was one of the first example of Banach spaces with the norm defined implicitly. Later in the same year, Figiel and Johnson [8] gave an inductive definition of the norm of the dual of the space constructed by Tsirelson, and this is what is known as the Tsirelson's space. Their construction is as follows: Writing $E < F$ for $\max(E) < \min(F)$ where $E, F \subseteq \mathbb{Z}^+$ are finite and nonempty, we say $\{E_i\}_{i=1}^k \subseteq [\mathbb{Z}^+]^{<\infty}$ is admissible if $\{k\} < E_1 < E_2 < \cdots < E_k$. Define a sequence of norms for $x \in c_{00}$ as

$$\|x\|_0 = \|x\|_\infty,$$
$$\|x\|_{n+1} = \max\left\{\|x\|_n, \frac{1}{2}\max\left\{\sum_{i=1}^{k}\left\|x\restriction_{E_i}\right\|_n : \{E_i\}_{i=1}^k \text{ is admissible}\right\}\right\}, \qquad (\star)$$

and finally set $\|x\|_{\mathcal{T}} = \lim_{n\longrightarrow\infty}\|x\|_n$. The Tsirelson space is the completion of c_{00} with respect to $\|\cdot\|_{\mathcal{T}}$.

A few years later, Bellenot [4] (1985) modified Figiel and Johnson's construction by looking at a different definition of admissible sets and by changing the constant $^1/_2$ in equation ($\star$) to a more general θ with $0 < \theta < 1$. Bellenot fixed a constant $d \in \mathbb{N}$ with $d \geq 2$ and defined the set $\{E_1, \ldots, E_d\}$ to be admissible if $E_1 < E_2 < \cdots < E_d$. He defined the norm as Figiel and Johnson did in [8] and proved the surprising result that if $d\theta > 1$, then the space generated is isomorphic to ℓ_p for p satisfying $d\theta = d^{1/p}$. Argyros and Deliyanni obtained the same result and some generalizations a few years later [1].

Recently, Arias, Dobrinen, Giron-Garnica, and Mijares [3] used similar ideas to construct spaces $S_k(d, \theta), k \in \mathbb{N} \setminus \{0\}$ using a special type of finitely branching tree with unbounded heights defined by Dobrinen in [5]. When $k = 1$, $S_1(d, \theta)$ is the space constructed by Bellenot. They proved that for $k > 1$, every infinite dimensional subspace of $S_k(d, \theta)$ has a copy of ℓ_p, where p is given by $d\theta = d^{1/p}$, that the $S_k(d, \theta)$'s are non-isomorphic for different values of k, in the sense that there is no linear homeomorphism in between the spaces, and that $S_k(d, \theta)$ embeds isometrically into $S_{k+1}(d, \theta)$.

Motivated by similar ideas on constructions of Banach spaces, we construct in this work a Banach space $T_1(d, \theta)$ based on the topological Ramsey Space T_1 defined by Dobrinen and Todorcevich [6] (shown below).

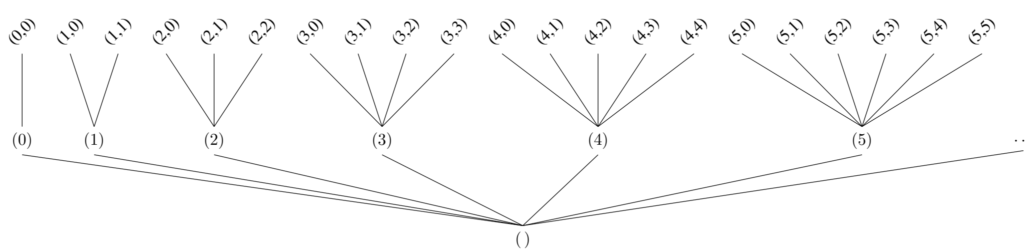

Figure 1: The structure of the tree T_1.

We will show that the top of the tree T_1 is a topological Ramsey space. The Banach space $T_1(d, \theta)$ has an unconditional Schauder basis indexed by the top nodes of the tree.

We will show that $T_1(d,\theta)$ is isomorphic to $\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p$, where each component ℓ_∞^{n+1} corresponds to the span of the top of the tree on the branch of (n).

The proof utilises continuous linear functionals on the space of $T_1(d,\theta)$. They are linear functions from $T_1(d,\theta)$ to the scalar field $\mathbb{R}$ that are continuous. The continuity of a linear function is equivalent to the condition that the image of the function on the closed unit ball is bounded. In other words, a continuous linear functional, in general, is a linear function $f : X \longrightarrow \mathbb{R}$ such that $\sup\left\{\left|f(x)\right| : \|x\| \leq 1\right\} < \infty$ where X is a normed space over the scalar field $\mathbb{R}$. The collection of continuous linear functionals on normed space is itself a Banach space with the norm $\|f\| = \sup_{\|x\|\leq 1}\left|f(x)\right|$. For each element in the space, the Hahn-Banach theorem gives us a continuous linear functional of norm 1 whose value at x is $\|x\|$. In exploration of the isomorphism type of the space $T_1(d,\theta)$, we construct, for each $x \in T_1(d,\theta)$, a set of linear functionals that attains the norm of x.

Finally we explore how our ideas can be applied to other trees with greater heights. Each branch in T_1 has a height of 2. We will extend the heights of the branches of the tree and construct Banach spaces on them. The isomorphism types of the Banach spaces on those extensions are to be examined.

CHAPTER 2: TOPOLOGICAL RAMSEY SPACES

We will set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and denote, for a set A, $A^{[n]} = \big\{B \subseteq A : |B| = n\big\}$ and $A^{[<\infty]} = \bigcup_{n\in\mathbb{N}} A^{[n]}$.

We begin with illustrating an idea that we will use in this chapter. One day infinitely many people decide to travel and stay at a hotel. Well, there are only finitely many hotels in the world, and so we figure that somewhere in the world, there has to be a Hilbert's hotel that can fit infinitely many travelers in it. This is known as the pigeonhole principle. Formally the pigeonhole principle states that for every $d \in \mathbb{N}$, given a partition $\big\{X_j\big\}_{j=0}^{d}$ of $\mathbb{N}$, meaning $X_j \neq \varnothing$, $\bigcup_{j=0}^{d} X_j = \mathbb{N}$ and $X_i \cap X_j = \varnothing$ for $i \neq j$, there is $k \in \{0, 1, 2, \ldots, d\}$ such that X_k is infinite. Ramsey [14] extended this idea to consider a partition on all n-element subsets of a countable set and stated that, for $n \in \mathbb{N}$, given a finite partition of the collection of all n-element subsets of $\mathbb{N}$, there is an infinite subset M of $\mathbb{N}$ such that all n-element subsets of M are contained in one of the sets from the partition. Thinking of a partition as assigning a number (or a colour) to each set, we formulate the Ramsey's theorem in the following form:

Theorem 1 (Ramsey)**.** [14] Let $k, d \in \mathbb{N}$ with $k > 0$. Then for every $f : \mathbb{N}^{[k]} \longrightarrow \{0, 1, 2, \ldots, d\}$, there is an $M \in \mathbb{N}^{[\infty]}$ such that $f \restriction M^{[k]}$ is constant.

As a corollary to the Ramsey's theorem, we have the finite version of this theorem.

Corollary 2. [16] For each $k, m, d \in \mathbb{Z}^+$, there is $n \in \mathbb{Z}^+$ such that for every n-element set N and every $f : N^{[k]} \longrightarrow \{0, 1, 2, \ldots, d\}$, there is $M \in N^{[m]}$ such that $f \restriction M^{[k]}$ is constant.

One generalisation of the Ramsey's theorem is to consider finite colourings on infinite subsets of naturals numbers. Starting with 2-colouring, we may consider the question: given $\mathcal{U} \subseteq \mathbb{N}^{[\infty]}$, is there an $X \in \mathbb{N}^{[\infty]}$ such that $X^{[\infty]} \subseteq \mathcal{U}$ or $X^{[\infty]} \cap \mathcal{U} = \varnothing$? This is not in general true for any collection of infinite subsets of $\mathbb{N}$, so we may characterise a collection of the natural numbers with this property as follows: a collection $\mathcal{U} \subseteq \mathbb{N}^{[\infty]}$ is said to be Ramsey if for every $X \in \mathbb{N}^{[\infty]}$, there is $Y \in X^{[\infty]}$ such that either $Y^{[\infty]} \subseteq \mathcal{U}$ or $Y^{[\infty]} \cap \mathcal{U} = \varnothing$.

For each $A \in \mathbb{N}^{[\infty]}$, we may list the elements of A in the increasing order $A = \{a_n\}_{n\in\mathbb{N}}$ Thus A corresponds to a sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. Thinking of infinite sets of Natural numbers as increasing sequences in $\mathbb{N}$, we induce a metric d on $\mathbb{N}^{[\infty]}$, namely, for $A = \{a_n\}_{n\in\mathbb{N}}$ and $B = \{b_n\}_{n\in\mathbb{N}}$ in increasing order, if $A \neq B$, then $d(A,B) = \frac{1}{2^m}$, where $m = \min\{n \in \mathbb{N} : a_n \neq b_n\}$ ($d(A,B) = 0$ if $A = B$). On the other hand, for each $a \in \mathbb{N}^{[<\infty]}$, we consider the set $[a] = \big\{B \in \mathbb{N}^{[\infty]} : a \sqsubseteq B\big\}$, where $a \sqsubseteq B$ means a is an initial segment of B. It is known that the collection $\big\{[a] : a \in \mathbb{N}^{[<\infty]}\big\}$ forms a basis for the metric topology mentioned previously. In this topology, Nash-Williams showed that clopen sets are Ramsey [12], Galvin showed that open sets are Ramsey [9], Silver showed that analytic sets are Ramsey [15], and Galvin and Prikry showed that Borel sets are Ramsey [10]. These prompted the explorations of the connections between topological notions and Ramsey sets.

In contrast we may consider the set $[a,B] = \big\{A \in \mathbb{N}^{[\infty]} : a \sqsubseteq A \subseteq B\big\}$ for $a \in \mathbb{N}^{[<\infty]}$ and $B \in \mathbb{N}^{[\infty]}$. The collection $\big\{[a,B] : a \in \mathbb{N}^{[<\infty]} \text{ and } B \in \mathbb{N}^{[\infty]}\big\}$ also forms a basis of a topology, which we call the Ellentuck topology on $\mathbb{N}^{[\infty]}$. Under these notions, a set $\mathcal{U} \subseteq \mathbb{N}^{[\infty]}$ is said to have the Ramsey property if for every nonempty set $[a,B]$, there is $A \in [a,B]$ such that $[a,A] \subseteq \mathcal{U}$ or $[a,A] \cap \mathcal{U} = \varnothing$. $\mathcal{U}$ is said to be Ramsey null if for every nonempty set $[a,B]$, there is $A \in [a,B]$ such that $[a,A] \cap \mathcal{U} = \varnothing$. Note that a Ramsey set mentioned previously is a special case of a set having the Ramsey property by taking

$a = \varnothing$ for the set $[a, B]$. Ellentuck also characterised a set having the Ramsey property as the following: a set $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ has the Ramsey property if and only if it has the Baire property in the Ellentuck topology, and a set $\mathcal{X}$ is Ramsey null if and only if it is meager in the Ellentuck topology [7]

Since then, Ramsey methods have provided fruitful results in various directions of analysis [2]. Some notions such as fronts and barriers that were brought up in the proofs of the Ramseyness of some topological sets also provides application in Banach spaces [13], [3]. In addition, Todorcevic formulated the axioms of an (abstract) Ramsey space [16] to generalise the connections between a topological space with the Ramsey property. Under this framework, given a set $\mathcal{R}$ with a quasi-order (a relation that is reflexive and transitive) that satisfies these axioms, a collection of subsets of $\mathcal{R}$ has the Ramsey property if and only if it has the property of Baire in its Ellentuck topology, and a collection of subsets of $\mathcal{R}$ is Ramsey null if and only if it is meager in its Ellentuck topology. Therefore abstract Ramsey spaces axiomatise the works of Ellentuck in [7]. Ramsey spaces, in particular, include a notion of finite approximation which generalises the idea of finite approximation of infinite sequences in $\mathbb{N}$. We will utilise these approximations in constructing new norms for our spaces in later chapters.

Following the abstractions by Todorcevic [16], a Ramsey space is a structure of the form $\left(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s\right)$ that satisfies certain axioms which we will give below. $\mathcal{R}$ and $\mathcal{S}$ can be identified with collections of infinite sequences. $\leq$ is a quasi-order on $\mathcal{S}$, i.e. $\leq$ is a subset of $\mathcal{S} \times \mathcal{S}$ that satisfies reflexivity and transitivity. This means that for each $X \in \mathcal{S}$, $X \leq X$, and for each $X, Y, Z \in \mathcal{S}$, $X \leq Y$ and $Y \leq Z$ implies $X \leq Z$. $\leq^0$ is a relation on $\mathcal{R} \times \mathcal{S}$ such that $\forall\, A \in \mathcal{R}$, $\forall\, X, Y \in \mathcal{S}$, $A \leq^0 X$ and $X \leq Y$ implies $A \leq^0 Y$. In addition $r : \mathcal{R} \times \mathbb{N} \longrightarrow \mathcal{AR}$ and $s : \mathcal{S} \times \mathbb{N} \longrightarrow \mathcal{AS}$ are surjective functions onto some sets $\mathcal{AR}$ and $\mathcal{AS}$. We will denote $r_n(A) = r(A, n)$ and $s_m(X) = s(X, m)$. For each

$n, m \in \mathbb{N}$, set $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$ and $\mathcal{AS}_m = \{s_m(X) : X \in \mathcal{S}\}$. We note that $\mathcal{AR} = \bigcup_{n\in\mathbb{N}} \mathcal{AR}_n$ and $\mathcal{AS} = \bigcup_{m\in\mathbb{N}} \mathcal{AS}_m$.

The following axioms give us the structures of a Ramsey space so that $\mathcal{AR}$ and $\mathcal{AS}$ contain finite sequences that approximate the elements of $\mathcal{R}$ and $\mathcal{S}$ respectively.

4 Sets of Axioms

A.1 (Sequencing)

(1) $r_0(A) = r_0(B)$ for every $A, B \in \mathcal{R}$, and $s_0(X) = s_0(Y)$ for every $X, Y \in \mathcal{S}$.

(2) For every $A, B \in \mathcal{R}$, $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some $n \in \mathbb{N}$, and for every $X, Y \in \mathcal{S}$, $X \neq Y$ implies $s_m(X) \neq s_m(Y)$ for some $m \in \mathbb{N}$.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for each $k \leq n$. $s_i(X) = s_j(Y)$ implies $i = j$ and $s_l(X) = s_l(Y)$ for each $l \leq i$.

Definition 1. Let $a, b \in \mathcal{AR}$ and $x, y \in \mathcal{AS}$. $a \sqsubseteq b$ if there are $m \leq n$ and $A \in \mathcal{R}$ such that $a = r_m(A)$ and $b = r_n(A)$. Similarly $x \sqsubseteq y$ if there are $j \leq i$ and $X \in \mathcal{S}$ such that $x = s_j(X)$ and $y = s_i(X)$. We say x is an initial segment of y if $x \sqsubseteq y$.

We call n the length of $a \in \mathcal{AR}$ if $a = r_n(A)$ for some $A \in \mathcal{R}$, and denote $\mathrm{length}(a) = n$ or simply $|a|$. Define and denote similarly the length of $x \in \mathcal{AS}$.

Under the first set of axioms, we can identify elements $A \in \mathcal{R}$ and $X \in \mathcal{S}$ with sequences $\big(r_n(A)\big)_{n\in\mathbb{N}} \in \mathcal{AR}^{\mathbb{N}}$ and $\big(s_n(X)\big)_{n\in\mathbb{N}} \in \mathcal{AS}^{\mathbb{N}}$ respectively.

A.2 (Finitisation)

There is a relation $\leq_{\mathrm{fin}}^0 \subseteq \mathcal{AR} \times \mathcal{AS}$ and a quasi-order $\leq_{\mathrm{fin}} \subseteq \mathcal{AS} \times \mathcal{AS}$ such that the following are satisfied.

(1) $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}}^0 x\}$ and $\{y \in \mathcal{AS} : y \leq_{\mathrm{fin}} x\}$ are both finite for each $x \in \mathcal{AS}$.

(2) $A \leq^0 X$ if and only if for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $r_n(A) \leq^0_{\text{fin}} s_m(X)$.

(3) $X \leq Y$ if and only if for each $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $s_n(X) \leq_{\text{fin}} s_m(Y)$.

(4) For every $a \in \mathcal{AR}$ and for every $x, y \in \mathcal{AS}$, $a \leq^0_{\text{fin}} x \leq_{\text{fin}} y$ implies $a \leq^0_{\text{fin}} y$.

(5) For every $a, b \in \mathcal{AR}$ and for every $x \in \mathcal{AS}$, $a \sqsubseteq b$ and $b \leq^0_{\text{fin}} x$ implies there is $y \sqsubseteq x$ such that $a \leq^0_{\text{fin}} y$.

Notation. For $a \in \mathcal{AR}$, $x \in \mathcal{AS}$, $m \in \mathbb{N}$ and $Y \in \mathcal{S}$, set

$$[a, Y] = \{A \in \mathcal{R} : A \leq^0 Y \text{ and } \exists\, n \in \mathbb{N} \; s.t. \; r_n(A) = a\},$$

$$[x, Y] = \{X \in \mathcal{S} : X \leq Y \text{ and } \exists\, n \in \mathbb{N} \; s.t. \; s_n(X) = x\},$$

and $[m, Y] = [s_m(Y), Y]$.

For $a \in \mathcal{AR}$, let $[a] = \{A \in \mathcal{R} : r_{|a|}(A) = a\}$.

Lemma 3. $\{[a] : a \in \mathcal{AR}\}$ forms a basis of the metric topology on $\mathcal{R}$, where for $A, B \in \mathcal{R}$, the metric d is defined as

$$d(A, B) = \begin{cases} \frac{1}{2^m} \text{ where } m = \min\{n : r_n(A) \neq r_n(B)\} & , \text{ if } \{n : r_n(A) \neq r_n(B)\} \neq \varnothing \\ 0 & , \text{ if } \{n : r_n(A) \neq r_n(B)\} = \varnothing \end{cases}.$$

Proof. We first show that d defined above is indeed a metric. Let $A, B, C \in \mathcal{R}$. If $A = B$, then $r_n(A) = r_n(B)$ for each $n \in \mathbb{N}$. So $\{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} = \varnothing$, which means $d(A, B) = 0$. If $A \neq B$, then $r_n(A) \neq r_n(B)$ for some $n \in \mathbb{N}$ by axiom A.1(2). So $\{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} \neq \varnothing$, and $m = \min\{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} \in \mathbb{N}$ exists by the well-ordering of $\mathbb{N}$, and so $d(A, B) = \frac{1}{2^m} > 0$. $d(A, B) = d(B, A)$ is clear. Suppose $d(A, B) = \frac{1}{2^m}$ and $d(A, C) = \frac{1}{2^p}$. If $p \leq m$, then $d(A, B) = \frac{1}{2^m} \leq \frac{1}{2^p} \leq d(A, C) + d(B, C)$.

Assume $p > m$. Since $r_n(A) = r_n(C)$ for all $n < p$, we have $r_m(A) = r_m(C)$. But $r_m(A) \neq r_m(B)$. So $d(B, C) \geq \frac{1}{2^m}$. This means that $d(A, B) = \frac{1}{2^m} \leq d(A, C) + d(B, C)$.

Take an open ball in the metric topology $\mathcal{OB}_\varepsilon(A) = \{B \in \mathcal{R} : d(A, B) < \varepsilon\}$ for $A \in \mathcal{R}$ and $\varepsilon > 0$. Let $B \in \mathcal{OB}_\varepsilon(A)$. Then $d(A, B) = \frac{1}{2^n} < \varepsilon$ for some $n \in \mathbb{N}$. Take $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon - \frac{1}{2^n}$. If $C \in [r_m(B)]$, then $r_k(C) = r_k(B)$ for each $k \leq m$. So $d(B, C) = \frac{1}{2^p}$ for some $p > m$. So $d(A, C) \leq d(A, B) + d(B, C) < \frac{1}{2^n} + \frac{1}{2^m} < \varepsilon$. This means that for every $B \in \mathcal{OB}_\varepsilon(A)$, there is $m \in \mathbb{N}$ such that $B \in [r_m(B)] \subseteq \mathcal{OB}_\varepsilon(A)$. Conversely for each element $[a]$, let $A \in [a]$. Take $\varepsilon < \frac{1}{2^{|a|}}$. Then for each $B \in \mathcal{OB}_\varepsilon(A)$, $d(A, B) = \frac{1}{2^m} < \varepsilon < \frac{1}{2^{|a|}}$ for some $m \in \mathbb{N}$. This means that $m = \min\{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} > |a|$, and so $r_k(A) = r_k(B)$ for each $k \leq |a|$. Hence $B \in [a]$ and we have $\mathcal{OB}_\varepsilon(A) \subseteq [a]$. Therefore $\{[a] : a \in \mathcal{AR}\}$ generate the same topology as the metric. $\square$

Lemma 4. Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ satisfies axioms A.1 and A.2. Then the set $[a, X]$ is closed with respect to the metric topology of $\mathcal{R}$ for every $a \in \mathcal{AR}$ and $X \in \mathcal{S}$.

Proof. $\mathcal{R} \setminus [a, X] = \{A \in \mathcal{R} : A \not\leq^0 X\} \cup \{A \in \mathcal{R} : r_n(A) \neq a \,\forall n \in \mathbb{N}\} = \left(\bigcup_{n\in\mathbb{N}} \{[r_n(A)] : r_n(A) \not\leq^0_{\text{fin}} s_m(X) \,\forall m \in \mathbb{N}\}\right) \cup \left(\bigcup_{b\in\mathcal{AR}} \{[b] : b \not\sqsubseteq a \text{ and } a \not\sqsubseteq b\}\right)$, which is a union of basis elements of the metric topology. $\square$

Before we get into the third set of axioms, we give the definition of the depth of an element in $\mathcal{AR}$ into a sequence in $\mathcal{S}$. The definition simplifies the expressions in the next sets of axioms that relate a specific finite sequence with some other elements in $\mathcal{S}$ than the one it approximates towards the Ramseyness of the space.

Definition 2. For $a \in \mathcal{AR}$ and $Y \in \mathcal{S}$,

$$\text{depth}_Y(a) = \begin{cases} \min\{k : a \leq^0_{\text{fin}} s_k(Y)\} & \text{, if } \exists\, k \text{ } s.t. \text{ } a \leq^0_{\text{fin}} s_k(Y) \\ \infty & \text{, otherwise.} \end{cases}$$

A.3 (Amalgamation)

(1) For every $a \in \mathcal{AR}$ and for every $Y \in \mathcal{S}$, if $\mathrm{depth}_Y(a) < \infty$, then $[a, X] \neq \varnothing$ for each $X \in [\mathrm{depth}_Y(a), Y]$.

(2) For every $a \in \mathcal{AR}$ and for every $X, Y \in \mathcal{S}$, if $X \leq Y$ and $[a, X] \neq \varnothing$, then there is $Y' \in [\mathrm{depth}_Y(a), Y]$ such that $[a, Y'] \subseteq [a, X]$.

Lemma 5. Suppose $\left(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s\right)$ satisfies A.1 and A.2. If $a \sqsubseteq b$ and if $[b, Y] \neq \varnothing$, then $[a, Y] \neq \varnothing$ and $\mathrm{depth}_Y(a) \leq \mathrm{depth}_Y(b) < \infty$.

Proof. Since $[b, Y] \neq \varnothing$, there is $A \in \mathcal{R}$ such that $A \leq^0 Y$ and $r_n(A) = b$ for some $n \in \mathbb{N}$. Since $a \sqsubseteq b$, there is $m \leq n$ such that $a = r_m(A)$ by definition. So $A \in [a, Y] \neq \varnothing$. Also there is $p \in \mathbb{N}$ such that $b = r_n(A) \leq^0_{\text{fin}} s_p(Y)$ by axiom A.2(3). So $\mathrm{depth}_Y(b) < \infty$. Set $p_0 = \min\{p : b \leq^0_{\text{fin}} s_p(Y)\}$. Then $b \leq^0_{\text{fin}} s_{p_0}(Y)$. By axiom A.2(5), there exists $q \leq p$ such that $a \leq^0_{\text{fin}} s_q(Y)$. So $\mathrm{depth}_Y(a) \leq \mathrm{depth}_Y(b)$. □

A.4 (Pigeonhole)

Suppose $a \in \mathcal{AR}$ has length j and $\mathcal{O}$ is a subset of $\mathcal{AR}_{j+1}$. Then for every $Y \in \mathcal{S}$ such that $[a, Y] \neq \varnothing$, there is an $X \in [\mathrm{depth}_Y(a), Y]$ such that $r_{j+1}[a, X] \subseteq \mathcal{O}$ or $r_{j+1}[a, X] \cap \mathcal{O} = \varnothing$.

Theorem 6. [16] $\left\{[a, Y] : a \in \mathcal{AR}, Y \in \mathcal{S}\right\}$ forms a basis for a topology.

Definition 3 (abstract Ellentuck topology)**.** The topology that the basis $\left\{[a, Y] : a \in \mathcal{AR}, Y \in \mathcal{S}\right\}$ generates is called the (abstract) Ellentuck topology on $\mathcal{R}$.

By lemma 4, the Ellentuck topology is finer than the metric topology since for each $a \in \mathcal{AR}$, $[a] = \bigcup_{X \in \mathcal{AS}} \left\{[a, X] : a \leq^0_{\text{fin}} s_m(X) \text{ for some } m \in \mathbb{N}\right\}$. In fact, it is strictly finer since for each $[a, X]$, $X < Y$ for some $Y \in \mathcal{S}$ means that $[a, X]$ is not a union of sets of the form $[b]$ for some $b \in \mathcal{AR}$.

In the special case that $\mathcal{R} = \mathcal{S}$, $\leq$ is $\leq^0$, and $r = s$, we have the space on the triple $(\mathcal{R}, \leq, r)$ that satisfies the axioms given below, where $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-order on $\mathcal{R}$, and $r : \mathcal{R} \times \mathbb{N} \longrightarrow \mathcal{AR}$. We will still denote $r_n(A) = r(A, n)$ for $A \in \mathcal{R}$ and $n \in \mathbb{N}$. Setting $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$, we have $\mathcal{AR} = \bigcup_{n\in\mathbb{N}} \mathcal{AR}_n$.

Since the set of approximations $\mathcal{AR}$ in this special case is going to define our norm, we will also give the axioms that are specified in this special case.

<u>Axiom 1</u>

(1) $r_0(A) = \varnothing$ for every $A \in \mathcal{R}$.

(2) For every $A, B \in \mathcal{R}$, $A \neq B$ implies that $r_n(A) \neq r_n(B)$ for some $n \in \mathbb{N}$.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for each $k \leq n$.

<u>Axiom 2</u>

There is a quasi-order $\leq_{\text{fin}}$ on $\mathcal{AR}$ that satisfies the following.

(1) For each $b \in \mathcal{AR}$, $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite.

(2) For every $A, B \in \mathcal{R}$, $A \leq B$ if and only if for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $r_n(A) \leq_{\text{fin}} r_m(B)$.

(3) For every $a, b, c \in \mathcal{AR}$, if $a \sqsubseteq b$ and $b \leq_{\text{fin}} c$, then there is $d \in \mathcal{AR}$ such that $d \sqsubseteq c$ and $a \leq_{\text{fin}} d$.

Recall the definition

$$\operatorname{depth}_B(a) = \begin{cases} \min\{k : a \leq_{\text{fin}} r_k(B)\} & \text{, if } \exists\, k \ s.t.\ a \leq_{\text{fin}} r_k(B) \\ \infty & \text{, otherwise.} \end{cases}$$

for $a \in \mathcal{AR}$ and $B \in \mathcal{R}$.

Axiom 3

(1) For every $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, $\mathrm{depth}_B(a) < \infty$ implies that $[a, A] \neq \varnothing$ for each $A \in [\mathrm{depth}_B(a), B]$.

(2) For every $a \in \mathcal{AR}$, $A, B \in \mathcal{R}$, if $A \leq B$ and $[a, A] \neq \varnothing$, then there is $A' \in [\mathrm{depth}_B(a), B]$ such that $[a, A'] \neq \varnothing$ and $[a, A'] \subseteq [a, A]$.

Axiom 4

Suppose $a \in \mathcal{AR}$ is of length j and $\mathcal{O}$ is a subset of $\mathcal{AR}_{j+1}$. Then for every $B \in \mathcal{R}$ such that $\mathrm{depth}_B(a) < \infty$, there is $A \in [\mathrm{depth}_B(a), B]$ such that $r_{j+1}[a, A] \subseteq \mathcal{O}$ or $r_{j+1}[a, A] \cap \mathcal{O} = \varnothing$.

If $[b, B] \cap [c, C] \neq \varnothing$, take $A \in [b, B] \cap [c, C]$. Then there are $p, q \in \mathbb{N}$ such that $r_p(A) = b$ and $r_q(A) = c$. Also $A \leq B$ and $A \leq C$. Take $m = \max\{p, q\}$. Certainly $A \in [r_m(A), A]$ by the reflexivity of the quasi-order $\leq$ on $\mathcal{R}$. Let $X \in [r_m(A), A]$. Then there is $n \in \mathbb{N}$ such that $r_n(X) = r_m(A)$ and $X \leq A$. By axiom 1(3), $n = m$ and $r_k(X) = r_k(A)$ for every $k \leq m$. This means that $r_p(X) = r_p(A) = b$ and $r_q(X) = r_q(A) = c$ since $p \leq m$ and $q \leq m$. Also by the transitivities of the quasi-order, we have $X \leq B$ and $X \leq C$. So $X \in [b, B] \cap [c, C]$. Therefore we get $A \in [r_m(A), A] \subseteq [b, B] \cap [c, C]$.

From the above we see that the collection of the sets of the form

$$[a, B] = \big\{A \in \mathcal{R} : A \leq B \text{ and } \exists\, n \in \mathbb{N}\ s.t.\ r_n(A) = a\big\},$$

where $a \in \mathcal{AR}$ and $B \in \mathcal{R}$ forms a basis of a topology, which we still call the Ellentuck topology.

Definition 4. A subset S of a topological space X is nowhere dense if for every open subset $\mathcal{O}$ in X, there is a nonempty open subet $\mathcal{U} \subseteq \mathcal{O}$ such that $S \cap \mathcal{U} = \varnothing$.

A subset $\mathcal{M}$ of X is meager if $\mathcal{M} \subseteq \bigcup_{n\in\mathbb{N}} S_n$ where $\{S_n\}_{n\in\mathbb{N}}$ is a countable collection of nowhere dense subsets of X.

The Ramseyness is generalised to an infinite quasi-ordered set satisfying the axioms A.1 through A.4 as the following definition.

Definition 5. A set $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey if for every nonempty basic open set $[a, A]$, there is $B \in [a, A]$ such that either $[a, B] \subseteq \mathcal{X}$, or $[a, B] \cap \mathcal{X} = \varnothing$.

A set $\mathcal{N} \subseteq \mathcal{R}$ is Ramsey null if for every nonempty basic open set $[a, A]$, there is $B \in [a, A]$ such that $[a, B] \cap \mathcal{N} = \varnothing$.

Definition 6. A subset $\mathcal{X}$ of $\mathcal{R}$ has the property of Baire if $\mathcal{X} = \mathcal{O} \triangle \mathcal{M}$ for some open set $\mathcal{O} \subseteq \mathcal{R}$ and some meager set $\mathcal{M} \subseteq \mathcal{R}$. We call a set that has the property of Baire and Baire set.

Definition 7. $(\mathcal{R}, \leq, r)$ is a topological Ramsey space if every Baire set of $\mathcal{R}$ is Ramsey and every meager set of $\mathcal{R}$ is Ramsey null.

We may identify the elements in $\mathcal{R}$ with the elements in $\mathcal{AR}^{\mathbb{N}}$ through the embedding $A \in \mathcal{R} \longmapsto \big(r_n(A)\big)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{AR}_n$. Note that the Ellentuck topology extends to the topology on $\mathcal{AR}^{\mathbb{N}}$ induced by the first difference metric

$$d\big((r_n(A))_{n\in\mathbb{N}}, (r_n(B))_{n\in\mathbb{N}}\big) = \begin{cases} \frac{1}{2^m} & \text{, if } m = \min\{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} < \infty \\ 0 & \text{, if } \{n \in \mathbb{N} : r_n(A) \neq r_n(B)\} = \varnothing \end{cases}.$$

Under these identifications, We have the following theorem, which will be utilised in the next chapter. Recall that for $a \in \mathcal{AR}$, $[a] = \{A \in \mathcal{R} : a \sqsubseteq A\}$.

Theorem 7 (Abstract Ellentuck Theorem)**.** [16] If $(\mathcal{R}, \leq, r)$ is closed as an embedding into $\mathcal{AR}^{\mathbb{N}}$ in the Ellentuck topology induced by the collection $\{[a, B] : a \in \mathcal{AR}$ and $B \in \mathcal{R}\}$ and if it satisfies axions A.1 through A.4, then $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.

CHAPTER 3: $\big(\mathcal{R}(T_1), \leq, r\big)$ AS A TOPOLOGICAL RAMSEY SPACE

Let X be a set. Denote $X^{<\mathbb{N}}$ as the collection of all finite sequences of elements of X. For $s = (s_0, s_1, s_2, \ldots, s_n) \in X^{<\mathbb{N}}$, we say n is the length of s and is denoted by $|\, s\,|$. If $i \in \mathbb{N}$ with $0 \leq i \leq n$, then $\alpha_i(s)$ denote the sequence of first i elements of s, namely $\alpha_i(s) = (s_0, s_1, \ldots, s_i)$. For $s, t \in X^{<\mathbb{N}}$, s is an initial segment of t if there exists $1 \leq i \leq |\, t\,|$ such that $s = \alpha_i(t)$.

For a set T of finite sequences, $\mathrm{cl}(T)$ is the set containing T and all the initial segments of elements of T. T is a tree if $T = \mathrm{cl}(T)$, and in this case we say T is closed under initial segments. We identify each sequence in T as a node of the tree. A terminal node s is one that whenever s is an initial segment of some node t, we have $s = t$. Here we look at a tree that is bounded on height but is unboundedly branched. Below we give the definition of a tree that we will call it T_1.

Definition 8 (the tree of T_1)**.** [17] For $i \in \mathbb{N}$, let

$$T_1(i) = \{(\,), (i), (i, j) : 0 \leq j \leq i\}\,,$$
$$\text{and set } T_1 = \bigcup_{i \in \mathbb{N}} T_1(i).$$

By the definitions and descriptions above, there is a simpler way to define a tree. Namely, we may give the collection of terminal nodes of a tree and close the branches downwards. By closing its branches, we mean including all of the initial segments of the terminal nodes. Therefore T_1 can also be defined by

$$T_1 = \mathrm{cl}\big\{(i, j) : i, j \in \mathbb{N} \text{ and } 0 \leq j \leq i\big\}.$$

We will be mainly working with the top of the tree, or the collection of the terminal nodes of T_1, which we denote by $[T_1]$. There is a natural order on $[T_1]$, the lexicographic order: $(i,j) < (k,l)$ if either $i < k$, or if $i = k$ and $j < l$. Note that this defines a linear order on $[T_1]$ of order type ω.

We look at the subtrees of T_1 isomorphic to T_1, i.e. those that have the same structure as T_1 itself. They are described by injective maps $\hat{S} : T_1 \longrightarrow T_1$ that preserve the initial segments and the lexicographic order, and often we will identify $\hat{S}$ by its range $\hat{S}(T_1)$. An example of a subtree isomorphic to T_1 is illustrated below.

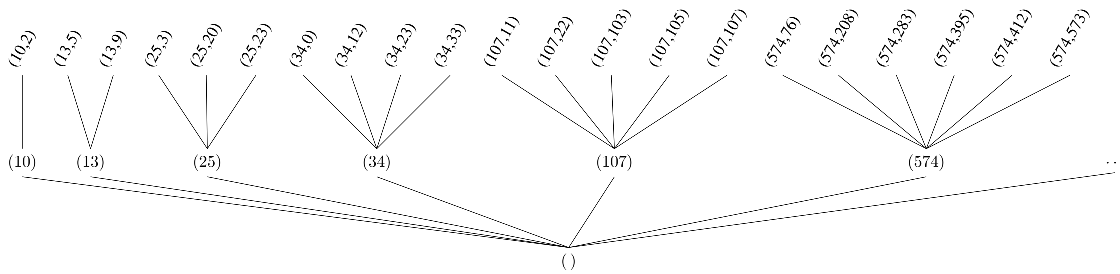

Figure 2: An initial structure of a subtree of T_1 isomorphic to T_1.

Since the map preserves initial segments, it also preserves the length of the sequences. Namely, $\hat{S}\big(()\big) = ()$, $\hat{S}\big((i)\big) = (j)$ and $\hat{S}\big((i,j)\big) = (k,l)$. Moreover, if $\hat{S}\big((i,j)\big) = (k,l)$ then $\hat{S}\big((i)\big) = (k)$.

The collection of all subtrees isomorphic to T_1 is denoted by

$$\mathcal{R}(T_1) = \big\{ \hat{S}(T_1) \,\big|\, \hat{S} : T_1 \longrightarrow T_1 \text{ is injective and preserves } \sqsubseteq \text{ and } \leq \big\}.$$

On $\mathcal{R}(T_1)$, define $\hat{S} \leq \hat{U}$ if $\hat{S} \subseteq \hat{U}$ as their ranges. This subset relation is clearly reflexive and transitive, and so $\leq$ is a quasi-order on $\mathcal{R}(T_1)$.

Dobrinen and Todorcevic [6] proved that $\big(\mathcal{R}(T_1), \leq, r\big)$ is a topological Ramsey space with $r_n(\hat{S}) = \hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)$. Here we will only restrict our interest on the top of the tree, namely $[T_1]$. Therefore we take $r : \mathcal{R}(T_1) \times \mathbb{N} \longrightarrow \big([T_1]\big)^{[<\infty]}$ by $(\hat{S}, n) \longmapsto \big[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\big]$. Analogously we will denote $r_n(\cdot) = r(\cdot, n)$. Then $\mathcal{AR}(T_1) = \{r_n(\hat{S}) : n \in \mathbb{N}, \hat{S} \in \mathcal{R}(T_1)\}$. For $a, b \in \mathcal{AR}(T_1)$, $a \sqsubseteq b$ if and only if there is $\hat{S} \in \mathcal{R}(T_1)$ and $m \leq n$ such that $a = r_m(\hat{S})$ and $b = r_n(\hat{S})$. Also $[a, \hat{U}] = \big\{\hat{S} \in \mathcal{R}(T_1) : \hat{S} \subseteq \hat{U}$ and $\exists\, n \in \mathbb{N}\ s.t.\ r_n(\hat{S}) = a\big\}$. An example of an approximation would be $\{(10,2), (13,5), (13,9), (25,3), (25,20), (25,23)\}$ for $n = 2$ for the subtree in figure 2 above. In general, if the first point of an approximation is (i, j), then it cannot have any other point with first coordinate as i. If the second point of the approximation is (k, l), then it contains exactly two points that have k as their first coordinate. This restriction makes approximations "thin" on large number branches of T_1.

We will show that $\big(\mathcal{R}(T_1), \leq, r\big)$ is a topological Ramsey space with basic open sets of the form $[a, \hat{U}]$. By the abstract Ellentuck theorem formulated in the previous chapter, it is enough to show that $\big(\mathcal{R}(T_1), \leq, r\big)$ satisfies the axioms A.1 through A.4, and is closed as an embedding into the product space $\big(\mathcal{AR}(T_1)\big)^{\mathbb{N}}$.

Proof. A.1.

(1) $r_0(\hat{S}) = \varnothing$, for every $\hat{S} \in \mathcal{R}(T_1)$.

(2) $\hat{S} \neq \hat{U}$ implies that there is $s \in T_1$ such that $\hat{S}(s) \neq \hat{U}(s)$. Suppose $s \in T_1(n)$ for some $n \in \mathbb{N}$. Then $\big[\hat{S} \restriction \bigcup_{i=0}^{n} T_1(i)\big] \neq \big[\hat{U} \restriction \bigcup_{i=0}^{n} T_1(i)\big]$. So $r_{n+1}(\hat{S}) \neq r_{n+1}(\hat{U})$.

(3) Suppose $r_n(\hat{S}) = r_m(\hat{U})$. Then $\big[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\big] = \big[\hat{U} \restriction \bigcup_{j=0}^{m-1} T_1(j)\big]$. By the cardinality of either set, we must have $n = m$. Also for every $k \leq n$, we have $\hat{S} \restriction \bigcup_{i=0}^{k-1} T_1(i) = \hat{U} \restriction \bigcup_{j=0}^{k-1} T_1(j)$. Hence we derive $r_k(\hat{S}) = r_k(\hat{U})$.

A.2.

On $\mathcal{AR}(T_1)$, define $a \leq_{\text{fin}} b$ if there are $\hat{S}, \hat{U} \in \mathcal{R}(T_1)$ and $n, m \in \mathbb{N}$ such that $a = \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right] = b$. Note that if $\left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right]$, we must have $n \leq m$ by an argument on the cardinalities of the sets. Since for each $\hat{S} \in \mathcal{R}(T_1)$ and for each $n \in \mathbb{N}$, $\left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right]$, we have $a \leq_{\text{fin}} a$ for every $a \in \mathcal{AR}(T_1)$. Suppose $a \leq_{\text{fin}} b$ and $b \leq_{\text{fin}} c$ for $a, b, c \in \mathcal{AR}(T_1)$. Then there are $\hat{S}, \hat{U}, \hat{V} \in \mathcal{R}(T_1)$ and $n, m, p \in \mathbb{N}$ such that $a = \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right] = b$ and $b = \left[\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right] \subseteq \left[\hat{V} \restriction \bigcup_{i=0}^{p-1} T_1(i)\right] = c$. By the previous note, we have $n \leq m$ and $m \leq p$. Since the usual order $\leq$ on $\mathbb{N}$ and the subset relation are both transitive, we have $a = \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{V} \restriction \bigcup_{i=0}^{p-1} T_1(i)\right] = c$ and thus $a \leq_{\text{fin}} c$. Therefore we see that $\leq_{\text{fin}}$ is a quasi-order on $\mathcal{AR}(T_1)$.

(1) For each $b \in \mathcal{AR}(T_1)$, since b is finite, there are at most finitely many subsets of b. Therefore $\{a \in \mathcal{AR}(T_1) : a \leq_{\text{fin}} b\}$ is finite.

(2) Suppose $\hat{S} \subseteq \hat{U}$ and let $n \in \mathbb{N}$. Then $r_n(\hat{S}) = \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \hat{U}$ as a restriction of $\hat{S}$. Since $\hat{U}$ is isomorphic to T_1 and $\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)$ is isomorphic to an initial segment of T_1, there is $m \in \mathbb{N}$ such that $\left(\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right) \subseteq \left(\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right)$, as $\bigcup_{j=0}^{n-1} T_1(j)$ is finite. So $\left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right]$ and we have $r_n(\hat{S}) \leq_{\text{fin}} r_m(\hat{U})$. Conversely suppose for each n, there is $m(n)$ such that $\left[\hat{S} \restriction \bigcup_{j=0}^{n-1} T_1(j)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{m(n-1)} T_1(i)\right]$. This gives us $\left(\hat{S} \restriction \bigcup_{j=0}^{n-1} T_1(j)\right) \subseteq \left(\hat{U} \restriction \bigcup_{i=0}^{m(n-1)} T_1(i)\right)$ as $\hat{S}$ and $\hat{U}$ preserves initial segments. So $\hat{S} = \bigcup_{n \in \mathbb{N}} \left[\hat{S} \restriction \bigcup_{j=0}^{n-1} T_1(j)\right] \subseteq \bigcup_{n \in \mathbb{N}} \left[\hat{U} \restriction \bigcup_{i=0}^{m(n-1)} T_1(j)\right] \subseteq \hat{U}$.

(3) Let $a, b, c \in \mathcal{AR}(T_1)$ and suppose $a \sqsubseteq b$ and $b \leq_{\text{fin}} c$. Then there are $\hat{S}, \hat{U} \in \mathcal{R}(T_1)$ and $n \leq m \leq p$ such that $a = \left[\hat{S} \restriction \bigcup_{i=0}^{n-1} T_1(i)\right]$ and $b = \left[\hat{S} \restriction \bigcup_{i=0}^{m-1} T_1(i)\right] \subseteq \left[\hat{U} \restriction \bigcup_{i=0}^{p-1} T_1(i)\right] = c$. This means that $a \subseteq b$ and $b \subseteq c$, and thus we have $a \leq_{\text{fin}} c$.

A.3.

Define $\operatorname{depth}_{\hat{S}}(a) = \min\{n \in \mathbb{N} : a \leq_{\text{fin}} r_n(\hat{S})\}$ if the minimum exists, and ∞ otherwise.

(1) Assume $a = \big[\hat{V} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big]$ for some $m \in \mathbb{N}$. Suppose $\operatorname{depth}_{\hat{S}}(a) < \infty$. Set $d = \operatorname{depth}_{\hat{S}}(a)$. Let $\hat{U} \in \big[r_d(\hat{S}), \hat{S}\big]$. Then $\hat{U} \subseteq \hat{S}$ and there is $n \in \mathbb{N}$ such that $\big[\hat{U} \restriction \bigcup_{i=0}^{n-1} T_1(i)\big] = r_d(\hat{S})$. By the definition of r, $n = d$ and $r_d(\hat{U}) = r_d(\hat{S})$, and since $a \leq_{\text{fin}} r_d(\hat{S})$, we have $m \leq d$ and $\big[\hat{V} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big] \subseteq \big[\hat{U} \restriction \bigcup_{i=0}^{d-1} T_1(i)\big]$. For each $p \geq m$, we may take $W_p \subseteq \hat{U} \restriction T_1\big(p + (d-m)\big)$ with $\big|W_p\big| = p$. Then by setting $\hat{W} = \big[\hat{V} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big] \cup \big(\bigcup_{p=m}^{\infty} W_p\big)$, we have $\hat{W} \in \mathcal{R}(T_1)$ and $\hat{W} \subseteq \hat{U}$. So $[a, \hat{U}] \neq \varnothing$.

(2) Let $a \in \mathcal{AR}(T_1)$. Assume $a = \big[\hat{V} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big]$ for some $m \in \mathbb{N}$. Suppose $\hat{U} \subseteq \hat{S}$ and $[a, \hat{U}] \neq \varnothing$. Since $[a, \hat{U}] \neq \varnothing$, there is $n \geq m$ such that $\big[\hat{V} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big] \subseteq \big[\hat{U} \restriction \bigcup_{i=0}^{n-1} T_1(i)\big]$. Set $d = \operatorname{depth}_{\hat{S}}(a)$ and $e = \operatorname{depth}_{\hat{U}}(a)$. Since $\hat{U} \subseteq \hat{S}$, there is $q \in \mathbb{N}$ such that $\big[\hat{U} \restriction \bigcup_{i=0}^{e-1} T_1(i)\big] \subseteq \big[\hat{S} \restriction \bigcup_{i=0}^{q-1} T_1(i)\big]$. So $d < \infty$ and we have $e \leq d$. Take $W_p = \big[\hat{U} \restriction T_1(p)\big]$ for each $p \geq d$. Then $\hat{W} = \big[\hat{S} \restriction \bigcup_{i=0}^{d-1} T_1(i)\big] \cup \big(\bigcup_{p=d}^{\infty} W_p\big) \in \mathcal{R}(T_1)$. So $\hat{W} \in [r_d(\hat{S}), \hat{S}]$ since $W_p \subseteq \hat{S} \restriction T_1\big(k(p)\big)$ for each $p \geq d$, where $k : \mathbb{N} \longrightarrow \mathbb{N}$ is increasing. Since $\operatorname{depth}_{\hat{W}}(a) = d$ and $\hat{W} \subseteq \hat{S}$, we have $[a, \hat{W}] \neq \varnothing$ by A.3 (1). Take $\hat{Y} \in [a, \hat{W}]$. Then $\big[\hat{Y} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big] = a$ and $\hat{Y} \subseteq \hat{W}$. So for every $j \in \mathbb{N}$, we have an increasing $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\big[\hat{Y} \restriction \bigcup_{i=0}^{j} T_1(i)\big] \subseteq \big[\hat{W} \restriction \bigcup_{i=0}^{\varphi(j)} T_1(i)\big]$. Since $\big[\hat{Y} \restriction \bigcup_{i=0}^{m-1} T_1(i)\big] \subseteq \big[\hat{W} \restriction \bigcup_{i=0}^{d-1} T_1(i)\big]$, we have $d = \varphi(m)$. So for every $j \geq m$, $\big[\hat{Y} \restriction T_1(j)\big] \subseteq \big[\hat{W} \restriction T_1\big(\varphi(j)\big)\big] \subseteq \big[\hat{U} \restriction T_1\big(\varphi(j)\big)\big]$. Therefore $\hat{Y} \subseteq \hat{U}$.

A.4.

Let $a \in \mathcal{AR}(T_1)$ with $\operatorname{length}(a) = j$ and let $\hat{S} \in \mathcal{R}(T_1)$. Suppose $\operatorname{depth}_{\hat{S}}(a) < \infty$ and let $\mathcal{O} \subseteq \mathcal{AR}_{j+1}(T_1)$. Set $d = \operatorname{depth}_{\hat{S}}(a)$. For each $m \in \mathbb{N}$, for each $u \subseteq \big[\hat{S} \restriction T_1(d+m)\big]$

with $|u| = j+1$, define

$$\varphi_{d+m}(u) = \begin{cases} 0 & \text{, if } a \cup u \in \mathcal{O} \\ 1 & \text{, if } a \cup u \notin \mathcal{O} \end{cases}.$$

Then this is a 2-colouring on the $j+1$-element subsets of $\left[\hat{S} \restriction T_1(d+m)\right]$. By the finite Ramsey theorem (corollary 2), there is $N(0) > d+1$ such that for the set $\left[\hat{S} \restriction T_1\big(N(0)-1\big)\right]$, there is $W_{N(0)} \subseteq \left[\hat{S} \restriction T_1\big(N(0)-1\big)\right]$ with $\left|W_{N(1)}\right| = d+1$ such that $\varphi_{N(0)-1} \restriction \big(W_{N(0)}\big)^{[j+1]}$ is constant. Recursively having chosen $N(n) > N(n-1) > \cdots > N(1) > N(0) > d+1$, there is $N(n+1) > N(n)$ such that for the set $\left[\hat{S} \restriction T_1\big(N(n+1)-1\big)\right]$, there is $W_{N(n+1)} \subseteq \left[\hat{S} \restriction T_1\big(N(n+1)-1\big)\right]$ with $\left|W_{N(n+1)}\right| = N(n)$ such that $\varphi_{N(n+1)-1} \restriction \big(W_{N(n+1)}\big)^{[j+1]}$ is constant. These are possible by the finite Ramsey theorem. Now, define

$$\varphi(n) = \begin{cases} 0 & \text{, if } \varphi_{N(n)-1} \restriction W_{N(n)} = \{0\} \\ 1 & \text{, if } \varphi_{N(n)-1} \restriction W_{N(n)} = \{1\} \end{cases}.$$

By the pigeonhole principle, there is $M \in \mathbb{N}^{[\infty]}$ such that $\varphi \restriction M$ is constant. Set $M = \{p(n)\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ in the increasing order. For each $n \in \mathbb{N}$, take $V_n \subseteq W_{p(n)}$ with $\left|V_n\right| = d+n+1$. Set $\hat{U} = a \cup \big(\bigcup_{n\in\mathbb{N}} V_n\big)$. Then $\hat{U} \in \mathcal{R}(T_1)$ and note that $\hat{U} \subseteq \hat{S}$ by construction. For each $y \in r_{j+1}\big([a, \hat{U}]\big)$, $y = a \cup v$ for some $v \in V_n$ for some $n \in \mathbb{N}$ and $|v| = j+1$. By the construction above, either all of such y are in $\mathcal{O}$ or all of such y are in $\mathcal{O}^c$. This means that either $r_{j+1}\big([a, \hat{U}]\big) \subseteq \mathcal{O}$ or $r_{j+1}\big([a, \hat{U}]\big) \cap \mathcal{O} = \varnothing$.

Lastly take $(x_n)_{n\in\mathbb{N}} \in \big(\mathcal{AR}(T_1)\big)^{\mathbb{N}} \setminus \prod_{n\in\mathbb{N}} \mathcal{AR}_n(T_1)$. Then there is $j \in \mathbb{N}$ such that $x_j \neq r_j\big(\hat{S}\big)$ for any $\hat{S} \in \mathcal{R}(T_1)$. This means that $x_j = r_k\big(\hat{U}\big)$ for some $k \neq j$ and $\hat{U} \in \mathcal{R}(T_1)$. So the ball centered at $(x_n)_{n\in\mathbb{N}}$ with the radius ${}^1\!/_{2^j}$ contains no elements in $\prod_{n\in\mathbb{N}} \mathcal{AR}_n(T_1)$ since for every element $(y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{AR}_n(T_1)$, $y_j = r_j(\hat{S})$ for some $\hat{S} \in \mathcal{R}(T_1)$. Therefore we see that $\prod_{n\in\mathbb{N}} \mathcal{AR}_n(T_1)$ is closed in $\big(\mathcal{AR}(T_1)\big)^{\mathbb{N}}$, and correspondingly $\mathcal{R}(T_1)$ is closed as an embedding in $\big(\mathcal{AR}(T_1)\big)^{\mathbb{N}}$. $\square$

CHAPTER 4: THE BANACH SPACE OF $T_1(d,\theta)$

The space of $T_1(d,\theta)$ is a Banach space with the norm that will be defined in this chapter.

For $d \in \mathbb{N}$ with $d \geq 2$ and for $m \leq d$, a collection of nonempty sets $\{E_i\}_{i=1}^m \subseteq \mathcal{AR}(T_1)$ is said to be admissible if for each $j = 1, 2, \ldots, m-1$, we have $\max E_j < \min E_{j+1}$, where $<$ is the lexicographic order on $[T_1]$, the top of the tree of T_1 described in the previous chapter.

Definition 9. By $c_{00}([T_1])$, we denote the set of all functions $x : [T_1] \longrightarrow \mathbb{R}$ such that the support set of each function x, denoted as $\mathrm{supp}(x) = \{s \in [T_1] : x(s) \neq 0\}$, is finite. Let $\{e_t\}_{t\in[T_1]}$ denote the canonical basis of $c_{00}([T_1])$: for each $x \in c_{00}([T_1])$, $x = \sum_{t\in[T_1]} x_t e_t = \sum_{t=\langle 0,0\rangle}^{u} x_t e_t$ for some $u \in [T_1]$, where $x_t \in \mathbb{R}$ for each t. If $E \in \mathcal{AR}(T_1)$, we set $Ex = \sum_{t\in E} x_t e_t$.

Definition 10. Take $0 < \theta < 1$ with $d\theta > 1$. For $x = \sum_{t\in[T_1]} x_t e_t \in c_{00}([T_1])$, we define the following sequence of norms:

$$\|x\|_0 = \sup_{t\in[T_1]} |x_t| = \max_{t\in[T_1]} |x_t|,$$

and for each $j \in \mathbb{N}$,

$$\|x\|_{j+1} = \max\left\{ \|x\|_j, \theta \max\left\{ \sum_{i=1}^{m} \left\|E_i x\right\|_j : 1 \leq m \leq d, \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}\right\}.$$

Finally set $\|x\|_{T_1(d,\theta)} = \sup_{j\in\mathbb{N}} \|x\|_j$.

Note that since $d \geq 2$, the support set is indeed split on each stage of the recursive process in taking admissible sets.

From now on, the underscript of the norm $\|\cdot\|_j$ would mean the j-th step in the inductive definition of taking norms in Definition 10. If we ever want to indicate the norm in the ℓ_p space, we will use the convention $\|\cdot\|_{\ell_p}$.

Now we show that this definition induces a norm on $c_{00}([T_1])$.

Lemma 8. $\|\cdot\|_{T_1(d,\theta)}$ defines a norm on $c_{00}([T_1])$.

Proof. We first show that $\|\cdot\|_j$ is a norm on $c_{00}([T_1])$ for each $j \in \mathbb{N}$ by induction on j. Throughout the proof we let x, y be arbitrary elements from $c_{00}([T_1])$.

$\|\cdot\|_0$ is the usual sup-norm on $c_{00}([T_1])$, so is a norm.

Suppose for some $j \in \mathbb{N}$, $\|\cdot\|_j$ is a norm defined on $c_{00}([T_1])$. Then $\|x\|_{j+1}$ is chosen as the maximum of nonnegative values, so is nonnegative. $\|x\|_{j+1} = 0$ means both $\|x\|_j = 0$ and $\left\|E_i x\right\|_j = 0$ for each $i = 1, 2, \cdots, m$ and for every choice of admissible collection $\{E_i\}_{i=1}^m$, which in turn implies $x = 0$ since $\|\cdot\|_j$ is a norm. If $x = 0$, then $\|x\|_j = 0$ and $\left\|Ex\right\|_j = 0$ for each $E \in \mathcal{AR}(T_1)$. So $\|x\|_{j+1} = 0$ as it is chosen from either a value of 0 or a sum of 0. For $\alpha \in \mathbb{R}$, either $\|\alpha x\|_{j+1} = \|\alpha x\|_j$ or $\|\alpha x\|_{j+1} = \theta \max\left\{\sum_{i=1}^m \left\|E_i x\right\|_j : \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}$. For the first case, $\|\alpha x\|_{j+1} = \|\alpha x\|_j = |\alpha|\,\|x\|_j$ as $\|\cdot\|_j$ is a norm. For the latter case, since we have $E(\alpha x) = \sum_{t\in E} \alpha x_t e_t = \alpha \sum_{t \in E} x_t e_t = \alpha\big(Ex\big)$ for each $E \in \mathcal{AR}(T_1)$, we see that

$$\begin{aligned}
\|\alpha x\|_{j+1} &= \theta \max\left\{\sum_{i=1}^m \left\|E_i(\alpha x)\right\|_j : \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}\\
&= \theta \max\left\{\sum_{i=1}^m \left\|\alpha\big(E_i x\big)\right\|_j : \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}\\
&= \theta \max\left\{|\alpha| \sum_{i=1}^m \left\|E_i x\right\|_j : \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}\\
&= |\alpha| \cdot \theta \max\left\{\sum_{i=1}^m \left\|E_i x\right\|_j : \left\{E_i\right\}_{i=1}^m \text{ is admissible}\right\}\\
&= |\alpha|\,\|x\|_{j+1}.
\end{aligned}$$

In addition, either $\|x+y\|_{j+1} = \|x+y\|_j \leq \|x\|_j + \|y\|_j \leq \|x\|_{j+1} + \|y\|_{j+1}$, or $\|x+y\|_{j+1} = \theta \max \Big\{ \sum_{i=1}^m \big\| E_i(x+y) \big\|_j : \big\{ E_i \big\}_{i=1}^m \text{ is admissible} \Big\}$. Since we also have $E(x+y) = \sum_{t \in E} (x_t + y_t) e_t = \sum_{t \in E} x_t e_t + \sum_{t \in E} y_t e_t = Ex + Ey$, we see that

$$
\begin{aligned}
\|x+y\|_{j+1} &= \theta \max \left\{ \sum_{i=1}^m \big\| E_i(x+y) \big\|_j : \big\{ E_i \big\}_{i=1}^m \text{ is admissible} \right\} \\
&= \theta \max \left\{ \sum_{i=1}^m \big\| E_i x + E_i y \big\|_j : \big\{ E_i \big\}_{i=1}^m \text{ is admissible} \right\} \\
&\leq \theta \max \left\{ \sum_{i=1}^m \big\| E_i x \big\|_j + \sum_{i=1}^m \big\| E_i y \big\|_j : \big\{ E_i \big\}_{i=1}^m \text{ is admissible} \right\} \\
&\leq \theta \Bigg(\max \left\{ \sum_{i=1}^m \big\| E_i x \big\|_j \right\} + \max \left\{ \sum_{i=1}^m \big\| E_i y \big\|_j] \right\} \Bigg) \\
&\leq \|x\|_{j+1} + \|y\|_{j+1}.
\end{aligned}
$$

Therefore $\| \cdot \|_{j+1}$ is a norm, and by induction $\| \cdot \|_j$ is a norm for each $j \in \mathbb{N}$.

Lastly, for each $x \in c_{00}([T_1])$, the sequence $\big(\|x\|_j \big)_{j \in \mathbb{N}}$ is increasing and bounded above by the $\ell_1([T_1])$-norm of x since on each stage of the recursion, the cardinality $\mathrm{car}\Big(\big\{ E_i \cap \mathrm{supp}(x) \neq \varnothing : \big\{ E_i \big\}_{i=1}^m \text{ is admissible} \big\} \Big)$ has to be less than or equal to $\mathrm{car}\big(\mathrm{supp}(x)\big)$. Hence $\|x\|_{T_1(d,\theta)} = \sup_{j \in \mathbb{N}} \|x\|_j < \infty$. Also $\|x\|_{T_1(d,\theta)} = \sup_{j \in \mathbb{N}} \|x\|_j \geq 0$ as $\|x\|_j \geq 0$ for each $j \in \mathbb{N}$. $\|x\|_{T_1(d,\theta)} = 0$ if and only if $\|x\|_j = 0$ for each $j \in \mathbb{N}$ if and only if $x = 0$ since $\| \cdot \|_j$ is a norm for each $j \in \mathbb{N}$. For $\alpha \in \mathbb{R}$, $\|\alpha x\|_{T_1(d,\theta)} = \lim_{j \longrightarrow \infty} \|\alpha x\|_j = \lim_{j \longrightarrow \infty} |\alpha| \, \|x\|_j = |\alpha| \lim_{j \longrightarrow \infty} \|x\|_j = |\alpha| \, \|x\|_{T_1(d,\theta)}$, and $\|x+y\|_{T_1(d,\theta)} = \lim_{j \longrightarrow \infty} \|x+y\|_j \leq \lim_{j \longrightarrow \infty} \|x\|_j + \lim_{j \longrightarrow \infty} \|y\|_j = \|x\|_{T_1(d,\theta)} + \|y\|_{T_1(d,\theta)}$ by the continuities of addition, scalar multiplication, and taking norms.

□

The completion of $c_{00}([T_1])$ with respect to $\|x\|_{T_1(d,\theta)}$ is a Banach space which is denoted by $\Big(T_1(d,\theta), \|\cdot\|_{T_1(d,\theta)}\Big)$, or simply by $T_1(d,\theta)$. Often enough we drop the subscript $T_1(d,\theta)$ of the norm and simply denote it by $\|\cdot\|$ in this context.

Notice that if $x = \sum_{t\in[T_1]} x_t e_t \in T_1(d,\theta)$, then

$$\|x\| = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^{m} \big\|E_i x\big\| : 1 \le m \le d, \big\{E_i\big\}_{i=1}^{m} \text{ admissible}\right\}\right\},$$

where $\|x\|_\infty = \sup_{t\in[T_1]} |x_t|$.

Notation. For $k \in \mathbb{N}$, denote

$$T_1[k] = \{t \in [T_1] : \alpha_1(t) = k\},$$

where $\alpha_1(t)$ gives us the length 1 initial segment of the node t, and set

$$T_1^{(k)} = \overline{\text{span}}\{e_t : t \in T_1[k]\}.$$

The following lemma provides a critical observation for a method to find maximal norms on admissible sets, and will be utilised in the proofs of the later properties.

Lemma 9. Let $\{E_i\}_{i=1}^m \subseteq \mathcal{AR}(T_1)$ be admissible and suppose for some $k \in \mathbb{N}$, we have $E_j \cap T_1[k] \neq \varnothing$ for some $j < m$. Then $E_{j+1} \cap T_1[k]$ is either empty or a singleton.

Proof. By the admissibility of $\{E_i\}_{i=1}^m$, we have $\max(E_j) < \min(E_{j+1})$. So we have $\alpha_1\big(\max(E_j)\big) \le \alpha_1\big(\min(E_{j+1})\big)$. If $k < \alpha_1\big(\max(E_j)\big) \le \alpha_1\big(\min(E_{j+1})\big)$ or $k \le \alpha_1\big(\max(E_j)\big) < \alpha_1\big(\min(E_{j+1})\big)$, then as $\alpha_1\big(\min(E_{j+1})\big) \le \alpha_1(t)$ for every $t \in E_{j+1}$, we have $k < \alpha_1(t)$ for every $t \in E_{j+1}$. Hence $E_{j+1} \cap T_1[k] = \varnothing$. Otherwise $k = \alpha_1\big(\max(E_j)\big) = \alpha_1\big(\min(E_{j+1})\big)$. Since $E_{j+1} \in \mathcal{AR}(T_1)$, $\min(E_{j+1})$ is the only node of E_{j+1} that is in $T_1[k]$. So $E_{j+1} \cap T_1[k]$ is a singleton. $\square$

Proposition 10. For a fixed $k \in \mathbb{N}$, if $x = \sum_{t \in T_1[k]} x_t e_t \in T_1^{(k)}$, then

$$\max_{t \in T_1[k]} |x_t| \leq \|x\| \leq \frac{d-1}{1-\theta} \cdot \max_{t \in T_1[k]} |x_t|.$$

And therefore we have $T_1^{(k)} \cong \ell_\infty^k$.

Proof. We show that $\max_{t \in T_1[k]} |x_t| \leq \|x\|_j \leq \left(\sum_{n=0}^{j} \theta^n\right)\left(d-1\right) \max_{t \in T_1[k]} |x_t|$ for each $j \in \mathbb{N}$ by induction. Note that since $\left(\sum_{n=0}^{j} \theta^n\right) \geq \theta^0 = 1$ and $d \geq 2$, the inequalities make sense.

The base case $j = 0$ is clear: $\max_{t \in T_1[k]} |x_t| = \|x\|_0 \leq \theta^0(d-1) \cdot \max_{t \in T_1[k]} |x_t|$ since $d - 1 \geq 1$.

Now suppose $\max_{t \in T_1[k]} |x_t| \leq \|x\|_p \leq \left(\sum_{n=0}^{p} \theta^n\right)\left(d-1\right) \cdot \max_{t \in T_1[k]} |x_t|$ for each $p \leq j$ for some $j \in \mathbb{N}$. Then since $\{\|x\|_j\}_{j=0}^{\infty}$ is an increasing sequence for each x, we have $\max_{t \in T_1[k]} |x_t| \leq \|x\|_{j+1}$. On the other hand, if $\|x\|_{j+1} = \|x\|_j$, then this means that $\|x\|_s = \|x\|_{s+1} = \cdots = \|x\|_j = \|x\|_{j+1} = \cdots$ for some $s \leq j$. Hence by induction hypothesis we have

$$\|x\|_{j+1} = \|x\|_s \leq \left(\sum_{n=0}^{s} \theta^n\right)\left(d-1\right) \cdot \max_{t \in T_1[k]} |x_t| \leq \left(\sum_{n=0}^{j+1} \theta^n\right)\left(d-1\right) \cdot \max_{t \in T_1[k]} |x_t|,$$

since $s \leq j$. Otherwise $\|x\|_{j+1} = \theta \cdot \max\left\{ \sum_{i=1}^{m} \|E_i x\|_j : \{E_i\}_{i=1}^{m} \text{ admissible}\right\}$. In order to attain the maximum, we may assume that $E_i \cap T_1[k] \neq \varnothing$ for all $i = 1, 2, \ldots, m$. By lemma 9, the support set $\operatorname{supp}\left(E_i x\right)$ are singletons for $i = 2, 3, \ldots, m$. Set $E_i = \{t_i\}$ where $t_i \in T_1[k]$ for each $i = 2, 3, \ldots, m$. Then we have

$$\begin{aligned}
\|x\|_{j+1} &= \theta \max\left\{ \left\|E_1 x\right\|_j + \sum_{i=2}^{m} |x_{t_i}| \right\} \\
&\leq \theta \left(\left(\sum_{n=0}^{j} \theta^n\right)\left(d-1\right) \cdot \max_{t \in T_1[k]} |x_t| + \left(d-1\right) \cdot \max_{i=2,\cdots,m} |x_{t_i}| \right)
\end{aligned}$$

$$\le \left(\sum_{n=0}^{j} \theta^{n+1} + \theta\right)\bigl(d-1\bigr) \cdot \max_{t \in T_1[k]} |x_t|$$

$$\le \left(\sum_{n=1}^{j+1} \theta^{n} + 1\right)\bigl(d-1\bigr) \cdot \max_{t \in T_1[k]} |x_t|$$

$$= \left(\sum_{n=0}^{j+1} \theta^{n}\right)\bigl(d-1\bigr) \cdot \max_{t \in T_1[k]} |x_t|.$$

Thus by induction we have the claim.

Now since $\|x\| = \sup_{j\in\mathbb{N}} \|x\|_j$, we see that

$$\max_{t \in T_1[k]} |x_t| = \|x\|_0 \le \sup_{j\in\mathbb{N}} \|x\|_j \le \left(\sum_{n=0}^{\infty} \theta^{n}\right)\bigl(d-1\bigr) \cdot \max_{t \in T_1[k]} |x_t| = \frac{d-1}{1-\theta} \cdot \max_{t \in T_1[k]} |x_t|.$$

$\square$

If $x \in T_1(d,\theta)$, we can write it as $x = \sum_{n=0}^{\infty} a_n x_n$ where for each $n \in \mathbb{N}$, $x_n \in T_1^{(n)}$, normalised. So $\{x_n\}_{n\in\mathbb{N}}$ denote the normalised block sequences on $T_1^{(n)}$ for each $n \in \mathbb{N}$. We will show that $\|x\| \approx \left(\sum_{i=0}^{\infty} \|a_n\|^p\right)^{1/p}$. We divide the proof in two parts, using Bellenot's result (see below) to prove the lower bound, and Argyros and Deliyanni's method to prove the upper bound.

Let $0 < \theta < 1$ and $1 \le p < \infty$. Take

$$x \in c_{00}(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} : a_n \neq 0 \text{ for finitely many } n \in \mathbb{N}\}.$$

Let $\{b_n\}_{n\in\mathbb{N}}$ be the canonical basis of $c_{00}(\mathbb{N})$, so that $x = \sum_{n=0}^{N} a_n b_n = \sum_{n\in\mathbb{N}} a_n b_n$ for each $x \in c_{00}(\mathbb{N})$ where $a_n = 0$ for $n \ge N+1$. For a sequence of subsets $\{D_i\}_{i=1}^{d} \subseteq \mathbb{N}^{[<\infty]}$, we say $\{D_i\}_{i=1}^{d}$ is Bel-admissible if $\max D_i < \min D_{i+1}$ for $i = 1, 2, \ldots, d-1$. Bellenot defines the norm $\|\cdot\|_{\text{Bel}}$ as the following: for $x = \sum_{n\in\mathbb{N}} a_n b_n$,

$$
\begin{aligned}
\|x\|_0 &= \sup_{n\in\mathbb{N}} |a_n|, \\
\|x\|_{j+1} &= \sup\left\{ \|x\|_j, \left(\theta \sum_{i=1}^{d} \|D_i x\|_j^p\right)^{1/p} \right\},
\end{aligned}
$$

where the supremum is taken over all Bel-admissible sequences $\{D_i\}_{i=1}^d$, and finally

$$
\|x\|_{\text{Bel}} = \lim_{j\to\infty} \|x\|_j = \sup\left\{ \|x\|_0, \left(\theta \sum_{i=1}^{d} \|D_i x\|_{\text{Bel}}^p\right)^{1/p} \right\}.
$$

In addtion we have the following inequality that Bellenot derived to characterise the norm with the ℓ_p-space.

Lemma 11 (Bellenot [4])**.** If $d\theta = d^{1/p} > 1$, then

$$
\theta^2 \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{N} a_n b_n \right\|_{\text{Bel}} \leq \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/p} \text{ for each } N \in \mathbb{N}.
$$

Lemma 12. If for each $k \in \mathbb{N}$, we take a specific $e_k \in T_1^{(k)}$, an element of the canonical basis, then $\overline{\text{span}}\{e_k : k \in \mathbb{Z}^+\} \cong \ell_p$.

Proof. Let $x = \sum_{k=0}^{N} a_k e_k$. Given $\{E_i\}_{i=1}^m \subseteq \mathcal{AR}(T_1)$ admissible, assume for each $i = 1, 2, \ldots, m$ that E_i consists of nodes of $[T_1]$ from $T_1[k_1^i], T_1[k_2^i], \ldots, T_1[k_{l_i}^i]$. As $\{E_i\}_{i=1}^m$ is admissible, we see that $k_{l_i}^i < k_1^{i+1}$. Hence $\{E_i \cap \text{supp}(x)\}_{i=1}^m$ forms a sequence of Bel-admissible sets on $\{k : k \in \mathbb{N}\}$. On the other hand, it is easy to see that for each Bel-admissible sequence on $\{k : k \in \mathbb{N}\}$, we can find an admissible sequence in $\mathcal{AR}(T_1)$ that, when restricted to the support of x, corresponds to the Bel-admissible sequence. Hence we get that $\left\|\sum_{k=0}^{N} a_k e_k\right\| = \left\|\sum_{k=0}^{N} a_k e_k\right\|_{\text{Bel}}$. Therefore, we obtain

$$\theta^2\Big(\sum_{n=0}^{N} |a_n|^p\Big)^{1/p} \leq \Big\|\sum_{n=0}^{N} a_n e_n\Big\| \leq \Big(\sum_{n=0}^{N} |a_n|^p\Big)^{1/p} \text{ for each } N \in \mathbb{N},$$

and this completes the proof. $\square$

The goal here is to show that the space $T_1(d,\theta)$ is isomorphic to $\Big(\sum_{n\in\mathbb{N}}\oplus\, \ell_\infty^{n+1}\Big)_p$. To this end we consider sequences in blocks where each block has support set on a single branch of T_1. Let $x = \sum_{n=0}^{N} x_n \in c_{00}([T_1])$ where for each $n = 0, 1, 2, \ldots, N$, $x_n \in T_1^{(n)}$. Then we derive a lower bound of the norm of x in the following lemma.

Lemma 13.

$$\|x\| \geq \frac{\theta^2(1-\theta)}{d-1}\left(\sum_{n=0}^{N} \|x_n\|^p\right)^{1/p}.$$

Proof. For each $n = 0, 1, 2, \ldots, N$, suppose $x_n = \sum_{t=0}^{n} a_{n_t} e_{n_t}$ and $\max\{|a_{n_t}|\}_{t\in T_1[n]} = |a_{n_{\max}}|$. Take $Q_{n_{\max}} : T_1(d,\theta) \longrightarrow \overline{\text{span}}\{e_{n_{\max}}\}$ as the canonical surjection onto the closed span of the basic element with maximal coefficient in each block. Set $Q = \sum_{n=0}^{N} Q_{n_{\max}}$. Since $T_1(d,\theta)$ has a 1-unconditional basis, Q is a contractive surjection. By a previous lemma we have $|a_{n_{\max}}| \geq \frac{1-\theta}{d-1}\|x_n\|$. Therefore we get

$$\begin{aligned}
\|x\| &= \Big\|\sum_{n=0}^{N} x_n\Big\| \geq \Big\|Q\Big(\sum_{n=0}^{N} x_n\Big)\Big\| = \Big\|\sum_{n=0}^{N} Q_{n_{\max}}(x_n)\Big\| = \Big\|\sum_{n=0}^{N} a_{n_{\max}} e_{n_{\max}}\Big\| \\
&\geq \theta^2\Big(\sum_{n=0}^{N} |a_{n_{\max}}|^p\Big)^{1/p} \geq \theta^2\Big(\sum_{n=0}^{N}\Big(\frac{1-\theta}{d-1}\|x_n\|\Big)^p\Big)^{1/p} \\
&= \theta^2\Big(\Big(\frac{1-\theta}{d-1}\Big)^p \sum_{n=0}^{N} \|x_n\|^p\Big)^{1/p} = \frac{\theta^2(1-\theta)}{d-1}\left(\sum_{n=0}^{N} \|x_n\|^p\right)^{1/p}.
\end{aligned}$$

$\square$

We now use techniques from Argyros and Delyanni [1] to show that

$$\|x\| \leq \frac{2}{\theta}\left(\sum_{n\in\mathbb{N}}\|x_n\|^p\right)^{1/p}.$$

Definition 11. We say that a sequence of finite sets $\{F_i\}_{i=1}^m \subseteq 2^{([T_1])}$, $m \leq d$, is almost admissible if there is admissible $\{E_i\}_{i=1}^m \subseteq \mathcal{AR}(T_1)$ such that $F_i \subseteq E_i$ for each $i \in \{1, 2, \ldots, m\}$.

For a linear functional $f : T_1(d, \theta) \longrightarrow \mathbb{R}$, we will call $\mathrm{supp}(f) = \{t \in [T_1] : f(e_t) \neq 0\}$ for simplicity.

Definition 12. Define inductively a sequence $\{K_s\}_{s\in\mathbb{N}}$ of subsets of linear functionals from the dual space of $T_1(d, \theta)$ such that $K_0 = \big\{\pm e_t^* : t \in [T_1]\big\}$, where e_t^* is linear such that $e_t^*(e_t) = 1$ and $e_t^*(e_u) = 0$ for each $u \neq t$ and that for each $s \in \mathbb{N}$

$$K_{s+1} = K_s \cup \left\{\theta\sum_{i=1}^m f_i : m \leq d, f_i \in K_s, \Big\{\text{supp}\big(f_i\big)\Big\}_{i=1}^m \text{almost admissible}\right\}.$$

Set $K = \bigcup_{s\in\mathbb{N}} K_s$. For convenience we call a sequence of functionals $\{f_i\}_{i=1}^m$ successive if $\{\text{supp}(f_i)\}_{i=1}^m$ is almost admissible.

Lemma 14. For each $f \in K$, we have $\|f\| \leq 1$. For each $x \in c_{00}([T_1])$, $\|x\| = \sup_{f\in K} f(x)$.

Proof. For the first statement, we show that for every $f \in K_s$, $s \in \mathbb{N}$, we have $|f(x)| \leq \|x\|$. Let $x \in c_{00}([T_1])$. If $f \in K_0$, then $f = \pm e_t^*$ for some $t \in [T_1]$. So $|f(x)| = \big|\pm e_t^*(x)\big| \leq \|x\|$ by unconditionality. Suppose $|f(x)| \leq \|x\|$ for every $f \in K_s$. Take $f \in K_{s+1}$. Either $f \in K_s$, in which case we are done, or $f = \theta\sum_{i=1}^m f_i$ for $m \leq d$, where $f_i \in K_s$ for $1 \leq i \leq m$, and $\big\{\text{supp}(f_i)\big\}_{i=1}^m$ almost admissible. Take $\{F_i\}_{i=1}^m \subseteq \mathcal{AR}(T_1)$

admissible such that $\operatorname{supp}(f_i) \subseteq F_i$ for each i. Then we have

$$\begin{aligned} \left|f(x)\right| = \left|\theta \sum_{i=1}^{m} f_i(x)\right| = \left|\theta \sum_{i=1}^{m} f_i(F_i x)\right| \leq \theta \left(\sum_{i=1}^{m} |f_i(F_i x)|\right) \leq \theta \left(\sum_{i=1}^{m} \left\|F_i x\right\|\right) \\ \leq \theta \sup \left\{\sum_{i=1}^{M} \left\|E_i x\right\| : 1 \leq M \leq d, \{E_i\}_{i=1}^{M} \text{ admissible}\right\} \leq \|x\|. \end{aligned}$$

By induction we establish that $\|f\| = \sup_{\|x\|\leq 1} \left|f(x)\right| \leq \sup_{\|x\|\leq 1} \|x\| \leq 1$ for each $f \in K$, and that $\sup_{f\in K} f(x) \leq \|x\|$.

For the second statement, we show that $\|x\|_s \leq \sup_{f\in K_s} f(x)$ for each $s \in \mathbb{N}$. For the base case we have $\|x\|_0 = \sup_{t\in[T_1]} |x_t| = \sup_{t\in[T_1]} \left| \pm e_t^*(x)\right| = \sup_{f\in K_0} f(x)$. Now suppose $\|x\|_s \leq \sup_{f\in K_s} f(x)$ for each $x \in c_{00}([T_1])$. Then either $\|x\|_{s+1} = \|x\|_s \leq \sup_{f\in K_s} f(x) \leq \sup_{f\in K_{s+1}} f(x)$ as $K_s \subseteq K_{s+1}$, or $\|x\|_{s+1} = \theta \sum_{i=1}^{m} \left\|E_i x\right\|_s$ for $\{E_i\}_{i=1}^{m}$ admissible. By the induction hypothesis and the fact that the supremum is actually attained, there is $\{f_i\}_{i=1}^{m} \subseteq K_s$ such that $\operatorname{supp}(f_i) \subseteq E_i$ and $\|E_i x\|_s \leq f_i(E_i x)$ for each i. Setting $g = \theta \sum_{i=1}^{m} f_i$, we have $g \in K_{s+1}$, and so $\|x\|_{s+1} = \theta \sum_{i=1}^{m} \|E_i x\|_s \leq \theta \sum_{i=1}^{m} f_i(E_i x) = \theta \sum_{i=1}^{m} f_i(x) = g(x) \leq \sup_{f\in K_{s+1}} f(x)$. Hence we have $\|x\| \leq \sup_{f\in K} f(x)$. □

Definition 13. For every $\varphi \in K_l \setminus K_{l-1}$, we define an analysis of φ as a sequence $\{F^s(\varphi)\}_{s=0}^{l}$ of subsets of K such that

(1) for each s, $F^s(\varphi)$ consists of successive elements of K_s and $\bigcup_{f\in F^s(\varphi)} \operatorname{supp}(f) = \operatorname{supp}(\varphi)$,

(2) If $f \in F^{s+1}(\varphi)$ then either $f \in F^s(\varphi)$, or there are successive $f_1, f_2, \ldots, f_m \in F^s(\varphi)$, $m \leq d$, such that $f = \theta \sum_{i=1}^{m} f_i$, and

(3) $F^l(\varphi) = \{\varphi\}$.

Note. By the definition of K, every $\varphi \in K$ has an analysis. Also given φ, if $f \in F^s(\varphi)$ and $g \in F^{s+1}(\varphi)$, then either $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$ or $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$.

Lemma 15. Let $\varphi \in K$. Then φ can be represented as $\sum_{i=1}^{k} a_{t_i} e_{t_i}^*$ with $a_{t_i} \neq 0$, where $\{t_i\}_{i=1}^{k} \subseteq [T_1]$. So $\operatorname{supp}(\varphi) = \{t_1, t_2, \dots, t_k\}$. Let $E \subseteq [T_1]$ be finite and set $P_E(\varphi) = \sum_{t \in E \cap \operatorname{supp}(\varphi)} a_t e_t^*$. Then we have $P_E(\varphi) \in K$.

Proof. We show $P_E(\varphi) \in K$ by induction on $s \in \mathbb{N}$ where $\varphi \in K_s$. If $\varphi \in K_0$, this is clearly true as $\varphi = \pm e_t^*$ for some $t \in [T_1]$. Assume that for every $\varphi \in K_s$, $P_E(\varphi) \in K$. Take $\varphi \in K_{s+1}$. Then either $\varphi = f$ for some $f \in K_s$, and $P_E(f) \in K$ by induction hypothesis, or $\varphi = \theta \sum_{i=1}^{m} f_i$ where $f_i \in K_s$ for $i = 1, 2, \dots, m$ and $\{\operatorname{supp}(f_i)\}_{i=1}^{m}$ almost admissible. So $P_E(\varphi) = P_E\Big(\theta \sum_{i=1}^{m} f_i\Big) = \theta \sum_{i=1}^{m} P_E(f_i)$ by the linearity of P_E. By the induction hypothesis $P_E(f_i) \in K$ for each $i = 1, 2, \dots, m$, and so $P_E(\varphi) \in K$ by the definition. $\square$

The lemma allows us restricting the functional to any part that we want to focus on.

Lemma 16. Let $\{x_n\}_{n \in \mathbb{N}}$ be the normalised block sequence such that $\operatorname{supp}(x_n) \in T_1^{(n)}$ and $\|x_n\| = 1$ for each $n \in \mathbb{N}$. Let $x = \sum_{n=0}^{N} a_n x_n \in c_{00}([T_1])$. Then we have

$$\left\| \sum_{n=0}^{N} a_n x_n \right\| \leq \frac{2}{\theta} \left(\sum_{n=0}^{N} |a_n|^p \right)^{1/p}, \text{ where } d\theta = d^{1/p}.$$

Proof. Throughout the proof, let $\{e_t\}_{t \in [T_1]}$ denote the canonical basis of $c_{00}([T_1])$. Let $\varphi \in K$, say $\varphi \in K_l \setminus K_{l-1}$ for some $l \in \mathbb{N} \setminus \{0\}$. We want to find an upper bound for $\varphi(x)$. By lemma 15 we may assume that $\operatorname{supp}(\varphi) \subseteq \operatorname{supp}(x)$ since $P_{\operatorname{supp}(x)}(\varphi) \in K$. As $x \in c_{00}([T_1])$ and $\{e_t\}_{t \in [T_1]}$ is an unconditional basis, we may also assume that x and φ have nonnegative coordinates. For each $n = 0, 1, 2, \dots, N$, set

$$s_n = \max \Big\{ s \in \mathbb{N} : \Big| \Big\{ f \in F^s(\varphi) : \operatorname{supp}(f) \cap \operatorname{supp}(x_n) \neq \varnothing \Big\} \Big| \geq 2 \Big\},$$

and $s_n = 0$ if $\Big|\{f \in F^s(\varphi) : \text{supp}(f) \cap \text{supp}(x_n) \neq \varnothing\}\Big| \leq 1$ for every s. Hence for each $n = 0, 1, 2, \ldots, N$, there are $g_{n(1)}, g_{n(2)}, \ldots, g_{n(m)}$, where $2 \leq n(m) \leq d$, of successive elements of $F^{s_n}(\varphi)$ such that for each $i = 1, 2, \ldots, m$, $\text{supp}(g_{n(i)}) \cap \text{supp}(x_n) \neq \varnothing$. Define the initial and the final parts, x'_n and x''_n respectively, as

$$x'_n = x_n \restriction \text{supp}(g_{n(1)}) \text{ and } x''_n = x_n \restriction \bigcup_{i=2}^{m} \text{supp}(g_{n(i)}). \qquad (*)$$

We show by induction on s that for every $J \subseteq \{1, 2, \ldots, N\}$ and for each $f \in F^s(\varphi)$, we have

$$f\Big(\sum_{n\in J} a_n x'_n\Big) \leq \frac{1}{\theta}\Big\|\sum_{n\in J} a_n e_n\Big\|_{\text{Bel}} \text{ and } f\Big(\sum_{n\in J} a_n x''_n\Big) \leq \frac{1}{\theta}\Big\|\sum_{n\in J} a_n e_n\Big\|_{\text{Bel}}.$$

We first show the initial part. For $f \in F^0(\varphi)$ this is clear: by the unconditionality we have $f\big(\sum_{n\in J} a_n x'_n\big) = e^*_t\big(\sum_{n\in J} a_n x'_n\big) \leq \|a_t e_t\| \leq \|\sum_{n\in J} a_n e_n\|_{\text{Bel}}$ for some $t \in [T_1]$.

Suppose for every $k \leq s$ and for each $f \in F^k(\varphi)$, we have that $f\left(\sum_{n\in J} a_n x'_n\right) \leq \frac{1}{\theta}\|\sum_{n\in J} a_n e_n\|_{\text{Bel}}$ for every $J \subseteq \{0, 1, 2, \ldots, N\}$. Let $f \in F^{s+1}(\varphi)$ and take an arbitrary $J \subseteq \{0, 1, 2, \ldots, N\}$. Either $f \in F^s(\varphi)$, in which case we are done, or $f = \theta\big(\sum_{i=1}^m f_i\big)$ for $m \leq d$ and $\{f_i\}_{i=1}^m \subseteq F^s(\varphi)$ where the support set is almost admissible. Take $\{E_i\}_{i=1}^m$ admissible such that $\text{supp}(f_i) \subseteq E_i$ for each $i = 1, 2, \ldots, m$. Set

$$D = \Big\{n \in J : \Big|\{i : \text{supp}(f_i) \cap \text{supp}(x'_n) \neq \varnothing\}\Big| \leq 1\Big\}$$

and consider the sets

$$I = \Big\{i \in \{1, 2, \ldots, m\} : \text{supp}(f_i) \cap \text{supp}(x'_n) \neq \varnothing \text{ for some } n \in D\Big\}$$

and

$$H = \Big\{n \in J : \Big|\{i : \text{supp}(f_i) \cap \text{supp}(x'_n) \neq \varnothing\}\Big| \geq 2\Big\}.$$

Let $n \in H$ and set $i_0 = \min\big\{i \in \{1, 2, \ldots, m\} : \operatorname{supp}(f_i) \cap \operatorname{supp}(x'_n) \neq \varnothing\big\}$. We claim that $i_0 + 1 \notin I$. Indeed, since $n \in H$, other than f_{i_0}, there is $f_j \in F^s(\varphi)$ such that $\operatorname{supp}(f_j) \cap \operatorname{supp}(x'_n) \neq \varnothing$ for some $j \geq i_0 + 1$. By the definition of s_n, we see that $s \leq s_n$. So we may find $g, h \in F^{s_n}(\varphi)$ such that $\operatorname{supp}(f_{i_0}) \subseteq \operatorname{supp}(g)$ and $\operatorname{supp}(f_j) \subseteq \operatorname{supp}(h)$. By ($*$), we must have $g = h$, for otherwise $\operatorname{supp}(f_j) \cap \operatorname{supp}(x'_n) \subseteq \operatorname{supp}(h) \cap \operatorname{supp}(x'_n) = \varnothing$ as x'_n only takes on the support of one functional on F^{s_n}. ($*$) also tells us that $\max \operatorname{supp}(g) = \max \operatorname{supp}(x'_n)$. Note that we have assumed that $\operatorname{supp}(f) \subseteq \operatorname{supp}(\varphi) \subseteq \operatorname{supp}(x)$ for any $f \in F^s(\varphi)$. In particular the supports of $f_{i_0}, f_{i_0+1}, \ldots, f_j$ are nonempty and have nonempty intersection with $\operatorname{supp}(x'_n)$. So by the almost admissibility, $\max \operatorname{supp}(f_{i_0+1}) \leq \max \operatorname{supp}(f) = \max \operatorname{supp}(x'_n)$ and $\min \operatorname{supp}(f_{i_0+1}) > \max \operatorname{supp}(f_{i_0}) \geq \min \operatorname{supp}(x'_n)$. Therefore $\operatorname{supp}(f_{i_0+1}) \subseteq \operatorname{supp}(x'_n)$ and we have $i_0 + 1 \notin I$.

As a result, the function $G : H \longrightarrow \{1, 2, \ldots, m\} \setminus I$ defined by $n \in H \longmapsto \min\{i : \operatorname{supp}(f_i) \cap \operatorname{supp}(x'_n) \neq \varnothing\} + 1$ is an injection. Therefore $\big|H\big| \leq m - \big|I\big| \leq d - \big|I\big|$ and we have $\big|I\big| + \big|H\big| \leq d$.

Thus we have

$$
\begin{aligned}
f\Big(\sum_{n \in J} a_n x'_n\Big) &= \Big(\theta \sum_{i=1}^{m} f_i\Big)\Big(\sum_{n \in J} a_n x'_n\Big) = \theta \sum_{i=1}^{m} \bigg(\sum_{n \in J} f_i\Big(a_n x'_n\Big)\bigg) \\
&= \theta\left(\sum_{i=1}^{m} \bigg(\sum_{n \in D} f_i\Big(a_n x'_n\Big)\bigg) + \sum_{i=1}^{m} \bigg(\sum_{n \in H} f_i\Big(a_n x'_n\Big)\bigg)\right) \\
&= \theta\left(\sum_{i \in I} f_i\Big(E_i \sum_{n \in D} a_n x'_n\Big) + \sum_{n \in H} \Big(\sum_{i=1}^{m} f_i\Big)\Big(a_n x'_n\Big)\right) \\
&\leq \theta\left(\sum_{i \in I} \frac{1}{\theta} \bigg\| E_i\Big(\sum_{n \in D} a_n e_n\Big)\bigg\|_{\text{Bel}} + \sum_{n \in H} \Big(\frac{1}{\theta} f\Big)\Big(a_n x'_n\Big)\right) \\
&\leq \theta\left(\frac{1}{\theta} \sum_{i \in I} \bigg\| E_i\Big(\sum_{n \in D} a_n e_n\Big)\bigg\|_{\text{Bel}} + \frac{1}{\theta} \sum_{n \in H} \Big\| a_n x'_n \Big\|\right) \\
&= \sum_{i \in I} \bigg\| E_i\Big(\sum_{n \in D} a_n e_n\Big)\bigg\|_{\text{Bel}} + \sum_{n \in H} \Big\| a_n e_n \Big\|_{\text{Bel}} \leq \frac{1}{\theta} \bigg\| \sum_{n \in J} a_n e_n \bigg\|_{\text{Bel}}
\end{aligned}
$$

as $\{E_i\}_{i\in I} \cup \{\{e_n\}\}_{n\in H}$ forms a sequence of Bel-admissible sets.

Now we show the inequality for the final part. By the unconditionality we have $f\big(\sum_{n\in J} a_n x''_n\big) = e_t^*\big(\sum_{n\in J} a_n x''_n\big) \leq \|a_t e_t\| \leq \|\sum_{n\in J} a_n e_n\|_{\text{Bel}}$ for some $t \in [T_1]$ for $f \in F^0(\varphi)$.

Suppose for every $k \leq s$ and for all $f \in F^k(\varphi)$, we have that $f\left(\sum_{n\in J} a_n x''_n\right) \leq \frac{1}{\theta}\|\sum_{n\in J} a_n e_n\|_{\text{Bel}}$ for each $J \subseteq \{1, 2, \ldots, N\}$. Let $f \in F^{s+1}(\varphi)$ and $J \subseteq \{1, 2, \ldots, N\}$. Either $f \in F^s(\varphi)$, in which case we are done, or $f = \theta\big(\sum_{i=1}^m f_i\big)$ for $m \leq d$ and $\{f_i\}_{i=1}^m \subseteq F^s(\varphi)$ where the support set is almost admissible. Take $\{E_i\}_{i=1}^m$ admissible such that $\text{supp}(f_i) \subseteq E_i$ for each $i = 1, 2, \ldots, m$. Set

$$D = \Big\{n \in J : \Big|\big\{i : \text{supp}(f_i) \cap \text{supp}(x''_n) \neq \varnothing\big\}\Big| \leq 1\Big\}$$

and consider the sets

$$I = \Big\{i \in \big\{1, 2, \ldots, m\big\} : \text{supp}(f_i) \cap \text{supp}(x''_n) \neq \varnothing \text{ for some } n \in D\Big\}$$

and

$$H = \Big\{n \in J : \Big|\big\{i : \text{supp}(f_i) \cap \text{supp}(x''_n) \neq \varnothing\big\}\Big| \geq 2\Big\}.$$

Let $n \in H$ and set $i_0 = \min\big\{i \in \{1, 2, \ldots, m\} : \text{supp}(f_i) \cap \text{supp}(x''_n) \neq \varnothing\big\}$. Since $n \in H$, other than f_{i_0}, there is $j > i_0$ such that $f_j \in F^s(\varphi)$ with $\text{supp}(j) \cap \text{supp}(x''_n) \neq \varnothing$. By the definition of s_n, we see that $s \leq s_n$. So we may find $g, h \in F^{s_n}(\varphi)$ such that $\text{supp}(f_{i_0}) \subseteq \text{supp}(g)$ and $\text{supp}(f_j) \subseteq \text{supp}(h)$. By the definition of the final part, $\min \text{supp}(g) \geq \min \text{supp}(x''_n)$. So from the above and the almost admissibility we have $\min \text{supp}(x''_n) \leq \min \text{supp}(g) \leq \min \text{supp}(f_{i_0}) \leq \max \text{supp}(f_{i_0}) < \min \text{supp}(f_j) \leq \max \text{supp}(x''_n)$. Therefore $\text{supp}(f_{i_0}) \subseteq \text{supp}(x''_n)$ and we have $i_0 \notin I$.

As a result, the function $G : H \longrightarrow \{1, 2, \ldots, m\} \setminus I$ defined by $n \in H \longmapsto \min\{i : \text{supp}(f_i) \cap \text{supp}(x''_n) \neq \varnothing\}$ is an injection. Therefore $|H| \leq m - |I| \leq d - |I|$ and we have $|I| + |H| \leq d$.

Thus we have

$$
\begin{aligned}
f\Big(\sum_{n\in J} a_n x_n''\Big) &= \Big(\theta\sum_{i=1}^{m} f_i\Big)\Big(\sum_{n\in J} a_n x_n''\Big) = \theta\sum_{i=1}^{m}\Big(\sum_{n\in J} f_i\Big(a_n x_n''\Big)\Big)\\
&= \theta\left(\sum_{i=1}^{m}\Big(\sum_{n\in D} f_i\Big(a_n x_n''\Big)\Big) + \sum_{i=1}^{m}\Big(\sum_{n\in H} f_i\Big(a_n x_n''\Big)\Big)\right)\\
&= \theta\left(\sum_{i\in I} f_i\Big(E_i\sum_{n\in D} a_n x_n''\Big) + \sum_{n\in H}\left(\sum_{i=1}^{m} f_i\right)\Big(a_n x_n''\Big)\right)\\
&\le \theta\left(\sum_{i\in I}\frac{1}{\theta}\left\|E_i\Big(\sum_{n\in D} a_n e_n\Big)\right\|_{\text{Bel}} + \sum_{n\in H}\left(\frac{1}{\theta}f\right)\Big(a_n x_n''\Big)\right)\\
&\le \theta\left(\frac{1}{\theta}\sum_{i\in I}\left\|E_i\Big(\sum_{n\in D} a_n e_n\Big)\right\|_{\text{Bel}} + \frac{1}{\theta}\sum_{n\in H}\left\|a_n x_n''\right\|\right)\\
&= \sum_{i\in I}\left\|E_i\Big(\sum_{n\in D} a_n e_n\Big)\right\|_{\text{Bel}} + \sum_{n\in H}\left\|a_n e_n\right\|_{\text{Bel}} \le \frac{1}{\theta}\left\|\sum_{n\in J} a_n e_n\right\|_{\text{Bel}}
\end{aligned}
$$

as $\{E_i\}_{i\in I}\cup\{\{e_n\}\}_{n\in H}$ forms a sequence of Bel-admissible sets.

Therefore since

$$
\begin{aligned}
f\left(\sum_{n=0}^{N} a_n x_n\right) &= f\left(\sum_{n=0}^{N} a_n x_n' + \sum_{n=0}^{N} a_n x_n''\right)\\
&= f\left(\sum_{n=0}^{N} a_n x_n'\right) + f\left(\sum_{n=0}^{N} a_n x_n''\right) \le \frac{2}{\theta}\left(\sum_{n=0}^{N}|a_n|^p\right)^{1/p}
\end{aligned}
$$

for every $f\in F^s(\varphi)$ and for each $\varphi\in K$, we conclude that

$$
\|x\| = \sup_{\varphi\in K}\varphi\Big(\sum_{n=0}^{N} a_n x_n\Big) \le \frac{2}{\theta}\Big(\sum_{n=0}^{N}|a_n|^p\Big)^{1/p}.
$$

□

Combining proposition 10, lemma 13, and lemma 16, we understand the structure of $T_1(d,\theta)$ through the following theorem.

Theorem 17. If $d\theta = d^{1/p}$, then we have

$$T_1\big(d,\theta\big) \cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n+1}\right)_p .$$

CHAPTER 5: BANACH SPACE OF HIGHER HIERARCHY

There are two directions of extending the tree T_1 along the lines of Ramsey spaces constructed by Dobrinen and Todorcevic [6] and Trujillo in his dissertation [17]. One of the examples increases the height [6], and the other increases the dimension of each branch by taking tensor products [17]. Both extensions are of interests building Banach spaces with Tsirelson-type norms by taking admissible sets from the finite approximations of the trees. Here we give the definitions and look at their isomorphism types.

With the base case of T_1, we may define recursively the higher levels of trees that the maximal nodes attain. Recall that $T_1 = \mathrm{cl}\{(i,j) : i,j \in \mathbb{N} \text{ and } 1 \le j \le i\}$. Suppose for some $n \in \mathbb{N} \setminus \{0\}$, T_n has been defined. For each $i \in \mathbb{N}$, set

$$T_{n+1}(i) = \left\{ (\,), (i), (i)^\frown s : s \in T_n(j), \text{ where } \frac{i(i+1)}{2} \le j < \frac{(i+1)(i+2)}{2} \right\},$$

and $T_{n+1} = \bigcup_{i \in \mathbb{N}} T_{n+1}(i)$.

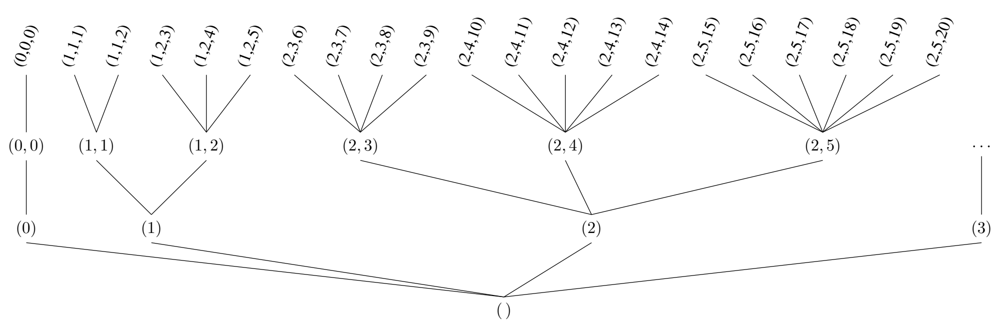

Figure 3: An Initial structure of T_2.

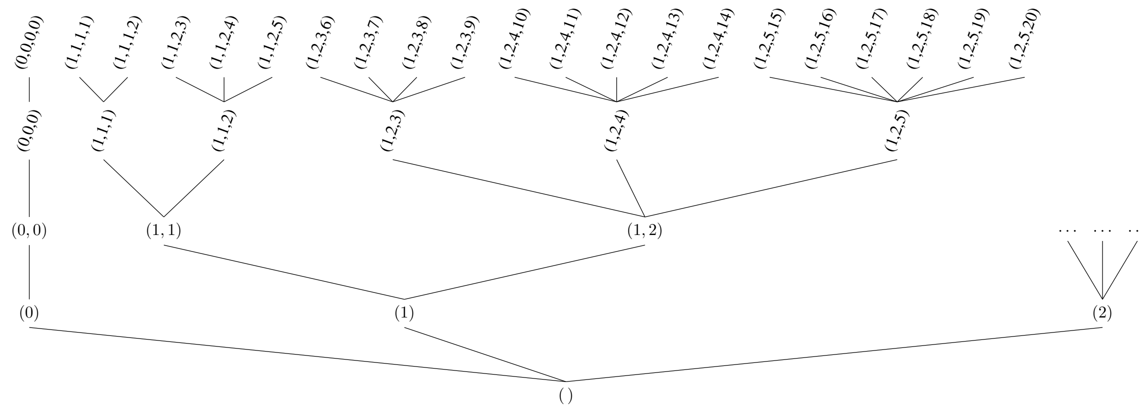

Figure 4: An Initial structure of T_3.

We construct a Banach space on $[T_n]$ (the set of terminal nodes of the tree) similarly for each $n \in \mathbb{N}\backslash\{0,1\}$. For $d \in \mathbb{N}$ with $d \geq 2$ and for $m \leq d$, we take a collection of admissible sets $\{E_i\}_{i=1}^{m} \subseteq \mathcal{AR}(T_n)$ such that $\max E_j < \min E_{j+1}$ for each $j = 1, 2, \ldots, m-1$, where $\mathcal{AR}(T_n)$ is the collection of the terminal nodes of subtrees isomorphic to T_n and $<$ is the lexicographic order on $[T_n]$. Set $c_{00}([T_n]) = \Big\{x : [T_n] \longrightarrow \mathbb{R} \;\Big|\; \operatorname{supp}(x) \text{ is finite}\Big\}$. Take the canonical basis $\{e_t\}_{t\in[T_n]}$ such that $x = \sum_{t\in[T_n]} x_t e_t$, $x_t \in \mathbb{R}$, for every $x \in c_{00}([T_n])$. For $0 < \theta < 1$ with $d\theta > 1$, define a sequence of norms recursively by

$$\|x\|_0 = \sup_{t\in[T_n]} |x_t| = \max_{t\in[T_n]} |x_t|,$$

and for each $j \in \mathbb{N}$,

$$\|x\|_{j+1} = \max\left\{\|x\|_j, \theta \max\left\{\sum_{i=1}^{m} \big\|E_i x\big\|_j : 1 \leq m \leq d, \{E_i\}_{i=1}^{m} \text{ is admissible}\right\}\right\}.$$

For each $x \in c_{00}([T_n])$, the sequence $\big(\|x\|_j\big)_{j\in\mathbb{N}}$ is increasing and bounded above by the $\ell_1([T_n])$-norm of x. Therefore we may set $\|x\|_{T_n(d,\theta)} = \sup_{j\in\mathbb{N}} \|x\|_j$. Completing $c_{00}(T_n)$ with respect to this norm, we get a Banach space $\big(T_n(d,\theta), \|\cdot\|_{T_n}\big)$

On the other hand we may also give

$$T_1 \otimes T_1 = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \left\{\big((n,n),(i,j)\big) : \frac{n(n+1)}{2} \le i,j < \frac{(n+1)(n+2)}{2}\right\}\right),$$

or in general for each $m \in \mathbb{N}$ with $m \geq 2$,

$$\otimes_m T_1 = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \left\{\big((n,\dots,n),(i_0,\dots,i_{m-1})\big) \in \mathbb{N}^m \times \mathbb{N}^m : \right\}\right), \quad \text{where } \frac{n(n+1)}{2} \le i_0,\dots,i_{m-1} < \frac{(n+1)(n+2)}{2}.$$

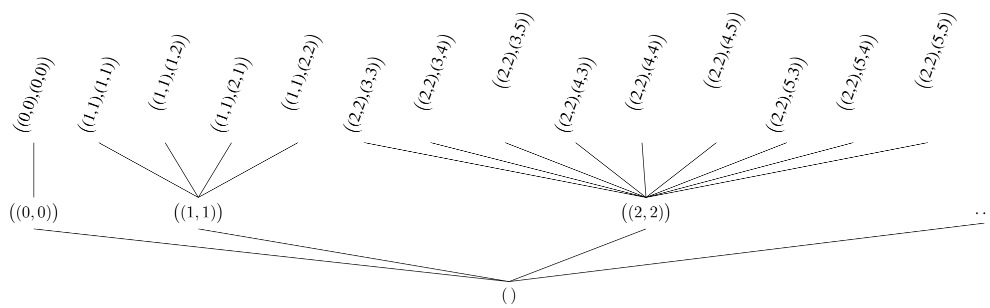

Figure 5: An Initial structure of $T_1 \otimes T_1$.

Note that we could still define similarly the recursive definition of the norm on $[\otimes_m T_1]$ for each $m \in \mathbb{N} \setminus \{0,1\}$. The completions also induce Banach spaces on the top of these trees. We will examine the Banach spaces on these two different types of extensions of the tree T_1.

Let T be a fixed tree of one of these types. We denote for each $k \in \mathbb{N}$ that

$$T[k] = \{t \in [T] : \alpha_1(t) = k\},$$
$$\text{and } T^{(k)} = \overline{\mathrm{span}}\{e_t : t \in T[k]\}.$$

For a fixed $k \in \mathbb{N}$, we have $\max_{t\in T[k]} |x_t| \leq \|x\|_j$ for each $j \in \mathbb{N}$ for every $x \in T^{(k)}$. For the upper bound of the norm, in the inductive step, the admissible sets $E_i \cap T[k]$ would need to be singletons for $i = 2, 3, \ldots, m$ by similar arguments as in Lemma 9 in order to achieve maximal norm on $\|\cdot\|_{j+1}$. So analogous arguments still give us that $\|x\|_j \leq \frac{d-1}{1-\theta} \cdot \max_{t\in T[k]} |x_t|$. Due to the definition of α_1, there are more terminal nodes on $T[k]$ than $T_1[k]$ for each $k \in \mathbb{N}$. For instances $\Big|T_2[3]\Big| = 65$, $\Big|\big(\otimes_2 T\big)[3]\Big| = 16$. In each extension, there is an increasing function $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $g(0) = 1$ and $g(k)$ is the number of terminal nodes on $T[k]$ for each $k \geq 1$. Therefore combining the lower bound and the upper bound, we obtain

$$T^{(k)} \cong \ell_\infty^{g(k)}.$$

Considering for each $x \in T(d,\theta)$ that $x = \sum_{k\in\mathbb{N}} a_k x_k$ as block sequences where $x_k \in T^{(k)}$ for each $k \in \mathbb{N}$, the norm of the linear sum of the block sequences is greater than or equal to the norm of the sum of the canonical surjections from each block onto the linear span of the node with the maximal coefficient in that block. Also the collection of the nodes each of which is from a specific $T[k]$ is Bel-admissible, and so $\|x\| = \|x\|_{\text{Bel}}$ if x is with non-zero coefficients on a single node from each $T[k]$. Since $\|x\|_{\text{Bel}}$ is bounded below by the p-norm of those coefficients, we have a lower bound for the block sequence:

$$\|x\| \geq \frac{\theta^2(1-\theta)}{d-1} \left(\sum_{k\in\mathbb{N}} \|x_n\|^p \right)^{1/p}.$$

For the upper bound, we will again make use of the set K of linear functionals such that each functional is the θ multiple of some of others in K that the support sets are almost admissible . Going through similar proofs, we see that the norm can still be represented by these functionals, namely $\|x\| = \sup_{f\in K} f(x)$. For each $\varphi \in K$, we can still utilise the analysis on φ to show that every functional f in the analysis satisfies that $f\Big(\sum_{n=0}^{N} a_n x_n\Big) \leq \frac{2}{\theta}\Big(\sum_{n=0}^{N} |a_n|^p\Big)^{1/p}$.

Therefore for a tree T of one of these types of extensions, we may induce a Banach space $T(d,\theta)$, and there is an increasing function $g : \mathbb{N} \longrightarrow \mathbb{N}$ with such that

$$T\big(d,\theta\big) \cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p .$$

We will now apply Pelczynsky's decomposition method to obtain the result that each of the Banach spaces constructed on the extensions of the tree T_1 is in fact isomorphic to the Banach space of $T_1(d,\theta)$.

Theorem 18 (Pelczysky decomposition theorem)**.** [11] Let X, Y be Banach spaces such that X can be imbedded complementedly into Y and Y can be imbedded complementedly into X. Suppose $X \cong \Big(\sum_{n\in\mathbb{N}} \oplus X\Big)_p$. Then $X \cong Y$.

Proof. Since X is imbedded complementedly into Y, we have $Y \cong X \oplus W \cong X \oplus_p W$ for some complete subspace $W \subseteq Y$. Also since Y is imbedded complementedly into X, we have $X \cong Y \oplus V \cong Y \oplus_p V$ for some complete subspace $V \subseteq X$. Therefore we have

$$\begin{aligned}
Y &\cong X \oplus_p W \\
&\cong \left(\sum_{n\in\mathbb{N}} \oplus X\right)_p \oplus_p W \cong X \oplus_p W \oplus_p \left(\sum_{n\in\mathbb{N}} \oplus X\right)_p \\
&\cong Y \oplus_p \left(\sum_{n\in\mathbb{N}} \oplus X\right)_p \cong Y \oplus_p \left(\sum_{n\in\mathbb{N}} \oplus \Big(V \oplus_p Y\Big)\right)_p \\
&\cong \left(\sum_{n\in\mathbb{N}} \oplus \Big(Y \oplus_p V\Big)\right)_p \cong \left(\sum_{n\in\mathbb{N}} \oplus X\right)_p \\
&\cong X.
\end{aligned}$$

$\square$

Lemma 19. Let X be a Banach space and suppose $Z = \left(\sum_{n\in\mathbb{N}} \oplus X\right)_p$. Then $Z \cong \left(\sum_{n\in\mathbb{N}} \oplus Z\right)_p$.

Proof. Let $\mathbb{N} = \bigcup_{j\in\mathbb{N}} \sigma_j$ be a partition of $\mathbb{N}$ such that σ_j is infinite for each $j \in \mathbb{N}$. Set

$$\begin{array}{rccc} \Phi : & \left(\sum_{n\in\mathbb{N}} \oplus \left(\sum_{m\in\sigma_n} \oplus X\right)_p\right)_p & \longrightarrow & \left(\sum_{k\in\mathbb{N}} \oplus X\right)_p \\ & \sum_{n\in\mathbb{N}} \left(\sum_{m\in\sigma_n} x_m^{(n)}\right) & \longmapsto & \sum_{n\in\mathbb{N}} x_n\,. \end{array}$$

by taking $x_n = \sum_{m\in\sigma_n} x_m^{(n)}$ for each $n \in \mathbb{N}$. Then Φ is a linear bijection. Moreover

$$\begin{aligned} \left\|\Phi\Big(\sum_{n\in\mathbb{N}}\Big(\sum_{m\in\sigma_n} x_m^{(n)}\Big)\Big)\right\| &= \left\|\sum_{n\in\mathbb{N}} x_n\right\| = \Big(\sum_{n\in\mathbb{N}} \|x_n\|^p\Big)^{1/p} \\ &= \Big(\sum_{n\in\mathbb{N}} \Big\|\sum_{m\in\sigma_n} x_m^{(n)}\Big\|^p\Big)^{1/p} \\ &= \Big(\sum_{n\in\mathbb{N}} \Big(\Big(\sum_{m\in\sigma_n} \|x_m^{(n)}\|^p\Big)^{1/p}\Big)^p\Big)^{1/p} \\ &= \Big(\sum_{n\in\mathbb{N}} \Big(\sum_{m\in\sigma_n} \|x_m^{(n)}\|^p\Big)\Big)^{1/p} = \left\|\sum_{n\in\mathbb{N}}\Big(\sum_{m\in\sigma_n} x_m^{(n)}\Big)\right\|. \end{aligned}$$

So Φ provides an isometric bijection between $\left(\sum_{n\in\mathbb{N}} \oplus Z\right)_p$ and Z. $\square$

Lemma 20.

$$T_1(d,\theta) \cong \left(\sum_{n\in\mathbb{N}} \oplus T_1(d,\theta)\right)_p.$$

Proof. We will apply the Pelczynsky decomposition method to show this isomorphism.

First $T_1(d,\theta)$ imbeds complementedly into $\left(\sum_{n\in\mathbb{N}} \oplus T_1(d,\theta)\right)_p$ since

$$\Big(\sum_{n\in\mathbb{N}} \oplus T_1(d,\theta)\Big)_p = T_1(d,\theta) \oplus_p \left(\sum_{n\in\mathbb{N}} \oplus T_1(d,\theta)\right)_p.$$

Secondly to imbed $\Big(\sum\oplus_{n\in\mathbb{N}} T_1(d,\theta)\Big)_p$ into $T_1(d,\theta)$, we note that

$$\left(\sum_{n\in\mathbb{N}}\oplus\, T_1(d,\theta)\right)_p \cong \left(\sum_{m\in\mathbb{N}}\oplus\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p\right)_p \cong \left(\sum_{m\in\mathbb{N}}\oplus\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}(m)\right)_p\right)_p,$$

where the m in the last expression is only to give each copy of ℓ_∞^{n+1} an index $\ell_\infty^{n+1}(m)$ to indicate the number of direct summand of that particular copy is in. We may order linearly the elements $(m,n)\in\mathbb{N}^2$ by the following: $(m_1,n_1)<(m_2,n_2)$ if $m_1+n_1<m_2+n_2$ or if $m_1+n_1=m_2+n_2$ and $m_1<m_2$. Suppose $\varphi:\mathbb{N}^2\longrightarrow\mathbb{N}$ provides this order with $\varphi(0,0)=0$. Then $\ell_\infty^{n+1}(m)$ is imbedded complenemtedly into $\ell_\infty^{\varphi(m,n)+1}$. (This is similar to the diagonalisation argument in proving the bijectivity between $\mathbb{N}^2$ and $\mathbb{N}$. Since

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}}\oplus\, T_1(d,\theta)\right)_p &\cong \left(\sum_{m\in\mathbb{N}}\oplus\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}(m)\right)_p\right)_p\\ &\cong \left(\ell_\infty^1(0)\oplus_p\ell_\infty^2(0)\oplus_p\ell_\infty^3(0)\oplus_p\cdots\right)\oplus_p\left(\ell_\infty^1(1)\oplus_p\ell_\infty^2(1)\oplus_p\ell_\infty^3(1)\oplus_p\cdots\right)\\ &\qquad\oplus_p\left(\ell_\infty^1(2)\oplus_p\ell_\infty^2(1)\oplus_p\ell_\infty^3(1)\oplus_p\cdots\right)\oplus_p\cdots,\end{aligned}$$

we may embed $\ell_\infty^1(0)$ into ℓ_∞^1, $\ell_\infty^2(0)$ into ℓ_∞^2, $\ell_\infty^1(1)$ into ℓ_∞^3, $\ell_\infty^3(0)$ into ℓ_∞^4, $\ell_\infty^2(1)$ into ℓ_∞^5, $\ell_\infty^1(2)$ into ℓ_∞^6, etc..)

Lastly by Lemma 19, we have

$$\Big(\sum_{n\in\mathbb{N}}\oplus\, T_1(d,\theta)\Big)_p \cong \left(\sum_{n\in\mathbb{N}}\oplus\Big(\sum_{m\in\mathbb{N}}\oplus\, T_1(d,\theta)\Big)_p\right)_p.$$

Therefore by the Pelczysky decomposition theorem, we have the result. $\square$

Let $T(d,\theta)$ be a the Banach space constructed on one of the aforementioned extension tree T. Note that we have

$$T_1(d,\theta) \cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n+1}\right)_p,$$
$$\text{and } \ T(d,\theta) \cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p$$

for some increasing function $g : \mathbb{N} \longrightarrow \mathbb{N}$.

Theorem 21. Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be an increasing function with $g(0) \geq 1$. Then

$$\left(\sum_{n\in\mathbb{N}} \oplus \ell_\infty^{g(n)}\right)_p \cong \left(\sum_{n\in\mathbb{N}} \oplus \ell_\infty^{n+1}\right)_p.$$

Proof. Since g is increasing, $\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n+1}\right)_p$ can be embedded complementedly into $\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p$ by embedding ℓ_∞^{n+1} into $\ell_\infty^{g(n)}$ naturally and the remaining parts as the complements for each $n \in \mathbb{N}$, namely

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p &\cong \left(\sum_{n\in\mathbb{N}} \oplus \left(\ell_\infty^{n+1} \oplus_p \ell_\infty^{g(n)-(n+1)}\right)\right)_p \\ &\cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n+1}\right)_p \oplus_p \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)-(n+1)}\right)_p.\end{aligned}$$

Also $\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p$ can be embedded complemented into $\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n}\right)_p$ by combining those dimensions in the range of g together and the others as the complement, specifically

$$\left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{n+1}\right)_p \cong \left(\sum_{n\in\mathbb{N}} \oplus\ \ell_\infty^{g(n)}\right)_p \oplus_p \left(\sum_{n\in\mathbb{N}\setminus \mathrm{rng}(g)} \oplus\ \ \ell_\infty^{n+1}\right)_p.$$

Moreover we have

$$\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p \cong \left(\sum_{m\in\mathbb{N}}\oplus\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p\right)_p$$

by Lemma 20.

Therefore we induce that

$$\left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{g(n)}\right)_p \cong \left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p$$

by the decomposition theorem. □

In conclusion, we see that

$$T(d,\theta) \cong \left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{g(n)}\right)_p \cong \left(\sum_{n\in\mathbb{N}}\oplus\,\ell_\infty^{n+1}\right)_p \cong T_1(d,\theta).$$

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