8-2023

Unilinear Residuated Lattices

Xiao Zhuang

University of Denver

Unilinear Residuated Lattices

Abstract

We characterize all residuated lattices that have height equal to 3 and show that the variety they generate has continuum-many subvarieties. More generally, we study unilinear residuated lattices: their lattice is a union of disjoint incomparable chains, with bounds added. We give the characterization of all unilinear residuated lattices. By presenting the constructions and axiomatizations for different classes of unilinear residuated lattices, we conclude that the study of unilinear residuated lattices can be reduced to the study of the $\top$-unital ones. Using the classification of unilinear residuated lattices, the idempotent unilinear residuated lattices are studied and amalgamation property and strong amalgamation properties are proved or disproved, depending on if there are extra constants in the language. We give two general constructions of $\top$-unital unilinear residuated lattices, provide an axiomatization and a proof-theoretic calculus for the variety they generate, and prove the finite embeddability property for various subvarieties. Finally, we study the involutive unilinear residuated lattices and give the characterization of a class of commutative 1-involutive compact unilinear residuated lattices. We present some open problems and future work at the end.

Document Type

Dissertation

Degree Name

Ph.D.

First Advisor

Nikolaos Galatos

Second Advisor

Michael Kinyon

Third Advisor

Andrew Linshaw

Keywords

Amalgamation, Axiomatization, Classification, Involutive, Subvarieties, Unilinear residuated lattices

Subject Categories

Algebra | Mathematics | Other Mathematics | Physical Sciences and Mathematics

Unilinear residuated lattices

A Dissertation

Presented to

the Faculty of the College of Natural Sciences and Mathematics

University of Denver

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Xiao Zhuang

August 2023

Advisor: Nikolaos Galatos

Author: Xiao Zhuang
Title: Unilinear residuated lattices
Advisor: Nikolaos Galatos
Degree Date: August 2023

ABSTRACT

We characterize all residuated lattices that have height equal to 3 and show that the variety they generate has continuum-many subvarieties. More generally, we study unilinear residuated lattices: their lattice is a union of disjoint incomparable chains, with bounds added. We give the characterization of all unilinear residuated lattices. By presenting the constructions and axiomatizations for different classes of unilinear residuated lattices, we conclude that the study of unilinear residuated lattices can be reduced to the study of the $\top$-unital ones. Using the classification of unilinear residuated lattices, the idempotent unilinear residuated lattices are studied and amalgamation property and strong amalgamation properties are proved or disproved, depending on if there are extra constants in the language. We give two general constructions of $\top$-unital unilinear residuated lattices, provide an axiomatization and a proof-theoretic calculus for the variety they generate, and prove the finite embeddability property for various subvarieties. Finally, we study the involutive unilinear residuated lattices and give the characterization of a class of commutative 1-involutive compact unilinear residuated lattices. We present some open problems and future work at the end.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my profound gratitude to my advisor, Nikolaos Galatos, whose patience, guidance, and insightful wisdom have been invaluable. His steadfast mentorship and faith in my abilities were instrumental in the completion of this project and dissertation, and for that, I am eternally grateful.

I extend my heartfelt thanks to the members of my dissertation committee: Michael Kinyon, Andrew Linshaw, Mark Siemens, and Petr Vojtechovsky. Their insightful comments, constructive criticism, and unwavering support provided much-needed encouragement throughout this endeavor.

My sincere appreciation goes to the entire Department of Mathematics of University of Denver, with a special mention to the graduate students, including Di Qin, Cheng-chih Ko, Vladimir Kovalchuk, Jesse Parrish, Evans Hedges, Isis Gallardo and Alex Stevens. The intellectual camaraderie and collaborative spirit that exists within this department have played a crucial role in my academic journey.

I owe a deep sense of gratitude to my parents, whose unwavering love, support and belief in my potential have been my backbone throughout this program. Their encouragement in times of doubt and their celebration in times of success have been instrumental in reaching this milestone.

Last but not least, I dedicate this dissertation to God, the true source of all wisdom and knowledge. This academic journey, punctuated with challenges and triumphs, has stood as a testament to His steadfast love and grace. It is in Him that I found the strength and guidance to embark on and complete this journey.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1: Introduction

Residuated lattices are prominent ordered algebraic structures that generalize various well-known structures such as lattice-ordered groups, the ideals of a unital ring, and relation algebras, among others. They also form algebraic semantics for various substructural logics, such as classical, intuitionistic, relevance, linear and many-valued logic; as a result further examples of residuated lattices include Boolean, Heyting, MV and BL-algebras. We refer the readers to [12] for an introduction to residuated lattices and substructural logics.

A substantial amount of work has focused on the study of totally-ordered residuated lattices (residuated chains) and the variety they generate (semilinear residuated lattices). Here, we start our study by exploring the other extreme: residuated lattices whose elements form an antichain, with two bounds added to obtain a lattice, which are called residuated lattices on M_X. Then we generalize the class and study the unilinear residuated lattices (URL). URL allows us to combine the study of residuated chains (e.g. see [5]) and the study of algebras of larger "width", but in a controlled manner. Also, it provides a new context to study the direct product $\mathbf{H} \times \mathbf{K}$ of a residuated chain $\mathbf{H}$ and a cancellative monoid $\mathbf{K}$, which is not a lattice.

The variety of residuated lattices is vast and it's usually difficult to do general exploration. So it's helpful to study the subclasses of residuated lattices. URL provides concrete examples for residuated lattices and allows the combination of other substructures into it. For example, generally the union of two residuated lattices doesn't give a universe of another residuated lattice, but during the study of URL, we find it works with a $\top$-unital residuated lattice and an integral residuated lattice as ingredients. This construction is like the antithesis of the ordinal sum of integral residuated lattices.

The whole study involves a lot of aspects of unilinear residuated lattices and some of them are of independent interest. As a guide to the reader, we outline the content as follows.

In Chapter 3, we show that all residuated lattices of height 3 are precisely the ones consisting of two parts: a zero-cancellative monoid and a semigroup of at most three elements, and we specify the process for putting these two parts together.

In Chapter 4 we provide an axiomatization for the positive universal class of residuated lattices of height up to three and of the variety M it generates. More generally, we consider the class URL of *unilinear* residuated lattices: they are based on disjoint unions of incomparable chains with two additional bounds. We axiomatize the positive universal class URL and the variety SRL of *semiunilinear* residuated lattices it generates. Moreover, we show that the finitely subdirectly irreducible members of SRL are precisely the unilinear ones. In the particular case of M, the simplicity of height-3 lattices directly gives the semisimplicity of M, but we further show that the variety bM, containing algebras on the expanded language that includes the bounds, is a discriminator variety. We conclude the chapter with a discussion of the proof-theory of SRL. In particular we present a hypersequent calculus for SRL that enjoys the cut-elimination property, thus resulting in an analytic system for SRL.

In Chapter 5 we show that there are continuum-many subvarieties of M. These are actually subvarieties of $\mathsf{CM_G}$, the variety generated by height-3 unilinear residuated lattices where the middle layer is an abelian group. In fact we show that subvarieties of $\mathsf{CM_G}$ correspond to $\mathsf{ISP_U}$-classes of abelian groups and we further present a completely combinatorial characterization of the subvariety lattice of $\mathsf{CM_G}$ (without any reference to group theory). We extend this characterization a little further, by allowing the middle layer of the residuated lattice to also include some semigroup elements, coming from the characterization in Chapter 3.

Chapter 6 contains a proof of the finite embeddability property (FEP) for the variety $\mathsf{CM_G}$, thus contrasting the complexity coming from the continuum-many subvarieties with the fact that the universal theory of $\mathsf{CM_G}$ is decidable. We also establish the FEP for more subvarieties of SRL, which do not have the height-3 restriction. To be more precise, the FEP holds for every subvariety of SRL that is axiomatized by equations in the language of multiplication, join and 1, and satisfies any weak commutativity axiom and any knotted rule; we establish this result by using the method of residuated frames.

In Chapter 7, we focus our attention on unilinear residuated lattices $\mathbf{R}$ where $M := R \setminus \{\bot, \top\}$ is a submonoid and the bounds are absorbing with respect to the elements of M; we call such unilinear residuated lattices *compact*. We provide two constructions of compact residuated lattices, with the first one coming from a finite cyclic monoid. In the second one M is the Cartesian product of a residuated chain and a cancellative monoid, relative to a 2-cocycle; thus it is a generalization of the semidirect product of monoids.

Chapter 8 classifies the (bounded) URLs into various classes based on the structure of the non-linear members of each class. These classes, which together cover all the URLs, will be: $\mathsf{B}4$, $\top\mathsf{unital}$, B, TW and LW. Furthermore, by providing axiomatizations and constructions of these classes, we show how the algebras in the three latter classes can be constructed from algebras in $\top\mathsf{unital}$, thus reducing the study of URLs to the study of the $\top$-unital ones. At the end of this chapter, we present an application of the the characterization of URL to a class called URL of type $h4.1$.

In Chapter 9 we focus on the class of idempotent URLs. We apply the classification in Chapter 8 to this class. Moreover, since our characterization has no restriction on the residuated chains, we apply the result in [5] about $\star$-involutive idempotent residuated chains to study the amalgamation property (AP) and strong amalgamation property (sAP) of each class. During the study, we realize the presence of constants $\bot$ and $\top$ in the language matters since AP and sAP are sensitive to the structures of subalgebras. We conclude this chapter by showing any join of two of the varieties generated by the classes above fails AP.

As we found in Chapter 8, the study of URLs can be reduced to the study of $\top$-unital URLs. So in Chapter 10 we provide the classification of the commutative 1-involutive URLs and then focus on the $\top$-unital ones. In the presence of involutivity, these URLs are not just $\top$-unital, they are compact. If further the negation constant is 1, then the disjoint chains form a group with the identity being the chain of 1. We give the characterization of a class of commutative 1-involutive compact URL, whose chain of 1 is an odd Sugihara chain and each chain is of finite order in the group. It turns out such URLs are precisely the subalgebras of conucleus images of the direct product constructed by the chain of 1 and the group, as mentioned in Chapter 7, up to isomorphism. Using this result, we characterize all finite commutative 1-involutive $\top$-unital URLs.

Finally, in Chapter 11 we list some open problems and future work.

Chapter 2: Preliminaries

The readers can find all the terminologies of this chapter in any textbook of Universal Algebra and paper about residuated lattices and residuated frames. Here we use [12] and [9] for reference.

2.1 Concepts from Universal Algebra

A *language* (or *signature*) $\mathcal{L}$ is the disjoint union of a set $\mathcal{L}^o$ of *operation symbols* and $\mathcal{L}^r$ of *relation symbols*, each with a fixed non-negative *arity*. Operation symbols of arity 0 are called *constant symbols*.

For a set X, the $\mathcal{L}$-*terms* over X is denoted by $Tm_{\mathcal{L}}(X)$ and defined as the smallest set T such that $X \subseteq T$, and if $f \in \mathcal{L}^o$ has arity n and $t_1, \ldots, t_n \in T$ then $f(t_1, \ldots, t_n) \in T$. Note the terms are simply strings of symbols. We fix a countable set of symbols (disjoint from $\mathcal{L}$) called *variables*, and we denote the set of all $\mathcal{L}$-terms over this set of symbols simply as $Tm_{\mathcal{L}}$.

An $\mathcal{L}$-structure $\mathbf{A} = (A, (l^{\mathbf{A}})_{l\in\mathcal{L}})$ is a nonempty set A, called the *universe*, together with an $\mathcal{L}$-tuple of operations and relations defined on A, where $l^{\mathbf{A}}$ has the same arity as l. An operation of arity n on a set A is simply a map from A^n to A and that a relation of arity n on a set A is a subset of A^n. An *algebra* is a structure without any relations. Two algebras $\mathbf{A}$ and $\mathbf{B}$ are of the *same type* when both of them are $\mathcal{L}$-structures for some language $\mathcal{L}$. An algebra is *finite* if the universe is a finite set, and is *trivial* if the universe is a singleton set.

A *sublanguage* $\mathcal{K}$ of a language $\mathcal{L}$ is simply a subset of $\mathcal{L}$, where every symbol retains its arity. The $\mathcal{K}$-*reduct* of an $\mathcal{L}$-structure $\mathbf{A} = (A, (l^{\mathbf{A}}))_{l\in\mathcal{L}})$ is the $\mathcal{K}$-structure $(A, (l^{\mathbf{A}}))_{l\in\mathcal{K}})$ on the same universe, where $\mathcal{K}$ is a sublanguage of $\mathcal{L}$. In this case $\mathbf{A}$ is called an *expansion* of $(A, (l^{\mathbf{A}}))_{l\in\mathcal{K}})$. The $\mathcal{K}$-reduct of $\mathbf{A}$ is also the $\mathcal{M}$-*free reduct* of $\mathbf{A}$, where $\mathcal{M}$ is the complement of $\mathcal{K}$ of $\mathcal{L}$. If $\mathcal{K}$ is clear from the context, we simply refer to the $\mathcal{K}$-reduct as the reduct of $\mathbf{A}$.

A structure $\mathbf{Q} = (Q, \leq)$ is a preordered set if $\leq$ is a binary relation on Q such that for all $x, y, z \in Q$, the following hold:

$$x \leq x \qquad \text{(reflexivity)}$$

$$x \leq y \text{ and } y \leq z \implies x \leq z \qquad \text{(transitivity)}$$

A structure $\mathbf{P} = (P, \leq)$ is a partially ordered set, or a *poset*, if it is a preordered set and for all $x, y, z \in P$

$$x \leq y \text{ and } y \leq x \implies x = y \qquad \text{(antisymmetry)}$$

An algebra $\mathbf{A} = (A, \wedge, \vee)$ is a *lattice*, if the binary operations, called *meet* and *join* respectively, are *commutative*, *associative* and mutually *absorptive*, i.e., for all $x, y, z \in A$ the following hold:

$$x \wedge y = y \wedge x \qquad \text{(commutativity of meet)}$$

$$x \vee y = y \vee x \qquad \text{(commutativity of join)}$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad \text{(associativity of meet)}$$

$$x \vee (y \vee z) = (x \vee y) \vee z \qquad \text{(associativity of join)}$$

$$x \vee (x \wedge y) = x \qquad \text{(absorption of meet by join)}$$

$$x \wedge (x \vee y) = x \qquad \text{(absorption of join by meet)}$$

An *equivalence relation* θ on a set A is a reflexive, symmetric and transitive binary relation on A. For symmetry, we mean for all $x, y \in A$

$$x\theta y \implies y\theta x. \qquad \text{(symmetry)}$$

The set A is partitioned into *equivalence classes* $[a]_\theta = \{x \in A : a\theta x\}$. If for every n-ary basic operation f, θ satisfies that

$$a_1\theta b_1, \ldots, a_n\theta b_n \implies f(a_1, \ldots, a_n)\theta f(b_1, \ldots, b_n),$$

then we say that the operations are *compatible* with the equivalence classes. An equivalence relation with this property is called a *congruence*, and the equivalence classes are called *congruence classes*.

An *assignment* or *valuation* into an algebra $\mathbf{A}$ is a function h from the set of variables to A. Any such function extends uniquely to a function (also denoted by h) from $Tm_{\mathcal{L}}$ to A by defining $h(f(t_1, \ldots, t_n)) = f^{\mathbf{A}}(h(t_1), \ldots, h(t_n))$ for each operation symbol $f \in \mathcal{L}^o$ with arity n.

Let Q be a set. A subset $\vdash$ of $\mathcal{P}(Q) \times Q$ is called *consequence relation* over Q, if for every subset $X \cup Y \cup \{x, z\}$ of Q

- if $x \in X$, then $X \vdash x$, and
- if $X \vdash Y$ and $Y \vdash z$, then $X \vdash z$,

where $X \vdash x$ stands for $(X, x) \in \vdash$ and $X \vdash Y$ stands for $X \vdash y$ for all $y \in Y$.

2.2 Residuated lattices

A *residuated lattice* is an algebra $(R, \wedge, \vee, \cdot, \backslash, /, 1)$ where

- $(R, \wedge, \vee)$ is a lattice,

- $(R, \cdot, 1)$ is a monoid, and
- $xy \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$ for all $x, y, z \in R$.

The last condition above is called *residuation*. Given posets $\mathbf{P}$ and $\mathbf{Q}$, a map $f : \mathbf{P} \to \mathbf{Q}$ is said to be *residuated* if there exists a map $f^* : \mathbf{Q} \to \mathbf{P}$ such that

$$f(x) \leq y \text{ iff } x \leq f^*(y)$$

for all $x \in P$ and $y \in Q$.

The following result is folklore in the theory of residuated maps.

Lemma 2.1. A function g from a poset $\mathbf{P}$ to a poset $\mathbf{Q}$ is residuated if and only if the set $\{x \in P : g(x) \leq y\}$ has a maximum for all $y \in Q$ and g is order-preserving.

Proof. Let $S_y = \{x \in P : g(x) \leq y\}$ and we assume that g is residuated with residual g^*. Note that $g^*(y) \leq g^*(y)$ yields $g(g^*(y)) \leq y$ so $g^*(y) \in S_y$. Also, for all $x \in S_y$, $g(x) \leq y$ hence $x \leq g^*(y)$. Therefore, $g^*(y) = \max S_y$.

If $x_1 \leq x_2$, then since $g(x_2) \leq g(x_2)$ yields $x_2 \leq g^*(g(x_2))$, we get $x_1 \leq g^*(g(x_2))$; hence $g(x_1) \leq g(x_2)$. Therefore, g is order-preserving.

Now suppose S_y has a maximum for all $y \in Q$ and g preserves the order. We define $g^* : Q \to P$ by $g^*(y) = \max S_y$; clearly g^* is order-preserving. If $g(x) \leq y$ for some $x \in P$, $y \in Q$, then $x \in S_y$ and $x \leq g^*(y)$ by definition. Conversely, if $x \leq g^*(y)$, then $g(x) \leq g(g^*(y))$ since g is order-preserving. Moreover, $g^*(y) \in S_y$ so $g(g^*(y)) \leq y$; thus $g(x) \leq y$. □

We mention that if the assumption that $\{x \in P : g(x) \leq y\}$ has a maximum is replaced by the demand that it has a join, then the order-preservation of g is not enough to give residuation.

Note that a lattice-ordered monoid supports a residuated lattice iff left and right multiplication are residuated. So Lemma 2.1 yields the following fact.

Corollary 2.2. A lattice-ordered monoid $\mathbf{R}$ is a reduct of a residuated lattice iff multiplication is order-preserving and for all $x, z \in R$, the sets $\{y \in R : xy \leq z\}$ and $\{y \in R : yx \leq z\}$ have maximum elements. In such a case the expansion to a residuated lattice is unique by $x\backslash z = \max\{y \in R : xy \leq z\}$ and $z/x = \max\{y \in R : yx \leq z\}$.

Corollary 2.3 ([12])**.** A complete lattice-ordered monoid $\mathbf{R}$ is a reduct of a residuated lattice iff multiplication distributes over arbitrary joins.

In particular, multiplication distributes over the empty join, if it exists; so if there is a bottom element $\bot$, then $x \cdot \bot = \bot = \bot \cdot x$, for all x. For convenience, we set $x \backslash\!\backslash z := \{y \in R : xy \leq z\}$ and $z /\!/ x := \{y \in R : yx \leq z\}$ for $x, z \in R$.

Remark 2.4. Let $\mathbf{P} = (P, \wedge, \vee, \cdot, \bot, \top)$ be a bounded lattice-ordered semigroup such that $\bot x = \bot$ for all $x \in P$. Then we have $\bot \backslash\!\backslash x = P$, so $\bot\backslash x = \max \bot \backslash\!\backslash x = \top$ for all $x \in P$. Also, since $x \backslash\!\backslash \top = P$, $x\backslash \top = \top$ for all $x \in P$. Similarly, $x/\bot = \top$ and $\top/x = \top$ for all $x \in P$.

A residuated lattice with bounds $\bot$ and $\top$ is called *rigorously compact* if $\top x = x\top = \top$ for all $x \neq \bot$. In this case we also have that $xy = \bot \Rightarrow x = \bot$ or $y = \bot$, since otherwise we get $x \neq \bot \neq y$, so $\bot = \bot\top = xy\top = x\top = \top$, a contradiction. Note that in rigorously compact residuated lattices we have $\bot\backslash x = x/\bot = \top = x\backslash\top = \top/x$, $\top\backslash y = y/\top = \bot$, $z\backslash\bot = \bot = \bot/z$ for all $x \in R$, $y \neq \top$, $z \neq \bot$.

Let $\vdash$ be a consequence relation on the set of $\mathcal{L}$-formulas, for some language $\mathcal{L}$, then a *matrix model* of $\vdash$ is a pair $(\mathbf{A}, F)$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a subset of A such that for every assignment f into $\mathbf{A}$, and for every set $\Phi \cup \{\psi\}$ of $\mathcal{L}$-formulas, such that $\Phi \vdash \psi$, if $f[\Phi] \in F$ then $f(\psi) \in F$. In this case F called a $\vdash$-deductive filter of $\mathbf{A}$, or a deductive filter of $\mathbf{A}$ with respect to $\vdash$.

Let $\mathbf{A}$ be a residuated lattice. For $a, x \in A$, we define the *left conjugate* $\lambda_a(x) = a\backslash xa \wedge 1$ and the *right conjugate* $\rho_a(x) = ax/a \wedge 1$ of x with respect to a. An *iterated conjugate* of x is a composition $\gamma_{a_1}(\gamma_{a_2}(\dots\gamma_{a_n}(x)\dots))$, where n is a positive integer, $a_1, a_2, \dots, a_n \in A$ and $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$, for all $i \in \{1, 2, \dots, n\}$. We denote the set of all iterated conjugates of elements of $X \subset A$ by $\Gamma(X)$. In analogy with groups, a subset X of A is called *normal* if for all $x \in X$ and $a \in A$, $\lambda_a(x), \rho_a(x) \in X$.

We use $[x, y]$ to denote the closed interval $\{u \in A : x \leq u \leq y\}$. As for posets, we call S *convex* if $[x, y] \subseteq S$ for all $x, y \in S$. Note that for a sublattice $\mathbf{S}$ of $\mathbf{A}$ the property of being convex is equivalent to $\kappa_u(x, y) \in S$ for all $u \in A$ and $x, y \in S$, where $\kappa_u(x, y) = (u \wedge x) \vee y$. Thus a convex normal subalgebra is precisely a subalgebra of $\mathbf{A}$ that is closed under the terms λ, ρ and κ.

For $a, b \in A$, we denote $a \leftrightarrow b = a\backslash b \wedge b\backslash a \wedge 1$ and $a \leftrightarrow' b = b/a \wedge a/b \wedge 1$; clearly $a \leftrightarrow 1 = a\backslash 1 \wedge a \wedge 1$. Moreover, for every subset X of A, we define the sets

$$X \wedge 1 = \{x \wedge 1 : x \in X\}$$

$$\Delta(X) = \{x \leftrightarrow 1 : x \in X\}$$

$$\Pi(X) = \{x_1 x_2 \cdots x_n : n \geq 1, x_i \in X\} \cup \{1\}$$

$$\Xi(X) = \{a \in A : x \leq a \leq x\backslash 1, \text{ for some } x \in X\}$$

$$\Xi^-(X) = \{a \in A : x \leq a \leq 1, \text{ for some } x \in X\}$$

Note that the negative part $A^- = \{a \in A : a \leq 1\}$ of A is closed under multiplication and it contains 1, so it's a submonoid of $\mathbf{A}$.

Theorem 2.5 ([12])**.** For every residuated lattice $\mathbf{A}$, the following properties hold.

1. If $\mathbf{S}$ is a convex normal subalgebra of $\mathbf{A}$, $\mathbf{M}$ convex normal in $\mathbf{A}$ submonoid of A^-, θ a congruence on $\mathbf{A}$ and F a deductive filter of $\mathbf{A}$, then

(a) $M_s(S) = S^-$, $M_c(\theta) = [1]_\theta^-$ and $M_f(F) = F^-$ are convex, normal in $\mathbf{A}$ submonoids of $\mathbf{A}^-$,

(b) $S_m(M) = \Xi(M)$, $S_c(\theta) = [1]_\theta$ and $S_f(F) = \Xi(F^-)$ are convex normal subalgebras of $\mathbf{A}$,

(c) $F_s(S) = \uparrow S, F_m(M) = \uparrow M$, and $F_c(\theta) = \uparrow [1]_\theta$ are deductive filters of $\mathbf{A}$,

(d) $\Theta_s(S) = \{(a, b) : a \leftrightarrow b \in S\}$, $\Theta_m(M) = \{(a, b) : a \leftrightarrow b \in M\}$ and $\Theta_f(F) = \{(a, b) : a \leftrightarrow b \in F\} = \{(a, b) : a\backslash b, b\backslash a \in F\}$ are congruences on $\mathbf{A}$.

2. (a) The convex, normal subalgebras of $\mathbf{A}$, the convex, normal in $\mathbf{A}$ submonoids of A^- and deductive filters of $\mathbf{A}$ form lattices, denoted by $\mathbf{CNS}(\mathbf{A})$, $\mathbf{CNM}(\mathbf{A})$ and $\mathbf{Fil}(\mathbf{A})$, respectively.

(b) All the above lattices are isomorphic to the congruence lattice $\mathbf{Con}(\mathbf{A})$ of $\mathbf{A}$ via the appropriate pairs of maps defined above.

(c) The composition (whenever defined) of any two of the above maps gives the corresponding map; e.g., $M_s(S_c(\theta)) = M_c(\theta)$.

3. If X is a subset of A^- and Y is a subset of A, then

(a) the convex, normal in $\mathbf{A}$ submonoid $M(X)$ of A^- generated by X is equal to $\Xi^-\Pi\Gamma(X)$;

(b) the convex, normal subalgebra $S(Y)$ of $\mathbf{A}$ generated by Y is equal to $\Xi\Pi\Gamma\Delta(Y)$;

(c) the deductive filter $F(Y)$ of $\mathbf{A}$ generated by $Y \subseteq A$ is equal to $\uparrow\Pi\Gamma(Y) = \uparrow\Pi\Gamma(Y \wedge 1)$;

(d) the congruence $\Theta(P)$ on $\mathbf{A}$ generated by a set of pairs $P \subseteq A^2$ is equal to $\Theta_m(M(P'))$, where $P' = \{a \leftrightarrow b : (a, b) \in P\}$.

2.3 Finite embeddability property and decidability

A class $\mathcal{K}$ of similar algebras is said to have the *finite embeddability property* (FEP) if for every algebra $\mathbf{A} \in \mathcal{K}$ and a finite subset B of A, there exists a finite algebra $\mathbf{C} \in \mathcal{K}$ such that the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ induced by B embeds in $\mathbf{C}$. Note that a class satisfying the finite embeddability property is generated by its finite members. Another consequence of FEP is that every universal sentence that fails in the class also fails in a finite member of the class. To see this, let $\mathcal{K}$ be a class satisfying FEP and φ be a universal sentence and $\mathbf{A} \in \mathcal{K}$ an algebra falsifying φ under a valuation v. Let P be the set of subterms of φ. Clearly the image $v[P] \subseteq A$ is finite. Moreover, it gives rise to a partial algebra of $\mathbf{A}$ with $v(s) * v(t)$ defined, as $v(s * t)$, if $s * t$ is a subterm of φ, where $*$ ranges over the operations occurring in φ. By FEP, $v[P]$ can be embedded in a finite algebra $\mathbf{D} \in \mathcal{K}$ and it is easy to see that the valuation defined by sending each variable x occurring in φ to the image of $v(x)$ in D (and arbitrary for other variables) falsifies φ in $\mathbf{D}$. Thus, if $\mathcal{K}$ is finitely axiomatizable, then its universal theories are decidable.

2.4 Amalgamation property

Let $\mathcal{K}$ be a class of similar algebras. A *V-formation in* $\mathcal{K}$ is an ordered quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_{\mathbf{B}}, \varphi_{\mathbf{C}})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $\varphi_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$ and $\varphi_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$ are embeddings. Given a V-formation $V = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_{\mathbf{B}}, \varphi_{\mathbf{C}})$ in $\mathcal{K}$ and a class $\mathcal{M}$ of algebras in the type of $\mathcal{K}$, an *amalgam* of V in $\mathcal{M}$ is an ordered triple $(\mathbf{D}, \psi_{\mathbf{B}}, \psi_{\mathbf{C}})$, where $\mathbf{D} \in \mathcal{M}$ and $\psi_{\mathbf{B}} : \mathbf{B} \to \mathbf{D}$ and $\psi_{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$ are embeddings such that $\psi_{\mathbf{B}} \circ \varphi_{\mathbf{B}} = \psi_{\mathbf{C}} \circ \varphi_{\mathbf{C}}$. A class $\mathcal{K}$ of similar algebras is said to *have the amalgamation property in* $\mathcal{M}$ if every V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{M}$. The class $\mathcal{K}$ is said to *have the amalgamation property* if $\mathcal{K}$ has the amalgamation property in $\mathcal{K}$.

Note that for classes closed under isomorphisms, we can assume without loss of generality that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$, and $\varphi_{\mathbf{B}}$ and $\varphi_{\mathbf{C}}$ are inclusion maps.

A class of similar algebras $\mathcal{K}$ has *the strong amalgamation property* if it has the amalgamation property and amalgams $(\mathbf{D}, \psi_{\mathbf{B}}, \psi_{\mathbf{C}})$ may be selected so that $(\psi_{\mathbf{B}} \circ \varphi_{\mathbf{B}})[A] = \psi_{\mathbf{B}}[B] \cap \psi_{\mathbf{C}}[C]$. In this case, we say $(\mathbf{D}, \psi_{\mathbf{B}}, \psi_{\mathbf{C}})$ is a strong amalgam.

It's easy to see if the class $\mathcal{K}$ of similar algebras is closed under isomorphisms, then the strong amalgamation property has an easier formulation: if $\mathbf{B}, \mathbf{C} \in \mathcal{K}$ intersect at a common subalgebra $\mathbf{A}$, there exists an algebra $\mathbf{D} \in \mathcal{K}$ such that $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{D}$.

2.5 Residuated frames

A *pogroupoid* is a structure $\mathbf{G} = (G, \leq, \cdot)$, where $\leq$ is a partial order on G and the binary operation $\cdot$ is order-preserving. A *residuated groupoid* is a structure $\mathbf{G} = (G, \leq, \cdot, \backslash, /)$ where $\leq$ is a partial order on G and the residuation property holds. It follows that multiplication is order-preserving. If $\leq$ is a lattice order, then $(G, \wedge, \vee, \cdot, \backslash, /)$ is said to be an rl-groupoid, and if this algebra is extended with a constant 1 that is a multiplicative unit, then it is said to be an rlu-groupoid.

For posets $\mathbf{P}$ and $\mathbf{Q}$, the maps $\triangleright : P \to Q$ and $\triangleleft : Q \to P$ form a *Galois connection* if for all $p \in P$ and $q \in Q$, $q \leq p^{\triangleright}$ iff $p \leq q^{\triangleleft}$. A *closure operator* γ on $\mathbf{P}$ is an increasing, monotone and idempotent map, i.e., $x \leq \gamma(x)$, $x \leq y$ implies $\gamma(x) \leq \gamma(y)$, and $\gamma(\gamma(x)) = \gamma(x)$ for all $x, y \in P$. $\mathbf{P}_\gamma$ denotes the poset of γ-*closed* sets, with underlying set the image $P_\gamma = \gamma[P] = \{\gamma(p) : p \in P\}$.

Given a relation $R \subseteq A \times B$ between sets A and B, for $X \subseteq A$ and $Y \subseteq B$ define

$$XRY \text{ iff } xRy \text{ for all } x \in X, y \in Y.$$

Note that a pair of maps ${}^{\triangleright} : \mathcal{P}(A) \to \mathcal{P}(B)$ and ${}^{\triangleleft} : \mathcal{P}(B) \to \mathcal{P}(A)$ forms a Galois connection iff there exists a relation $R \subseteq A \times B$ such that $X^{\triangleright} = \{y \in B : XRy\}$ and $Y^{\triangleleft} = \{x \in A : xRY\}$. In this case we have xRy iff $x \in \{y\}^{\triangleleft}$ (iff $y \in \{x\}^{\triangleright}$) and $({}^{\triangleright}, {}^{\triangleleft})$ is called the Galois connection *induced* by R. The closure operator $\gamma_R : \mathcal{P}(A) \to \mathcal{P}(A)$ is defined by $\gamma_R(X) = X^{\triangleright\triangleleft}$. For a closure operator γ on a complete lattice $\mathbf{P}$, $D \subseteq P$ is a *basis* for γ if the elements in $\gamma[\mathbf{P}]$ are exactly the meets of elements in D.

Lemma 2.6 ([9])**.** Let A and B be sets.

1. If R is a relation between A and B, then γ_R is a closure operator on $\mathcal{P}(A)$.

2. If γ is a closure operator on $\mathcal{P}(A)$, then $\gamma = \gamma_R$ for some R with domain A.

A *nucleus* on a pogroupoid $\mathbf{G}$ is a closure operator γ on $\mathbf{G}$ such that $\gamma(x)\gamma(y) \leq \gamma(xy)$ for all $x, y \in G$. Now let $\mathbf{G} = (G, \leq, \cdot)$ be a residuated pogroupoid, γ a nucleus on $\mathbf{G}$, and for all $x, y \in G$ define $x \cdot_\gamma y = \gamma(x \cdot y)$. $\mathbf{G}_\gamma = (G_\gamma, \leq, \cdot_\gamma)$ is called the γ-*image* of $\mathbf{G}$.

Lemma 2.7 ([9])**.** (i) The nucleus image G_γ of a pogroupoid $\mathbf{G}$ is a pogroupoid and the properties of lattice-ordering, being residuated and having a unit are preserved.

(ii) All equations and inequations involving $\{\cdot, \vee, 1\}$ are preserved.

(iii) If $\mathbf{G}$ is a residuated lattice and γ is a nucleus on it, then the γ-image $\mathbf{G}_\gamma$ of $\mathbf{G}$ is a residuated lattice.

Let $(W, \circ)$ and $(W', \cdot)$ be ternary relation structures. A relation $N \subseteq W \times W'$ is called *nuclear* on $(W, \circ)$ if there exist ternary relations $\bbslash \subseteq W \times W' \times W'$ and $/\!\!/ \subseteq W' \times W \times W'$ such that for all $u, v \in W$, $w \in W'$,

$$u \circ v N w \text{ iff } v N u \bbslash w \text{ iff } u N w /\!\!/ v.$$

Lemma 2.8 ([9])**.** If $(W, \circ)$ and $(W', \cdot)$ are ternary relation structures and $N \subseteq W \times W'$, then γ_N is a nucleus on $\mathcal{P}(W, \circ)$ iff N is a nuclear relation.

Now we introduce the concept of residuated frames.

A *residuated frame* is a structure of the form $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/)$ where $(W, \circ)$ is a ternary relation structure and $N \subseteq W \times W'$ is a nuclear relation on $(W, \circ)$ with respect to $\backslash\!\!\backslash, /\!\!/$. Concretely, this means

- N is a binary relation from W to W', called the *Galois relation*,
- $\circ \subseteq W^3$, $\backslash\!\!\backslash \subseteq W \times W' \times W'$, $/\!\!/ \subseteq W' \times W \times W'$ and
- $(u \circ v)Nw$ iff $vN(u \backslash\!\!\backslash w)$ iff $uN(w /\!\!/ v)$ for all $u, v \in W$ and $w \in W'$.

It follows above lemmas that $\mathcal{P}(W, \circ)_{\gamma_N}$ is an rl-groupoid, called *Galois algebra of* $\mathbf{W}$.

Here is an instance of the application of residuated frame.

Let $\mathbf{A}$ be a residuated lattice and $\mathbf{B}$ be a partial subalgebra of $\mathbf{A}$. Define $(W, \circ, 1)$ to be the submonoid of $\mathbf{A}$ generated by B. A *unary linear polynomial* of $(W, \circ, 1)$ is a map u on W of the form $u(x) = v \circ x \circ w$, for $v, w \in W$. Such polynomials are also known as *sections* and we denote the set of all sections by S_W. Let $W' = S_W \times B$ and define $xN(u, b)$ by $u(x) \leq^{\mathbf{A}} b$. Given $x, y \in W$ and $u \in S_W$, define sections $u'(x) = u(x \circ y)$ and $u''(y) = u(x \circ y)$. We also use the notation $u' = u(\_ \circ y)$ and $u'' = u(x \circ \_)$. Now define $x \backslash\!\!\backslash (u, b) = \{(u(x \circ \_), b)\}$ and $(u, b) /\!\!/ y = \{(u(\_ \circ y), b)\}$. Then it's easy to see that $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/)$ is a residuated frame and the map $b \mapsto \{(\mathrm{id}, b)\}^{\triangleleft}$ is an embedding of the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ into the residuated lattice $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$.

Chapter 3: Residuated Lattices on M_X

Residuated lattices based on chains have been studied extensively. We start by looking into residuated lattices based on an antichain, with extra top and bottom elements.

3.1 Properties

Given a set X, we denote by $\mathbf{M}_X$ the lattice over the set $X \cup \{\bot, \top\}$, where $\top$ is the top element, $\bot$ is the bottom element, and $x \vee y = \top$ and $x \wedge y = \bot$, for distinct $x, y \in X$.

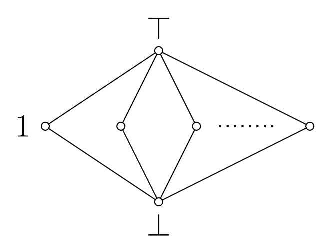

Figure 3.1: A residuated lattice over M_X

The characterization of all residuated lattices based on $\mathbf{M}_X$ where X is non-empty and closed under multiplication, $\bot$ is absorbing in M_X and $\top$ is absorbing in $X \cup \{\top\}$ is known ([12] p. 205): X is a cancellative monoid. We will characterize all residuated lattices based on $\mathbf{M}_X$, even when X is not closed under multiplication.

Recall that in every bounded residuated lattice the bottom element is absorbing. Also, in a residuated lattice based on $\mathbf{M}_X$ we have $\top x, x\top \in \{x, \top\}$ for all x, since $1 \leq \top$ implies $x \leq \top x$ and $x \leq x\top$.

In a residuated lattice $\mathbf{R}$ on $\mathbf{M}_X$, we define

$$U_R = \{x \in R \setminus \{\bot, \top\} : x\top = \top\} \text{ and } Z_R = \{x \in R \setminus \{\bot, \top\} : x\top = x\},$$

the set of elements that behave as units for $\top$ and the set of elements that behave as zeros for $\top$; when the residuated lattice is clear from the context we drop the subscript in U_R and Z_R. Note that $1 \in U$ and $U \cap Z = \emptyset$.

A monoid $\mathbf{S}$ with a zero (absorbing element) 0 is called 0-*cancellative* if for all $x, y, z \in S$,

$$xy = xz \neq 0 \;\Rightarrow\; y = z$$
$$yx = zx \neq 0 \;\Rightarrow\; y = z.$$

An element c in a residuated $\mathbf{R}$ lattice is called *central* if $xc = cx$, for all $x \in R$. Also, we denote by $\sqcup$ the disjoint union operation.

Theorem 3.1. If $\mathbf{R}$ is a residuated lattice based on $\mathbf{M}_X$, then

1. $\top$ is central in $\mathbf{R}$ and $R = U \sqcup Z \sqcup \{\bot, \top\}$.

2. $U_\top = U \cup \{\top\}$ is a $\top$-cancellative submonoid of $\mathbf{R}$.

3. $Z_\bot = Z \cup \{\bot\}$ is a subsemigroup of $\mathbf{R}$ with zero $\bot$, $|Z_\bot| \leq 3$ and $xy = \bot$ for all distinct $x, y \in Z_\bot$.

   Also, either $Z_\bot$ is idempotent, or $Z_\bot = \{b, \bot\}$ with $b^2 = \bot$.

4. $ab = ba = b$ for all $a \in U$ and $b \in Z$.

Proof. (1) We will show that $\top x = x\top$, for all $x \in R$. If x is $\top, \bot$ or $\top$, then $\top x$ and $x\top$ both are equal to $\top, \bot, x$, respectively. Also, if x is incomparable to 1, then $x \vee 1 = \top$, so $\top x = (1 \vee x)x = x \vee x^2 = x(1 \vee x) = x\top$.

Since $\top$ is central and $x\top \in \{x, \top\}$ for all x, we have that for every $x \in R \setminus \{\bot, \top\}$ either $x \in U$ or $x \in Z$, but not both.

(2) If $a, b \in U_\top$, then $ab \cdot \top = a \cdot b\top = a\top = \top$, so $ab \in U_\top$. Similarly, $ba \in U_\top$ and $\top$ is a zero for $U_\top$.

If $x, y, z \in U_\top$ and $xy = xz \neq \top$, then $x(y \vee z) = xy \vee xz = xy \neq \top$. So $y \vee z \neq \top$, because $x\top = \top$; in particular, $y \neq \top \neq z$. Also, since $y, z \in U_\top$ and $\bot \notin U_\top$, we get $y \neq \bot \neq z$; hence $y, z \in X$ and $y \vee z \neq \top$. Since, $\mathbf{R}$ is based on $\mathbf{M}_X$, we get that $y = z$. Similarly, we obtain the other implication of $\top$-cancellativity.

(3) If $c, d \in Z_\bot$, then $cd \cdot \top = c \cdot d\top = cd$. Also, $cd \leq c\top = c < \top$; hence $cd \in Z_\bot$. Clearly, $\bot$ is a zero for $Z_\bot$.

Since $Z_\bot \subseteq X \cup \{\bot\}$, for distinct $x, y \in Z_\bot$, we have $xy = xy \wedge xy \leq x\top \wedge \top y = x \wedge y = \bot$. So, if there were distinct $x, y, z \in Z$, then $y \vee z = \top$ and $x = x\top = x(y \vee z) = xy \vee xz = \bot \vee \bot = \bot$, a contradiction. Therefore $|Z_\bot| \leq 3$.

If b is a non-idempotent element of $Z_\bot \subseteq X \cup \{\bot\}$, then $b \neq \bot$ and $b^2 \leq b\top = b$, so $b^2 = \bot$. If c is an element of $Z_\bot$ distinct from b and $\bot$, then $b^2 = b^2 \vee \bot = b^2 \vee bc = b(b \vee c) = b\top = b$, a contradiction. So, if $Z_\bot$ is not idempotent, then $Z_\bot = \{b, \bot\}$ and $b^2 = \bot$.

(4) For $a \in U$ and $b \in Z$, using the centrality of $\top$, we get

$$b = \top b = \top a \cdot b = \top \cdot ab = ab \cdot \top = a \cdot b\top = ab.$$

Similarly, we get $ba = b$. □

It is straight-forward to see that that the possible options for the subsemigroup $Z_\bot$, mentioned in Theorem 3.1(3) are precisely the ones in Figure 3.2.

Note that if a residuated lattice based on $\mathbf{M}_X$ is integral (i.e., it satisfies $x \leq 1$), then $U = \emptyset$. By taking into account all of the possibilities for $Z_\bot$, it follows that the only integral residuated lattices based on $\mathbf{M}_X$ are the 2-element and 4-element Boolean algebras, the 3-

$$\begin{array}{c|c} & \bot \\ \hline \bot & \bot \end{array}\ ,\quad \begin{array}{c|cc} & \bot & b \\ \hline \bot & \bot & \bot \\ b & \bot & b \end{array}\ ,\quad \begin{array}{c|cc} & \bot & b \\ \hline \bot & \bot & \bot \\ b & \bot & \bot \end{array}\ ,\quad \begin{array}{c|ccc} & \bot & b_1 & b_2 \\ \hline \bot & \bot & \bot & \bot \\ b_1 & \bot & b_1 & \bot \\ b_2 & \bot & \bot & b_2 \end{array}\ .$$

Figure 3.2: Four multiplication tables

element Heyting algebra and the 3-element MV-algebra. The latter two, together with the 3-element Sugihara monoid, are the only 3-element residuated chains.

3.2 Construction and characterization

We now prove the converse of Theorem 3.1. Let $\mathbf{A}$ be a $\top$-cancellative monoid with zero $\top$ and $\mathbf{B}$ a semigroup with zero $\bot$, whose multiplication table is one of those in Figure 3.2.

We define the lattice structure $\mathbf{M}_X$ on the set $R = A \cup B$, where $X = R \setminus \{\bot, \top\}$, $\bot$ is the bottom and $\top$ is the top. Also, we define a multiplication on R that extends the multiplications on $\mathbf{A}$ and $\mathbf{B}$ by: $xy = yx = y$, for all $x \in A$ and $y \in B$. We denote by $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ the resulting algebra.

Theorem 3.2. If $\mathbf{A}$ is a $\top$-cancellative monoid with zero $\top$ and $\mathbf{B}$ is a semigroup with zero $\bot$, whose multiplication table is one of those in Figure 3.2, then $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ is the reduct of a residuated lattice based on $\mathbf{M}_X$, where $X = (A \cup B) \setminus \{\bot, \top\}$.

Proof. Since associativity holds in $\mathbf{A}$ and $\mathbf{B}$ and every element of B is an absorbing element for A, we get that multiplication on $\mathbf{R}$ is associative.

Corollary 2.3 ensures that an expansion of $\mathbf{M}_X$ by a monoid structure is a residuated lattice iff multiplication distributes over arbitrary joins. Since $\bot x = x \bot = \bot$ for all $x \in R$, multiplication distributes over the empty join. Also, we observe every infinite join is equivalent to a finite join, so it suffices to show $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$ for all $x, y, z \in R$ and $y \neq z$. Here we prove $x(y \vee z) = xy \vee xz$.

If $\bot \in \{x, y, z\}$, then it is easy to check that this equation always holds, so we will assume that $\bot \notin \{x, y, z\}$. Since $y \neq z$, we get $y \vee z = \top$. Now we will verify that $x\top = xy \vee xz$.

If $x \in B$, then the left-hand side is x. If, further, $y \in A$ or $z \in A$, then the right-hand side is $x \vee xz = x$ or $xy \vee x = x$, since $xu \leq x$ for all $u \in R$. If $y, z \in B$, then since $|B| \leq 3$ and $y, z, \bot$ are distinct, we get $B = \{y, z, \bot\}$ and $x = y$ or $x = z$. In this case, $xy \vee xz = x \vee \bot = x$, so the equation holds.

If $x \in A$, then the left-hand side is equal to $\top$. If $y \in B$ and $z \in B$, then the right-hand side is $y \vee z = \top$, since $y \neq z$. If $y \in B$ and $z \in A$, then the right-hand side is $y \vee xz = \top$, since $y \in B$, $xz \in A$ and $\bot \notin \{x, y, z\}$. Likewise, if $y \in A$ and $z \in B$, then the right-hand side is $\top$. If $y \in A$ and $z \in A$, then the right-hand side is $xy \vee xz = \top$ since $\mathbf{A}$ is $\top$-cancellative.

Similarly, we can show $(y \vee z)x = yx \vee xz$ for all $x, y, z \in R$. $\square$

By Corollary 2.2 the divisions are uniquely determined by $x \backslash z = \max\{y \in R : xy \leq z\}$ and $z/x = \max\{y \in R : yx \leq z\}$, and we give the precise values below.

It turns out that $A \cup \{\bot\}$ and $B \cup \{\top\}$ are subalgebras of $\mathbf{R}_{\mathbf{A},\mathbf{B}}$. In particular, $B \cup \{\top\}$ is the 2-element generalized Boolean algebra, the 3-element generalized Brouwerian algebra, 3-element generalized MV-algebra, or the 4-element generalized Boolean algebra, corresponding to the tables in Figure 3.2. The divisions are given by Remark 2.4 and

$$a_1 \backslash a_2 = \begin{cases} a_3 & \text{if } a_1 a_3 = a_2 \\ \bot & \text{otherwise} \end{cases} \qquad a_2 / a_1 = \begin{cases} a_3 & \text{if } a_3 a_1 = a_2 \\ \bot & \text{otherwise} \end{cases}$$

for $a_1, a_2, a_3 \in A$, where the a_3 is guaranteed to be unique, when it exists. Finally, for $a \in A \setminus \{\top\}$ and $b \in B$, any operation between a and b works the same as the operation between 1 and b. For example, $b \backslash a = b \backslash 1$, $a \wedge b = 1 \wedge b$, $ab = 1b$, etc.

By combining Theorem 3.1 and Theorem 3.2, we obtain the following characterization.

Corollary 3.3. The residuated lattices based on $\mathbf{M}_X$ are precisely the ones of the form $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ and the integral ones: the 2-element generalized Boolean algebra, the 3-element generalized Brouwerian algebra, 3-element generalized MV-algebra, or the 4-element generalized Boolean algebra, where $\mathbf{A}$ is a $\top$-cancellative monoid with zero $\top$ and $\mathbf{B}$ is a semigroup with zero $\bot$, whose multiplication table is one of those in Figure 3.2.

Chapter 4: Axiomatizations

In this section we will provide axiomatizations for the various classes we will be considering and also discuss their proof theory.

4.1 Axiomatization of residuated lattices based on $\mathbf{M}_X$

We start by giving an axiomatization for the variety M generated by all residuated lattices based on $\mathbf{M}_X$, where X is a set; see Corollary 4.4. Since the lattice $\mathbf{M}_X$ is simple, when $|X| \geq 3$, residuated lattices based on $\mathbf{M}_X$ are also simple; if $|X| \leq 3$ the residuated lattice is simple, as well. It turns out (Corollary 4.7) that these are precisely the subdirectly irreducible algebras in M and we will provide an axiomatization for them.

Actually, we can also expand the language of residuated lattices to include constants which then evaluate as bounds. A *bounded residuated lattice* is an expansion of a residuated lattice that happens to be based on a bounded lattice, by the addition of constants $\bot$ and $\top$, evaluating at these bounds (so $\bot \leq x \leq \top$, for all x). We will consider both cases where the language includes the bounds or not, but opt for the axioms to be expressible without the need for bounds. We can arrange for the axioms we will be considering to be positive universal sentences, which is convenient for applying the correspondence provided in [8].

A (bounded) residuated lattice is called *unilinear* if it satisfies:

$$\forall x, y, z, w\, (x \leq y \text{ or } y \leq x \text{ or } (x \wedge y \leq w \text{ and } z \leq x \vee y)) \qquad \text{(URL)}$$

Since the axiom (URL) is equivalent to

$$\forall x, y,\, (x \leq y \text{ or } y \leq x \text{ or } \forall z, w\, (x \wedge y \leq w \text{ and } z \leq x \vee y)),$$

a residuated lattice is unilinear iff it is linear or else the lattice is actually bounded and every pair of incomparable elements joins to the top of the lattice and meets to the bottom of the lattice. In other words the non-linear residuated lattices consist of two bounds and the rest of the lattice is a disjoint union of totally incomparable chains; see Figure 4.1. For these non-linear unilinear residuated lattices, we will be denoting these bounds by $\bot$ and $\top$, even when the language does not include constants for the bounds. We denote by URL and bURL the (positive universal) classes of unilinear and bounded unilinear residuated lattices, respectively. Clearly, (bounded) residuated lattices on an $\mathbf{M}_X$ are unilinear.

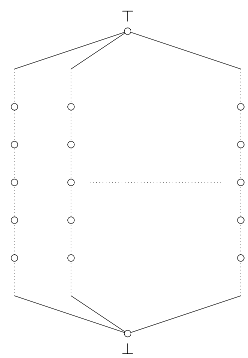

Figure 4.1: A non-linear unilinear residuated lattice

What distinguishes $\mathbf{M}_X$ from other lattices is its height, so we axiomatize unilinear residuated lattices whose height is no greater than a given number. We are careful to formulate the first-order sentence so it has no implication in it and it remains a positive sentence.

Proposition 4.1. Given a natural number n, a (bounded) unilinear residuated lattice has height at most n if and only if it satisfies

$$\forall x_1, \ldots, x_{n+1} \, (\underset{1 \leq m \leq n}{\mathrm{OR}} \, x_1 \vee \cdots \vee x_m = x_1 \vee \cdots \vee x_{m+1}). \tag{h_n}$$

Also, it has width at most n if and only if it satisfies

$$\forall x_1, \dots, x_{n+1} \, (\underset{1 \leq i \neq j \leq n+1}{\text{OR}} x_i \leq x_j). \tag{w_n}$$

Proof. Having height at most n is equivalent to saying that every subchain has at most n elements. Now, every subchain always has the form $a_1 \leq a_1 \vee a_2 \leq a_1 \vee a_2 \vee a_3 \leq \cdots \leq a_1 \vee \cdots \vee a_k$, where $a_1, \dots, a_k$ are elements of the lattice and where the number of the inequalities that are equalities determines the number of elements in the chain. So, having height at most n is equivalent to stipulating that in every chain $a_1, a_1 \vee a_2, \dots, a_1 \vee \cdots \vee a_{n+1}$, at least two adjacent elements are equal.

Having width at most n is equivalent to having at most n pairwise incomparable elements. □

We denote by URL_n the subclass of URL axiomatized by (h_n). In particular, (h_3) is the universal closure (which we often suppress) of

$$x_1 = x_1 \vee x_2 \text{ or } x_1 \vee x_2 = x_1 \vee x_2 \vee x_3 \text{ or } x_1 \vee x_2 \vee x_3 = x_1 \vee x_2 \vee x_3 \vee x_4.$$

Corollary 4.2. The (bounded) residuated lattices that are based on $\mathbf{M}_X$, for some X, together with the trivial algebra, are precisely the ones in the class URL_3 (bURL_3).

4.2 Equational basis for M

The class URL_3 is axiomatized by positive universal sentences. We note that [8] provides a general method for axiomatizing the variety of residuated lattices generated by a positive universal class. In detail, if

$$1 \leq p_1 \text{ or } \cdots \text{ or } 1 \leq p_n$$

is a positive universal formula, then the variety generated by the residuated lattices satisfying the universal closure of the formula is axiomatized by the infinitely many equations

$$1 = \gamma_1(p_1) \vee \cdots \vee \gamma_n(p_n)$$

where $\gamma_1, \ldots, \gamma_n \in \Gamma(Var)$, the set of all iterated conjugates. The *left conjugate* of a by x is the term $x\backslash ax \wedge 1$ and the *right conjugate* is $xa/x \wedge 1$; *iterated conjugates* are obtained by repeated applications of left and right conjugates by various conjugating elements from the set Var of variables. If ϕ is a set of positive universal formulas, we denote by V_ϕ the variety axiomatized by the set Γ_ϕ of all the equations corresponding to the positive universal formulas in ϕ.

We consider the variety SRL generated by the class URL and we call its elements *semi-unilinear*. Since URL is axiomatized by

$$x \leq y \text{ or } y \leq x \text{ or } (x \wedge y \leq z \text{ and } w \leq x \vee y),$$

which can be written as the conjunction of the two sentences

$$x \leq y \text{ or } y \leq x \text{ or } x \wedge y \leq z, \qquad x \leq y \text{ or } y \leq x \text{ or } w \leq x \vee y,$$

and, in turn, as

$$1 \leq x\backslash y \text{ or } 1 \leq y\backslash x \text{ or } 1 \leq (x \wedge y)\backslash z, \;\; 1 \leq x\backslash y \text{ or } 1 \leq y\backslash x \text{ or } 1 \leq w\backslash(x \vee y),$$

we get the following result.

Corollary 4.3. The variety SRL of semiunilinear residuated lattices is axiomatized by the infinitely many equations

$$1 = \gamma_1(x\backslash y) \vee \gamma_2(y\backslash x) \vee \gamma_3((x \wedge y)\backslash z) \qquad 1 = \gamma_4(x\backslash y) \vee \gamma_5(y\backslash x) \vee \gamma_6(w\backslash(x \vee y)),$$

for all $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \Gamma(Var)$.

Corollary 4.4. The variety M generated by the class URL_3, of residuated lattices on an $\mathbf{M}_X$, is axiomatized relative to SRL by : $1 =$

$$\gamma_1((x_1 \vee x_2)\backslash x_1) \vee \gamma_2((x_1 \vee x_2 \vee x_3)\backslash(x_1 \vee x_2)) \vee \gamma_3((x_1 \vee x_2 \vee x_3 \vee x_4)\backslash(x_1 \vee x_2 \vee x_3))$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(Var)$.

The equational bases are not necessarily infinite. [12] gives an example of searching for a finite equational basis of the variety RRL of representable residuated lattices. However, the method there doesn't work for our case. It's still unknown if there are finite equational bases for varieties SRL or M.

We denote by bM the corresponding variety of bounded residuated lattices. Also, we can characterize the finitely subdirectly irreducible algebras in these varieties.

Theorem 4.5. The finitely subdirectly irreducible (FSI) semiunilinear residuated lattices are precisely the unilinear residuated lattices: $\mathsf{SRL}_{FSI} = \mathsf{URL}$. More generally, if ϕ is a set of positive universal sentences, then the FSIs in $\mathsf{SRL} \cap \mathsf{V}_\phi$ are precisely the unilinear residuated lattices that satisfy ϕ.

Proof. It follows from the proof of Theorem 9.73(2) of [8] that an FSI algebra satisfies the unilinearity condition iff it satisfies the equations of Corollary 4.3, i.e., iff it is semiunilinear. So, the semiunilinear FSIs are actually unilinear.

Conversely, if an algebra is unilinear, then its negative cone $\downarrow 1$ is a chain. Therefore, the convex normal submonoids of the negative cone are nested and $\{1\}$ cannot be the intersection of two non-trivial convex normal submonoids; see [12] for the correspondence between congruences and convex normal submonoids of the negative cone of residuated lattices. Therefore, the trivial congruence is meet-irreducible and the algebra is FSI (and semiunilinear, as it is unilinear). □

Corollary 4.6. Every semiunilinear residuated lattice is a subdirect product of unilinear ones.

Corollary 4.7. The subdirectly irreducibles in M are the same as the finitely subdirectly irreducibles in M and as the simple ones in M and they are precisely the non-trivial residuated lattices based on $\mathbf{M}_X$, for some X. The same holds for bM.

That every subdirectly irreducible in each of the varieties M and bM is actually simple follows from the fact that its negative cone has two elements. Consequently, these varieties are semisimple. For bM we can say a bit more.

We define the following terms

$$r(x) = (1 \vee x)(1 \wedge x) \wedge (1 \vee 1/x)(1 \wedge 1/x) \qquad x \leftrightarrow y = x\backslash y \wedge y\backslash x \wedge 1$$

$$t(x, y, z) = r(x \leftrightarrow y) \cdot z \vee (r(x \leftrightarrow y)\backslash \bot \wedge 1) \cdot x$$

Lemma 4.8. bM is a discriminator variety with discriminator term t.

Proof. If $\mathbf{R} \in \mathsf{bM}_{SI}$ then, by Corollary 4.7, $\mathbf{R}$ is a non-trivial bounded residuated lattice based on $\mathbf{M}_X$ for some X. Note that if x is incomparable to 1, then also $1/x$ is incomparable to 1 or is equal to $\bot$, so $1 \wedge x = 1 \wedge 1/x = \bot$, hence $r(x) = \bot$. Also, if $x \in \{\bot, \top\}$, then $\{x, 1/x\} = \{\bot, \top\}$, so $1 \wedge x = \bot$ or $1 \wedge 1/x = \bot$, hence $r(x) = \bot$. Finally, since $1/1 = 1$, we have $r(x) = 1$, if $x = 1$ and $r(x) = \bot$ otherwise.

Note that for all $x, y \in R$, we have $x \leftrightarrow y \leq 1$, i.e., $x \leftrightarrow y \in \{\bot, 1\}$. Moreover, $x \leftrightarrow y = 1$ iff $1 = x\backslash y \wedge y\backslash x \wedge 1$ iff $1 \leq x\backslash y \wedge y\backslash x$ iff ($1 \leq x\backslash y$ and $1 \leq \wedge y\backslash x$) iff ($x \leq y$ and $y \leq x$) iff $x = y$. Thus we have $x \leftrightarrow y = 1$ if $x = y$ and $x \leftrightarrow y = \bot$ if $x \neq y$.

Therefore, $t(x, y, z) = r(1)\cdot z \vee (r(1)\backslash\bot \wedge 1)\cdot x = 1\cdot z \vee (1\backslash\bot\wedge 1)\cdot x = z \vee \bot\cdot x = z$, if $x = y$; and $t(x, y, z) = r(\bot)\cdot z \vee (r(\bot)\backslash\bot\wedge 1)\cdot x = \bot\cdot z \vee (\bot\backslash\bot\wedge 1)\cdot x = (\top\wedge 1)\cdot x = 1\cdot x = x$, if $x \neq y$. □

4.3 Including (or not) the bounds in the signature

Note that when axiomatizing classes of unilinear residuated lattices for which the non-linear members are asked to satisfy a certain positive universal sentence, oftentimes the axiomatization looks nicer in the case where the language includes constants for the bounds. For example, the class of URLs whose non-linear members satisfy $\top x = x\top$ is axiomatized by the positive universal formula

$$u \leq v \text{ or } v \leq v \text{ or } x(u \vee v) = (u \vee v)x.$$

For non-linear bURLs this formula is equivalent to

$$x\top = \top x.$$

For the sake of readability, we will allow ourselves to denote the first of these sentences as the more pleasing to the eye:

$$x\overline{\top} = \overline{\top}x.$$

We call a (bounded) unilinear residuated lattice $\top$-*central*, if it satisfies this formula.

More generally, if Φ is the sentence $\forall\vec{x}\,(\varphi(\vec{x},\top,\bot))$, where φ is in the language of URL, we denote by $\overline{\Phi}$ the sentence

$$\forall\vec{x}\,(\varphi(\vec{x},\overline{\top},\overline{\bot})) := \forall u, \forall v, \forall\vec{x}\,(u \le v \text{ or } v \le u \text{ or } \varphi(\vec{x}, u \vee v, u \wedge v))$$

where u, v are fresh variables.

Likewise, we call a (bounded) unilinear residuated lattice $\top$-*unital*, if it satisfies the formula

$$x = \overline{\bot} \text{ or } x\overline{\top} = \overline{\top} = \overline{\top}x,$$

since in the non-linear models every non-bottom element acts as a unit for the top. Note that for non-linear bURLs being $\top$-unital is the same as being rigorously compact.

Lemma 4.9. Let φ be a positive universal formula in the language of URLs, let Φ be $\forall\vec{x}\,(\varphi(\vec{x},\top,\bot))$ and let $\overline{\Phi}$ be $\forall\vec{x}\,(\varphi(\vec{x},\overline{\top},\overline{\bot}))$.

1. The non-linear bURLs that satisfy Φ are precisely the non-linear bURLs that satisfy $\overline{\Phi}$.

2. The non-linear URLs that satisfy $\overline{\Phi}$ are precisely the bound-free reducts of the non-linear bURLs that satisfy $\overline{\Phi}$.

3. The linear (bounded) URLs that satisfy $\overline{\Phi}$ are precisely the (bounded) residuated chains.

Proof. (1) If $\mathbf{R}$ is a non-linear bURL, then it satisfies $\overline{\Phi}$ iff it satisfies it for all incomparable elements u, v (as $\overline{\Phi}$ automatically holds for comparable elements u, v) iff it satisfies Φ (since when u, v are incomparable, we have $u \vee v = \top$ and $u \wedge v = \bot$).

(2) follows from the fact that all non-linear URLs are bounded, say b and t are the bounds, and that for bounded non-linear URLs $\overline{\Phi}$ is equivalent to $\forall\vec{x}\,(\varphi(\vec{x}, t, b))$.

(3) follows from the fact that $\overline{\Phi}$ holds in all totally ordered algebras. □

We note that there might be linear bURLs that satisfy $\overline{\Phi}$, but fail to satisfy Φ. This happens for example when Φ is $\top x = x\top$.

4.4 Proof theory for SRL

Certain varieties of residuated lattices admit a proof-theoretic analysis, which is often complementary to their algebraic study and which often yields interesting results. Not all varieties of residuated lattices admit a proof-theoretic calculus, but we show that SRL does admit a hypersequent calculus. We present the hypersequent system, but we do not pursue any further applications in this paper.

As a motivating example, we mention the equational theory of lattices, which is axiomatized by the standard basis of the semilattice and the absorption laws. New valid equations can be derived from these axioms using the derivational system of equational logic, which includes the rules of reflexivity, symmetry, transitivity, and replacement/congruence. This system is not amenable to an inverse proof search analysis as, given an equation $s = t$, to determine if it is derivable in the system one cannot simply go through all applications of these derivational rules that could have the equation as a conclusion and proceed recursively: the transitivity rule $\frac{s=t \quad t=r}{s=r}$ introduces (read upward) a new term that does not appear in the equation. Also, using inequational reasoning, where for example $\frac{s\leq t \quad t\leq r}{s\leq r}$ is used instead and the axioms are replaced by inequational axioms such as $s \leq s\vee t$, does not make the problem go away: simply omitting this transitivity rule from the system changes the set of derivable inequalities. However, a way to bypass this problem is to replace the lattice axioms by inference rules; for example we replace $s \leq s \vee t$ by the inference rule $\frac{r\leq s}{r\leq s\vee t}$. The axiom and the rule are equivalent in the presence of transitivity, but the rule has elements of transitivity *injected* in it when compared to the axiom: the rule implies the axiom by instantiation, but the axiom implies the rule only with the help of transitivity. Moreover, the new rule does not suffer from the problem of transitivity as all terms in the numerator are already contained in the denominator; so it is safe to replace the axiom by the rule. There is a way to inject transitivity into all the axioms, converting them to innocent inference rules, such that in the new system the transitivity rule itself becomes completely redundant. The resulting system can be used to show the decidability of lattice equations.

A similar approach works for certain subvarieties of residuated lattices; the axioms in the subvariety may or may not be amenable to injecting transitivity to them. Also, since there are more operations than in lattices, the above inequalities have to be replaced by *sequents*. These are expressions of the form $s_1, s_2, \ldots, s_n \Rightarrow s_0$, where the s_i's are residuated-lattice terms, and their interpretation is given by $s_1 \cdot s_2 \cdots s_n \leq s_0$. The transitivity rule itself at the level of sequents takes the form or a rule called (cut) and the goal is cut-elimination, in the same spirit as above, for lattices; we often write $\Gamma \Rightarrow \Pi$ for sequents, where Γ is a sequence of formulas and Π is a single formula. The corresponding derivational systems/calculi define different types of *substructural logics* and varieties of residuated lattices serve as algebraic semantics for them; see [12].

The variety of all residuated lattices admits a sequent derivation system, which leads to the decidability of the equational theory of residuated lattices, among other things. The variety of semilinear residuated lattices (generated by residuated chains) however, provably does not admit a sequent calculus, due to the shape of its axioms. It does, however, admit a hypersequent calculus. Hypersequents are more complex syntactic objects of the form $\Gamma_1 \Rightarrow \Pi_1 \mid \Gamma_2 \Rightarrow \Pi_2 \mid \cdots \mid \Gamma_m \Rightarrow \Pi_m$, i.e., they are multisets of sequents. We denote by $\mathbf{HRL}$ the basis hypersequent system for the variety of residuated lattices; additional inference rules can be added in order to obtain systems for subvarieties.

We follow [3], which describes the process of injecting transitivity into hypersequents, and we obtain a hypersequent system for the variety SRL that admits cut elimination. We start with the axioms of URL, the positive universal class that generates SRL.

First we convert the first axiom $\forall x, y, z(x \leq y$ or $y \leq x$ or $z \leq (x \vee y))$ to the equivalent form $\forall x, y, z, t_1, t_2, t_3, s_1, s_2, s_3$

$$t_1 \leq x \text{ and } y \leq s_1 \text{ and } t_2 \leq y \text{ and } x \leq s_2 \text{ and } t_3 \leq z \text{ and } (x \vee y) \leq s_3$$
$$\Rightarrow t_1 \leq s_1 \text{ or } t_2 \leq s_2 \text{ or } t_3 \leq s_3$$

by injecting some transitivity. This then allows to remove the $\vee$ from the axiom, by rewriting it as $\forall x, y, z, t_1, t_2, t_3, s_1, s_2, s_3$

$$t_1 \leq x \text{ and } y \leq s_1 \text{ and } t_2 \leq y \text{ and } x \leq s_2 \text{ and } t_3 \leq z \text{ and } x \leq s_3 \text{ and } y \leq s_3$$
$$\Rightarrow t_1 \leq s_1 \text{ or } t_2 \leq s_2 \text{ or } t_3 \leq s_3$$

In the terminology of [3], the clause is *linear* and *exclusive*, so we eliminate the redundant variables in the premise (noting that z appears only on the right side of inequations, while x and y appear on both sides): we apply transitivity closure and removal of variables in the premise of the clause. The procedure yields the equivalent clause $\forall t_1, t_2, t_3, s_1, s_2, s_3$

$$t_1 \leq s_2 \text{ and } t_1 \leq s_3 \text{ and } t_2 \leq s_1 \text{ and } t_2 \leq s_3$$
$$\Rightarrow t_1 \leq s_1 \text{ or } t_2 \leq s_2 \text{ or } t_3 \leq s_3$$

We now instantiate s_j by $c\backslash p_j/d$ and use residuation to rewrite $t_i \leq s_j$ as $t_i \leq c\backslash p_j/d$ and as $ct_id \leq p_j$. This results in the equivalent clause $\forall t_1, t_2, t_3, c, p_1, p_2, p_3, d$

$$ct_1d \leq p_2 \text{ and } ct_1d \leq p_3 \text{ and } ct_2d \leq p_1 \text{ and } ct_2d \leq p_3$$
$$\Rightarrow ct_1d \leq p_1 \text{ or } ct_2d \leq p_2 \text{ or } ct_3d \leq p_3$$

Converting the clause to the corresponding hypersequent rule we get

$$\frac{\Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi_2 \quad \Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi_3 \quad \Xi \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi_1 \quad \Xi \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi_3}{\Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi_1 \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi_2 \mid \Gamma, \Sigma_3, \Delta \Rightarrow \Pi_3}$$

Likewise the second axiom of unilinearity gives the hypersequent rule

$$\frac{\Xi \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi_1 \quad \Xi \mid \Gamma, \Sigma_3, \Delta \Rightarrow \Pi_1 \quad \Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi_2 \quad \Xi \mid \Gamma, \Sigma_3, \Delta \Rightarrow \Pi_2}{\Xi \mid \Gamma, \Sigma_1, \Delta \Rightarrow \Pi_1 \mid \Gamma, \Sigma_2, \Delta \Rightarrow \Pi_2 \mid \Gamma, \Sigma_3, \Delta \Rightarrow \Pi_3}$$

We refer to these hypersequent rules as (URL1) and (URL2), respectively.

Corollary 4.10. The extension of **HFL** with the rules (URL1) and (URL2) provides a cut-free hypersequent calculus for the variety SRL by [3].

It is notable, that even though SRL has an infinite equational axiomatization involving iterated conjugates, there are only two inference rules needed for the hypersequent calculus. This is because hypersequent calculi have the ability to go directly to the level of (finitely) subdirectly irreducibles ($\mathsf{SRL}_{FSI} = \mathsf{URL}$ in this case) and read off the axiomatization from there.

Chapter 5: Continuum-many subvarieties of M

Even though we have a fairly good understanding of the residuated lattices based on $\mathbf{M}_X$, where X is a set, we now show that there are continuum-many subvarieties of M. More precisely, we will prove that the variety M_G generated by all the residuated lattices of the form $\mathbf{M}_\mathbf{G}$, where $\mathbf{G}$ is an (abelian) group, has continuum-many subvarieties. We start with an equational basis for M_G.

Proposition 5.1. The variety M_G is axiomatized by the equations $1 = \gamma_1(u\backslash v) \vee \gamma_2(v\backslash u) \vee \gamma_3(x\backslash(u \wedge v)) \vee \gamma_4((u \vee v)\backslash x) \vee \gamma_5(x(x\backslash 1))$, where γ_1,γ_2, γ_3, γ_4, $\gamma_5 \in \Gamma(Var)$.

Proof. The formula $x = \overline{\bot}$ or $x = \overline{\top}$ or $x(x\backslash 1) = 1$ axiomatizes the FSIs in the variety, so the result follows by Theorem 4.5. □

It is known that there are continuum-many varieties of groups (for example, see [15]) and we can use this fact to show that there is a continuum of subvarieties of $\mathsf{M}_\mathbf{G}$, as follows. Starting with two varieties $\mathcal{V}_1 \neq \mathcal{V}_2$ of groups, we can consider the free groups $\mathbf{F}_1$ and $\mathbf{F}_2$ on countably many generators in these varieties; hence we have $\mathsf{V}(\mathbf{F}_1) = \mathcal{V}_1 \neq \mathcal{V}_2 = \mathsf{V}(\mathbf{F}_2)$. Then, it is possible to show that $\mathsf{V}(\mathbf{M}_{\mathbf{F}_1}) \neq \mathsf{V}(\mathbf{M}_{\mathbf{F}_2})$.

It is also well known that there are only countably-many varieties of abelian groups. However, we are still able to show that the variety CM_G of the commutative algebras in $\mathsf{M}_\mathbf{G}$ also has continuum-many subvarieties. Actually, we give a full description of the subvariety lattice of CM_G.

We consider the direct power $\mathbb{N}^\omega$ of countably many copies of the chain $(\mathbb{N}, \leq)$ and its subset I of (not necessarily strictly) decreasing sequences that are eventually zero, such as $(4, 2, 1, 1, 0, 0, \dots)$, $(3, 2, 1, 1, 1, 0, 0, \dots)$ etc. We will also denote these sequences by

$(4,2,1,1)$ and $(3,2,1,1,1)$, respectively. It is easy to see that I defines a sublattice $\mathbf{I}$ of the direct product. We also consider the subset $I^{\oplus\omega}$ of the direct product $\mathbf{I}^{\omega}$ of all sequences of elements of I that are eventually the zero sequence. It is easy to see that this defines a sublattice $\mathbf{I}^{\oplus\omega}$ of the direct product $\mathbf{I}^{\omega}$; it makes sense to call $\mathbf{I}^{\oplus\omega}$ the *direct sum* of ω copies of $\mathbf{I}$. We use commas to separate the numbers in each sequence in I, but we use semicolons to separate the sequences in each element of $I^{\oplus\omega}$; this allows for dropping parenthesis, if desired. Therefore, $(2,1;3,1,1;0;2,1,1;0;\dots)$ is an example of an element of $I^{\oplus\omega}$.

Now let $\mathbf{P} = \mathbf{2} \times \mathbf{I}^{\oplus\omega}$, where $\mathbf{2}$ is the two-element lattice on $\{0,1\}$. For $a \in P$, we define $\exp(a)$ to be the maximum number appearing in a; e.g., $\exp(1;3,1;0;2;0;\dots) = 3$ and $\exp(0;1,1,1;4,1;3,2;0;\dots) = 4$. Also, for $a \in P$ we write $a = (a_0;a_1;a_2;\ldots)$, where $a_0 \in \{0,1\}$ and $a_n \in I$, for $n > 0$; we define $\operatorname{primes}(a) = \{n \in \mathbb{N} : a_n \neq \overline{0}\}$. For $T \subseteq P$, we define $\exp(T) = \{\exp(a) : a \in T\}$ and $\operatorname{primes}(T) = \bigcup\{\operatorname{primes}(a) : a \in T\}$.

A downset D of $\mathbf{P}$ is said to be $\mathbb{Z}$*-closed* if for all $a \in P$,

$$\exp(D \cap {\uparrow}a) \text{ or } \operatorname{primes}(D \cap {\uparrow}a) \text{ is unbounded implies } a \vee (1;0;0;\ldots) \in D.$$

For example, for $a = (0;1;0;0;0;...)$, this condition has the following consequences:

$$(0;1;1;0;0;\ldots),(0;1;2;0;0;\ldots),(0;1;3;0;0;\ldots),\ldots \in D$$

or

$$(0;1,1;0;0;\ldots),(0;2,1;0;0;\ldots),(0;3,1;0;0;\ldots),\ldots \in D$$

implies $(1;1;0;0;0;\ldots) \in D$, because $\exp(D \cap {\uparrow}a)$ is unbounded. Also,

$$(0;1;1;0;0;\ldots),(0;1;0;1;0;\ldots),(0;1;0;0;1;\ldots),\ldots \in D$$

implies $(1;1;0;0;0;\ldots) \in D$, because $\operatorname{primes}(D \cap {\uparrow}a)$ is unbounded. However,

$$(0;1;1;0;0;\ldots),(0;1;1,1;0;0;\ldots),(0;1;1,1,1;0;0;\ldots),\ldots \in D$$

does not imply $(1;1;0;0;0;\ldots) \in D$.

We denote the lattice of all $\mathbb{Z}$-closed downsets of $\mathbf{P}$ by $\mathcal{O}_{\mathbb{Z}}(\mathbf{P})$.

Theorem 5.2. The subvariety lattice of $\mathsf{CM_G}$ is isomorphic to $\mathcal{O}_{\mathbb{Z}}(\mathbf{P})$.

Proof. Recall that a class of algebras is closed under $\mathsf{HSP_U}$ iff it is axiomatizable by positive universal sentences. In other words, $\mathsf{HSP_U}$-classes coincide with positive universal classes.

Let $\mathcal{F}$ be a congruence-distributive variety such that $\mathcal{F}_{FSI}$ is a positive universal class. We claim that the subvarieties of $\mathcal{F}$ are in bijective correspondence with $\mathsf{HSP_U}$-subclasses of $\mathcal{F}_{FSI}$, where the correspondence is given by $\mathcal{V} \mapsto \mathcal{V}_{FSI}$ and $\mathcal{K} \mapsto \mathsf{HSP}(\mathcal{K})$; furthermore, it is clear that this correspondence preserves and reflects the inclusion order. Indeed, $\mathcal{V}_{FSI} = \mathcal{V} \cap \mathcal{F}_{FSI}$, so $\mathcal{V}_{FSI}$ is axiomatized by positive universal sentences and the forward map of the correspondence is well defined. To show that the two maps are inverses of each other note that $\mathsf{HSP}(\mathcal{V}_{FSI}) \subseteq \mathcal{V} \subseteq \mathsf{SP}(\mathcal{V}_{SI}) \subseteq \mathsf{HSP}(\mathcal{V}_{FSI})$ and by Jónsson's Lemma $\mathcal{K} = \mathcal{K}_{FSI} \subseteq \mathsf{HSP}(\mathcal{K})_{FSI} \subseteq \mathsf{HSP_U}(\mathcal{K}) = \mathcal{K}$.

Note that residuated lattices form a congruence distributive variety by [12] and, by Theorem 4.5 and Proposition 5.1, $(\mathsf{CM_G})_{FSI} = \mathsf{CM_G} \cap \mathsf{SRL}_{FSI}$ is axiomatized by positive universal sentences. So, by the preceding paragraph, the lattice of subvarieties of $\mathsf{CM_G}$ is isomorphic to the lattice of $\mathsf{HSP_U}$-classes of FSIs in $\mathsf{CM_G}$, which by Theorem 4.5 and Proposition 5.1 are $\mathsf{HSP_U}$-classes of algebras of the form $\mathbf{M_G}$, where $\mathbf{G}$ is an abelian group.

Further note that H can be replaced by I. Indeed, every ultrapower of algebras of the form $\mathbf{M_G}$, where $\mathbf{G}$ is an abelian group, is also an algebra of the same form (it satisfies the same first-order sentences, hence also all positive universal sentences). Also, subalgebras are also of the same form (where we also include the trivial algebra). Finally, since every algebra of this form is simple (since their lattice reducts are simple), H does not contribute any new algebras. So we are interested in $\mathsf{ISP_U}$-classes of algebras of the form $\mathbf{M_G}$, where $\mathbf{G}$ is an abelian group.

We now prove that such classes are in bijective correspondence with $\mathsf{ISP_U}$-classes of abelian groups, by showing that for every class $\mathcal{K}$ of abelian groups, we have $\mathsf{ISP_U}(\{\mathbf{M_H} :$

$\mathbf{H} \in \mathcal{K}\}) = \mathsf{I}\{\mathbf{M_G} : \mathbf{G} \in \mathsf{SP_U}(\mathcal{K})\}$ and thus this class can be associated with $\mathsf{ISP_U}(\mathcal{K})$; clearly this correspondence preserves and reflects the order.

First we show $\mathsf{IP_U}(\{\mathbf{M_H} : \mathbf{H} \in \mathcal{K}\}) = \mathsf{I}\{\mathbf{M_G} : \mathbf{G} \in \mathsf{IP_U}(\mathcal{K})\}$. For a residuated lattice $\mathbf{R}$, if $\mathbf{R} \in \mathsf{IP_U}(\{\mathbf{M_H} : \mathbf{H} \in \mathcal{K}\})$, then $\mathbf{R}$ satisfies all first-order sentences that hold in the $\mathbf{M_H}$'s, where $\mathbf{H} \in \mathcal{K}$. In particular, $\mathbf{R}$ is commutative, unilinear, has height at most 3, and all of its non-bound elements are invertible, closed under multiplication and serve as units for the top. Therefore, $\mathbf{R}$ is isomorphic to $\mathbf{M_G}$ for some abelian group $\mathbf{G}$. Also, clearly, all algebras in $\mathsf{P_U}(\mathcal{K})$ are abelian groups. Therefore the classes on both sides of the equation contain only algebras isomorphic to $\mathbf{M_G}$ for some abelian group $\mathbf{G}$, and it is enough to focus on such algebras: we show that for every abelian group $\mathbf{G}$, $\mathbf{M_G} \in \mathsf{IP_U}(\{\mathbf{M_H} : \mathbf{H} \in \mathcal{K}\})$ iff $\mathbf{G} \in \mathsf{IP_U}(\mathcal{K})$; we will identify the bounds in all algebras to omit I.

If $\mathbf{M_G} \in \mathsf{P_U}(\{\mathbf{M_H} : \mathbf{H} \in \mathcal{K}\})$, there exists an index set I, an ultrafilter U on I and $\mathbf{H}_i \in \mathcal{K}$, $i \in I$, such that $\mathbf{M_G} = \prod \mathbf{M_{H_i}}/U$. So, for every $g \in G$ there exists $x_g \in \prod \mathbf{M_{H_i}}$ such that $g = [x_g]$, the equivalence class of x_g. We will use $\overline{\top}$ and $\overline{\bot}$ to denote the tuples $(\top)_{i\in I}$ and $(\bot)_{i\in I}$ in $\prod \mathbf{M_{H_i}}$ respectively. Then for all $g \in G$, we have $g \neq [\overline{\top}]$ and $g \neq [\overline{\bot}]$, since g is invertible while $[\overline{\top}]$ and $[\overline{\bot}]$ are idempotents different than the identity. So we know $\{i \in I : x_g(i) \neq \top\} \in U$ and $\{i \in I : x_g(i) \neq \bot\} \in U$, hence $\{i \in I : x_g(i) \in H\} = \{i \in I : x_g(i) \neq \top\} \cap \{i \in I : x_g(i) \neq \bot\} \in U$. Now define a tuple x in $\prod \mathbf{H_i}$ by $x(i) = x_g(i)$, if $x_g(i) \in H_i$, and $x(i) = 1$ otherwise. Then we have $g = [x_g] = [x] \in \prod H_i/U$, so $\mathbf{G} \in \mathsf{P_U}(\mathcal{K})$.

If $\mathbf{G} \in \mathsf{P_U}(\mathcal{K})$, then there exists an index set I, an ultrafilter U on I and $\mathbf{H}_i \in \mathcal{K}$, $i \in I$, such that $\mathbf{G} = \prod \mathbf{H_i}/U$. Using the same index set I and ultrafilter U on I, we know $\prod \mathbf{M_{H_i}}/U$ is also of the form $\mathbf{M_K}$, where $\mathbf{K}$ is an abelian group. Since $[\overline{\top}] \vee [x] = [\overline{\top} \vee x] = [\overline{\top}]$ and $[\overline{\bot}] \wedge [x] = [\overline{\bot} \wedge x] = [\overline{\bot}]$, we get $[\overline{\top_{\mathbf{M_{H_i}}}}] = \top_{\prod \mathbf{M_{H_i}}/U}$ and $[\overline{\bot_{\mathbf{M_{H_i}}}}] = \bot_{\prod \mathbf{M_{H_i}}/U}$. For $[x] \in K$, we have $[x] \neq [\overline{\top_{\mathbf{M_{H_i}}}}]$ and $[x] \neq [\overline{\bot_{\mathbf{M_{H_i}}}}]$. So $\{i \in I : x(i) \neq \top_{\mathbf{M_{H_i}}}\} \in U$

and $\{i \in I : x(i) \neq \bot_{\mathbf{M_{H_i}}}\} \in U$, hence $\{i \in I : x(i) \in H_i\} = \{i \in I : x(i) \neq \top_{\mathbf{M_{H_i}}}\} \cap \{i \in I : x(i) \neq \bot_{\mathbf{M_{H_i}}}\} \in U$; so $[x] \in \prod H_i/U = \mathbf{G}$ and $K \subseteq G$. Conversely, if $[x] \in \prod H_i/U = \mathbf{G}$ then $[x] \in K$, so $G \subseteq K$. Therefore $\mathbf{M_G} \in \mathsf{P_U}(\{\mathbf{M_H} : \mathbf{H} \in \mathcal{K}\})$.

Again note that to show $\mathsf{S}(\mathbf{M_H}) = \{\mathbf{M_G} : \mathbf{G} \in \mathsf{S}(\mathbf{H})\}$ it is enough to focus on algebras of the form $\mathbf{M_G}$, where $\mathbf{G}$ is an abelian group. If $\mathbf{M_G} \in \mathsf{S}(\mathbf{M_H})$, then for all $x, y \in G$, we have $x \cdot_{\mathbf{G}} y = x \cdot_{\mathbf{M_G}} y = x \cdot_{\mathbf{M_H}} y = x \cdot_{\mathbf{H}} y$ and $x^{-1_{\mathbf{G}}} = x \backslash_{\mathbf{M_G}} 1 = x \backslash_{\mathbf{M_H}} 1 = x^{-1_{\mathbf{H}}}$; so $\mathbf{G} \in \mathsf{S}(\mathbf{H})$. Conversely, if $\mathbf{G} \in \mathsf{S}(\mathbf{H})$, then for all $x, y \in M_G \setminus \{\bot, \top\}$ we have $x \cdot_{\mathbf{M}_G} y = x \cdot_{\mathbf{G}} y = x \cdot_{\mathbf{H}} y = x \cdot_{\mathbf{M}_H} y$, $x \backslash_{\mathbf{M}_G} y = x^{-1_{\mathbf{G}}} \cdot_{\mathbf{G}} y = x^{-1_{\mathbf{H}}} \cdot_{\mathbf{H}} y = x \backslash_{\mathbf{M}_H} y$ and $y /_{\mathbf{M}_G} x = y \cdot_{m_G} x^{-1_{\mathbf{G}}} = y \cdot_{m_H} x^{-1_{\mathbf{H}}} = y /_{\mathbf{M}_H} x$. Also, since $\mathbf{M}_G$ is rigorously compact, the operations on $\mathbf{G}$ and $\mathbf{H}$ also agree if one of x, y is in $\{\bot, \top\}$. So $\mathbf{M_G} \in \mathsf{S}(\mathbf{M_H})$.

Actually, given that every algebra is an ultraproduct of its finitely generated subalgebras, $\mathsf{ISP_U}$-classes of abelian groups are fully determined by their intersection with the class of finitely generated abelian groups. Therefore, we are interested only in such intersections; clearly this correspondence preserves and reflects the order.

By the fundamental theorem of finitely generated abelian groups we know that every finitely generated abelian group is isomorphic to exactly one group of the form

$$\mathbb{Z}^m \times (\mathbb{Z}_{p_1^{n_{1,1}}} \times \cdots \times \mathbb{Z}_{p_1^{n_{1,m_1}}}) \times \cdots \times (\mathbb{Z}_{p_k^{n_{k,1}}} \times \cdots \times \mathbb{Z}_{p_k^{n_{k,m_k}}})$$

for some $m, k, m_1, \ldots, m_k, n_{i,j} \in \mathbb{N}$, where $n_{i,j} \geq n_{i,j+1}$ for all suitable i, j, and $p_1 < p_2 < \cdots < p_k < \ldots$ is the listing of all primes. We denote by $\mathcal{FA}$ the set of all groups of this form; also by $f\mathcal{A}$ we denote all the finite algebras in $\mathcal{FA}$ (i.e., where $m = 0$).

Since $\mathcal{FA}$ is a full set of representatives of the isomorphism classes of finitely generated abelian groups, instead of considering intersections of $\mathsf{ISP_U}$-classes of abelian groups with the class of finitely generated abelian groups, we can instead focus on intersections of $\mathsf{ISP_U}$-classes of abelian groups with $\mathcal{FA}$. In other words, we have established that the subvariety lattice of $\mathsf{CM_G}$ is isomorphic to $\{\mathcal{K} \cap \mathcal{FA} : \mathcal{K}$ is an $\mathsf{ISP_U}$-class of abelian groups$\}$, where the order is given by: $\mathcal{K} \cap \mathcal{FA} \leq \mathcal{L} \cap \mathcal{FA}$ iff $\mathsf{ISP_U}(\mathcal{K} \cap \mathcal{FA}) \subseteq \mathsf{ISP_U}(\mathcal{L} \cap \mathcal{FA})$. In the following, we will write $\mathcal{K}_{\mathcal{FA}}$ for $\mathcal{K} \cap \mathcal{FA}$.

To the abelian group displayed above, we associate the sequence

$$(m; (n_{1,1}, \ldots, n_{1,m_1}, 0, \ldots); \ldots; (n_{k,1}, \ldots, n_{k,m_k}, 0, \ldots); (0, \ldots); \ldots)$$

which is an element of the lattice $\mathbb{N} \times \mathbf{I}^{\oplus\omega}$. Also, note that the bijective correspondence from $\mathcal{FA}$ to $\mathbb{N} \times I^{\oplus\omega}$ is actually a lattice isomorphism between $\mathbb{N} \times \mathbf{I}^{\oplus\omega}$ and $\mathcal{FA}$ under the order given by: $\mathbf{G} \leq_{\mathcal{FA}} \mathbf{H}$ iff $\mathbf{G} \in \mathsf{IS}(\mathbf{H})$.

Now, sets of the form $\mathcal{K}_{\mathcal{FA}}$, where $\mathcal{K}$ is an $\mathsf{ISP_U}$-class of abelian groups, are of course downsets of $\mathcal{FA}$, but unfortunately not all downsets of $\mathcal{FA}$ are of this form. For example, note that for $r, s \in \mathbb{Z}^+$, $\mathbf{G} \in f\mathcal{A}$ and $\mathcal{K}$ an $\mathsf{ISP_U}$-class of abelian groups, we have: $\mathbf{G} \times \mathbb{Z}^r \in \mathcal{K}$ iff $\mathbf{G} \times \mathbb{Z}^s \in \mathcal{K}$. (So, for example $\downarrow\{\mathbb{Z}^2\} = \{\{1\}, \mathbb{Z}^2, \mathbb{Z}\}$ is a downset of $\mathcal{FA}$ that is not of the form $\mathcal{K}_{\mathcal{FA}}$.)

To prove this, it suffices to prove: if $\mathbf{G} \times \mathbb{Z} \in \mathcal{K}$ then $\mathbf{G} \times \mathbb{Z}^t \in \mathcal{K}$ for all $t \in \mathbb{Z}^+$. Let U be a non-principal ultrafilter on $\mathbb{N}$ and consider the elements $a = [\bar{1}]_U$ and $b = [(2, 2^2, 2^3, \ldots)]_U$ of $\mathbb{Z}^{\mathbb{N}}/U$; each has infinite order. Note that for all $m, n \in \mathbb{N}$, the set $\{i \in \mathbb{N} : m \cdot 1 = n \cdot 2^i\}$ contains at most one element. Since U is not principal, we get $\{i \in \mathbb{N} : m \cdot 1 = n \cdot 2^i\} \notin U$, so $ma \neq nb$. Thus $\langle a, b\rangle \cong \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \in \mathsf{P_U}(\mathbb{Z})$. Similarly, to show $\mathbb{Z}^t \in \mathsf{P_U}(\mathbb{Z})$, it suffices to take $a_{p_1} = [(p_1, p_1^2, p_1^3, \ldots)]_U$, $a_{p_2} = [(p_2, p_2^2, p_2^3, \ldots)]_U$, $\ldots$, $a_{p_t} = [(p_t, p_t^2, p_t^3, \ldots)]_U$, where $p_1, p_2, \ldots, p_t$ are distinct primes, and we have $\langle a_{p_1}, \ldots, a_{p_t}\rangle \cong \mathbb{Z}^t$. More generally, we can show $\{\mathbf{G} \times \mathbb{Z}^t : t \in \mathbb{Z}^+\} \subseteq \mathsf{P_U}(\mathbf{G} \times \mathbb{Z})$ for any $\mathbf{G} \in f\mathcal{A}$.

For this reason, it makes sense to identify $\mathbf{G} \times \mathbb{Z}^r$ and $\mathbf{G} \times \mathbb{Z}^s$ whenever r and s are both non-zero. This can be done by considering the subset $\mathcal{FA}' = f\mathcal{A} \cup \{\mathbb{Z} \times \mathbf{G} : \mathbf{G} \in f\mathcal{A}\}$

of $\mathcal{FA}$. The set $\mathcal{FA}'$ also forms a lattice (actually a sublattice of $\mathcal{FA}$) isomorphic to $\mathbf{P} = \mathbf{2} \times \mathbf{I}^{\oplus\omega}$. Therefore, moving through the isomorphism, we can apply the definitions of $\exp$ and primes also to downsets of $\mathcal{FA}'$. To be more specific, a downset D of $\mathcal{FA}'$ is $\mathbb{Z}$-closed if for all $\mathbf{G} \in f\mathcal{A}$, $\exp(D \cap \uparrow\mathbf{G})$ or $\mathrm{primes}(D \cap \uparrow\mathbf{G})$ being unbounded implies that $\mathbb{Z} \times \mathbf{G} \in D$. Also, by the fact established in the last paragraph we have a lattice isomorphism between $\{\mathcal{K}_{\mathcal{FA}} : \mathcal{K} \text{ is a } \mathsf{ISP_U}\text{-class}\}$ and $\{\mathcal{K}_{\mathcal{FA}'} : \mathcal{K} \text{ is a } \mathsf{ISP_U}\text{-class}\}$, where $\mathcal{K}_{\mathcal{FA}'} = \mathcal{K} \cap \mathcal{FA}'$.

Clearly, if $\mathcal{K}$ is an $\mathsf{ISP_U}$-class of abelian groups, then $\mathcal{K}_{\mathcal{FA}'}$ is a downset of $\mathcal{FA}'$. Unfortunately, still not every downset of $\mathcal{FA}'$ is of this form. For example, $\{\mathbb{Z}_p : p \text{ is prime}\}$ is a downset of $\mathcal{FA}'$, but since $\mathbb{Z} \in \mathsf{P_U}(\{\mathbb{Z}_p : p \text{ is prime}\})$, $\{\mathbb{Z}_p : p \text{ is prime}\}$ is not of the form $\mathcal{K}_{\mathcal{FA}'}$. In the following we show that $\{\mathcal{K}_{\mathcal{FA}'} : \mathcal{K} \text{ is an } \mathsf{ISP_U}\text{-class}\}$ is equal to the lattice of $\mathbb{Z}$-closed downsets of $\mathcal{FA}'$.

First we note that for $X \subseteq P$, we have that $\exp(X)$ and $\mathrm{primes}(X)$ are bounded iff there exist $K, N \in \mathbb{N}$ such that for all $a \in X$, $k > K$, $n, m \in \mathbb{N}$, we have $a_k = \overline{0}$ and $a_{n,m} \leq N$. Therefore, for $X \subseteq \mathcal{FA}'$, we have that $\exp(X)$ and $\mathrm{primes}(X)$ are bounded iff there exist $K, N \in \mathbb{N}$ such that the cyclic groups in the decomposition of groups in X are among the $\mathbf{Z}_{p_k^n}$, where $k \leq K$ and $n \leq N$. This is in turn equivalent to asking that there is $M \in \mathbb{N}$ such that all elements in all the finite groups in X have order at most M (by taking $M = (p_1 \cdots p_K)^N$).

Now, for an $\mathsf{ISP_U}$-class $\mathcal{K}$ of abelian groups, $\mathcal{K}_{\mathcal{FA}'}$ is a downset of $\mathcal{FA}'$. To show that it is $\mathbb{Z}$-closed, let $\mathbf{G} \in f\mathcal{A}$. If one of $\exp(\mathcal{K}_{\mathcal{FA}'} \cap \uparrow\mathbf{G})$, $\mathrm{primes}(\mathcal{K}_{\mathcal{FA}'} \cap \uparrow\mathbf{G})$ is unbounded, there is no uniform bound in the order of the elements in the groups from $\mathcal{K}_{\mathcal{FA}'}$; so, there is an infinite subset $\{\mathbf{H}_n : n \in \mathbb{N}\}$ of $\mathcal{K}_{\mathcal{FA}'} \cap \uparrow\mathbf{G}$ such that $\mathbf{H}_n$ contains an element of order greater than n, say h_n. Therefore, the element $[(h_n)]$ in any fixed non-principal ultraproduct $\mathbf{H}$ of $\{\mathbf{H}_n : n \in \mathbb{N}\}$ has infinite order, and consequently $\mathbf{H}$ contains a copy of $\mathbb{Z}$.

On the other hand, note that if $\mathbf{G} = \{g_1, \ldots, g_k\}$, then for every group $\mathbf{A}$ we have $\mathbf{G} \in \mathsf{IS}(\mathbf{A})$ iff $\mathbf{A} \vDash \phi_{\mathbf{G}}$, where $\phi_{\mathbf{G}}$ encodes the multiplication of $\mathbf{G}$: $\exists x_{g_1}, \ldots, x_{g_k} (\bigwedge\{x_{g_i} \neq x_{g_j} : i \neq j\} \wedge \bigwedge\{x_{g_i}x_{g_j} = x_{g_i g_j} : 1 \leq i, j \leq n\})$. Since, for all n, $\mathbf{H}_n$ contains a copy of $\mathbf{G}$, $\mathbf{H}_n$ satisfies $\phi_{\mathbf{G}}$; hence $\mathbf{H}$ also satisfies $\phi_{\mathbf{G}}$ and $\mathbf{H}$ contains a subgroup isomorphic to $\mathbf{G}$. Therefore, $\mathbb{Z} \times \mathbf{G} \in \mathsf{IS}(\mathbf{H}) \subseteq \mathsf{IS}(\mathcal{K}) = \mathcal{K}$ and so $\mathbb{Z} \times \mathbf{G} \in \mathcal{K}_{\mathcal{FA}'}$.

Conversely, for a $\mathbb{Z}$-closed downset D of $\mathcal{FA}'$, we define $\mathcal{K}_D = \mathsf{ISP_U}(D)$ and prove that $\mathcal{K}_D \cap \mathcal{FA}' = D$. Since $D \subseteq \mathcal{K}_D$ and $D \subseteq \mathcal{FA}'$, it suffices to prove $\mathcal{K}_D \cap \mathcal{FA}' \subseteq D$. If $\mathbb{Z}^m \times \mathbf{G} \in \mathcal{K}_D \cap \mathcal{FA}'$, where $m \in \{0, 1\}$ and $\mathbf{G} \in f\mathcal{A}$, then a copy of $\mathbb{Z}^m \times \mathbf{G}$ is contained in the ultraproduct $\prod \mathbf{A}_i/U$ of some $\{\mathbf{A}_i : i \in I\} \subseteq D$. Since $\prod \mathbf{A}_i/U$ contains a copy of $\mathbf{G}$, it satisfies the sentence $\phi_{\mathbf{G}}$, so $I_{\mathbf{G}} := \{i \in I : \mathbf{G} \in \mathsf{IS}(\mathbf{A}_i)\} = \{i \in I : \mathbf{A}_i \vDash \phi_{\mathbf{G}}\} \in U$. If $m = 1$, then $\prod \mathbf{A}_i/U$ contains a copy of $\mathbb{Z}$, so it has an element of infinite order. Therefore, there is no M such that $\prod \mathbf{A}_i/U$ satisfies the sentence $(\forall x)(Mx = 0)$, so there is no M such that $\{\mathbf{A}_i : i \in I_{\mathbf{G}}\}$ satisfy the sentence, so there is no uniform bound on the orders of the elements of $\{\mathbf{A}_i : i \in I_{\mathbf{G}}\}$; thus $\exp(\{\mathbf{A}_i : i \in I_G\})$ or $\mathrm{primes}(\{\mathbf{A}_i : i \in I_{\mathbf{G}}\})$ is unbounded. Since, $\exp(\{\mathbf{A}_i : i \in I_{\mathbf{G}}\}) \subseteq \exp(D \cap \uparrow\mathbf{G})$, $\mathrm{primes}(\{\mathbf{A}_i : i \in I_{\mathbf{G}}\}) \subseteq \mathrm{primes}(D \cap \uparrow\mathbf{G})$ and D is a $\mathbb{Z}$-closed downset, we get $\mathbb{Z}^m \times \mathbf{G} = \mathbb{Z} \times \mathbf{G} \in D$. If $m = 0$, then we also have $\mathbb{Z}^m \times \mathbf{G} = \mathbf{G} \in D$.

Thus the lattice $\{\mathcal{K}_{\mathcal{FA}'} : \mathcal{K}$ is a $\mathsf{ISP_U}$-class$\}$ is isomorphic to $\mathcal{O}_{\mathbb{Z}}(\mathbf{P})$, and hence the lattice $\Lambda(\mathsf{CM_G})$ of subvarieties of $\mathsf{CM_G}$ is isomorphic to the lattice $\mathcal{O}_{\mathbb{Z}}(\mathbf{P})$. $\square$

Corollary 5.3. The variety generated by $\{\mathbf{M}_{\mathbb{Z}_p} : p$ is prime$\}$ has continuum-many subvarieties. Therefore the subvariety lattices of $\mathsf{M_G}$ and of M have size continuum.

Proof. For every prime p, the variety $\mathsf{V}(\mathbf{M}_{\mathbb{Z}_p})$ corresponds to the principal downset of the sequence $(0; 0; \ldots; 0; 1; 0; \ldots)$ in $\mathbf{P}$, where the 1 is at the position of the prime p. The variety generated by all $\mathbf{M}_{\mathbb{Z}_p}$'s is the join of all of the $\mathsf{V}(\mathbf{M}_{\mathbb{Z}_p})$, where p is prime, and corresponds to the $\mathbb{Z}$-closed downset $\overline{P\mathbb{N}} = \{(1; 0; 0; \ldots), (0; 1; 0; \ldots), \ldots, (0; \ldots; 1; \ldots), \ldots\}$

in $\mathbf{P}$. The $\mathbb{Z}$-closed subdownsets of $\overline{P\mathbb{N}}$ in the lattice $\mathcal{O}_\mathbb{Z}(\mathbf{P})$ is clearly isomorphic, as a lattice, to $\mathcal{P}(\mathbb{N})$. □

We denote by $\mathsf{CM}_{\mathbf{G}Z}$ the variety generated by the algebras in M that satisfy the formula

$$x\overline{\top} = x \text{ or } x(x\backslash 1) = 1. \qquad \text{(ZGroup)}$$

Let $\mathbf{F}$ be the poset on $\{0, 1, 2, 3\}$, where $0 < 1, 2, 3$ and $1, 2, 3$ are incomparable. For a downset D of $\mathbf{P} \times \mathbf{F}$ and $i \in F$, we set $D_i = \{a : (a, i) \in D\}$. A downset D of $\mathbf{P} \times \mathbf{F}$ is called $\mathbb{Z}$-closed if D_0, D_1, D_2 and D_3 are $\mathbb{Z}$-closed downsets of $\mathbf{P}$; we denote by $\mathcal{O}_\mathbb{Z}(\mathbf{P} \times \mathbf{F})$ the lattice of all $\mathbb{Z}$-closed downsets of $\mathbf{P} \times \mathbf{F}$.

Theorem 5.4. The subvariety lattice of $\mathsf{CM}_{\mathsf{GZ}}$ is isomorphic to $\mathcal{O}_\mathbb{Z}(\mathbf{P} \times \mathbf{F})$.

Proof. By Theorem 4.5 and Corollary 3.3 the FSI members of $\mathsf{CM}_{\mathsf{GZ}}$ are unilinear residuated lattices of the form $\mathbf{R}$, $\mathbf{R}+1$, $\mathbf{R}+2$ or $\mathbf{R}+3$, where $\mathbf{R} = \mathbf{M_G}$ and $\mathbf{G}$ is an abelian group, $\mathbf{A}$ is the $\top$-cancellative monoid on $G \cup \{\top\}$; $\mathbf{R}+1 = \mathbf{R}_{\mathbf{A},\mathbf{B}_1}$, where $\mathbf{B}_1$ is the $\bot$-semigroup based on $\{\bot, b\}$ given in Figure 3.2 with $b^2 = \bot$; $\mathbf{R}+2 = \mathbf{R}_{\mathbf{A},\mathbf{B}_2}$, where $\mathbf{B}_2$ is the $\bot$-semigroup based on $\{\bot, b_1, b_2\}$ given in Figure 3.2; and $\mathbf{R}+3 = \mathbf{R}_{\mathbf{A},\mathbf{B}_3}$, where $\mathbf{B}_3$ is the $\bot$-semigroup based on $\{\bot, b\}$ given in Figure 3.2 with $b^2 = b$; we define $\mathbf{R}+0 = \mathbf{R}$. Note that $\mathbf{R}$ is a subalgebra of $\mathbf{R}+i$, for all $i \in \{0, 1, 2, 3\}$.

In the proof of Theorem 5.2, we saw that subvarieties of CM_{G} are determined by the $\mathbb{Z}$-closed downsets of $\mathcal{FA}'$. We now sketch how subvarieties of $\mathsf{CM}_{\mathsf{GZ}}$ are determined by the $\mathbb{Z}$-closed downsets of the poset $\mathbf{M}_{\mathcal{FA}'} + \mathbf{F} := \{\mathbf{M_G} + i : \mathbf{G} \in \mathcal{FA}', i \in F\}$, where the order is given by $\mathbf{M_G} + i \leq \mathbf{M_H} + j$ iff $\mathbf{G} \leq_{\mathcal{FA}'} \mathbf{H}$ and $i \leq_\mathbf{F} j$; this poset is clearly isomorphic to $\mathbf{P} \times \mathbf{F}$, so the definition of $\mathbb{Z}$-closed downsets of $\mathbf{P} \times \mathbf{F}$ can be transferred here. More specifically, a downset D of $\mathbf{M}_{\mathcal{FA}'} + \mathbf{F}$ is $\mathbb{Z}$-closed iff for all $0 \leq i \leq 3$, $D \cap (\mathbf{M}_{\mathcal{FA}'} + \{i\})$ is isomorphic to a $\mathbb{Z}$-closed downset of $\mathcal{FA}'$.

Every subvariety $\mathcal{V}$ of $\mathsf{CM_{GZ}}$ is determined by its finitely generated FSI algebras. These are finitely generated algebras of the form $\mathbf{R}$, $\mathbf{R}+1$, $\mathbf{R}+2$ or $\mathbf{R}+3$, where $\mathbf{R} \in (\mathsf{CM_G})_{FSI}$, i.e., $\mathbf{R} = \mathbf{M_G}$, and $\mathbf{G}$ is a finitely generated abelian group. So, $\mathcal{V}_{FSI}$ is a downset of $\mathbf{M}_{\mathcal{FA}'} + \mathbf{F}$.

For $0 \leq i \leq 3$, if $\mathbf{G} \in f\mathcal{A}$ and $\exp(D_i \cap \uparrow G)$ or $\mathrm{primes}(D_i \cap \uparrow G)$ is unbounded, where $D_i = \{\mathbf{K} \in \mathcal{FA}' : \mathbf{M_K} + i \in \mathcal{V}_{FSI}\}$, then by the proof of Theorem 5.2, we have $\mathbb{Z} \times \mathbf{G} \in D_i$. So D_i is a $\mathbb{Z}$-closed downset of $\mathcal{FA}'$ for $0 \leq i \leq 3$ and hence $\mathcal{V}_{FSI}$ is a $\mathbb{Z}$-closed downset of $M_{\mathcal{FA}'} + \mathbf{F}$.

By Corollary 3.3, for every downset D of $\mathbf{M}_{\mathcal{FA}'} + \mathbf{F}$, the ultraproducts of algebras from D are isomorphic to $\mathbf{M_G} + i$, for some $0 \leq i \leq 3$. It can be easily shown that for such ultraproduct $\mathbf{M_G} + i$, $\mathbf{G}$ is an ultraproduct of $\{\mathbf{H} : i \leq_{\mathbf{F}} j, \mathbf{M_H} + j \in D\}$; since D is a downset, actually $\mathbf{G}$ is an ultraproduct of $\{\mathbf{H} : \mathbf{M_H} + i \in D\}$. (Also, conversely, if $\mathbf{G}$ is an ultraproduct of $\{\mathbf{H}_j : j \in J\}$ and $i \in F$, then $\mathbf{M_G} + i$ is isomorphic to an ultraproduct of algebras in the downset $\{\mathbf{M}_{\mathbf{K}_j} + k : j \in J, \mathbf{K}_j \leq_{\mathcal{FA}'} \mathbf{H}_j, k \leq_{\mathbf{F}} i\}$ of $\mathbf{M}_{\mathcal{FA}'} + \mathbf{F}$.) So if D is a $\mathbb{Z}$-closed, then $\mathbf{G} \in D_i$; hence $\mathbf{M_G} + i \in D$. Consequently, we have $\mathsf{ISP_U}(D) \cap (M_{\mathcal{FA}'} + \mathbf{F}) = D$, hence the subvariety lattice of $\mathsf{CM_{GZ}}$ is isomorphic to $\mathcal{O}_{\mathbb{Z}}(\mathbf{P} \times \mathbf{F})$. □

Chapter 6: The finite embeddability property

In this section we establish the finite embeddability property for certain subvarieties of SRL.

Recall that a class $\mathcal{K}$ is said to have the *finite embeddability property* (FEP) if for every algebra $\mathbf{A} \in \mathcal{K}$ and a finite subset B of A, there exists a finite algebra $\mathbf{C} \in \mathcal{K}$ such that the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ induced by B embeds in $\mathbf{C}$.

For varieties axiomatized by a finite set of equations, the valid universal sentences form a recursively enumerable set. Also, if the variety has the FEP, then any universal sentence that is not valid will fail in a finite algebra of the variety. By enumerating these finite algebras (using the finite axiomatizability of the variety) we can thus enumerate the universal sentences that fail in the variety. Therefore, recursively axiomatizable varieties with the FEP have a decidable universal theory; moreover, they are generated as universal classes (thus also as quasivarieties and as varieties) by their finite algebras.

Theorem 6.1. The variety $\mathsf{CM_G}$ has the FEP.

Proof. First note that since the algebras in $\mathsf{CM_G}$ are commutative, the conjugates in the equational basis are not needed, so $\mathsf{CM_G}$ has a finite equational basis.

To prove the FEP for $\mathsf{CM_G}$, we claim that the variety of abelian groups has FEP first. By Theorem 5.1 of [14], an abelian group is subdirectly irreducible if and only if it is a subgroup of a p-cyclic group, i.e., either it is a p^∞-group or a cyclic group of order p^n, where p is a prime. So every finitely generated subdirectly irreducible abelian group is finite. By Corollary 2 in [2] every finitely generated abelian group is residually finite. By

Theorem 1 in [4] this is equivalent to having the FEP, so the variety of abelian groups has the FEP.

Note that the above characterization of the finitely generated subdirectly irreducibles does not extend to algebras in $\mathsf{CM_G}$, since the notion of subdirectly irreducible is different. Nevertheless, we can make use of the FEP for abelian groups.

It suffices to prove the FEP for the subdirectly irreducible algebras in $\mathsf{CM_G}$. Let $\mathbf{G}$ be an abelian group and $\mathbf{B}$ a finite subset of $\mathbf{M_G}$. Without loss of generality, we can assume $\bot, \top \in B$, where $\top$ and $\bot$ denote the bounds of $\mathbf{M_G}$, so $(B, \wedge, \vee)$ is a sublattice of $\mathbf{M}_G$. Then $(B', \cdot, 1)$ is a finite partial subgroup of $\mathbf{G}$, where $B' = B \setminus \{\top, \bot\}$. By the FEP for abelian groups, there exists a finite abelian group $\mathbf{C}'$ such that $(B', \cdot, 1)$ can be embedded into $\mathbf{C}'$; without loss of generality we assume that $B' \subseteq C'$.

We consider the set $C = C' \cup \{\top, \bot\}$ and define an order keeping the elements of C' incomparable and setting $\bot < x < \top$, for all $x \in C'$. Also, we extend the multiplication of $\mathbf{C}'$ by stipulating that $\top$ is absorbing for $C \cup \{\top\}$ and $\bot$ is absorbing for C'. Finally, we define $x \to y = x^{-1} \cdot y$ for $x \in C'$, $\top \to u = \bot = v \to \bot$ for $u \neq \top$ and $v \neq \bot$, and $w \to \top = \top = \bot \to w$, for all w.

Since $(B, \wedge, \vee)$ is a sublattice of $\mathbf{M}_G$ and $B' \subseteq C'$, $(B, \wedge, \vee)$ is a sublattice of $(C, \wedge, \vee)$. For all $x, y \in B'$, if $x \cdot_{\mathbf{B}} y \in B$, then $x \cdot_{\mathbf{B}} y = x \cdot_{\mathbf{B'}} y = x \cdot_{\mathbf{C'}} y = x \cdot_{\mathbf{C}} y$, since $\mathbf{G}$ is closed under multiplication; if $x \to_{\mathbf{B}} y \in B$, then $x^{-1_{\mathbf{B}}} \in B$ and $x \to_{\mathbf{B}} y = x^{-1_{\mathbf{B}}} \cdot_{\mathbf{B}} y = x^{-1_{\mathbf{B'}}} \cdot_{\mathbf{B'}} y = x^{-1_{\mathbf{C'}}} \cdot_{\mathbf{C'}} y = x \to_{\mathbf{C}} y$, since $\mathbf{G}$ is also closed under inverses. Finally, if $x, y \in B$ and $x \in \{\bot, \top\}$ or $y \in \{\bot, \top\}$, then the embedding works since $\bot \to_{\mathbf{M_G}} a = \top = a \to_{\mathbf{M_G}} \top$, $a\bot = \bot = \bot a$ for all $a \in M_G$ and $b \to_{\mathbf{M_G}} \bot = \bot = \top \to_{\mathbf{M_G}} c$, $b\top = \top = \top b$ for all $b \neq \bot$ and $c \neq \top$. □

Corollary 6.2. The universal theory of the variety $\mathsf{CM_G}$ is decidable.

We can actually prove the FEP for many more subvarieties of SRL, unrelated to $\mathsf{GM_G}$, using a construction based on residuated frames.

An equation is called *knotted* if it is of the form $x^m \leq x^n$, where $n \neq m$. Also, we consider the following weak versions of commutativity. For every $n \in \mathbb{Z}^+$ and non-constant *partition* a of $n+1$ (i.e., $a = (a_0, a_1, \ldots, a_n)$, where $a_0 + a_1 + \cdots + a_n = n+1$ and not all a_i's are 1), we consider the $(n+1)$-variable identity (a):

$$xy_1xy_2\cdots y_nx = x^{a_0}y_1x^{a_1}y_2\cdots y_nx^{a_n}.$$

For example, $(2,0)$ is the identity $xyx = xxy$ and $(2,0,1)$ is the identity $xyxzx = xxyzx$. We call all of these identities *weak commutativity* identities.

Theorem 6.3. If a subvariety of SRL is axiomatized by a knotted identity, a weak commutativity identity and any additional (possibly empty) set of equations over $\{\vee, \cdot, 1\}$, then it has the FEP.

Proof. If $\mathcal{V}$ is such a variety, it suffices to prove the FEP for the subdirectly irreducible algebras in $\mathcal{V}$; so it suffices to prove it for unilinear residuated lattices. Let $\mathbf{A}$ be a unilinear residuated lattice in $\mathcal{V}$ and $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$.

Let $\mathbf{W}$ be the submonoid of $\mathbf{A}$ generated by B, $W' = W \times B \times W$ and let $N \subseteq W \times W'$ be defined by: $x\ N\ (y,b,z)$ if $yxz \leq b$, then $\mathbf{W_{A,B}} = (W, W', N, \cdot, 1)$ is a residuated frame in the sense of [9] and the Galois algebra $\mathbf{W_{A,B}}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N}, \cdot_{\gamma_N}, \gamma(\{1\}), \backslash, /)$ is a residuated lattice, where $X \cup_\gamma Y = \gamma(X \cup Y)$, $X \cdot_\gamma Y = \gamma(X \cdot Y)$, $X \backslash Y = \{z \in W : zX \subseteq Y\}$ and $Y/X = \{z \in W : Xz \subseteq Y\}$. Moreover, [9] shows that $\mathbf{W_{A,B}}^+$ satisfies all $\{\vee, \cdot, 1\}$-equations that $\mathbf{A}$ satisfies and that $\mathbf{B}$ embeds in $\mathbf{W_{A,B}}^+$. Also, [1] shows that such $\mathbf{W_{A,B}}^+$ is finite, due to the knotted rule and the weak commutativity. So it suffices to show that it is in SRL; we will show that $\mathbf{W_{A,B}}^+$ is actually unilinear.

Note that for all $(y,b,z) \in W'$, we have $a \in \{(y,b,z)\}^{\triangleleft}$ iff $a\ N\ (y,b,z)$ iff $yaz \leq b$ iff $a \leq y\backslash b/z$. Therefore, $\{(y,b,z)\}^{\triangleleft} = \downarrow (y\backslash b/z)$. By basic properties of Galois connections, every element X of $\gamma_N[\mathcal{P}(W)]$ is an intersection of sets of the form $\{(y,b,z)\}^{\triangleleft}$; actually

$X = \bigcap\{\{w\}^{\triangleleft} : w \in X^{\triangleright}\}$. Therefore, X is an intersection of principal downsets of $\mathbf{A}$. Since $\mathbf{A}$ is unilinear, X is either equal to A itself or a linear downset of A.

Now, let $X, Y \in \gamma_N[\mathcal{P}(W)]$; hence each of them is either equal to A or a linear subset of A. If $X \nsubseteq Y$ and $Y \nsubseteq X$, then none of them equals A, hence they are both linear downsets. Since $X \nsubseteq Y$, there is an $x \in X$ such that $x \notin Y$. Since, $Y \nsubseteq X$, not every element of Y is below x, so there exists $y \in Y$ with $y \nleq x$. Since $x \notin Y$ and Y is a downset, we get $x \nleq y$; therefore in this case $\mathbf{A}$ is not linear. By unilinearity of $\mathbf{A}$, it has a top $\top$ and $\top = x \vee y \in X \cup_\gamma Y$, which is also a downset; hence $X \cup_\gamma Y = A$. Also, if $z \in X \cap Y$, then $z \leq x, y$ and by the unilinearity of $\mathbf{A}$, we get $z = \bot$; so $X \cap Y = \{\bot\}$. Consequently, $\gamma_N[\mathcal{P}(W)]$ is unilinear. □

Note that all knotted identities and all weak commutativity identities are equations over $\{\vee, \cdot, 1\}$. So, the theorem includes cases where multiple knotted and/or multiple weak commutativity equations are included in the axiomatization.

Corollary 6.4. If a subvariety of SCRL is axiomatized by a knotted identity and any finite (possibly empty) set of equations over $\{\vee, \cdot, 1\}$, then its universal theory is decidable.

Proof. As for the case $\mathsf{CM_G}$, the variety SCRL is finitely axiomatizable, and so for these subvarieties, then the results follows. □

Chapter 7: Constructing Compact URLs

A unilinear residuated lattice $\mathbf{R}$ is called *compact* if it is $\top$-*unital* (i.e., it satisfies: $x = \bot$ or $x\top = \top = \top x$) and $R \setminus \{\top, \bot\}$ is closed under multiplication. In other words, non-linear compact URLs are obtained by a partially-ordered monoid $\mathbf{M}$ that is a union of chains by adding bounds that absorb all elements of M. We will provide some constructions of compact URLs, but first we start by giving an axiomatization.

Lemma 7.1. The class of compact URLs is axiomatized by the sentences

$$\forall x\, (x = \bot \text{ or } x\top = \top = \top x) \text{ and } \forall x, y, z\, (x = \top \text{ or } x(y \wedge z) = xy \wedge xz).$$

Proof. By the definition of compactness, it suffices to show that, for every $\top$-unital non-linear unilinear residuated lattice $\mathbf{R}$, the second formula captures the fact that $R \setminus \{\top, \bot\}$ is closed under multiplication. Note that if $a, b \notin \{\top, \bot\}$, then $ab\top = a\top = \top$, so $ab \neq \bot$.

Assume first that $\mathbf{R}$ satisfies the second formula, but there exist $a_1, a_2 \in R \setminus \{\bot, \top\}$ such that $a_1a_2 = \top$. Since $\mathbf{R}$ is not linear, there exists an element a_3 that is incomparable to a_1 or to a_2; without loss of generality, a_3 is incomparable to a_2, so $a_3 \in R \setminus \{\bot, \top\}$. Hence

$$\bot = a_1\bot = a_1(a_2 \wedge a_3) = a_1a_2 \wedge a_1a_3 = \top \wedge a_1a_3 = a_1a_3,$$

a contradiction. Thus $R \setminus \{\top, \bot\}$ is closed under multiplication.

Now assume $R \setminus \{\top, \bot\}$ is closed under multiplication and that $x, y, z \in R$ with $x \neq \top$. If $x = \bot$, then the formula holds, so we assume that $x \neq \bot$. Also, if y and z are comparable, then $x(y \wedge z) = xy \wedge xz$ holds since multiplication preserves the order; so we assume that y and z are incomparable. In this case, $xy \vee xz = x(y \vee z) = x\top = \top$. Since $R \setminus \{\top, \bot\}$ is closed under multiplication, xy and xz are incomparable, hence $x(y \wedge z) = x \cdot \bot = \bot = xy \wedge xz$. $\square$

It follows that an alternative second formula is $\forall x, y, z\, (x = \top$ or $(y \wedge z)x = yx \wedge zx)$.

Corollary 7.2. The variety generated by the class of compact URL is axiomatized by

$$1 = \gamma_1(u\backslash v) \vee \gamma_2(v\backslash u) \vee \gamma_3(x\backslash(u \wedge v)) \vee \gamma_4((u \vee v)\backslash(x(u \vee v) \wedge (u \vee v)x))$$
$$1 = \gamma_5(u\backslash v) \vee \gamma_6(v\backslash u) \vee \gamma_7((u \vee v)\backslash x) \vee \gamma_8((xu \wedge xv)\backslash x(u \wedge v))$$

where γ_1,γ_2, γ_3, γ_4, γ_5, γ_6,γ_7, $\gamma_8 \in \Gamma(Var)$.

Lemma 7.3. If $\mathbf{R}$ is a compact URL, then the comparability relation $\equiv$ on $\mathbf{M}$, where $M = R \setminus \{\bot, \top\}$, is a congruence relation and the quotient monoid $\mathbf{M}/\equiv$ is cancellative. Also, $[1]_\equiv$ defines a totally-ordered submonoid of $\mathbf{M}$.

Proof. That the comparability relation $\equiv$ is a congruence on $\mathbf{M}$ follows from the order-preservation of multiplication and the unilinear order. For the cancellativity of $\mathbf{M}/\equiv$, note that if for $x, y, z \in M$ and $y \parallel z$, we have $\top = x(y \vee z) = xy \vee xz$, and since $\mathbf{M}$ is closed under multiplication, we get $xy \parallel xz$. Finally, $[1]_\equiv$ is a totally-ordered submonoid of $\mathbf{M}$ since $x \equiv 1$ and $y \equiv 1$ implies $xy \equiv 1 \cdot 1 = 1$. $\square$

7.1 From a finite cyclic monoid

We show how to construct a compact URL starting from a finite cyclic monoid.

Given a finite cyclic monoid $\mathbf{M}$ generated by an element a of M, there is a smallest natural number r, called the *index*, such that $a^r = a^{r+s}$ for some positive integer s; the smallest such s then is called the *period*. So $M = \{1, a, \ldots, a^r, \ldots, a^{r+s-1}\}$ and $|M| =$

$r + s$. Note that every natural number $n > r$ can be written as $n = r + ms + k$ for unique $m \in \mathbb{N}$ and $0 \leq k < s$; we define $[n]_r^s := r + k$ for $n \geq r + s$ and $[n]_r^s := n$ for $0 \leq n < r + s$. (We will write $[n]$, when r, s are clear from the context.) Then the multiplication on $\mathbf{M}$ is given by $a^i \cdot a^j = a^{[i+j]_r^s}$.

In particular, $\{a^r, \ldots, a^{r+s-1}\}$ is a subsemigroup of $\mathbf{M}$ and it is a group in its own right with identity element a^t such that $t \equiv 0 \pmod{s}$; so it is isomorphic to $\mathbb{Z}_s$.

We extend the multiplication of $\mathbf{M}$ to the set $R = M \cup \{\bot, \top\}$ by $\bot x = x\bot = \bot$ for all $x \in R$, and $\top x = x\top = \top$ for all $x \neq \bot$. Also we define an order on R by $\bot \leq x \leq \top$ for all $x \in R$ and $a^i \leq a^j$ if and only if $j = i + ns$ for some $n \in \mathbb{N}$, where $0 \leq i, j \leq r + s - 1$; see Figure 7.1(left). It is easy to see that this yields a unilinear lattice order; we denote by $\mathbf{R_M}$ the resulting lattice-ordered monoid.

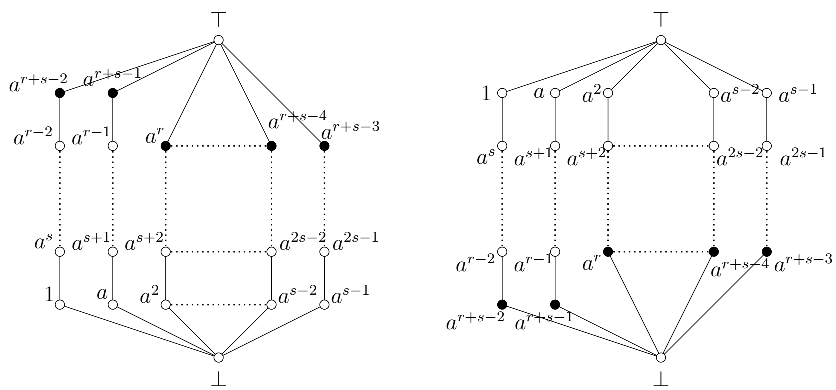

Figure 7.1: The two URLs based on a finite cyclic monoid

Theorem 7.4. If M is a finite cyclic monoid, then $\mathbf{R_M}$ is the reduct of a residuated lattice.

Proof. Since both $\top$ and $\bot$ are zero elements for $\mathbf{M}$ and $\bot\top = \top\bot = \bot$, the associativity of $\mathbf{M}$ easily extends to the associativity of $\mathbf{R_M}$. Since R is finite, by Corollary 2.3, it suffices to show that multiplication distributes over binary joins; we will show distribution from the left: $x(y \vee z) = xy \vee xz$, for all $x, y, z \in R$.

If any of x, y, z is $\top$ or $\bot$, it easy to see that the equation holds, so we assume that $x, y, z \in M$: $x = a^i, y = a^j$ and $z = a^k$ for some $0 \leq i, j, k \leq r + s - 1$. If $y = a^j$

and $z = a^k$ are incomparable, then $j \not\equiv k \,(\mathrm{mod}\, s)$ by definition, so we have $i + j \not\equiv i + k \,(\mathrm{mod}\, s)$ and hence $xy \parallel xz$. Thus we have $x(y \vee z) = x\top = \top = xy \vee xz$. If $a^j = y \leq z = a_k$, we have $k = j + ns$ for some $0 \leq n \leq \lfloor (r + s - 1 - j)/s \rfloor$; we will show that $xy = a^{[i+j]} \leq a^{[i+j+ns]} = xz$. This is true since for $\ell = i + j$, we have $[\ell + ns] = [\ell] + ms$, where $m = n$ if $\ell + ns < r + s$ and $m = ([\ell + ns] - [\ell])/s$ if $\ell + ns \geq r + s$. □

The (commutative) residuated lattice based on $\mathbf{R_M}$ is compact so we have $\bot \to x = \top = x \to \top$, $\top \to y = \bot$, $z \to \bot = \bot$ for all $x \in R_M$, $y \neq \top$, $z \neq \bot$. Also, the remaining implications can be easily calculated to be as follows:

$$a^i \to a^j = \begin{cases} \bot & \text{if } j < i \leq r \text{ or } j < r \leq i \leq r + s - 1 \\ a^{j-i} & \text{if } i \leq j < r \\ a^{j-i+\lfloor \frac{r+s-1+i-j}{s} \rfloor s} & \text{if } i < r \leq j \leq r + s - 1 \\ a^k & \text{if } r \leq i, j, k \leq r + s - 1 \text{ and } a^i a^k = a^j. \end{cases}$$

In particular, the subsemigroup $\{a^r, \ldots, a^{r+s-1}\}$ is closed under implication, but M is not.

It is easy to see that if we impose the dual order on the elements of $\mathbf{M}$ instead, then we can obtain a different unilinear residuated lattice; see Figure 7.1(right). Residuation in this second example works differently:

$$a^i \to a^j = \begin{cases} a^{j-i} & \text{if } i \leq j \leq r + s - 1 \\ a^{j-i+\lceil \frac{i-j}{s} \rceil s} & \text{if } j < i \leq r + s - 1 \end{cases}$$

In this case, M is closed under implication, but $\{a^r, \ldots, a^{r+s-1}\}$ is not.

Remark 7.5. Actually, we can prove that given a finite cyclic monoid M, these are the only two ways where $M \cup \{\bot, \top\}$ is the monoid reduct of a compact unilinear residuated lattice.

Suppose $M \cup \{\bot, \top\}$ is the monoid reduct of a compact URL $\mathbf{R}$. Let a^i and a^j be distinct group elements in M. If $a^i < a^j$, then $e = a^i a^k < a^j a^k$, where e is the identity for the group elements in M and a^k is the inverse of a^i in the group. Then $e < a^j a^k < (a^j a^k)^2 < \cdots$, so M contains an infinite ascending chain, contradicting the fact that M is finite. Thus the group elements in M are pairwise incomparable.

We also observe that given $0 \leq i < j < r + s$,

$$\begin{aligned} a^i < a^j &\text{ iff for all } 0 \leq k \leq i,\ a^{i-k} < a^{j-k} \\ a^j < a^i &\text{ iff for all } 0 \leq k \leq i,\ a^{j-k} < a^{i-k} \end{aligned} \tag{*}$$

The backward direction is trivial, so we just show the forward direction. Given $0 \leq i < j < r + s$ such that $a^i < a^j$ and $0 \leq k \leq i$, if $a^{i-k} \parallel a^{j-k}$, then $\top = a^k \top = a^k(a^{i-k} \vee a^{j-k}) = a^i \vee a^j$, so $a^i \parallel a^j$, a contradiction; if $a^{i-k} > a^{j-k}$, then $a^i > a^j$ since multiplication is order-preserving and a^i is distinct from a^j.

Finally, we know $1 \equiv e$, since otherwise we would have $\top = e(1 \vee e) = e \vee e^2 = e$, a contradiction.

Now let t be the smallest natural number such that $a^t \equiv 1$. If $t = 0$, then by (*), $a^i \parallel a^j$ for all $0 \leq i < j < r + s$; otherwise $1 \equiv a^{j-i}$ where $j - i > 0$, a contradiction. Especially we have $e = 1$ in this case, so M is a group and $\mathbf{R}$ is based on $\mathbf{M}_X$. Now we assume $t > 0$. If $1 < a^t$, then we have $a^r \leq a^{r+t}$ and both of them are group elements in M. Since all group elements are pairwise incomparable, we know $a^r = a^{r+t}$, so $t = s$. Since $s = t$ is the smallest integer such that $1 < a^s$, we know $1 \parallel a^k$ for all $1 < k < s$, thus by (*) $a^k \parallel a^l$ for all $0 \leq k \neq l \leq s - 1$. Since $1 < a^s$, we have $1 < a^s < a^{2s} < \cdots < a^{ms}$, where $ms < r + s \leq (m + 1)s$. Hence $a^i < a^j$ iff $1 < a^{j-i}$ iff $j = i + ns$ for some $n \in \mathbb{Z}^+$, so

$a^i \leq a^j$ iff $j = i + ns$ for some $n \in \mathbb{N}$ and $\mathbf{R}$ is of the form as the left in Figure 7.1(left). Similarly we can prove $\mathbf{R}$ is of the form as the right in Figure 7.1(right) if $a^t < 1$.

7.2 From a semidirect product of a residuated chain and a cancellative monoid; monoid extensions with 2-cocycles

We first provide a general construction of compact residuated lattices and then show that under certain assumptions a compact residuated lattice is exactly of this form.

Let $\mathbf{A}$ be a residuated chain, $\mathbf{K}$ a cancellative monoid and $\varphi : \mathbf{K} \to \mathbf{ResEnd}(\mathbf{A})$ a monoid homomorphism, where $\mathbf{ResEnd}(\mathbf{A})$ is the monoid of residuated maps on the chain $(A, \leq)$ which are also endomorphisms of the monoid $(A, \cdot, 1)$. If φ and ψ are in $\mathbf{ResEnd}(\mathbf{A})$ with residuals φ^* and ψ^* respectively, then $(\psi \circ \varphi)(a) \leq b$ iff $\varphi(a) \leq \psi^*(b)$ iff $a \leq (\varphi^* \circ \psi^*)(b)$ for all $a, b \in A$; so $\psi \circ \varphi$ is also residuated. Thus, $\mathbf{ResEnd}(\mathbf{A})$ is a submonoid of $\mathbf{End}(\mathbf{A})$. Consequently, the semidirect product $\mathbf{A} \rtimes_\varphi \mathbf{K}$ of the monoid reduct of $\mathbf{A}$ and $\mathbf{K}$ with respect to φ is also a monoid with multiplication given by

$$(a_1, k_1) \cdot (a_2, k_2) = (a_1 \varphi_{k_1}(a_2), k_1 k_2),$$

for all $(a_1, k_1), (a_2, k_2) \in A \times K$, and identity $(1_\mathbf{A}, 1_\mathbf{K})$. We define an order on $\mathbf{A} \rtimes_\varphi \mathbf{K}$ by: for all $(a_1, k_1), (a_2, k_2) \in A \times K$,

$$(a_1, k_1) \leq (a_2, k_2) \text{ if and only if } k_1 = k_2 \text{ and } a_1 \leq a_2.$$

Also, we extend the multiplication and order of $\mathbf{A} \rtimes_\varphi \mathbf{K}$ to $R = (A \times K) \cup \{\top, \bot\}$ by: $\bot \leq x \leq \top$, $\bot x = x \bot = \bot$ and $\top y = y \top = \top$ for all $x \in R$, $y \neq \bot$. It is clear that this defines a lattice order; see Figure 7.2. We denote by $\mathbf{A} \rtimes_\varphi^b \mathbf{K}$ the resulting bounded lattice-ordered monoid.

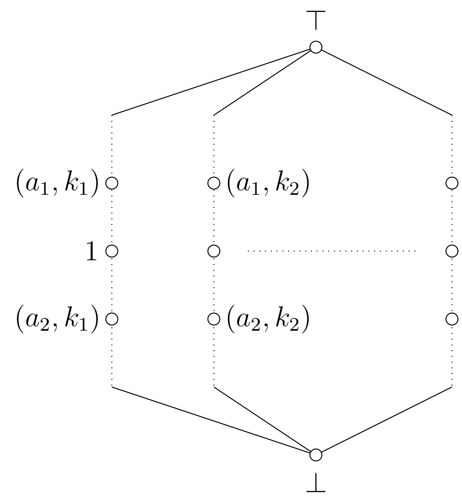

Figure 7.2: A URL based on a semidirect product

Theorem 7.6. If $\mathbf{A}$ is a residuated chain, $\mathbf{K}$ is a cancellative monoid and $\varphi : \mathbf{K} \to \mathbf{ResEnd}(\mathbf{A})$ is a monoid homomorphism, then $\mathbf{A} \rtimes_{\varphi}^{b} \mathbf{K}$ is a residuated lattice.

The proof of the above theorem follows from a more general construction. Given a monoid $\mathbf{K}$, a totally-ordered monoid $\mathbf{A}$ and a map $\varphi : \mathbf{K} \to \mathbf{ResEnd}(\mathbf{A})$, then a function $f : K \times K \to A$ is called a 2-*cocycle* with respect to $\mathbf{K}, \mathbf{A}, \varphi$, if it satisfies the following conditions:

1. $f(k_1, k_2)$ is invertible, for all $k_1, k_2 \in K$.

2. $f(k, 1) = f(1, k) = 1$, for all $k \in K$.

3. $\varphi_{1_{\mathbf{K}}} = \mathrm{id}_{\mathbf{A}}$ and $\varphi_{k_1 k_2}(a) = f(k_1, k_2) \cdot \varphi_{k_1}\varphi_{k_2}(a) \cdot f(k_1, k_2)^{-1}$ for all $k_1, k_2 \in K$ and $a \in A$.

4. $f(k_1, k_2 k_3)\varphi_{k_1}(f(k_2, k_3)) = f(k_1 k_2, k_3) f(k_1, k_2)$, for $k_1, k_2, k_3 \in K$.

Now, given a cancellative monoid $\mathbf{K}$, a residuated chain $\mathbf{A}$, a map φ from $\mathbf{K}$ into $\mathbf{ResEnd}(\mathbf{A})$ and a 2-cocycle $f : K \times K \to A$, we define multiplication on $A \times K$ by

$$(a_1, k_1) \cdot (a_2, k_2) = (a_1 \varphi_{k_1}(a_2) f(k_1, k_2)^{-1}, k_1 k_2)$$

Also, we extend the multiplication to $R = A \times K \cup \{\bot, \top\}$ by making $\bot$ absorbing for R and $\top$ absorbing for $R \setminus \{\bot\}$, and we define a lattice ordering $\leq$ by: for all $a, a_1, a_2 \in A$ and $k, k_1, k_2 \in K$, $\bot = \bot < (a, k) < \top = \top$ and

$$(a_1, k_1) \leq (a_2, k_2) \text{ iff } a_1 \leq_{\mathbf{A}} a_2 \text{ and } k_1 = k_2.$$

We denote the resulting algebra by $\mathbf{R}_{\varphi,f}$.

Theorem 7.7. If $\mathbf{K}$ is a cancellative monoid, $\mathbf{A}$ is a residuated chain, φ is a map from $\mathbf{K}$ into $\mathbf{ResEnd}(\mathbf{A})$, and $f : K \times K \to A$ is a 2-cocycle with respect to $\mathbf{K}$, $\mathbf{A}$ and φ, then $\mathbf{R}_{\varphi,f}$ is the reduct of a residuated lattice.

Proof. In the following we use $\mathbf{R}$ for $\mathbf{R}_{\varphi,f}$ and M for $A \times K$. Clearly, M is closed under multiplication and $(1, 1)$ is the identity. Also,

$$\begin{aligned}
&(a_1, k_1)(a_2, k_2) \cdot (a_3, k_3) \\
=&(a_1\varphi_{k_1}(a_2)f(k_1, k_2)^{-1}, k_1k_2) \cdot (a_3, k_3) \\
=&(a_1\varphi_{k_1}(a_2)f(k_1, k_2)^{-1}\varphi_{k_1k_2}(a_3)f(k_1k_2, k_3)^{-1}, k_1k_2 \cdot k_3) \\
=&(a_1\varphi_{k_1}(a_2)f(k_1, k_2)^{-1} \cdot f(k_1, k_2)\varphi_{k_1}\varphi_{k_2}(a_3)f(k_1, k_2)^{-1} \cdot f(k_1k_2, k_3)^{-1}, \\
&k_1k_2 \cdot k_3) \\
=&(a_1\varphi_{k_1}(a_2)\varphi_{k_1}\varphi_{k_2}(a_3)f(k_1, k_2)^{-1}f(k_1k_2, k_3)^{-1}, k_1k_2 \cdot k_3) \\
=&(a_1\varphi_{k_1}(a_2)\varphi_{k_1}\varphi_{k_2}(a_3)\varphi_{k_1}(f(k_2, k_3)^{-1})f(k_1, k_2k_3)^{-1}, k_1k_2 \cdot k_3) \\
=&(a_1\varphi_{k_1}(a_2\varphi_{k_2}(a_3)f(k_2, k_3)^{-1})f(k_1, k_2k_3)^{-1}, k_1 \cdot k_2k_3) \\
=&(a_1, k_1) \cdot (a_2\varphi_{k_2}(a_3)f(k_2, k_3)^{-1}, k_2k_3) \\
=&(a_1, k_1) \cdot (a_2, k_2)(a_3, k_3)
\end{aligned}$$

where we used the identities

$$\varphi_{k_1k_2}(a) = f(k_1,k_2)\cdot\varphi_{k_1}\varphi_{k_2}(a)\cdot f(k_1,k_2)^{-1}$$
$$f(k_1,k_2k_3)\varphi_{k_1}(f(k_2,k_3)) = f(k_1k_2,k_3)f(k_1,k_2)$$

and the assumption that φ_k is an endomorphism. Therefore $\mathbf{M} = (M,\cdot,(1,1))$ is a monoid. Since both $\top$ and $\bot$ are absorbing elements for $\mathbf{M}$ and $\top\bot = \bot\top = \bot$, associativity holds on $\mathbf{R}$.

We now prove that multiplication is order-preserving: $y \leq z \implies (xy \leq xz$ and $yx \leq zx)$ for all $x,y,z \in R$. If $y = z$ or x,y,z is $\bot$ or $\top$, then it it easy to see that the implication holds; so we assume that $\bot < x < \top$ and $\bot < y < z < \top$. Also, we assume that $x = (a_1,k_1)$, $y = (a_2,k_2)$ and $z = (a_3,k_2)$ with $a_2 < a_3$. Using the order preservation of φ_{k_1} (it is a residuated map) and of multiplication in $\mathbf{A}$, we get

$$\begin{aligned}
(a_1,k_1)(a_2,k_2) &= (a_1\varphi_{k_1}(a_2)f(k_1,k_2)^{-1},k_1k_2)\\
&\leq (a_1\varphi_{k_1}(a_3)f(k_1,k_2)^{-1},k_1k_2)\\
&= (a_1,k_1)(a_3,k_2)\\
(a_2,k_2)(a_1,k_1) &= (a_2\varphi_{k_2}(a_1)f(k_2,k_1)^{-1},k_2k_1)\\
&\leq (a_3\varphi_{k_2}(a_1)f(k_2,k_1)^{-1},k_2k_1)\\
&= (a_3,k_2)(a_1,k_1)
\end{aligned}$$

Next we show that the sets $x\backslash\!\backslash z$ and $z/\!\!/x$ have maximum elements for all $x,z \in R$. By Remark 2.4, we know $\bot\backslash\!\backslash z = z/\!\!/\bot = x\backslash\!\backslash\top = \top/\!\!/x = R$ for all $x,z \in R$, so the maximum element of all of these sets is $\top$. Also, by construction, $x\backslash\!\backslash\bot = \bot/\!\!/x = \top\backslash\!\backslash z = z/\!\!/\top = \{\bot\}$ for all $x \in R\setminus\{\bot\}$ and $z \in R\setminus\{\top\}$, so the maximum for all these sets is $\bot$. We now assume that $\bot < x, z < \top$ and that $x = (a, k)$ and $z = (a', k')$ for some $(a, k), (a', k') \in A \times K$.

For all $(a_1, k_1), (a_2, k_2) \in x \backslash\backslash z$, we have $(a\varphi_k(a_1)f(k, k_1)^{-1}, kk_1) = (a, k)(a_1, k_1) \leq (a', k')$ and $(a\varphi_k(a_2)f(k, k_2)^{-1}, kk_2) = (a, k)(a_2, k_2) \leq (a', k')$, so $kk_1 = k' = kk_2$ and $k_1 = k_2$, by the cancellativity of $\mathbf{K}$. Since, $\mathbf{A}$ is a chain, we get that (a_1, k_1) and (a_2, k_2) are comparable; hence $x \backslash\backslash z$ is a chain.

For all (a'', k''), we have that $(a'', k'') \in x \backslash\backslash z$ if and only if $(a, k)(a'', k'') \leq (a', k')$ if and only if $(a\varphi_k(a'')f(k, k'')^{-1}, kk'') \leq (a', k')$ if and only if $a\varphi_k(a'')f(k, k'')^{-1} \leq a'$ and $k' = kk''$. Since multiplication is residuated, φ_k is residuated, say with residual φ_k^*, and $f(k, k'')$ is invertible, we have: $a\varphi_k(a'')f(k, k'')^{-1} \leq a'$ if and only if $\varphi_k(a'') \leq a \backslash_{\mathbf{A}} a' f(k, k'')$ if and only if $a'' \leq \varphi_k^*(a \backslash_{\mathbf{A}} a' f(k, k''))$. Therefore, we have $(a'', k'') \in x \backslash\backslash z$ if and only if $(a'', k'') \leq (\varphi_k^*(a \backslash_{\mathbf{A}} a' f(k, k'')), k'')$. Consequently, $\max(x \backslash\backslash z)$ exists and it is one of the elements $\bot, (\varphi_k^*(a \backslash_{\mathbf{A}} a' f(k, k'')), k''), \top$. Likewise, $\max(z /\!/ x)$ is one of the elements $\bot, (a' f(k'', k)/_{\mathbf{A}} \varphi_{k''}(a), k''), \top$. By Corollary 2.2, $\mathbf{R}_{\varphi,f}$ is the reduct of a compact residuated lattice. □

So $\mathbf{R}_{\varphi,f}$ is the reduct of a compact residuated lattice, which we will also denote by $\mathbf{R}_{\varphi,f}$ and whose divisions are given by

$$x \backslash y = \begin{cases} \bot & \text{if } x = (a_1, k_1), y = (a_2, k_2) \text{ and } k_2 \notin k_1 K \\ (\varphi_{k_1}^*(a_1 \backslash_{\mathbf{A}} a_2 f(k_1, k)), k) & \text{if } x = (a_1, k_1), y = (a_2, k_1 k) \end{cases}$$

$$y / x = \begin{cases} \bot & \text{if } x = (a_1, k_1), y = (a_2, k_2) \text{ and } k_2 \notin K k_1 \\ (a_2 f(k, k_1)/_{\mathbf{A}} \varphi_k(a_1), k) & \text{if } x = (a_1, k_1), y = (a_2, k k_1) \end{cases}$$

and the standard divisions involving $\bot$ and $\top$ are given by Remark 2.4.

Theorem 7.6 follows as the special case where the 2-cocycle is trivial, thus implying that φ is a monoid homomorphism.

Corollary 7.8. If $\mathbf{K}$ is a cancellative monoid, $\mathbf{A}$ is a residuated chain, φ is a map from $\mathbf{K}$ into $\mathbf{ResEnd}(\mathbf{A})$ and $f : K \times K \to A$ is the trivial 2-cocycle with respect to $\mathbf{K}$, $\mathbf{A}$ and φ, then φ is a homomorphism and $\mathbf{R}_{\varphi,f} = \mathbf{A} \rtimes^b_\varphi \mathbf{K}$.

In particular, when φ is trivial we get $\mathbf{A} \times^b \mathbf{K}$, where $\mathbf{A}$ is a residuated chain and $\mathbf{K}$ is a cancellative monoid.

Note that the examples of section 7.1 are not embeddable into a residuated lattice of the form $\mathbf{A} \times^b \mathbf{K}$. For example, consider the URL $\mathbf{R}$ where $R = \{\bot, 1, a, a^2, \top\}$ with $a^3 = a$ and $1 < a^2$. If $\mathbf{R}$ were embeddable then we would have $1 \mapsto (1, 1)$, $a \mapsto (a_1, k)$, $a^2 \mapsto (a_1^2, k^2)$ and $a^3 \mapsto (a_1^3, k^3)$. So, $(1, 1) < (a_1^2, k^2)$ implies $1 < a_1^2$ and $k^2 = 1$; thus $1 < a_1$ and $k = 1$. But then $a_1 \leq a_1^2 \leq a_1^3 = a_1$, so $a_1^2 = a_1$, hence $(a_1^2, k^2) = (a_1, k)$, a contradiction.

Even though not all compact URLs are of the form $\mathbf{R}_{\varphi,f}$, we show that this holds when the comparability relation on $R \setminus \{\bot, \top\}$ is an *admissible* congruence and the chain of 1 is *cancellative with respect to* the factor monoid.

We say that the congruence $\equiv$ on $\mathbf{M}$ is *admissible* if $x[1]_\equiv = [x]_\equiv = [1]_\equiv x$, for all $x \in M$. Also, we say $\mathbf{H}$ is $\mathbf{K}$-*cancellative* if there exists a selection of representatives $^- : K \to M$ (i.e., for all $x \in M$, if $\overline{k} \equiv x$ then $x \in k$) satisfying $\overline{1_{\mathbf{K}}} = 1_{\mathbf{M}}$ and the left and right multiplications by $\overline{k}$ are injective on H. The terminology $\mathbf{K}$-cancellative and 2-cocycle come from [13].

Proposition 7.9. If $\mathbf{R}$ is a compact unilinear residuated lattice, the comparability relation $\equiv$ is an admissible congruence of $\mathbf{M}$, where $M = R \setminus \{\bot, \top\}$, and $\mathbf{H}$ is $\mathbf{K}$-cancellative, where $H = [1]_\equiv$ and $\mathbf{K} = \mathbf{M}/\equiv$, then $\mathbf{R} \cong \mathbf{R}_{\varphi,f}$ for some map $\varphi : K \to \mathbf{ResAut}(\mathbf{H})$ and 2-cocycle $f : K \times K \to H$ with respect to $\mathbf{H}$, $\mathbf{K}$ and φ.

Proof. Since $\mathbf{H}$ is $\mathbf{K}$-cancellative, there exists a selection of representatives $^{-} : K \to M$. We denote by L_x and R_x the left and right multiplication by $x \in M$, respectively. We know that for all $k \in K$, the maps $R_{\overline{k}}, L_{\overline{k}} : H \to k$ are injective and since $\equiv$ is an admissible congruence on $\mathbf{M}$ and $H = [1]_{\equiv}$, they are also surjective. So, for any $k \in K$, the map $\varphi_k : \mathbf{H} \to \mathbf{H}$ given by $\varphi_k(h) = R_{\overline{k}}^{-1} L_{\overline{k}}(h)$ is a well-defined bijection on H; hence $\overline{k}h = \varphi_k(h)\overline{k}$.

Note that

$$\begin{aligned}\varphi_k(h_1h_2)\overline{k} &= \overline{k} \cdot h_1h_2\\ &= \overline{k}h_1 \cdot h_2\\ &= \varphi_k(h_1)\overline{k} \cdot h_2\\ &= \varphi_k(h_1) \cdot \overline{k}h_2\\ &= \varphi_k(h_1) \cdot \varphi_k(h_2)\overline{k}\\ &= \varphi_k(h_1)\varphi_k(h_2) \cdot \overline{k}.\end{aligned}$$

Since $\mathbf{H}$ is $\mathbf{K}$-cancellative, we have $\varphi_k(h_1h_2) = \varphi_k(h_1)\varphi_k(h_2)$. Now suppose $h_1 \leq h_2$ for some $h_1, h_2 \in H$. Since $\mathbf{R}$ is residuated, we get

$$\begin{aligned}\varphi_k(h_1) \leq \varphi_k(h_2) &\text{ iff } R_{\overline{k}}^{-1} L_{\overline{k}}(h_1) \leq R_{\overline{k}}^{-1} L_{\overline{k}}(h_2)\\ &\text{ iff } R_{\overline{k}}(R_{\overline{k}}^{-1} L_{\overline{k}}(h_1)) \leq L_{\overline{k}}(h_2)\\ &\text{ iff } L_{\overline{k}}(h_1) \leq L_{\overline{k}}(h_2).\end{aligned}$$

It follows from the order-preservation of $L_{\overline{k}}$ that φ_k is order-preserving. So φ_k is an automorphism of the totally-ordered monoid $\mathbf{H}$, and $\overline{1_{\mathbf{K}}} = 1_{\mathbf{H}}$ yields $\varphi_1 = \mathrm{id}_{\mathbf{H}}$.

Since $\equiv$ is admissible on $\mathbf{M}$ and $\mathbf{K} = \mathbf{M}/\equiv$, we have

$$\mathbf{H}\overline{k_1 k_2} = k_1 k_2 = \mathbf{H}\overline{k_1}\mathbf{H}\overline{k_2} = \mathbf{H}\overline{k_1}\,\overline{k_2}.$$

Therefore there exist $f(k_1, k_2)$ and $g(k_1, k_2)$ in H such that

$$\overline{k_1 k_2} = f(k_1, k_2)\overline{k_1}\,\overline{k_2}, \qquad \overline{k_1}\,\overline{k_2} = g(k_1, k_2)\overline{k_1 k_2}$$

for all $k_1, k_2 \in K$. Since $\mathbf{H}$ is $\mathbf{K}$-cancellative, it follows that f and g are well-defined functions from $K \times K$ to H. Moreover, since $f(k_1, k_2)g(k_1, k_2) = g(k_1, k_2)f(k_1 k_2) = 1$ for all $k_1, k_2 \in K$, we get that $f(k_1, k_2)$ and $g(k_1, k_2)$ are invertible. By definition, we have

$$\overline{k_2} = f(1_{\mathbf{K}}, k_2)\overline{1_{\mathbf{K}}}\,\overline{k_2}, \qquad \overline{k_1} = f(k_1, 1_{\mathbf{K}})\overline{k_1}\,\overline{1_{\mathbf{K}}}.$$

Again by the $\mathbf{K}$-cancellativity of $\mathbf{H}$, we get $f(1_{\mathbf{K}}, k) = f(k, 1_{\mathbf{K}}) = 1_{\mathbf{H}}$ for all $k \in K$. Also, by the definition of f, we know

$$L_{\overline{k_1 k_2}} = L_{f(k_1,k_2)} L_{\overline{k_1}} L_{\overline{k_2}}, \qquad R_{\overline{k_1 k_2}} = R_{\overline{k_2}} R_{\overline{k_1}} R_{f(k_1,k_2)}.$$

Thus by the $\mathbf{K}$-cancellativity of $\mathbf{H}$ we have

$$\begin{aligned}
\varphi_{k_1 k_2} &= R^{-1}_{\overline{k_1 k_2}} L_{\overline{k_1 k_2}} \\
&= R^{-1}_{f(k_1,k_2)} R^{-1}_{\overline{k_1}} R^{-1}_{\overline{k_2}} L_{f(k_1,k_2)} L_{\overline{k_1}} L_{\overline{k_2}} \\
&= R^{-1}_{f(k_1,k_2)} L_{f(k_1,k_2)} R^{-1}_{\overline{k_1}} L_{\overline{k_1}} R^{-1}_{\overline{k_2}} L_{\overline{k_2}} \\
&= R^{-1}_{f(k_1,k_2)} L_{f(k_1,k_2)} \varphi_{k_1} \varphi_{k_2}
\end{aligned}$$

for all $k_1, k_2 \in K$. So we get

$$\varphi_{k_1k_2}(h) = f(k_1, k_2) \cdot_{\mathbf{H}} \varphi_{k_1}\varphi_{k_2}(h) \cdot_{\mathbf{H}} f(k_1, k_2)^{-1}$$

for all $h \in H$.

Finally, we observe that

$$\begin{aligned}
&\overline{k_1 \cdot k_2k_3} = \overline{k_1k_2 \cdot k_3} \\
\text{iff } &f(k_1, k_2k_3)\overline{k_1}\,\overline{k_2k_3} = f(k_1k_2, k_3)\overline{k_1k_2}\,\overline{k_3} \\
\text{iff } &f(k_1, k_2k_3)\overline{k_1}f(k_2, k_3)\overline{k_2}\,\overline{k_3} = f(k_1k_2, k_3)f(k_1, k_2)\overline{k_1}\,\overline{k_2} \cdot \overline{k_3} \\
\text{iff } &f(k_1, k_2k_3)\varphi_{k_1}(f(k_2, k_3))\overline{k_1} \cdot \overline{k_2}\,\overline{k_3} = f(k_1k_2, k_3)f(k_1, k_2)\overline{k_1}\,\overline{k_2} \cdot \overline{k_3}.
\end{aligned}$$

So by the associativity of $\mathbf{K}$ and the $\mathbf{K}$-cancellativity of $\mathbf{H}$, we get

$$f(k_1, k_2k_3)\varphi_{k_1}(f(k_2, k_3)) = f(k_1k_2, k_3)f(k_1, k_2)$$

for all $k_1, k_2, k_3 \in K$. Therefore f is a 2-cocycle with respect to $\mathbf{H}$, $\mathbf{K}$ and φ.

Finally, we define the map $\psi : \mathbf{R} \to \mathbf{R}_{\varphi,f}$, given by $\psi(\bot) = \bot$, $\psi(\top) = \top$ and $\psi(x) = (h_x, k_x)$, where $k_x = [x]_{\equiv}$ is the chain to which x belongs and $h_x = R^{-1}_{\overline{k_x}}(x)$. Since $\equiv$ is admissible, $\mathbf{H}$ is $\mathbf{K}$-cancellative and H is totally-ordered, $L_{\overline{k_x}}$ and $R_{\overline{k_x}}$ are order isomorphisms between the sets H and k_x, so ψ is well-defined. We will show that ψ is a residuated-lattice isomorphism.

Suppose $\psi(x) = \psi(y)$ for some $x, y \in M$. Then $k_x = k_y$ and $h_x = h_y$, i.e., $x \equiv y$ and $R^{-1}_{\overline{k_x}}(x) = R^{-1}_{\overline{k_y}}(y)$. Since $R_{\overline{k_x}} = R_{\overline{k_y}}$ is a bijection between H and k_x, we have $x = y$. For $(h, k) \in H \times K$, let $x = h\overline{k}$. Since $R_{\overline{k}}$ is a bijection, we know $h = R^{-1}_{\overline{k}}(x)$, so $\psi(x) = (h, k)$. Since $\psi(\bot) = \bot$ and $\psi(\top) = \top$ are uniquely defined, ψ is a bijection between R and $R_{\varphi,f}$.

Since $R_{\overline{k_x}}$ is an order isomorphism between $\mathbf{H}$ and the chain k_x, $x \leq_{\mathbf{R}} y$ iff $k_x = k_y$ and $R^{-1}_{\overline{k_x}}(x) \leq_{\mathbf{H}} R^{-1}_{\overline{k_y}}(y)$, hence $x \leq_{\mathbf{R}} y$ iff $\psi(x) \leq_{\mathbf{R}_{\varphi,f}} \psi(y)$ for all $x, y \in M$. Since $\psi(\bot) = \bot$ and $\psi(\top) = \top$, ψ is a lattice isomorphism between $\mathbf{R}$ and $\mathbf{R}_{\varphi,f}$.

Since $k_{xy} = k_x \cdot_{\mathbf{K}} k_y$, we have $\overline{k_{xy}} = \overline{k_x k_y} = f(k_x, k_y)\overline{k_x}\,\overline{k_y}$, so for all $x, y \in M$

$$\psi(xy) = (R^{-1}_{\overline{k_{xy}}}(xy), k_{xy}) = (R^{-1}_{\overline{k_x}} R^{-1}_{\overline{k_y}}(xy) f^{-1}(k_x, k_y), k_x k_y).$$

On the other hand,

$$\begin{aligned}
\psi(x)\psi(y) =& (R^{-1}_{\overline{k_x}}(x), k_x)(R^{-1}_{\overline{k_y}}(y), k_y) \\
=& (R^{-1}_{\overline{k_x}}(x)\varphi_{k_x}(R^{-1}_{\overline{k_y}}(y)) f^{-1}(k_x, k_y), k_x k_y) \\
=& (\varphi_{k_x}(L^{-1}_{\overline{k_x}}(x))\varphi_{k_x}(R^{-1}_{\overline{k_y}}(y)) f^{-1}(k_x, k_y), k_x k_y) \\
=& (\varphi_{k_x}(L^{-1}_{\overline{k_x}}(x) R^{-1}_{\overline{k_y}}(y)) f^{-1}(k_x, k_y), k_x k_y)
\end{aligned}$$

Since

$$xy = L_{\overline{k_x}} L^{-1}_{\overline{k_x}}(x) \cdot R_{\overline{k_y}} R^{-1}_{\overline{k_y}}(y) = R_{\overline{k_y}} L_{\overline{k_x}}(L^{-1}_{\overline{k_x}}(x) R^{-1}_{\overline{k_y}}(y)),$$

we have

$$L^{-1}_{\overline{k_x}}(x) R^{-1}_{\overline{k_y}}(y) = L^{-1}_{\overline{k_x}} R^{-1}_{\overline{k_y}}(xy).$$

So

$$R^{-1}_{\overline{k_x}} R^{-1}_{\overline{k_y}}(xy) = R^{-1}_{\overline{k_x}}(L_{\overline{k_x}} L^{-1}_{\overline{k_x}}) R^{-1}_{\overline{k_y}}(xy) = \varphi_{k_x}(L^{-1}_{\overline{k_x}} R^{-1}_{\overline{k_y}}(xy)) = \varphi_{k_x}(L^{-1}_{\overline{k_x}}(x) R^{-1}_{\overline{k_y}}(y)),$$

hence

$$\psi(xy) = \psi(x)\psi(y).$$

Since $\mathbf{R}$ is compact, we know $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in R$. So ψ is a lattice-ordered monoid isomorphism. Since both of $\mathbf{R}$ and $\mathbf{R}_{\varphi,f}$ are residuated lattices, ψ is a lattice and monoid isomorphism, and the divisions are definable by the order and multiplication, we get that ψ is a residuated-lattice isomorphism. $\square$

Chapter 8: Unilinear Residuated Lattices

We will now undertake a classification of unilinear residuated lattices. We will identify four natural classes of URLs and show that these classes cover all URLs. We will provide axiomatizations for these classes and, via a series of constructions, show that algebras in three of these classes can be constructed from algebras in the fourth one.

Recall that a (bounded) URL is called $\top$-central if it satisfies

$$\forall x\,(\overline{\top}x = x\overline{\top}) \qquad (\top\text{-central})$$

which is short for the formula

$$\forall u_1, u_2, x\,(u_1 \leq u_2 \text{ or } u_2 \leq u_1 \text{ or } (u_1 \vee u_2)x = x(u_1 \vee u_2)).$$

By Lemma 4.9(3), $\top$-centrality imposes no restriction on the linear models, so we will focus only on the non-linear models. Also, by Lemma 4.9(2) for these non-linear models there is no real distinction between including the bounds in the language or not.

The following result shows all non-linear unilinear residuated lattices are $\top$-central.

Proposition 8.1. Every non-linear (bounded) unilinear residuated lattice is $\top$-central.

Proof. Let $\mathbf{R}$ be a non-linear unilinear residuated lattice. First we observe that $\top$ is central in $\mathbf{R}$ iff $\top x = x\top$ for all $\bot < x < 1$.

To prove this, we see that if $x \geq 1$, then $\top = \top 1 \leq \top x$ and $\top = 1\top \leq x\top$, so $\top x = \top = x\top$. Also, we know that $\bot\top = \top\bot = \bot$. Finally, for $x \parallel 1$, we have $\top x = (1 \vee x)x = x \vee x^2 = x(1 \vee x) = x\top$.

Since $\mathbf{R}$ is non-linear, there exists $d \in R$ such that $d \parallel 1$. Now let $\bot < x < 1$, then we know $dx \leq d$ by order-preserving of multiplication. If $dx = \bot$, then we have $\top x = (1 \vee d)x = x \vee dx = x$; if $dx > \bot$, then we have $dx \equiv d \parallel 1$ and $\top x = (1 \vee d)x = x \vee dx = \top$. So if $\bot < x < 1$, either $\top x = x$ or $\top x = \top$; similarly we can show if $\bot < x < 1$, either $x\top = x$ or $x\top = \top$.

Suppose there exists $\bot < x, y < 1$ such that $\top x = x$ and $\top y = y$, then by $x = \top x = (1 \vee d)x = x \vee dx$ we have $dx \leq x$. Since $dx \leq d$ and $d \parallel x$, we know $dx = \bot$ by unilinearity. Likewise we can show $dy = \bot$. Now we have $x = \top x = (y \vee d)x = yx \vee dx = yx$ and $y = \top y = (x \vee d)y = xy \vee dy = xy$. Since $x, y < 1$, we get $x = yx \leq y \cdot 1 = y$ and $y = xy \leq x \cdot 1 = x$, so $x = y$, i.e., if there exists such element x that $\bot < x < 1$ and $\top x = x$, then it's unique. Similarly we can show if there exists such element x that $\bot < x < 1$ and $x\top = x$, then it's unique.

Finally, let x be the unique element such that $\bot < x < 1$ and $\top x = x$, then we know $dx = \bot$. By above proof we also know either $x\top = \top$ or $x\top = x$. By way of contradiction, assume that $x\top = \top$, then $\bot = \bot\top = dx \cdot \top = d \cdot x\top = d\top \geq d$, a contradiction since $d \parallel 1$. Thus $x\top = x$ and hence x is also the unique such element that $\bot < x < 1$ and $x\top = x$.

Therefore for all $\bot < x < 1$, either $\top x = x\top = \top$ or $\top x = x\top = x$, and the latter case is unique. $\square$

8.1 Properties of non-linear unilinear residuated lattices

We focus on (bounded) unilinear residuated lattices whose lattice reduct is *non-linear*. Here we use the notation $a \equiv b$ to indicate that two elements a and b are comparable, i.e., that $a \leq b$ or $b \leq a$.

Given a non-linear unilinear residuated lattice $\mathbf{R}$, we define

$$U_R = U := \{x \in R \setminus \{\top, \bot\} \mid \top x = \top\}$$

$$Z_R = Z := \{x \in R \setminus \{\top, \bot\} \mid \top x = x\}$$

$$W_R = W := \{x \in R \setminus \{\top, \bot\} \mid x < \top x < \top\}.$$

We will be dropping the subscript R, when it is clear from the context. Note that $U \sqcup Z \sqcup W = R \setminus \{\top, \bot\}$.

The following result follows from [11].

Proposition 8.2. In any residuated lattice $\mathbf{R}$ with top $\top$, the set $R_\top = Z \cup \{\top, \bot\}$ is a subalgebra of $\mathbf{R}$ with respect to all operations other than 1, and $\top$ is a multiplicative identity for $R_\top$. Hence, $\mathbf{R}_\top$ is an integral residuated lattice. If $\mathbf{R}$ is unilinear, then also $\mathbf{R}_\top$ is unilinear.

An element a of a residuated lattice is called *invertible* if there is an element b such $ab = ba = 1$. The following theorem provides useful insight in the structure of unilinear residuated lattices.

Theorem 8.3. Let $\mathbf{R}$ be a non-linear unilinear residuated lattice.

1. $U \cup \{\top\}$, $Z \cup \{\bot, \top\}$ and $Z \cup \{\bot\}$ are closed under multiplication.

   Furthermore, $ab = ba = b$ for all $a \in U, b \in Z$.

2. Z is either a chain or a 2-element antichain. Also, if $Z = \{b_1, b_2\}$ is a 2-element antichain, then $b_1^2 = b_1$, $b_2^2 = b_2$ and $b_1 b_2 = b_2 b_1 = \bot$. In this case, $Z \cup \{\bot, \top\}$ is a 1-free subalgebra of $\mathbf{R}$ and isomorphic to a 4-element Boolean algebra.

3. There is at most one $b \in Z$ with $b \equiv 1$. If there is such an element b, then $bx = xb = \bot$ for all $x \parallel 1$ and $U = {\uparrow} b \setminus \{b, \top\}$. If there is no such b, then $a \parallel b'$ for all $a \in U$ and $b' \in Z$.

4. The chain of 1 (i.e., $\updownarrow 1$) is disjoint from W.

5. W forms a chain and $c < b_0 := c\top = cb_0 = b_0c = cc' \in Z$, for all $c, c' \in W$. In other words, $W = {\downarrow} b_0 \setminus \{\bot, b_0\}$. Also, if $W \neq \emptyset$, then and $(W \cup \{b_0\}, \cdot, \leq)$ is a totally-ordered null semigroup with zero element b_0.

   Moreover, $xc, cx \leq b_0$ for all $x \in R$, $c \in W$; in particular, $xc = cx = b_0$ when $x \parallel 1$.

6. For all $a \in U$ and $c \in W$, we have $a \parallel c$.

7. If $W \neq \emptyset$, then the set $\{b \in Z : b \parallel 1\} \cup \{\top\}$ equals ${\uparrow} b_0$ and is closed under multiplication and divisions.

8. If $W \neq \emptyset$, then all invertible elements are comparable with 1.

Proof. Below, we prove the statements in a convenient order.

(1) Note that since $\top\top = \top$ and $\top\bot = \bot$, we have $U \cup \{\top\} = \{x \in R : \top x = \top\}$ and $Z \cup \{\bot, \top\} = \{x \in R : \top x = x\}$. If $a_1, a_2 \in U \cup \{\top\}$, then $\top \cdot a_1a_2 = \top a_1 \cdot a_2 = \top a_2 = \top$, so $a_1a_2 \in U \cup \{\top\}$. Also if $b_1, b_2 \in Z \cup \{\bot, \top\}$, then $\top \cdot b_1b_2 = \top b_1 \cdot b_2 = b_1b_2$, hence $b_1b_2 \in Z \cup \{\bot, \top\}$. Finally, if $b_1, b_2 \in Z \cup \{\bot\}$, then $b_1b_2 \in Z \cup \{\bot, \top\}$ and $b_1b_2 \leq b_1\top = b_1 < \top$, so $b_1b_2 \in Z \cup \{\bot\}$.

Now, for $a \in U$ and $b \in Z$, using the centrality of $\top$, we have $ab = a \cdot b\top = ab \cdot \top = \top \cdot ab = \top a \cdot b = \top b = b$. Similarly we show that $ba = b$.

(4) If $x \geq 1$, then $\top x \geq \top \cdot 1 = \top$, so $x \in U \cup \{\top\}$. If $x < 1$, then $dx \leq d$, where d is some fixed element in $R \setminus \{\bot, \top\}$, incomparable to 1; such a d exists, since R is non-linear. If $dx = \bot$, then $\top x = (1 \vee d)x = x \vee dx = x \vee \bot = x$, so $x \in Z \cup \{\bot\}$. If $dx \neq \bot$, then since $dx \leq d$, we get that dx is incomparable with 1 and so also with x. In this case $\top x = (1 \vee d)x = x \vee dx = \top$, thus $x \in U$. Therefore, every element of $\updownarrow 1$ is outside W.

(3) Let $b \in Z$ such that $b \equiv 1$. Since $\mathbf{R}$ is non-linear, there exists $x \in R$ such that $x \parallel 1$. Since $b = b\top = b(1 \vee x) = b \vee bx$, we have $bx \leq b$. Also, note that $b < 1$, since

otherwise $\top b = \top$, so we get $bx \leq x$; hence $bx \leq b \wedge x$. Since $x \parallel 1 \equiv b \notin \{\top, \bot\}$, we get $x \parallel b$ by unilinearity, hence $bx \leq b \wedge x = \bot$; likewise, we have $xb = \bot$. Likewise, for every b' with $1 \equiv b' \in Z$, we get $xb' = b'x = \bot$, so $b' = \top b' = (b \vee x)b' = bb' \vee xb' = bb'$ and $b = b\top = b(b' \vee x) = bb' \vee bx = bb'$. Thus $b' = b$.

Furthermore, every element of U is comparable to 1 in this case, since if there were an $a \in U$ with $a \parallel 1$, we would get $ab = ba = \bot$ by the preceding paragraph, which contradicts the fact $ab = ba = b$. Also, since U is upward-closed, we get $b < a$; hence $U \subseteq {\uparrow} b \setminus \{b, \top\}$. Conversely, by (4), no element of W is comparable to b, so since b is the only element of Z that is comparable to 1, we get ${\uparrow} b \setminus \{b, \top\} \subseteq U$.

Now suppose $b \parallel 1$ for all $b \in Z$. By way of contradiction, assume that $a \equiv b'$, for some $a \in U$ and $b' \in Z$. Since $b' \parallel 1$, we have $a \parallel 1$. Since U is upward-closed, we get $b' < a$. So, using (1), we get $\top = a\top = a(1 \vee b') = a \vee ab' = a \vee b' = a$, a contradiction.

(5) If c, c' are in W, they are incomparable to 1, by (4). By the definition of W, we have $c < c\top = c(1 \vee c') = c \vee cc' = c\top < \top$. By unilinearity, $c\top$ is join-irreducible, and using $c < c \vee cc' = c\top < \top$, we get $c < cc' = c\top < \top$; likewise, using the centrality of $\top$, we have $c' < cc' = \top c'$. So, cc' is comparable to both c and c' and since none of them is $\bot$ or $\top$, by the unilinearity of $\mathbf{R}$, we get that c and c' are comparable; hence W is a chain. Moreover, it follows from the above calculations that for any $c_1, c_2, c_3, c_4 \in W$ we have $c_1c_2 = c_1\top = c_1c_4 = \top c_4 = c_3c_4$ and we use b_0 to denote the common value of all these products. Furthermore, we have $\top b_0 = \top \cdot \top c_1 = \top c_1 = b_0$, so $b_0 \in Z \cup \{\top\}$. Actually, $b_0 \in Z$ because $b_0 = \top$ implies $c_1\top = \top$, contrary to the definition of W. Finally, we have $b_0 c = \top c \cdot c = \top \cdot cc = \top b_0 = b_0$ and $cb_0 = c \cdot c\top = c^2\top = b_0\top = b_0$, where $c \in W$.

Note that for all $x \in R$ and $c \in W$, we have $xc \leq \top c = b_0$ and $cx \leq c\top = b_0$. Finally, if $x \parallel 1$ and $c \in W$, then $b_0 = c(1 \vee x) = c \vee cx$ and since b_0 is join-irreducible, we get $b_0 = c \in W$ or $b_0 = cx$, so $b_0 = cx$; likewise $b_0 = xc$.

(6) Given $a \in U$ and $c \in W$, we have $c \parallel 1$ according to (3). So $a \vee ca = (1 \vee c)a = \top a = \top$. Since $a \neq \top$ and $ca \leq b_0 a = b_0 < \top$, we have $a \parallel ca$ by unilinearity. Finally, since $\bot < ca \equiv b_0 \equiv c < \top$, we get $a \parallel c$ again by unilinearity.

(7) Suppose there exists $c \in W$ and set $Z' = \{b \in Z : b \parallel 1\}$. By (4) and (5), we have $c^2 = b_0 \in Z'$, so Z' is not empty. If $b \in Z'$, then $b \parallel 1$, so $bc = b_0$ by (5). By (4), we have $c \parallel 1$, so $b = b\top = b(1 \vee c) = b \vee bc = b \vee b_0$ and $b_0 \leq b$; thus $Z' \subseteq {\uparrow} b_0$. Conversely, for $b_0 \leq x < \top$, we have $x \parallel 1$, since $1 \parallel b_0$ and $\mathbf{R}$ is unilinear. Also, $b_0 x \leq b_0 \top = b_0 \leq x$, so $\top x = (1 \vee b_0)x = x \vee b_0 x = x$ and $x \in Z \cup \{\top\}$; hence $x \in Z' \cup \{\top\}$. Therefore $Z' \cup \{\top\} = {\uparrow} b_0$.

Now we show ${\uparrow} b_0$ is closed under multiplication and divisions. For $b, b' \geq b_0$, we have $bb' \geq b_0^2 = \top c \cdot c \top = \top c^2 \cdot \top = \top b_0 \cdot \top = b_0$. Also, since $b_0 = b_0 \top = b_0(1 \vee b) = b_0 \vee b_0 b$, we have $b_0 b \leq b_0 \leq b'$ and $b_0 \leq b'/b$, by residuation; likewise $b_0 \leq b \backslash b'$.

(8) Assume that a_1 is invertible: $a_1 a_2 = a_2 a_1 = 1$, for some a_2. Note that if $\top a_1 = a_1$, then $\top = \top 1 = \top \cdot a_1 a_2 = a_1 a_2 = 1$, which is a contradiction, so $a_1 \notin Z \cup \{\top, \bot\}$. If $a_1 \in W$, then also $a_1 = a_1 \cdot a_1 a_2 = a_1^2 a_2 = b_0 a_2 = b_0 a_1 \cdot a_2 = b_0$, which is also a contradiction, so $a_1 \notin W$. Therefore, invertible element belong to U and $a_1, a_2 \in U$. If $a_1 \parallel 1$, then $a_2 \parallel 1$ since $\top = \top a_2 = (a_1 \vee 1)a_2 = a_1 a_2 \vee a_2 = 1 \vee a_2$. If, further, $c \in W$, by (5) we get $ca_1 = ca_2 = b_0$, so $c = c \cdot 1 = c \cdot a_1 a_2 = ca_1 \cdot a_2 = b_0 a_2 = b_0$, which is a contradiction. Therefore $a_1 \equiv 1$ and $a_2 \equiv 1$ when $W \neq \emptyset$.

(2) First note that for $x, y \in Z$ we have $xy \leq x\top = x$ and $xy \leq \top y = y$, so $xy \leq x \wedge y$; in particular, if x, y are incomparable, then $xy = \bot$. Now suppose Z is not a chain and x, y are incomparable elements of Z. Now, any $z \in Z$ has to be incomparable to x or to y; without loss of generality, $z \parallel y$. So $x = x\top = x(y \vee z) = xy \vee xz = \bot \vee xz = xz \leq \top z = z$ and $z = z\top = z(y \vee x) = zy \vee zx = \bot \vee zx = zx \leq \top x = x$, hence $x = z$. Thus, if Z is not a chain, then it is a 2-element antichain: $Z = \{x, y\}$. In such a case, $x^2 = x^2 \vee \bot = x^2 \vee xy = x(x \vee y) = x\top = x$ and likewise $y^2 = y$. Also, from

above we know that $xy = yx = \bot$. To show that $Z \cup \{\bot, \top\}$ is a 1-free subalgebra of $\mathbf{R}$ and isomorphic to a 4-element Boolean algebra, we compute the divisions involving x and y, by distinguishing two cases.

If both x and y are incomparable to 1, then $W = \emptyset$ by (7). Since $xa = ax = x$ and $ya = ay = y$ for all $a \in U \cup \{\top\}$, we know $x\backslash\bot = x\backslash y = y/x = \bot/x = y$ and $y\backslash\bot = y\backslash x = x/y = \bot/y = x$.

If x is comparable with 1, then y is not comparable to x or 1. So, $x\backslash\bot = x\backslash y = y/x = \bot/x = y$ since $xa = ax = x$ for all $a \in U \cup \{\top\}$; $xc = cx = \bot$ for all $c \in W \cup \{y\}$ and if $W \neq \emptyset$, then $c < y$ for all $c \in W$ by proof in (4). Also, $y\backslash\bot = y\backslash x = x/y = \bot/y = x$ since $ay = ya = y$ for all $a \in U \cup \{\top\}$ and $cy = yc = y$ for all $c \in W \cup \{y\}$ by proof in (4). $\square$

8.2 Classification of unilinear residuated lattices

In the following we will make use of Theorem 8.3 to classify the (bounded) URLs into various classes (along the lines of the properties mentioned in the theorem). As we mentioned, it suffices to describe the structure of the non-linear members of each class. These classes, which together cover all the URLs, will be: $\mathsf{B4}$ (containing only the 4-element Boolean algebra), $\top\mathsf{unital}$, B, TW, LW. Furthermore, we show how the algebras in the three latter classes can be constructed from algebras in $\top\mathsf{unital}$, thus reducing the study of URLs to the study of the $\top$-unital ones. Moreover, we provide axiomatizations for each class.

We will also identify a subclass T of TW and a subclass L of LW that will play a role later. Also, bounded versions of all of these classes can be considered without any change in the axiomatization.

The culmination of the following exhaustive list of configurations of (bounded) URLs, together with specific constructions presented in the following subsections, will result to the following result.

Corollary 8.4. Every (bounded) URL belongs to one of the classes: B4, ⊤unital, B, TW, LW. Moreover, the algebras in the last three classes can be constructed from algebras in the class ⊤unital.

8.2.1 The class Lin. The class Lin consists of all the residuated chains; we denote by bLin the class of bounded residuated chains. Figure 8.1 illustrates integral and non-integral chains. Clearly Lin is axiomatized by the sentence $(\forall u, v)(u \leq v$ or $v \leq u)$. Moreover, this class is contained in all of the others, as for the other classes we only pose restrictions on their non-linear members.

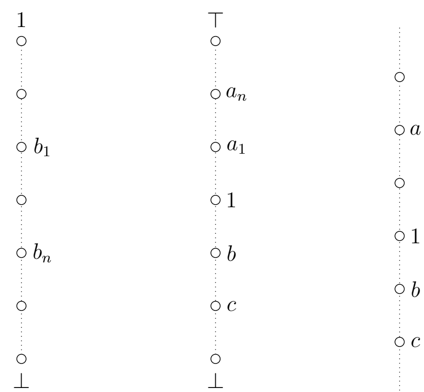

Figure 8.1: Lin: $\mathbf{R}$ is linear, ⊤ is not necessarily central

8.2.2 The class B4. We consider the class where the non-linear members are integral. If $\mathbf{R}$ is integral, then $\top = 1$, so $x\top = \top x = x$ for all $x \in R$; hence $R = Z_R \cup \{\bot, \top\}$. Since $\mathbf{R}$ is non-linear, by Theorem 8.3(2) we get that $\mathbf{R}$ is isomorphic to the 4-element generalized Boolean algebra; see Figure 8.2(a).

This class is axiomatized by the formula: $\overline{\top} x = x = x\overline{\top}$, which is short for

$$\forall u, v, x\,(u \leq v \text{ or } v \leq u \text{ or } (u \vee v)x = x = x(u \vee v)).$$

8.2.3 The class ⊤unital. We consider the class of URLs whose non-linear members $\mathbf{R}$ are non-integral, and have $W_R = \emptyset$ and $Z_R = \emptyset$. This is equivalent to the fact that $\mathbf{R}$

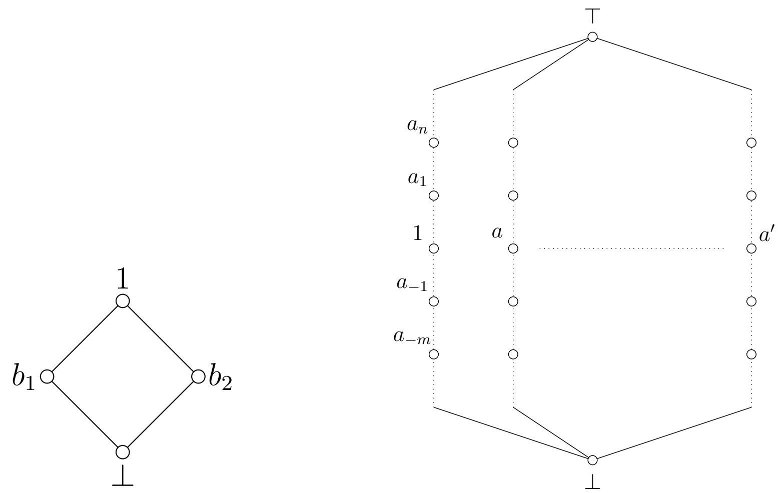

Figure 8.2: (a) B4: non-linear and integral;
(b)$\top$unital : $W = \emptyset$ and $Z = \emptyset$

satisfies $x\top = \top x = \top$ for all $x \in R \setminus \{\bot\}$, i.e., $\mathbf{R}$ is rigorously compact. Figure 8.2(b) shows the general form of $\mathbf{R}$. Therefore, the non-linear members of the class are exactly the rigorously compact URLs, hence the class is axiomatized by the $\top$-unital axiom. We use b$\top$unital for the bounded version.

Chapter 7 contains general constructions that give algebras in this class. In the following, we will prove that all of the remaining cases can be reduced to the $\top$-unital case, by providing specific constructions for each class.

8.2.4 The class T**.** We denote by T the class of URLs whose non-linear members are non-integral and satisfy: $W_R = \emptyset$, $Z_R = \{b \neq b'\}$ and $b \equiv 1$; all of the results also hold for the corresponding class bT of bURLs. The name of the class is motivated that there are elements of Z_R and U_R that are together in the same chain. Let $\mathbf{R}$ be a non-linear member of T. Since $W_R = \emptyset$, we have that $\mathbf{R}$ satisfies the formula

$$\forall x\,(\overline{\top}x = \overline{\top} \text{ or } \overline{\top}x = x) \tag{$W\emptyset$}$$

Since $b \equiv 1$ for some $b \in Z_R$, Theorem 8.3(3) implies that $U_R = {\uparrow}b \setminus \{b, \top\}$ is a chain. By Theorem 8.3(3), b is the only element of Z_R that is comparable to 1, so b' is incomparable to both 1 and b. By Theorem 8.3(2), $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra of $\mathbf{R}$ and it isomorphic to the 4-element generalized Boolean algebra. Therefore, $\mathbf{R}$ has to be ordered as in Figure 8.3(a). In particular, $\mathbf{R}$ satisfies (w_2), as it has width at most 2.

By Theorem 8.3(1), b is the multiplicative zero for $U_R \cup \{\top\}$ and by Theorem 8.3(2), $b^2 = b$. So, ${\uparrow}b$ is totally-ordered, closed under multiplication and contains 1. Moreover, it is closed under divisions: since $xb = bx = b \leq a$ for all $x \in {\uparrow}b$ and $a \in U_R$, we get $b \leq x\backslash a, a/x \leq \top$; also $b \leq a\backslash b$ and $a' \not\leq a\backslash b$, for all $a' \in U_R$, hence $a\backslash b = b$; and $b\backslash b = b/b = \top$. So ${\uparrow}b$ is a $\bot$-free subalgebra of $\mathbf{R}$. By Theorem 8.3(1), $xy = y = yx$, for $x \in U_R$ and $y \in Z_R$. Therefore, we have $a\backslash b' = b'/a = b'$ and $b'\backslash a = a/b' = b$, for all $a \in U_R$. Note that given two incomparable elements in R, at least one of them is b' and we know $b'\backslash\bot \vee b' = b \vee b' = \top$. Therefore, $\mathbf{R}$ satisfies the sentence:

$$\forall x, y\, (x \leq y \text{ or } y \leq x \text{ or } x\backslash\overline{\bot} \vee x = \overline{\top} \text{ or } y\backslash\overline{\bot} \vee y = \overline{\top}) \qquad \text{(compl)}$$

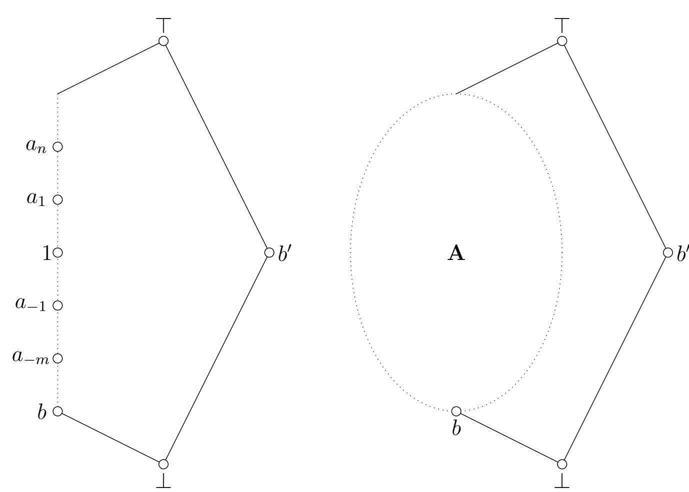

Figure 8.3: (a) T: $W = \emptyset, Z = \{b, b'\}$ and $b \parallel b'$;
(b) the construction

We have shown that the non-linear members of the class T satisfy (URL), $(W\emptyset)$, (w_2) and (compl). We show now that, conversely, these sentences provide an axiomatization for the class.

Theorem 8.5. The class T is axiomatized by (URL), $(W\emptyset)$, (w_2) and (compl).

Proof. Let $\mathbf{R}$ be a non-linear residuated lattice satisfying (URL), $(W\emptyset)$, (w_2) and (compl); let $d \in R$ such that $1 \parallel d$, then $d \notin \{\bot, \top\}$. Since $1\backslash\bot \vee 1 = \bot \vee 1 = 1 \neq \top$, by (compl) we get $d\backslash\bot \vee d = \top$. Since $d \neq \bot$, we know $d\backslash\bot \neq \top$, so $d\backslash\bot \parallel d$. Since U_R is closed under multiplication and elements in U_R are multiplicative identities for Z_R, $d \in U_R$ implies $d\backslash\bot = \bot$ then $d\backslash\bot \equiv d$, a contradiction; so we know $d \notin U_R$. Now, $(W\emptyset)$ gives $W_R = \emptyset$, hence $d \in Z_R$. Also, by $d\backslash\bot \leq (d\backslash\bot)\top \leq d\backslash(\bot\top) = d\backslash\bot$, we have $d\backslash\bot \in Z_R$. By Theorem 8.3(2), we know now $Z_R = \{d, d\backslash\bot\}$ being a 2-element antichain. By (w_2) we know the width of $\mathbf{R}$ is at most 2, and since $d \parallel 1$, $d \parallel d\backslash\bot$, we can conclude that $d\backslash\bot \equiv 1$. □

We have shown that if $\mathbf{R}$ is a non-linear member of the class T, then $\uparrow b$ is a bounded residuated chain with top $\top$ and bottom b satisfying $\top x = x\top = \top$ for all $x \neq b$ (rigorous compactness). Also, we know $R = \uparrow b \sqcup \{\bot\} \sqcup \{b'\}$, $\bot$ is absorbing for R, $bb' = b'b = \bot$, $xb' = b'x = b'$ for all $x \in R \setminus \{\bot, b\}$, and b' is only comparable to $\top$ and $\bot$. We will show in Corollary 8.7 that, conversely, if we have these ingredients ($\uparrow b$, $\bot$ and b') subject to the above restrictions, there is an algebra in T whose ingredients are the given ones. Actually, the construction we will describe is slightly more general even: it produces a residuated lattice even if $\uparrow b$ is not totally ordered (Theorem 8.6); the residuated lattice is unilinear iff $\uparrow b$ is totally ordered.

Let $\mathbf{A}$ be a bounded residuated lattice with top $\top$ and bottom b that is rigorously compact ($\top x = x\top = \top$ for all $x \in A \setminus \{b\}$); then $A \setminus \{b\}$ is closed under multiplication by associativity. We consider the set $R_{\mathbf{A},\{b,b'\}} = A \cup \{\bot, b'\}$, where $\bot$ and b' are new elements, and we define a lattice ordering on it by: $\bot < b' < \top$ and b' is incomparable to all other elements; see Figure 8.3(b). Also, we extend the multiplication of $\mathbf{A}$ by

$$xb' = b'x = b' \quad \text{if } x \in R_{\mathbf{A},\{b,b'\}} \setminus \{\bot, b\}$$

$$bb' = b'b = \bot$$

$$x\bot = \bot x = \bot \quad \text{for all } x \in R.$$

We denote the resulting algebra by $\mathbf{R}_{\mathbf{A},\{b,b'\}}$.

Theorem 8.6. Let $\mathbf{A}$ be a bounded residuated lattice with top $\top$ and bottom b satisfying $\top x = x\top = \top$ for all $x \in A \setminus \{b\}$, and $\bot, b' \notin A$ are distinct elements. Then $\mathbf{R}_{\mathbf{A},\{b,b'\}}$ is the reduct of a (unique) residuated lattice, which denote in the same way. Furthermore, if $\mathbf{A}$ is linear, then $\mathbf{R}_{\mathbf{A},\{b,b'\}}$ is unilinear.

Proof. We set $\mathbf{R} := \mathbf{R}_{\mathbf{A},\{b,b'\}}$ for convenience. Since $A \setminus \{b\}$ itself is a monoid, $\bot$ and b' are absorbing elements for $A \setminus \{b\}$, $\bot$ absorbs b', and $\{b, b', \bot\}$ is a semigroup, associativity holds in $\mathbf{R}$.

We first prove that multiplication of $\mathbf{R}$ is order-preserving. Given that $A \cup \{\bot\}$ and $\{\bot, \top, b, b'\}$ are residuated lattices, order preservation holds there. Also, and $\bot$ is absorbing, so the verification for the remaining cases is: $b \leq a \Rightarrow bb' = \bot \leq b' = ab'$ and $b' \leq \top \Rightarrow ab' \leq \top = a\top$, for all $a \in A \setminus \{b\}$, and likewise for multiplication on the left.

Now by way of contradiction, suppose $x \backslash\!\!\backslash z$ does not have maximum, for some $x, z \in R$. In particular, $\top \notin x \backslash\!\!\backslash z$. By Remark 2.4, $\max \bot \backslash\!\!\backslash y = \max y \backslash\!\!\backslash \top = \top$ for all $y \in R$, hence $x \neq \bot$ and $z \neq \top$.

Assume that there exists $a \in A \setminus \{b, \top\}$ such that $a \in x \backslash\!\!\backslash z$; then $x\top \neq xa$, since otherwise we would have $\top \in x \backslash\!\!\backslash z$. So $x \notin \{b, b', \bot, \top\}$, i.e., $x \in A \setminus \{b, \top\}$. Since $xa \leq z$, $A \setminus \{b\}$ is closed under multiplication and $b < a$, $b' \parallel a$ for all $a \in A \setminus \{b, \top\}$, we have $z \in A \setminus \{b, \top\}$. By definition of $x \backslash\!\!\backslash z$, we know $x \backslash\!\!\backslash z = \{y \in A : xy \leq z\} \cup \{\bot\}$.

Thus $\max x\mathbin{\backslash\!\!\backslash} z = x\backslash_{\mathbf{A}} z$, which is a contradiction. So we get $x\mathbin{\backslash\!\!\backslash} z \subseteq \{\bot, b, b'\}$. Since $\bot \in x\mathbin{\backslash\!\!\backslash} z$ for all $x, z \in R$ and $x\mathbin{\backslash\!\!\backslash} z$ has no maximum, we know $x\mathbin{\backslash\!\!\backslash} z = \{\bot, b, b'\}$. Since $\top \notin x\mathbin{\backslash\!\!\backslash} z$, we get $x\top \neq xb$ and $x\top \neq xb'$, so $x \notin \{\bot, b, b'\}$, or equivalently, $x \in A \setminus \{b\}$. Since $b \in x\mathbin{\backslash\!\!\backslash} z$ and $b' \in x\mathbin{\backslash\!\!\backslash} z$, we get $xb = b \leq z$ and $xb' = b' \leq z$, thus $z = \top$, which is a contradiction. Therefore the maximum of $x\mathbin{\backslash\!\!\backslash} z$ exists for all $x, z \in R$, and likewise $z\mathbin{/\!\!/} x$ has a maximum. By Corollary 2.2, we get that $\mathbf{R}$ is a residuated lattice. □

Corollary 8.7. The residuated lattices of the form $\mathbf{R}_{\mathbf{A},\{b,b'\}}$, where $\mathbf{A}$ is a rigorously compact residuated chain, are up to isomorphism precisely the non-linear algebras in T.

8.2.5 The class B**.** We denote by B the class of URLs whose non-linear members satisfy: $\mathbf{R}$ is non-integral, W_R is empty, $Z_R = \{b, b'\}$ and $\{b, b', 1\}$ is a 3-element antichain. Let $\mathbf{R}$ be a non-linear member of B; the results below hold also for bB, the bounded version. The name of the class is motivated by the fact that $Z_R \cup \{\bot, \top\}$ forms a Boolean algebra. Since $W_R = \emptyset$, $\mathbf{R}$ satisfies the formula ($W\emptyset$). Since $Z_R = \{b, b'\}$ and $b \parallel b'$, Theorem 8.3(2) implies that $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra satisfying $b^2 = b, b'^2 = b'$ and $bb' = b'b = \bot$. Actually $Z_R \cup \{\bot, \top\}$ itself is a 4-element Boolean algebra, so $\mathbf{R}$ satisfies

$$\forall x\, (\overline{\top} x = \overline{\top} \text{ or } x\backslash\overline{\bot} \vee x = \overline{\top}), \qquad \text{(ZBoolean)}$$

which is short for

$$\forall u \forall v \forall x\, (u \leq v \text{ or } v \leq u \text{ or } (u \vee v)x = u \vee v \text{ or } x\backslash(u \wedge v) \vee x = u \vee v).$$

Since $\{b, b', 1\}$ is a 3-element antichain, Theorem 8.3(3) yields that $a \parallel b$ for all $a \in U_R$ and $b \in Z_R$. By Theorem 8.3(1) we have $ab = ba = b$ and also we have $W_R = \emptyset$, so $a_1\backslash a_2, a_2/a_1 \in U_R \cup \{\bot\}$ for all $a_1, a_2 \in U_R$; thus $U_R \cup \{\bot, \top\}$ is a subalgebra of $\mathbf{R}$. For the same reason $\mathbf{R}$ satisfies the formula

$$\overline{\top}\backslash 1 = \overline{\bot}, \tag{$b \parallel 1$}$$

which is equivalent to

$$\forall u, \forall v\, (u \leq v \text{ or } v \leq u \text{ or } (u \vee v)\backslash 1 = u \wedge v).$$

Finally, the remaining divisions are given by

$$\begin{array}{ll} a\backslash b = b/a = b & a\backslash b' = b'/a = b' \\ b\backslash a = a/b = b' & b'\backslash a = a/b' = b, \end{array}$$

for all $a \in U_R$. The lattice structure of $\mathbf{R}$ is given in Figure 8.4(a). We now show that these sentences provide an axiomatization.

Theorem 8.8. The class B is axiomatized by (URL), ($W\emptyset$), (ZBoolean) and ($b \parallel 1$).

Proof. Suppose $\mathbf{R}$ is a non-linear residuated lattice that satisfies theses axioms. By ($W\emptyset$) we have $W_R = \emptyset$ and by (ZBoolean) we know $Z_R \neq \emptyset$ and $x \in Z_R$ implies $x \parallel x\backslash\bot$. Similarly to the proof of Theorem 8.5, we can show that $x\backslash\bot \in Z_R$ for $x \in Z_R$, so Z_R is a 2-element antichain. Finally, since $\mathbf{R}$ satisfies ($b \parallel 1$), if there exists $b \in Z_R$ such that $b \equiv 1$, then $\top\backslash 1 = b > \bot$ by Theorem 8.3(3), a contradiction, so $b \parallel 1$ and $b' \parallel 1$, and $\{1, b, b'\}$ is a 3-element antichain in this case. □

We now work in the converse direction and describe a general construction that will help us characterize all non-linear algebras in B.

Let $\mathbf{A}$ be a bounded residuated lattice with top $\top$ and bottom $\bot$ satisfying $\top x = x\top = \top$ for all $x \in A\backslash\{\bot\}$ and $\mathbf{B}$ be a bounded integral residuated lattice with top $\top$ and bottom

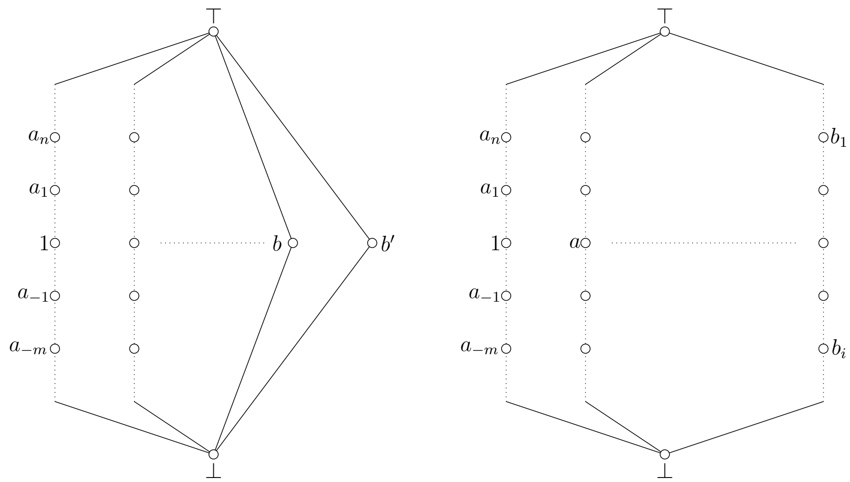

Figure 8.4: (a) In B; (b) In L

$\bot$, such that $A \cap B = \{\bot, \top\}$. We consider the set $R_{\mathbf{A},\mathbf{B}} = A \cup B$ and the union of the orders of $\mathbf{A}$ and of $\mathbf{B}$, resulting in the lattice order of Figure 8.5.

Also, we extend the multiplication on $\mathbf{A}$ and $\mathbf{B}$ by stipulating that each element of B is absorbing for $A \setminus \{\bot, \top\}$. We denote the resulting algebra as $\mathbf{R}_{\mathbf{A},\mathbf{B}}$.

Theorem 8.9. Let $\mathbf{A}$ be a bounded residuated lattice with top $\top$ and bottom $\bot$ satisfying $\top x = x\top = \top$ for all $x \in A \setminus \{\bot\}$, and $\mathbf{B}$ be a bounded integral residuated lattice with top $\top$ and bottom $\bot$. Then $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ is the reduct of a (unique) residuated lattice which we denote the same way. If $\mathbf{A}$ and $\mathbf{B}$ are linear, then $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ is unilinear.

Proof. We abbreviate $R_{\mathbf{A},\mathbf{B}}$ as R for convenience. Since all elements in B are zero elements for those in $A \setminus \{\bot\}$, associativity holds.

Since both $\mathbf{A}$ and $\mathbf{B}$ are residuated, multiplication is order-preserving inside each of them. Since elements in $A \setminus \{\bot, \top\}$ are multiplicative identities for those in B, multiplication between elements of A and B is order-preserving as well. So the multiplication on $\mathbf{R}$ is order-preserving.

To show that $\mathbf{R}$ is a reduct of a residuated lattice, by Corollary 2.2 it suffices to show that $x\backslash\!\!\backslash z = \{y \in R : xy \leq z\}$ has a maximum for all $x, z \in R$. By Remark 2.4, we know

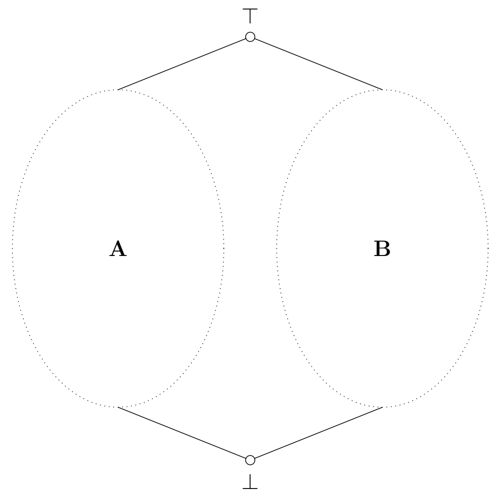

Figure 8.5: The algebra $\mathbf{R}_{\mathbf{A},\mathbf{B}}$

$\bot\backslash\!\!\backslash z = x\backslash\!\!\backslash\top = R$ for all $x, z \in R$, so $\max \bot\backslash\!\!\backslash z = \max x\backslash\!\!\backslash\top = \top$. In the following we assume that $x > \bot$ and $z < \top$. Working toward a contradiction we assume that $x\backslash\!\!\backslash z$ does not have a maximum for some $x, z \in R$; in particular, $\top \notin x\backslash\!\!\backslash z$. Note that if there were an $a \in A \setminus \{\bot, \top\}$ with $a \in x\backslash\!\!\backslash z$, then $x\top \neq xa$ (since otherwise we would get $\top \in x\backslash\!\!\backslash z$), so $x \notin B$, or equivalently, $x \in A \setminus \{\bot, \top\}$. Then, since $a \in x\backslash\!\!\backslash z$, $A \setminus \{\bot\}$ is closed under multiplication and elements in $A \setminus \{\bot, \top\}$ are incomparable with those in $B \setminus \{\bot, \top\}$, we get $z \in A \setminus \{\bot\}$, thus $x\backslash\!\!\backslash z = \{y \in R : xy \leq z\} = \{y \in A : xy \leq z\}$, and the maximum of $x\backslash\!\!\backslash z$ is $x\backslash_{\mathbf{A}} z$, which is a contradiction. Hence $x\backslash\!\!\backslash z$ is fully contained in $B \setminus \{\top\}$. If $x \in A \setminus \{\bot, \top\}$, then $x\backslash\!\!\backslash z = \{y \in B : xy \leq z\} = \{y \in B : y \leq z\}$ and since the maximum does not exist, we get that $z \in A$ and then $x\backslash\!\!\backslash z = \{y \in B : y \leq z\} = \{\bot\}$, a contradiction; so $x \in B$. Now, if $z \in A$, then $x\backslash\!\!\backslash z = \{y \in B : xy \leq z\} = \{\bot\}$ since B is closed under multiplication, a contradiction; so $z \in B$. In this case $x\backslash\!\!\backslash z$ has maximum $x\backslash_{\mathbf{B}} z$, which again is a contradiction. Similarly for $z/\!\!/x = \{y \in R : yx \leq z\}$. □

Corollary 8.10. If $\mathbf{A}$ is a bounded unilinear residuated lattice with top $\top$ and bottom $\bot$ satisfying $\top x = x\top = \top$ for all $x \in A \setminus \{\bot\}$, and $\mathbf{B}$ is the 4-element Boolean algebra, then $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ is a non-linear member in B. Moreover, all non-linear members of B are of this form.

8.2.6 The class L. We denote by L the URL class in which the non-linear members are non-integral with $W_R = \emptyset$ and Z_R is linear; the same results below hold for the corresponding bURL class bL. The name of the class is motivated from the fact that Z_R is linear. Let $\mathbf{R}$ be a non-linear member in L. Since $W_R = \emptyset$, we know that $\mathbf{R}$ satisfies ($W\emptyset$). Since $\mathbf{R}$ is non-linear and Z_R is linear, $b \parallel 1$ for all $b \in Z_R$, otherwise by Theorem 8.3(3) $U_R = \uparrow b \setminus \{\bot, \top\}$ and $\mathbf{R}$ is linear. Also by Theorem 8.3(3) we know $b \parallel a$ for all $a \in U$. Since Z_R is linear and $W_R = \emptyset$, $\mathbf{R}$ satisfies

$$\forall x\,(\overline{\top}x = \overline{\top} \text{ or } x\backslash\overline{\bot} \leq x \text{ or } x \leq x\backslash\overline{\bot}). \qquad (Z\text{linear})$$

Similar to the case for B, we can show that $U_R \cup \{\bot, \top\}$ is a subalgebra and $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra, and that the divisions are given by

$$a\backslash b = b/a = b, \quad b\backslash a = b\backslash\bot, \quad a/b = \bot/b$$

for all $a \in U_R$ and $b \in Z_R$.

Theorem 8.11. The class L is axiomatized by (URL), ($W\emptyset$) and (Zlinear).

Proof. Suppose $\mathbf{R}$ is a non-linear residuated lattice satisfying these axioms. By ($W\emptyset$) we have $W_R = \emptyset$. By (Zlinear) we get that if $x \in Z_R$ then $x \equiv x\backslash\bot$. Since by Theorem 8.3(2) $x\backslash\bot \parallel x$ for all $x \in Z_R$ when Z_R is a 2-element antichain, we know if $Z_R \neq \emptyset$ then it is linear in this case; otherwise $Z_R = \emptyset$ and $\mathbf{R}$ is in the class $\top\mathsf{unital}$. □

Note that the construction in Theorem 8.9 is general enough to apply to this case as well, thus yielding the next characterization of all the nonlinear algebras in L.

Corollary 8.12. If $\mathbf{A}$ is a bounded unilinear residuated lattice with top $\top$ and bottom $\bot$ satisfying $\top x = x\top = \top$ for all $x \in A \setminus \{\bot\}$, and $\mathbf{B}$ is an integral residuated chain, then $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ is a nonlinear member in L. Moreover, all nonlinear members of L are of this form.

8.2.7 The class TW. We denote by TW the class of URL whose non-linear members $\mathbf{R}$ are non-integral, $Z_R = \{b \neq b_0\}$ and $b \equiv 1$; the results below hold also for the bounded version bTW. Let $\mathbf{R}$ be a non-linear member of TW. Since $Z_R = \{b \neq b_0\}$ and $b \equiv 1$, Theorem 8.3(3) yields $U_R = {\uparrow}b \setminus \{b, \top\}$ and that b is the unique element in Z_R which is comparable with 1, so $b \parallel b_0$ and Z_R is a 2-element antichain.

If $W_R = \emptyset$, then $\mathbf{R}$ satisfies ($W\emptyset$). Since $U_R = {\uparrow}b \setminus \{b, \top\}$ and $Z_R = \{b \neq b_0\}$, $\mathbf{R}$ also satisfies (w_2) and $a \parallel b_0$ for all $a \in U_R$, so $x \parallel y$ implies $x = b_0$ or $y = b_0$ for all $x, y \in R$. Since $b_0 \backslash \bot = b$ and $b_0 \parallel b$, we get that $\mathbf{R}$ satisfies (compl), so $\mathbf{R}$ is a member of the class T by Theorem 8.5.

If $W_R \neq \emptyset$, then $c^2 \in Z_R$, $c^2 \parallel 1$ and $c \parallel 1$ for all $c \in W_R$ by Theorem 8.3(4) and (5), so $c^2 = b_0$. Also by Theorem 8.3(5) we know $W_R = {\downarrow}b_0 \setminus \{\bot, b_0\}$, thus $\mathbf{R}$ satisfies (w_2). By Theorem 8.3(3), $bc = \bot$ for all $c \in W_R$. Since $ac, ca \leq b_0$ by Theorem 8.3(5) and $\bot < ac, ca$ by associativity for all $a \in U_R = {\uparrow}b \setminus \{b, \top\}$, $c \backslash \bot = b$ for all $c \in W_R$. Since $c \parallel b$, we know that $c\backslash\bot \vee c = b \vee c = \top$, hence $\mathbf{R}$ satisfies (compl).

Theorem 8.13. The class TW is axiomatized by (URL), (w_2) and (compl).

Proof. Suppose $\mathbf{R}$ is a non-linear residuated lattice satisfying these axioms. Since $\mathbf{R}$ satisfies (compl), $d \parallel 1$ implies that $d \notin U_R \cup \{\bot, \top\}$ since $a\backslash\bot \vee a = a < \top$ for all $a \in U_R$; such d exists since $\mathbf{R}$ is non-linear. Also, since $d\backslash\bot \leq d\backslash\bot \cdot \top \leq d\backslash(\bot\top) = d\backslash\bot$ and $d\backslash\bot \vee d = \top$, we know $d\backslash\bot \in Z_R$. Now if $d \in Z_R$, then $Z_R = \{d, d\backslash\bot\}$ is a 2-element antichain. Since $\mathbf{R}$ satisfies (w_2) and $1 \parallel d$, we know $d\backslash\bot \equiv 1$, so $\mathbf{R}$ is a nonlinear member of TW. On the other hand, if $d \in W_R$, then $d^2 = b_0 \in Z_R$ by Theorem 8.3(5). Since $d < b_0$ and $d\backslash\bot \vee d = \top$, we know $d\backslash\bot \vee b_0 = \top$. Since $d\backslash\bot \in Z_R$, again Z_R is a 2-element antichain. Since $\mathbf{R}$ satisfies (w_2) and $b_0 \parallel 1$ by Theorem 8.3(7), we know $d\backslash\bot < 1$, so $\mathbf{R}$ is a non-linear member of TW. □

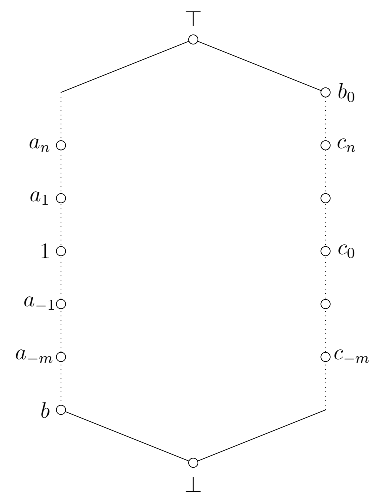

Figure 8.6: TW: $W \neq \emptyset, Z = \{b, b_0\}$ and $b \equiv 1$

Since all non-linear members of T satisfy the above axioms, we have that T is a subclass of TW. As in the case T, we can show that $\uparrow b$ is a $\bot$-free subalgebra of $\mathbf{R} \in \mathsf{TW}$ and that it is rigorously compact. Similarly, $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra and itself is a 4-element generalized Boolean algebra. If $W_R \neq \emptyset$, then $W_R \cup \{\bot, b_0\}$ is a totally-ordered semigroup subreduct of $\mathbf{R}$ satisfying $xy = b_0$ for all $x, y \in W_R \cup \{b_0\}$ by Theorem 8.3(5). As we mentioned in the proof above, we have that $\bot < ac, ca \leq b_0$ and $bc = cb = \bot$ for all $a \in U_R, c \in W_R$, so the divisions among U_R, Z_R and W_R are given by

$$a\backslash b_0 = b_0/a = b_0, \quad b_0\backslash a = a/b_0 = b$$

$$a\backslash c = c/a \in W_R \cup \{\bot\}, \quad c\backslash a = a/c = b$$

$$b_0\backslash c = c/b_0 = b, \quad c\backslash b_0 = b_0/c = \top$$

$$b\backslash c = c/b = b_0, \quad c\backslash b = b/c = b,$$

for all $a \in U_R, c \in W_R$.

Moreover, since $W_R \cup \{\bot, b_0\}$ satisfies $xy = b_0$ for all $x, y \in W_R \cup \{b_0\}$, $ac, ca \in W_R \cup \{b_0\}$ and b_0 is a multiplicative zero for $U_R \cup \{\top\}$, combined with the fact $bb_0 = b_0 b = \bot$, we have that $\uparrow b$ acts on $W_R \cup \{\bot, b_0\}$ from the left and the right. Also, since the multiplication of $\mathbf{R}$ is residuated, the following properties hold.

- $\top \cdot c = c \cdot \top = b_0$ for all $c \in W_R \cup \{b_0\}$.
- $a \cdot b_0 = b_0 \cdot a = b_0$ for all $a \in U_R \cup \{\top\}$.
- If $a \cdot c = \bot$ or $c \cdot a = \bot$, then either $a = b$ or $c = \bot$ for all $a \in \uparrow b$ and $c \in \downarrow b_0$.

Conversely, we construct a residuated lattice based on a rigorously compact residuated lattice, a bounded semigroup which is almost null and a bi-residuated bi-action. When the given algebras are totally-ordered, we obtain a non-linear member of TW, as we show below.

Let $\mathbf{A}$ be a rigorously compact residuated lattice with bottom b and top $\top$ and let $\mathbf{C}$ be a bounded lattice-ordered semigroup with top b_0 and bottom $\bot$ satisfying $xy = b_0$ for all $x, y \in C \setminus \{\bot\}$ and $\bot z = z \bot = z$ for all $z \in C$. Suppose $*$ is a *bi-residuated bi-action* of $\mathbf{A}$ on $\mathbf{C}$, i.e.,

$$1 * c = c,\ (a_1 a_2) * c = a_1 * (a_2 * c),\ (a_1 * c) * a_2 = a_1 * (c * a_2),\ c * 1 = c,\ c * (a_2 a_1) = (c * a_2) * a_1,$$

and there exist functions $\backslash^l : A \times C \to C$, $/^l : C \times C \to A$, $/^r : C \times A \to C$ and $\backslash^r : C \times C \to A$ such that

$$a * c_1 \leq c_2 \text{ iff } c_1 \leq a \backslash^l c_2 \text{ iff } a \leq c_2 /^l c_1$$

$$c_1 * a \leq c_2 \text{ iff } c_1 \leq c_2 /^r a \text{ iff } a \leq c_1 \backslash^r c_2$$

for all $a, a_1, a_2 \in A$ and $c, c_1, c_2 \in C$. Here we write $*$ for both the left $* : A \times C \to C$ and right $* : C \times A \to C$ aspects of the action. Also, assume that the action *respects top elements* and *has no zero-divisors*:

$\top * c = c * \top = b_0$ for all $c \in C \setminus \{\bot\}$

$a * b_0 = b_0 * a = b_0$ for all $a \in A \setminus \{b\}$

If $a * c = \bot$ or $c * a = \bot$, then either $a = b$ or $c = \bot$ for all $a \in A$ and $c \in C$

Note that the converse of the third condition above holds, given that $*$ is residuated.

Now we let $R_{\mathbf{A},\mathbf{C},*} = A \cup C$ and define as order the union of the orders of $\mathbf{A}$ and $\mathbf{C}$ and also making $\bot$ and $\top$ the new bounds. This results in a lattice ordering as shown below, where in the picture A includes $\top$ and b, and C includes $\bot$ and b_0.

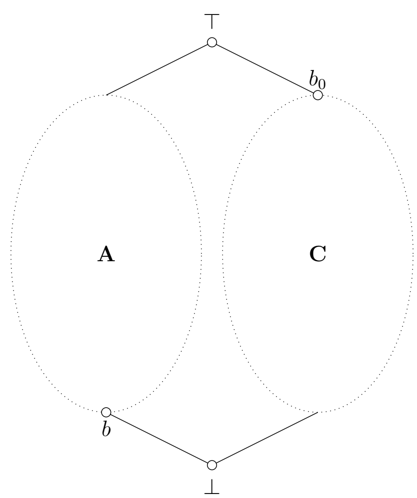

Figure 8.7: $\mathbf{R}_{\mathbf{A},\mathbf{C},*}$

Also, we define multiplication on $R_{\mathbf{A},\mathbf{C},*}$ by

$$x \cdot y = \begin{cases} x \cdot_{\mathbf{A}} y & \text{if } x, y \in A \\ x \cdot_{\mathbf{C}} y & \text{if } x, y \in C \\ x * y & \text{if } x \in A, y \in C \text{ or } x \in C, y \in A. \end{cases}$$

We denote the resulting algebra by $\mathbf{R}_{\mathbf{A},\mathbf{C},*}$.

Theorem 8.14. Let $\mathbf{A}$ be a rigorously compact residuated lattice with top $\top$ and bottom b and $\mathbf{C}$ be a bounded lattice-ordered semigroup with top b_0 and bottom $\bot$ satisfying $xy = b_0$ for all $x, y \in C \setminus \{\bot\}$ and $\bot z = z\bot = z$ for all $z \in C$. Suppose that $\mathbf{A}$ has a bi-residuated bi-action on $\mathbf{C}$ that respects tops and has no zero divisors. Then $\mathbf{R}_{\mathbf{A},\mathbf{C},*}$ is the reduct of a residuated lattice, which we denote in the same way.

Proof. In the proof we use $\mathbf{R}$ as short for $\mathbf{R}_{\mathbf{A},\mathbf{C},*}$.

For associativity, we first observe that $\{\bot, \top, b, b_0\}$ is a 4-element Boolean algebra. Also, by the definition of multiplication we have $bc = cb = \bot$ for all $c \in C$ and $\bot$ is the multiplicative zero for $\mathbf{R}$. So we may focus on the multiplications between $\mathbf{A}$ and $\mathbf{C}$ that do not involve b and $\bot$. For $a, a_1, a_2 \in A \setminus \{b\}$ and $c, c_1, c_2 \in C \setminus \{\bot\}$, the verification of the nontrivial cases is as follows:

$$a_1 a_2 \cdot c = a_1 a_2 * c = a_1 * (a_2 * c) = a_1 \cdot a_2 c$$

$$a_1 c \cdot a_2 = (a_1 * c) * a_2 = a_1 * (c * a_2) = a_1 \cdot c a_2$$

$$c_1 c_2 \cdot a = b_0 a = b_0 = c_1 \cdot c_2 a$$

$$c_1 a \cdot c_2 = b_0 = c_1 \cdot a c_2$$

The remaining cases are similar or straightforward.

Since $\mathbf{A}$ is residuated, multiplication on $\mathbf{A}$ is order-preserving. Since $c_1c_2 = b_0$ for all $c_1, c_2 \in C\setminus\{\bot\}$ and $\bot x = x\bot = \bot$ for all $x \in C$, multiplication on $\mathbf{C}$ is order-preserving. For other cases we show $y \le z \implies xy \le xz$. Since $\mathbf{A}$ has a bi-residuated bi-action $*$ on $\mathbf{C}$, the multiplication is order-preserving if $x \in A$, $y, z \in C$ or $x \in C$, $y, z \in A$. So the remaining cases are

- $y = \bot$, $z \in A$: $xy = \bot \le xz$ for all $x \in R$.
- $y \in C \setminus \{\bot\}$ and $z = \top$: If $x \in A \setminus \{b\}$, then $xy \le b_0 < \top = xz$; if $x \in C \setminus \{\bot\}$, then $xy = b_0 = xz$; otherwise $xy = \bot \le xz$.

Similarly we can show $y \le z \implies yx \le zx$. Therefore the multiplication of $\mathbf{R}$ is order-preserving.

To show that $\mathbf{R}$ is the reduct of a residuated lattice, we will show that $x\backslash\!\backslash z = \{y \in R : xy \le z\}$ has maximum for all $x, z \in R$. Since $*$ is a bi-residuated action, there exist functions $\backslash^l : A \times C \to C$, $/^l : C \times C \to A$, $/^r : C \times A \to C$ and $\backslash^r : C \times C \to A$ such that

$$a * c_1 \le c_2 \text{ iff } c_1 \le a\backslash^l c_2 \text{ iff } a \le c_2/^l c_1$$

$$c_1 * a \le c_2 \text{ iff } c_1 \le c_2/^r a \text{ iff } a \le c_1\backslash^r c_2$$

for all $a \in A$, $c_1, c_2 \in C$. By Remark 2.4, we know $\bot\backslash\!\backslash z = x\backslash\!\backslash\top = R$ for all x and z, so $\max \bot\backslash\!\backslash z = \max x\backslash\!\backslash\top = \top$ in this case. In the following we assume that $x > \bot$ and $z < \top$. Working toward a contradiction we assume that $x\backslash\!\backslash z$ has no maximum for some $x, z \in R$, then $\top \notin x\backslash\!\backslash z$. First, since $\bot \in x\backslash\!\backslash z$, we know $x\backslash\!\backslash z$ is not empty. If there exists $c \in C \setminus \{\bot\}$ such that $c \in x\backslash\!\backslash z$, then we have $x\top \ne xc$, so $x \notin C$, or equivalently, $x \in A$. Since $xc \le z < \top$, $xc \in C$ and elements in $A \setminus \{\top\}$ are incomparable with those in $C \setminus \{\bot\}$, we get $z \in C$. According to the definition of $x\backslash\!\backslash z$ and $*$, we know the maximum of $x\mathbin{\backslash\!\!\backslash} z$ is $x\backslash^l z$, which is a contradiction. Thus $x\mathbin{\backslash\!\!\backslash} z \subseteq (A \cup \{\bot\}) \setminus \{\top\}$. Since $x\mathbin{\backslash\!\!\backslash} z$ does not have a maximum and $x\mathbin{\backslash\!\!\backslash} z$ is downward closed, $\{\bot, b\}$ is a proper subset of $x\mathbin{\backslash\!\!\backslash} z$, so there exists $a \in A \setminus \{b, \top\}$ such that $a \in x\mathbin{\backslash\!\!\backslash} z$. Thus $x\top \neq xa$ implies that $x \notin \{\bot, \top, b, b_0\}$. If $x \in A \setminus \{b, \top\}$, then we know $z \in A \setminus \{b, \top\}$ since $A \setminus \{b\}$ is closed under multiplication and elements in $C \setminus \{\bot\}$ is incomparable with those in $A \setminus \{\top\}$. In this case, $x\mathbin{\backslash\!\!\backslash} z$ has a maximum $x\backslash_{\mathbf{A}} z$ by the residuation of $\mathbf{A}$, which is a contradiction, so we know $x \in C \setminus \{\bot, b_0\}$. Furthermore, since $xa \leq z$, $xa \in C$ and elements in $A \setminus \{\top\}$ are incomparable with those in $C \setminus \{\bot\}$, we know $z \in C$. In this case, $x\mathbin{\backslash\!\!\backslash} z$ has a maximum $x\backslash^r z$ since $*$ is bi-residuated, which is again a contradiction. Therefore, $x\mathbin{\backslash\!\!\backslash} z$ has a maximum for all $x, z \in R$. Similarly for the existing of the maximum of $z\mathbin{/\!\!/} x = \{y \in R : yx \leq z\}$. □

Corollary 8.15. If $\mathbf{A}$ is a rigorously compact residuated chain and $\mathbf{C}$ is a totally-ordered semigroup with top b_0 and bottom $\bot$ satisfying $xy = b_0$ for all $x, y \in C \setminus \{\bot\}$ and $\bot z = z\bot = z$ for all $z \in C$, then $\mathbf{R}_{\mathbf{A},\mathbf{C},*}$ is in TW. Also, all non-linear members of TW are either of this form when $W \neq \emptyset$ or of the form $\mathbf{R}_{\mathbf{A},\{b,b'\}}$ in Theorem 8.6 when $W = \emptyset$.

8.2.8 The class LW. We denote by LW the class of URLs whose non-linear members $\mathbf{R}$ are non-integral, Z_R is linear. Let $\mathbf{R}$ be a non-linear member of LW; the bounded version is denoted by bLW. Since $\mathbf{R}$ is non-linear and Z_R is linear, if $W_R \neq \emptyset$, then $Z_R = {\uparrow} b_0 \setminus \{\top\}$ by Theorem 8.3(7), so $b \parallel 1$ for all $b \in Z_R$; if $W_R = \emptyset$, then $\mathbf{R}$ satisfies (URL), ($W\emptyset$) and (Zlinear), so $\mathbf{R} \in \mathsf{L}$ by Theorem 8.11 and $b \parallel 1$ for all $b \in Z_R$; thus $a \parallel b$ for all $a \in U_R$ and $b \in Z_R$ by Theorem 8.3(3). Note that in this case Z_R does not need to have a least element. If $W_R \neq \emptyset$, then $\mathbf{R}$ satisfies

$$\forall x, \forall y\, (\overline{\top} x = \overline{\top} \text{ or } \overline{\top} y = \overline{\top} \text{ or } x \leq y \text{ or } y \leq x). \qquad \text{(ZWlinear)}$$

Since $\bot < ac, ca \leq b_0$ for all $a \in U_R$ and $c \in W_R$, every $b \in Z_R$ is a multiplicative zero for U_R and elements in U_R are incomparable with those in $Z_R \cup W_R$, we know $U_R \cup \{\bot, \top\}$ is a subalgebra of $\mathbf{R}$. By Theorem 8.3(7), $\uparrow b_0 = Z_R \cup \{\top\}$ is a $(\bot, 1)$-free subalgebra of $\mathbf{R}$ and by Theorem 8.3(5) $W_R \cup \{\bot, b_0\}$ is a totally-ordered semigroup subreduct satisfying $xy = b_0$ for all $x, y \in W_R \cup \{b_0\}$; in particular, in this case Z_R has a least element b_0. Also, by Theorem 8.3(5), $bc = cb = b_0$ and $ca = ac = b_0$ for all $b \in Z_R$, $c \in W_R$ and $a \in U_R$ with $a \parallel 1$. For divisions, we know

$$\begin{aligned} a\backslash b = b/a = b, &\quad b\backslash a = a/b = \bot \\ a\backslash c = c/a \in W_R \cup \{\bot\}, &\quad c\backslash a = a/c = \bot \\ b\backslash c = c/b = \bot, &\quad c\backslash b = b/c = \top, \end{aligned}$$

for all $a \in U_R$, $b \in Z_R$ and $c \in W_R$.

Theorem 8.16. The class LW is axiomatized by (URL) and (ZWlinear).

Proof. Suppose $\mathbf{R}$ is a non-linear residuated lattice satisfying (URL) and (ZWlinear). By (ZWlinear) we know that if $x, y \notin U_R \cup \{\top\}$, then $x \equiv y$. So if $W_R = \emptyset$ then $\mathbf{R}$ satisfies (Zlinear). If, further, we have $Z_R = \emptyset$, then $\mathbf{R}$ is in $\top\mathsf{unital}$; if $Z_R \neq \emptyset$, then $\mathbf{R} \in \mathsf{L}$. If $W_R \neq \emptyset$, then $Z_R \neq \emptyset$ and $Z_R \cup W_R = \updownarrow b_0 \setminus \{\bot, \top\}$ by Theorem 8.3(5) and (7), so $Z_R \cup W_R$ is linear. $\square$

Notice that L is a subclass of LW, and just as the class L, $U_R \cup \{\bot, \top\}$ has a bi-residuated bi-action on $W_R \cup \{\bot, b_0\}$ satisfying

- $\top * c = c * \top = b_0$ for all $c \in W_R \cup \{b_0\}$;
- $a * b_0 = b_0 * a = b_0$ for all $a \in U_R \cup \{\top\}$;

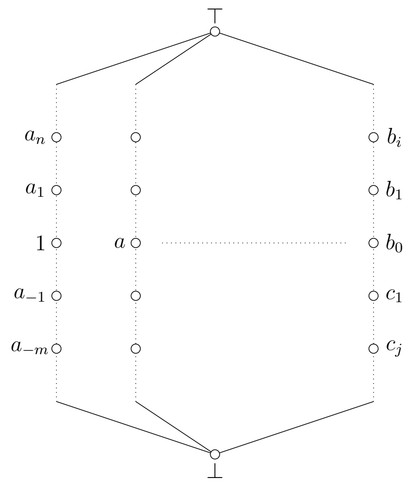

Figure 8.8: LW: $W \neq \emptyset$ and $(Z \cup W, \wedge, \vee)$ is a single chain

- If $a * c = \bot$ or $c * a = \bot$, then either $a = \bot$ or $c = \bot$ for all $a \in U_R \cup \{\bot\}$ and $c \in W_R \cup \{\bot\}$.

Similar to the case of TW, we also give a construction based on a rigorously compact residuated lattice, an integral residuated lattice and a totally-ordered bounded semigroup to obtain a residuated lattice. As a special case we obtain an algebra in LW.

Let $\mathbf{A}$ be a rigorously compact residuated lattice with bounds $\bot$ and $\top$, $\mathbf{C}$ be a bounded lattice-ordered semigroup with bottom $\bot$ and top b_0 satisfying $xy = b_0$ for all $x, y \in C \setminus \{\bot\}$ and $\bot z = z\bot = \bot$ for all $z \in C$ and let $\mathbf{B}$ be an integral residuated lattice with bottom b_0 and top $\top$. Also assume that $\mathbf{A}$ has a bi-residuated bi-action $*$ on $\mathbf{C}$ respecting tops and without zero divisors:

$\top * c = c * \top = b_0$ for all $c \in C \setminus \{\bot\}$

$a * b_0 = b_0 * a = b_0$ for all $a \in A \setminus \{\bot\}$

If $a * c = \bot$ or $c * a = \bot,$ then either $a = \bot$ or $c = \bot$ for all $a \in A$ and $c \in C.$

We consider the set $R_{\mathbf{A},\mathbf{B},\mathbf{C},*} = A \cup B \cup C$ and define a lattice order on it by extending the orders of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and setting all of $\mathbf{C}$ below $\mathbf{B}$ and having $\bot$ and $\top$ as the new bounds, as can be seen in Figure 8.9.

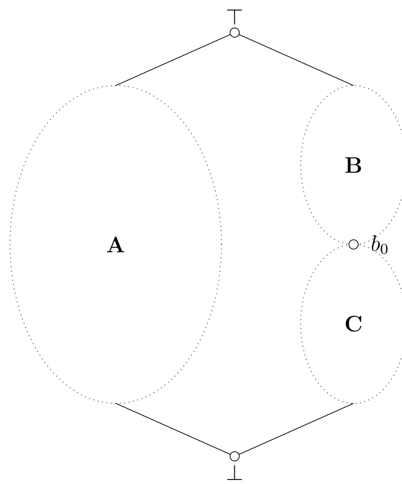

Figure 8.9: $\mathbf{R}_{\mathbf{A},\mathbf{B},\mathbf{C},*}$

Also, we extend the multiplications on $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ by

$$x \cdot y = \begin{cases} x * y & \text{if } x \in A, y \in C \text{ or } x \in C, y \in A \\ x & \text{if } x \in B, y \in A \\ y & \text{if } x \in A, y \in B \\ b_0 & \text{if } x \in B, y \in C \setminus \{\bot\} \text{ or } x \in C \setminus \{\bot\}, y \in B \\ \bot & \text{if } x = \bot \text{ or } y = \bot. \end{cases}$$

We denote the resulting algebra by $\mathbf{R}_{\mathbf{A},\mathbf{B},\mathbf{C},*}$.

Theorem 8.17. Let $\mathbf{A}$ be a rigorously compact residuated lattice with bounds $\bot$ and $\top$, $\mathbf{C}$ be a bounded lattice-ordered semigroup with bottom $\bot$ and top b_0 satisfying $xy = b_0$ for all $x, y \in C \setminus \{\bot\}$ and $\bot z = z\bot = \bot$ for all $z \in C$ and let $\mathbf{B}$ be an integral residuated lattice with bottom b_0 and top $\top$. Also assume that $\mathbf{A}$ has a bi-residuated bi-action $*$ on

$\mathbf{C}$ respecting tops and without zero divisors. Then $\mathbf{R}_{\mathbf{A},\mathbf{B},\mathbf{C},*}$ is the reduct of a residuated lattice, which we denote also the same way. Furthermore, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are linear, then $\mathbf{R}_{\mathbf{A},\mathbf{B},\mathbf{C},*}$ is in LW.

Proof. In the following we use R in short for $R(\mathbf{A}, \mathbf{B}, \mathbf{C}, *)$.

For associativity, since $\perp$ is the multiplicative zero by definition, we focus on the cases not involving $\perp$. Among those cases, we observe that elements in A are multiplicative identities for those in B and the products between elements of $\mathbf{B}$ and $\mathbf{C}$ are constant b_0, multiplicative zero for $B \cup C$, so multiplication between $\mathbf{B}$ and $\mathbf{C}$ is associative. Also, since $*$ is a bi-action and $\mathbf{C} \setminus \{\perp\}$ is a null-semigroup with zero b_0, which also a zero for $A \setminus \{\perp\}$, multiplication between $\mathbf{A}$ and $\mathbf{C}$ is associative. The left cases are

$$ab \cdot c = bc = b_0 = ab_0 = a \cdot bc$$

$$ac \cdot b = b_0 = ab_0 = a \cdot bc$$

$$ba \cdot c = bc = b_0 = b \cdot ac$$

$$bc \cdot a = b_0 a = b_0 = b \cdot ca$$

$$ca \cdot b = b_0 = cb = c \cdot ab$$

$$cb \cdot a = b_0 a = b_0 = cb = c \cdot ba$$

for all $a, a_1, a_2 \in A \setminus \{\perp\}$, $b, b_1, b_2 \in B$ and $c, c_1, c_2 \in C \setminus \{\perp\}$. Other cases are similar or straightforward. So the associativity holds on $\mathbf{R}$.

For residuation, first we prove the multiplication is order-preserving. Since $\perp$ is the multiplicative zero for all elements in R, multiplication involving $\perp$ is order-preserving. Since $\mathbf{A}$ and $\mathbf{B}$ are residuated respectively, multiplications inside themselves are order-preserving respectively. Also, according to the definition of multiplication, elements in $A \setminus \{\perp, \top\}$ are multiplicative identities of those in B. So the multiplication between $\mathbf{A}$ and

$\mathbf{B}$ is order-preserving. Since $\mathbf{C} \setminus \{\bot\}$ is a null semigroup with zero b_0, multiplication of $\mathbf{C}$ is order-preserving. Since the action $*$ is bi-residuated, the multiplication between $\mathbf{A}$ and $\mathbf{C}$ is order-preserving. Also, since the products between elements in B and $C \setminus \{\bot\}$ are constant b_0, multiplication between $\mathbf{B}$ and $\mathbf{C}$ is order-preserving. Finally we show $xy \leq xz$ for all $x \in R \setminus \{\bot\}$ when $\bot < y < b_0 < z < \top$. If $x \in A \setminus \{\bot\}$, then $xy \leq b_0 < z = xz$; if $x \in C \setminus \{\bot\}$, then $xy = b_0 = xz$; otherwise $x \in B$, then $xy = b_0 \leq xz$. Similarly we can show $yx \leq zx$ for all $x \in R \setminus \{\bot\}$ when $\bot < y < b_0 < z < \top$. Therefore the multiplication is order-preserving on $\mathbf{R}$.

Now we want to prove $x \backslash\!\!\backslash z = \{y \in R : xy \leq z\}$ has maximum for all $x, z \in R$. First we observe that there exist functions $\backslash^l : A \times C \to C$, $/^l : C \times C \to A$, $/^r : C \times A \to C$ and $\backslash^r : C \times C \to A$ such that

$$a * c_1 \leq c_2 \text{ iff } c_1 \leq a \backslash^l c_2 \text{ iff } a \leq c_2 /^l c_1$$

$$c_1 * a \leq c_2 \text{ iff } c_1 \leq c_2 /^r a \text{ iff } a \leq c_1 \backslash^r c_2$$

for all $a \in A$, $c_1, c_2 \in C$ since $*$ is a bi-residuated bi-action. By Remark 2.4 we know $\bot \backslash\!\!\backslash z = x \backslash\!\!\backslash \top = R$ for all $x, z \in R$, so $\max \bot \backslash\!\!\backslash z = \max x \backslash\!\!\backslash \top = \top$ in this case. In the following we assume that $x > \bot$ and $z < \top$. To prove toward contradiction we assume that $x \backslash\!\!\backslash z$ doesn't have maximum for some $x, z \in R$, then we know $\top \notin x \backslash\!\!\backslash z$. Since $\bot \in x \backslash\!\!\backslash z$, we know $x \backslash\!\!\backslash z$ is not empty. If there exists $a \in A \setminus \{\bot, \top\}$ such that $a \in x \backslash\!\!\backslash z$, then $x\top \neq xa$, so we know $x \notin B \cup \{\bot\}$. In this case, if $x \in A \setminus \{\bot, \top\}$, then $z \in A \setminus \{\bot\}$ since $A \setminus \{\bot\}$ is closed under multiplication and elements in $A \setminus \{\bot, \top\}$ is incomparable with those in $(B \cup C) \setminus \{\bot, \top\}$. So $x \backslash\!\!\backslash z$ has maximum $x \backslash_{\mathbf{A}} z$, which is a contradiction.

Thus if there exists $a \in A \setminus \{\bot, \top\}$, we know $x \in C \setminus \{\bot, b_0\}$. Again by $xa \leq z$ we know $z \in (B \cup C) \setminus \{\top\}$. If $z \in C \setminus \{\bot\}$, then $x \backslash\!\!\backslash z \subseteq A \setminus \{\top\}$ has maximum $x \backslash^r z$ by the definition of $*$, which is a contradiction. Thus we know $z \in B \setminus \{b_0, \top\}$. However, then we have $x\top = b_0 \leq z$ and $\top \in x\backslash\!\!\backslash z$, which is another contradiction. Hence we know $x\backslash\!\!\backslash z \subseteq (B \cup C) \setminus \{\top\}$.

Since $x\backslash\!\!\backslash z$ doesn't have a maximum, $\{\bot\}$ is a proper subset of $x\backslash\!\!\backslash z$, so there exists $u \in (B \cup C) \setminus \{\bot, \top\}$ such that $u \in x\backslash\!\!\backslash z$. Then $x\top \neq xu$ implies that $x \notin C$. If we know further that $b_0 \notin x\backslash\!\!\backslash z$, then $x\backslash\!\!\backslash z \subseteq C \setminus \{b_0\}$ since $x\backslash\!\!\backslash z$ is downward closed. In this case $xb_0 \neq xc$ for some $c \in C \setminus \{\bot, b_0\}$ implies that $x \notin B \cup C$, so $x \in A \setminus \{\bot\}$. Thus we can tell $z \in C$ in this case, then $x\backslash\!\!\backslash z \subseteq C$ has maximum $x\backslash^l z$, which is a contradiction. Hence there exists $b \in B \setminus \{\top\}$ such that $b \in x\backslash\!\!\backslash z$. In this case, if $x \in A \setminus \{\bot\}$, then $xb = b \leq z$ implies $z \in B \setminus \{\top\}$, so $x\backslash\!\!\backslash z$ has maximum z, which is a contradiction; if $x \in C \setminus \{\bot\}$, then $xb = b_0 \leq z$ implies that $z \in B \setminus \{\top\}$, so $\top \in x\backslash\!\!\backslash z$ and $x\backslash\!\!\backslash z$ has maximum $\top$, which is another contradiction. Thus we know $x \in B$ and $z \in B \setminus \{\top\}$ since $\mathbf{B}$ is closed under multiplication. In this case $x\backslash\!\!\backslash z = \{y \in B : xy \leq z\} \cup C$, so $x\backslash\!\!\backslash z$ has maximum $x\backslash_{\mathbf{B}} z$, a contradiction.

Therefore $x\backslash\!\!\backslash z$ has maximum for all $x, z \in R$. Similarly for $z/\!\!/x = \{y \in R : yx \leq z\}$. Hence $\mathbf{R}$ is a residuated lattice. $\square$

Corollary 8.18. All non-linear members of LW have the form $\mathbf{R}_{\mathbf{A},\mathbf{B},\mathbf{C},*}$ in Theorem 8.17 when $W \neq \emptyset$ or have the form $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ in Theorem 8.9 when $W = \emptyset$.

8.3 Some $\top$-unital unilinear residuated lattices of height 4

As a small application of the the characterization of unilinear residuated lattices, we can describe all residuated lattices whose lattice reducts are of the following form: i.e., all maximal chains have 3 elements, except for one that has 4 elements. We refer to these (residuated) lattices as of *type* h4.1. In Figure 8.11 we list all types of residuated lattices coming from the characterization given in Corollary 8.4, except for the linear and 4-element Boolean algebra case.

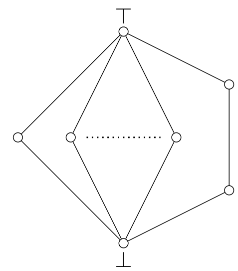

Figure 8.10: Type $h4.1$

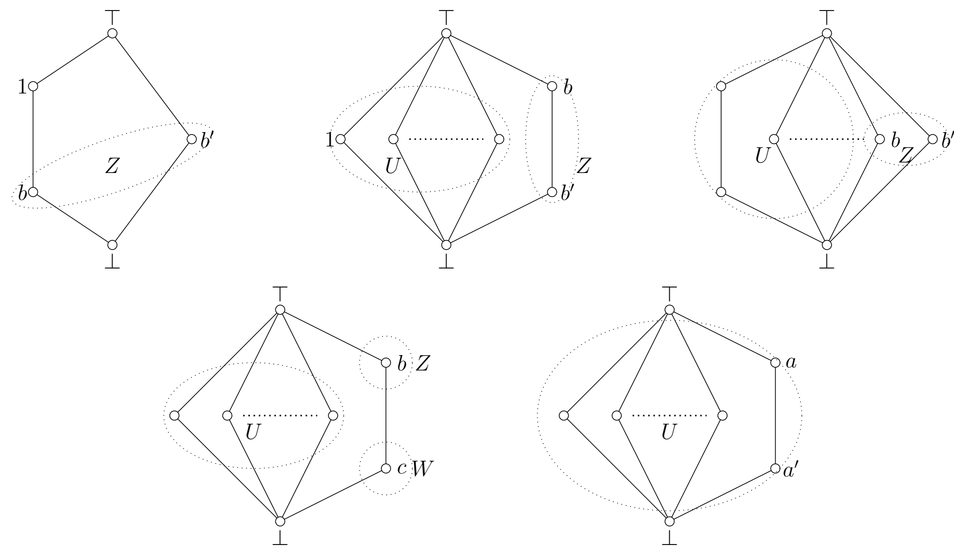

Figure 8.11: The algebras of type h4.1 are in T, L, B, LW and ⊤unital, respectively.

We know that in the first four cases the algebra can be obtained by a ⊤unital algebra, which actually has height 3, hence it is characterized in Chapter 3. So, we will give a characterization of residuated lattices only of the latter type, where the algebra is ⊤unital.

We distinguish 3 subcases depending on the location of the multiplicative identity 1. But first we give a property enjoyed by all subcases.

Proposition 8.19. If $\mathbf{R}$ is a non-linear $\top$-unital residuated lattice whose subchains are finite, then $R \setminus \{\bot, 1\}$ is closed under multiplication.

Proof. Since $R \setminus \{\bot\}$ is closed under multiplication, to prove toward contradiction we assume that there exist $x, y \in R \setminus \{\bot, 1\}$ such that $xy = 1$, then both x and y are invertible.

Since every subchain of $\mathbf{R}$ is finite, we know $x \parallel 1$ and if x and y are distinct then they are incomparable; otherwise $\mathbf{R}$ contains a totally-ordered group, which is infinite. Since $\mathbf{R}$ is $\top$-unital, we have $\top = \top y = (1 \vee x)y = y \vee xy = y \vee 1$, so $y \parallel 1$. Let $\bot < a' < a < \top$ be the unique long chain in $\mathbf{R}$. Since $a' > a$ and y is invertible, we have $ya > ya'$ by order-preservation of the multiplication. According to the lattice reduct and the fact that $R \setminus \{\bot\}$ is closed under multiplication, we know either $ya = \top$ or $ya = a$, $ya' = a'$. If $ya = \top$, then $a = xy \cdot a = x \cdot ya = x\top = \top$, a contradiction. If $ya = a$, then $\top = (1 \vee y)a = a \vee ya = a$, which is another contradiction. Therefore $R \setminus \{\bot, 1\}$ is closed under multiplication. □

Now we give the characterization of all the subcases.

8.3.1 1 **is low in the long chain.** We start with a $\top$unital residuated lattice of type h4.1 and 1 is low on the long chain, and characterize its structure.

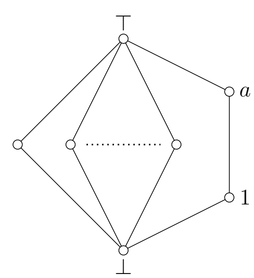

Figure 8.12: 1 is low in the long chain

Proposition 8.20. If $\mathbf{R}$ is a $\top$unital residuated lattice of type h4.1 and 1 is low and a is high on the long chain, then

(i) $S = R \setminus \{\bot, 1\}$ gives a $\top$-cancellative semigroup;

(ii) for all $x \in S$, we have $xa, ax \in \{x, \top\}$;

(iii) for all $x \in S$ and $y \in S \setminus \{a\}$, we have

$$(x, xy), (xy, x), (x, yx), (yx, x) \in S^2 \setminus \Delta_{S\setminus\{\top\}}$$

where $\Delta_{S\setminus\{\top\}} = \{(s, s) : s \in S \setminus \{\top\}\}$.

Proof. (i) By Proposition 8.19, we know S is closed under multiplication. Let $x, y, z \in S$ such that $y \neq z$. Since elements in $S \setminus \{\top\}$ are incomparable iff they are distinct, we know $y \vee z = \top$. Since $\mathbf{R} \in \top\mathsf{unital}$, we get $\top = x\top = x(y \vee z) = xy \vee xz$, so $xy = \top$ or $xz = \top$ or $xy \parallel xz$. If $xy \parallel xz$, then $xy \neq xz$ since S is closed under multiplication and elements in $S \setminus \{\top\}$ are incomparable iff they are distinct. Thus we get $y \neq z \implies xy = \top$ or $xz = \top$ or $xy \neq xz$, which is equivalent to $xy = xz \neq \top \implies y = z$. Similarly for $yx = zx \neq \top \implies y = z$.

(ii) Since $a > 1$ and $\top$ is the upper cover for all $x \in S$, we get $xa, ax \in \{x, \top\}$ by order-preserving of the multiplication.

(iii) Let $x \in S$ and $y \in S \setminus \{a\}$, then either $y = \top$ or $y \parallel 1$ and hence $y \vee 1 = \top$. Since $\mathbf{R}$ is $\top$-unital, we know $\top = x\top = x(y \vee 1) = xy \vee x$, so $x = \top$ or $xy = \top$ or $xy \parallel x$, the last of which implies to $xy \neq x$. Thus $(x, xy) \in (S^2 \setminus \Delta_S) \cup (\{\top\} \times S) \cup (S \times \{\top\}) = S^2 \setminus \Delta_{S\setminus\{\top\}}$. Similarly for $(xy, x), (x, yx)$ and (yx, x). □

Conversely we can construct a residuated lattice as above, given a special semigroup. Let $\mathbf{S}$ be a semigroup with zero $\top$ satisfying

(i) $\mathbf{S}$ is $\top$-cancellative;

(ii) there exists $a \in S \setminus \{\top\}$ such that for all $x \in S$, we have $ax, xa \in \{x, \top\}$;

(iii) for all $x \in S$ and $y \in S \setminus \{a\}$, we have

$$(x, xy), (xy, x), (x, yx), (yx, x) \in S^2 \setminus \Delta_{S\setminus\{\top\}}$$

where $\Delta_{S\setminus\{\top\}} = \{(s,s) : s \in S \setminus \{\top\}\}$.

We consider the set $R_{\mathbf{S},\ell} = S \cup \{\bot, 1\}$ and define a lattice structure of type h4.1 on it so that 1 is low and a is high in the long chain. Also, we extend the multiplication on $\mathbf{S}$ by making 1 the unit and $\bot$ an absorbing element. We denote by $\mathbf{R}_{\mathbf{S},\ell}$ the resulting algebra.

Theorem 8.21. If $\mathbf{S}$ is a semigroup with above properties, then $\mathbf{R}_{\mathbf{S},\ell}$ is the reduct of a $\top$unital residuated lattice of type h4.1 and 1 is low on the long chain. Moreover, this is a characterization of the latter.

Proof. Since 1 is the multiplicative identity and $\bot$ is the multiplicative zero, associativity holds for the multiplication on $\mathbf{R}$. For residuation, since the lattice reduct of $\mathbf{R}$ is complete, it suffices for us to show the multiplication distributes over join. Since an arbitrary join in $\mathbf{R}$ is equivalent to a finite joint, it suffices to show

$$x(y \vee z) = xy \vee xz$$

$$(y \vee z)x = yx \vee zx$$

for all $x, y, z \in R$ with $y \neq z$.

First, it's easy to see the equation always holds if $x \in \{\bot, 1, \top\}$ or $y \in \{\bot, \top\}$ or $z \in \{\bot, \top\}$. So in the following we assume that $x \notin \{\bot, 1, \top\}$ and $y, z \notin \{\bot, \top\}$. Also we can tell that elements in $S \setminus \{\top\}$ form an antichain in $\mathbf{R}$. If $y = 1$ and $z = a$, then $y \vee z = a$ and $xy = x, xz = xa$. Since $xa \in \{x, \top\}$, we know $xa \geq x$ for all $x \in S$, so $x(y \vee z) = xa = xy \vee xz$. Similarly for the case when $y = a$, $z = 1$. Now assume $y \parallel z$, then $y \vee z = \top$, so we know $x(y \vee z) = \top$. If $y = 1$, then $xy \vee xz = x \vee xz$. Since $z \parallel 1$, we know $z \neq a$. By assumption (iii) for $\mathbf{S}$ we get $(x, xz) \in S^2 \setminus \Delta_{S\setminus\{\top\}}$, so $x \neq xz$ and hence $x \vee xz = \top$ since S is closed under multiplication. Similarly for the case when $z = 1$. Finally, if $y \neq 1$ and $z \neq 1$, then $xy \vee xz = \top$ by the $\top$-cancellativity of $\mathbf{S}$.

Similarly we can prove $(y \vee z)x = yx \vee zx$. Therefore $\mathbf{R}$ is a residuated lattice. □

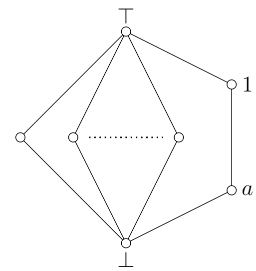

Figure 8.13: 1 is high in the long chain

8.3.2 1 is high in the long chain.

Proposition 8.22. If $\mathbf{R}$ is a $\top$unital residuated lattice of type h4.1 and 1 is high and a is low on the long chain, then

(i) $S = R \setminus \{\bot, 1\}$ is a $\top$-cancellative semigroup;

(ii) for all $x \in S$, we have $xa = ax = x$;

(iii) for all $x \in S$ and $y \in S \setminus \{a\}$, we have

$$(x, xy), (xy, x), (x, yx), (yx, x) \in S^2 \setminus \Delta_{S\setminus\{\top\}}$$

where $\Delta_{S\setminus\{\top\}} = \{(s, s) : \, s \in S \setminus \{\top\}\}$.

Proof. The proof is almost the same as the case when 1 is low in the long chain. The only difference is that $xa = ax = x$ for all $x \in S$, which is the result of the facts that $\bot < a < 1$, that multiplication is order-preserving and S is closed under multiplication. □

Conversely, let $\mathbf{S}$ be a semigroup with zero $\top$ satisfying

(i) $\mathbf{S}$ is $\top$-cancellative;

(ii) there exists $a \in S \setminus \{\top\}$ such that for all $x \in S$, we have $xa = ax = x$;

(iii) for all $x \in S$ and $y \in S \setminus \{a\}$, we have

$$(x, xy), (xy, x), (x, yx), (yx, x) \in S^2 \setminus \Delta_{S \setminus \{\top\}}$$

where $\Delta_{S \setminus \{\top\}} = \{(s, s) : s \in S \setminus \{\top\}\}$.

As before we define an algebra $\mathbf{R}_{\mathbf{S},h}$, based on the set $R = S \cup \{\bot, 1\}$ with multiplication extending that of $\mathbf{S}$ by making 1 a unit and $\bot$ bottom, but with a lattice structure where 1 is high and a is low on the long chain.

Theorem 8.23. If $\mathbf{S}$ is a semigroup with the above properties, then $\mathbf{R}_{\mathbf{S},h}$ is the reduct of a $\top$unital residuated lattice of type h4.1 and 1 is low on the long chain. Moreover, this is a characterization of the latter.

Proof. Similar to the proof of the case when 1 is low on the long chain. ☐

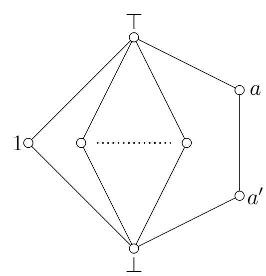

Figure 8.14: 1 is on a short chain

8.3.3 1 is on a short chain.

Proposition 8.24. If $\mathbf{R}$ is a $\top$unital residuated lattice of type h4.1 and 1 is on a short chain, then

(i) for all $x \in S = R \setminus \{\bot, 1\}$,

$$xa \neq xa' \implies xa = \top$$
$$ax \neq a'x \implies ax = \top;$$

(ii) for all $x, y \in S$, we have

$$(xy, x), (xy, y), (yx, x), (yx, y) \in T$$
$$(x, xy), (y, xy), (x, yx), (y, yx) \in T,$$

where $T = S^2 \setminus (\{a, a'\}^2 \cup \Delta_{S\setminus\{\top\}})$ and $\Delta_{S\setminus\{\top\}} = \{(s, s) : s \in S \setminus \{\top\}\}$;

(iii) for all $x \in S$ and $(y, z) \in T$, we have $(xy, xz) \in T$ and $(yx, zx) \in T$.

Proof. (i) Let $x \in S$, then either $x = \top$ or $x \parallel 1$, so $1 \vee x = \top$. Since multiplication on $\mathbf{R}$ is order-preserving and $a \geq a'$, we have $xa \geq xa'$. Suppose $xa > xa'$. Since S is closed under multiplication, we know either $xa = \top$ or $xa = a, xa' = a'$. If $xa = a, xa' = a'$, then we get $\top = \top a = (1 \vee x)a = a \vee xa = a$, a contradiction. Thus $xa \neq xa' \implies xa = \top$. Similarly for $ax \neq a'x \implies ax = \top$.

(ii) First we show that $s_1 \vee s_2 = \top$ is equivalent to $(s_1, s_2) \in T$ for all $s_1, s_2 \in S$.

$$\begin{aligned} s_1 \vee s_2 = \top &\Leftrightarrow s_1 = \top \text{ or } s_2 = \top \text{ or } s_1 \parallel s_2 \\ &\Leftrightarrow s_1 = \top \text{ or } s_2 = \top \text{ or } (s_1, s_2) \in S^2 \setminus (\{(a, a'), (a', a)\} \cup \Delta_S) \\ &\Leftrightarrow (s_1, s_2) \in T. \end{aligned}$$

Now let $x, y \in S$, then we have $x \vee 1 = y \vee 1 = \top$, so $\top = \top y = (x \vee 1)y = xy \vee y$. By the equivalence above, we know $(xy, y) \in T$. Similarly for $(xy, x) \in T, (yx, x) \in T$ and $(yx, y) \in T$. The other parts come from the symmetry of T.

(iii) Let $x \in S$ and $(y, z) \in T$, then we have $y \vee z = \top$ by the equivalence in (ii). Since $\top = x\top = x(y \vee z) = xy \vee xz$, we get $(xy, xz) \in T$. Similarly for $(yx, zx) \in T$. $\square$

Conversely, let $\mathbf{S}$ be a semigroup with zero $\top$ and assume there exist $a, a' \in S \setminus \{\top\}$ such that

(i) for all $x \in S$,

$$xa \neq xa' \implies xa = \top$$
$$ax \neq a'x \implies ax = \top;$$

(ii) for all $x, y \in S$, we have

$$(xy, x), (xy, y), (yx, x), (yx, y) \in T$$
$$(x, xy), (y, xy), (x, yx), (y, yx) \in T,$$

where $T = S^2 \setminus (\{a, a'\}^2 \cup \Delta_{S \setminus \{\top\}})$ and $\Delta_{S\setminus\{\top\}} = \{(s, s) : s \in S \setminus \{\top\}\}$;

(iii) for all $x \in S$ and $(y, z) \in T$, we know $(xy, xz) \in T$ and $(yx, zx) \in T$.

We consider the set $R_{\mathbf{S}} = S \cup \{\bot, 1\}$ and define a lattice order of type h4.1 where 1 is on a short chain, a' is low and a is high on the long chain. Also, extend the multiplication on $\mathbf{S}$ by making 1 a unit and $\bot$ a bottom element.

Theorem 8.25. If $\mathbf{S}$ is a semigroup with the above properties, then $\mathbf{R_S}$ is the reduct of a $\top\mathsf{unital}$ residuated lattice of type h4.1 and 1 is on a short chain. Moreover, this is a characterization of the latter.

Proof. As previous cases, it's easy to see the associativity of multiplication on $\mathbf{R}$ holds. For residuation, since the lattice reduct is complete and all infinite join is equivalent to a finite join, it suffices to show

$$x(y \vee z) = xy \vee xz$$
$$(y \vee z)x = yx \vee zx$$

for all $x, y, z \in R$ with $y \neq z$.

We assume that $x \notin \{\bot, 1, \top\}$ and $y, z \notin \{\bot, \top\}$. If $y = a', z = a$, then $y \vee z = a$ and $x(y \vee z) = xa$. In this case, if $xa = xa'$, then the equation holds; otherwise, by condition (i), we have $x(y \vee z) = xa' \vee xa = xa' \vee \top = \top = xy \vee xz$. Similarly for the case when $y = a$ and $z = a'$. Now suppose $y \parallel z$, then we know $y \vee z = \top$ and $x(y \vee z) = \top$. By the definition of the lattice reduct, we know $s_1 \vee s_2 = \top$ iff $(s_1, s_2) \in T$ for all $s_1, s_2 \in S$. If $y = 1$, then $z \in S$ and $xy \vee xz = x \vee xz$. Since $(x, xz) \in T$ by condition (ii), $x \vee xz = \top$ holds. Similarly when $z = 1$ and $y \in S$. Otherwise $y, z \in S$ and $(y, z) \in T$. By condition (iii), we have $(xy, xz) \in T$, which is equivalent to $xy \vee xz = \top$. Therefore $x(y \vee z) = xy \vee xz$ holds. Similarly for $(y \vee z)x = yx \vee zx$. $\square$

8.4 Distributivity of multiplication over meet

In this short section, we look into how different levels of distribution of multiplication over meet in a unilinear residuated lattice $\mathbf{R}$ impacts the sets U_R, Z_R and W_R.

Lemma 8.26. If multiplication distributes over meet in a non-linear URL $\mathbf{R}$, then $\{x \in R : \top x = \top\}$ is a chain.

Proof. If there exist a_1, a_2 in the set such that $a_1 \parallel a_2$, then $\bot = \top(a_1 \wedge a_2) = \top a_1 \wedge \top a_2 = \top$, a contradiction. $\square$

Therefore, distribution of multiplication over meet is a very strong condition. We consider the weaker sentence

$$\forall x_1, x_2, x_3\, (\underset{1 \leq i \leq 3}{\mathrm{OR}}\ \overline{\top} x_i = x_i \text{ or } x_1(x_2 \wedge x_3) = x_1x_2 \wedge x_1x_3). \tag{$\cdot_Z\wedge$}$$

Theorem 8.27. Let $\mathbf{R} \in \mathsf{URL}$ such that U_R is non-linear. Then ($\cdot_Z\wedge$) holds if and only if U_R is closed under multiplication and $W_R = \emptyset$.

Proof. Assume ($\cdot_Z\wedge$) holds first. To prove toward contradiction suppose there exists $c \in W_R$. Since $\mathbf{R}$ is non-linear, we know $c \parallel 1$ and $c^2 = b_0$ by Theorem 8.3 (4) and (5). Then we get

$$\bot = c(1 \wedge c) = c \wedge c^2 = c \wedge b_0 = c,$$

which is a contradiction. Now suppose there exist $a_1, a_2 \in U_R$ such that $a_1 a_2 = \top$. Since U_R is non-linear, there exists $a_3 \in U_R$ such that $a_1 \parallel a_3$ or $a_2 \parallel a_2$. Without loss of generality, $a_2 \parallel a_3$, so

$$\bot = a_1(a_2 \wedge a_3) = a_1 a_2 \wedge a_1 a_3 = \top \wedge a_1 a_3 = a_1 a_3.$$

But this yields $\top = \top a_3 = \top a_1 \cdot a_3 = \top \cdot a_1 a_3 = \top\bot = \bot$, another contradiction. Thus U is closed multiplication and $W_R = \emptyset$.

Now assume U_R is closed multiplication and $W_R = \emptyset$. Let $x, y, z \in R$ such that $\top x \neq x$ and $\top y \neq y$ and $\top z \neq z$, then x, y, z are not in $Z_R \cup \{\bot, \top\}$. Since $W_R = \emptyset$, we know $x, y, z \in U_R$. Without loss of generality, we show $x(y \wedge z) = xy \wedge xz$ here. If $y \equiv z$, then $x(y \wedge z) = xy \wedge xz$ holds since multiplication preserves ordering. Otherwise $y \parallel z$. Since $x(y \vee z) = xy \vee xz$ always holds, we know $xy \vee xz = \top$. Since U is closed under multiplication, we know $xy \parallel xz$. Thus $x(y \wedge z) = x\bot = \bot = xy \wedge xz$. $\square$

Remark 8.28. By a similar proof we can show U_R being closed under multiplication and $W_R = \emptyset$ iff

$$\forall x_1, x_2, x_3 \, (\underset{1\leq i\leq 3}{\mathrm{OR}} \, \overline{\top} x_i = x_i \text{ or } (x_2 \wedge x_3)x_1 = x_2 x_1 \wedge x_3 x_1), \qquad (\wedge\cdot_Z)$$

given that U_R is non-linear in $\mathbf{R}$. Therefore ($\cdot_Z\wedge$) is also equivalent to ($\wedge\cdot_Z$) when U_R is non-linear.

Chapter 9: Idempotent unilinear residuated lattices

In this chapter we will focus on unilinear and semiunilinear residuated lattices that are idempotent and study their structure. We denote the corresponding classes by IdURL and IdSRL and their bounded analogues by bIdURL and bIdSRL. We will prove the congruence extension property and identify natural subclasses that have the (strong) amalgamation property. Recall that up to now the distinction between having the bounds in the language or not was unimportant. However, since the amalgamation property is sensitive to subalgebra generation, we will see that the inclusion or not of the bounds in the language leads to different amalgamation results.

We start with an important simplification on the unilinear structure when we restrict to idempotent algebras.

Proposition 9.1. If $\mathbf{R}$ is an idempotent non-linear unilinear residuated lattice, then U_R is a subchain of the chain of 1 and W_R is empty.

Proof. Since $\mathbf{R}$ is non-linear, there exists $b \in R$ such that $b \parallel 1$, then we have $b = b \vee b^2 = (1 \vee b)b = \top b$, so $b \in Z_R$. Therefore $U_R \subseteq \updownarrow 1$ and $W_R = \emptyset$ by Theorem 8.3(4). □

So by Theorem 8.3 every idempotent URL is the 4-element Boolean algebra, or it is contained in one of the classes T, B and L; see Figure 9.1.

9.1 Congruence extension property

Given the language of bURL and variable y, first we define terms s and t by $s(y) = y \wedge y^{\ell\ell} \wedge y^{rr}$ and $t(y) = s(y) \wedge 1$ where $y^{\ell} = 1/y$ and $y^{r} = y\backslash 1$.

Lemma 9.2. If $\mathbf{A}$ is an algebra in $\mathsf{(b)IdSRL}$, then

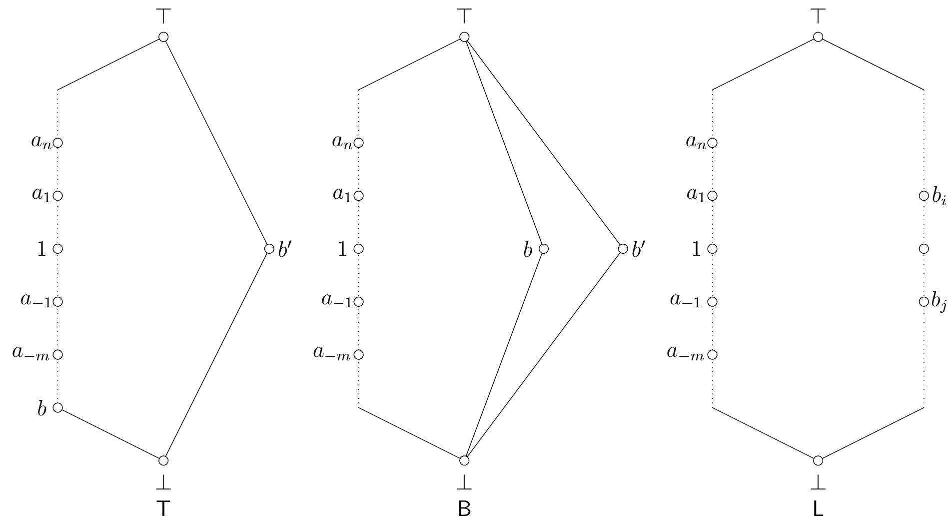

Figure 9.1: Non-linear non-integral idempotent unilinear residuated lattices

1. $s(y) \leq xy/x, x\backslash yx$ for all $x, y \in A$.

2. A deductive filter of $\mathbf{A}$ is closed under the term operations s and t.

Proof. (1) If $\mathbf{A}$ is linear then $\mathbf{A}$ is semiconic and Lemma 3.1 in [5] implies that $s(y) \leq xy/x, x\backslash yx$ for all $x, y \in A$. We will now assume that $\mathbf{A}$ is nonlinear and consider cases for the class that $\mathbf{A}$ belongs to.

If $\mathbf{A}$ is the 4-element generalized Boolean algebra, then $Z_A = \{b, b'\}$ and $b \parallel b'$. So $s(b) = b \wedge b^{\ell\ell} \wedge b^{rr} = b$ and $b'b/b' = b'\backslash bb' = b$, hence $s(b) \leq b'b/b', b'\backslash bb'$; similarly for $s(b') \leq bb'/b, b\backslash b'b$. Thus the equations hold for all $x, y \in A$.

If $\mathbf{A} \in \mathsf{T}$, then $Z_A = \{b, b'\}$ and $b \leq 1$, $b' \parallel 1$ by Theorem 8.5. Since $\updownarrow 1$ is the universe of a subalgebra of $\mathbf{A}$ and $\updownarrow b'$ itself is an integral residuated chain, by Lemma 3.1 in [5] we know the equations hold when $x, y \in \updownarrow 1$ or $x, y \in \updownarrow b'$. Now let $y = b'$, then $s(b') = b' \wedge b'^{\ell\ell} \wedge b'^{rr} = b'$ since $b'^{\ell\ell} = b'^{rr} = b'$. If $x \in U_A \cup \{b, \top\}$, then $xb'/x = x\backslash b'x = b'$; if $x = \bot$, then $\bot b'/\bot = \bot\backslash b'\bot = \top$, so $s(b') \leq xb'/x, x\backslash b'x$ for all $x \in \updownarrow 1$. Then let $x = b'$. If $y \in U_A \cup \{\top\}$, then $b'y/b' = b'\backslash yb' = \top$ and the equations trivially hold. If

$y = b$, then $s(y) = b \wedge b^{\ell\ell} \wedge b^{rr} = b$ and $b'b/b' = b'\backslash bb' = b$, so the equations also hold. If $y = \bot$, then $s(y) = \bot$ so the equations trivially hold.

If $\mathbf{A} \in \mathsf{B}$, then $Z_A = \{b, b'\}$ and $1, b, b'$ is an antichain by Theorem 8.8. Since $\updownarrow 1$ is the universe of a subalgebra of $\mathbf{A}$, by Lemma 3.1 in [5] we know the equations hold when $x, y \in \updownarrow 1$. Let $y = b$, then $s(b) = b \wedge b^{\ell\ell} \wedge b^{rr} = b$ since $b^{\ell\ell} = b^{rr} = b$. If $x \in \{\bot, b\}$, then $xb/x = x\backslash bx = \top$, so the equations trivially hold; otherwise $x \in U \cup \{b', \top\}$, then $xb/x = x\backslash bx = b$ and equations also hold. Similarly for the case when $y = b'$. Now let $x \in \{b, b'\}$ and $y \in \updownarrow 1$. If $y \in U \cup \{\top\}$, then $xy/x = x\backslash yx = \top$ for all $x \in \{b, b'\}$, so the equation hold trivially. If $y = \bot$, then $s(\bot) = \bot$ and equations also trivially hold.

If $\mathbf{A} \in \mathsf{L}$, then Z_A is totally-ordered. Since $\updownarrow 1$ is the universe of a subalgebra of $\mathbf{A}$ and $Z_A \cup \{\bot, \top\}$ itself is an integral residuated chain, we know the equations hold when $x, y \in \updownarrow 1$ or $x, y \in Z_A \cup \{\bot, \top\}$ by [5] Lemma 3.1. For the rest cases, first let $x \in \updownarrow 1$ and $y \in Z_A$. Then $s(y) = y \wedge y^{\ell\ell} \wedge y^{rr} = y \wedge \top \wedge \top = y = xy/x = x\backslash yx$. Finally let $x \in Z_A$ and $y \in \updownarrow 1$. Then if $y \neq \bot$ then $xy/x = x\backslash yx = \top$, so the equations trivially hold; if $y = \bot$ then $s(y) = \bot$ and the equations also trivially hold.

(2) let $F \subseteq A$ be a deductive filter. If $\mathbf{A}$ is linear, then by Lemma 3.1 in [5] F is closed under s and t. So assume that $\mathbf{A}$ is nonlinear. First we observe that if there exists $b \in F$ such that $b \parallel 1$, then $F = A$ since $1 \wedge b$, so F is closed under s. So we assume that F is a linear. Since $1 \in F$, we know $F \subseteq \updownarrow 1$. Since $\updownarrow 1$ is a subalgebra of $\mathbf{A}$, $F = F \cap \updownarrow 1$ is a deductive filter on $\updownarrow 1$, so by Lemma 3.1 in [5] again F is closed under s and t. $\square$

An algebra $\mathbf{B}$ has *congruence extension property* (CEP) if for any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and $\Theta \in \mathrm{Con}(\mathbf{A})$, there exists $\Phi \in \mathrm{Con}(\mathbf{B})$ such that $\Phi \cap A^2 = \Theta$. A class of algebras is said to have CEP if each of its member has CEP.

Theorem 9.3. The varieties IdSRL and bIdSRL have the congruence extension property.

Proof. By Theorem 3.47(2) in [12], we know the lattice of congruence of a residuated lattice is isomorphic to the lattice of deductive filters. So it suffices for us to show that for algebras $\mathbf{A}, \mathbf{B} \in \mathsf{IdSRL}$ (bIdSRL) with $\mathbf{A} \leq \mathbf{B}$ and a deductive filter $F \subseteq A$, there exists deductive filter $G \subseteq B$ such that $F = G \cap A$. Let G be the deductive filter of $\mathbf{B}$ generated by F, then we know $F \subseteq G \cap A$. Now Let $x \in G \cap A$, then $x \in A$ and x is in an upset of a product of iterated conjugates of elements in F. By Lemma 9.2(1), we know $y \wedge y^{\ell\ell} \wedge y^{rr} \leq zy/z, z\backslash yz$ for all $y \in F$ and $z \in B$, so x is in an upset of a product of compositions of term operations t of elements in F. Since F is closed under meet, multiplication and t, the product of those compositions of t's is also in F. Since $x \in A$ and F is upward-closed in $\mathbf{A}$, we know $x \in F$. Therefore $F = G \cap A$ and CEP holds in the class IdSRL (bIdSRL). ☐

9.2 Subclass of $\mathsf{B4}$

It is shown in [5] that the class of idempotent residuated chains does not have the amalgamation property (AP) but the class of $\star$-involutive idempotent residuated chains has the strong amalgamation property (sAP). By Theorem 8.3 we know that in a non-linear members of B and L the chain of 1 is a subalgebra and also that in non-linear members of T the set $\uparrow b$ is a $\bot$-free subalgebra (in the language of bURL) or a subalgebra (in the language of URL); also, $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra of the non-linear algebras in all cases. This makes some of these subalgebras amenable to the results in [5] and our goal is to extend them to the whole algebra.

In the following part we will axiomatize the subclasses of idempotent members in $\mathsf{B4}$, T, B and L which satisfy the natural condition given in [5]. Then we will give proofs and counterexamples for the (strong) amalgamation property (AP/sAP) in each class.

The subclass of idempotent members of $\mathsf{B4}$ is axiomatized by $\forall x\,(\overline{\top}x = x = x\overline{\top})$ and

$$\forall x\,(x^2 = x). \tag{idem}$$

So all the linear members in this class are Gödel chains and the only non-linear member is the 4-element generalized Boolean algebra. We use $\mathsf{B4_i}$ for the subclass of IdURL and $\mathsf{bB4_i}$ for the subclass of bIdURL. Let $\mathbf{A}$ be the 3-element Gödel chain, $\mathbf{B}$ be the 4-element generalized Boolean algebra and $\mathbf{C}$ be the 4-element Gödel chain, then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{B4_i}$ ($\mathsf{bB4_i}$). Since the 4-element generalized Boolean algebra is the only non-linear member in $\mathsf{B4_i}$ ($\mathsf{bB4_i}$), it's impossible to find an amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Thus both $\mathsf{B4_i}$ and $\mathsf{bB4_i}$ fail amalgamation property.

Unlike other cases following, we don't explore the $\star$-involutive subclass of $\mathsf{B4_i}$ and $\mathsf{bB4_i}$, since 1 is the only $\star$-involutive element in an algebra in $\mathsf{B4_i}$ ($\mathsf{bB4_i}$).

9.3 Subclass of T

Given a residuated lattice $\mathbf{R}$ and $x \in R$, we define

$$x^\ell = 1/x,\ x^r = x\backslash 1 \text{ and } x^\star = x^\ell \wedge x^r.$$

$\mathbf{R}$ is called $\star$-*involutive* if $x^{\star\star} = x$ for all $x \in R$. In non-linear algebras in T we have $b'^{\star\star} = \top \neq b'$, so $\star$-involutivity is not automatically satisfied for all elements. It turns out that for extending the amalgamation results from chains to unilinear residuated lattices, we only need to stipulate the $\star$-involutivity condition only for the elements comparable to 1, so we consider this restricted notion of $\star$-involutivity, so as not to excluding algebras where involutivity fails for some elements incomparable to 1.

We denote by $\mathsf{T_i}$ the class of idempotent algebras in T that satisfy involutivity for all elements comparable to 1:

$$\forall x, y\, (x \leq y \text{ or } ((x \vee 1)^{\star\star} = x \vee 1 \text{ and } (x \wedge 1)^{\star\star} = x \wedge 1)) \qquad (\star\text{-inv}\updownarrow 1)$$

This axiom means if an element is comparable with 1, then it is either the $\bot$ or $\star$-involutive. For the convenience of following proofs, we prove a lemma first.

Lemma 9.4. Let $\mathbf{R}$ be an idempotent residuated chain satisfying ($\star$-inv$\updownarrow$1), then either $R \setminus \{\bot\}$ is the universe of a $\star$-involutive subalgebra of $\mathbf{R}$ or $\mathbf{R}$ is $\star$-involutive.

Proof. Since $\mathbf{R}$ is a residuated chain and satisfies ($\star$-inv$\updownarrow$1), we know every $x \neq \bot$ is $\star$-involutive. Suppose $R\setminus\{\bot\}$ is not the universe of a subalgebra of $\mathbf{R}$, then there exist $x, y \in R \setminus \{\bot\}$ such that $x\backslash y = \bot$ or $x/y = \bot$, since the multiplication on $\mathbf{R}$ is conservative. Without loss of generality, assume $x\backslash y = \bot$, then $x > y$ and by Corollary 4.1 in [5] we get $\bot = x\backslash y = x^r \wedge y$. Since $y \in R \setminus \{\bot\}$ and $\mathbf{R}$ is linear, we know $x^r = \bot$, so $x^\star = x^\ell \wedge x^r = \bot$. Since x is $\star$-involutive, we have $x = x^{\star\star} = \bot^\star = \bot^\ell \wedge \bot^r = \top$, thus $\bot^{\star\star} = \top^\star = x^\star = \bot$ and hence $\mathbf{R}$ is $\star$-involutive. $\square$

9.3.1 Amalgamation property of $\mathsf{bT_i}$ and $\mathsf{T_i}$.

Now we will explore the amalgamation property on the class $\mathsf{T_i}$. By Lemma 4.9, the non-linear members of URL and bURL are identical. However, the subalgebras of them are not. In our case such difference matters. We will see the existence of constants $\bot$ and $\top$ help build sAP and missing of them leads to the failure of AP. To distinguish these two cases, now we use $\mathsf{T_i}$ for the subclass of IdURL and $\mathsf{bT_i}$ for the subclass of bIdURL, both of which are axiomatized by axioms of T, (idem) and ($\star$-inv$\updownarrow$1).

First let's check $\mathsf{bT_i}$. Since $\bot$ and $\top$ are constants in the language of bURL and $b = \top\backslash 1, b' = b\backslash\bot$, a subalgebra of a non-linear member of $\mathsf{bT_i}$ is also a non-linear member of $\mathsf{bT_i}$. The linear members of $\mathsf{bT_i}$ are bounded residuated chains satisfying (idem) and ($\star$-inv$\updownarrow$1).

Theorem 9.5. The strong amalgamation property (sAP) holds on $\mathsf{bT_i}$.

Proof. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be a V-formation in $\mathsf{bT_i}$. Since $\mathsf{bT_i}$ is a positive universal class, it is closed under isomorphisms, so we can assume $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively and $A = B \cap C$.

First assume that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are linear members of $\mathsf{bT_i}$. Since $\mathsf{bT_i}$ satisfies ($\star$-inv$\updownarrow$1), either all of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $\star$-involutive or $\bot$ is not $\star$-involutive in one of them. In the latter case, since $\bot$ is a constant in the language, $A = B \cap C$ and A is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively, we know $\bot$ is not $\star$-involutive in all of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, so by Lemma 9.4 $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ are $\star$-involutive idempotent residuated chains, where $A' = A \setminus \{\bot\}$, $B' = B \setminus \{\bot\}$ and $C' = C \setminus \{\bot\}$.

Assume $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation of $\star$-involutive idempotent residuated chains, then by Theorem 5.7 in [5], there exists a $\star$-involutive idempotent residuated chain $\mathbf{D}$ such that $D = B \cup C$ and $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{D}$ respectively. Since $\bot, \top \in B \cap C$, $\mathbf{D}$ is also bounded by them, so $\mathbf{D} \in \mathsf{bT_i}$ and it is a strong amalgam for the V-formation. Now suppose $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ is a V-formation of $\star$-involutive idempotent residuated chains, then again by Theorem 5.7 in [5], there exists a $\star$-involutive idempotent residuated chain $\mathbf{D}'$ such that $D' = B' \cup C'$ and $\mathbf{B}'$ and $\mathbf{C}'$ are subalgebras of $\mathbf{D}'$ respectively. Let $D = D' \cup \{\bot\}$ and define $\bot \leq x$ and $\bot x = x \bot = \bot$ for all $x \in D'$, then $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{D}$ respectively since $\bot \in B \cap C$ and $\bot b = \bot$ and $\bot c = \bot$ for all $b \in B, c \in C$; thus $\mathbf{D} \in \mathsf{bT_i}$ is a strong amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Next we suppose one of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is non-linear. As mentioned above, the subalgebra of a non-linear member of $\mathsf{bT_i}$ is also a non-linear member of $\mathsf{bT_i}$, so we know $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are non-linear since $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively. Since $\mathsf{bT_i}$ satisfies the axioms for T, we know $b = \top \backslash 1 < 1$, b is a cover of $\bot$ and $b' = b \backslash \bot$ is incomparable with 1; these facts hold in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Let $A' = {\uparrow_{\mathbf{A}}} b$, $B' = {\uparrow_{\mathbf{B}}} b$ and $C' = {\uparrow_{\mathbf{C}}} b$, then $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ are $\bot$-free subalgebras of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively, so $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ is a V-formation of $\star$-involutive idempotent residuated chains. By Theorem 5.7 in [5], there exists a $\star$-involutive idempotent residuated chain $\mathbf{D}'$ such that $D' = B' \cup C'$ and $\mathbf{B}'$, $\mathbf{C}'$ are subalgebras of $\mathbf{D}'$. Since $b, \top \in B \cap C$, $\mathbf{D}'$ is also bounded by b and $\top$. Define $\mathbf{D} = \mathbf{R}_{\mathbf{D}', \{b, b'\}}$ given in Theorem 8.6, then $\mathbf{D}$ is a non-linear member of $\mathsf{bT_i}$. Since $b', \bot \in B \cap C$ and elements in

$U_B \cup \{\top\}, U_C \cup \{\top\}$ are multiplicative identities for $\bot$ and b', $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{D}$, so $\mathbf{D}$ is a strong amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ in $\mathsf{bT_i}$. See Figure 9.2 for the visualization of the process.

□

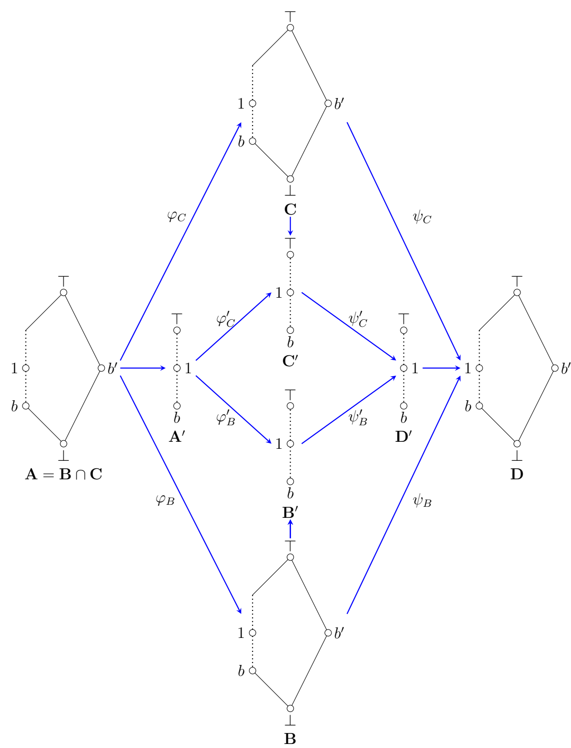

Figure 9.2: sAP of $\mathsf{bT_i}$

Corollary 9.6. The variety $\mathsf{V}(\mathsf{bT_i})$ generated by $\mathsf{bT_i}$ has strong amalgamation property.

Proof. By Theorem 4.5, we know the class of finitely subdirectly irreducible algebras in $\mathsf{V}(\mathsf{bT_i})$ is exactly $\mathsf{bT_i}$. Since the variety bIdSRL has the congruence extension property by Theorem 9.3 and $\mathsf{V}(\mathsf{bT_i})$ is arithmetical, $\mathsf{V}(\mathsf{bT_i})$ has strong amalgamation property by Theorem 4.8 in [6]. □

For the class $\mathsf{T_i}$, first we note that a subalgebra of a non-linear member in $\mathsf{T_i}$ is not necessarily non-linear, e.g., $\uparrow b$ is a subalgebra of a non-linear member $\mathbf{R} \in \mathsf{T_i}$, and $\uparrow b$ is isomorphic to a linear member of $\mathsf{T_i}$. Based on this fact, we give a counter example for AP in $\mathsf{T_i}$.

Let $\mathbf{A}$ be the 3-element Sugihara monoid and $\mathbf{B} = \mathbf{R}^{6,5}_{3,1}$, $\mathbf{C} = \mathbf{R}^{5,4}_{2,2}$ in the list [10], as shown in the Figure 9.3. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively, so $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{T_i}$. Suppose there exists a tuple $(\mathbf{D}, \psi_B, \psi_C)$ such that $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Since $1 \parallel e$ in $\mathbf{C}$ and ψ_C is an embedding, we know $\psi_C(1) \parallel \psi_C(e)$, so $\mathbf{D}$ is a non-linear member of $\mathsf{T_i}$. Since $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, we know $\psi_B(a) = \psi_C(a)$. However, since $a = 1 \vee e$ in $\mathbf{C}$, we have $\psi_B(a) = \psi_C(a) = \top_{\mathbf{D}} = \psi_B(g)$, contradicting that ψ_B is an embedding. Therefore AP fails in $\mathsf{T_i}$.

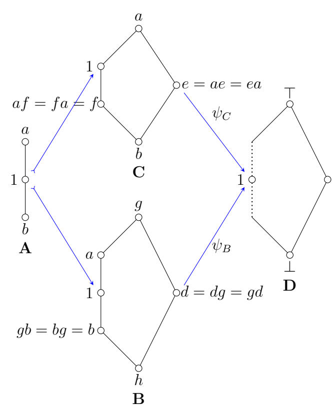

Figure 9.3: Counterexample for AP of $\mathsf{T_i}$

9.4 Subclass of B

We denote by $\mathsf{B_i}$ and $\mathsf{bB_i}$ the subclasses of bIdURL and IdURL, respectively, that are axiomatized by B, (idem) and ($\star$-inv$\updownarrow$1).

Since $\mathsf{B_i}$ and $\mathsf{bB_i}$ are subclasses of B, we know given the non-linear members $\mathbf{R}$ of these classes, we have $U_R \cup \{\bot, \top\} = \updownarrow 1$ is a subalgebra and $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra, which is isomorphic to the 4-element Boolean algebra.

9.4.1 Amalgamation property of $\mathsf{bB_i}$ and $\mathsf{B_i}$.

Theorem 9.7. The amalgamation property holds in $\mathsf{bB_i}$.

Proof. We consider a V-formation in $\mathsf{bB_i}$, suitable for the AP. Since $\mathsf{bB_i}$ is a positive universal class, it is closed under isomorphisms, the V-formation consists of a common subalgebra $\mathbf{A}$ of $\mathbf{B}$ and $\mathbf{C}$ with $A = B \cap C$. Let 1 be the identity element of $\mathbf{A}$, which is also the identity of $\mathbf{B}$ and $\mathbf{C}$.

Let $A' = \updownarrow_{\mathbf{A}} 1$, $B' = \updownarrow_{\mathbf{B}} 1$ and $C' = \updownarrow_{\mathbf{C}} 1$. Since $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively and $A = B \cap C$, we know $\updownarrow_{\mathbf{A}} 1 = \updownarrow_{\mathbf{B}} 1 \cap A = \updownarrow_{\mathbf{B}} 1 \cap B \cap C = \updownarrow_{\mathbf{B}} 1 \cap C = \updownarrow_{\mathbf{B}} 1 \cap \updownarrow_{\mathbf{C}} 1$, i.e., $A' = B' \cap C'$ and $\mathbf{A}'$ is a subalgebra of $\mathbf{B}'$ and $\mathbf{C}'$ respectively. Similarly to the proof of Theorem 9.5, there exists a strong amalgam $D' \in \mathsf{bT_i}$ for $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ such that $D' = B' \cup C'$ and $\mathbf{B}'$ and $\mathbf{C}'$ are subalgebras of $\mathbf{D}'$ respectively.

We consider a copy $Z \cup \{\bot, \top\}$ of the the 4-element Boolean algebra, where $Z = \{b, b'\}$ and define $\mathbf{D} = \mathbf{R}_{\mathbf{D}', Z \cup \{\bot, \top\}}$ as the non-linear member of $\mathsf{bB_i}$ given by Theorem 8.9. Also, we define $g_B : B \to D$ and $g_C : C \to D$ by extending the inclusions $B' \subseteq D' \subseteq D$ and $C' \subseteq D' \subseteq D$, respectively, with $g_B(b_B) = b = g_C(b_C)$ and $g_B(b'_B) = b' = g_C(b'_C)$, where b_B, b'_B and also b_C, b'_C are the elements of B and C, respectively, that are not in B' and C', if any. Since elements in $Z_D = Z$ are multiplicative zeros for those in $U_D \cup \{\top\}$ and $1 \parallel x$ for all $x \in Z_D$, we know the multiplications and divisions in $\mathbf{B}$ and $\mathbf{C}$ correspond to those in $\mathbf{D}$. Thus g_B and g_C are embedding. The process is shown in Figure 9.4. □

On the other hand the strong amalgamation property fails in $\mathsf{bB_i}$. If we take $\mathbf{A}$ to be the 3-element Sugihara monoid and $\mathbf{B}$ both $\mathbf{C}$ isomorphic to $\mathbf{R}_{\mathbf{A}, Z \cup \{\bot, \top\}}$, with $Z = \{b, b'\}$, but set-theoretically $B \neq C$, then every strong amalgam would contain $B \cup C$, so it would contain the antichain $\{1, b_B, b'_B, b_C, B'_C\}$, so it cannot be an element of the class $\mathsf{bB_i}$. See Figure 9.5.

For $\mathsf{B_i}$, we give a counter example for AP. Let $\mathbf{A}$ be the 3-element Sugihara monoid and $\mathbf{B}$, $\mathbf{C}$ be non-linear idempotent $\star$-involutive as shown in the Figure 9.6. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively, and $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{B_i}$. Suppose there exists a tuple $(\mathbf{D}, \psi_B, \psi_C)$ in $\mathsf{B_i}$ such that $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Since $1 \parallel d$ in

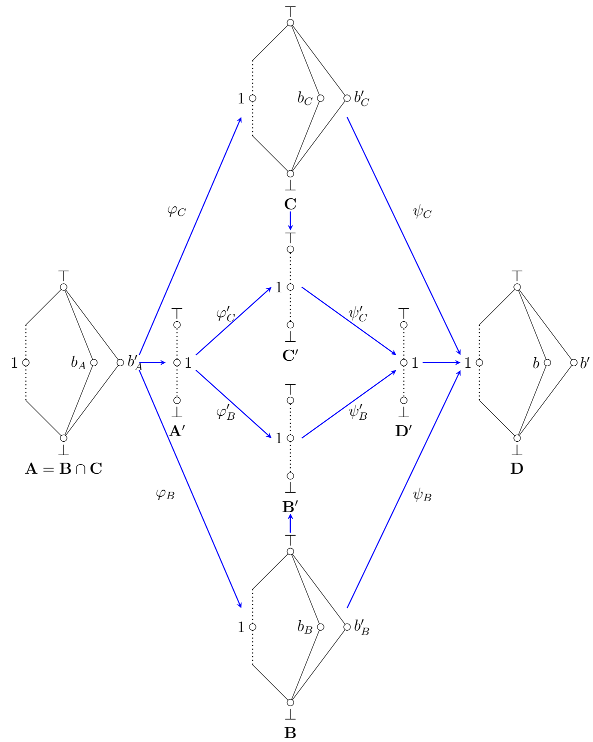

Figure 9.4: AP of $\mathsf{bB_i}$

$\mathbf{C}$ and ψ_C is an embedding, we know $\psi_C(1) \parallel \psi_C(d)$, so $\mathbf{D}$ is a non-linear member of $\mathsf{B_i}$. Since $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, we have $\psi_B(a) = \psi_C(a)$. However, since $a = 1 \vee d$ in $\mathbf{C}$, we have $\psi_B(a) = \psi_C(a) = \top_{\mathbf{D}} = \psi_B(f)$, contradicting that ψ_B is an embedding. Therefore AP fails in $\mathsf{B_i}$.

9.5 Subclass of L

Similar to previous case, we denote the subclass of bIdURL by $\mathsf{bL_i}$ and the subclass of IdURL by $\mathsf{L_i}$, both of which are axiomatized by the axioms for L, (idem) and ($\star$-inv$\updownarrow$1)

Since $\mathsf{bL_i}$ and $\mathsf{L_i}$ are subclasses of L, we know given a non-linear member $\mathbf{R}$ of them (their non-linear members identify), $U_R \cup \{\bot, \top\} = \updownarrow 1$ is a subalgebra and $Z_R \cup \{\bot, \top\}$ is a totally-ordered 1-free subalgebra of $\mathbf{R}$.

9.5.1 Amalgamation property of $\mathsf{bL_i}$ and $\mathsf{L_i}$.

Theorem 9.8. The strong amalgamation property holds on $\mathsf{bL_i}$.

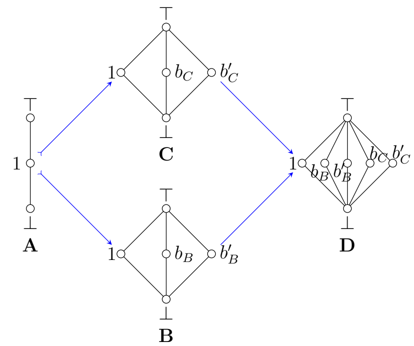

Figure 9.5: Counterexample for sAP of $\mathsf{bB_i}$

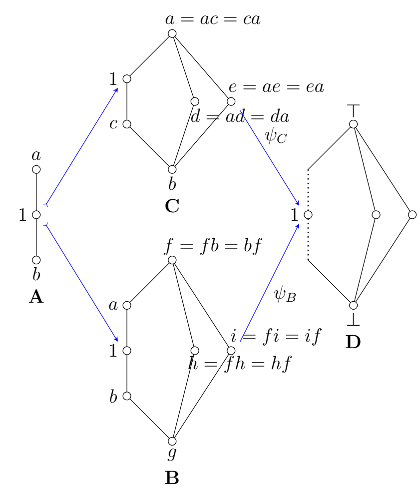

Figure 9.6: Counterexample for AP of $\mathsf{B_i}$

Proof. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be a V-formation in $\mathsf{bL_i}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively and $A = B \cap C$. Let 1 be the identity element of $\mathbf{A}$, which is also the identity of $\mathbf{B}$ and $\mathbf{C}$.

Let $A_1 = \updownarrow_{\mathbf{A}} 1$, $B_1 = \updownarrow_{\mathbf{B}} 1$ and $C_1 = \updownarrow_{\mathbf{C}} 1$. As mentioned before, $\mathbf{A_1}$, $\mathbf{B_1}$ and $\mathbf{C_1}$ are subalgebras of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively. Since $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively and $A = B \cap C$, $\mathbf{A_1}$ is also a subalgebra of $\mathbf{B_1}$ and $\mathbf{C_1}$ respectively and $\updownarrow_{\mathbf{A}} 1 = \updownarrow_{\mathbf{B}} 1 \cap A = \updownarrow_{\mathbf{B}} 1 \cap B \cap C = \updownarrow_{\mathbf{B}} 1 \cap C = \updownarrow_{\mathbf{B}} 1 \cap \updownarrow_{\mathbf{C}} 1$, i.e., $A_1 = B_1 \cap C_1$. Similarly to the proof of Theorem 9.5, there exists a strong amalgam $D_1 \in \mathsf{bL_i}$ for $(\mathbf{A_1}, \mathbf{B_1}, \mathbf{C_1})$ such that $D_1 = B_1 \cup C_1$ and $\mathbf{B_1}$ and $\mathbf{C_1}$ are subalgebras of $\mathbf{D_1}$ respectively.

On the other hand, let $A_2 = Z_A \cup \{\bot, \top\}$, $B_2 = Z_B \cup \{\bot, \top\}$ and $C_2 = Z_C \cup \{\bot, \top\}$. Since $\mathbf{A_2}$, $\mathbf{B_2}$ and $\mathbf{C_2}$ are 1-free subalgebras of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively and $\top$ is the multiplicative identity for elements in them, $(\mathbf{A_2}, \mathbf{B_2}, \mathbf{C_2})$ is a V-formation of Gödel chains. Since $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively and $A = B \cap C$, we know $\mathbf{A_2}$ is a subalgebra of $\mathbf{B_2}$ and $\mathbf{C_2}$ respectively and $A_2 = B_2 \cap C_2$. Since the class of Gödel chains has strong amalgamation property, there exists a Gödel chain $\mathbf{D_2}$ such that $\mathbf{B_2}$ and $\mathbf{C_2}$ are subalgebras of $\mathbf{D_2}$. Since $\bot, \top \in B_2 \cap C_2$, $\mathbf{D_2}$ is also bounded by them, so $\mathbf{D_2}$ is a strong amalgam for $(\mathbf{A_2}, \mathbf{B_2}, \mathbf{C_2})$.

Now let $\mathbf{D} = \mathbf{R_{D_1,D_2}}$ be the non-linear member of $\mathsf{bL_i}$ given in Theorem 8.9. Since $U_D = U_{D_1} = U_B \cup U_C$, $Z_B \subseteq Z_D$, $Z_C \subseteq Z_D$ and elements in U are the multiplicative identities for Z, we know B and C are closed under multiplications and divisions in $\mathbf{D}$, thus $\mathbf{D}$ is a strong amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ in $\mathsf{bL_i}$. The progress is shown in Figure 9.7.

□

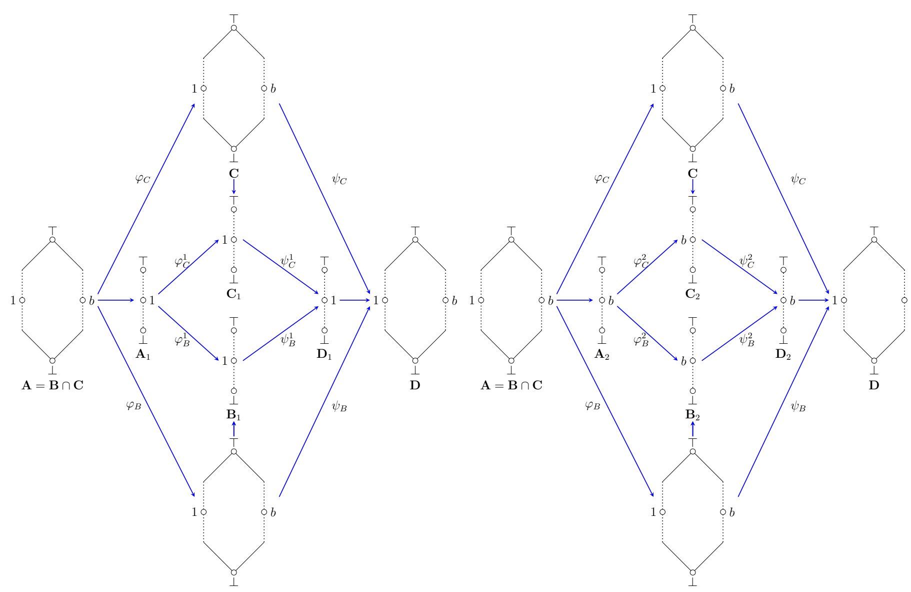

Figure 9.7: sAP for $\mathsf{bL_i}$

Corollary 9.9. The variety $\mathsf{V}(\mathsf{bL_i})$ generated by $\mathsf{bL_i}$ has strong amalgamation property.

For $\mathsf{L_i}$, we give a counter example for AP here.

Let $\mathbf{A}$ be the 3-element Sugihara monoid and $\mathbf{B} = \mathbf{R}^{6,5}_{3,2}$, $\mathbf{C} = \mathbf{R}^{5,4}_{2,3}$ in the list [10] as shown in the Figure 9.8. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ respectively, so $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{L_i}$. Suppose there exists a tuple $(\mathbf{D}, \psi_B, \psi_C)$ in $\mathsf{L_i}$ such that $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Since $1 \parallel e$ in $\mathbf{C}$ and ψ_C is an embedding, we know $\psi_C(1) \parallel \psi_C(e)$, so $\mathbf{D}$ is a non-linear member of $\mathsf{L_i}$. Since $\mathbf{D}$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, we know $\psi_B(a) = \psi_C(a)$. However, since $a = 1 \vee e$ in $\mathbf{C}$, we have $\psi_B(a) = \psi_C(a) = \top_{\mathbf{D}} = \psi_B(g)$, contradicting that ψ_B is an embedding. Therefore AP fails in $\mathsf{L_i}$.

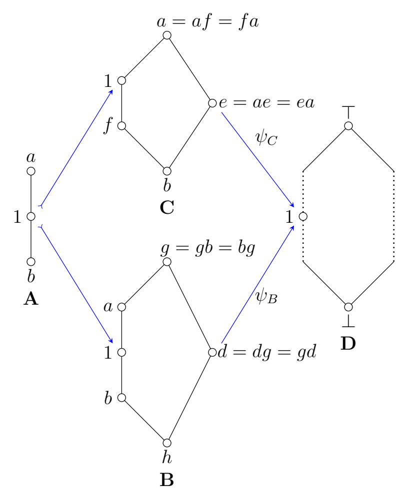

Figure 9.8: Counterexample for AP of $\mathsf{L_i}$

9.6 The joins of $\mathsf{V(bT_i)}$, $\mathsf{V(bB_i)}$ and $\mathsf{V(bL_i)}$

Note first that with the process mentioned in Section 4.2, all the varieties $\mathsf{V(bT_i)}$, $\mathsf{V(bB_i)}$ and $\mathsf{V(bL_i)}$ are equationally axiomatizable. So by [8], any join of them is also equationally axiomatizable.

Theorem 9.10. The joins of any two of the varieties $\mathsf{V(bT_i)}$, $\mathsf{V(bB_i)}$ and $\mathsf{V(bL_i)}$ fail amalgamation property. Even stronger, if a subvariety of bIdSRL that contains non-linear members from at least two different classes from $\mathsf{T}, \mathsf{L}, \mathsf{B}, \mathsf{B4}$, then it does not have the AP.

Proof. By Theorem 9.3, we know every join of the varieties above has the congruence extension property. By Theorem 4.5, we know the class of finitely subdirectly irreducible algebras of the varieties are precisely $\mathsf{bT_i}$, $\mathsf{bB_i}$ and $\mathsf{bL_i}$. So by Theorem 3.4 of [6], to show joins of $\mathsf{V}(\mathsf{bT_i})$, $\mathsf{V}(\mathsf{bB_i})$ and $\mathsf{V}(\mathsf{bL_i})$ fail AP, it suffices to show they fail one-side amalgamation property (1AP). Recall that a class $\mathcal{K}$ has 1AP if for every V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C)$ there exists $\mathbf{D} \in \mathcal{K}$, homomorphism $\psi_B : B \to D$ and embedding $\psi_C : C \to D$ such that $\psi_B\varphi_B = \psi_C\varphi_C$.

Now we give counterexamples for 1AP of the 3 binary joins of $\mathsf{V}(\mathsf{bT_i})$, $\mathsf{V}(\mathsf{bB_i})$ and $\mathsf{V}(\mathsf{bL_i})$.

For $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bB_i})$, let $\mathbf{A}$ be the 3-element Sugihara monoid, $\mathbf{B} \in \mathsf{bB_i}$ and $\mathbf{C} \in \mathsf{bT_i}$ be as show in Figure 9.9(a), then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bB_i})$. Suppose there exist $(\mathbf{D}, \psi_B, \psi_C)$ to be a one-side amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. If ψ_B is a homomorphism and ψ_C is an embedding, then ψ_B is also an embedding since $\mathbf{B}$ is simple, by the isomorphism between the lattice of congruence and that of convex normal submonoids of negative cones of a residuated lattice proved in [12]. In this case, $\{\psi_B(b_B), \psi(b'_B), \psi_C(b_C)\} \subseteq Z_D$ is a 3-element antichain, which is impossible. If ψ_C is a homomorphism and ψ_C is an embedding, then the only non-injective non-trivial ψ_C has $\psi_C(1) = \psi_C(b_C)$. However, in this case we get $\psi_C(\top) = \psi_C(b_C \backslash b_C) = \psi_C(b_C) \backslash \psi_C(b_C) = \psi_C(1) \backslash \psi_C(1) = \psi_C(1 \backslash 1) = 1$, so $\psi_B(\top) = \top \neq 1 = \psi_C(\top)$, a contradiction. Thus $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bB_i})$ fails 1AP.

For $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bL_i})$, let $\mathbf{A}$ be the 3-element Sugihara monoid, both $\mathbf{B} \in \mathsf{bL_i}$ and $\mathbf{C} \in \mathsf{bT_i}$ be as shown in Figure 9.9(b), then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bL_i})$. Suppose $(\mathbf{D}, \psi_B, \psi_C)$ is a one-side amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. If ψ_B is a homomorphism and ψ_C is an embedding, then ψ_B is also an embedding since $\mathbf{B}$ is simple, so $\{\psi_B(b_B), \psi_B(b'_B), \psi_C(b_C)\} \subseteq Z_D$ with $\psi_B(b_B) < \psi_B(b'_B)$ and $\psi_C(b_C) \parallel \psi_B(b_B)$, which is impossible. If ψ_C is a homomorphism and ψ_C is an embedding, then as the case for $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bB_i})$, the only non-injective non-trivial ψ_C has $\psi_C(1) = \psi_C(b_C)$, then $\psi_C(\top) = \psi_C(b_C \backslash b_C) = \psi_C(b_C) \backslash \psi_C(b_C) = \psi_C(1) \backslash \psi_C(1) = \psi_C(1 \backslash 1) = 1$, so $\psi_B(\top) = \top \neq 1 = \psi_C(\top)$, a contradiction. Thus $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bL_i})$ fails 1AP.

For $\mathsf{V}(\mathsf{bB_i}) \vee \mathsf{V}(\mathsf{bL_i})$, let $\mathbf{A}$ be the 3-element Sugihara monoid, $\mathbf{B} \in \mathsf{bL_i}$ and $\mathbf{C} \in \mathsf{bB_i}$ be as shown in Figure 9.9(c), then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a V-formation in $\mathsf{V}(\mathsf{bB_i}) \vee \mathsf{V}(\mathsf{bL_i})$. Suppose $(\mathbf{D}, \psi_B, \psi_C)$ is a one-side amalgam for $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Since both $\mathbf{B}$ and $\mathbf{C}$ are simple in this case, $(\mathbf{D}, \psi_B, \psi_C)$ is an amalgam, so $\{\psi_B(b_B), \psi_B(b'_B), \psi_C(b_C), \psi_C(b'_C)\} \subseteq Z_D$ such that $\psi_B(b_B) < \psi_B(b'_B)$ and $\psi_C(b_C) \parallel \psi_B(b_B)$ or $\psi_C(b'_C) \parallel \psi_B(b_B)$, which is impossible. Thus $\mathsf{V}(\mathsf{bB_i}) \vee \mathsf{V}(\mathsf{bL_i})$ fails 1AP. □

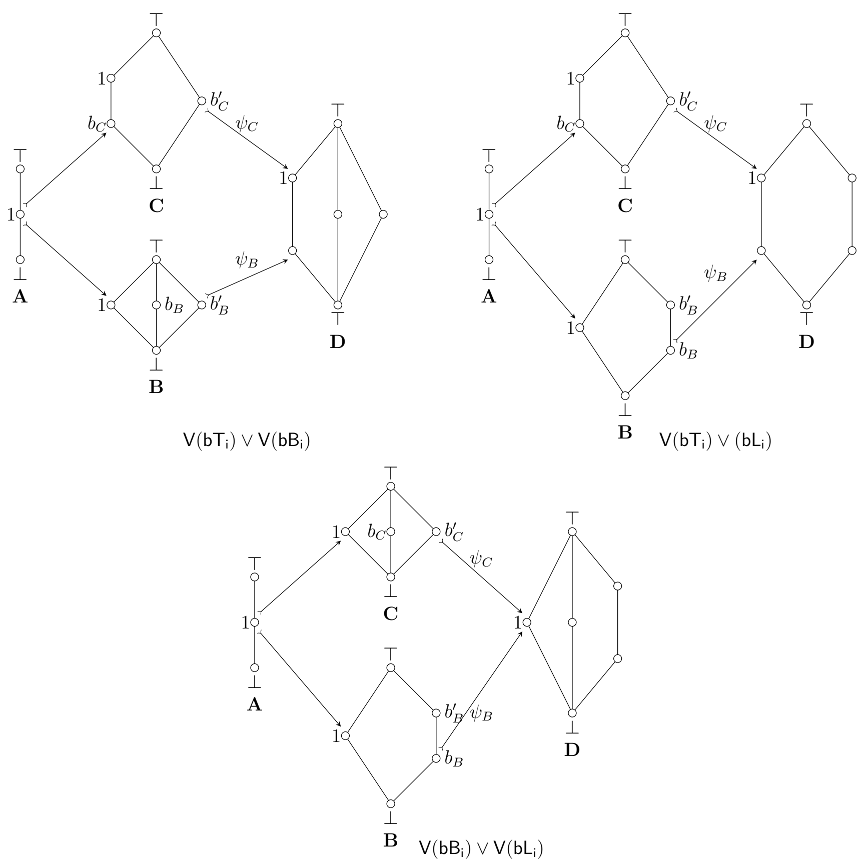

Figure 9.9: Counterexamples for AP of (a) $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bB_i})$; (b) $\mathsf{V}(\mathsf{bT_i}) \vee \mathsf{V}(\mathsf{bL_i})$; (C) $\mathsf{V}(\mathsf{bB_i}) \vee \mathsf{V}(\mathsf{bL_i})$

Chapter 10: Involutive semiunilinear residuated lattices

A (bounded) residuated lattices called 1-*involutive* if it satisfies $x^{\ell r} = x = x^{r\ell}$, where $x^{\ell} = 1/x$ and $x^{r} = x\backslash 1$. In this chapter we study compact commutative 1-involutive unilinear residuated lattices multiplication and in particular, we characterize the finite ones.

Recall that a residuated lattice $\mathbf{R}$ based on $\mathbf{M}_X$, for some X, satisfies $\top x = x$ for all x iff $R = U_R \cup \{\bot, \top\}$.

Theorem 10.1. In a residuated lattice $\mathbf{R}$ based on an $\mathbf{M}_X$, we have $b^{\ell r} = b = b^{r\ell}$ for all non-idempotent $b \in Z_R$. Also, a residuated lattice $\mathbf{R}$ based on an $\mathbf{M}_X$ is 1-involutive and satisfies $\top x = \top$ iff X is closed under multiplication and it forms a group.

Proof. Let $\mathbf{R}$ be a residuated lattice based on an $\mathbf{M}_X$, then by Theorem 3.1(1) we have $X = U_R \cup Z_R$. If $b \in Z_R$, then by Theorem 3.1(4) $ab = b \not\leq 1$, for all $a \in U_R \cup \{\top\}$; hence $b^r \in Z_R \cup \{\bot\}$. Also, since b is non-idempotent, by Theorem 3.1(3) there exists $b' \in Z_R$ (possibly $b' = b$) such that $bb' = \bot \leq 1$, hence $b^r = b'$ and likewise $b^{\ell} = b'$. In both cases where Z_R has one or two elements, we get $b^{\ell r} = b = b^{r\ell}$ for all $b \in Z_R$.

If $\mathbf{M}$ is 1-involutive and it satisfies $\top x = \top$, then Z_R is empty and $X = U_R$. For all $a \in U_R$, if $a^r = \bot$, then we would get $a^{r\ell} = \bot^{\ell} = \top \neq a$, a contradiction; so $a^r \neq \bot$. Also, if $a^r = \top$, then $a^{r\ell} = \top^{\ell} = \bot \neq a$, a contradiction. So, $a^r \in U_R$ and likewise for a^{ℓ}. Also, by $aa^r \leq 1$ and the fact that no product of elements of U_R gives $\bot$, we get that $aa^r = 1$, for all $a \in U_R$, i.e., every element of $X = U_R$ is invertible, so X is closed under multiplication and forms a group. Also, it is easy to see that if X is a group, then $\mathbf{R}$ is 1-involutive. $\square$

Corollary 10.2. The 1-involutive residuated lattices on $\mathbf{M}_X$ are precisely those of the form $\mathbf{R}_{\mathbf{A},\mathbf{B}}$ in Corollary 3.3, where $\mathbf{A}$ is a monoid with zero $\top$ such that $A \setminus \{\top\}$ is a group and $\mathbf{B}$ is the semigroup with zero $\bot$ such that either $B = \{\bot, b\}$ with $b^2 = \bot$ or $B = \{\bot, b, b'\}$ with $b^2 = b$, $b'^2 = b'$ and $bb' = b'b = \bot$.

The residuated lattice in the previous theorem is a non-linear 1-involutive compact URL. However, it is not the case that in every finite non-linear 1-involutive compact URL $(U, \cdot, 1)$ is a subgroup. Here is a counterexample.

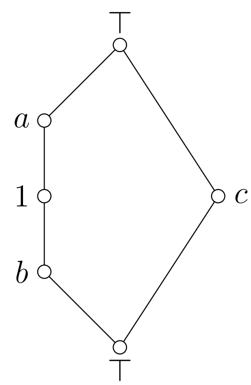

Figure 10.1: Counterexample: non-linear 1-involutive compact URL without subgroups

It satisfies $\top x = \top$ for all $x \neq \bot$ and the multiplication table is

·	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

From the table we can tell $a^\ell = a^r = b, b^\ell = b^r = a, c^\ell = c^r = c$, so it is 1-involutive.

An element a in a residuated lattice is called *n-potent*, for some $n \in \mathbb{Z}^+$, if it satisfies $a^{n+1} = a^n$; it is called *potent* if it is n-potent for some n. A residuated lattice is called *n-potent*, for some $n \in \mathbb{Z}^+$, if every element is n-potent; it is called *potent* if it is n-potent for some n. Note that if a residuated lattice is potent, then each of its elements is potent (but not conversely, as no universal n might exist). Clearly, finite residuated lattices are potent. The following result is not about unilinear residuated lattices, but rather about residuated chains. However, it seems to have gone unnoticed up to now.

Lemma 10.3. Every 1-involutive residuated chain where every element is potent is actually idempotent. In particular, every potent (hence also every finite) 1-involutive residuated chain is idempotent. Consequently, the only commutative 1-involutive residuated chains that are potent/finite/every element is potent are the odd Sugihara chains.

Proof. Let $a \leq 1$ such that $a^{n+2} = a^{n+1}$ for some $n \in \mathbb{Z}^+$, so a^{n+1} is idempotent. If $1 < a(a^{n+1})^r$, then $a^n \leq a^n a(a^{n+1})^r = a^{n+1}(a^{n+1})^r = a^{2(n+1)}(a^{n+1})^r \leq a^{n+1}$. If $a(a^{n+1})^r \leq 1$, then $a \leq (a^{n+1})^{r\ell} = a^{n+1} \leq a^2$ and so $a^n \leq a^{n+1}$ by order-preserving. Thus if a is negative, $a^{n+2} = a^{n+1}$ implies $a^n = a^{n+1}$ for all n. Inductively, we get that for negative a, we have $a = a^2$.

Now for $1 \leq a$, we have $a^\ell = 1/a \leq 1/1 = 1$, hence a^ℓ is idempotent. Therefore, $a\backslash a = (a^\ell a)^r = (a^\ell(a^\ell a))^r \geq (a^\ell 1)^r = a$, hence $a^2 \leq a$. Since $1 \leq a$ implies $a \leq a^2$, we get that a is idempotent. $\square$

Here we introduce some properties of a class of involutive rigorously compact unilinear residuated lattice.

Theorem 10.4. Let $\mathbf{R}$ be a non-linear rigorously compact URL satisfying involutivity and $(\cdot_Z\wedge)$, then

1. $M = R \setminus \{\bot, \top\}$ is closed under multiplication and divisions, so $\mathbf{R}$ is compact;
2. the comparability relation $\equiv$ is compatible with multiplication and divisions.

Proof. (1) Since $\mathbf{R}$ is rigorously compact, we know $M = U_R$. Since $\mathbf{R}$ is nonlinear and satisfies $(\cdot_Z\wedge)$, M is closed under multiplication by Theorem 8.27. Let 0 denote the negation constant in $\mathbf{R}$ and let $x^\ell = 0/x, x^r = x\backslash 0$. Since $\mathbf{R}$ is involutive, $x^\ell, x^r \notin \{\top, \bot\}$ for all $x \in M$, so M is closed under $^\ell$ and ^{r}. Since $x\backslash y = x^r + y = (y^\ell x)^r$ for all $x, y \in M$, we know M is closed under left divisions; and similarly for right divisions.

(2) Since multiplication preserves the ordering in $\mathbf{R}$, $\equiv$ is compatible with the multiplication. Since $\mathbf{R}$ is involutive, we know $x \leq y$ iff $y^\ell \leq x^\ell$ iff $y^r \leq x^r$ for all $x, y \in R$, thus $\equiv$ is compatible with $^\ell$ and ^{r}. Since $x\backslash y = (y^\ell x)^r$ for all $x, y \in R$, $\equiv$ is compatible with divisions. □

In an involutive residuated lattice, an element a is called *periodic* if there exists $n \in \mathbb{Z}^+$ such that $x^{\ell^n} = x^{r^n}$.

Theorem 10.5. Let $\mathbf{R}$ be a non-linear 1-involutive $\top$-unital URL, then

1. $\mathbf{R}$ is compact;
2. $\mathbf{G} = (M, \cdot, 1)/\equiv$ is a group, where $M = R \setminus \{\bot, \top\}$;
3. if every element is periodic, then $\mathbf{R}$ is cyclic;
4. if every element in the chain of 1, H, is potent then H is isomorphic to an odd Sugihara chain.

Proof. (1) Since $\mathbf{R}$ is non-linear and $\top$-unital, $R \setminus \{\bot\}$ is closed under multiplication by associativity. Since $\mathbf{R}$ is 1-involutive, $x^\ell = 1/x, x^r = x\backslash 1 \notin \{\top, \bot\}$ for all $x \in M$, so M is closed under $^\ell$ and ^{r}. Now suppose there exists $x, y \in M$ such that $xy = \top$, then $x^\ell, y^\ell \in M$. Since $R \setminus \{\bot\}$ is closed under multiplication, we have $\bot < x^\ell x \leq 1$, so $\top = x^\ell \top = x^\ell(xy) = (x^\ell x)y \leq y$, contradicting the assumption that $y \in M$, hence $\mathbf{R}$ is compact.

(2) By Theorem 10.4(2) we know the comparable relation is compatible with multiplication. Since M is closed under multiplication, $\mathbf{G}$ is a monoid. Since $[x]_\equiv \cdot_{\mathbf{G}} [x^r]_\equiv = [x \cdot_{\mathbf{M}} x^r] = [1]_\equiv$ for all $x \in M$, $\mathbf{G}$ is a group.

(3) For every $x \in M$ we have $xx^\ell \leq x(x^\ell \vee x^r) = xx^\ell \vee xx^r \leq x/x \vee 1 = x/x = (xx^r)^\ell$. By Theorem 10.4(2) we know M is closed under multiplication and divisions, so by unilinearity we get $x(x^\ell \vee x^r) \in M$. Therefore, $x^\ell \vee x^r \neq \top$, hence $x^\ell \equiv x^r$. Now if $x^\ell < x^r$, then $x^{\ell\ell\ell} < x^\ell$ by 1-involution. Hence we can get $\ldots < x^{\ell\ell\ell\ell\ell} < x^{\ell\ell\ell} < x^\ell < x^r < x^{rrr} < x^{rrrrr} < \ldots$ inductively, contradicting the assumption that every element in R is periodic. Likewise, $x^r < x^\ell$ leads to a contradiction, so $x^r = x^\ell$ for all $x \in M$.

(4) Since 1 is the multiplicative identity and the lattice reduct is unilinear, H is closed under multiplication, $^\ell$ and ^{r}, so $\mathbf{H}$ is a subalgebra of $\mathbf{M}$. By Lemma 10.3, $\mathbf{H}$ is idempotent; and by (3), it is cyclic. Since $\mathbf{H}$ is a cyclic idempotent chain, it follows [5] that $\mathbf{H}$ is isomorphic to an odd Sugihara chain. □

Now we give a characterization of the commutative 1-involutive compact URLs whose chain of 1 itself is a bounded odd Sugihara chain and the chains form a group in which every element is of finite order. Let $\mathbf{R}$ be such a URL. We say an element x in $M = R \setminus \{\bot, \top\}$ is *lower* if $x^{n_x} \leq 1$; it is *upper* if $x^{n_x} > 1$, where $n_x \in \mathbb{Z}^+$ is the order of the chain of x, viewed as an element in the comparability group. We omit the subscript when it is clear in the context.

Lemma 10.6. Let $\mathbf{R}$ be a commutative 1-involutive compact URL satisfying that the chain of 1, $H \cup \{\bot, \top\}$, is a bounded odd Sugihara chain and that $\mathbf{G} := \mathbf{M}/{\equiv}$ is a group whose elements are of finite orders, where $M = R \setminus \{\bot, \top\}$. In the following we use x^ℓ for $x \to 1$. We use n and m for the orders of the chains of x and y respectively if they are not comparable to 1, and make $n = 2$ if x is comparable to 1 (so that $n - 1$ is defined).

1. If the chain of x is g, then the chain of x^ℓ is g^{-1}; both g and g^{-1} are of the same order.

2. Every chain in $\mathbf{M}$ contains a lower element.

3. If $x \in M$ is a lower element such that $(x^\ell)^n \leq 1$, then $(x^\ell)^n = x^n$.

4. If $x \in M$ is a lower element, then $x^\ell x = x^n$; if x is an upper element, then x^ℓ is a lower element and $x^\ell x = (x^\ell)^n$.

5. If $x \in M$ is an upper element, then $(x^n)^\ell = (x^\ell)^n$.

6. If there exist elements $x, y \in M$ such that $x \equiv y$ and $x^n = y^n$, where n is the order of x, then $x = y$. So for every element $x \in M$, the $(\cdot)$-subreduct generated by x is $\langle x \rangle_\cdot = \{x, x^2, \ldots, x^n\}$, so it is a group isomorphic to $\mathbb{Z}_n$ with identity x^n. Consequently, if $x \equiv y$ and $x^s = y^s$, for some $x, y \in M$ and $s \in \mathbb{Z}^+$, then $x = y$.

7. For every element $x \in M$, $(x^k)^\ell x^k = x^\ell x$ for all $1 \leq k \leq n$.

8. If $x \in M$ is a lower element, then $(x^\ell)^n > 1$ iff $x^\ell > x^{n-1}$ iff $(x^{n-1})^\ell > x$; if x is an upper element, then $(x^\ell)^n < 1$.

9. For a lower element $x \in M$, if $x^{n-1} < x^\ell$, then $(x^k)^\ell = (x^\ell)^k$ for all $1 \leq k \leq n$, so the $(\cdot)$-subreduct generated by x^ℓ is $\langle x^\ell \rangle_\cdot = \{x^\ell, (x^2)^\ell, \ldots, (x^n)^\ell\}$ and the $(\cdot, ^\ell)$-subreduct generated by x is $\langle x \rangle = \langle x \rangle_\cdot \cup \langle x \rangle_\cdot^\ell = \{x, x^2, \ldots, x^n, x^\ell, (x^2)^\ell, \ldots, (x^n)^\ell\}$. If $x^{n-1} = x^\ell$, then $\langle x^\ell \rangle_\cdot = \{x, x^2, \ldots, x^n\}$ and $\langle x \rangle = \langle x \rangle_\cdot \cup \langle x \rangle_\cdot^\ell$.

10. For an upper element $x \in M$, we have $(x^k)^\ell = (x^\ell)^k$ for all $1 \leq k \leq n$, so the $(\cdot, ^\ell)$-subreduct generated by x is $\langle x \rangle = \langle x \rangle_\cdot \cup \langle x \rangle_\cdot^\ell$.

11. For all $x, y \in M$, we have $(x^i y^j)^\ell = (x^i)^\ell (y^j)^\ell$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

12. For all $x \in M$, we have $\langle x \rangle = \langle x^\ell \rangle = \langle x \rangle_\cdot \cup \langle x \rangle_\cdot^\ell$.

13. Let $x, y \in M$ be lower elements such that $y^m \leq x^n \leq 1$, then

    $x^{n-1} < x^\ell$ and $y^{m-1} < y^\ell \implies (xy)^{s-1} < (xy)^\ell$;

    $x^{n-1} = x^\ell$ and $y^{m-1} < y^\ell \implies (xy)^{s-1} < (xy)^\ell$;

    $x^{n-1} < x^\ell$ and $y^{m-1} = y^\ell \implies (xy)^{s-1} = (xy)^\ell$;

$x^{n-1} = x^\ell$ and $y^{m-1} = y^\ell \implies (xy)^{s-1} = (xy)^\ell$;

where s is the least common multiple of n and m.

14. If $x, y \in M$ are lower elements and their chains g^n and g^m are in the same cyclic subgroup $\langle g \rangle$ of $\mathbf{G}$, then the $(\cdot, {}^\ell)$-subreduct generated by x and y is $\langle x, y \rangle = \langle x^i y^j \rangle \cup \langle x \rangle \cup \langle y \rangle$, for some $i, j \in \mathbb{Z}_s$, where $s \in \mathbb{Z}^+$ is the order of g.

15. Let $x, y \in M$ be lower elements in chains g_1 and g_2 respectively. If $\langle g_1 \rangle \cap \langle g_2 \rangle = \{1_\mathbf{G}\}$, then the $(\cdot, {}^\ell)$-subreduct generated by x and y is $\langle x, y \rangle = (\langle x \rangle_. \times \langle y \rangle_.) \cup (\langle x \rangle_. \times \langle y \rangle_.)^\ell$.

16. For each element $x \in M$, if $x^{n-1} = x^\ell$, then x is a maximum lower element in its chain.

    If x is the maximum lower element in its chain, then x^ℓ is the maximum lower element in its chain when $x^{n-1} = x^\ell$ and the minimum upper element in its chain when $x^{n-1} < x^\ell$.

Proof. (1) Let x be in chain g and x^ℓ be in chain g'. Since $x^\ell x \leq 1$, we know $g'g = 1_\mathbf{G}$, so $g' = g^{-1}$; g and g^{-1} are of the same order by group theory.

(2) Let $x \in M$ be an upper element, then we know $x^n > 1$. Since $x^\ell x \in H$, $x(x^\ell x) \leq x$, so $(x(x^\ell x))^n \in H$. Since $\mathbf{H}$ itself is an odd Sugihara chain, we have $(x(x^\ell x))^n = (x^\ell)^n (x^n)^2 \overset{id}{=} (x^\ell)^n x^n = (x^\ell x)^n \overset{id}{=} x^\ell x \leq 1$, where $\overset{id}{=}$ denotes the implication of idempotency, so $x(x^\ell x)$ is a lower element in the same chain as x. Thus every chain contains a lower element.

(3) Let $x \in M$ be lower element such that $(x^\ell)^n \leq 1$, then $(x^\ell)^{n-1} \leq x$ by residuation. By idempotency of $(x^\ell)^n$ and order-preservation of multiplication, we get $(x^\ell)^n = ((x^\ell)^n)^{n-1} = ((x^\ell)^{n-1})^n \leq x^n$. Since x is a lower element, we know $x^n \leq 1$, so $x^{n-1} \leq x^\ell$ by residuation, and thus $x^n \overset{id}{=} (x^n)^{n-1} = (x^{n-1})^n \leq (x^\ell)^n$. Therefore $(x^\ell)^n = x^n$.

(4) Let $x \in M$ be a lower element. Since $x^\ell x \leq 1$ and $\mathbf{H}$ is an odd Sugihara chain, we know $x^\ell x \overset{id}{=} (x^\ell x)^n = (x^\ell)^n x^n \in \{(x^\ell)^n, x^n\}$. If $x^\ell x = (x^\ell)^n$, then $(x^\ell)^n \leq 1$, so by (3) we know $x^\ell x = (x^\ell)^n = x^n$, therefore $x^\ell x = x^n$ always holds.

Now assume x is an upper element. Since $x^\ell x \overset{id}{=} (x^\ell x)^n = (x^\ell)^n x^n \in \{(x^\ell)^n, x^n\}$, $x^n > 1$ and $x^\ell x \leq 1$, we get that $(x^\ell)^n = x^\ell x$, and x^ℓ is a lower element.

(5) Since $(x^\ell)^n x^n = (x^\ell x)^n \overset{id}{=} x^\ell x \leq 1$, we know $(x^\ell)^n \leq (x^n)^\ell$ by residuation. Since $x(x \to x) = x$ for all $x \in M$, we have $x^n \to x^\ell x = (x^n (x^\ell x)^\ell)^\ell = (x^n(x \to x))^\ell = (x^n)^\ell$. Since $\mathbf{H}$ is an odd Sugihara chain and $x^n > 1$, we get $(x^n)^\ell = x^n (x^n)^\ell = x^n(x^n \to x^\ell x) \leq x^\ell x$. Since $(x^\ell)^n = x^\ell x$ by (4), we have $(x^n)^\ell \leq (x^\ell)^n$, hence $(x^n)^\ell = (x^\ell)^n$.

(6) First assume that x is a lower element. Since $y^n = x^n$, y is also a lower element. Since $x^\ell x = x^n$ by (4), we have $(x^n)^\ell \overset{(4)}{=} (x^\ell x)^\ell = x^\ell \to x^\ell$, so $x^\ell = (x^\ell \to x^\ell) x^\ell = (x^n)^\ell x^\ell$, which is equivalent to $x \overset{inv}{=} ((x^n)^\ell x^\ell)^\ell = x^\ell \to x^n$, where $\overset{(4)}{=}$ and $\overset{inv}{=}$ denote the application of (4) and 1-involution respectively; similarly we have $y = y^\ell \to y^n$. Since $x \equiv y$, we know $1 \equiv x^\ell y$, so $x^\ell y \overset{id}{=} (x^\ell y)^n = (x^\ell)^n y^n = (x^\ell)^n x^n = (x^\ell x)^n \overset{id}{=} x^\ell x = x^n$, hence $y \leq x^\ell \to x^n = x$; Similarly we have $y^\ell x = y^n$, thus $x \leq y^\ell \to y^n = y$, therefore $x = y$.

If x is an upper element, then x^ℓ is lower by (4) and $(x^n)^\ell = (x^\ell)^n$ by (5). Since $y^n = x^n$, y is also an upper element and thus y^ℓ is lower and $(y^\ell)^n \overset{(5)}{=} (y^n)^\ell = (x^n)^\ell \overset{(5)}{=} (x^\ell)^n$. By above proof, we conclude that $x^\ell = y^\ell$, so $y = x$.

If x is a lower element, then $x^{n+k} = x^n x^k \leq x^k$ for all $1 \leq k \leq n$; if x is an upper element, then $x^k \leq x^n x^k = x^{n+k}$. So, in both cases we get $x^{n+k} \equiv x^k$. Using the idempotency of x^n, we have $(x^{n+k})^n = (x^n)^{n+k} \overset{id}{=} x^n \overset{id}{=} (x^n)^k = (x^k)^n$, so $x^{n+k} = x^k$ for all $1 \leq k \leq n$ by above proof.

Thus $\langle x \rangle_\cdot = \{x, x^2, \ldots, x^n\}$ and x^n is the identity element. Since $x^k x^{n-k} = x^n$ for all $1 \leq k \leq n$, we know $\langle x \rangle_\cdot$ is isomorphic to the group $\mathbb{Z}_n$.

Finally, if $x \equiv y$ and $x^s = y^s$, for some $x, y \in M$ and $s \in \mathbb{Z}^+$, where $s = qn + k$, with $0 \leq k < n$, then $x^k = x^n x^k \overset{id}{=} x^{qn} x^k = x^s = y^s = y^{qn} y^k \overset{id}{=} y^n y^k = y^k$. Since x^{n-k} and y^{n-k} are the group inverses of x^k and y^k respectively, $x^{n-k} = y^{n-k}$ and $x^n = x^{k+(n-k)} = x^k x^{n-k} = y^k y^{n-k} = y^{k+(n-k)} = y^n$, hence $x = y$.

(7) Since $x(x \to x) = x$ for all $x \in M$, we have $(x^k)^\ell = (x^k(x \to x))^\ell = (x^k(x^\ell x)^\ell)^\ell = x^k \to x^\ell x$, so $(x^k)^\ell x^k \leq x^\ell x$ for all $1 \leq k \leq n$. Since $\langle x \rangle_\cdot$ is isomorphic to $\mathbb{Z}_n$ by (6), we have $x^\ell = (x^{n+1})^\ell = (x^{n+1}(x^k \to x^k))^\ell = (x(x^k \to x^k))^\ell = (x((x^k)^\ell x^k)^\ell)^\ell = x \to ((x^k)^\ell x^k)$, so $x^\ell x = (x \to ((x^k)^\ell x^k))x \leq (x^k)^\ell x^k$ and hence $x^\ell x = (x^k)^\ell x^k$ for all $1 \leq k \leq n$.

(8) If x is a lower element, then $x^n \leq 1$, so $x^{n-1} \leq x^\ell$ and $x \leq (x^{n-1})^\ell$ by residuation. By further invoking 1-involution, we get: $x^\ell > x^{n-1}$ iff $(x^{n-1})^\ell > x$.

Now assume $x^\ell = x^{n-1}$, then $(x^\ell)^n = (x^{n-1})^n = (x^n)^{n-1} \overset{id}{=} x^n \leq 1$, so $x^\ell = x^{n-1}$ implies $x^n \leq 1$. Since $x^{n-1} \leq x^\ell$, we know $x^n > 1$ implies $x^{n-1} < x^\ell$ by contraposition.

If $(x^{n-1})^\ell > x$, then $(x^{n-1})^\ell \to x \equiv x \to x \equiv 1$. Since the set $\{y \in R : (x^{n-1})^\ell y \leq x\}$ is closed downward, $(x^{n-1})^\ell > x$ and $\mathbf{R}$ is unilinear, we know $1 \notin \{y \in R : (x^{n-1})^\ell y \leq x\}$, so $(x^\ell (x^{n-1})^\ell)^\ell = (x^{n-1})^\ell \to x < 1$, which is equivalent to $1 < x^\ell (x^{n-1})^\ell$ by 1-involution. Since $x(x^{n-1})^\ell \overset{(6)}{=} x^{n+1}(x^{n-1})^\ell = x^2(x^{n-1}(x^{n-1})^\ell) \overset{(7)}{=} x^2(x^\ell x) \overset{(4)}{=} x^2 x^n \overset{(6)}{=} x^2$, we get $x^n((x^{n-1})^\ell)^n = (x(x^{n-1})^\ell)^n = (x^2)^n = (x^n)^2 \overset{id}{=} x^n \leq 1$, so $((x^{n-1})^\ell)^n \leq (x^n)^\ell$ by residuation. By way of contradiction, if $(x^\ell)^n \leq 1$, then $(x^\ell)^n = x^n = x^\ell x$ by (3), so $1 < x^\ell (x^{n-1})^\ell$ and the idempotency of $\mathbf{H}$ implies that $1 < x^\ell (x^{n-1})^\ell \overset{id}{=} (x^\ell)^n ((x^{n-1})^\ell)^n = x^n((x^{n-1})^\ell)^n \leq x^n (x^n)^\ell \leq 1$, a contradiction. Therefore $(x^\ell)^n > 1$.

If x is an upper element, then $(x^\ell)^n = x^\ell x \leq 1$ by (4). If $(x^\ell)^n = 1$, then by (5) we know $(x^n)^\ell = (x^\ell)^n = 1$, so $x^n = 1$ and x is a lower element, a contradiction. Thus $(x^\ell)^n < 1$ when x is an upper element.

(9) Let x be a lower element. First we show $x^{n-1} < x^\ell$ implies $(x^n)^\ell = (x^\ell)^n$. By (4) and the fact that $x^\ell (x^\ell \to x^\ell)$ we get $(x^\ell)^n (x^n)^\ell \overset{(4)}{=} (x^\ell)^n (x^\ell x)^\ell = (x^\ell)^n (x^\ell \to x^\ell) = (x^\ell)^n$.

Since $x^{n-1} < x^\ell$, we get $(x^\ell)^n > 1$ by (8), so $1 \leq (x^n)^\ell \leq (x^\ell)^n$, because $1 \leq (x^n)^\ell$ and $\mathbf{H}$ is an odd Sugihara chain. On the other hand, since $(x^\ell)^n x^n = (x^\ell x)^n \stackrel{id}{=} x^\ell x \leq 1$, we have $(x^\ell)^n \leq (x^n)^\ell$ by residuation, thus $(x^n)^\ell = (x^\ell)^n$.

Now let $1 \leq k \leq n$. Since $(x^\ell)^{k-1}x^k = (x^\ell x)^{k-1}x \stackrel{id}{=} (x^\ell x)x \stackrel{(4)}{=} x^n x \stackrel{(6)}{=} x$, we obtain $x^k \leq (x^\ell)^{k-1} \to x$ by residuation. Set $y := (x^\ell)^{k-1} \to x$ and note that $x^k \leq y$ and $(x^\ell)^{k-1}y \leq x$. So, $(x^n)^\ell y^n = (x^\ell)^n y^n \stackrel{id}{=} ((x^\ell)^n)^{k-1}y^n = ((x^\ell)^{k-1})^n y^n = ((x^\ell)^{k-1}y)^n \leq x^n$, thus $y^n \leq (x^n)^\ell \to x^n = x^n \stackrel{id}{=} (x^k)^n \leq y^n$. Hence $y^n = (x^k)^n$ and by (6) we conclude that $y = x^k$. Therefore $x^k = (x^\ell)^{k-1} \to x = ((x^\ell)^k)^\ell$ and thus $(x^k)^\ell = (x^\ell)^k$ by 1-involution.

Since $x^{n-1} < x^\ell$, we know $(x^\ell)^n > 1$ by (8), so x^ℓ is an upper element. By (6) we know $\langle x^\ell \rangle_{\cdot}$ is a group isomorphic to $\mathbb{Z}_n$, thus $(x^i)^\ell (x^j)^\ell = (x^\ell)^i (x^\ell)^j = (x^\ell)^{i+_n j} = (x^{i+_n j})^\ell$ for all $i, j \in \mathbb{Z}_n$. Finally, since $(x^i)^\ell x^j = (x^\ell)^i x^j = (x^\ell x)^i x^{n+j-i} \stackrel{id}{=} (x^\ell x)x^{j-_n i} \stackrel{(4)}{=} x^n x^{j-_n i} \stackrel{(6)}{=} x^{j-_n i}$, the $(\cdot, {}^\ell)$-subreduct generated by x is $\langle x \rangle = \{x, x^2, \ldots, x^n, x^\ell, (x^2)^\ell, \ldots, (x^n)^\ell\} = \langle x \rangle_{\cdot} \cup \langle x \rangle^\ell_{\cdot}$.

Now if $x^{n-1} = x^\ell$, then $(x^\ell)^k = (x^{n-1})^k = x^{n-k}$ for all $1 \leq k \leq n$ by (6), so $\langle x^\ell \rangle_{\cdot} = \{x, x^2, \ldots, x^n\}$. Also we have $(x^i)^\ell x^j = ((x^i)^\ell x^i)x^{j-_n i} \stackrel{(7)}{=} (x^\ell x)x^{j-_n i} \stackrel{(4)}{=} x^n x^{j-_n i} \stackrel{(6)}{=} x^{j-_n i}$. Therefore we know $\langle x \rangle \subseteq \langle x \rangle_{\cdot} \cup \langle x \rangle^\ell_{\cdot}$ and $\langle x \rangle = \langle x \rangle_{\cdot} \cup \langle x \rangle^\ell_{\cdot}$.

(10) Since x is an upper element, we know x^ℓ is a lower element by (4) and $((x^\ell)^\ell)^n = x^n > 1$, so $((x^\ell)^k)^\ell \stackrel{(9)}{=} ((x^\ell)^\ell)^k \stackrel{inv}{=} x^k$ and hence $(x^\ell)^k = (x^k)^\ell$ for all $1 \leq k \leq n$ by (9). Thus $(x^i)^\ell (x^j)^\ell = (x^\ell)^i (x^\ell)^j = (x^\ell)^{i+_n j} = (x^{i+_n j})^\ell$ for all $i, j \in \mathbb{Z}_n$. By (6) we know $\langle x^\ell \rangle_{\cdot}$ is also a group isomorphic to $\mathbb{Z}_n$ with identity $(x^\ell)^n \stackrel{(9)}{=} (x^n)^\ell$. So we have $(x^i)^\ell x^j \stackrel{(6)}{=} (x^\ell)^i x^j = (x^\ell)^{n+i} x^j = (x^\ell)^{n+i-j}(x^\ell x)^j \stackrel{id}{=} (x^\ell)^{i-_n j}(x^\ell x) \stackrel{(4)}{=} (x^\ell)^{i-_n j}(x^\ell)^n \stackrel{(6)}{=} (x^\ell)^{i-_n j} = (x^{i-_n j})^\ell$, hence the $(\cdot, {}^\ell)$-subreduct generated by x is $\langle x \rangle = \langle x \rangle_{\cdot} \cup \langle x \rangle^\ell_{\cdot}$.

(11) Without loss of generality, assume that the interval bounded by x^n and $(x^n)^\ell$ is a subset of that bounded by y^m and $(y^m)^\ell$, then $x^n \to y^m = (x^n)^\ell \to y^m = y^m$ and $x^n \to (y^m)^\ell = (x^n)^\ell \to (y^m)^\ell = (y^m)^\ell$. Let $1 \leq i \leq n$ and $1 \leq j \leq m$. Since

$(x^i)^\ell(y^j)^\ell(x^iy^j) = ((x^i)^\ell x^i)((y^j)^\ell y^j) \leq 1$, we have $(x^i)^\ell(y^j)^\ell \leq (x^iy^j)^\ell$ by residuation. Since $(x^i)^\ell(y^j)^\ell = (x^iy^j)^\ell$ is equivalent to $x^iy^j = (x^i)^\ell \to y^j$ by 1-involution, to prove the former identity, it suffices to show x^iy^j is the maximum such element u that $(x^i)^\ell u \leq y^j$. First we observe that $(x^i)^\ell(x^iy^j) = ((x^i)^\ell x^i)y^j \leq y^j$, so $x^iy^j \leq (x^i)^\ell \to y^j$ by residuation. Let $z \geq x^iy^j$ such that $(x^i)^\ell z \leq y^j$. Let s be the least common multiple of $n' = n/(n,i)$ and $m' = m/(m,j)$, where (n,i) and (m,j) are the greatest common divisors, then $(x^iy^j)^s = (x^i)^s(y^j)^s \overset{id}{=} x^ny^m = y^m$. Since $(x^i)^\ell z \leq y^j$ and multiplication is order-preserving, we get $((x^i)^\ell)^s z^s = ((x^i)^\ell z)^s \leq (y^j)^s$, so $z^s \leq ((x^i)^\ell)^s \to (y^j)^s \overset{id}{=} ((x^i)^\ell)^{n'} \to y^m = y^m$ since $((x^i)^\ell)^{n'} = x^n$ or $(x^n)^\ell$ by (9) and (10). Since $z \geq x^iy^j$, we know $z^s \geq (x^iy^j)^s = y^m$, hence $z^s = (x^iy^j)^s$. By (6), we know $z = x^iy^j$ and hence $x^iy^j = (x^i)^\ell \to y^j$ and $(x^i)^\ell(y^j)^\ell = (x^iy^j)^\ell$ by 1-involution.

(12) By (9) and (10), we know if $x^\ell \neq x^{n-1}$, then $(x^k)^\ell = (x^\ell)^k$, so $x^k = ((x^\ell)^\ell)^k = ((x^\ell)^k)^\ell$ for all $1 \leq k \leq n$ by 1-involution, hence $\langle x^\ell\rangle = \langle x^\ell\rangle_{\cdot} \cup \langle x^\ell\rangle_{\cdot}^\ell = \langle x\rangle_{\cdot}^\ell \cup \langle x\rangle_{\cdot} = \langle x\rangle$. If $x^\ell = x^{n-1}$, then $\langle x\rangle_{\cdot} = \langle x^\ell\rangle_{\cdot}$ and $\langle x\rangle_{\cdot}^\ell = \langle x^\ell\rangle_{\cdot}^\ell$, so $\langle x\rangle = \langle x^\ell\rangle$.

(13) Let x and y be such lower elements. Since the chains of x^ℓ and y^ℓ are also of orders n and m respectively, so by (11) we have $((xy)^\ell)^s \overset{(11)}{=} (x^\ell y^\ell)^s = (x^\ell)^s(y^\ell)^s \overset{id}{=} (x^\ell)^n(y^\ell)^m = (y^\ell)^m$. The last equation comes from the assumption that $y^m \leq x^n \leq 1$. Then the results follows.

(14) We know the subgroup generated by g^n and g^m is $\langle g^n, g^m\rangle = \langle g^{(n,m)}\rangle$, where (n,m) is the greatest common divisor of n and m. By Bézout's identity, there exists $i,j \in \mathbb{Z}$ such that $(n,m) = in + jm$. Note that Bézout's identity also holds in $\mathbb{Z}_s$, so without loss of generality we assume that $i,j \in \mathbb{Z}_s$. In the following we show $\langle x,y\rangle = \langle x^iy^j\rangle \cup \langle x\rangle \cup \langle y\rangle$. First we prove for the $(\cdot)$-subreduct we have $\langle x,y\rangle_{\cdot} = \langle x^iy^j\rangle_{\cdot} \cup \langle x\rangle_{\cdot} \cup \langle y\rangle_{\cdot}$. Since $x \in g^n$ and $y \in g^m$, we know $x^{s/(s,n)} \equiv y^{s/(s,m)} \equiv 1$. First we assume that $y^{s/(s,m)} \leq x^{s/(s,n)} \leq 1$. Since $in +_s jm = (n,m)$, we know $x^iy^j \in g^{(n,m)}$, so $(x^iy^j)^{m/(n,m)} \in g^m$ and hence $(x^iy^j)^{m/(n,m)} \equiv y$. To show $(x^iy^j)^{m/(n,m)} = y$, it

suffices to show $((x^iy^j)^{m/(n,m)})^s = y^s$ by (6). For this we have $((x^iy^j)^{m/(n,m)})^s = (x^s)^{im/(n,m)}(y^s)^{jm/(n,m)} \overset{id}{=} x^sy^s \overset{id}{=} x^{s/(s,n)}y^{s/(s,m)} = y^{s/(s,m)} \overset{id}{=} y^s$, thus $y \in \langle x^iy^j\rangle_.$. Let $p,q \in \mathbb{Z}_p$, then $x^p(x^iy^j)^q \in g^{pn}g^{q(n,m)} = g^{pn+q(n,m)} = g^{(pn/(n,m)+q)(n,m)}$. Since $(x^iy^j)^{pn/(n,m)+q} \in g^{(pn/(n,m)+q)(n,m)}$, we know $(x^iy^j)^{pn/(n,m)+q} \equiv x^p(x^iy^j)^q$. Since we know $(x^p(x^iy^j)^q)^s = (x^s)^{p+iq}(y^s)^{jq} \overset{id}{=} x^sy^s = y^s$ and similarly $((x^iy^j)^{pn/(n,m)+q})^s = x^sy^s = y^s$, $(x^iy^j)^{pn/(n,m)+q} = x^p(x^iy^j)^q$ by (6), so $x^p(x^iy^j)^q \in \langle x^iy^j\rangle_.$ and hence $\langle x,y\rangle_. = \langle x\rangle_. \cup \langle x^iy^j\rangle_.$.

Since $y \in \langle x^iy^j\rangle_.$, we know $y^\ell \in \langle x^iy^j\rangle_. \cup \langle x^iy^j\rangle_.^\ell = \langle x^iy^j\rangle$. Let $p,q,r \in \mathbb{Z}_s$ and denote x^iy^j by z. Since $(x^p)^\ell x^q = ((x^p)^\ell x^p)x^{q-sp} \overset{(7)}{=} (x^\ell x)x^{q-sp} \overset{(4)}{=} x^nx^{q-sp} \overset{(6)}{=} x^{q-sp} \in \langle x\rangle_.$ by (7) and $x^qz^r \in \langle z\rangle_.$ by proof above, we know $\langle x\rangle_.^\ell \cdot \langle x\rangle_. \subseteq \langle x\rangle_.$ and $\langle x\rangle_. \cdot \langle z\rangle_. \subseteq \langle x\rangle_.$, so $\langle x\rangle_.^\ell \cdot \langle z\rangle_. \subseteq \langle z\rangle_.$ and hence $(x^r)^\ell z^t \in \langle z\rangle_.$. Since $(z^p)^\ell x^q = (z^{p-_snq/(n,m)}z^{nq/(n,m)})^\ell x^q = (z^{p-_snq/(n,m)})^\ell(z^{nq/(n,m)})^\ell x^q$, $z^{nq/(n,m)} \equiv x^q$ and $(z^{nq/(n,m)})^s = y^s \leq x^s = (x^q)^s$, we know $(z^{nq/(n,m)})^\ell x^q \in \langle z\rangle_.^\ell$, thus $(z^p)^\ell x^q \in \langle z\rangle_.^\ell$. Since $(x^p)^\ell(z^q)^\ell = (x^pz^q)^\ell$ by (11) and $x^pz^q \in \langle z\rangle_.$, we know $(x^p)^\ell(z^q)^\ell \in \langle z\rangle_.^\ell$. Since $y, y^\ell \in \langle z\rangle$, we know $x^py^q, x^p(y^q)^\ell, (x^p)^\ell y^q$ and $(x^p)^\ell(y^q)^\ell$ are in $\langle z\rangle$. Therefore $\langle x,y\rangle = \langle x^iy^j\rangle \cup \langle x\rangle$.

If $y^{s/(s,m)} \leq x^{s/(s,n)} \leq 1$, then similarly we can show $\langle x,y\rangle = \langle x^iy^j\rangle \cup \langle y\rangle$.

(15) Since $\langle g_1\rangle \cap \langle g_2\rangle = \{1_{\mathbf{G}}\}$, we know $\langle x\rangle_. \cap \langle y\rangle_.$ is empty if $x^n \neq y^m$ and is singleton otherwise, thus $\langle x,y\rangle_. = \langle x\rangle_. \times \langle y\rangle_.$. Without loss of generality assume that $y^m \leq x^n \leq 1$, then $(x^i)^\ell y^j \equiv x^{n-i}y^j$. Since $[x^n,(x^n)^\ell] \subseteq [y^m,(y^m)^\ell]$ in $\mathbf{H}$, we know $((x^i)^\ell y^j)^s = ((x^i)^\ell)^sy^s = y^s \overset{id}{=} (x^{n-i})^s(y^j)^s = (x^{n-i}y^j)^s$, where s is the least common multiple of n and m, thus $(x^i)^\ell y^j = x^{n-i}y^j \in \langle x,y\rangle_.$. Similarly we can show $(y^i)^\ell x^j \in \langle x,y\rangle_.^\ell$. Therefore $\langle x,y\rangle = \langle x,y\rangle_. \cup \langle x,y\rangle_.^\ell = (\langle x\rangle_. \times \langle y\rangle_.) \cup (\langle x\rangle_. \times \langle y\rangle_.)^\ell$.

(16) Let $x \in M$ satisfying $x^{n-1} = x^\ell$, then $x^n = x^{n-1}x = x^\ell x \leq 1$, so x is a lower element. Suppose there exists a lower element $y \in M$ such that $y \geq x$, then $y^\ell \leq x^\ell = x^{n-1} \leq y^{n-1}$ by 1-involution and order-preservation of multiplication. Since y is lower, we know $y^{n-1} \leq y^\ell$, so $y^\ell = x^\ell$ and $y = x$. Thus x is the maximum lower element in its chain.

Now assume that x is the maximum lower element in its chain. If $x \in H$, then $x = 1$, so the statement holds. So we assume that $x \parallel 1$. Suppose $x^{n-1} = x^{\ell}$, then x^{ℓ} is a lower element by (4) and $x = (x^{n-1})^{n-1} = (x^{\ell})^{n-1}$ by (6). Let y be a lower element such that $y \geq x^{\ell}$, then $y^{n-1} \geq (x^{\ell})^{n-1} = x$. Since y is a lower element, so is y^{n-1}. Since x is the maximum lower element in its chain, we know $y^{n-1} = x$, thus $y \overset{(6)}{=} (y^{n-1})^{n-1} = x^{n-1} = x^{\ell}$ and hence x^{ℓ} is the maximum lower element in its chain. Now suppose $x^{n-1} < x^{\ell}$, then x^{ℓ} is an upper element by (8). Let y be an upper element such that $y \leq x^{\ell}$, then y^{ℓ} is a lower element by (4) and $y^{\ell} \geq x$ by 1-involution. Since x is the maximum lower element in its chain, we know $y^{\ell} = x$, so $y = x^{\ell}$ and x^{ℓ} is the minimum upper element in its chain. $\square$

Recall that Corollary 7.8 shows in particular that if $\mathbf{H}$ is a 1-involutive residuated chain and $\mathbf{G}$ is a group then $\mathbf{H} \times^b \mathbf{G}$ is a compact URL; see Section 7.2. We improve this result, by observing that in this case the result is 1-involutive.

Lemma 10.7. If $\mathbf{H}$ is a 1-involutive residuated chain and $\mathbf{G}$ is a group then $\mathbf{H} \times^b \mathbf{G}$ is a 1-involutive compact URL. In particular, if $\mathbf{H}$ is an odd Sugihara chain and $\mathbf{G}$ an abelian group, then $\mathbf{H} \times^b \mathbf{G}$ is a commutative 1-involutive compact URL.

The next theorem shows that every commutative 1-involutive compact URL can be obtained from such an $\mathbf{H} \times^b \mathbf{G}$ by a small modification. In a later result we get an even more precise characterization.

Recall that a *conucleus* σ on a residuated lattice $\mathbf{R}$ is a contracting ($\sigma(x) \leq x$) monotone ($x \leq y \implies \sigma(x) \leq \sigma(y)$) idempotent ($\sigma(\sigma(x)) = \sigma(x)$) operator that satisfies $\sigma(x)\sigma(1) = \sigma(1)\sigma(x) = \sigma(x)$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$, the latter of which is equivalent to $\sigma(\sigma(x)\sigma(y)) = \sigma(xy)$.

Theorem 10.8. Let $\mathbf{R}$ be a commutative 1-involutive compact URL such that the chain of 1, $H \cup \{\bot, \top\}$, is a bounded odd Sugihara chain and $\mathbf{M}/\equiv \cong \mathbf{G}$, where $\mathbf{G}$ is an abelian group where every element has finite order and $M = R \setminus \{\bot, \top\}$. Then $\mathbf{R}$ is isomorphic to a subalgebra of a conucleus image of $\mathbf{H} \times^b \mathbf{G}$.

Proof. Since, by Theorem 10.4(1), $\mathbf{M}$ is a $(\cdot, \to, 1)$-subreduct of $\mathbf{R}$ and since $(R, \wedge, \vee)$ is unilinear, in the following we focus on $\mathbf{M}$. By Lemma 10.6(2) we know every chain in $\mathbf{M}$ contains lower elements, so for each chain in $\mathbf{M}$ either all lower elements satisfy $x^{n-1} < x^\ell$ or there exists a lower element satisfying $x^{n-1} = x^\ell$, in which case that x is the maximum lower element in its chain, by Lemma 10.6(16). If G_1 denotes the set of chains whose lower elements satisfy $x^{n-1} < x^\ell$ and G_2 denotes the set of chains which contain a lower element x satisfying $x^{n-1} = x^\ell$, then $G = G_1 \sqcup G_2$. By Lemma 10.6(13), each one of G_1 and G_2 is closed under multiplication. Also by Lemma 10.6(16), G_2 is closed under inverses, so G_1 is also closed under inverses. Since $1 \in G_2$, we know $\mathbf{G}_2$ is a subgroup of $\mathbf{G}$. Now we show that for every chain $g \in G_1$, the map $\psi_g : g \to H$ defined by $\psi_g(x) = x^n$, where n is the order of g in the group $\mathbf{G}$, is an order embedding and $\psi_g[g]$ a multiplicative ideal of $\mathbf{H}$ which is also closed under $^\ell$, so $I_g := H \setminus \psi_g[g]$ is equal to (e_g, e_g^ℓ) or to $[e_g, e_g^\ell]$ for some negative $e_g \in H$.

Since the multiplication on $\mathbf{M}$ is order-preserving, if $x, y \in g$ with $x \leq y$ then $\psi_g(x) = x^n \leq y^n = \psi_g(y)$. Now suppose $\psi_g(x) \leq \psi_g(y)$ for some $x, y \in g$; then $x^n \leq y^n$. If $y < x$, then $y^n \leq x^n$, so $x^n = y^n$ and using Lemma 10.6(6), we get $x = y$, contradicting $y < x$; thus $\psi_g(x) \leq \psi_g(y)$ implies $x \leq y$, so ψ_g is an order-embedding. Now for $x \in g$ and $a \in H$, we have $a \equiv 1$ so $ax \equiv x$, i.e. $ax \in g$. Therefore, $a\psi_g(x) = ax^n \overset{id}{=} a^n x^n = (ax)^n = \psi_g(ax) \in \psi_g[g]$, hence $\psi_g[g]$ is a multiplicative ideal of $\mathbf{H}$. Finally, since all lower elements in g satisfy $x^{n-1} < x^\ell$, Lemma 10.6(9) and (10) entail that for all $x \in g$ and all $1 \leq k \leq n$, we have $(x^k)^\ell = (x^\ell)^k$ and also that $\langle x \rangle_\cdot$ and $\langle x^\ell \rangle_\cdot$ are both isomorphic to $\mathbb{Z}_n$. Also, $x^{n-1} < x^\ell$ implies $x < (x^{n-1})^\ell$ by 1-involutivity, so $(x^{n-1})^\ell \in g$ and $\psi_g(x)^\ell = (x^n)^\ell \overset{(9)}{=} (x^\ell)^n \overset{id}{=} ((x^\ell)^{n-1})^n \overset{(9)}{=} ((x^{n-1})^\ell)^n = \psi_g((x^{n-1})^\ell)$, hence $\psi_g[g]$ is closed under $^\ell$.

Now let $I_g = H \setminus \psi_g[g]$ and $a, b, c \in I_g$ such that $a < b < c$ and $a, c \in H$. If $b \notin I_g$, then $b \in \psi_g[g]$. Since $\psi_g[g]$ is an ideal of $\mathbf{H}$, we know $ab, bc \in \psi_g[g]$. Since the multiplication on $\mathbf{H}$ is conservative, we get $ab = b = bc$, contradicting that $\mathbf{H}$ is an odd Sugihara chain; so I_g is convex. Since $\psi_g[g]$ is closed under $^\ell$, I_g is also closed under $^\ell$. Finally, $1 \in I_g$, since otherwise $1 \in \psi_g[g]$, so there exists $x \in g$ such that $1 = x^n$, hence $x^\ell = 1 \cdot x^\ell = x^n x^\ell = x^{n-1}(x^\ell x) \overset{(4)}{=} x^{n-1}x^n \overset{(6)}{=} x^{n-1}$, contradicting the assumption that $x^{n-1} < x^\ell$ for all $x \in g$. Therefore $I_g = (e_g, (e_g)^\ell)$ or $I_g = [e_g, (e_g)^\ell]$ for some negative $e_g \in H$.

For $g \in G_2$, let u_g denote an element in g satisfying $u_g^{n_g - 1} = u^\ell$, where n_g is the order of chain of g; note that there is a unique such element because, by Lemma 10.6(16), u_g is the maximum lower element in g. Note that $J_g := (u_g^{n_g}, (u_g^{n_g})^\ell]$ is a subset of H. We define $M_0 := (H \times G) \setminus ((\bigcup_{g \in G_1} I_g^\circ \times \{g\}) \cup (\bigcup_{g \in G_2} J_g \times \{g\}))$, where $I_g^\circ = (e_g, (e_g)^\ell)$ for all $g \in G_1$. Now we show that $M_0^b = M_0 \cup \{\bot, \top\}$ is a conucleus image of $H \times^b G$.

To show that M_0^b is closed under multiplication, we consider $(h_1, g_1), (h_2, g_2) \in M_0$. If $g_1, g_2 \in G_1$, then $h_1 \in H \setminus I_1^\circ$ and $h_2 \in H \setminus I_2^\circ$, where I_1 and I_2 are short for I_{g_1} and I_{g_2} respectively. By the definition of I_1 and I_2, there exist $x \in g_1$ and $y \in g_2$ such that $h_1 = x^n$ and $h_2 = y^m$, where n and m are the orders of g_1 and g_2 respectively. Since G_1 is closed under multiplication, $\psi_{g_1 g_2}$ is well-defined and $\psi_{g_1 g_2}(xy) = (xy)^s = x^s y^s \overset{id}{=} x^n y^m = h_1 h_2$, where s is the least common multiple of n and m, thus $(h_1 h_2, g_1 g_2) \in M_0$. If $g_1, g_2 \in G_2$, then $h_1 \in H \setminus (u_1^n, (u_1^n)^\ell]$ and $h_2 \in H \setminus (u_2^m, (u_2^m)^\ell]$. Since G_2 is a subgroup of $\mathbf{G}$, $J_{g_1 g_2}$ is well-defined and $J_{g_1 g_2} = ((u_1 u_2)^p, ((u_1 u_2)^s)^\ell]$ by Lemma 10.6(13). Since $(u_1 u_2)^s = u_1^s u_2^s \overset{id}{=} u_1^n u_2^m \in \{u_1^n, u_2^m\}$, we know $J_{g_1 g_2} = J_{g_1}$ or J_{g_2}. Since $h_1 h_2 \in \{h_1, h_2\}$, we know $h_1 h_2 \notin J_{g_1}$ and $h_1 h_2 \notin J_{g_2}$, thus $h_1 h_2 \notin J_{g_1 g_2}$ and hence $(h_1 h_2, g_1 g_2) \in M_0$. If $g_1 \in G_1$ and $g \in G_2$, then $h_1 \in H \setminus (e_1, e_1^\ell)$ and $h_2 \in H \setminus (u^m, (u^m)^\ell]$. When $e_1 \leq u^m$, $h_1 h_2 \notin (e_1, e_1^\ell)$ and by Lemma 10.6(13) we know $g_1 g_2 \in G_1$ and $I_{g_1 g_2}^\circ = (e_1, e_1^\ell)$, so $h_1 h_2 \in H \setminus I_{g_1 g_2}^\circ$ and $(h_1 h_2, g_1 g_2) \in M_0$. When $u^m < e_1$, $h_1 h_2 \notin (u^m, (u^m)^\ell]$ and again by Lemma 10.6(13) we know $g_1g_2 \in G_2$ and $J_{g_1g_2} = (u^m, (u^m)^\ell]$, so $h_1h_2 \in H \setminus J_{g_1g_2}$ and $(h_1h_2, g_1g_2) \in M_0$. Finally, if $g_1 \in G_2$ and $g_2 \in G_1$, then similarly to the previous case we can show $(h_1h_2, g_1g_2) \in M_0$.

Since the lattice reduct of $\mathbf{M}$ is unilinear and $\bot, \top \in M_0^b$, M_0^b is closed under $\vee$. Finally, for all $(h, g) \in H \times G$, if $(h, g) \notin M_0$, then either $h \in I_g^\circ$ for some $g \in G_1$ or $h \in J_g$ for some $g \in G_2$. If $h \in I_g^\circ$ for some $g \in G_1$, then $\downarrow(h, g) \cap M_0$ has maximum element $(e_g, g) \in M_0$; if $h \in J_g$ for some $g \in G_2$, then $\downarrow(h, g) \cap M_0$ has maximum element $(u_g^{n_g}, g) \in M_0$. Therefore M_0^b is a conucleus image of $H \times^b G$ and the conucleus σ is defined by $\sigma(\top) = \top$ and $\sigma(\bot) = \bot$ and if $g \in G_1$, then $\sigma(h, g) = \begin{cases} (h, g) & \text{if } h \notin (e_g, e_g^\ell) \\ (e_g, g) & \text{if } h \in (e_g, e_g^\ell) \end{cases}$; if $g \in G_2$, then $\sigma(h, g) = \begin{cases} (h, g) & \text{if } h \notin [u_g^{n_g}, (u_g^{n_g})^\ell] \\ (u_g^{n_g}, g) & \text{if } h \in [u_g^{n_g}, (u_g^{n_g})^\ell] \end{cases}$. By [7], we know M_0^b is the universe of a residuated lattice.

Now we show that $\mathbf{R}$ can be embedded into $\mathbf{M}_0^b$. Define $\varphi : R \to M_0^b$ by $\varphi(\top) = \top$, $\varphi(\bot) = \bot$ and $\varphi(x) = (x^{n_g}, g)$, where g is the chain of x and n_g is the order of g. We omit the subscript when it is clear in the context. For all $x, y \in M$, if $x \leq y$, then x and y belong to the same chain and $x^n \leq y^n$ by the order-preserving of multiplication on $\mathbf{M}$, so $\varphi(x) = (x^n, g) \leq (y^n, g) = \varphi(y)$. Conversely if $\varphi(x) \leq \varphi(y)$, then $x \equiv y$ and $x^n \leq y^n$. If $y < x$, then $y^n \leq x^n$, so $y^n = x^n$, so by Lemma 10.6(6), we get $x = y$, a contradiction; thus $x \leq y$ and φ is an order-embedding. Since multiplication on $\mathbf{M}$ is compatible with $\equiv$, we know $g_x g_y = g_{xy}$, so $\varphi(x)\varphi(y) = (x^n, g_x)(y^m, g_y) = (x^n y^m, g_{xy})$. Since the order s of $g_x g_y$ is the least common multiple of those of g_x and g_y, we know $\varphi(xy) = ((xy)^s, g_{xy}) = (x^s y^s, g_{xy}) \overset{id}{=} (x^n y^m, g_{xy}) = \varphi(x)\varphi(y)$. Since G_1 and G_2 are closed under multiplication and inverse respectively, we know if the chain of x is in a chain of G_1, so is the chain of x^ℓ; similarly for x in a chain of G_2. First assume that x is in a chain of G_1, then

$\varphi(x^\ell) = ((x^\ell)^n, g_{x^\ell}) = ((x^n)^\ell, g^{-1})$. Since $g \in G_1$, we know $x^n \notin I_g^\circ$ by the definition of I_g, so $\varphi(x)^{\ell_\sigma} = \sigma((x^n, g)^{\ell_{\mathbf{H}\times^b\mathbf{G}}}) = \sigma((x^n)^\ell, g^{-1}) = ((x^n)^\ell, g^{-1}) = \varphi(x^\ell)$. Now assume x is in a chain of G_2. If x satisfies $x^{n-1} < x^\ell$, then by the proof in the previous case we can show $\varphi(x^\ell) = \varphi(x)^{\ell_\sigma}$. So we assume that x satisfies $x^{n-1} = x^\ell$, then by Lemma 10.6(16), x is the maximum lower element u_g in its chain. Now $\varphi(u_g^\ell) = \varphi(u_g^{n-1}) = ((u_g^{n-1})^n, g^{n-1}) = (u_g^n, g^{n-1})$ and $\varphi(u_g)^{\ell_\sigma} = \sigma((u_g^n, g)^{\ell_{\mathbf{H}\times^b\mathbf{G}}}) = \sigma((u_g^n)^\ell, g^{-1}) = (u_g^n, g^{n-1}) = \varphi(u_g^\ell)$ by the definition of σ. Therefore φ is an embedding from $\mathbf{R}$ into $\mathbf{M}_0^b$. $\square$

The following theorem strengthens Theorem 10.8 by showing that we only need to consider conuclei that fix the chain of 1.

Theorem 10.9. Let $\mathbf{H}$ be an idempotent residuated chain, $\mathbf{G}$ a group whose elements are of finite orders and σ a conucleus on $\mathbf{H} \times^b \mathbf{G}$ such that $\sigma(\bot) = \bot$, and $\sigma(\top) = \top$. Then $\sigma[H \times G] = \sigma[\sigma'[H] \times G]$, where $\sigma'[H]$ is the projection of $\sigma[H \times \{1_\mathbf{G}\}]$. If $\mathbf{H}$ is an odd Sugihara chain, $\mathbf{G}$ is abelian and $\sigma[H \times G]$ is $\sigma(1_\mathbf{H}, 1_\mathbf{G})$-involutive, then $\sigma'[H]$ is also an odd Sugihara chain.

Proof. Let $(h, g) \in \sigma[H \times^b G]$ such that the chain g is of order n. Since σ is a conucleus on $H \times^b G$ and $\mathbf{H}$ is idempotent, we have $(h, 1) = (h^n, g^n) = (h, g)^n \in \sigma[H \times^b G]$. By the definition of σ', we know $h \in \sigma'[H]$, so $(h, g) \in \sigma'[H] \times^b G$. Since σ is idempotent, we get $(h, g) \in \sigma[\sigma'[H] \times^b G]$. Thus if $\sigma(k, g) > \bot$ for some $(k, g) \in H \times G$, then $\sigma(k, g) \in \sigma[\sigma'[H] \times^b G]$, therefore $\sigma[H \times G] \subseteq \sigma[\sigma'[H] \times G]$. The other direction is trivial.

Now assume that $\mathbf{H}$ is an odd Sugihara chain, $\mathbf{G}$ is abelian and $\sigma[H \times G]$ is $\sigma(1_\mathbf{H}, 1_\mathbf{G})$-involutive. Since $\mathbf{H}$ is a subalgebra of $\mathbf{H} \times^b \mathbf{G}$ and $\sigma'[\mathbf{H}]$ is the projection of $\sigma[\mathbf{H} \times \{1_\mathbf{G}\}]$, we know $\sigma'[\mathbf{H}]$ is also a subalgebra of $\sigma[\mathbf{H} \times^b \mathbf{G}]$. Since $\sigma[\mathbf{H} \times^b \mathbf{G}]$ is $\sigma(1_\mathbf{H}, 1_\mathbf{G})$-involutive, $\sigma'[H]$ is $\sigma'(1_\mathbf{H})$-involutive by definition of σ'. Since $\mathbf{H}$ is commutative and idempotent, $\sigma'[H]$ is also commutative and idempotent. Hence $\sigma'[\mathbf{H}]$ is an odd Sugihara chain. $\square$

Note that Theorem 10.8 and Theorem 10.9 do not provide a converse to Lemma 10.7, because conuclei images of $\mathbf{H} \times^b \mathbf{G}$ may fail to be 1-involutive. Interestingly, connecting to Theorem 10.11, if the conucleus fixes the chain of 1, then the resulting URL is 1-involutive.

Lemma 10.10. Assume $\mathbf{H}$ is a commutative 1-involutive chain, $\mathbf{G}$ is a group where every element has finite order and σ is a conucleus on $\mathbf{H} \times^b \mathbf{G}$ such that σ is the identity on $H \times \{1\} \cup \{\bot, \top\}$. If $\sigma(h_0, g_0) = \bot$ for some $h_0 \in H$ and $g_0 \neq 1$, then $\sigma[H \times \{g_0\}] = \{\bot\}$. Also, $\mathbf{G}'$ is a subgroup of $\mathbf{G}$ and $G_\bot G' \subseteq G_\bot$, where $G_\bot := \{g \in G : \sigma[H \times \{g\}] = \{\bot\}\}$ and $G' := G \setminus G_\bot$. Therefore $\sigma[H \times G] = \sigma[H \times G']$.

Proof. Suppose there exists $(h_0, g_0) \in H \times G$ such that $\sigma(h_0, g_0) = \bot$. Since σ is monotone, we know $\sigma(h, g_0) \leq \sigma(h_0, g_0)$, so $\sigma(h, g_0) = \bot$ for all $h \leq h_0$. Without loss of generality, assume that $h_0 \leq 1$. Since σ is the identity map on $\mathbf{H}$, we have $(h_0, 1)\sigma(h, g_0) = \sigma(h_0, 1)\sigma(h, g_0) \leq \sigma(h_0 h, g_0) = \sigma(h_0, g_0) = \bot$ for all $h \in [h_0, h_0^\ell]$. Since $H \times G$ is closed under multiplication, we get $\sigma(h, g_0) = \bot$ for all $h \in [h_0, h_0^\ell]$. If $h > h_0^\ell$, then $h^\ell < h_0$ and so $\sigma(h^\ell, g_0) = \bot$. Hence $(h^\ell, 1)\sigma(h, g_0) = \sigma(h^\ell, 1)\sigma(h, g_0) \leq \sigma(hh^\ell, g_0) = \sigma(h^\ell, g_0) = \bot$, thus $\sigma(h, g_0) = \bot$ for all $h \in H$, i.e., $\sigma[H \times \{g_0\}] = \{\bot\}$. Since g_0 is of finite order $n \in \mathbb{Z}^+$, we know $g_0^{-1} = g_0^{n-1}$ and $g_0 = (g_0^{-1})^{n-1}$, so $(1, g_0) = (1, (g_0^{-1})^{n-1}) = (1, g_0^{-1})^{n-1}$. Thus $(\sigma(1, g_0^{-1}))^{n-1} \leq \sigma((1, g_0^{-1})^{n-1}) = \sigma(1, g_0) = \bot$, therefore $\sigma(1, g_0^{-1}) = \bot$ and $\sigma[H \times \{g_0^{-1}\}] = \{\bot\}$. Hence $G_\bot$ is closed under inverse.

Since $\sigma \restriction_{H \times \{1\}} = \mathrm{id}_{H \times \{1\}}$, we know $1 \in G'$. Since $G_\bot$ is closed under inverse, G' is also closed under inverse. Now let $g_1, g_2 \in G'$, then $\sigma[H \times \{g_1\}] \cap \{\bot\} = \emptyset$ and $\sigma[H \times \{g_2\}] \cap \{\bot\} = \emptyset$. Since $\sigma(h, g_1)\sigma(1, g_2) \leq \sigma(h, g_1 g_2)$, we know $\sigma(h, g_1 g_2) > \bot$ for all $h \in H$, so $\mathbf{G}'$ is a subgroup of $\mathbf{G}$.

Finally, let $g_1 \in G_\bot$ and $g_2 \in G'$, then $\sigma[H \times \{g_1\}] = \{\bot\}$ and $\sigma[H \times \{g_2\}] \cap \{\bot\} = \emptyset$. Since G' is closed under inverse, we know $\sigma[H \times \{g_2^{-1}\}] \cap \{\bot\} = \emptyset$, so $\sigma(h, g_1 g_2)\sigma(1, g_2^{-1}) \leq \sigma(h, g_1) = \bot$ implies that $\sigma(h, g_1 g_2) = \bot$. Therefore $G_\bot G' \subseteq G_\bot$. ∎

Theorem 10.11. If $\mathbf{H}$ is an odd Sugihara chain, $\mathbf{G}$ is an abelian group where every element has finite order and σ is a conucleus on $\mathbf{H} \times^b \mathbf{G}$ such that σ is the identity on $H \times \{1\} \cup \{\bot, \top\}$, then $\sigma[\mathbf{H} \times^b \mathbf{G}]$ is a commutative 1-involutive compact URL.

Proof. By Lemma 10.10, we can assume that $\sigma(h, g) > \bot$ for all $(h, g) \in H \times G$. Since $\mathbf{H} \times^b \mathbf{G}$ is a commutative 1-involutive compact URL and σ is a conucleus, we know $\sigma[\mathbf{H} \times^b \mathbf{G}]$ is a commutative URL. Since $\sigma(\top) = \top$, $\sigma(\bot) = \bot$ and $\sigma(h, g) > \bot$ for all $(h, g) \in H \times G$, we know $\sigma[H \times G]$ is closed under multiplication since $H \times G$ is closed under multiplication, hence $\sigma[\mathbf{H} \times^b \mathbf{G}]$ is compact. Now we show that $\sigma[\mathbf{H} \times^b \mathbf{G}]$ is 1-involutive.

Since $\sigma(h, g) > \bot$ for all $(h, g) \in H \times G$, there exists $e_g \leq 1$ such that $\sigma(1, g) = (e_g, g)$ for all $g \neq 1$. Since σ is idempotent, we know $\sigma(e_g, g) = (e_g, g)$. Since g is of order n for some $n \in \mathbb{Z}^+$, we know $g^{-1} = g^{n-1}$ and $g = (g^{-1})^{n-1}$. Let $\sigma(1, g^{-1}) = (e, g^{-1})$ for some $e \leq 1$, then $(e, g^{-1}) = \sigma(1, g^{-1}) = \sigma(1, g^{n-1}) = \sigma((1, g)^{n-1}) \geq (\sigma(1, g))^{n-1} = (e_g, g)^{n-1} = (e_g, g^{-1})$, so $e \geq e_g$. By $(e_g, g) = \sigma(1, g) = \sigma(1, (g^{-1})^{n-1}) \geq (\sigma(1, g^{-1}))^{n-1} = (e, g^{-1})^{n-1} = (e, g)$, we have $e_g \geq e$, thus $e = e_g$.

Now for $h \in H \backslash [e_g, e_g^\ell]$, we have $(h, g) = (h, 1)(e_g, g)$, so $\sigma(h, g) \geq \sigma(h, 1)\sigma(e_g, g) = (h, 1)(e_g, g) = (h, g) \geq \sigma(h, g)$, thus $\sigma(h, g) = (h, g)$ for $h \in H \setminus [e_g, e_g^\ell]$. Also, since σ is monotone, we have $(e_g, g) = \sigma(e_g, g) \leq \sigma(h, g) \leq \sigma(1, g) = (e_g, g)$, so $\sigma(h, g) = (e_g, g)$ for all $h \in [e_g, 1]$. Now let $h \in (1, e_g^\ell)$, then $(e_g, g) = \sigma(1, g) \leq (h', g) = \sigma(h, g) \leq \sigma(e_g^\ell, g)$, so $e_g \leq h' \leq h < e_g^\ell$. Suppose $h' > e_g$, then $h' > 1$ by the monotonicity of σ. In this case we have $(h, g) = (h, 1)(h', g)$, so $(h', g) = \sigma(h, g) \geq \sigma(h, 1)\sigma(h', g) = (h, 1)(h', g) = (h, g) \geq \sigma(h, g) = (h', g)$, thus $\sigma(h, g) = (h, g)$. On the other hand, since $1 < h < e_g^\ell$, we get $e < h^\ell < 1$ by 1-involution, so $(h^\ell, g) = (h^\ell, 1)(h, g)$ and hence $(e_g, g) = \sigma(h^\ell, g) \geq \sigma(h^\ell, 1)\sigma(h, g) = (h^\ell, 1)(h, g) = (h^\ell, g) \geq \sigma(h^\ell, g) = (e_g, g)$, thus $h^\ell = e_g$, contradicting $e_g < h^\ell$. Therefore $h' = e_g$ and $\sigma(h, g) = (e_g, g)$ for all $h \in [e_g, e_g^\ell)$. Since $\sigma(1, g^{-1}) = (e_g, g^{-1})$, similarly we can show $\sigma(h, g^{-1}) = (e_g, g^{-1})$ for all $h \in [e_g, e_g^\ell)$. Also, by above proof we can tell $\sigma(e_g^\ell, g)$ is either (e_g, g)

or (e_g^ℓ, g) and $\sigma(e_g^\ell, g^{-1})$ is either (e_g, g^{-1}) or (e_g^ℓ, g^{-1}). Since g is of finite order in $\mathbf{G}$, $g^{-1} = g^{n-1}$ and $g = (g^{-1})^{n-1}$ for some $n \in \mathbb{Z}^+$. So $(e_g^\ell, g) = (e_g^\ell, g^{-1})^{n-1}$ and hence $\sigma(e_g^\ell, g) \geq (\sigma(e_g^\ell, g^{-1}))^{n-1}$; similarly we have $\sigma(e_g^\ell, g^{-1}) \geq (\sigma(e_g^\ell, g))^{n-1}$. Thus if $\sigma(e_g^\ell, g) = (e_g, g)$ then $\sigma(e_g^\ell, g^{-1}) = (e_g, g^{-1})$; if $\sigma(e_g^\ell, g^{-1}) = (e_g, g^{-1})$ then $\sigma(e_g^\ell, g) = (e_g, g)$, i.e., $\sigma(e_g^\ell, g) = (e_g, g)$ iff $\sigma(e_g^\ell, g^{-1}) = (e_g, g^{-1})$; hence $\sigma(e_g^\ell, g) = (e_g^\ell, g)$ iff $\sigma(e_g^\ell, g^{-1}) = (e_g^\ell, g^{-1})$. If $\sigma(e_g^\ell, g) = (e_g, g)$, then $\sigma(h, g) = \begin{cases} (h, g) & \text{if } h \notin [e_g, e_g^\ell] \\ (e_g, g) & \text{if } h \in [e_g, e_g^\ell] \end{cases}$ and $\sigma(h, g^{-1}) = \begin{cases} (h, g^{-1}) & \text{if } h \notin [e_g, e_g^\ell] \\ (e_g, g^{-1}) & \text{if } h \in [e_g, e_g^\ell] \end{cases}$. In this case, if $h \notin [e_g, e_g^\ell]$, then $(\sigma(h, g))^{\ell_\sigma \ell_\sigma} = (h, g)^{\ell_\sigma \ell_\sigma} = (\sigma(h^\ell, g^{-1}))^{\ell_\sigma} = (h^\ell, g^{-1})^{\ell_\sigma} = \sigma(h, g)$. If $h \in [e_g, e_g^\ell]$, then $(\sigma(h, g))^{\ell_\sigma \ell_\sigma} = (e_g, g)^{\ell_\sigma \ell_\sigma} = (\sigma(e_g^\ell, g^{-1}))^{\ell_\sigma} = (e_g, g^{-1})^{\ell_\sigma} = \sigma(e_g^\ell, g) = (e_g, g) = \sigma(h, g)$.

Finally, if $\sigma(e_g^\ell, g) = (e_g^\ell, g)$, then $\sigma(h, g) = \begin{cases} (h, g) & \text{if } h \notin (e_g, e_g^\ell) \\ (e_g, g) & \text{if } h \in (e_g, e_g^\ell) \end{cases}$, $\sigma(h, g^{-1}) = \begin{cases} (h, g^{-1}) & \text{if } h \notin (e_g, e_g^\ell) \\ (e_g, g^{-1}) & \text{if } h \in (e_g, e_g^\ell) \end{cases}$. In this case, if $h \notin (e_g, e_g^\ell)$, then $(\sigma(h, g))^{\ell_\sigma \ell_\sigma} = (h, g)^{\ell_\sigma \ell_\sigma} = (\sigma(h^\ell, g^{-1}))^{\ell_\sigma} = (h^\ell, g^{-1})^{\ell_\sigma} = \sigma(h, g)$. If $h \in (e_g, e_g^\ell)$, then $(\sigma(h, g))^{\ell_\sigma \ell_\sigma} = (e_g, g)^{\ell_\sigma \ell_\sigma} = (\sigma(e_g^\ell, g^{-1}))^{\ell_\sigma} = (e_g^\ell, g^{-1})^{\ell_\sigma} = \sigma(e_g, g) = (e_g, g) = \sigma(h, g)$. □

Corollary 10.12. The commutative 1-involutive $\top$-unital URLs $\mathbf{R}$ such that the chain of 1, $H \cup \{\bot, \top\}$, is a bounded odd Sugihara chain and $\mathbf{M}/\equiv \cong \mathbf{G}$, where $\mathbf{G}$ is an abelian group whose each element has finite order and $M = R \setminus \{\bot, \top\}$ are (up to isomorphism) precisely the subalgebras of conucleus images of $\mathbf{H} \times^b \mathbf{G}$, where the conucleus fixes the chain of 1.

Now we say a commutative 1-involutive $\top$-unital URL is (n, k)-*potent* if there exist $n \in \mathbb{Z}^+$ such that every element satisfies $x^{n+k} = x^k$ for all $1 \leq k \leq n$. Since the chain of 1 in such URL is a subalgebra, by Lemma 10.3 we know the chain of 1 itself is an odd Sugihara chain. Also, such URLs include those whose $\mathbf{G}$ is an abelian group of order n.

Corollary 10.13. Every (n, k)-potent/finite commutative 1-involutive compact URL $\mathbf{R}$ is isomorphic to a subalgebra of a conucleus image of $\mathbf{H} \times^b \mathbf{G}$, where $\mathbf{H}$ is the chain of 1 in $M = R \setminus \{\bot, \top\}$, $\mathbf{G} = \mathbf{M}/\equiv$ and the conucleus fixes the chain of 1; these are precisely all the (n, k)-potent/finite commutative 1-involutive compact URLs.

Corollary 10.14. Here are all the commutative 1-involutive URL: the $\top$-unital ones; the non-$\top$-unital ones in the class LW in which $W = \emptyset$, $Z \cup \{\bot, \top\}$ is an integral residuated chain, such that $\top$-unital subalgebra is 1-involutive and the 1-free subalgebra $Z \cup \{\bot, \top\}$ is $\bot$-involutive; the ones in B whose $\top$-unital subalgebra is 1-involutive.

Proof. Here we show the non-linear non-$\top$-unital ones are as described above. Let $\mathbf{R}$ be a non-linear commutative URL in TW such that $\mathbf{R}$ is not $\top$-unital, then there exists $b, b_0 \in Z_R$ such that $b < 1$ and $b_0 \parallel b$ by Theorem 8.13. Since $b_0^\ell = b$ and $b^\ell = \top$, $\mathbf{R}$ is not 1-involutive.

Let $\mathbf{R}$ be a non-linear commutative URL in LW such that $\mathbf{R}$ is not $\top$-unital and $W_R \neq \emptyset$, then there exists an idempotent $b_0 \in Z_R$ such that $b_0 = c^2$ for all $c \in W_R$ by Theorem 8.3(5). Since $b_0^\ell = \bot$ and $\bot^\ell = \top$, such $\mathbf{R}$ is not 1-involutive.

Let $\mathbf{R}$ be a non-linear commutative URL in LW such that $\mathbf{R}$ is not $\top$-unital and $W_R = \emptyset$, then $\mathbf{R} \in \mathsf{L}$ and $U_R \cup \{\bot, \top\}$ is a subalgebra, $Z_R \cup \{\bot, \top\}$ is a 1-free subalgebra and $b \to 1 = b \to \bot$. Thus $\mathbf{R}$ is 1-involutive iff $U_R \cup \{\bot, \top\}$ is 1-involutive and $Z_R \cup \{\bot, \top\}$ is $\bot$-involutive.

Finally, let $\mathbf{R}$ be a non-linear commutative URL in B such that $\mathbf{R}$ is not $\top$-unital, then $U_R \cup \{\bot, \top\}$ is a subalgebra and $Z_R \cup \{\bot, \top\} = \{b, b', \bot, \top\}$ is isomorphic to the 4-element generalized Boolean algebra by Theorem 8.8. Since $b^\ell = b'$ and $b'^\ell = b$, $\mathbf{R}$ is 1-involutive iff $U_R \cup \{\bot, \top\}$ is 1-involutive. □

Chapter 11: Open problems and future work

Finally we end this thesis by listing some open problems.

1. The forgoing chapters classify the class of unilinear residuated lattices into classes B4, Tunital, B, TW, and LW and give the axiomatizations and constructions respectively. We can conclude that the class Tunital is at the center of the study of unilinear residuated lattices. Even though we present 2 constructions of $\top$-unital URLs in Chapter 7, we hope to find more constructions.

2. In the chapter of involutive SRLs, we focus on the class of commutative 1-involutive $\top$-unital URLs whose chain of 1 is an odd Sugihara chain and the chains are of finite orders in the group of chains. We hope to generalize the characterization there. For example, what if the chains form an arbitrary abelian group, like $\mathbb{Z}$? We have see such an example, $M_{\mathbb{Z}}$, in previous chapters and we hope to generalize the characterization.

3. We characterize the URLs axiomatized by the equation $x^2 = x$ in Chapter 9. We would to explore the classes axiomatized by the equations like $x \leq x^2 \vee 1$ or $x^n \leq x^m$ for some $m, n \in \mathbb{Z}^+$.

4. Finally, note that the non-bound elements in a URL are join-irreducible and meet-irreducible. We hope to understand the residuated lattices whose non-bound elements are only join-irreducible (or meet-irreducible). This is motivated by the fact that in the proof of characterization of URLs, we only use the join-irreducibility of non-bound elements in the most cases.

Bibliography

[1] R. Cardona and N. Galatos. The FEP for some varieties of fully distributive knotted residuated lattices. *Algebra Universalis*, 78(3):363–376, 2017.

[2] W. H. Carlisle. Residual finiteness of finitely generated commutative semigroups. *Pacific J. Math.*, 36:99–101, 1971.

[3] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory: hypersequents and hypercompletions. *Ann. Pure Appl. Logic*, 168(3):693–737, 2017.

[4] T. Evans. Some connections between residual finiteness, finite embeddability and the word problem. *J. London Math. Soc. (2)*, 1:399–403, 1969.

[5] W. Fussner and N. Galatos. Conic idempotent residuated lattices. *arXiv preprint arXiv:2208.09724*, 2022.

[6] W. Fussner and G. Metcalfe. Transfer theorems for finitely subdirectly irreducible algebras. *arXiv preprint arXiv:2205.05148*, 2022.

[7] N. Galatos. *Varieties of residuated lattices*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Vanderbilt University.

[8] N. Galatos. Equational bases for joins of residuated-lattice varieties. *Studia Logica*, 76(2):227–240, 2004.

[9] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Trans. Amer. Math. Soc.*, 365(3):1219–1249, 2013.

[10] N. Galatos and P. Jipsen. Residuated lattices of size up to 6, 2017.

[11] N. Galatos and P. Jipsen. The structure of generalized BI-algebras and weakening relation algebras. *Algebra Universalis*, 81(3):Paper No. 35, 35, 2020.

[12] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.

[13] L. Pavel and T. Sund. Monoid extensions admitting cocycles. *Semigroup Forum*, 65(1):1–32, 2002.

[14] B. M. Schein. Homomorphisms and subdirect decompositions of semi-groups. *Pacific J. Math.*, 17:529–547, 1966.

[15] M. R. Vaughan-Lee. Uncountably many varieties of groups. *Bull. London Math. Soc.*, 2:280–286, 1970.