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Combinatorial Problems on the Integers: Colorings, Games, and Permutations

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Abstract

This dissertation consists of several combinatorial problems on the integers. These problems fit inside the areas of extremal combinatorics and enumerative combinatorics.

We first study monochromatic solutions to equations when integers are colored with finitely many colors in Chapter 2. By looking at subsets of $\{1, 2, \ldots, n\}$ whose least common multiple is small, we improved a result of Brown and Rödl on the smallest integer n such that every 2-coloring of $\{1, 2, \ldots, n\}$ has a monochromatic solution to equations with unit fractions. Using a recent result of Boza, Marín, Revuelta, and Sanz, this technique also allows us to show a polynomial upper bound for the same problem, but with three colors.

We then study Maker-Breaker positional games for equations with fractional powers in Chapter 3. In these games, Maker and Breaker take turns to select a previously unclaimed number in $\{1, 2, \ldots, n\}$, Maker wins if they can form a solution to a given equation, and Breaker wins if they can stop Maker. Using combinatorial arguments and results from number theory and arithmetic Ramsey theory, we found exact expressions or strong bounds for the smallest n such that Maker has a winning strategy.

Finally, we study permutations of integers in Chapters 4 to 6. In Chapter 4, we provide an alternative proof of a result by Miner and Pak which says that 123- and 132-avoiding permutations with a fixed leading term are enumerated by the ballot numbers. We then study the number of pattern-avoiding permutations with a fixed prefix of length $t \geq 1$, generalizing the $t = 1$ case. We find exact expressions for single and pairs of patterns of length three as well as the pair 3412 and 3421. These expressions depend on t, the extrema, and the order statistics. In Chapter 5, we define rotations of permutations and study permutations such that they and their rotations avoid certain patterns. We obtain many enumerative results for patterns of length three and several of them are related to existing results on permutations avoiding other patterns. In Chapter 6, we look at subsequences with certain arithemtic properties that exist in all permutations of a given length. For example, we prove that for all positive integers $k \geq 3$ and sufficiently large n, every permutation of $\{1, 2, \ldots, n\}$ has a subsequence $(a_1, a_2, \ldots, a_k)$ such that either $\sum_{i=1}^{k} a_i = 2a_1$ or $\sum_{i=1}^{k} a_i = 2a_k$.

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DECLARATIONS

Several chapters of this dissertation are based on joint work with other mathematicians. These chapters are as follows:

1. Chapter 3 is based on joint work with Paul Horn which is published in *Integers* [48].

2. Chapters 4 and 5 are based on joint works with Ömer Eğecioğlu and Mei Yin.

3. Chapter 6 is based on joint work with Paul Horn.

None of the results herein have appeared in any other dissertation or thesis, and all coauthors have agreed to the inclusion of these joint works in this dissertation.

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NOTATION

Throughout this dissertation, we use $\mathbb{N}$ to denote the set of positive integers $\{1, 2, \ldots\}$. For any $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$. For a set A, $|A|$ is the size/cardinality of the set A and $\mathcal{P}(A)$ is the power set of A.

For functions $f(k)$ and $g(k)$, $f(k) = O(g(k))$ if there exist constants K and C such that $|f(k)| \leq C|g(k)|$ for all $k \geq K$; $f(k) = \Omega(g(k))$ if there exist constants K' and c such that $|f(k)| \geq c|g(k)|$ for all $k \geq K'$; $f(k) = \Theta(g(k))$ if $f(k) = O(g(k))$ and $f(k) = \Omega(g(k))$; and $f(k) = o(g(k))$ if $\lim_{k\to\infty} f(k)/g(k) = 0$.

Chapter 1: Introduction

We study several combinatorial problems on the integers. Due to the arithmetic operations and the total order, integers have special structures which are not present in other combinatorial objects, such as graphs and sets. These special structures allow us to utilize results from number theory, such as the prime number theorem and the linear independence of integers with fractional powers, to solve combinatorial problems on the integers.

While benefiting from results in number theory, combinaotrial results on integers also provide insights/tools for solving problems in number theory. For example, Szemerédi's Theorem, which will be stated in Section 1.1, was used to prove the Green-Tao Theorem [56] which says that the primes contain arbitrarily long arithmetic progressions. Another example is *Euler's sum of powers conjecture*. In 1769, Euler conjectured that for all positive integers $k, \ell \geq 2$, if the equation

$$x_1^\ell + x_2^\ell + \cdots + x_k^\ell = y^\ell \tag{1.1}$$

has a solution consists of positive integers, then $k \geq \ell$. This is true for $k = 3$ by Fermat's Last Theorem [122], but is known to be false for $k = 4$ [40] and $k = 5$ [79]; however, it is unknown whether this is true for any $k \geq 6$. Even though this number-theoretic problem is still unsolved, recently, Chow, Lindqvist, and Prendiville [27] proved a related result in arithmetic Ramsey theory: for all $\ell \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, if $\mathbb{N}$ is finitely colored, then one of them contains a monochromatic solutions to Equation (1.1). Colorings and monochromatic solutions to equations will be defined in Section 1.1.

Combinatorial results on the integers are also related computer science. One of the central problems of computer science is the task of sorting, that is, arranging distinct numbers in increasing or decreasing order. One of the sorting algorithms is *stack-sorting* [18, Chapter 8]. Knuth [76] proved that a permutation is stack-sortable if and only if it avoids the pattern 231. Patterns and pattern-avoiding permutations will be defined in Section 1.3. Due to this connection between stack-sortable permutations and permutation patterns, we now know many results on pattern-avoiding permutations and many new problems are still being studied to this day. There is now a large database on the enumeration of permutations avoiding certain patterns [2].

This pure mathematical exploration on pattern-avoiding permutations has also provided new insights for the sorting problem in computer science. In 1990, West [119] defined t-stack-sortable permutations which are permutations that can be sorted after running the stack-sorting algorithm t times. It turns out that, for $t \geq 2$, t-sortable permutations are also related to permutation patterns. For example, West [119] proved that a permutation is 2-stack-sortable if and only if it does not contain a 2341 pattern, and it does not contain a 3241 pattern, except possibly as part of a 35241 pattern. See also [18, p. 354] for this result.

In the following sections of this chapter, we define the objects we study, survey important results in each area, and discuss how our main results in later chapters fit in these areas.

1.1 Arithmetic Ramsey theory

Let X be a set, $\mathcal{F} \subseteq \mathcal{P}(X)$, and $r \in \mathbb{N}$. An r-coloring of X is a function

$$\Delta : X \to [r],$$

and $F \in \mathcal{F}$ is said to be *monochromatic* for Δ if $\Delta(a) = \Delta(b)$ for all $a, b \in F$. In Ramsey theory, given X, one is interested in certain $\mathcal{F}$ such that every coloring of X contains a monochromatic $F \in \mathcal{F}$. The name Ramsey theory is due to Ramsey's Theorem [101] proved by philosopher, mathematician, and economist Frank Ramsey in 1930.

Theorem 1.1.1 (Ramsey's Theorem [101])**.** *For all $k, r \in \mathbb{N}$ and any r-coloring*

$$\Delta : \begin{bmatrix} \mathbb{N} \\ k \end{bmatrix} \to [r]$$

of the k-element subsets of $\mathbb{N}$, there is always an infinite subset $S \subset \mathbb{N}$ with all its k-element subsets having the same color.

In Theorem 1.1.1, we have $X = \begin{bmatrix} \mathbb{N} \\ k \end{bmatrix}$ and

$$\mathcal{F} = \left\{ \begin{bmatrix} S \\ k \end{bmatrix} : S \subseteq \mathbb{N} \text{ is infinite} \right\}.$$

Even though Ramsey theory was named after Frank Ramsey, Schur in 1916 already proved a Ramsey-type result [108].

Theorem 1.1.2 (Schur's Theorem [108])**.** *Every finite coloring of $\mathbb{N}$ has a monochromatic solution to the equation $x + y = z$.*

In Theorem 1.1.2, $X = \mathbb{N}$ and $\mathcal{F}$ is the set of solutions to $x + y = z$ in $\mathbb{N}$ where x, y, z are not necessarily distinct. As pointed out by Graham and Butler [54, p. 62], Schur needed this theorem to prove a result related to Fermat's Last Theorem [122].

In Schur's theorem, $\mathcal{F}$ contains sets which satisfy certain arithmetic properties. Arithmetic Ramsey theory deals with results of this type. In 1933, Rado [100] generalized Schur's result to systems of homogeneous linear equations. The general result on systems of homogeneous linear equations depends on the so-called "column condition," however, we will focus on the single equation version of Rado's theorem. To state this theorem, we say that an equation e is *partition regular* if every finite coloring of $\mathbb{N}$ has a monochromatic solution to e, where in each solution, the variables are not necessary distinct.

Theorem 1.1.3 (Rado's Single Equation Theorem [54])**.** *Let* $a_1x_1+a_2x_2+\cdots+a_kx_k = 0$ *be an equation where* $x_1, x_2, \ldots, x_k$ *are variables and* $a_1, a_2, \ldots, a_k \in \mathbb{Z}\backslash\{0\}$ *are constants. Then* $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ *is partition regular if and only if for some nonempty* $S \subseteq [k]$, $\sum_{i\in S} a_i = 0$.

By Theorem 1.1.3, for all positive integers $k \geq 2$, every finite coloring of $\mathbb{N}$ has a monochromatic solution to the linear equation

$$\sum_{i=1}^{k} x_i = y. \tag{1.2}$$

We remind the reader that there are $k + 1$ variables $x_1, x_2, \ldots, x_{k-1}, y$ in Equation (1.2).

In Theorems 1.1.2 and 1.1.3, we color all the positive integers. One might wonder, what if we instead color $[n]$ for some $n \in \mathbb{N}$? If n is large enough, do we still obtain monochromatic solutions? The compactness theorem provides an affirmative answer to this question.

Theorem 1.1.4 (Compactness Theorem [54, 80])**.** *Let* $\mathcal{F}$ *be a set of finite subsets of* $\mathbb{N}$ *and let* r *be a positive integer. If every* r*-coloring of* $\mathbb{N}$ *has a monochromatic* $F \in \mathcal{F}$*, then there exists* $N \in \mathbb{N}$ *such that every* r*-coloring of* $[N]$ *has a monochromatic* $F \in \mathcal{F}$.

Due to Theorem 1.1.4, it is natural to determine the smallest integer $n \in \mathbb{N}$ such that every r-coloring of $[n]$ contains a solution to a given equation. Let $R_r(k)$ be the smallest positive integer n such that every r-coloring of $[n]$ has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$, where $x_1, x_2, \ldots, x_k$ are not necessarily distinct. Beutelspacher and Brestovansky [17] proved that $R_2(k) = k^2 + k - 1$ and, recently, Boza, Marín, Revuelta, and Sanz [21] completed the proof that $R_3(k) = k^3 + 2k^2 - 2$.

Let $R(a_1, a_2, \ldots, a_k)$ be the smallest positive integer n such that every 2-coloring of $[n]$ has a monochromatic solution to the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = y$, given that the condition in Rado's theorem is satisfied. Hopkins and Schaal [70], and Guo and Sun [57] prove that

$$R(a_1, a_2, \ldots, a_k) = av^2 + v - a$$

where $a = \min\{a_1, a_2, \ldots, a_k\}$ and $v = a_1 + a_2 + \cdots + a_k$.

In Rado's theorem, we specified that the variables in the equation are not necessarily distinct. We say that an equation is *strongly partition regular* if every finite coloring of $\mathbb{N}$ has a monochromatic solution to this equation when the variables are required to be distinct. It turns out that, for homogeneous linear equation, the same result holds if the variables are required to be distinct.

Theorem 1.1.5 (Hindman and Leader [69])**.** *If a homogeneous linear equation is partition regular, then it is strongly partition regular.*

Another important line of research in arithmetic Ramsey theory is about arithmetic progressions (APs). A set of integers $\{a_1, a_2, \ldots, a_k\}$ is called a k-term AP if there exists $d > 0$ such that $a_i = a_{i-1} + kd$ for all $i = 2, 3, \ldots, k$. In 1937, van der Waerden proved the renowned van der Waerden's theorem.

Theorem 1.1.6 (van der Waerden's Theorem, see [54, 80])**.** *For $k, r \in \mathbb{N}$, there exists an integer $W(k, r)$ so that if $[W(k, r)]$ is colored with r colors then there is a monochromatic k-term AP.*

In 1975, Szemerédi proved a density version of van der Wader's theorem which implies that the monochromatic APs will occur in the "most frequently" occuring color.

Theorem 1.1.7 (Szemerédi's Theorem [110])**.** *If A is a set of positive integers with positive upper density, that is,*

$$\limsup_{N} \frac{|A \cap [N]|}{N} > 0,$$

then A contains arbitrarily long arithmetic progression.

Szemerédi's proof of Theorem 1.1.7 is purely combinatorial which uses the well-known "Szemerédi's Regularity Lemma." Since the publication of Szemerédi's proof, other proofs of Theorem 1.1.7 have also been discovered. For example, ergodic theory by Furstenberg [47] and Fourier analysis by Gowers [53]. Nowadays, ergodic theory and Fourtier analysis have become standard tools in combinatorial problems on the integers [37, 85, 115].

Notice that, for a fixed integer $k \geq 3$, all k-term APs are solutions to the following system of linear equations: $x_1 + x_3 = 2x_2$, $x_2 + x_4 = 2x_3, \ldots, x_{k-2} + x_k = 2x_{k-1}$. Hence, one could say that all the results we have discussed so far are about linear equations. In 1991, Lefmann [81] and Brown and Rödl [24] proved similar results on whether homogeneous equations involving unit fractions are partition regular. Brown and Rödl [24] also proved a quantitative result which says that, for all $k \geq 2$, if $\{1, 2, \ldots, k^2(k^2 - k + 1)(k^2 + k - 1)\}$ is colored with two colors, then there exists a monochromatic solutions to the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = \frac{1}{y}. \tag{1.3}$$

In Chapter 2, we improve Brown and Rödl's quantitative result significantly by showing that if $\{1, 2, \ldots, 6k(k + 1)(k + 2)\}$ is colored with two colors, then there exists a monochromatic solution to Equation (1.3). The key idea in our proof is to find a subset A of $[n]$ such that every 2-coloring of A has a monochromatic solution to the linear equation $x_1 + x_2 + \cdots + x_k = y$ and the least common multiple of A is small. This idea also allows us to handle the case with three colors.

The (systems of) equations studied by Brown, Lefmann, and Rödl are nonlinear, but still homogeneous. Several equations that are nonlinear and nonhomogeneous have also been studied recently. To state these results, we say that an equation is r-regular if every r-coloring of $\mathbb{N}$ has a monochromatic solution to this equation such that not all variables are assigned the same number. We imposed the extra condition here to exclude trivial solutions. Csikvári, Gyarmati, and Sárközy [32] showed that $x + y = z^2$ is not 16-regular. Improving upon this result, Green and Lindqvist [55] showed that $x + y = z^2$ is 2-regular, but not 3-regular. Green and Lindqvist's proof that $x + y = z^2$ is 2-regular uses Fourier analysis. Pach [95] proved a stronger result using purely combinatorial arguments. In [14] and [83], Pach's combinatorial argument are then used to characterize when equations of the form $ax + by = p(z)$ are 2-regular, where $p(z)$ is a polynomial in the variable z. Bergelson [15, p. 53] showed that if $p(t) \in \mathbb{Z}[t]$ and $p(0) = 0$, then $x - y = p(z)$ is regular. Doss, Saracino, and Vestal [34] proved that the smallest n such that every 2-coloring of $[n]$ has a monochromatic solution to $x - y = z^\ell$ is $1 + 2^{\ell+1}$ and Sanders [106] proved that the smallest n such that every r-coloring of $[n]$ has a monochromatic solution to $x - y = z^2$ is greater than $2^{2^{r-1}}$ but less than or equal to $2^{2^{2^{O(r)}}}$.

1.2 Positional games

Positional games are combinatorial games where players (usually two) take turns to select elements in a given set X (usually finite). In these games, the players focus on a set $\mathcal{F} \subseteq \mathcal{P}(X)$, which is the set of *winning sets*. One of the first results on positional games was in the paper by Hales and Jewett [58] in 1963. In that paper, Hales and Jewett generalized van der Waerden's Theorem to higher dimensions and then considered the corresponding positional games. This is related the higher-dimensional Tic-Tac-Toe. For more details related to Hales and Jewett's work, see the books by Beck [13] and by Hefetz, Krivelevich, Stojaković, and Szabó [61].

Two types of two-player positional games have been extensively studied in the literature. The first one deals with *strong games*, where First Player and Second Player take turns to select elements in X and the first player to complete some $F \in \mathcal{F}$ wins the game. In strong games, there are three possible outcomes:

(1) First Player has a winning strategy;

(2) Second Player has a winning strategy;

(3) both players have drawing strategies.

A *strategy* is a set of instructions which tells the player what to do each round given what had been previously played by both players. It turns out that, in a strong game, First Player can guarantee at least a draw. This is partially due to *strategy stealing*, which says that if Second Player had a winning strategy then, after the first move, First Player can pretend to be Second Player and steal their winning strategy. However, strong games are in general hard to analyze and if $\mathcal{F} \subset \mathcal{F}'$, having a winning strategy for $\mathcal{F}$ does not gurantee that a player has a winning strategy for $\mathcal{F}'$ [61, p. 11].

Due to these difficulties, most work in positional games have focused on weak games. Weak games are also called Maker-Breaker games. In Maker-Breaker games, Maker's goal is to claim at least one set $F \in \mathcal{F}$ and Breaker's goal is to stop Maker. This simplification makes the games much easier to analyze and a lot of other tools in combinatorics are able to be utilized. One of the first results on general Maker-Breaker games is the Erdős-Selfridge criterion [43] which provides a sufficient condition for Breaker to win.

Theorem 1.2.1 (Erdős-Selfridge criterion [43])**.** *Let X be a finite set and $\mathcal{F} \subset \mathcal{P}(X)$. If*

$$\sum_{F \in \mathcal{F}} 2^{-|F|} < \frac{1}{2},$$

then Breaker has a winning strategy for the Maker-Breaker game played on X with winning sets $\mathcal{F}$.

Chvátal and Erdős [28] studied biased Maker-Breaker games on graphs. In biased Maker-Breaker games, Maker selects p elements and Breaker selects q elements each turn. We call them (p, q)-Maker-Breaker games. Beck [11] generalized Erdős-Selfridge criterion to (p, q)-Maker-Breaker games and also proved a sufficient condition for Maker to win.

Theorem 1.2.2 ([11]). *Let X be a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$. If*

$$\sum_{F \in \mathcal{F}} (1+q)^{-|F|/p} < \frac{1}{1+q},$$

then Breaker has a winning strategy for the (p, q)-Maker-Breaker game played on X with winning sets $\mathcal{F}$.

Theorem 1.2.3 ([11]). *Let X be a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$. If*

$$\sum_{F \in \mathcal{F}} \left(1 + \frac{q}{p}\right)^{-|F|} > p^2 q^2 (p+q)^{-2} d_2(\mathcal{F}) |\mathcal{F}|,$$

where $d_2(\mathcal{F}) = \max\{|\{F \in \mathcal{F} : \{u, v\} \subseteq F\}| : u, v \in X, u \neq v\}$, then Maker has a winning strategy for the (p, q)-Maker-Breaker game played on X with winning sets $\mathcal{F}$.

In addition to the general criteria for Maker and Breaker to win, a fundamental question in biased Maker-Breaker games is the so-called *threshold bias*.

Definition 1.2.4. Let X be a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $\mathcal{F} \neq \emptyset$ and $\min\{|F| : F \in \mathcal{F}\} \geq 2$. The smallest positive integer q such that Breaker wins the $(1, q)$-Maker-Breaker game on X with winning set $\mathcal{F}$ is called the threshold bias of the game.

Now we described some results on a family of well-studied biased Maker-Breaker games on graphs, called triangle games.

Example 1.2.5. The triangle game. The board of the game is the edge set of the complete graph K_k on k vertices, and the winning sets are all copies of K_3 of K_k. Maker selects one edge each round and Breaker selects q edges each round. Chvátal and Erdős [28] showed that Breaker has a winning strategy if $q \geq 2\sqrt{k}$. Balogh and Samotij [9] improved Chvátal and Erdős's result and showed that Breaker has a winning strategy if $q \geq (2 - 1/24)\sqrt{k}$. Recently, Glazik and Srivastav [52] improved the result even further and showed that Breaker has a winning strategy if $q \geq \sqrt{(8/3 + o(1))k}$.

Like Example 1.2.5, in most Maker-Breaker games studied in this area, the board X is either the edges or the vertices of a given graph. In 1981, Beck [10] introduced Maker-Breaker games where the board X is $[n]$ and the winning sets $\mathcal{F}$ are k-term arithmetic progressions for a fixed k. These games were motivated by a result of van der Waerden's Theorem as stated in Section 1.1. By the Compactness Theorem stated in Section 1.1 and strategy stealing [13, Section 5] (see also [61, Chapter 1]), Maker can win the van der Waerden games if n is large enough. Therefore, one would naturally want to find the smallest n such that Maker can win the van der Waerden games. Beck [10] proved that, for any given k, the smallest n such that Maker has a winning strategy for the van der Waerden games is between $2^{k-7k^{7/8}}$ and $k^3 2^{k-4}$.

Recently, Kusch, Rué, Spiegel, and Szabó [78] studied a generalization of van der Waerden games called Rado games. In Rado games, $\mathcal{F}$ is the set of solutions to a system of linear equations. By Rado's theorem [100], if n is large enough, then Maker is guaranteed to win the Rado games if the system of linear equations satisfies the so-called column condition [54, Chapter 10]. Kusch, Rué, Spiegel, and Szabó studied the biased Rado games and derived asymptotic threshold bias for Breaker to win. Their result on 3-term arithmetic progressions was later improved by Cao et al. [26]. Hancock [59] replaced $[n]$ with a random subset of $[n]$ where each number is included with probability p and proved asymptotic thresholds of p for Breaker or/and Maker to win. However, unlike the van der Waerden games, the smallest n such that Maker wins for the unbiased and deterministic Rado games are left unstudied.

In Chapter 3, we study unbiased Maker-Breaker Rado games for the equation

$$x_1^{1/\ell} + x_2^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$$

where k and ℓ are integers with $k \geq 2$ and $\ell \neq 0$. Let $f(k, \ell)$ be the smallest positive integer n such that Maker has a winning strategy when $x_1, x_2, \ldots, x_k$ are not necessarily distinct, and let $f^*(k, \ell)$ be the smallest positive integer n such that Maker has a winning strategy when $x_1, x_2, \ldots, x_k$ are distinct. When $\ell \geq 1$, we prove that, for all $k \geq 2$, $f(k, \ell) = (k+2)^\ell$ and $f^*(k, \ell) = (k^2+3)^\ell$; when $\ell \leq -1$, we prove that $f(k, \ell) = [k + \Theta(1)]^{-\ell}$ and $f^*(k, \ell) = [\exp(O(k \log k))]^{-\ell}$. Our proofs use combinatorial arguments and a result of Besicovitch [16] on the linear independence of integers with fractional powers. We also prove a game variant of a theorem of Brown and Rödl [24] which is used to handle $f^*(k, \ell)$ with $\ell \leq -1$.

In addition to Maker-Breaker games, other positional games have also been studied recently. Here we briefly describe two of them. In Picker-Chooser games [12] on the board X with winning sets $\mathcal{F}$, Picker selects $p + q$ unselected elements from the board X each round. After Picker has picked $p + q$ elements, Chooser selects q elements out of these $p + q$ elements and the rest p elements belong to Picker. Picker wins if they can complete a set in $\mathcal{F}$ and Chooser wins otherwise. In Avoider-Enforcer games [62, 63], Avoider wins if they can avoid completing a winning set in $\mathcal{F}$ and Enforcer wins if Avoider fails to do so. This is the *misére* version of the Maker-Breaker games. Misére games are common in combinatorial games which do not belong to positional games [109].

1.3 Pattern-avoiding permutations

Let $A \subseteq \mathbb{N}$ be a finite set. A permutation σ on A is a sequence $(\sigma(1), \sigma(2), \ldots, \sigma(|A|))$ of length $|A|$ consisting of distinct numbers in A. When $A \subseteq \{1, 2, \ldots, 9\}$ or when there is no confusion, we simply write a permutation/sequence without commas or parentheses in single-line notation. The first several terms of a sequence is called the leading terms of the sequence. We use S_A to denote the set of permutations on A. When $A = [n] := \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, we write S_n for $S_{[n]}$.

For any $\tau \in S_n$ and $\sigma \in S_k$, if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that for all $1 \leq a < b \leq k$, $\tau(i_a) < \tau(i_b)$ if and only if $\sigma(a) < \sigma(b)$, then we say that τ contains σ as a pattern and that $(\tau(i_1), \tau(i_2), \ldots, \tau(i_k))$ is a σ pattern. A permutation τ is said to avoid σ if τ does not contain σ as a pattern. For example, the permutation $\tau = 12453 \in S_5$ contains the pattern 132 because $\tau(1)\tau(3)\tau(5) = 143$ is a 132 pattern; however, τ avoids the pattern 321. For any $m, n, k \in \mathbb{N}$ and $\sigma_1, \sigma_2, \ldots, \sigma_m \in S_k$, we use $S_n(\sigma_1, \sigma_2, \ldots, \sigma_m)$ to denote the set of permutations on $[n]$ which avoid all of the patterns $\sigma_1, \sigma_2, \ldots, \sigma_m$.

The interest in the study of pattern avoidance can be traced back to stack-sortable permutations in computer science [74, Section 2.1]. One of the earliest results is the enumeration of permutations avoiding $\sigma \in S_3$, i.e., patterns of length three. D. Knuth proved that the number of permutations in S_n avoiding any given pattern of length three is counted by the Catalan numbers C_n (see also [18, Theorem 4.7]).

Theorem 1.3.1. [76, p. 238] *For all $n \geq 1$ and $\sigma \in S_3$, we have*

$$|S_n(\sigma)| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is well known that the Catalan numbers count many combinatorial objects and they have the following recursive relation (see[36, Section 3.2] and [114, Section 1.2]): for

$n > 0$,

$$C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i},$$

and $C_0 = 1$.

Using the recurrence relation for Catalan numbers, one can see that $|S_n(231)| = C_n$ for all n. Let $\tau \in S_n(231)$ such that $\tau(i) = n$. Then for all j, k with $j < i < k$, $\tau(j) < \tau(k)$. Otherwise $\tau(j)\tau(i)\tau(k)$ would be a 231 pattern. So $\{\tau(1), \tau(2), \ldots, \tau(i-1)\} = \{1, 2, \ldots, i-1\}$ and $\{\tau(i+1), \tau(i+2), \ldots, \tau(n)\} = \{i, i+1, \ldots, n-1\}$. Since $(\tau(1), \tau(2), \ldots, \tau(i-1))$ and $(\tau(i+1), \tau(i+2), \ldots, \tau(n))$ both avoid 231, we have $|S_{i-1}(231)|$ possibilities for $(\tau(1), \tau(2), \ldots, \tau(i-1))$ and $|S_{n-i}(231)|$ possibilities for $(\tau(i+1), \tau(i+2), \ldots, \tau(n))$. It is easy to check all these possibilities can guarantee that τ avoids 231. Hence, we have

$$|S_n(231)| = \sum_{i=1}^{n} |S_{i-1}(231)||S_{n-i}(231)|.$$

This is exactly the recurrence relation for the Catalan numbers. Hence $|S_n(231)| = C_n$. Now define the complement of any $\tau \in S_n$ as $\tau^c = (n+1-\tau(1), n+1-\tau(2), \ldots, n+1-\tau(n))$ and the reverse of any $\tau \in S_n$ as $\tau^r = (\tau(n), \tau(n-1), \ldots, \tau(1))$. One can check that τ^c avoids σ if and only if τ avoids σ, and that τ^r avoids σ if and only if τ avoids σ. Since $(231)^c = 213$, $(231)^r = 132$, and $(132)^c = 312$, we have

$$|S_n(312)| = |S_n(132)| = |S_n(213)| = |S_n(231)| = C_n.$$

Since $(321)^c = 123$, we also have $|S_n(123)| = |S_n(321)|$. Hence we only need to prove that $|S_n(123)| = |S_n(132)|$ in order to establish Theorem 1.3.1. As pointed out by Bóna [18, p. 151], this is the first nontrivial result for pattern-avoiding permutations. Several bijections between the sets $S_n(123)$ and $S_n(132)$ have been discoverd [39, 102, 113, 120].

Exact expressions for some patterns of length four are also known. For example, Gessel [51, p. 281] prove that

$$|S_n(1234)| = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}.$$

However, in general, patterns of length greater than three are much more complicated and the results are often proved using algebraic methods such as generating functions.

Instead of finding exact expressions for $S_n(\sigma)$, Wilf equivalence classes are also an important topic in pattern avoidance for permutations. Two permutation patterns σ and σ' are said to be *Wilf equivalent*, denoted $\sigma \sim \sigma'$, if $|S_n(\sigma)| = |S_n(\sigma')|$ for all $n \in \mathbb{N}$. By Theorem 1.3.1, all permutation patterns of length three are Wilf equivalent: $123 \sim 132 \sim 213 \sim 231 \sim 312 \sim 321$. In other words, there is only one Wilf-equivalence class for permutation patterns of length three. For patterns of length four, it is known that there are three Wilf-equivalence classes [18, p. 158].

Another well-studied question in pattern-avoiding permutations is the *Stanley-Wilf limit*. In the 1980s, Richard P. Stanley and Herbert Wilf independently conjectured the growth of the number of permutations avoiding a given pattern. It has two different forms and Arratia [6] proved that these two are equivalent.

Conjecture 1.3.2 (Stanley-Wilf Conjecture Version I)**.** Let σ be any pattern. Then there exists a constant c_σ so that for all positive integers n,

$$|S_n(\sigma)| \leq c_\sigma^n.$$

Conjecture 1.3.3 (Stanley-Wilf Conjecture Version II)**.** Let σ be any pattern. Then the following limit exists:

$$L(\sigma) = \lim_{n\to\infty} \sqrt[n]{|S_n(\sigma)|}$$

The Stanley-Wilf conjecture was proved by Marcus and Tardos [84]. Due to this, $L(\sigma)$ is called the Stanley-Wilf limit. Claesson, Jelínek, and Steingrímsson [30] conjectuerd that for any permutation σ of length k, $|S_n(\sigma)| \leq (2k)^{2n}$. However, Fox [46] disproved this by proving that there exists an infinite sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ of patterns so that σ_k is a pattern of length k and $L(\sigma_k) = 2^{\Omega(k^{1/4})}$.

Pattern avoidance has also been studied for other combinatorial objects, such as set partitions [104, 71], words and parking functions [1, 72], and rooted labeled forests [3, 49, 97].

In Chapter 4, we study pattern-avoiding permutations with fixed leading terms. When only one leading term is fixed, we used direct counting arguments to provide a different proof of a result of Miner and Pak [88] which says that 123- and 132-avoiding permutations with one leading term fixed are enumerated by the ballot numbers. When t leading terms (we call then a prefix of length t) are fixed, we obtained many enumerative results for single patterns of length three, pairs of patterns of length three, and the pair 3412 and 3421. These results depend on the extrema, the order statistics, and the number of fixed leading terms. For example, the number of 231-avoding permutations in S_n with fixed leading terms $(c_1, c_2, \ldots, c_t)$ is either 0 or a product of Catalan numbers. We also define r-Wilf equivalence for permutations with a single fixed leading term r, and classify the r-Wilf-equivalence classes for both classical and vincular patterns of length three.

In Chapter 5, we introduce the concept of the rotations of permutations and study permutations such that they and their rotations avoid certain patterns. The rotations of a permutation $p = (p(1), p(2), \ldots, p(n)) \in S_n$ are of the form

$$(p(k), p(k+1), \ldots, p(n), p(1), p(2), \ldots, p(k-1))$$

where $k \in [n]$. We resolved the enumerative question on the number of permutations such that they and their rotations all avoid a given pattern of length three. One key feature of our results on this problem is that the enumeration depends on whether the pattern is monotone. We also enumerate permutations $p \in S_n$ such that p and $(p(2), p(3), \ldots, p(n), p(1))$ avoid different patterns. In addition to being interesting objects themselves, our enumerative results for permutation rotations are also related to other results in the pattern avoidance. For example, the number of permutations $p \in S_n$ such that p avoids 123 and $(p(2), p(3), \ldots, p(n), p(1))$ avoids 231 is the same as the number of permutations avoiding the patterns 321, 2143, and 3142.

1.4 Extremal and arithmetic structures in permutations

In Sections 1.1 and 1.2, we reviewed some combinatorial problems on the integers where extremal phenomena and arithmetic are the center of the discussion. In contrast, all of the results on pattern-avoiding permutations reviewed in Section 1.3 involve the order properties of the permutation, not the arithmetic properties. It is still natural, however, to consider extremal problems and arithmetic properties in permutations.

There are several studies on extremal problems in permutations. Wilf considered the maximum number of patterns, regardless of length, which can be contained by a permutation of fixed length n. Wilf's results were future improved by Coleman [31], Eriksson et al. [45], and Miller [87]. Hegarty [65] studied the largest n such that there exists $p \in S_n$ such that all patterns of length k contained in p are distinct. Engen and Vatter [41] surveyed different versions of the question "what is the shortest object containing all permutations of a given length?"

Arithmetic structures in permutations have also been studied. Hegarty, Martinsson, Sawhney, and Stoner [64, 66, 107] studied whether a given abelian group G has the following property: there exists a permutation τ of G such that there does not exist a triple (a, b, c) of elements G, not all equal, with $c - b = b - a$ and $\tau(c) - \tau(b) = \tau(b) - \tau(a)$.

There are also many results on Costas arrays which are permutations $\tau \in S_n$ such that if a, b, c, d are distinct integers in $[n]$ and $a - b = c - d$ then $\tau(a) - \tau(b) \neq \tau(c) - \tau(d)$. For results on Costas arrays, see [118] and references therein. Very recently, Pomerance, Sah, and Sawhney [98, 105] studied coprime permutations which are permutations $\tau \in S_n$ such that $p(m)$ and m are coprime for all $m \in [n]$.

In Chapter 6, we added more results on permutations with arithmetic properties. The main question we study is whether there exists $n \in \mathbb{N}$ such that for all permutation $p \in S_n$, p has a subsequence whose sum satisfies certain properties. More specifically, we say that a sequence of positive integers $(a_1, a_2, \ldots, a_k)$ is ℓ-additive if $\sum_{i=1}^{k} a_i = \ell a_1$ or $\sum_{i=1}^{k} a_i = \ell a_k$. We prove that for all positive $k \geq 3$ and sufficiently large n, every $p \in S_n$ has a 2-additive subsequence of length k. By routine calculation, we also show that for all $n \geq 48$, every $p \in S_n$ has a montone subsequence of length three which is 2-additive. We also conjecture that, for all $k \geq 3$ with $k \neq \ell$ and sufficiently large n, every $p \in S_n$ has an ℓ-additive subsequence of length k. Some related extremal problems are also studied.

Chapter 2: Rado numbers for equations with unit fractions

2.1 Introduction

In 1991, Brown and Rödl [24], and Lefmann [81], extended Rado's theorem (Theorem 1.1.3) to some nonlinear homogeneous equations. One of their results is that for all positive integers $k \geq 2$, every finite coloring of $\mathbb{N}$ has a monochromatic solution to the nonlinear equation

$$\sum_{i=1}^{k} \frac{1}{x_i} = \frac{1}{y}. \tag{2.1}$$

Let $f_r(k)$ be the smallest positive integer n such that every r-coloring of $\{1, 2, \ldots, n\}$ has a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$, where $x_1, x_2, \ldots, x_k$ are not necessarily distinct. Tejaswi and Thangdurai [116] proved some recursive lower bounds for $f_r(2)$ and that, for $r \geq 3$, $f_r(2) \geq 2^{r-3} \cdot 192$. Recently, Myers and Parrish [90] computed that $f_2(2) = 60$, $f_2(3) = 40$, $f_2(4) = 48$, $f_2(5) = 39$, $f_3(2) = 3276$, and $f_4(2) > 87,000$. For general k, the only known result is the upper bound $f_2(k) = O(k^6)$ proved by Brown and Rödl [24] in 1991.

Theorem 2.1.1 (Brown and Rödl [24])**.** *For all positive integers $k \geq 2$,*

$$f_2(k) \leq k^2(k^2 - k + 1)(k^2 + k - 1).$$

In this chapter, we make significant improvement on Theorem 2.1.1 by showing that $f_2(k) = O(k^3)$.

Theorem 2.1.2. *For all positive integers $k \geq 2$,*

$$f_2(k) \leq 6k(k+1)(k+2).$$

The constant factor 6 in the above expression can be reduced to 2 when $k \geq 4$ is even or not divisible by 3. Our proof of Theorem 2.1.2 uses a variant of a theorem by Brown and Rödl [24]. To illustrate our idea, we first state a quantitative version of their theorem.

Theorem 2.1.3 (Brown and Rödl [24])**.** *Let $r, k, T \in \mathbb{N}$ and $G(x_1, x_2, \ldots, x_k, y) = 0$ a system of homogeneous equations such that every r-coloring of $\{1, 2, \ldots, T\}$ has a monochromatic solution to $G(x_1, x_2, \ldots, x_k, y) = 0$. Let S be the least common multiple of $\{1, 2, \ldots, T\}$. Then every r-coloring of $\{1, 2, \ldots, S\}$ has a monochromatic solution to the system of equations $G(1/x_1, 1/x_2, \ldots, 1/x_k, 1/y) = 0$.*

By Theorem 2.1.3, since $R_2(k) = k^2 + k - 1$, we have $f_2(k) \leq \text{lcm}\{1, 2, \ldots, k^2 + k - 1\}$. It is well known that $\text{lcm}\{1, 2, \ldots, k^2 + k - 1\} = \exp((1 + o(1))k^2)$ (see, for example, [91, Chapter 8]). So we have $f_2(k) \leq \exp((1 + o(1))k^2)$ which does not help us improve the upper bound for $f_2(k)$ in Theorem 2.1.1. Our key observation is that the discrete interval $\{1, 2, \ldots, T\}$ in Theorem 2.1.3 can be replaced with a finite subset of $\mathbb{N}$ whose least common multiple is smaller. Hence, to obtain a better upper bound for $f_2(k)$, it suffices to find a finite set $A \subseteq \mathbb{N}$ such that (1) every 2-coloring of A has a monochromatic solution to the linear equation $x_1 + x_2 + \cdots + x_k = y$ and (2) the least common multiple of the integers in A is small.

This chapter is organized as follows. In Section 2.2, we will first show that $f_2(k) = O(k^5)$ in two different ways to illustrate our method. In Section 2.3, we use this method to prove Theorem 2.1.2. A generalization of Theorem 2.1.2 is shown in Section 2.4. Finally, we prove a polynomial upper bound for $f_3(k)$ and a lower bound for $f_r(k)$ in Section 2.5.

2.2 A weaker upper bound for $f_2(k)$

Theorem 2.2.1. *For all positive integers $k \geq 2$,*

$$f_2(k) \leq k^2(k+1)(k^2+k-1).$$

Remark 2.2.2. Theorem 2.2.1 is already an improvement on Theorem 2.1.1. Since $f_2(2) = 60$, the inequalities in Theorems 2.1.1 and 2.2.1 both become equality for $k = 2$.

Our first proof of Theorem 2.2.1 is a direct proof. This is similar to the proof of Theorem 2.1.1 by Brown and Rödl [24].

First Proof of 2.2.1. Let $k \geq 2$ be an integer. It suffices to show that every 2-coloring of $\{1, 2, \ldots, k^2(k+1)(k^2+k-1)\}$ has a monochromatic solution to $1/x_1+1/x_2+\cdots+1/x_k = 1/y$. Suppose, for a contradiction, that there exists a 2-coloring

$$\Delta : \{1, 2, \ldots, k^2(k+1)(k^2+k-1)\} \to \{R, B\}$$

without a monochromatic solution to $1/x_1+1/x_2+\cdots+1/x_k = 1/y$. WLOG, we assume that $\Delta(1) = R$.

Claim 1: If $a \in \{1, 2, \ldots, k^2(k+1)(k^2+k-1)\}$ such that $k^2 a \in \{1, 2, \ldots, k^2(k+1)(k^2+k-1)\}$, then $\Delta(a) = \Delta(k^2 a) \neq \Delta(ka)$. This is because

$$\underbrace{\frac{1}{ka} + \cdots + \frac{1}{ka}}_{k} = k \cdot \frac{1}{ka} = \frac{1}{a},$$

$$\underbrace{\frac{1}{k^2 a} + \cdots + \frac{1}{k^2 a}}_{k} = k \cdot \frac{1}{k^2 a} = \frac{1}{ka},$$

and Δ does not have a monochromatic solution to $1/x_1 + \cdots + 1/x_k = 1/y$.

By Claim 1, we have $\Delta(k^2) = \Delta(1) = R$, $\Delta(k) = B$, and $\Delta(k+1) = \Delta(k^2(k+1)) \neq \Delta(k(k+1))$. Since

$$\underbrace{\frac{1}{k^2} + \cdots + \frac{1}{k^2}}_{k-1} + \frac{1}{k^2(k+1)} = (k-1) \cdot \frac{1}{k^2} + \frac{1}{k^2(k+1)} = \frac{1}{k+1}$$

and $\Delta(k^2) = R$, we have $\Delta(k+1) = \Delta(k^2(k+1)) = B$ and hence $\Delta(k(k+1)) = R$.

By Claim 1, we have $\Delta(k^2+k-1) = \Delta(k^2(k^2+k-1)) \neq \Delta(k(k^2+k-1))$. Since

$$\begin{aligned}
&\frac{1}{k^2+k-1} + \underbrace{\frac{1}{k^2(k^2+k-1)} + \cdots + \frac{1}{k^2(k^2+k-1)}}_{k-1} \\
=&\frac{1}{k^2+k-1} + (k-1) \cdot \frac{1}{k^2(k^2+k-1)} = \frac{1}{k^2}
\end{aligned}$$

and $\Delta(k^2) = R$, we have $\Delta(k^2+k-1) = \Delta(k^2(k^2+k-1)) = B$ and hence $\Delta(k(k^2+k-1)) = R$.

By Claim 1 again, we have $\Delta((k+1)(k^2+k-1)) = \Delta(k^2(k+1)(k^2+k-1)) \neq \Delta(k(k+1)(k^2+k-1))$. Since

$$\begin{aligned}
&\frac{1}{(k+1)(k^2+k-1)} + \underbrace{\frac{1}{k^2(k+1)(k^2+k-1)} + \cdots + \frac{1}{k^2(k+1)(k^2+k-1)}}_{k-1} \\
&= \frac{1}{(k+1)(k^2+k-1)} + (k-1) \cdot \frac{1}{k^2(k+1)(k^2+k-1)} = \frac{1}{k^2(k+1)}
\end{aligned} \tag{2.2}$$

and $\Delta(k^2(k+1)) = B$, we have $\Delta((k+1)(k^2+k-1)) = \Delta(k^2(k+1)(k^2+k-1)) = R$.

Now we have $\Delta((k+1)(k^2+k-1)) = \Delta(k(k^2+k-1)) = \Delta(k(k+1)) = R$. Since

$$\begin{aligned}
&\underbrace{\frac{1}{k(k^2+k-1)} + \cdots + \frac{1}{k(k^2+k-1)}}_{k-1} + \frac{1}{(k+1)(k^2+k-1)} \\
&= (k-1) \cdot \frac{1}{k(k^2+k-1)} + \frac{1}{(k+1)(k^2+k-1)} = \frac{1}{k(k+1)},
\end{aligned}$$

we have a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$ which is a contradiction. $\square$

Now we state a variant of Theorem 2.1.3 and then use it to give a second proof for Theorem 2.2.1.

Theorem 2.2.3. *Let $r, k \geq 2$ be integers, A a finite subset of $\mathbb{N}$, L the least common multiple of the integers in A, and $G(x_1, x_2, \ldots, x_k, y) = 0$ a system of homogeneous equations. If every r-coloring of A has a monochromatic solution to $G(x_1, x_2, \ldots, x_k, y) = 0$, then every r-coloring of $\{1, 2, \ldots, L\}$ has a monochromatic solution to the system of equations*

$$G(1/x_1, 1/x_2, \ldots, 1/x_k, 1/y) = 0.$$

Proof. Let $r, k \geq 2$ be integers, A a finite subset of $\mathbb{N}$, L the least common multiple of the integers in A, and $G(x_1, x_2, \ldots, x_k, y) = 0$ a system of homogeneous equations. Suppose that every r-coloring of A has a monochromatic solution to $G(x_1, x_2, \ldots, x_k, y) = 0$. Let

$$\Delta : \{1, 2, \ldots, L\} \to [r]$$

be an r-coloring. We define

$$\overline{\Delta} : A \to [r]$$

where $\overline{\Delta}(x) = \Delta(L/x)$ for all $x \in A$. Since L is the least common multiple of the integers in A, $\overline{\Delta}$ is a well-defined function and hence an r-coloring of A.

By assumption, there exist $a_1, a_2, \ldots, a_k, b \in A$ such that $\overline{\Delta}(a_1) = \overline{\Delta}(a_2) = \cdots = \overline{\Delta}(a_k) = \overline{\Delta}(b)$ and $G(a_1, a_2, \ldots, a_k, b) = 0$. Hence, by our construction, $\Delta(L/a_1) = \Delta(L/a_2) = \cdots = \Delta(L/a_k) = \Delta(L/b)$. Since $G(x_1, x_2, \ldots, x_k, y) = 0$ is homogeneous, we have $G(a_1/L, a_2/L, \ldots, a_k/L, b/L) = 0$. This can be rewritten as

$$G(1/(L/a_1), 1/(L/a_2), \ldots, 1/(L/a_k), 1/(L/b)) = 0.$$

So $(x_1, x_2, \ldots, x_k, y) = (L/a_1, L/a_2, \ldots, L/a_k, L/b)$ is a monochromatic solution to the equation $G(1/x_1, 1/x_2, \ldots, 1/x_k, 1/y) = 0$. □

In order to prove Theorem 2.2.1 using Theorem 2.2.3, we first find a finite set $A \subseteq \mathbb{N}$ such that every 2-coloring of A has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$ and the least common multiple of the integers in A is at most $k^2(k+1)(k^2+k-1)$.

Lemma 2.2.4. *For all positive integers $k \geq 2$, every 2-coloring of*

$$\{1, k, k+1, k^2, k^2+k-1\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$.

Proof. Let $k \geq 2$ be a positive integer. Suppose, for a contradiction, that

$$\Delta : \{1, k, k+1, k^2, k^2+k-1\} \to \{R, B\}$$

is a 2-coloring without a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. WLOG, we assume that $\Delta(1) = R$. Since

$$\underbrace{1 + \cdots + 1}_{k} = k \cdot 1 = k,$$

we have $\Delta(k) = B$. Since

$$\underbrace{k + \cdots + k}_{k} = k \cdot k = k^2,$$

we have $\Delta(k^2) = R$. Since

$$\underbrace{1 + \cdots + 1}_{k-1} + k^2 = (k-1) \cdot 1 + k^2 = k^2 + k - 1,$$

we have $\Delta(k^2 + k - 1) = B$. Since

$$\underbrace{(k+1) + \cdots + (k+1)}_{k-1} + 1 = (k-1) \cdot (k+1) + 1 = k^2,$$

we have $\Delta(k+1) = B$. Now we have $\Delta(k) = \Delta(k+1) = \Delta(k^2 + k - 1) = B$. Since

$$\underbrace{(k+1) + \cdots + (k+1)}_{k-1} + k = (k-1) \cdot (k+1) + k = k^2 + k - 1, \qquad (2.3)$$

we have a monochromatic solution which is a contradiction. □

Remark 2.2.5. The largest integer in $\{1, k, k+1, k^2, k^2 + k - 1\}$ is equal to $R_2(k)$. As defined in Section 1.1, $R_2(k)$ is the smallest integers n such that every 2-coloring of $[n]$ contains a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$.

Second Proof of Theorem 2.2.1. Let $k \geq 2$ be a positive integer. By Lemma 2.2.4, every 2-coloring of $\{1, k, k+1, k^2, k^2+k-1\}$ has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. The least common multiple of 1, k, $k+1$, k^2, and $k^2 + k - 1$ is at most $k^2(k+1)(k^2 + k - 1)$. So by Theorem 2.2.3, every 2-coloring of $\{1, 2, \ldots, k^2(k+1)(k^2 + k - 1)\}$ has a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. Hence $f_2(k) \leq k^2(k+1)(k^2 + k - 1)$. □

Remark 2.2.6. The identities used in the two proofs for Theorem 2.1 are related. For example, we used Equation (2.2) in the first proof and identity Equation (2.3) in the second proof. Equation (2.3) can be easily obtained from Equation (2.2). While the first proof works with the nonlinear equation $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$ directly, the second proof uses Theorem 2.2.3 and hence transforms the problem for a nonlinear equation to a problem for a linear equation.

2.3 Proof of Theorem 2.1.2

We start by finding a finite subset A of $\mathbb{N}$ such that every 2-coloring of A has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$ and the least common multiple of the integers in A is smaller than $k^2(k+1)(k^2+k-1)$. In order to achieve this goal, unlike Lemma 2.3, some elements of A are larger than $R_2(k) = k^2 + k - 1$.

Lemma 2.3.1. *For all integers $k \geq 4$, every 2-coloring of*

$$\{1, 2, 3, k+1, k+2, 2(k+1), 2(k+2), 3k, k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$.

Proof. Let $k \geq 4$ be an integer and write $\mathcal{A} = \{1, 2, 3, k+1, k+2, 2(k+1), 2(k+2), 3k, k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$. Suppose, for a contradiction, that

$$\Delta : \mathcal{A} \to \{R, B\}$$

is a two-coloring without a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. WLOG, we assume that $\Delta(1) = R$. We have three cases depending on $\Delta(2)$ and $\Delta(3)$.

Case 1: $\Delta(2) = R$. Since

$$\underbrace{1 + \cdots + 1}_{k-1} + 2 = (k-1) \cdot 1 + 2 = k+1$$

and

$$\underbrace{1 + \cdots + 1}_{k-2} + 2 + 2 = (k-2) \cdot 1 + 2 + 2 = k+2,$$

we have $\Delta(k+1) = \Delta(k+2) = B$. Since

$$\underbrace{(k+1) + \cdots + (k+1)}_{k} = k \cdot (k+1) = k(k+1)$$

and

$$\underbrace{(k+2) + \cdots + (k+2)}_{k} = k \cdot (k+2) = k(k+2),$$

we have $\Delta(k(k+1)) = \Delta(k(k+2)) = R$. Now we have

$$\Delta(1) = \Delta(2) = \Delta(k(k+1)) = \Delta(k(k+2)) = R.$$

Since

$$k(k+1) + \underbrace{1 + \cdots + 1}_{k-2} + 2 = k(k+1) + (k-2) \cdot 1 + 2 = k(k+2),$$

we have a monochromatic solution which is a contradiction.

Case 2: $\Delta(2) = B$ and $\Delta(3) = R$. Since

$$\underbrace{3 + \cdots + 3}_{k} = k \cdot 3 = 3k$$

and

$$\underbrace{1 + \cdots + 1}_{k-1} + 3 = (k-1) \cdot 1 + 3 = k+2,$$

we have $\Delta(3k) = \Delta(k+2) = B = \Delta(2)$. Now since

$$(k+2) + \underbrace{2 + \cdots + 2}_{k-1} = (k+2) + (k-1) \cdot 2 = 3k,$$

we have a monochromatic solution which is a contradiction.

Case 3: $\Delta(2) = \Delta(3) = B$. Since

$$3 + 3 + \underbrace{2 + \cdots + 2}_{k-2} = 3 + 3 + (k-2) \cdot 2 = 2(k+1)$$

and

$$3 + 3 + 3 + 3 + \underbrace{2 + \cdots + 2}_{k-4} = 3 + 3 + 3 + 3 + (k-4) \cdot 2 = 2(k+2),$$

we have $\Delta(2(k+1)) = \Delta(2(k+2)) = R$. Since

$$\underbrace{2(k+1) + \cdots + 2(k+1)}_{k} = k \cdot 2(k+1) = 2k(k+1)$$

and

$$\underbrace{2(k+2) + \cdots + 2(k+2)}_{k} = k \cdot 2(k+2) = 2k(k+2),$$

we have $\Delta(2(k+1)) = \Delta(2(k+2)) = B$. Now we have $\Delta(2) = \Delta(3) = \Delta(2(k+1)) = \Delta(2(k+2)) = B$. Since

$$2k(k+1) + 3 + 3 + \underbrace{2 + \cdots + 2}_{k-3} = 2k(k+1) + 3 + 3 + (k-3) \cdot 2 = 2k(k+2),$$

we have a monochromatic solution which is a contradiction. $\square$

Proof of Theorem 1.2. Suppose first that $k \geq 4$ is an integer. By Lemma 2.3.1, every 2-coloring of

$$\{1, 2, 3, k+1, k+2, 2(k+1), 2(k+2), 3k, k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. The least common multiple of this set of integers is at most $6k(k+1)(k+2)$. Hence $f_2(k) \leq 6k(k+1)(k+2)$ for all $k \geq 4$.

By Myers and Parrish [90], we also have that $f_2(2) = 60 < 6 \cdot 2(2+1)(2+2)$ and $f_2(3) = 40 < 6 \cdot 3(3+1)(3+2)$. Hence we have $f_2(k) \leq 6k(k+1)(k+2)$ for all $k \geq 2$. □

When $k \geq 4$ is even or not divisible by 3, we can improve the constant factor of the upper bound for $f_2(k)$ in Theorem 2.1.2 to 2. To see this, we first need a lemma.

Lemma 2.3.2. *For all even integers $k \geq 4$, every 2-coloring of*

$$\{1, 2, 3, k+1, k+2, 2k, 2(k+1), 2(k+2), k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$.

Proof. Let $k \geq 4$ be an even integer and write $\mathcal{B} = \{1, 2, 3, k+1, k+2, 2k, 2(k+1), 2(k+2), k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$. Suppose, for a contradiction, that

$$\Delta : \mathcal{B} \to \{R, B\}$$

is a two-coloring without a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. WLOG, we assume that $\Delta(1) = R$. We have two cases depending on $\Delta(2)$.

Case 1: $\Delta(2) = R$. The proof of this case is the same as the proof of Case 1 in Lemma 2.3.1.

Case 2: $\Delta(2) = B$. Since

$$\underbrace{2 + \cdots + 2}_{k} = k \cdot 2 = 2k,$$

we have $\Delta(2k) = R$. Since

$$\underbrace{1 + \cdots + 1}_{k/2} + \underbrace{3 + \cdots + 3}_{k/2} = (k/2) \cdot 1 + (k/2) \cdot 3 = 2k,$$

we have $\Delta(3) = B$. Now we have $\Delta(2) = \Delta(3) = B$ and hence the rest of the proof is the same as the proof of Case 3 in Lemma 2.3.1. □

Theorem 2.3.3. *If $k \geq 4$ is even or not divisible by 3, then*

$$f_2(k) \leq 2k(k+1)(k+2).$$

Proof. We first consider that $k \geq 4$ is an even integer. By Lemma 2.3.2, every 2-coloring of

$$\{1, 2, 3, k+1, k+2, 2k, 2(k+1), 2(k+2), k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. Since 3 divides $k(k+1)(k+2)$, the least common multiple of this set of integers is $2k(k+1)(k+2)$. Therefore, by Theorem 2.2.3, we have $f_2(k) \leq 2k(k+1)(k+2)$.

Now we suppose that $k \geq 4$ and k is not divisible by 3. By Lemma 2.3.1, every 2-coloring of

$$\{1, 2, 3, k+1, k+2, 2(k+1), 2(k+2), 3k, k(k+1), k(k+2), 2k(k+1), 2k(k+2)\}$$

has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$. Since 3 does not divide k, we have that 3 divides $(k+1)(k+2)$ and hence the least common multiple of this set of integers is $2k(k+1)(k+2)$. Therefore, by Theorem 2.2.3, we have $f_2(k) \leq 2k(k+1)(k+2)$. □

2.4 A generalization of Theorem 2.1.2

Theorem 2.1.2 can be used to obtain an upper bound for Rado numbers of a larger family of equations. Lefmann [81] proved that for all integers $r, k \geq 2$ and $\ell \geq 1$, every r-coloring of $\mathbb{N}$ has a monochromatic solution to the equation

$$\sum_{i=1}^{k} \frac{1}{x_i^{1/\ell}} = \frac{1}{y^{1/\ell}}. \tag{2.4}$$

The following result is a special case of a theorem by Lefmann [81]. For completeness, we provide a short proof for this result.

Lemma 2.4.1 (Lefmann [81])**.** *Let $k, r \geq 2$ and $\ell \geq 1$ be integers, and A a finite subset of $\mathbb{N}$. If every r-coloring of A has a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$, then every r-coloring of $A' := \{a^\ell : a \in A\}$ has a monochromatic solution to $1/x_1^{1/\ell} + \cdots + 1/x_k^{1/\ell} = 1/y^{1/\ell}$.*

Proof. Suppose that every r-coloring of A has a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. Let

$$\Delta : A^\ell \to [r]$$

be an r-coloring of A^ℓ. Define

$$\overline{\Delta} : A' \to [r]$$

where $\overline{\Delta}(x) = \Delta(x^\ell)$ for all $x \in A$. By the definition of A', $\overline{\Delta}$ a well-defined function and hence an r-coloring of A. By assumption, there exist $a_1, a_2, \ldots, a_k, b \in A$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_k = 1/b$ and $\overline{\Delta}(a_1) = \overline{\Delta}(a_2) = \cdots = \overline{\Delta}(a_k) = \overline{\Delta}(b)$. So we have $1/(a_1^\ell)^{1/\ell} + 1/(a_2^\ell)^{1/\ell} + \cdots + 1/(a_k^\ell)^{1/\ell} = 1/(b^\ell)^{1/\ell}$ and, by the definition of $\overline{\Delta}$, $\Delta(a_1^\ell) = \Delta(a_2^\ell) = \cdots = \Delta(a_k^\ell) = \Delta(b^\ell)$. Hence $(x_1, x_2, \ldots, x_k, y) = (a_1^\ell, a_2^\ell, \ldots, a_k^\ell, b^\ell)$ is a monochromatic solution to $1/x_1^{1/\ell} + 1/x_2^{1/\ell} + \cdots + 1/x_k^{1/\ell} = 1/y^{1/\ell}$. $\square$

Let $f_r(k,\ell)$ be the smallest positive integer n such that every r-coloring of $[n]$ has a monochromatic solution to $1/x_1^{1/\ell} + 1/x_2^{1/\ell} + \cdots + 1/x_k^{1/\ell} = 1/y^{1/\ell}$, where $x_1, x_2, \ldots, x_k$ are not necessarily distinct. Note that we have $f_r(k,1) = f_r(k)$. The following result is a generalization of Theorem 2.1.2.

Theorem 2.4.2. *For all positive integers $k, r \geq 2$ and $\ell \geq 1$,*

$$f_2(k,\ell) \leq 6^\ell k^\ell (k+1)^\ell (k+2)^\ell.$$

Proof. Let $k \geq 2$ and $\ell \geq 1$ be integers. By Theorem 2.1.2, every 2-coloring of

$$\{1, 2, \ldots, 6k(k+1)(k+2)\}$$

has a monochromatic solution to $1/x_1 + \cdots + 1/x_k = 1/y$. So, by Lemma 2.4.1, every 2-coloring of $\{1^\ell, 2^\ell, \ldots, [6k(k+1)(k+2)]^\ell\}$ has a monochromatic solution to $1/x_1^{1/\ell} + 1/x_2^{1/\ell} + \cdots + 1/x_k^{1/\ell} = 1/y^{1/\ell}$. Hence we have $f_2(k,\ell) \leq [6k(k+1)(k+2)]^\ell$. □

Similar to Theorem 2.3.3, we have a slightly better upper bound for $f_2(k,\ell)$ when $k \geq 4$ is even or not divisible by 3.

Theorem 2.4.3. *For all positive integers $k \geq 4$, $r \geq 2$, and $\ell \geq 1$, if k is even or not divisible by 3, then*

$$f_2(k,\ell) \leq 2^\ell k^\ell (k+1)^\ell (k+2)^\ell.$$

Proof. Let $k \geq 4$, $r \geq 2$, and $\ell \geq 1$ be integers with k even or not divisible by 3. By Theorem 2.3.3, every 2-coloring of $\{1, 2, \ldots, 2k(k+1)(k+2)\}$ has a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. So, by Lemma 2.4.1, every 2-coloring of $\{1^\ell, 2^\ell, \ldots, [2k(k+1)(k+2)]^\ell\}$ has a monochromatic solution to $1/x_1^{1/\ell} + 1/x_2^{1/\ell} + \cdots + 1/x_k^{1/\ell} = 1/y^{1/\ell}$. Hence we have $f_2(k,\ell) \leq [2k(k+1)(k+2)]^\ell$. □

2.5 Other bounds for $f_r(k)$

By Theorem 2.2.3, as more Ramsey-type results for linear homogeneous equations are discovered, often times related results on nonlinear homogeneous equations become direct consequences. As an example, we show that a recent result by Boza, Marín, Revuelta, and Sanz [21] implies a polynomial upper bound for $f_3(k)$.

Lemma 2.5.1 (Boza, Marín, Revuelta, and Sanz [21])**.** *Let $k \geq 3$ and*

$$\begin{aligned}\chi(k) = \{&1, 2, k, k+1, k+2, 2k, k^2-k+1, k^2-1, k^2, k^2+1, k^2+k-1,\\ &k^2+k, k^2+k+1, 2k^2-2k+1, 2k^2-k, 2k^2-k+1, 2k^2-1, 2k^2+k-2,\\ &3k^2-2k, 3k^2-k-1, 3k^2-2, k^3, k^3+1, k^3+k-1, k^3+k, k^3+k^2-k,\\ &k^3+k^2-1, k^3+k^2+k-2, k^3+2k^2-k-1, k^3+2k^2-2\}.\end{aligned}$$

Then every 3-coloring of $\chi(k)$ has a monochromatic solution to $x_1 + x_2 + \cdots + x_k = y$.

Theorem 2.5.2. $f_3(k) = O(k^{43})$.

Proof. Let $k \geq 3$. The least common multiple of $\chi(k)$ as defined in Lemma 2.5.1 is at most

$$\begin{aligned}&k^3(k+1)(k+2)(k^2-k+1)(k-1)(k^2+1)(k^2+k-1)(k^2+k+1)(2k^2-2k+1)\\ &\times(2k-1)(2k^2-k+1)(2k^2-1)(2k^2+k-2)(3k-2)(3k^2-k-1)(3k^2-2)\\ &\times(k^3+k-1)(k^3+k^2-1)(k^3+k^2+k-2)(k^3+2k^2-k-1)\\ &\times(k^3+2k^2-2) = \Theta(k^{43}).\end{aligned}$$

So by Theorem 2.2.3, we have $f_3(k) = O(k^{43})$. $\square$

The method in this paper does not provide lower bounds for $f_r(k)$, at least not directly. Nevertheless, we note that we have the following easy lower bound for $f_r(k)$.

Theorem 2.5.3. *For all positive integers $k, r \geq 2$,*

$$f_r(k) \geq k^r.$$

Proof. Let $k, r \geq 2$ be positive integers. It suffices to show that there exists an r-coloring of $\{1, 2, \ldots, k^r - 1\}$ which does not have a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. Consider the r-coloring

$$\Delta : \{1, 2, \ldots, k^r - 1\} \to \{0, 1, \ldots, r - 1\}$$

where $\Delta(x) = i$ if $x \in \{k^i, k^i + 1, \ldots, k^{i+1} - 1\}$ for some $i \in \{0, 1, \ldots, r - 1\}$. That is, for all $i \in \{0, 1, \ldots, r - 1\}$, $\Delta(\{k^i, k^i + 1, \ldots, k^{i+1} - 1\}) = i$.

Suppose that $(x_1, x_2, \ldots, x_k, y) = (a_1, a_2, \ldots, a_k, b)$, with $b < a_1 \leq a_2 \leq \cdots \leq a_k < k^r$, is a solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. Then we have

$$\frac{1}{b} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \geq \frac{1}{a_k} + \frac{1}{a_k} + \cdots + \frac{1}{a_k} = \frac{k}{a_k}$$

and hence $a_k \geq kb$. So $b < k^{r-1}$. It follows that $b \in \{k^j, k^j + 1, \ldots, k^{j+1} - 1\}$ for some $j \in \{0, 1, \ldots, r-2\}$. Then $a_k \geq kb \geq kk^j = k^{j+1}$. Now by our definition, $\Delta(a) \geq j+1 \neq j = \Delta(b)$. So $(x_1, x_2, \ldots, x_k, y) = (a_1, a_2, \ldots, a_k, b)$ is not a monochromatic solution. Therefore, Δ does not have a monochromatic solution to $1/x_1 + 1/x_2 + \cdots + 1/x_k = 1/y$. □

By Theorem 2.5.3, we have $f_2(k) = \Omega(k^2)$. Considering this and that $f_2(k) = O(k^3)$, $f_2(4) = 48$, and $f_2(5) = 39$, we ask the following question:

Question 2.5.4. Is it true that $f_2(k) = \Theta(k^2)$?

Chapter 3: Maker-Breaker Rado games for equations with radicals

3.1 Introduction

In this chapter, we study the smallest positive integer n such that Maker wins the Rado games on $[n]$ when $\mathcal{F}$ is the set of solutions to the equation

$$x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell} \tag{3.1}$$

where k and ℓ are integers with $k \geq 2$ and $\ell \neq 0$. Equation (3.1) is connected with results in arithmetic Ramsey theory [54, 80]. In arithmetic Ramsey theory, a system of equations $E(x_1, \ldots, x_k, y) = 0$ in variables $x_1, \ldots, x_k, y$ is called *partition regular* if whenever $\mathbb{N}$ is partitioned into a finite number of classes, one of them contains a solution to $E(x_1, \ldots, x_k, y) = 0$. In 1991, Lefmann [81] proved that, among other things, Equation (3.1) is partition regular for all $\ell \in \mathbb{Z}\backslash\{0\}$. In the same year, Brown and Rödl [24] proved that if a system $E(x_1, \ldots, x_k, y) = 0$ of homogeneous equations is partition regular, then the system $E(1/x_1, \ldots, 1/x_k, 1/y) = 0$ is also partition regular.

To state our results, we first define the games we study in detail. Let $A \subseteq \mathbb{N}$ be a finite set and let $e(x_1, \ldots, x_k, y) = 0$ be an equation in variables $x_1, \ldots, x_k, y$. The Maker-Breaker Rado games denoted

$$G(A, e(x_1, \ldots, x_k, y) = 0) \text{ and } G^*(A, e(x_1, \ldots, x_k, y) = 0)$$

have the following rules:

(1) Maker and Breaker take turns to select a number from A. Once a number is selected by a player, neither player can select that number again. Maker starts the game.

(2) Maker wins the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game if a collection of the numbers chosen by Maker form a solution to $e(x_1, \ldots, x_k, y) = 0$ where $x_1, \ldots, x_k$ are *not* necessarily distinct; and Maker wins the $G^*(A, e(x_1, \ldots, x_k, y) = 0)$ game if a collection of the numbers chosen by Maker form a solution to $e(x_1, \ldots, x_k, y) = 0$ where $x_1, \ldots, x_k$ are distinct.

(3) Breaker wins if Maker fails to occupy a solution to $e(x_1, \ldots, x_k, y) = 0$.

We use the following shorter notations for games with Equation (3.1):

$$G([n], k, \ell) := G\left([n], x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}\right)$$

and

$$G^*([n], k, \ell) := G^*\left([n], x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}\right).$$

We say that a player wins a game if there is a winning strategy which guarantees that this player wins no matter what the other player does. A *winning strategy* is a set of instructions which tells the player what to do each round given what had been previously played by both players. Let $f(k, \ell)$ be the smallest positive integer n such that Maker wins the $G([n], k, \ell)$ game and let $f^*(k, \ell)$ be the smallest positive integer n such that Maker wins the $G^*([n], k, \ell)$ game.

For $\ell \geq 1$, we are able to find exact formulas for $f(k, \ell)$ and $f^*(k, \ell)$.

Theorem 3.1.1. *For all integers $k \geq 2$ and $\ell \geq 1$, we have $f(k, \ell) = (k+2)^\ell$.*

Theorem 3.1.2. *For all integers $k \geq 2$ and $\ell \geq 1$, we have $f^*(k, \ell) = (k^2+3)^\ell$.*

Our proofs of Theorems 3.1.1 and 3.1.2 involve showing that $f(k,1) = k+2$ and $f^*(k,1) = k^2+3$ using elementary combinatorial arguments, and that $f(k,\ell) \le [f(k,1)]^\ell$ and $f^*(k,\ell) \le [f^*(k,1)]^\ell$ using a result of Besicovitch [16] on the linear independence of integers with fractional powers.

For $\ell \le -1$, our main results are the following:

Theorem 3.1.3. *Let k, ℓ be integers with $k \ge 2$ and $\ell \le -1$. Then $f(k,\ell) = [k+\Theta(1)]^{-\ell}$. More specifically, if $k \ge 1/(2^{-1/\ell}-1)$, then $f(k,\ell) \ge (k+1)^{-\ell}$; and if $k \ge 4$, then $f(k,\ell) \le (k+2)^{-\ell}$.*

Theorem 3.1.4. *Let k, ℓ be integers with $k \ge 2$ and $\ell \le -1$. Then*

$$f^*(k,\ell) = [\exp(O(k \log k))]^{-\ell}.$$

The proof of Theorem 3.1.4 involves showing that $f^*(k,-1) = \exp(O(k\log k))$ using a game theoretic variant of a theorem in arithmetic Ramsey theory by Brown and Rödl [24].

Our results indicate that it is "easier" to form a solution to Equation (3.1) strategically compared to their counterparts in arithmetic Ramsey theory. To illustrate this, let $R(k,\ell)$ be the smallest positive integer n such that if $[n]$ is partitioned into two classes then one of them has a solution to Equation (3.1) with $x_1, \ldots, x_k$ not necessarily distinct, and let $R^*(k,\ell)$ be the smallest positive integer n such that if $[n]$ is partitioned into two classes then one of them has a solution to Equation (3.1) with $x_1, \ldots, x_k$ distinct. Note that if Maker and Breaker choose numbers in $[n]$, with $n \ge R(k,\ell)$ (respectively, $n \ge R^*(k,\ell)$), until there is no number left to choose, then the sets of numbers chosen by Maker and Breaker form a partition of $[n]$. If Maker does not win the game, then it means that the set of numbers chosen by Breaker contains a solution to Equation (3.1). Since Maker goes first, by strategy stealing, Maker could follow Breaker's strategy and win the game. Therefore, we have $f(k,\ell) \leq R(k,\ell)$ and $f^*(k,\ell) \leq R^*(k,\ell)$. When $\ell \in \{-1,1\}$, some results on $R(k,\ell)$ and $R^*(k,\ell)$ are known.

For $\ell = 1$, Beutelsapacher and Brestovansky [17] proved that $R(k,1) = k^2 + k - 1$. The exact formula for $R^*(k,1)$ is not known, but Boza, Revuelta, and Sanz [22] proved that, for $k \geq 6$, $R^*(k,1) \geq (k^3 + 3k^2 - 2k)/2$. Hence, by Theorems 3.1.1 and 3.1.2, we have

$$\lim_{k\to\infty} \frac{f(k,1)}{R(k,1)} = \lim_{k\to\infty} \frac{f^*(k,1)}{R^*(k,1)} = 0.$$

For $\ell = -1$, Myers and Parrish [90] calculated that $R(2,-1) = 60$, $R(3,-1) = 40$, $R(4,-1) = 48$, and $R(5,-1) = 39$; and in Chapter 2, we proved that $R(k,-1) \geq k^2$. So by Theorem 3.1.3, we have

$$\lim_{k\to\infty} \frac{f(k,-1)}{R(k,-1)} = 0. \tag{3.2}$$

Unfortunately, we do not know a similar lower bound for $R^*(k,-1)$. However, we believe that Maker can still do better by selecting numbers strategically.

Conjecture 3.1.5. $\lim_{k\to\infty} f^*(k,-1)/R^*(k,-1) = 0$.

This chapter is organized as follows. We first prove some preliminary results in Section 3.2. The next four sections are devoted to proving Theorems 3.1.1 to 3.1.4. In Section 3.7, we study Rado games for linear equations with arbitrary coefficients. We discuss some future research directions in Section 3.8.

We remind the reader that, throughout this chapter, *we only use asymptotic notation for functions of* k where ℓ is neither a parameter nor a constant.

3.2 Preliminaries

We prove some results which will be used to prove Theorems 3.1.1 to 3.1.4. Our first result shows that the games for equations with radicals can be partially reduced to games for equation without radicals, i.e., $\ell = 1$ or $\ell = -1$.

Lemma 3.2.1. *Let k and ℓ be integers with $k \geq 2$ and $\ell \neq 0$. If $\ell \geq 1$, then*

$$f(k,\ell) \leq [f(k,1)]^{\ell} \text{ and } f^*(k,\ell) \leq [f(k,1)]^{\ell}.$$

If $\ell \leq -1$, then

$$f(k,\ell) \leq [f(k,-1)]^{-\ell} \text{ and } f^*(k,\ell) \leq [f(k,-1)]^{-\ell}.$$

Proof. We prove that if $\ell \geq 1$, then $f(k,\ell) \leq [f(k,1)]^{\ell}$. The other inequalities can be proved similarly.

Write $M = f(k,1)$ and let $\mathcal{M}$ be a Maker's winning strategy for the $G([M],k,1)$ game. Notice that if $(x_1,\ldots,x_k,y) = (a_1,\ldots,a_k,b)$ is a solution to $x_1 + \cdots + x_k = y$, then $(x_1,\ldots,x_k,y) = (a_1^{\ell},\ldots,a_k^{\ell},b^{\ell})$ is a solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$.

For $i = 1,2,\ldots$, let $m_i \in [M^{\ell}]$ be the number chosen by Maker and let $b_i \in [M^{\ell}]$ be the number chosen by Breaker in round i. We define a strategy for Maker recursively. We note that Maker focuses on the set $\{1^{\ell},2^{\ell},\ldots,M^{\ell}\}$ in this strategy. In round 1, if $\mathcal{M}$ tells Maker to choose a_1 for the $G([M],k,1)$ game, then set $m_1 = a_1^{\ell}$. If $b_1 = z_1^{\ell}$ for some $z_1 \in [M]$, then set $b_1' = z_1$; otherwise, arbitrarily set b_1' equal to some number in $M \backslash \{a_1\}$. In round $i \geq 2$, given $a_1,a_2,\ldots,a_{i-1},b_1',b_2',\ldots,b_{i-1}'$, if $\mathcal{M}$ tells Maker to choose a_i, then set $m_i = a_i$. This is possible because $\mathcal{M}$ is a winning strategy. If $b_i = z_i^{\ell}$ for some $z_i \in [M]$, then set $b_i' = z_i$; otherwise, arbitrarily set b_i' equal to some number in $M \backslash \{a_1,a_2,\ldots,a_{i-1},a_i,b_1',b_2',\ldots,b_{i-1}'\}$.

Now since $\mathcal{M}$ is a winning strategy, there exists t such that $\{a_1,a_2,\ldots,a_t\}$ has a solution to $x_1 + \cdots + x_k = y$. Hence $\{m_1,m_2,\ldots,m_t\} = \{a_1^{\ell},a_2^{\ell},\ldots,a_t^{\ell}\}$ has a solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$. Therefore, Maker wins the $G([M^{\ell}],k,\ell)$ game. $\square$

Theorems 3.1.1 and 3.1.2 indicate that these inequalities in Lemma 3.2.1 are actually equalities when $\ell \geq 2$. This is due to a result of Besicovitch [16]. To state this result, we first need the following definition.

Definition 3.2.2. Let $a \in \mathbb{N}\backslash\{1\}$. We say that a is *power-ℓ free* if $a = b^\ell c$, with $b, c \in \mathbb{N}$, implies $b = 1$.

Theorem 3.2.3 (Besicovitch [16])**.** *For all positive integers $\ell \geq 2$, the set*

$$A(\ell) := \{a^{1/\ell} : a \in \mathbb{N}\backslash\{1\} \textit{ and } a \textit{ is power-}\ell \textit{ free}\}$$

is linearly independent over $\mathbb{Z}$. That is, if $a_1, \ldots, a_m \in A(\ell)$ and $c_1, \ldots, c_m \in \mathbb{N}$ satisfy $c_1 a_1 + \cdots + c_m a_m = 0$, then $c_1 = \cdots = c_m = 0$.

Besicovitch [16] actually provided an elementary proof of a stronger result, but Theorem 3.2.3 is enough for our purposes. For interested readers, we note that Richards [103] proved a similar result to the one in [16], but using Galois theory instead. A direct consequence of Theorem 3.2.3 is the following result which will be used in proving Theorems 3.1.1 and 3.1.2.

Corollary 3.2.4. *Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. The solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ are of the form $(x_1, \ldots, x_k, y) = (ca_1^\ell, \ldots, ca_k^\ell, cb^\ell)$ where $a_1, \ldots, a_k, b, c \in \mathbb{N}$, $a_1 + \cdots + a_k = b$, and c is power-ℓ free.*

Proof. Suppose that $\alpha_1, \ldots, \alpha_k, \beta \in \mathbb{N}$ satisfy

$$\alpha_1^{1/\ell} + \cdots + \alpha_k^{1/\ell} = \beta^{1/\ell}.$$

We write $\alpha_i = c_i a_i^\ell$ for all $i = 1, ..., k$, and $\beta = db^\ell$ where $a_1, \ldots, a_k, c_1, \ldots, c_k, b, d \in \mathbb{N}$ and $c_1, \ldots, c_k, d$ are power-ℓ free. Then we have

$$a_1 c_1^{1/\ell} + \cdots + a_k c_k^{1/\ell} - bd^{1/\ell} = 0. \tag{3.3}$$

We first show that $c_1 = \cdots = c_k = d$. Suppose, for a contradiction, that $c_1, \ldots, c_k, d$ are not all the same. We split this into two cases.

Case 1: $d \neq c_i$ for all $i \in [k]$. After combining terms with the same ℓ-th roots, the left-hand side of Equation (3.3) has at least two terms where one of them is $-bd^{1/\ell}$. Now by Theorem 3.2.3, $b = 0$ which is a contradiction.

Case 2: $d = c_i$ for some $i \in [k]$. Then there exists $j \in [k]\backslash\{i\}$ such that $c_j \neq c_i$. After combining terms with the same ℓ-th roots, the left-hand side of Equation (3.3) has a term with $c_j^{1/\ell}$. This is because all the terms with $c_j^{1/\ell}$ contain only positive coefficients. By Theorem 3.2.3, the coefficient of $c_j^{1/\ell}$ is zero after combining like terms. But this is impossible because the coefficient of $c_j^{1/\ell}$ is the sum of a subset of $\{a_1, ..., a_k\}$ consisting only positive integers.

Hence we have $c_1 = \cdots = c_k = d$. Therefore, $a_1 + \cdots + a_k = b$. □

We note that Newman [92] proved Corollary 3.2.4 for the case $k = 2$ without using Theorem 3.2.3.

Next, we prove a game theoretic variant of a result by Brown and Rödl [24, Theorem 2.1]. We note that an equation $e(x_1, \ldots, x_k, y) = 0$ is *homogeneous* if whenever $(x_1, \ldots, x_k, y) = (a_1, \ldots, a_k, b)$ is a solution to $e(x_1, \ldots, x_k, y) = 0$, for all $m \in \mathbb{N}$, $(x_1, \ldots, x_k, y) = (ma_1, \ldots, ma_k, mb)$ is a also a solution to $e(x_1, \ldots, x_k, y) = 0$.

Theorem 3.2.5. *Let A be a finite subset of $\mathbb{N}$, L the least common multiple of A, $k \in \mathbb{N}$, and $e(x_1, \ldots, x_k, y) = 0$ a homogeneous equation. If Maker wins the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game, then Maker wins the $G([L], e(1/x_1, \ldots, 1/x_k, 1/y) = 0)$ game. Similarly, if*

Maker wins the $G^*(A, e(x_1, \ldots, x_k, y) = 0)$ *game, then Maker wins the*

$$G^*([L], e(1/x_1, \ldots, 1/x_k, 1/y) = 0)$$

game.

Proof. Suppose that Maker wins the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game. Let $\mathcal{M}$ be a Maker's winning strategy. We consider the following Maker's strategy for the

$$G([L], e(1/x_1, \ldots, x_k, 1/y) = 0)$$

game. In round 1, if $\mathcal{M}$ tells Maker to choose m_1 for the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game, then Maker chooses $L/m_1 \in \{1, \ldots, L\}$. The rest of the strategy is defined inductively. For all rounds i, let L/b_i be the number chosen by Breaker and L/m_i be the number chosen by Maker where $m_i \in \{1, \ldots, L\}$. If $b_i \in A$, then we set $b'_i = b_i$; if $b_i \notin A$, then arbitrarily set b'_i equal to some number in $A \backslash \{m_1, \ldots, m_i, b'_1, \ldots, b'_{i-1}\}$. For all rounds $i \geq 2$, given $\{m_1, \ldots, m_{i-1}, b'_1, \ldots, b'_{i-1}\}$, if $\mathcal{M}$ tells Maker to choose m_i for the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game, then Maker chooses L/m_i for the

$$G([L], e(1/x_1, \ldots, 1/x_k, 1/y) = 0)$$

game. This process is possible because $\mathcal{M}$ is a winning strategy.

Since $\mathcal{M}$ is a winning strategy, in some round t, there exists a subset $\{a_1, \ldots, a_s\}$ of $\{m_1, \ldots, m_t\}$ which form a solution to $e(x_1, \ldots, x_k, y) = 0$. By homogeneity, the set $\{L/a_1, \ldots, L/a_s\}$ form a solution to $e(1/x_1, \ldots, 1/x_k, 1/y) = 0$. So Maker wins the $G([L], e(1/x_1, \ldots, 1/x_k, 1/y) = 0)$ game.

The case for the $G^*([L], e(1/x_1, \ldots, 1/x_k, 1/y) = 0)$ game can be proved in a similar way. $\square$

The key feature of Theorem 3.2.5 is that one can choose a set A whose least common multiple L is small. This was not used by Brown and Rödl [24, Theorem 2.1]. For interested readers, we note that, in Chapter 2, we improved a quantitative result by Brown and Rödl [24, Theorem 2.5] with the help of this observation.

Finally, we also need the following definitions.

Definition 3.2.6. Given $m \in \mathbb{N}$ mutually disjoint subsets $\{s_1, t_1\}$, $\{s_2, t_2\}$, …, $\{s_m, t_m\}$ of $\mathbb{N}$ with size 2, the *pairing strategy* over those disjoint subsets for a player is defined as follows: if their opponent chooses s_i for some $i = 1, 2, \ldots, m$, then this player chooses t_i.

Definition 3.2.7. Let $k \geq 2$ be an integer and $a_1x_1 + \cdots + a_kx_k = y$ a linear equation. Suppose, at some point of the $G^*([n], a_1x_1 + \cdots + a_kx_k = y)$ game, Maker has claimed a set A of at least k integers. Then we call $a_1\alpha_1 + \cdots + a_k\alpha_k$ a k-*sum* for any k distinct integers $\alpha_1, \ldots, \alpha_k \in A$.

3.3 Proof of Theorem 3.1.1

We first prove Theorem 3.1.1 for the case $\ell = 1$.

Lemma 3.3.1. *For all integers $k \geq 2$, we have $f(k, 1) = k + 2$.*

Proof. We first show that Maker wins the $G([k + 2], k, 1)$ game. Note that this will be proved in more full generality later in Theorem 3.7.1. We consider two cases.

Case 1: $k = 2$. Maker starts by choosing 2. Since $2 + 2 = 4$ and $1 + 1 = 2$, Maker wins the game in the next round by choosing either 1 or 4, whichever is available.

Case 2: $k > 2$. Maker starts by selecting 1. Notice that

$$\underbrace{1 + 1 + \cdots + 1}_{k} = k \cdot 1 = k,$$

$$\underbrace{1 + 1 + \cdots + 1}_{k-1} + 2 = (k - 1) \cdot 1 + 2 = k + 1,$$

and

$$\underbrace{1+1+\cdots+1}_{k-2}+2+2=(k-2)\cdot 1+2\cdot 2=k+2.$$

If Breaker chooses k in the first round, then Maker chooses 2 in round 2 and wins the game in round 3 by choosing either $k+1$ or $k+2$. If Breaker does not choose k in round 1, then Maker can win the game in round 2 by choosing k.

Now we show that Breaker wins the $G([k+1],k,1)$ game. When $\ell=1$, the only possible solutions to Equation (3.1) in $\{1,\ldots,k+1\}$ are

$$(x_1,x_2,\ldots,x_{k-1},x_k,y)=(1,1,\ldots,1,1,k)$$

and

$$(x_1,x_2,\ldots,x_{k-1},x_k,y)=(1,1,\ldots,1,2,k+1).$$

If $k=2$, then Breaker wins the game by the pairing strategy over $\{1,2\}$. If $k\geq 3$, then Breaker wins the game by the pairing strategy over $\{1,k\}$ and $\{2,k+1\}$. □

We also need a result on the solutions to $x_1^{1/\ell}+\cdots+x_k^{1/\ell}=y^{1/\ell}$ in $\{1,2,\ldots,(k+2)^\ell-1\}$ when k,ℓ are integers with $k\geq 2$ and $\ell\geq 1$.

Lemma 3.3.2. *For all integers $k\geq 2$ and $\ell\geq 1$, the only solutions to $x_1^{1/\ell}+\cdots+x_k^{1/\ell}=y^{1/\ell}$ in $\{1,2,\ldots,(k+2)^\ell-1\}$ are*

$$(x_1,\ldots,x_{k-2},x_{k-1},x_k,y)=(a,\ldots,a,a,a,ak^\ell),$$

and

$$(x_1,\ldots,x_{k-2},x_{k-1},x_k,y)=(b,\ldots,b,b,b2^\ell,b(k+1)^\ell),$$

where $a,b\in\{1,2,\ldots,2^\ell-1\}$ and are power-ℓ free.

Proof. Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. By Corollary 3.2.4, the only solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ in $\mathbb{N}$ are $(x_1, \ldots, x_k, y) = (c\alpha_1^\ell, \ldots, c\alpha_k^\ell, c\beta^\ell)$ where $\alpha_1, \ldots, \alpha_k, \beta, c \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_k = \beta$, and c is power-ℓ free. Restricted to the set $\{1, 2, \ldots, (k+2)^\ell - 1\}$, we must have $c\alpha_1^\ell, \ldots, c\alpha_k^\ell, c\beta^\ell \leq (k+2)^\ell - 1$. It follows that $\alpha_1^\ell, \ldots, \alpha_k^\ell \in \{1^\ell, 2^\ell, \ldots, (k+1)^\ell\}$ and hence $\alpha_1, \ldots, \alpha_k, \beta \leq k+1$. So $\alpha_1, \ldots, \alpha_k, \beta$ form a solution to $x_1 + \cdots + x_k = y$ in $\{1, 2, \ldots, k+1\}$. Since the only solutions to $x_1 + \cdots + x_k = y$ in $\{1, 2, \ldots, k+1\}$ are

$$(x_1, \ldots, x_{k-1}, x_k, y) = (1, \ldots, 1, 1, k),$$

and

$$(x_1, \ldots, x_{k-1}, x_k, y) = (1, \ldots, 1, 2, k+1),$$

we have either

$$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \beta) = (1, \ldots, 1, 1, 1, k)$$

or

$$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \beta) = (1, \ldots, 1, 2, k+1).$$

Now since $c\beta^\ell \leq (k+2)^\ell - 1$, we have

$$c \leq \frac{(k+2)^\ell - 1}{\beta^\ell} \leq \frac{(k+2)^\ell - 1}{k^\ell} < \left(1 + \frac{2}{k}\right)^\ell \leq 2^\ell.$$

Hence $c \in \{1, 2, \ldots, 2^\ell - 1\}$. $\square$

Proof of Theorem 3.1.1. Let $k \geq 2$ and $\ell \geq 1$ be integers. By Lemmas 3.2.1 and 3.3.1, we have $f(k, \ell) \leq [f(k, 1)]^\ell = (k+2)^\ell$. It remains to show that $f(k, \ell) \geq (k+2)^\ell$. This is true for $\ell = 1$ by Lemma 3.3.1. So we assume $\ell \geq 2$. It suffices to show that Breaker wins the $G\left([(k+2)^\ell - 1], k, \ell\right)$ game.

To do this, we build a winning strategy for Breaker based on Lemma 3.3.2. If $k = 2$, then Breaker wins the game by the pairing strategy over the sets $\{a, a2^\ell\}$ where $a \in \{1, 2, \ldots, 2^\ell - 1\}$. If $k \geq 3$, then Breaker wins the game by the pairing strategy over the sets $\{a, ak^\ell\}$ and $\{b2^\ell, b(k+1)^\ell\}$ where $a, b \in \{1, 2, \ldots, 2^\ell - 1\}$. In these pairing strategies, if Maker chooses some a or $b2^\ell$ so that $ak^\ell > (k+2)^\ell - 1$ or $b(k+1)^\ell > (k+2)^\ell - 1$, then Breaker arbitrarily chooses an available number in $\{1, 2, \ldots, (k+2)^\ell - 1\}$. □

3.4 Proof of Theorem 3.1.2

We first use the following two lemmas to prove Theorem 3.1.2 for $\ell = 1$.

Lemma 3.4.1. *For all integers $k \geq 2$, we have $f^*(k, 1) \leq k^2 + 3$.*

Proof. To prove this, it suffices to show that Maker wins the $G^*([k^2 + 3], k, 1)$ game. For $i = 1, 2, \ldots, \lceil n/2 \rceil$, let m_i denote the number selected by Maker in round i. For $j = 1, 2, \ldots, \lfloor n/2 \rfloor$, let b_j denote the number selected by Breaker in round j.

We first consider the case that $k = 2$. Then $k^2 + 3 = 7$. Maker starts by choosing $m_1 = 1$. Then no matter what b_1 is, there are three consecutive numbers in $\{2, 3, 4, 5, 6, 7\}$ available to Maker, say $\{a, b, c\}$. Maker sets $m_2 = b$. Notice that $1 + a = b$ and $1 + b = c$. Since Breaker can only choose one of a and c, Maker wins in round 3 by setting $m_3 = a$ or $m_3 = c$.

Now suppose $k = 3$. Then $k^2 + 3 = 12$. Maker starts by choosing $m_1 = 1$. We have 4 cases based on Breaker's choices.

Case 1: If $b_1 \neq 2$, then Maker chooses $m_2 = 2$. Suppose Breaker has selected b_2. Now consider the 3-term arithmetic progressions of difference $m_1 + m_2 = 3$:

$$\{3, 6, 9\}, \{4, 7, 10\}, \text{ and } \{5, 8, 11\}.$$

At the start of round 3, Breaker has chosen two numbers and hence one of these 3-term arithmetic progressions is available to Maker. Maker can set m_3 equal to the middle number of the available 3-term arithmetic progression and win the game in round 4 by choosing either the smallest or the largest number of the same 3-term arithmetic progression.

Case 2: If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 \neq 4, 8, 12$. Since $\{4, 8, 12\}$ is a 3-term arithmetic progression of difference $m_1 + m_2 = 4$, Maker can set $m_3 = 8$ and win the game in round 4 by choosing either 4 or 12.

Case 3: If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 = 4$ or 8. Then Maker sets $m_3 = 5$. If $b_3 \neq 9$, then Maker sets $m_4 = 9$. Since $m_1 + m_2 + m_3 = 1 + 3 + 5 = 9 = m_4$, Maker wins the game. Suppose $b_3 = 9$. Then Maker sets $m_4 = 6$. Since $m_1 + m_2 + m_4 = 1 + 3 + 6 = 10$ and $m_1 + m_3 + m_4 = 1 + 5 + 6 = 12$, Maker wins in round 5 by choosing either 10 or 12.

Case 4: If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 = 12$. Then Maker sets $m_3 = 4$. If $b_3 \neq 8$, then Maker sets $m_4 = 8$. Since $m_1 + m_2 + m_3 = 1 + 3 + 4 = 8 = m_4$, Maker wins the game. Suppose $b_3 = 8$. Then Maker sets $m_4 = 5$. Since $m_1 + m_2 + m_4 = 1 + 3 + 5 = 9$ and $m_1 + m_3 + m_4 = 1 + 4 + 5 = 10$, Maker wins in round 5 by choosing either 9 or 10.

Finally, we consider that $k \geq 4$. First notice that, since $k \geq 4$, all the k-sums are at least

$$\sum_{i=1}^{k} i = \frac{1}{2}k^2 + \frac{1}{2}k > 2k.$$

To see this, consider the following strategy for Maker: if a k-sum is available to Maker, then Maker chooses the k-sum and wins the game; otherwise Maker selects the smallest number available. By this strategy, Maker will choose the smallest numbers possible for the first k rounds and the smallest k-sum is $m_1 + \cdots + m_k$.

Also notice that $m_i \leq 2i - 1$ for $i = 1, ..., k$. Indeed, at the start of round i, Maker and Breaker have together chosen $2(i - 1) = 2i - 2$ numbers. Hence, one of the numbers in $\{1, 2, \ldots, 2i - 1\}$ is still available to Maker. So by Maker's strategy, we have $m_i \leq 2i - 1$.

Since $m_i \leq 2i-1$ for $i = 1, ..., k$, we have

$$\sum_{i=1}^{k} m_i \leq 1 + 3 + \cdots + 2k - 1 = k^2 \leq k^2 + 3.$$

If Breaker did not choose $m_1 + \cdots + m_k$ during the first k rounds, then Maker chooses $m_1 + \cdots + m_k$ in round $k+1$ and wins the game.

Now suppose that Breaker has selected $m_1 + \cdots + m_k$ during the first k rounds. Consider the middle of round $k+1$ when Maker has chosen $k+1$ numbers but Breaker has only chosen k numbers where s, $1 \leq s \leq k$, of them are k-sums. Since there are $2k+1$ numbers in $\{1, 2, \ldots, 2k+1\}$ and Breaker has chosen only k numbers, we have $m_{k+1} \leq 2k+1$ by Maker's strategy. Since $m_1, \ldots, m_{k+1}$ are distinct, the total number of k-sums is $\binom{k+1}{k} = k+1$.

Notice that if Breaker has chosen s k-sums during the first k rounds and one of them is $\sum_{i=1}^{k} m_i$, then

$$m_{k+1-s+j} \leq 2(k+1-s+j) - 1 - j = 2(k+1-s) + j - 1$$

for $j = 1, 2, \ldots, s$. Indeed, since the k-sums are greater than $2k$, if Breaker has chosen s k-sums, then Breaker has chosen at most $k-s$ numbers in $\{1, 2, \ldots, 2k-s+1\}$. By Maker's strategy, Maker has chosen $k+1$ numbers in $\{1, 2, \ldots, 2k-s+1\}$. If $s = 1$, then we have $m_{k+1} \leq 2k$. If $s > 1$, then by Maker's strategy, we have $m_{k+1} > m_k > \cdots > m_{k+1-s+1}$. Since $m_{k+1}, \ldots, m_{k+1-s+1} \in \{1, 2, \ldots, 2k-s+1\}$, this is also true.

Now we split it into two cases based on the value of s and what Breaker chooses in round $k+1$.

Case 1: $1 \leq s \leq k-1$ or $s = k$ and Breaker does not choose a k-sum in round $k+1$. Then Breaker will have chosen at most k k-sums at the beginning of round $k+2$. Since

$m_i \leq 2i-1$ for $i = 1, ..., k$ and $m_{k+1-s+j} \leq 2(k+1-s)+j-1$ for $j = 1, 2, \ldots, s$, at the beginning of round $k+2$, there exists an unclaimed k-sum whose value is at most

$$\begin{aligned}\sum_{i=1}^{k+1-s-2} m_i + \sum_{i=k+1-s}^{k+1} m_i \leq & \sum_{i=1}^{k+1-s-2} (2i-1) + \sum_{j=0}^{s} [2(k+1-s)+j-1] \\ =&(k-s-1)^2 + (s+1)2(k+1-s) + \frac{s(s-1)}{2} - 1 \\ =&k^2 - \frac{1}{2}s^2 + \frac{3}{2}s + 2 \leq k^2 + 3.\end{aligned}$$

Hence Maker chooses this k-sum in round $k+2$ and wins the $G^*([k^2+3], k, 1)$ game.

Case 2: $s = k$ and Breaker chooses a k-sum in round $k+1$. In this cases, at the end of round $k+1$, Breaker has chosen all possible k-sums from $\{m_1, \ldots, m_{k+1}\}$. Recall that the k-sums are greater than $2k$. Since $k+2 \leq 2k$ for $k \geq 2$, Breaker did not choose any number in $\{1, 2, \ldots, k+2\}$. So $m_i = i$ for $i = 1, 2, \ldots, k+2$. Notice that the largest k-sum before round $k+2$ is

$$\sum_{i=2}^{k+1} m_i = \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k.$$

Setting $m_{k+2} = k+2$, Maker now has two larger k-sums which are untouched by Breaker:

$$m_{k+2} + \sum_{i=2}^{k} m_i = k+2 + \frac{k(k+1)}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k + 1$$

and

$$m_{k+1} + m_{k+2} + \sum_{i=2}^{k-1} m_i = k+1+k+2 + \frac{(k-1)k}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k + 2.$$

Since $k \geq 4$, we have $k^2 + 3 \geq \frac{1}{2}k^2 + \frac{3}{2}k + 2$. Hence Maker can win the $G^*([k^2+3], k, 1)$ game in round $k+3$. □

Lemma 3.4.2. *For all integers $k \geq 2$, we have $f^*(k,1) \geq k^2+3$.*

Proof. It suffices to show that Breaker wins the $G([k^2+2],k,1)$ game.

For $i = 1,2,\ldots,\lceil n/2 \rceil$, let m_i denote the number selected by Maker in round i. For $j = 1,2,\ldots,\lfloor n/2 \rfloor$, let b_j denote the number selected by Breaker in round j.

We first consider $k = 2$. Then $k^2+2 = 2^2+2 = 6$. If $m_1 = 1$, then Breaker chooses $b_1 = 4$. Now Breaker wins by the pairing strategy over $\{2,3\}$ and $\{5,6\}$. If $m_1 \neq 1$, then Breaker chooses $b_1 = 1$. Now there are only two solutions available to Maker: $2+3=5$ and $2+4=6$. There are three cases:

Case 1: $m_1 = 2$. Then Breaker wins by the pairing strategy over $\{3,5\}$ and $\{4,6\}$.

Case 2: $m_1 \neq 1,2$, $b_1 = 1$, $m_2 = 2$. Then Breaker wins by the pairing strategy over $\{3,5\}$ and $\{4,6\}$.

Case 3: $m_1 \neq 1,2$, $b_1 = 1$, $m_2 \neq 2$. Then by choosing $b_2 = 2$, Breaker wins because the smallest numbers now available to Maker are 3 and 4, and $3+4 = 7 > 6$.

Now we consider $k \geq 3$. Notice that we have $k^2-1 \geq 2k+2$ when $k \geq 3$. We will prove that Breaker wins with the following strategy:

(1) in each round $i \in [k-1]$, Breaker chooses smallest number available;

(2) and in round k, if there is an unclaimed number in $[2k-2]$, then Breaker chooses the unclaimed number; otherwise, Breaker's strategy depends on the sum of the numbers in $[2k-2]$ claimed by Maker, which is denoted by S:

- If $S = (k-1)^2+3$, then Breaker chooses the smallest numbers possible.
- If $S = (k-1)^2+2$, then Breaker plays the pairing strategy over $\{2k-1, k^2+2\}$.
- If $S = (k-1)^2+1$, then Breaker plays the pairing strategy over $\{2k-1, k^2+1\}$ and $\{2k, k^2+2\}$.

- If $S = (k-1)^2$, then Breaker plays the pairing strategy over $\{2k-1, k^2\}$, $\{2k, k^2+1\}$, and $\{2k+1, k^2+2\}$.

Let $a_1 < a_2 < a_3 < \cdots < a_s$ with $s \leq \lceil n/2 \rceil$ be the numbers chosen by Maker when the game ends. We claim the following hold:

(i) $a_i \geq 2i-1$ for $i = 1, 2, \ldots, k$, $a_{k+1} \geq 2k$, and $a_{k+2} \geq 2k+1$;

(ii) if $a_{k-1} > 2k-2$, then Breaker wins;

(iii) the smallest k-sum possible for Maker is $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k}(2i-1) = k^2$ and hence Maker needs one of k^2, k^2+1, and k^2+2 to win;

(iv) if a k-sum does not contain all $\{a_1, ..., a_{k-1}\}$, then Breaker wins.

Here is why (i) holds. Since $a_i \geq 1 = 2 \cdot 1 - 1$, this is true for $i = 1$. Now consider $2 \leq i \leq k$. By Breaker's strategy, Breaker can select at least $i-1$ numbers in $\{1, 2, \ldots, 2(i-1)\}$. So Maker can select at most $i-1$ numbers in $\{1, 2, \ldots, 2(i-1)\}$. Hence $a_i \geq 2(i-1)+1 = 2i-1$.

To see that (ii) holds, notice that if $a_{k-1} > 2k-2$, then $a_{k-1} \geq 2k-1$ and $a_k \geq 2k$. Hence the smallest k-sum possible for Maker is

$$\sum_{i=1}^{k} a_i \geq 2k-1+2k+\sum_{i=1}^{k-2}(2i-1) = 2k-1+2k+(k-2)^2 = k^2+3 > k^2+2$$

and hence Breaker wins.

The reason why (iv) holds is because if a k-sum does not contain all of $\{a_1, \ldots, a_{k-1}\}$, then the k-sum is at least

$$a_k + a_{k+1} + \sum_{i=1}^{k-2} a_i \geq 2k-1+2k+(k-2)^2 = k^2+3 > k^2+2.$$

We first suppose that after Maker has chosen $m_1, \ldots, m_k$, there is an unclaimed number in $[2k-2]$. In this case, Breaker sets b_k equal to some number in $[2k-2]$. Now Breaker has chosen k numbers in $[2k-2]$ which implies that Maker can choose at most $k-2$ numbers in $[2k-2]$. Hence $a_{k-1} > 2k-2$. It follows that, Breaker wins.

Now assume that all the numbers in $[2k-2]$ are claimed in the middle of round k when Breaker has chosen k numbers and Breaker has chosen $k-1$ numbers. In this case, we must have $a_1, \ldots, a_{k-1} \in [2k-2]$ and hence $\sum_{i=1}^{k-1} a_i = S$. We consider the solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them. Recall that if a k-sum does not contain all numbers in $\{a_1, \ldots, a_{k-1}\}$, then Breaker wins. So we have the following cases:

Case 1: If $S = \sum_{i=1}^{k-1} a_i = (k-1)^2$, then there are three solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them: $\{a_1, \ldots, a_{k-1}, 2k-1, k^2\}$, $\{a_1, \ldots, a_{k-1}, 2k, k^2+1\}$, and $\{a_1, \ldots, a_{k-1}, 2k+1, k^2+2\}$. This is because if $S = \sum_{i=1}^{k-1} a_i = (k-1)^2$, then

$$a_k + \sum_{i=1}^{k-1} a_i \geq 2k - 1 + (k-1)^2 = k^2,$$

$$a_{k+1} + \sum_{i=1}^{k-1} a_i \geq 2k + (k-1)^2 = k^2 + 1,$$

$$a_{k+2} + \sum_{i=1}^{k-1} a_i \geq 2k + 1 + (k-1)^2 = k^2 + 2,$$

and

$$a_s + \sum_{i=1}^{k-1} a_i \geq 2k + 1 + 1 + (k-1)^2 = k^2 + 3 > k^2 + 2$$

for $s \geq k+3$.

Case 2: If $S = \sum_{i=1}^{k-1} a_i = (k-1)^2 + 1$, then there are two solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them:

$\{a_1, \ldots, a_{k-1}, k^2+1\}$ and $\{a_1, \ldots, a_{k-1}, a_{k+1}, k^2+2\}$. This is because if $S = \sum_{i=1}^{k-1} a_i = (k-1)^2+1$, then

$$a_k + \sum_{i=1}^{k-1} a_i \geq 2k-1+(k-1)^2+1 = k^2+1,$$

$$a_{k+1} + \sum_{i=1}^{k-1} a_i \geq 2k+(k-1)^2+1 = k^2+2,$$

and

$$a_s + \sum_{i=1}^{k-1} a_i \geq 2k+1+(k-1)^2+1 = k^2+3 > k^2+2$$

for $s \geq k+2$.

Case 3: If $S = \sum_{i=1}^{k-1} a_i = (k-1)^2+2$, then there is only one solution to $x_1+\cdots+x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them: $\{a_1, \ldots, a_k, k^2+2\}$. This is because if $S = \sum_{i=1}^{k-1} a_i = (k-1)^2+2$, then

$$a_k + \sum_{i=1}^{k-1} a_i \geq 2k-1+(k-1)^2+2 = k^2+2,$$

and

$$a_s + \sum_{i=1}^{k-1} a_i \geq 2k+(k-1)^2+2 = k^2+3 > k^2+2$$

for $s \geq k+1$.

In Case 1, Breaker uses the pairing strategy over $\{2k-1, k^2\}$, $\{2k, k^2+1\}$, and $\{2k+1, k^2+2\}$. Since these sets are pairwise disjoint, Breaker wins. Similarly, in Case 2, Breaker uses the pairing strategy over $\{2k-1, k^2+1\}$ and $\{2k, k^2+2\}$; and in Case 3, Breaker uses the pairing strategy over $\{2k-1, k^2+2\}$. $\square$

Proof of Theorem 3.1.2. Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. By Lemmas 3.2.1, 3.4.1 and 3.4.2, we have $f^*(k,\ell) \leq [f^*(k,1)]^\ell = (k^2+3)^\ell$. It remains to show that

$f^*(k,\ell) \geq (k^2+3)^\ell$ for all $\ell \geq 2$. To do this, it suffices to show that Breaker wins the $G([(k^2+3)^\ell - 1], k, \ell)$ game. For all $c \in \{1, 2, \ldots, 2^\ell - 1\}$, let

$$A(c) = \{c \cdot 1^\ell, c \cdot 2^\ell, \ldots, c \cdot (k^2+2)^\ell\} \cap \{1, 2, \ldots, (k^2+3)^\ell - 1\}.$$

Notice that if $c, c' \in \{1, 2, \ldots, 2^\ell - 1\}$ with $c \neq c'$, then $A(c) \cap A(c') = \emptyset$.

By Corollary 3.2.4, every solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$, with $x_1, \ldots, x_k$ distinct, in $\{1, 2, \ldots, (k^2+3)^\ell - 1\}$ belongs to $A(c)$ for some $c \in \{1, 2, \ldots, 2^{\ell-1}\}$.

Let $\mathcal{B}$ be a Breaker's winning strategy for the $G^*([k^2+2], k, 1)$ game. We define a Breaker's strategy for the $G([(k^2+3)^\ell - 1], k, \ell)$ game recursively. For rounds $i = 1, 2, \ldots$, let m_i be the number chosen by Maker and let b_i be the number chosen by Breaker. Let $m_1 = c_1 a_1^\ell$ where c_1 is power-ℓ free. If $\mathcal{B}$ tells Breaker to choose α_1 for the $G^*([k^2+2], k, 1)$ game given that Maker has selected a_1, then Breaker sets $b_1 = c_1 \alpha_1^\ell$. Consider round $i \geq 2$. Suppose Maker has chosen $m_1 = c_1 a_1^\ell, m_2 = c_2 a_2^\ell, \ldots, m_i = c_i a_i^\ell$ and Breaker has selected $b_1 = c_1 \alpha_1^\ell, b_2 = c_2 \alpha_2^\ell, \ldots, b_{i-1} = c_{i-1} \alpha_{i-1}^\ell$. Let $c_{j_1}, c_{j_2}, \ldots, c_{j_s} \in \{1, \ldots, i-1\}$ be all the indices such that

$$c_{j_1} = c_{j_2} = \cdots = c_{j_s} = c_i.$$

If $\mathcal{B}$ tells Breaker to choose α_i for the $G^*([k^2+2], k, 1)$ game given that Maker has has selected $a_{j_1}, a_{j_2}, \ldots, a_{j_s},\ a_i$ and Breaker has selected $b_{j_1}, b_{j_2}, \ldots, b_{j_s}$, then Breaker sets $b_i = c_i \alpha_i^\ell$.

Since $\mathcal{B}$ is a winning strategy for Breaker, Breaker can stop Maker from completing a solution set from each $A(c)$ and hence wins the game. $\square$

3.5 Proof of Theorem 3.1.3

Lemma 3.5.1. *Let k, ℓ be integers with $k \geq 2$ and $\ell \leq -1$. If $n < 2k^{-\ell}$ and Maker does not choose 1 in the first round, then Breakers wins the $G([n], k, \ell)$ game.*

Proof. Suppose $n < 2k^{-\ell}$ and Maker does not choose 1 in the first round. We show that Breaker wins the $G([n], k, \ell)$ game by choosing 1 in the first round. Suppose, for a contradiction, that Maker wins. Let $(x_1, \dots, x_k, y) = (a_1, \dots, a_k, b)$ be a solution to Equation (3.1) in $\{1, 2, \dots, n\}$ completed by Maker. Then since $a_i \leq n < 2k^{-\ell}$ for all $i = 1, \dots, k$, we have

$$b^{1/\ell} = a_1^{1/\ell} + \cdots + a_k^{1/\ell} > k(2k^{-\ell})^{1/\ell} = 2^{1/\ell}.$$

So $b < 2$ which is impossible. □

Proof of Theorem 3.1.3. We first prove that, if $k \geq 1/(2^{-1/\ell} - 1)$, then $f(k, \ell) \geq (k+1)^{-\ell}$. To do this, it suffices to show that that Breaker wins the $G([(k+1)^{-\ell} - 1], k, \ell)$ game. By straightforward calculation, we have

$$(k+1)^{-\ell} - 1 < 2k^{-\ell}.$$

Hence, by Lemma 3.5.1, we can assume that Maker chooses 1 in the first round and $b = 1$. Now we show that the only solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = 1$ in $\{1, 2, \dots, (k+1)^{-\ell} - 1\}$ is $(x_1, \dots, x_k) = (k^{-\ell}, \dots, k^{-\ell})$. This would imply that Breaker can choose $k^{-\ell}$ in the first round and win the game. Let $a_1, \dots, a_k \in \{1, 2, \dots, (k+1)^{-\ell} - 1\}$ with

$$a_1^{1/\ell} + \cdots + a_k^{1/\ell} = 1,$$

and $a_1 \leq \cdots \leq a_k$. Since the sum a rational number and an irrational number is irrational, $a_1^{1/\ell}, \ldots, a_k^{1/\ell}$ are rational numbers. Since $a_1, \ldots, a_k \in \{1, 2, \ldots, (k+1)^{-\ell} - 1\}$, we have $a_1, \ldots, a_k \in \{1, 2^{-\ell}, \ldots, k^{-\ell}\}$. If $a_i < k^{-\ell}$ for some $i \in [k]$, then

$$1 = a_1^{1/\ell} + \cdots + a_k^{1/\ell} > k(k^{-\ell})^{1/\ell} = 1$$

which is impossible. Hence the only solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = 1$ in $\{1, 2, \ldots, (k+1)^{-\ell} - 1\}$ is $(x_1, \ldots, x_k) = (k^{-\ell}, \ldots, k^{-\ell})$ and Breaker wins the $G([(k+1)^{-\ell} - 1], k, \ell)$ game.

Now we prove that if $k \geq 4$, then $f(k, \ell) \leq (k+2)^{-\ell}$. By Lemma 3.2.1, $f(k, \ell) \leq [f(k, -1)]^{-\ell}$. Hence, it suffices to show that for all $k \geq 4$, $f(k, -1) \leq k + 2$. We split it into two cases.

Case 1: $k + 1 \neq p$ or p^2 for any prime p. We will prove that $f(k, -1) \leq k + 1$. To do this, we will prove that Maker wins the $G([k+1], k, -1)$ game. In this case, we have $k+1 = rs$ for some integers $r > 1$ and $s > 1$ with $r \neq s$. Then we have $(r-1)s \neq r(s-1)$, $(r-1)s < k < k+1$ and $r(s-1) < k < k+1$. Consider the following solutions in $\{1, 2, \ldots, k+1\}$:

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (k, k, \ldots, k, k, 1),$$

$$(x_1, \ldots, x_{(r-1)s}, x_{(r-1)s+1}, \ldots, x_k, y) = (rs, \ldots, rs, r(s-1), \ldots, r(s-1), 1),$$

and

$$(x_1, \ldots, x_{r(s-1)}, x_{r(s-1)+1}, \ldots, x_k, y) = (rs, \ldots, rs, (r-1)s, \ldots, (r-1)s, 1).$$

Based on these solutions, Maker wins the $G([k+1], k, -1)$ game using the following strategy: Maker chooses 1 in the first round; if Breaker does not choose k in the first round, then Maker chooses k in the second round to win the game; otherwise, Maker will choose $k + 1 = rs$ in the second round and win the game by choosing either $r(s-1)$ or $(r-1)s$ in the third round.

Case 2: $k + 1 = p$ or p^2 for some prime $p \geq 5$. We will show that Maker wins the $G([k+2], k, -1)$ game.

Since $k + 1 \geq 5$ is odd, k is even and $k \geq 4$. Hence $(k+2)/2 \neq k$. Consider the following solutions in $\{1, 2, \ldots, k+2\}$:

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (k, k, \ldots, k, k, 1),$$

$$(x_1, \ldots, x_{(k-2)/2}, x_{(k-2)/2+1}, \ldots, x_k, y) = (k-2, \ldots, k-2, k+2, \ldots, k+2, 1),$$

and

$$(x_1, x_2, x_3, \ldots, x_k, y) = ((k+2)/2, (k+2)/2, k+2, \ldots, k+2, 1).$$

Based on these solutions, Maker wins the $G([k+2], k, -1)$ game by the following strategy: Maker chooses 1 in the first round; if Breaker does not choose k in the first round, then Maker chooses k in the second round to win the game; otherwise, Maker will choose $k+2$ in the second round and win the game by choosing either $(k+2)/2$ or $k-2$ in the third round. □

3.5.1 Remarks. In the proof of Theorem 3.1.3, we showed that if $k+1 = p$ or p^2 for some prime $p \geq 5$, then $f(k, -1) \leq k+2$. This inequality becomes equality when $k+1 = p$ for some odd prime p.

Theorem 3.5.2. *If $k+1=p$ for some odd prime p, then $f(k,-1)=k+2$.*

Proof. Suppose $k+1=p$ for some odd prime. By Theorem 3.1.3, we have $f(k,-1)\leq k+2$. It remains to show that $f(k,-1)\geq k+2$. To do this, it suffices to show that Breaker wins the $G([k+1],k,-1)$ game.

Case 1: $k+1=3$. The only solution to $1/x_1+\cdots+1/x_k=1/y$ in $\{1,2,3\}$ with $x_1,\ldots,x_k$ not necessarily distinct is $(x_1,x_2,y)=(2,2,1)$. Hence Breaker can win by choosing either 1 or 2 in the first round.

Case 2: $k+1\geq 5$. By Lemma 3.5.1, if Maker does not choose 1 in the first round, then Breaker wins. So we assume that Maker chooses 1 in the first round. Now we show that Breaker wins by choosing k in the first round. It suffices to show that $\{1,2,\ldots,k-1,k+1\}$ does not have a solution to $1/x_1+\cdots+1/x_k=1/1$ where x_1, …, x_k are not necessarily distinct. Suppose $(x_1,x_2,\ldots,x_{k-1},x_k)=(a_1,a_2,\ldots,a_{k-1},a_k)$ is a solution in $\{1,2,\ldots,k-1,k+1\}$. We show that $a_k=k+1$. Suppose not. Then $a_i<k$ for all $i=1,2,\ldots,k$. So

$$\frac{1}{a_1}+\cdots+\frac{1}{a_k}>\frac{1}{k}+\cdots+\frac{1}{k}=\frac{1}{1}$$

which is a contradiction. Hence $a_k=k+1$. Now we have

$$1=\frac{r}{k+1}+\sum_{i=1}^{k-r}\frac{1}{a_i}$$

where $r\in\{1,2,\ldots,k-1\}$ and $a_i<k$ for all $i=1,\ldots,k-r$. Rearranging the equation, we get

$$\sum_{i=1}^{k-r}\frac{1}{a_i}=\frac{p-r}{p}.$$

Since p is prime, p divides the least common multiple of $a_1,\ldots,a_{k-r}$. Since p is prime, p divides a_i for some i which is a contradiction because $a_i<p$ for all i. Hence Breaker wins the game. □

We are unable to verify that $f(k,-1) = k+2$ when $k+1 = p^2$ for some odd prime p. However, we believe this should be the case.

Conjecture 3.5.3. If $k+1 = p^2$ for some odd prime p, then $f(k,-1) = k+2$.

3.6 Proof of Theorem 3.1.4

To prove Theorem 3.1.4, we need the following result.

Lemma 3.6.1. *Let $k \geq 4$ be an integer and let $A = \{1, 2, \ldots, 2k+1\} \cup \{k^2 - k + 1, k^2 - k + 2, \ldots, k^2 + 2k\}$. Then Maker wins the $G^*(A, x_1 + \cdots + x_k = y)$ game.*

Proof. Let $k \geq 4$. For $i = 1, \ldots, k+3$, let m_i be the number selected by Maker in round i and let b_i be the number selected by Breaker in round i.

Consider the following strategy for Maker:

(1) Set $m_1 = 1$ and $M_1 = \{\{2,3\}, \{4,5\}, \ldots, \{2k, 2k+1\}\}$.

(2) For $i = 2, \ldots, k+1$, if $b_{i-1} \in B$ for some $B \in M_{i-1}$, then set $m_i \in B \backslash \{b_{i-1}\}$ and $M_i = M_{i-1} \backslash \{B\}$; if $b_{i-1} \notin B$ for any $B \in M_{i-1}$, then set $m_i = \min_{S \in M_{i-1}} \min S$, $M_i = M_{i-1} \backslash S'$ where $m_i \in S'$.

(3) In round $k+2$, if there exists a subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k such that $a_1 + \cdots + a_k \in \{k^2 - k + 1, \ldots, k^2 + 2k\} \backslash \{b_1, \ldots, b_{k+1}\}$, then set $m_{k+2} = a_1 + \cdots + a_k$. Otherwise, set $m_{k+2} = 2k+1$, and then, in round $k+3$, set $m_{k+3} = a_1 + \cdots + a_k$ where $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+2}\}$ has size k with $a_1 + \cdots + a_k \in \{k^2 - k + 1, \ldots, k^2 + 2k\} \backslash \{b_1, \ldots, b_{k+2}\}$.

In Step (3), Maker wins for the first case. So it remains to show that if no subset

$$\{a_1, a_2, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$$

of size k satisfies $a_1+\cdots+a_k \in \{k^2-k+1,\ldots,k^2+2k\}\backslash\{b_1,\ldots,b_{k+1}\}$, then Maker can set $m_{k+2}=2k+1$ in round $k+2$ and there exists a subset $\{a_1,...,a_k\} \subseteq \{m_1,\ldots,m_{k+2}\}$ of size k such that $a_1+\cdots+a_k \in \{k^2-k+1,\ldots,k^2+2k\}\backslash\{b_1,\ldots,b_{k+2}\}$.

Suppose, at the beginning of round $k+2$, no subset $\{a_1,\ldots,a_k\} \subseteq \{m_1,\ldots,m_{k+1}\}$ of size k satisfies $a_1+\cdots+a_k \in \{k^2-k+1,\ldots,k^2+2k\}\backslash\{b_1,\ldots,b_{k+1}\}$. First note that by Maker's strategy, for all $i=2,\ldots,k+1$, $m_i=2(i-1)$ or $2(i-1)+1$. So for all subsets $\{a_1,\ldots,a_k\} \subseteq \{m_1,\ldots,m_{k+1}\}$ of size k, we have

$$a_1+\cdots+a_k \geq 1+2+4+\cdots+2(k-1) = k^2-k+1$$

and

$$a_1+\cdots+a_k \leq 3+5+\cdots+2k+1 = (k+1)^2-1 = k^2+2k.$$

So if no subset $\{a_1,\ldots,a_k\} \subseteq \{m_1,\ldots,m_{k+1}\}$ of size k satisfies $a_1+\cdots+a_k \in \{k^2-k+1,\ldots,k^2+2k\}\backslash\{b_1,\ldots,b_{k+1}\}$, then $b_1,\ldots,b_{k+1} \notin \{1,\ldots,2k+1\}$. Now according to Maker's strategy, we have, $m_1=1$, and $m_i=2(i-1)$ for all $i=2,\ldots,k+1$. This implies that at the beginning of round $k+2$, $2k+1$ is available to Maker and hence Maker can set $m_{k+2}=2k+1$. At the same time, for all subsets $\{a_1,\ldots,a_k\} \subseteq \{m_1,\ldots,m_{k+1}\}$ of size k, we have $a_1+\cdots+a_k \leq 2+4+\cdots+2k = k^2+k$ and hence $b_1,\ldots,b_{k+1} \leq k^2+k$. By setting $m_{k+2}=2k+1$, there are at least two subsets of $\{m_1,\ldots,m_{k+2}\}$ of size k whose sum is greater than k^2+k. They are $\{2,4,\ldots,2(k-1),2k+1\}$ and $\{2,4,\ldots,2(k-2),2k,2k+1\}$. The first subset sums to $k^2+k+1 < k^2+2k$ and the second one sums to $k^2+k+3 < k^2+2k$. Since Breaker can only occupy one of them in round $k+2$, there exists a subset $\{a_1,\ldots,a_k\} \subseteq \{m_1,...,m_{k+2}\}$ of size k such that $a_1+\cdots+a_k \in \{k^2-k+1,\ldots,k^2+2k\}\backslash\{b_1,\ldots,b_{k+2}\}$. This proves that Maker wins the $G^*(A, x_1+\cdots+x_k=y)$ game. $\square$

Proof of Theorem 3.1.4. By Lemma 3.2.1, we have $f^*(k,\ell) \leq [f^*(k,-1)]^{-\ell}$. It remains to show that $f^*(k,-1) = \exp(O(k \log k))$. By Theorem 3.2.5, it suffices to find a finite set $A \subseteq \mathbb{N}$ such that Maker wins the $G^*(A, x_1 + \cdots + x_k = y)$ game and the least common multiple of A is small.

Let $k \geq 4$ be an integer and let $A := \{1, \ldots, 2k+1\} \cup \{k^2 - k + 1, \ldots, k^2 + 2k\}$. By Theorem 3.2.5 and Lemma 3.6.1, we have

$$
\begin{aligned}
f^*(k,-1) \leq & \operatorname{lcm}\{n : n \in A\} \\
\leq & \operatorname{lcm}\{1, ..., 2k+1\} \operatorname{lcm}\{k^2 - k + 1, ..., k^2 + 2k\} \\
\leq & \operatorname{lcm}\{1, ..., 2k+1\} (k^2 + 2k)^{3k} \\
= & e^{(2+o(1))k} e^{3k \log(k^2+2k)}.
\end{aligned}
$$

Hence we have $f^*(k,-1) = \exp(O(k \log k))$. $\square$

3.6.1 Remarks. By exhaustive search, we are able to find the exact value of $f^*(k,-1)$ for $k = 2$.

Proposition 3.6.2. *We have* $f^*(2,-1) = 36$.

Proof. We first show that Maker wins the $G^*([36], 2, -1)$ game. Consider the following solutions to $1/x_1 + 1/x_2 = 1/y$ in $\{1, 2, \ldots, 36\}$ with $x_1 \neq x_2$: $(x_1, x_2, y) = (4, 12, 3)$, $(6, 12, 4)$, $(12, 36, 9)$, and $(18, 36, 12)$.

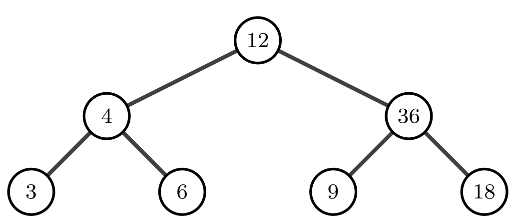

Figure 3.1: A rooted binary tree for some solutions to $1/x_1 + 1/x_2 = 1/y$.

In Figure 3.1, we constructed a rooted binary tree based on these solutions. Each path from the root 12 to a leaf is a solution set to $1/x_1 + 1/x_2 = 1/y$. It is easy to see that Maker can win this game by doing the following:

(1) Maker selects the root in round 1.

(2) In round 2, Maker selects a vertex that is adjacent to the root such that both of its children are untouched by Breaker.

(3) In round 3, Maker chooses a child of the vertex that Maker selected in round 2.

Now we show that Breaker wins the $G^*([35], 2, -1)$ game. By standard calculation, one can check that there are 13 solutions to $1/x_1 + 1/x_2 = 1/y$ in [35]: $\{2, 3, 6\}$, $\{3, 4, 12\}$, $\{4, 6, 12\}$, $\{4, 5, 20\}$, $\{5, 6, 30\}$, $\{6, 8, 24\}$, $\{6, 9, 18\}$, $\{6, 10, 15\}$, $\{8, 12, 24\}$, $\{10, 14, 35\}$, $\{10, 15, 30\}$, $\{12, 20, 30\}$, and $\{12, 21, 28\}$. Breaker wins the game using the pairing strategy over $\{4, 12\}$, $\{8, 24\}$, $\{10, 15\}$, $\{2, 3\}$, $\{5, 20\}$, $\{6, 30\}$, $\{9, 18\}$, $\{14, 35\}$, $\{20, 30\}$, and $\{21, 28\}$. $\square$

For general k, Theorem 3.1.4 only provides an upper bound for $f^*(k, -1)$. It is trivially true that $f^*(k, -1) \geq 2k + 1$ because Maker needs to occupy at least $k + 1$ numbers to win. However, we do not have a nontrivial lower bound.

Problem 3.6.3. Find a nontrivial lower bound for $f^*(k, -1)$.

3.7 Equations with arbitrary coefficients

In this section, we briefly discuss the Maker-Breaker Rado games for the equation

$$a_1 x_1 + \cdots + a_k x_k = y, \tag{3.4}$$

where $k, a_1, \ldots, a_k$ are positive integers with $k \geq 2$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. Write $w := a_1 + \cdots + a_k$, and $w^* := \sum_{i=1}^{k} (2i-1) a_i$. Let $f(a_1, \ldots, a_k; y)$ be the smallest positive integer

n such that Maker wins the $G([n], a_1x_1 + \cdots + a_kx_k = y)$ game and let $f^*(a_1, \ldots, a_k; y)$ be the smallest positive integer n such that Maker wins the $G^*([n], a_1x_1 + \cdots + a_kx_k = y)$ game.

Hopkins and Schaal [70], and Guo and Sun [57], proved that if $\{1, 2, \ldots, a_kw^2 + w - a_k\}$ is partitioned into two classes, then one of them contains a solution to Equation (3.4) with $x_1, \ldots, x_k$ not necessarily distinct; and there exists a partition of $\{1, 2, \ldots, a_kw^2 + w - a_k - 1\}$ into two classes such that neither contains a solution to Equation (3.4) with $x_1, \ldots, x_k$ not necessarily distinct. By these results and strategy stealing, we have $f(a_1, \ldots, a_k; y) \leq a_kw^2 + w - a_k$. The strategy stealing argument here is similar to the one in Section 3.1 where we explained that $f(k, \ell) \leq R(k, \ell)$ and $f^*(k, \ell) \leq R^*(k, \ell)$. The next theorem shows that, in fact, $f(a_1, \ldots, a_k; y)$ is much smaller than $a_kw^2 + w - a_k$.

Theorem 3.7.1. *For all integers $k \geq 2$, we have $w + 2a_k \leq f(a_1, \ldots, a_k; y) \leq w + a_{k-1} + a_k$.*

Proof. The case that $k = 2$ and $a_1 = a_2 = 1$ is a special case of Lemma 3.3.1. So we assume that $k > 2$ or $k = 2$ but $a_1 \geq 2$. Then $w > 2$.

We first show that Maker wins the $G([w + a_{k-1} + a_k], a_1x_1 + \cdots + a_kx_k = y)$ game. Maker chooses 1 in round 1. If Breaker does not choose w in round 1, then Maker wins in round 2 by choosing w. If Breaker chooses w in round 1, then Maker chooses 2 in round 2 and either $w + a_k$ or $w + a_{k-1} + a_k$ in round 3 to win the game.

Now we show that Breaker wins the $G([w + 2a_k - 1], a_1x_1 + \cdots + a_kx_k = y)$ game. The only solutions to Equation (3.4) in $\{1, 2, \ldots, w + 2a_k - 1\}$ are

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (1, 1, \ldots, 1, 1, w)$$

and

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (1, 1, \ldots, 1, 2, w + a_k).$$

Now Breaker wins by the pairing strategy over $\{1, w\}$ and $\{2, w+a_k\}$. Note that if $a_i = a_k$ for some $i \in \{1, 2, \ldots, k-1\}$, then

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k, y) = (1, \ldots, 1, 2, 1, \ldots, 1, w + a_1)$$

is also a solution, but Breaker can still win the game by the pairing strategy becuase $w+a_i = w + a_k$. □

The next theorem provides lower and upper bounds for $f^*(a_1, \ldots, a_k; y)$.

Theorem 3.7.2. *For all integers $k \geq 4$, we have*

$$w^* \leq f^*(a_1, \ldots, a_k; y) \leq w^* + (2k-2)(a_1 - a_k) + (k+3)a_{k-2}.$$

Proof. Let $k \geq 4$ be an integer and write $W = w^* + (2k-2)(a_1 - a_k) + (k+3)a_{k-2}$. We first show that Breaker wins the $G^*([w^*-1], a_1x_1 + \cdots + a_kx_k = y)$ game by choosing the smallest number available each round. Suppose, for a contradiction, that Maker wins. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, where $s \geq k+1$, be the numbers chosen by Maker after winning the game. Then by Breaker's strategy, we have $\alpha_i \geq 2i-1$ for all $i = 1, 2, \ldots, k$. By the rearrangement inequality [60], the smallest k-sum is

$$\sum_{i=1}^{k} a_i\alpha_i \geq \sum_{i=1}^{k}(2i-1)a_i = w^*$$

which is a contradiction.

Now we show that Maker wins the $G^*([W], a_1x_1 + \cdots + a_kx_k = y)$ game. We split it into two cases.

Case 1: $\alpha_1 = \alpha_k = c$ for some c. Since the coefficients of $x_1, \ldots, x_k$ are the same, Maker's strategy defined in the proof of Lemma 3.4.1 still applies by multiplying the k-

sums in the proof of Lemma 3.4.1 by c. So Maker wins the $G^*([ck^2+3c], a_1x_1+\cdots+a_kx_k=y)$ game. Since

$$W = w^* + (2k-2)(a_1-a_k) + (k+3)a_{k-2} = ck^2 + ck + 3c > ck^2 + 3c,$$

Maker wins the $G^*([W], a_1x_1+\cdots+a_kx_k=y)$ game.

Case 2: $a_1 > a_k$. We will show that Maker wins the game with the following strategy:

(1) Maker chooses the smallest number available each round for the first $k+1$ rounds;

(2) and then chooses an available k-sum in round $k+2$.

For $i = 1, 2, \ldots, k+1$, let m_i be the number chosen by Maker in round i. Then by Maker's strategy, we have $i \le m_i \le 2i-1$ for all $i = 1, 2, \ldots, k+1$.

Since $a_1 > a_k$, there exists $t \in \{2, 3, \ldots, k\}$ such that $\alpha_t < \alpha_{t-1}$. For $i = 1, \ldots, k+1$, let m_i be the number chosen by Maker in round i. By the rearrangement inequality, we have the following k distinct k-sums involving only $m_1, \ldots, m_k$:

$$(a_tm_{t+j} + a_{t+j}m_t) - (a_tm_t + a_{t+j}m_{t+j}) + \sum_{i=1}^{k} a_im_i, \text{ where } j = 0, 1, \ldots, k-t$$

and

$$(a_{t-j'}m_k + a_km_{t-j'}) - (a_{t-j'}m_{t-j'} + a_km_k) + \sum_{i=1}^{k} a_im_i, \text{ where } j' = 1, 2, \ldots, t-1.$$

Among these distinct k-sums, the smallest is $\sum_{i=1}^{k} a_im_i$ and the largest is

$$(a_1m_k + a_km_1) - (a_1m_1 + a_km_k) + \sum_{i=1}^{k} a_im_i = a_1m_k + \left(\sum_{i=2}^{k-1} a_im_i\right) + a_km_1.$$

Since $k \geq 4$, there are two terms of the form $a_i m_i$, $i \in \{2, \ldots, k-1\}$, in the middle of the right hand side of the equation above. Replacing m_{k-1} with m_{k+1} and replacing m_{k-2} with m_{k+1}, we get two larger and distinct k-sums:

$$a_1 m_k + \left(\sum_{i=2}^{k-2} a_i m_i \right) + a_{k-1} m_{k+1} + a_k m_1$$

and

$$a_1 m_k + \left(\sum_{i=2}^{k-3} a_i m_i \right) + a_{k-2} m_{k+1} + a_{k-1} m_{k-1} + a_k m_1.$$

The largest of these k-sums is

$$\begin{aligned}
& a_1 m_k + \left(\sum_{i=2}^{k-3} a_i m_i \right) + a_{k-2} m_{k+1} + a_{k-1} m_{k-1} + a_k m_1 \\
= & a_1 m_k + + a_{k-2} m_{k+1} + a_k m_1 - a_1 m_1 - a_{k-2} m_{k-2} - a_k m_k + \sum_{i=1}^{k} a_i m_i \\
= & (m_k - m_1)(a_1 - a_k) + a_{k-2}(m_{k+1} - m_{k-2}) + \sum_{i=1}^{k} a_i m_i \\
\leq & w^* + [(2k-1) - 1](a_1 - a_k) + [2k + 1 - (k-2)] a_{k-2} \\
= & w^* + (2k-2)(a_1 - a_k) + (k+3) a_{k-2} = W.
\end{aligned}$$

So there exists a k-sum unoccupied by Breaker in the beginning of round $k+2$ and hence Maker wins the $G^*([W], a_1 x_1 + \cdots + a_k x_k = y)$ game by choosing the available k-sum in round $k+2$. □

The bounds in Theorem 3.7.2 can be optimized using the technique in the proofs of Lemmas 3.4.1 and 3.4.2, but we do not attempt it here.

3.8 Concluding remarks

It would be interesting to study Rado games for other well-studied equations in arithmetic Ramsey theory. One direction is to study Rado games for

$$a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell}, \tag{3.5}$$

where $\ell, k, a_1, \ldots, a_k$ are positive integers with $k \geq 2$ and $\ell \neq 0$. Even though we studied the $G([n], a_1x_1 + \cdots + a_kx_k = y)$ and $G^*([n], a_1x_1 + \cdots + a_kx_k = y)$ games in Section 3.7, but how the fractional power $1/\ell$ interacts with the coefficients $a_1, \ldots, a_k$ is yet unknown.

Problem 3.8.1. What is the smallest integer n such that Maker wins the $G([n], a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell})$ game for $\ell \in \mathbb{Z}\backslash\{-1, 0, 1\}$? And what is the smallest integer n such that Maker wins the $G^*([n], a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell})$ game for $\ell \in \mathbb{Z}\backslash\{-1, 0, 1\}$?

Another direction is to study Rado games for the equation

$$x_1^\ell + \cdots + x_k^\ell = y^\ell, \tag{3.6}$$

where $\ell \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and $k \in \mathbb{N}\backslash\{1\}$. In 2016, Heule, Kullmann, and Marek [67] verified that if $\{1, 2, \ldots, 7825\}$ is partitioned into two classes, then one of them contains a solution to Equation (3.6) with $k = \ell = 2$ and that there exists a partition of $\{1, 2, \ldots, 7824\}$ into two classes so that neither contains a solution to Equation (3.6) with $k = \ell = 2$. It is easy to see that if $a_1, a_2, b \in \mathbb{N}$ with $a_1^2 + a_2^2 = b^2$, then $a_1 \neq a_2$. So the result of Heule, Kullmann, and Marek implies that Maker wins both the $G([7825], x_1^2 + x_2^2 = y^2)$ game and the $G^*([7825], x_1^2 + x_2^2 = y^2)$ game. It would be interesting to see if Maker can do better.

Problem 3.8.2. Does there exist $n < 7825$ such that Maker wins the $G^*([n], x_1^2 + x_2^2 = y^2)$ game?

The situation for Maker is more complicated when $\ell \geq 3$. By Fermat's last theorem [122], for all $n, \ell \in \mathbb{N}$ with $\ell \geq 3$, Breaker wins both the $G([n], x_1^\ell + x_2^\ell = y^\ell)$ game and the $G^*([n], x_1^\ell + x_2^\ell = y^\ell)$ for $\ell \geq 3$. By homogeneity, Breaker also wins the $G([n], x_1^\ell + x_2^\ell = y^\ell)$ game and the $G^*([n], x_1^\ell + x_2^\ell = y^\ell)$ game for all $n \in \mathbb{N}$ and $\ell \leq -3$. Hence, in order to study Rado games for Equation (3.6), one needs extra conditions on k and ℓ to make sure there are solutions to Equation (3.6) in $\mathbb{N}$. Recently, Chow, Lindqvist, and Prendiville [27] proved that, for all $\ell \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, if we partition of $\mathbb{N}$ into two classes, then one of them contains a solution to Equation (3.6) with $x_1, \ldots, x_k$ not necessarily distinct. By the result of Brown and Rödl [24] described in Section 3.1, the same result holds for $\ell \in \{-1, -2, \ldots\}$ as well. For example, if $|\ell| = 2$, then $k = 4$ suffices; and if $|\ell| = 3$, then $k = 7$ is enough.

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Chapter 4: Enumerating pattern-avoiding permutations by leading terms

4.1 Introduction

For $r, n \in \mathbb{N}$ with $r \leq n$, let $S_{n,r}$ denote the set of permutations $\tau \in S_n$ with $\tau(1) = r$. It is clear that $|S_{n,r}| = (n-1)!$ for all $r \in [n]$. While studying for the shapes of pattern-avoiding permutations, Miner and Pak [88] proved that $S_{n,r}(123)$ and $S_{n,r}(132)$ are enumerated by the ballot numbers (see Aval [7] for more details on the ballot numbers).

Theorem 4.1.1. [88, Lemmas 4.1 and 5.2] *For all* $1 \leq r \leq n$*, we have*

$$|S_{n,r}(123)| = |S_{n,r}(132)| = b_{n,r} = \frac{n-r+1}{n+r-1}\binom{n+r-1}{n}.$$

Miner and Pak [88] proved Theorem 4.1.1 via a bijection between $S_{n,r}(123)$ (respectively, $S_{n,r}(132)$) and certain types of Dyck paths. These bijections are based on the Robinson–Schensted–Knuth (RSK) correspondence. In this paper, we prove Theorem 4.1.1 using instead a direct counting argument. By the classical bijection between $S_n(123)$ and $S_n(132)$ [18, Lemma 4.4] which preserves the leading term, one only needs to prove that $|S_{n,r}(123)| = b_{n,r}$. We achieve this by utitlizing a result of Simion and Schmidt [113, Lemma 2] on the number of 123-avoiding permutations with a fixed initial decreasing streak.

It is natural to consider the case in which more than one leading term of the permutation is fixed. Motivated by this general case, we study pattern-avoiding permutations with a fixed prefix $(c_1, c_2, \ldots, c_t)$ of length $t \geq 1$. Here the $t = 1$ instance corresponds to the case studied by Miner and Pak.

Definition 4.1.2. For any $n, m \in \mathbb{N}$, distinct integers $c_1, c_2, \ldots, c_t \in [n]$, and permutation patterns $\sigma_1, \sigma_2, \ldots, \sigma_m$, we use $S_{n,(c_1,c_2,\ldots,c_t)}$ to denote the set of permutations $\tau \in S_n$ such that $(\tau(1), \tau(2), \ldots, \tau(t)) = (c_1, c_2, \ldots, c_t)$; and we use $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ to denote the set of permutations $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$ such that τ avoids all the patterns $\sigma_1, \sigma_2, \ldots, \sigma_m$ simultaneously.

Convention 4.1.3. Unless otherwise specified, for $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma_1, \sigma_2, \ldots, \sigma_m)$, we assume that the fixed prefix $(c_1, c_2, \ldots, c_t)$ itself avoids all of the patterns $\sigma_1, \sigma_2, \ldots, \sigma_m$, $n \geq 3$, and $1 \leq t < n$.

We first show that the size of $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)$ can be determined exactly for all $\sigma \in S_3$. For $\sigma \in \{123, 132, 321, 312\}$, if $|S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)| \neq 0$, then $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)$ is enumerated by ballot numbers. This is because there is a natural bijection between $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)$ and $S_{n-t+1,r}(\sigma)$ for some r related to $\{c_1, c_2, \ldots, c_t\}$.

For $\sigma \in \{213, 231\}$, if $|S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)| \neq 0$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)|$ is equal to a product of Catalan numbers. This is because, for all $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)$, the order statistics of $c_1, c_2, \cdots, c_t$ determine a "block" structure for $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$.

We then show that for all pairs of patterns of length three, the number of permutations avoiding these patterns and with a fixed prefix can also be determined exactly. The expressions for pairs of patterns of length three depend on the extrema, the order statistics, and the length of the prefix. We also enumerate $S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$. Similar to $S_{n,(c_1,c_2,\ldots,c_t)}(231)$, if $|S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)| \neq 0$, then for all permutation $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$, $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ has a "block" structure. However, since the patterns 3412 and 3421 are of length four, some overlap between the "blocks" are possible. The size of $S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$ is related to the large Schröder numbers which is then used to obtain a new combinatorial proof of a recurrence relation for large Schröder numbers $\mathbb{S}_n$ [18, p. 446].

In addition to exact enumeration, the notion of Wilf equivalence classes are also an important topic in pattern avoidance. Two permutation patterns σ and σ' are said to be Wilf equivalent, denoted $\sigma \sim \sigma'$, if $|S_n(\sigma)| = |S_n(\sigma')|$ for all $n \in \mathbb{N}$. By Theorem 1.3.1, all permutation patterns of length three are Wilf equivalent: $123 \sim 132 \sim 213 \sim 231 \sim 312 \sim 321$. In other words, there is only one Wilf-equivalence class for permutation patterns of length three. For patterns of length four, it is known that there are three Wilf-equivalence classes [18, p. 158].

We consider a similar Wilf-equivalence concept for permutations with a fixed leading term. For a fixed $r \in \mathbb{N}$, two patterns σ and σ' are called r-*Wilf equivalent* if $|S_{n,r}(\sigma)| = |S_{n,r}(\sigma')|$ for all $n \geq r$. We write $\sigma \overset{r}{\sim} \sigma'$ if σ and σ' are r-Wilf equivalent. As an example, two patterns σ and σ' are 2-Wilf equivalent, denoted $\sigma \overset{2}{\sim} \sigma'$, if for all $n \geq 2$, $|S_{n,2}(\sigma)| = |S_{n,2}(\sigma')|$. We show that there are two 1-Wilf-equivalence classes for patterns of length three, $123 \overset{1}{\sim} 132$ and $321 \overset{1}{\sim} 312 \overset{1}{\sim} 213 \overset{1}{\sim} 231$; and, for all $r \geq 2$, there are three r-Wilf-equivalence classes for patterns of length three, $213 \overset{r}{\sim} 231$, $123 \overset{r}{\sim} 132$, and $321 \overset{r}{\sim} 312$. We also show that for all $r \geq 5$, there are nine r-Wilf-equivalence classes for vincular patterns of length three as studied in [8, 29].

This chapter is organized as follows. In Section 4.2, we define basic concepts and state our preliminary results. We then provide a new proof of Theorem 4.1.1 in Section 4.3. In Section 4.4, we present results on the number of permutations with a fixed prefix that avoid a single pattern of length three. The case for the avoidance of pairs of patterns of length three is presented in Section 4.5. Permutations avoiding both 3412 and 3421 are then studied in Section 4.6. In Section 4.7, we classify r-Wilf-equivalence classes for classical and vincular patterns of length three.

4.2 Preliminaries

Definition 4.2.1. For a permutation $\tau \in S_n$, the *complement* τ^c of τ is the permutation in S_n defined by setting $\tau^c(i) = n + 1 - \tau(i)$.

The following result relates permutations avoiding certain patterns with permutations avoiding the complement of these patterns. Since the proof is elementary, we state it without proof.

Lemma 4.2.2. *Let t, n, m, and k be positive integers with $t, k \leq n$, $\sigma_1, \sigma_2, \ldots, \sigma_m \in S_k$ permutation patterns, and $c_1, c_2, \ldots, c_t \in [n]$. Then we have*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(\sigma_1, \sigma_2, \ldots, \sigma_m)| = |S_{n,(n+1-c_1,n+1-c_2,\ldots,n+1-c_t)}(\sigma_1^c, \sigma_2^c \ldots, \sigma_m^c)|.$$

Definition 4.2.3. Let A and B be two finite subsets of $\mathbb{N}$ with $A \subseteq B$, $\sigma \in S_A$, and $\tau \in S_B$. We say that σ is a *subpermutation* of τ on A if there exist indices $1 \leq i_1 < i_2 < \cdots < i_{|A|} \leq |B|$ such that

$$(\tau(i_1), \tau(i_2), \ldots, \tau(i_{|A|})) = (\sigma(1), \sigma(2), \ldots, \sigma(|A|)).$$

For example, if $\tau = 543621 \in S_6$, then $\sigma = 462 \in S_{\{2,4,6\}}$ is a subpermutation of τ on $\{2, 4, 6\}$.

Definition 4.2.4. Suppose σ is a permutation on a set A and τ is a permutation on a set B with $A \cap B = \emptyset$. A *shuffle* of σ and τ is a permutation α on $A \cup B$ such that σ is a subpermutation of α on A and τ is a subpermutation of α on B.

For example, if $A = \{4, 5, 7\}$, $B = \{1, 3, 6\}$, $\sigma = 457 \in S_A$, and $\tau = 631 \in S_B$, then $\alpha = 643571 \in S_{A\cup B}$ and $\alpha' = 456317 \in S_{A\cup B}$ are shuffles of σ and τ. The following simple observation is crucial for our later derivations. We state it without proof.

Lemma 4.2.5. *Suppose $A, B \subseteq \mathbb{N}$ with $|A \cap B| = \emptyset$, $|A| = k$, and $|B| = \ell$. If $\sigma \in S_A$ and $\tau \in S_B$, then the number of shuffles of σ and τ is $\binom{k+\ell}{k}$.*

We will use the following terminology.

Definition 4.2.6. Let $A \subseteq \mathbb{N}$ be a finite set and $\tau \in S_A$. If $\tau \in S_A$ and $\tau(i) = a$, then we use $\mathcal{A}_\tau(a) = \{\tau(1), \tau(2), \ldots, \tau(i-1)\}$ to denote the set of *ancestors* of a in τ and $\mathcal{D}_\tau(a) = \{\tau(i+1), \tau(i+2), \ldots, \tau(|A|)\}$ to denote the set of *descendants* of a in τ.

For example, if $\tau = 2785 \in S_{\{2,5,7,8\}}$, then $\mathcal{A}_\tau(8) = \{2, 7\}$ and $\mathcal{D}_\tau(7) = \{5, 8\}$.

Definition 4.2.7. Let $n \in \mathbb{N}$ and $A \subseteq [n]$. The *standardization* of

$$\tau = (\tau(1), \tau(2), \ldots, \tau(|A|)) \in S_A$$

is the permutation $s(\tau) \in S_{|A|}$ obtained by replacing the ith smallest entry in τ with i for all i.

For example, the standardization of $567832 \in S_{\{2,3,5,6,7,8\}}$ is $345621 \in S_6$. We include below a simple observation concerning standardization.

Lemma 4.2.8. *Let $n \in \mathbb{N}$, $A \subseteq [n]$, and $\tau \in S_A$. If τ avoids a pattern σ, then $s(\tau)$ also avoids the pattern σ.*

Definition 4.2.9. Let $n, n' \in \mathbb{N}$ with $n \leq n'$, $A \subseteq [n']$ with $|A| = n$, and $\tau \in S_n$. Then the *matching permutation* τ' of τ on A is defined as follows: if $\tau(i) = j$ where $i, j \in \{1, 2, \ldots, n\}$, then $\tau'(i)$ is the jth smallest integer in A.

For example, the matching permutation of $231 \in S_3$ on $\{2, 4, 7\}$ is $472 \in S_{\{2,4,7\}}$.

Notice that the matching permutation of a permutation also preserves pattern avoidance.

The first few Catalan numbers C_n, Bell numbers B_n, and large Schröder numbers $\mathbb{S}_n$ are listed in Table 4.1 for later reference.

n	0	1	2	3	4	5	6	7	8	9	OEIS [94]
C_n	1	1	2	5	14	42	132	429	1430	4862	A000108
B_n	1	1	2	5	15	52	203	877	4140	21147	A000110
$\mathbb{S}_n$	1	2	6	22	90	394	1806	8558	41586	206098	A006318

Table 4.1: C_n, B_n, and $\mathbb{S}_n$ for $n \leq 10$.

We also need the following elementary results on the Catalan and the Bell numbers.

Lemma 4.2.10. *For all $n \geq 4$, we have $C_n < B_n$.*

Proof. It is well-known that C_n counts the number of noncrossing partitions of $[n]$ and B_n counts the total number of partitions of $[n]$. For these facts and the definitions of partitions and noncrossing partitions, see for example [86, Section 1.1] and [112]. For $n \geq 4$, there is at least one crossing partition of $[n]$ and therefore $C_n < B_n$. □

Lemma 4.2.11. *For all $n \geq 3$, we have $B_n > 2B_{n-1}$.*

Proof. The Bell numbers B_n satisfy the following recurrence relation [36, p. 49]:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Let $n \geq 3$. Then we have

$$\begin{aligned} B_n &= B_{n-1} + \binom{n-1}{n-2} B_{n-2} + \binom{n-1}{n-3} B_{n-3} + \cdots + \binom{n-1}{0} B_0 \\ &> B_{n-1} + \binom{n-2}{n-2} B_{n-2} + \binom{n-2}{n-3} B_{n-3} + \cdots + \binom{n-2}{0} B_0 \\ &= B_{n-1} + \sum_{k=0}^{n-2} \binom{n-2}{k} B_k = 2B_{n-1}. \end{aligned}$$

□

4.3 A new proof of Theorem 4.1.1

Let $1 \leq r \leq n$. The well-known bijection between $S_n(123)$ and $S_n(132)$ by Simion and Schmidt [113] preserves the leading term. We refer to [18, Lemma 4.4] for a proof of this fact. Hence, we have $|S_{n,r}(123)| = |S_{n,r}(132)|$. So we only need to show that $|S_{n,r}(123)| = b_{n,r}$.

First, consider $r = 1$. If $\tau \in S_{n,1}(123)$, then $(\tau(2), \tau(3), \ldots, \tau(n))$ must be a decreasing sequence and hence $\tau = (1, n, n-1, \ldots, 2)$. Therefore, we have $|S_{n,1}(123)| = 1 = b_{n,1}$.

Now suppose $r \geq 2$. We will need the following definition.

Definition 4.3.1. For all $n \in \mathbb{N}$, define $a_n(i) : [n] \to [n]$ as follows:

(i) for all $i < n$, let $a_n(i)$ be the number of permutations $\tau \in S_n(123)$ such that i is the smallest index with $\tau(i) < \tau(i+1)$, and

(ii) set $a_n(n) = 1$.

That is, if $i < n$, then $a_n(i)$ is the number of permutations $\tau \in S_n$ avoiding 123 such that $\tau(1) > \tau(2) > \cdots > \tau(i-1) > \tau(i) < \tau(i+1)$. If $i = n$, then $a_n(n) = 1$ because there is exactly one decreasing sequence $(n, n-1, \ldots, 2, 1) \in S_n$. Simion and Schmidt [113, Lemma 2] proved the following result for $a_n(i)$:

Lemma 4.3.2. *For all* $1 \leq i \leq n$*,*

$$a_n(i) = \binom{2n-i-1}{n-1} - \binom{2n-i-1}{n}.$$

Let $\mathcal{P}$ be a subset of $S_{n,r}$ such that every $\tau \in \mathcal{P}$ has the following properties:

(i) the subpermutation τ' of τ on $\{1, 2, \ldots, r-1\}$ avoids 123;

(ii) the subpermutation τ'' of τ on $\{r+1, r+2, \ldots, n\}$ is $(n, n-1, \ldots, r+1)$;

(iii) if $r > 2$ and $i < r-1$ is the smallest index with $\tau'(1) > \tau'(2) > \cdots > \tau'(i) < \tau'(i+1)$, then $\{r+1, r+2, \ldots, n\} \subseteq \mathcal{A}_\tau(\tau'(i+1))$.

For Property (iii), we do not impose any extra condition on the positions of $\{r+1, r+2, \ldots, n\}$ when $\tau' = (r-1, r-2, \ldots, 1)$.

We first show that

$$|\mathcal{P}| = \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i).$$

Let $\tau \in \mathcal{P}$. If $r > 2$, then for all $i \in \{1, 2, \ldots, r-2\}$ and a fixed τ' with $\tau'(1) > \tau'(2) > \cdots > \tau'(i) < \tau'(i+1)$, the number of shuffles of $(\tau'(1), \tau'(2), \ldots, \tau'(i))$ and $\tau'' = (n, n-1, \ldots, r+1)$ is $\binom{i+n-r}{i}$. If $\tau' = (r-1, r-2, \ldots, 1)$, then the number of shuffles of τ' and τ'' is $\binom{r-1+n-r}{r-1}$. As the number of such τ' is $a_{r-1}(i)$ for all $i \in \{1, 2, \ldots, r-1\}$, we have $|\mathcal{P}| = \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i)$.

It remains to show that $S_{n,r}(123) = \mathcal{P}$. Let $\tau \in S_{n,r}(123)$, and let τ' be the subpermutation of τ on $\{1, 2, \ldots, r-1\}$ and τ'' be the subpermutation of τ on $\{r+1, r+2, \ldots, n\}$. Since τ avoids 123, τ' avoids 123 as well. We now show that τ'' avoids 12. If this is not the case then there exist $a < b \leq n-r$ such that $\tau''(a) < \tau''(b)$. Since $\tau''(a) > r$, $r\tau''(a)\tau''(b)$ is a 123 pattern, and this is a contradiction. Therefore $\tau'' = (n, n-1, \ldots, r+1)$. Now suppose $r > 2$ and let $i < r-1$ be the smallest index such that $\tau'(1) > \tau'(2) > \cdots > \tau'(i) < \tau'(i+1)$. We still need to show that $\{r+1, r+2, \ldots, n\} \subseteq \mathcal{A}_\tau(\tau'(i+1))$. Suppose, by way of contradiction, $\{r+1, r+2, \ldots, n\} \not\subseteq \mathcal{A}_\tau(\tau'(i+1))$. Then there exists $a \in \{r+1, r+2, \ldots, n\} \cap \mathcal{D}_\tau(\tau'(i+1))$. Since $\tau'(i)\tau'(i+1)a$ is a 123 pattern, this would be a contradiction. Hence, we have $S_{n,r}(123) \subseteq \mathcal{P}$.

Now let $\tau \in \mathcal{P}$. We need to show that $\tau \in S_{n,r}(123)$. Let τ' be the subpermutation of τ on $\{1, 2, \ldots, r-1\}$ and τ'' be the subpermutation of τ on $\{r+1, r+2, \ldots, n\}$. If $r > 2$, then let $i < r-1$ be the smallest index such that $\tau'(1) > \tau'(2) > \cdots > \tau'(i) < \tau'(i+1)$; if $\tau' = (r-1, r-2, \ldots, 1)$, then we set $i = r-1$. Since $\tau \in \mathcal{P}$, $\tau = (r, \alpha(1), \alpha(2), \ldots, \alpha(n-$

$r+i), \tau'(i+1), \tau'(i+2), \ldots, \tau'(r-1))$ where $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n-r+i))$ is a shuffle of $(\tau'(1), \tau'(2), \ldots, \tau'(i))$ and $\tau'' = (n, n-1, \ldots, r+1)$. We need to show that any subpermutation abc of τ is not a 123 pattern. Suppose, by way of contradiction, there exists a subpermutation abc of τ which is a 123 pattern. Then $a < b < c$. We split into two cases:

Case 1: $a \geq r$. Then $c > b > r$. It follows that bc is an increasing subpermutation of τ''. But this contradicts the fact that $\tau'' = (n, n-1, \ldots, r+1)$.

Case 2: $a < r$. Since τ' avoids 123, either $b > r$ or $c > r$. If $b > r$, then $c > b > r$. Using a similar argument as in Case 1, we have a contradiction. So we suppose $b < r$ and $c > r$. Here $b \neq r$ because $b \in D_\tau(a)$. Since ab is an increasing subpermutation of τ', we have $b = \tau'(j)$ for some $j \geq i+1$. Hence $c \in \mathcal{A}_\tau(\tau'(i+1)) \subseteq \mathcal{A}_\tau(b)$ which is again a contradiction.

This proves that $\mathcal{P} \subseteq S_{n,r}(123)$, and, consequently, $S_{n,r}(123) = \mathcal{P}$. Therefore, by Lemma 4.3.2,

$$
\begin{aligned}
|S_{n,r}(123)| =& |\mathcal{P}| = \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i) \\
=& \sum_{i=1}^{r-1} \binom{i+n-r}{i} \left[\binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} \right] \\
=& \frac{n-r+1}{n+r-1} \binom{n+r-1}{n} = b_{n,r}.
\end{aligned}
$$

4.4 Single patterns of length three

In this section, we enumerate $S_{n,(c_1,c_2,\ldots,c_t)}(\sigma)$ for $\sigma \in S_3$. By Lemma 4.2.2, it suffices to enumerate permutations avoiding the patterns 123, 132, and 231. We start with the pattern 231. The key features about $S_{n,(c_1,c_2,\ldots,c_t)}(231)$ are that the enumeration is related to the order statistics of $\{c_1, c_2, \ldots, c_t\}$ and there is a 'block' structure for all $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(231)$.

Theorem 4.4.1. *If $c_i < c_j$ for some $1 \le i < j \le t$ and there exists $\alpha < c_i$ such that $\alpha \notin \{c_1, c_2, \ldots, c_t\}$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(231)| = 0$; otherwise, we have*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(231)| = \prod_{k=1}^{t+1} C_{c_{(k)}-c_{(k-1)}-1}, \tag{4.1}$$

where C_i is the ith Catalan number, $c_{(0)} = 0$, $c_{(t+1)} = n+1$, and $c_{(1)} < c_{(2)} < \cdots < c_{(t)}$ are the order statistics of $\{c_1, c_2, \ldots, c_t\}$.

Proof. If $c_i < c_j$ for some $1 \le i < j \le t$ and there exists $\alpha < c_i$ such that $\alpha \notin \{c_1, c_2, \ldots, c_t\}$, then $c_i c_j \alpha$ is a 231 pattern. Therefore, $|S_{n,(c_1,c_2,\ldots,c_t)}(231)| = 0$.

Now suppose otherwise. We will build a set $\mathcal{Q}$ whose cardinality is given by the right hand side of (4.1) and then show that $\mathcal{Q} = S_{n,(c_1,c_2,\ldots,c_t)}(231)$. Let $\mathcal{Q}$ be the subset of $S_{n,(c_1,c_2,\ldots,c_t)}$ such that every $\tau \in \mathcal{Q}$ has the following properties:

(i) for all $k, \ell \in [t+1]$ with $k < \ell$, if $x \in \{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$ and $y \in \{c_{(\ell-1)}+1, c_{(\ell-1)}+2, \ldots, c_{(\ell)}-1\}$, then $x \in \mathcal{A}_\tau(y)$, and

(ii) for all $k \in [t+1]$, the subpermutation on $\{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$ avoids 231.

That is, for each $\tau \in \mathcal{Q}$, $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is the concatenation of $t+1$ (some possibly empty) 231-avoiding permutations, which we call 231-avoiding blocks. For all $k \in [t+1]$, the kth block is a 231-avoiding permutation on $\{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$. Since for all $k \in [t+1]$, the size of $\{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$ is $c_{(k)} - c_{(k-1)} - 1$, the number of 231-avoiding permutations on the set $\{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$ is $C_{c_{(k)}-c_{(k-1)}-1}$. Hence we have $|\mathcal{Q}| = \prod_{k=1}^{t+1} C_{c_{(k)}-c_{(k-1)}-1}$.

Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(231)$. Then every subpermutation of τ avoids 231. So, τ satisfies Property (ii). Now we show that τ satisfies Property (i). Suppose not. Then there exist

$k, \ell \in [t]$ with $k < \ell$, $x \in \left\{c_{(k-1)} + 1, c_{(k-1)} + 2, \ldots, c_{(k)} - 1\right\}$, and

$$y \in \left\{c_{(\ell-1)} + 1, c_{(\ell-1)} + 2, \ldots, c_{(\ell)} - 1\right\},$$

such that $x \in \mathcal{D}_\tau(y)$. It follows that $c_{(\ell-1)}yx$ is a 231 pattern, which is a contradiction. Hence $\tau \in \mathcal{Q}$. This proves that $S_{n,(c_1,c_2,\ldots,c_t)}(231) \subseteq \mathcal{Q}$.

Now let $\tau \in \mathcal{Q}$. To show that $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(231)$, it suffices to show that any subpermutation abc of τ is not a 231 pattern. Suppose, by way of contradiction, that a subpermutation abc of τ is a 231 pattern. Then we must have $c < a < b$ and $c \in D_\tau(b)$. We split into four cases:

Case 1: $a, b, c \notin \{c_1, c_2, \ldots, c_t\}$. Since $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is a concatenation of $t + 1$ (some possibly empty) 231-avoiding blocks, c must be in a block after the block a is in. Since $c < a$, this contradicts Property (i).

Case 2: $a \in \{c_1, c_2, \ldots, c_t\}$ and $b, c \notin \{c_1, c_2, \ldots, c_t\}$. Then we have $a = c_{(k)}$ for some $k \in [t]$. Since $c < a < b$, b and c must be in two different blocks. Since $c \in D_\tau(b)$, c is in a block after the block b is in which contradicts Property (i).

Case 3: $a, b \in \{c_1, c_2, \ldots, c_t\}$ and $c \notin \{c_1, c_2, \ldots, c_t\}$. Then $a = c_i$ and $b = c_j$ with $1 \leq i < j \leq t$, and $c < a$. This is a contradiction.

Case 4: $a, b, c \in \{c_1, c_2, \ldots, c_t\}$. Then abc is a subpermutation of $(c_1, c_2, \ldots, c_t)$ which contradicts our convention that $(c_1, c_2, \ldots, c_t)$ avoids 231.

Hence, we have $\mathcal{Q} \subseteq S_{n,(c_1,c_2,\ldots,c_t)}(231)$. □

Remark 4.4.2. For all $n \geq 3$, by Theorem 1.3.1 and Theorem 4.4.1 with $t = 1$, we have

$$C_n = |S_n(231)| = \sum_{r=1}^{n} |S_{n,r}(231)| = \sum_{r=1}^{n} C_{n-r}C_{r-1}.$$

This offers an alternative interpretation for the well-known recurrence relation for the Catalan numbers, see [36, Section 3.2] and [114, Section 1.2].

In contrast to the 231 pattern, the expressions for the 123 and 132 patterns are related to the minimum of $\{c_1, c_2, \ldots, c_t\}$. Recall that for a subset $A \subseteq [n]$, the *standardization* of a permutation $\tau = (\tau(1), \tau(2), \ldots, \tau(|A|))$ is the permutation $s(\tau) \in S_{|A|}$ obtained by replacing the ith smallest entry in τ with i for all i. We will need the following result:

Lemma 4.4.3. *Suppose $A \subseteq [n]$ such that there exists $r \in [n]$ with $[r] \subseteq A$, and $\sigma \in S_k$ with $k \leq |A|$. Then $s(S_{A,r}(\sigma)) = S_{|A|,r}(\sigma)$, where $s(S_{A,r}(\sigma)) = \{s(\tau) : \tau \in S_{A,r}(\sigma)\}$.*

Proof. Let $\tau \in S_{A,r}(\sigma)$. Since $[r] \subseteq A$, $\tau(1)$ is the rth smallest number in τ. So $s(\tau) \in S_{|A|,r}$. By Lemma 4.2.8, we have $s(S_{A,r}(\sigma)) \subseteq S_{|A|,r}(\sigma)$.

Now let $\tau \in S_{|A|,r}(\sigma)$. Let $a_1 < a_2 < \cdots < a_{|A|}$ be the elements in A. Since $[r] \subseteq A$, we have $a_r = r$. Let τ' be the matching permutation of τ on A. Since $\tau(1) = r$, we have $\tau'(1) = a_r = r$. It is not hard to see that τ' avoids σ. Hence $\tau' \in S_{A,r}(\sigma)$. By our construction, we also have $s(\tau') = \tau$. So $\tau \in s(S_{A,r}(\sigma))$. Hence, $S_{|A|,r}(\sigma) \subseteq s(S_{A,r}(\sigma))$. $\square$

Theorem 4.4.4. *If $c_i < c_j$ for some $1 \leq i < j \leq t$ and there exists $\alpha > c_j$ such that $\alpha \notin \{c_1, c_2, \ldots, c_t\}$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(123)| = 0$; otherwise, we have*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123)| = |S_{n-t+1,\min\{c_1,c_2,\ldots,c_t\}}(123)| = b_{n-t+1,\min\{c_1,c_2,\ldots,c_t\}}.$$

Proof. If $c_i < c_j$ for some $i < j$ and there exists $\alpha > c_j$ such that $\alpha \notin \{c_1, c_2, \ldots, c_t\}$, then $c_i c_j \alpha$ is a 123 pattern. Therefore, $|S_{n,(c_1,c_2,\ldots,c_t)}(123)| = 0$.

Now suppose otherwise. For simplicity, we write $x = \min\{c_1, c_2, \ldots, c_t\}$ and

$$A = ([n] \backslash \{c_1, c_2, \ldots, c_t\}) \cup \{x\}.$$

Consider the map

$$f : S_{n,(c_1,c_2,\ldots,c_t)}(123) \to S_{A,x}(123)$$

such that for all $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123)$,

$$f(\tau) = (x, \tau(t+1), \tau(t+2), \ldots, \tau(n)).$$

This is well defined since $(x, \tau(t+1), \tau(t+2), \ldots, \tau(n))$ is a subpermutation of τ. If $\tau, \tau' \in S_{n,(c_1,c_2,\ldots,c_t)}(123)$ with $\tau \neq \tau'$, then $\tau(i) \neq \tau'(i)$ for some $i \in \{t+1, t+2, \ldots, n\}$ and hence $(x, \tau(t+1), \tau(t+2), \ldots, \tau(n)) \neq (x, \tau'(t+1), \tau'(t+2), \ldots, \tau'(n))$. So f is injective. f is also surjective because for all $\tau = (x, \tau(2), \tau(3), \ldots, \tau(|A|)) \in S_{A,x}(123)$, we have

$$\tau' = (c_1, c_2, \ldots, c_t, \tau(2), \tau(3), \ldots, \tau(|A|)) \in S_{n,(c_1,c_2,\ldots,c_t)}(123)$$

and $f(\tau') = \tau$. Therefore, f is a bijection and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123)| = |S_{A,x}(123)|$.

Since $x = \min\{c_1, c_2, \ldots, c_t\}$, we have $[x] \subseteq A$.

Now by Lemma 4.4.3, we have $|S_{n,(c_1,c_2,\ldots,c_t)}(123)| = |S_{A,x}(123)| = |s(S_{A,x}(123))| = |S_{|A|,x}(123)| = |S_{n-t+1,x}(123)|$. □

The result for the 132 pattern is similar to the 123 pattern and hence we state the following result without proof.

Theorem 4.4.5. *If $c_i < c_j$ for some $i < j$ and there exists α such that $c_i < \alpha < c_j$ and $\alpha \notin \{c_1, c_2, \ldots, c_t\}$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(132)| = 0$. Otherwise, we have*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(132)| = |S_{n-t+1,\min\{c_1,c_2,\ldots,c_t\}}(132)| = b_{n-t+1,\min\{c_1,c_2,\ldots,c_t\}}.$$

4.5 Pairs of patterns of length three

In this section, we enumerate permutations with fixed prefix $(c_1, c_2, \ldots, c_t)$ which avoid a pair of patterns $\{\sigma_1, \sigma_2\}$ of length three. Recall that we use $S_n(\sigma_1, \sigma_2)$ to denote the set of permutations $\tau \in S_n$ such that τ avoids both σ_1 and σ_2. We need the following results by Simion and Schmidt [113, Section 3]:

Theorem 4.5.1. [113, Section 3] *For all* $n \geq 1$*,*

$$\begin{aligned} |S_n(123,132)| =& |S_n(321,312)| = |S_n(123,213)| = |S_n(321,231)| = |S_n(132,213)| \\ =& |S_n(312,231)| = |S_n(132,231)| = |S_n(312,213)| = |S_n(132,312)| \\ =& |S_n(213,231)| = 2^{n-1}, \end{aligned}$$

$$|S_n(123,312)| = |S_n(321,132)| = |S_n(123,231)| = |S_n(321,213)| = \binom{n}{2} + 1,$$

and

$$|S_n(123,321)| = \begin{cases} 0 & \textit{if} \;\; n \geq 5, \\ n & \textit{if} \;\; n = 1 \textit{ or } n = 2, \\ 4 & \textit{if} \;\; n = 3 \textit{ or } n = 4. \end{cases}$$

Out of the 15 pairs of patterns of length 3, there are three self-complementary pairs: $\{123, 321\}$, $\{132, 312\}$, and $\{213, 231\}$. That is,

$$\{123^c, 321^c\} = \{123, 321\},\ \{132^c, 312^c\} = \{132, 312\},\ \textit{and}\ \{213^c, 231^c\} = \{213, 231\}.$$

By the Erdős-Szekeres theorem [44, p. 467], for $n \geq 5$, every $\tau \in S_n$ has either an increasing or a decreasing subpermutation of length three. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 321)| = 0$ if $n \geq 5$. Since one could routinely calculate $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 321)|$ when $n \leq 4$, we do not include the exact results for the pair $\{123, 321\}$ here. We start with $\{132, 312\}$ and $\{213, 231\}$.

Theorem 4.5.2. *If $\{c_1, c_2, \ldots, c_t\}$ is a set of consecutive integers, then*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)| = \binom{n-t}{\min\{c_1, c_2, \ldots, c_t\} - 1};$$

otherwise, $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)| = 0$.

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$ and $A = [n] \backslash \{c_1, c_2, \ldots, c_t\}$.

Suppose that $\{c_1, c_2, \ldots, c_t\}$ is a set of consecutive integers. Then we have

$$\{c_1, c_2, \ldots, c_t\} = \{x, x + 1, \ldots, x + t - 1\}.$$

Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)$. Since τ avoids 132, the subpermutation on $\{x + t, x + t + 1, \ldots, n\}$, if $x + t - 1 \neq n$, is $(x + t, x + t + 1, \ldots, n)$; and since τ avoids 312, the subpermutation on $\{1, 2, \ldots, x - 1\}$, if $x \neq 1$, is $(x - 1, x - 2, \ldots, 1)$. The number of shuffles of $(x + t, x + t + 1, \ldots, n)$ and $(x - 1, x - 2, \ldots, 1)$ is $\binom{n-t}{x-1}$. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)| \leq \binom{n-t}{x-1}$. Now let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$ such that $(\tau(t + 1), \tau(t + 2), \ldots, \tau(n))$ is a shuffle of $(x + t, x + t + 1, \ldots, n)$ and $(x - 1, x - 2, \ldots, 1)$. It is easy to check that $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)$. Hence we have $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)| = \binom{n-t}{x-1}$.

Now suppose that $\{c_1, c_2, \ldots, c_t\}$ is not a set of consecutive integers. Then there exists $y \in A$ and $i, j \in [t]$ such that $i < j$, and $c_i < y < c_j$ or $c_j < y < c_i$. Then $c_i c_j y$ is either a 132 pattern or a 312 pattern. Hence, we have $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 312)| = 0$. □

Theorem 4.5.3. *If $(c_1, c_2, \ldots, c_t)$ is a shuffle of $(1, 2, \ldots, t-s)$ and $(n, n-1, \ldots, n-s+1)$ for some $s \in \{0, 1, \ldots, t\}$, then*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = 2^{n-t-1};$$

otherwise, $|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = 0$.

Note that, in Theorem 4.5.3, when $s = 0$, we mean that $(c_1, c_2, \ldots, c_t) = (1, 2, \ldots, t)$; and when $s = t$, we mean that $(c_1, c_2, \ldots, c_t) = (n, n-1, \ldots, n-t+1)$.

Proof. Write $A = [n] \backslash \{c_1, c_2, \ldots, c_t\}$.

Suppose $(c_1, c_2, \ldots, c_t)$ is a shuffle of $(1, 2, \ldots, t-s)$ and $(n, n-1, \ldots, n-s+1)$ for some $s \in \{0, 1, \ldots, t\}$. Consider the map $f : S_{n,(c_1,c_2,\ldots,c_t)}(213, 231) \to S_{n-t}(213, 231)$ such that for all $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)$,

$$f(\tau) = s(\tau(t+1), \tau(t+2), \ldots, \tau(n))$$

where $s(x)$ is the standardization of x. We will show that f is a bijection. It is easy to see that f is a well-defined function. Similarly to the logic in the proof of Theorem 4.4.4, one can also see that f is injective. It remains to show that f is surjective.

Let $\tau \in S_{n-t}(213, 231)$ and let τ' be the matching permutation of τ on A. Then $\pi := (c_1, c_2, \ldots, c_t, \tau'(1), \tau'(2), \ldots, \tau'(n-t)) \in S_{n,(c_1,c_2,\ldots,c_t)}$. We need to show that π avoids 213 and 231. Let xyz be a subpermutation of π. If xyz is a subpermutation of $(c_1, c_2, \ldots, c_t)$ or τ', then xyz is neither a 213 pattern nor a 231 pattern. So we assume that $x \in \{c_1, c_2, \ldots, c_t\}$ and $z \in A$. We split into two cases:

Case 1: $x > z$. Then xyz is not a 213 pattern. Since $(c_1, c_2, \ldots, c_t)$ is a shuffle of $(1, 2, \ldots, t-s)$ and $(n, n-1, \ldots, n-s+1)$, we have $x \geq n-s+1$. Since $y \in \mathcal{D}_\pi(x)$, we have $y < x$ and hence xyz is not a 231 pattern.

Case 2: $x < z$. Then xyz is not a 231 pattern. Since $(c_1, c_2, \ldots, c_t)$ is a shuffle of $(1, 2, \ldots, t-s)$ and $(n, n-1, \ldots, n-s+1)$, we have $x \leq t-s$. Since $y \in \mathcal{D}_\pi(x)$, we have $y > x$ and hence xyz is not a 213 pattern.

Hence $\pi \in S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)$ and $f(\pi) = \tau$ by our construction. This shows that f is surjective. Now f is a bijection and hence, by Theorem 4.5.1,

$$|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = |S_{n-t}(213, 231)| = 2^{n-t-1}.$$

Now suppose $(c_1, c_2, \ldots, c_t)$ is not a shuffle of $(1, 2, \ldots, t-s)$ and $(n, n-1, \ldots, n-s+1)$ for any $s \in \{0, 1, \ldots, t\}$. There are two scenarios where this could happen. The first one is when $\{c_1, c_2, \ldots, c_t\} \neq \{1, 2, \ldots, t-s, n-s+1, n-s+2, \ldots, n\}$, and the second one is when $\{c_1, c_2, \ldots, c_t\} = \{1, 2, \ldots, t-s, n-s+1, n-s+2, \ldots, n\}$ but for any $s \in \{0, 1, \ldots, t\}$, either the subpermutation of $(c_1, c_2, \ldots, c_t)$ on $\{1, 2, \ldots, t-s\}$ is not $(1, 2, \ldots, t-s)$ or the subpermutation of $(c_1, c_2, \ldots, c_t)$ on $\{n-s+1, n-s+2, \ldots, n\}$ is not $(n, n-1, \ldots, n-s+1)$. We split into two cases based on these scenarios. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$.

Case 3: $\{c_1, c_2, \ldots, c_t\} \neq \{1, 2, \ldots, t-s, n-s+1, n-s+2, \ldots, n\}$ for any $s \in \{0, 1, \ldots, t\}$. Then there exist $x \in \{c_1, c_2, \ldots, c_t\}$ and $y, z \in A$ such that $y > x$ and $z < x$. In this case, either xyz or xzy is a subpermutation of τ and hence τ contains either a 213 pattern or a 231 pattern. Hence, $|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = 0$.

Case 4: $\{c_1, c_2, \ldots, c_t\} = \{1, 2, \ldots, t-s, n-s+1, n-s+2, \ldots, n\}$ for some $s \in \{0, 1, \ldots, t\}$.

Subcase 4.1: The subpermutation of $(c_1, c_2, \ldots, c_t)$ on $\{1, 2, \ldots, t-s\}$ is not

$$(1, 2, \ldots, t-s).$$

Then there exist $x, y \in \{c_1, c_2, \ldots, c_t\}$ and $z \in A$ such that $x < y < z$ and $y \in \mathcal{A}_\tau(x)$. Now yxz is a 213 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = 0$.

Subcase 4.2: The subpermutation of $(c_1, c_2, \ldots, c_t)$ on $\{n-s+1, n-s+2, \ldots, n\}$ is not $(n, n-1, \ldots, n-s+1)$. Then there exist $x, y \in \{c_1, c_2, \ldots, c_t\}$ and $z \in A$ such that

$z < y < x$ and $y \in \mathcal{A}_\tau(x)$. Now yxz is a 231 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(213, 231)| = 0$. □

We have 12 pairs left to consider. By Lemma 4.2.2, it suffices to look at $\{123, 132\}$, $\{123, 213\}$, $\{132, 213\}$, $\{132, 231\}$, $\{123, 312\}$, and $\{123, 231\}$.

Theorem 4.5.4. *Write* $\alpha = \max([n]\backslash\{c_1, c_2, \ldots, c_t\})$. *If* $\{c_1, c_2, \ldots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \ldots, n\}$ *or* $(c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t) \neq (\alpha-1, \alpha-2, \ldots, n-t)$*, then*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 0;$$

otherwise,

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 2^{n-t-1}.$$

Note that, in Theorem 4.5.4, we must have $\alpha \geq n-t$, and if $\alpha = n-t$, then $\{c_1, c_2, \ldots, c_t\} = \{n-t+1, n-t+2, \ldots, n\}$.

Proof. Write $A = [n]\backslash\{c_1, c_2, \ldots, c_t\}$.

First suppose that $\{c_1, c_2, \ldots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \ldots, n\}$. As mentioned earlier, if $\alpha = n-t$, then $\{c_1, c_2, \ldots, c_{n-\alpha}\} = \{c_1, c_2, \ldots, c_t\} = \{n-t+1, n-t+2, \ldots, n\}$. So we must have $\alpha > n-t$. Since $\alpha = \max A$, there exist $i \in \{1, 2, \ldots, n-\alpha\}$ and $j \in \{n-\alpha+1, n-\alpha+2, \ldots, t\}$ such that $c_i < \alpha < c_j$. So $c_i c_j \alpha$ is a 132 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 0$.

Next suppose that $\{c_1, c_2, \ldots, c_{n-\alpha}\} = \{\alpha+1, \alpha+2, \ldots, n\}$ but

$$(c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t) \neq (\alpha-1, \alpha-2, \ldots, n-t).$$

Notice that this could only happen when $\alpha > n-t$ because otherwise $n-\alpha+1 = n-(n-t)+1 = t+1$. We split into three cases:

Case 1: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t\} \neq \{\alpha-1, \alpha-2, \ldots, n-t\}$ and there exists $i \in \{n-\alpha+1, n-\alpha+2, \ldots, t\}$ such that $c_i > \alpha$. Then $\{c_1, c_2, \ldots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \ldots, n\}$. This is a contradiction.

Case 2: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t\} \neq \{\alpha-1, \alpha-2, \ldots, n-t\}$ and $c_i < \alpha$ for all $i \in \{n-\alpha+1, n-\alpha+2, \ldots, t\}$. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$. Then there exist $y \in A$ and $c_i \in \{c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t\}$ such that $c_i < y < \alpha$. So either $c_i y \alpha$ or $c_i \alpha y$ is a subpermutation of τ. It follows that τ has either a 123 pattern or a 132 pattern. Hence, $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 0$.

Case 3: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t\} = \{\alpha-1, \alpha-2, \ldots, n-t\}$ but

$$(c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t) \neq (\alpha-1, \alpha-2, \ldots, x).$$

Then there exist $i, j \in \{n-\alpha+1, n-\alpha+2, \ldots, t\}$ with $i < j$ and $c_i < c_j$. Now $c_i c_j \alpha$ is a 123 pattern. Hence, $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 0$.

The proof that $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 132)| = 2^{n-t-1}$ when $\{c_1, c_2, \ldots, c_{n-\alpha}\} = \{\alpha+1, \alpha+2, \ldots, n\}$ and $(c_{n-\alpha+1}, c_{n-\alpha+2}, \ldots, c_t) = (\alpha-1, \alpha-2, \ldots, n-t-1)$ is similar to the proof of Theorem 4.5.3. □

Theorem 4.5.5. *If there exists* $\alpha \in [n]\backslash\{c_1, c_2, \ldots, c_t\}$ *and* $1 \leq i < j \leq t$ *with* $c_i, c_j < \alpha$, *then* $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| = 0$*; otherwise,*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| = 2^{\max\{0, \min\{c_1,c_2,\ldots,c_t\}-2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$ and $A = [n]\backslash\{c_1, c_2, \ldots, c_t\}$.

Suppose there exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i, c_j < \alpha$. Then $c_i c_j \alpha$ is either a 123 pattern or a 213 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| = 0$.

Suppose there do not exist $\alpha \in A$ and $1 \le i < j \le t$ with $c_i, c_j < \alpha$. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)$.

Case 1: $x = 1$. Then $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is a decreasing sequence. Otherwise, there would exist $y, z \in A$ such that $1yz$ is a 123 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| = 1$.

Case 2: $x \ge 2$. Since τ avoids 123, the subpermutation of τ on $A \cap \{x+1, x+2, \ldots, n\}$ is decreasing. Since τ avoids 213, for all $y \in \{1, 2, \ldots, x-1\}$ and for all $z \in A \cap \{x+1, x+2, \ldots, n\}$, $z \in \mathcal{A}_\pi(y)$. So by Theorem 4.5.1, we have

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| \le |S_{x-1}(123, 213)| = 2^{x-2}.$$

It is easy to check that for all $\tau' \in S_{x-1}(123, 213)$ and decreasing subpermutation τ'' on $A \backslash [x-1]$, we have $(c_1, c_2, \ldots, c_t, \tau''(1), \tau''(2), \ldots, \tau''(n-t-x+1), \tau'(1), \tau'(2), \ldots, \tau'(x-1)) \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)$. Hence, we have

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 213)| = |S_{x-1}(123, 213)| = 2^{x-2}. \qquad \square$$

Theorem 4.5.6. *If there exist $\alpha, \beta \in [n] \backslash \{c_1, c_2, \ldots, c_t\}$ and $i \in [t]$ with*

$$\min\{c_1, c_2, \ldots, c_t\} < \alpha < c_i < \beta,$$

then $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 213)| = 0$; if there exist $\alpha \in [n] \backslash \{c_1, c_2, \ldots, c_t\}$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $c_j < c_i < \alpha$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 213)| = 0$; otherwise,

$$|S_{n,(c_1,c_2,\ldots,c_t)}(132, 213)| = 2^{\max\{0, \min\{c_1,c_2,\ldots,c_t\}-2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$ and $A = [n] \backslash \{c_1, c_2, \ldots, c_t\}$.

Suppose there exist $\alpha, \beta \in A$ and $i \in [t]$ with $x < \alpha < c_i < \beta$. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$. If $\alpha \in \mathcal{A}_\tau(\beta)$, then $c_i\alpha\beta$ is a 213 pattern; and if $\alpha \in \mathcal{D}_\tau(\beta)$, then $x\beta\alpha$ is a 132 pattern. Hence, $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 213)| = 0$.

Now suppose there exist $\alpha \in A$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $c_j < c_i < \alpha$. If $c_i < \alpha < c_j$, then $c_ic_j\alpha$ is a 132 pattern; and if $c_j < c_i < \alpha$, then $c_ic_j\alpha$ is a 213 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 213)| = 0$.

The rest of the proof is similar to the proof of Theorem 4.5.5. □

Theorem 4.5.7. *If there exist $\alpha \in [n]\backslash\{c_1, c_2, \ldots, c_t\}$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $\alpha < c_i < c_j$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 231)| = 0$; otherwise,*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(132, 231)| = 2^{\max\{0,\min\{c_1,c_2,\ldots,c_t\}-2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$ and $A = [n]\backslash\{c_1, c_2, \ldots, c_t\}$.

Suppose there exist $\alpha \in A$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $\alpha < c_i < c_j$. If $c_i < \alpha < c_j$, then $c_ic_j\alpha$ is a 132 pattern; and if $\alpha < c_i < c_j$, then $c_ic_j\alpha$ is a 231 pattern. Hence, $|S_{n,(c_1,c_2,\ldots,c_t)}(132, 231)| = 0$.

The rest of the proof is similar to the proof of Theorem 4.5.5. □

Theorem 4.5.8. *If $\{c_1, c_2, \ldots, c_t\}$ is a set of consecutive integers, then*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 0,\ 1,\ \textit{or}\ \min\{c_1, c_2, \ldots, c_t\};$$

otherwise, $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 0$ or 1.

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$, $y = \max\{c_1, c_2, \ldots, c_t\}$, and

$$A = [n]\backslash\{c_1, c_2, \ldots, c_t\}.$$

Suppose $\{c_1, c_2, \ldots, c_t\}$ is a set of consecutive integers. Then $\{c_1, c_2, \ldots, c_t\} = \{x, x+1, \ldots, x+t-1\}$. We split into three cases:

Case 1: There exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. Then $c_i c_j \alpha$ is a 123 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 0$.

Case 2: $x = n-t+1$ and there do not exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)$. Then $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is a subpermutation on $\{1, 2, \ldots, x-1\}$. If $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is not decreasing, then we would have a 312 pattern. It is easy to check that $(c_1, c_2, \ldots, c_t, x-1, x-2, \ldots, 1)$ avoids both 123 and 312 patterns. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 1$.

Case 3: $x < n-t+1$ and there do not exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. In this case, we have $y < n$. Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)$. Let τ' be the subpermutation on $\{1, 2, \ldots, x-1\}$ and let τ'' be the subpermutation on $\{y+1, y+2, \ldots, n\}$. Since τ avoids 123, $\tau'' = (n, n-1, \ldots, y+1)$, and since τ avoids 312, $\tau' = (x-1, x-2, \ldots, 1)$. Moreover, if $\tau(i) \in \{1, 2, .., x-1\}$ and $\tau(j), \tau(k) \in \{y+1, y+2, \ldots, n\}$ with $j < k$, either $\tau(j), \tau(k) \in \mathcal{A}_\tau(\tau(i))$ or $\tau(j), \tau(k) \in \mathcal{D}_\tau(\tau(i))$. Otherwise, $\tau(j)\tau(i)\tau(k)$ would be a 312 pattern. Therefore the number of shuffles of τ' and τ'' that do not create a 312 pattern is simply $\binom{x-1+1}{1} = x$. It is easy to check that none of these shuffles creates a 123 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = x$.

Now suppose $\{c_1, c_2, \ldots, c_t\}$ is not a set of consecutive integers. There are three cases:

Case 4: If there exists $\alpha \in A$, $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$, then $c_i c_j \alpha$ is a 123 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 0$.

Case 5: If there exists $\alpha \in A$, $1 \leq i < j \leq t$ with $c_j < \alpha < c_i$, then $c_i c_j \alpha$ is a 312 pattern and hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 0$.

Case 6: The conditions in Case 4 and Case 5 are not met. Let

$$\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 312).$$

Let τ' be the subpermutation on $\{1, 2, \ldots, y-1\} \cap A$ and let τ'' be the subpermutation on $\{x+1, x+2, \ldots, n\} \cap A$. Since τ avoids both 123 and 312, both τ' and τ'' are decreasing. Since $\{c_1, c_2, \ldots, c_t\}$ is not a set of consecutive integers, there exists $\alpha \in A$ with $x < \alpha < y$. Since α is in both τ' and τ'', $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is a decreasing permutation on A. It is easy to see that if the conditions in Case 4 and Case 5 are not met and $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ is decreasing, then τ avoids both 123 and 312 patterns. Hence we have $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 312)| = 1$. $\square$

Theorem 4.5.9. *If* $(c_1, c_2, \ldots, c_t) = (n, n-1, \ldots, n-t+1)$*, then*

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = \binom{n-t}{2} + 1;$$

otherwise, $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = 0$ *or* 1.

Proof. Write $x = \min\{c_1, c_2, \ldots, c_t\}$ and $A = [n] \backslash \{c_1, c_2, \ldots, c_t\}$.

Suppose $(c_1, c_2, \ldots, c_t) = (n, n-1, \ldots, n-t+1)$. It is easy to see that $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$ avoids both 123 and 231 if and only if $(\tau(t+1), \tau(t+2), \ldots, \tau(n))$ avoids both 123 and 231. So by Theorem 4.5.1,

$$|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = |S_{n-t}(123, 231)| = \binom{n-t}{2} + 1.$$

Now suppose $(c_1, c_2, \ldots, c_t) \neq (n, n-1, \ldots, n-t+1)$. We split into three cases:

Case 1: $\{c_1, c_2, \ldots, c_t\} = \{n-t+1, n-t+2, \ldots, n\}$. Then there exist indices $1 \leq i < j \leq t$ and $\alpha \in A$ such that $\alpha < c_i < c_j$. So $c_i c_j \alpha$ is a 231 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = 0$.

Case 2: $\{c_1, c_2, \ldots, c_t\} \neq \{n-t+1, n-t+2, \ldots, n\}$ and there exist indices $1 \leq i < j \leq t$ and $\alpha \in A$ such that $c_i < c_j < \alpha$ or $\alpha < c_i < c_j$, then $c_i c_j \alpha$ is either a 123 pattern or a 231 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = 0$.

Case 3: The conditions for Case 1 and Case 2 are not met. Let

$$\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(123, 231).$$

Since τ avoids 123, the subpermutation τ' of τ on $\{x+1, x+2, \ldots, n\} \cap A$ is decreasing. Since τ avoids 231, $\{x+1, x+2, \ldots, n\} \cap A \subseteq \mathcal{D}_\tau(i)$ for all $i < x$. Notice that since $\{c_1, c_2, \ldots, c_t\} \neq \{n-t+1, n-t+2, \ldots, n\}$, there exists $\alpha \in A$ with $\alpha > x$. Now if the subpermutation τ'' of τ on $\{1, 2, \ldots, x-1\}$ is not decreasing, we would have a 123 pattern. So

$$\tau = (c_1, c_2, \ldots, c_t, x-1, x-2, \ldots, 1, \tau'(1), \tau'(2), \ldots, \tau'(n-t-x+1)).$$

It is easy to check that τ avoids both 123 and 231.

Hence, we have $|S_{n,(c_1,c_2,\ldots,c_t)}(123, 231)| = 1$. □

4.6 The pair 3412 and 3421

The goal of this section is to show that the counting argument used in the proof of Theorem 4.4.1 can be generalized to permutations avoiding both 3412 and 3421. We need the following result proved by Kremer [77, Corollary 9]:

Theorem 4.6.1. *For all $n \geq 1$,*

$$|S_n(3412, 3421)| = \mathbb{S}_{n-1},$$

where $\mathbb{S}_{n-1}$ is the $(n-1)$st large (big) Schröder number.

For the rest of this section, $\mathbb{S}_n$ is the nth large (big) Schröder number for all $n \in \mathbb{N}$.

We first present our result on $S_{n,r}(3412, 3421)$ since it has an easier presentation but still shows the subtle difference between this pair of pattern of length four and Theorem 4.4.1. In addition, our result on $S_{n,r}(3412, 3421)$ also allows us to provide an alternate proof of a recurrence relation on the large Schröder numbers.

Theorem 4.6.2. *For all $n \geq 2$ and $r \in \{1, 2, n\}$, we have*

$$|S_{n,r}(3412, 3421)| = \mathbb{S}_{n-2};$$

and for all $n \geq 4$ and $2 < r < n$, we have

$$|S_{n,r}(3412, 3421)| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}.$$

Proof. First, suppose $n \geq 1$ and $r \in \{1, n\}$. Let $\tau \in S_{n,r}$. If $r = 1$, then $r < a$ for all $a \in \mathcal{D}_\tau(r)$. If $r = n$, then $r > a$ for all $a \in \mathcal{D}_\tau(r)$. If $r = 2$, then there is exactly one $a \in \mathcal{D}_\tau(r)$ with $a < r$. In any case, $rxyz$ is not a 3412 pattern or a 3421 pattern for any $x, y, z \in \mathcal{D}_\tau(r)$. Hence $\tau \in S_{n,r}(3412, 3421)$ if and only if $(\tau(2), \tau(3), \ldots, \tau(n))$ avoids both 3412 and 3421. Therefore, by Theorem 4.6.1, we have $|S_{n,r}(3412, 3421)| = |S_{n-1}(3412, 3421)| = \mathbb{S}_{n-2}$.

Now suppose $n \geq 4$ and $2 < r < n$. Let $\mathcal{R}$ be a subset of $S_{n,r}$ such that every $\tau \in \mathcal{R}$ has the following properties:

(i) $\{\tau(2), \tau(3), \ldots, \tau(r-1)\} \subseteq \{1, 2, \ldots, r-1\}$;

(ii) the subpermutation τ' of τ on $\{1, 2, \ldots, r-1\}$ avoids both 3412 and 3421;

(iii) $\tau'' = (\tau(r), \tau(r+1), \ldots, \tau(n))$ avoids both 3412 and 3421.

Let $\tau \in \mathcal{R}$. By Theorem 4.6.1, there are $\mathbb{S}_{r-2}$ ways for τ' to avoid both 3412 and 3421, and for each fixed τ', there are $\mathbb{S}_{n-r}$ ways for τ'' to avoid both 3412 and 3421. Hence, we have $|\mathcal{R}| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}$.

Now we show that $S_{n,r}(3412, 3421) = \mathcal{R}$. Let $\tau \in S_{n,r}(3412, 3421)$, τ' the subpermutation of τ on $\{1, 2, \ldots, r-1\}$, and $\tau'' = (\tau(r), \tau(r+1), \ldots, \tau(n))$. Since τ avoids both 3412 and 3421, τ' avoids both 3412 and 3421 as well. Similarly, τ'' avoids both 3412 and 3421.

We now show that $\tau(i) \in \{1, 2, \ldots, r-1\}$ for all $i \in \{2, 3, \ldots, r-1\}$. Suppose, by way of contradiction, that $\tau(i) > r$ for some $i \in \{2, 3, \ldots, r-1\}$. Then, since $\tau(1) = r$, at most $r-3$ numbers in $\{\tau(1), \tau(2), \ldots, \tau(r-1)\}$ are less than r. So there exist $k > j > r-1$ such that $\tau(j), \tau(k) < r$. Now $r\tau(i)\tau(j)\tau(k)$ is either a 3412 pattern or a 3421 pattern. This is a contradiction. Hence, we have $S_{n,r}(3412, 3421) \subseteq \mathcal{R}$.

On the other hand, suppose $\tau \in \mathcal{R}$. We will show that $\tau \in S_{n,r}(3412, 3421)$. Suppose, by way of contradiction, that $xyzw$ is a subpermutation of τ which is a 3412 pattern or a 3421 pattern. Then we have $z, w < x < y$ and $z, w \in D_\tau(y)$. We split into three cases:

Case 1: $x = r$. Then $y > r$. Since $\{\tau(2), \tau(3), \ldots, \tau(r-1)\} \subseteq \{1, 2, \ldots r-1\}$, we must have $y = \tau(i)$ for some $i > r-1$ and at most one $j > i$ with $\tau(j) < r$. So either $z > r = x$ or $w > r = x$ which is a contradiction.

Case 2: $x < r$. Since the subpermutation on $\{1, 2, \ldots, r-1\}$ avoids both 3412 and 3421, we must have $y > r$. The rest of the argument is then the same as Case 1.

Case 3: $x > r$. Since $\{\tau(2), \tau(3), \ldots, \tau(r-1)\} \subseteq \{1, 2, \ldots, r-1\}$, $xyzw$ is a subpermutation of $(\tau(r), \tau(r+1), \ldots, \tau(n))$. Since $(\tau(r), \tau(r+1), \ldots, \tau(n))$ avoids both 3412 and 3421, $xyzw$ is not a 3412 or 3421 pattern. This is a contradiction.

This completes the proof that $\mathcal{R} \subseteq S_{n,r}(3412, 3421)$.

Hence we have $S_{n,r}(3412, 3421) = \mathcal{R}$, and therefore $|S_{n,r}(3412, 3421)| = |\mathcal{R}| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}$. □

Summing over r in Theorem 4.6.2, we have the following well-known recurrence relation for $\mathbb{S}_n$:

Corollary 4.6.3. *For all $n \geq 1$,*

$$\mathbb{S}_{n+1} = \mathbb{S}_n + \sum_{r=0}^{n} \mathbb{S}_r \mathbb{S}_{n-r}.$$

Proof. Let $n \geq 1$. Note that by Table 4.1, we have $\mathbb{S}_0 = 1$. So by Theorem 4.6.2, we have $|S_{n+2,2}(3412, 3421)| = |S_{n+2,n+2}(3412, 3421)| = \mathbb{S}_n = \mathbb{S}_n\mathbb{S}_0$. Now, by Theorems 4.6.1 and 4.6.2, we have $\mathbb{S}_{n+1} = |S_{n+2}(3412, 3421)| = \sum_{r=1}^{n+2} |S_{n+2,r}(3412, 3421)| = \mathbb{S}_n + \sum_{r=2}^{n+2} \mathbb{S}_{r-2}\mathbb{S}_{n+2-r} = \mathbb{S}_n + \sum_{r=0}^{n} \mathbb{S}_r\mathbb{S}_{n-r}$. □

Remark 4.6.4. Qi and Guo [99, Theorem 5] proved Corollary 4.6.3 using generating functions. In [18, p. 446], it is also noted that Corollary 4.6.3 can also be derived from the recurrence $\mathbb{S}_n = \sum_{i=0}^{n} \binom{2n-i}{i} C_{n-i}$ which was proved by West [121, p. 255]. Our proof of this identity does not use the Catalan numbers and is purely combinatorial.

Now we generalize our result for $S_{n,r}(3412, 3421)$ to $S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$. As before, we assume that $(c_1, c_2, \ldots, c_t)$ avoids both 3412 and 3421.

Theorem 4.6.5. *Let*

$$U = \{c_i : i \in [t] \text{ and there exist } j, k \in [t] \text{ such that } i < j < k \text{ and } c_ic_jc_k \text{ is a 231 pattern}\}$$

and

$$V = \{c_i : i \in [t] \text{ and there exists } j \in [t] \text{ such that } i < j \text{ and } c_i < c_j\}.$$

If $U \neq \emptyset$ and $|[\max U]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 1$ or $V \neq \emptyset$ and $|[\max V]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 2$, then $|S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)| = 0$; otherwise,

$$|S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)| = \mathbb{S}_{c_{(j)}-c_{(j-1)}-2} \prod_{i=j+1}^{t+1} \mathbb{S}_{c_{(i)}-c_{(i-1)}-1},$$

where $c_{(0)} = 0$*,* $c_{(t+1)} = n+1$*,* $c_{(1)} < c_{(2)} < \cdots < c_{(t)}$ *are the order statistics of* $\{c_1, c_2, \ldots, c_t\}$*, and* $j = \min\{i \in [t+1] : c_{(i)} - c_{(i-1)} > 1\}$*.*

Proof. For all $k \in [t+1]$, let $A_k = \{c_{(k-1)}+1, c_{(k-1)}+2, \ldots, c_{(k)}-1\}$. We note that it is possible that $A_k = \emptyset$ for some k. Also notice that

$$[n]\backslash\{c_1, c_2, \ldots, c_t\} = \bigcup_{k=1}^{t+1} A_k.$$

We first suppose that $U \neq \emptyset$ and $|[\max U]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 1$. Let $x, y, z \in [t]$ such that $c_x = \max U$, $x < y < z$, and $c_x c_y c_z$ is a 231 pattern. Since $|[c_x]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 1$, there exists $\alpha \in [n]\backslash\{c_1, c_2, \ldots, c_t\}$ such that $\alpha < c_x$. If $\alpha < c_z$, then $c_x c_y c_z \alpha$ is a 3421 pattern; and if $\alpha > c_z$, then $c_x c_y c_z \alpha$ is a 3412 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)| = 0$.

Next, we suppose that $V \neq \emptyset$ and $|[\max V]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 2$. Let $x, y \in [t]$ such that $c_x = \max V$, $x < y$, and $c_x < c_y$. Since $|[c_x]\backslash\{c_1, c_2, \ldots, c_t\}| \geq 2$, there exist $\alpha, \beta \in [n]\backslash\{c_1, c_2, \ldots, c_t\}$ such that $\alpha < \beta < c_x < c_y$. If $\alpha \in \mathcal{A}_\tau(\beta)$, then $c_x c_y \alpha\beta$ is a 3412 pattern; and if $\alpha \in \mathcal{D}_\tau(\beta)$, then $c_x c_y \beta\alpha$ is a 3421 pattern. Hence $|S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)| = 0$.

Now suppose otherwise. Let $j = \min\{i \in [t+1] : c_{(i)} - c_{(i-1)} > 1\}$. In other words, $j \in [t+1]$ is the smallest index such that $A_j \neq \emptyset$. For all $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}$, let τ_j be the subpermutation of τ on A_j, and for all $i \in \{j+1, j+2, \ldots, t+1\}$, let x_i be the last term of the subpmermutation of τ on $A_j \cup A_{j+1} \cup \cdots \cup A_{i-1}$ and τ_i be the subpermutation of τ on $A_i \cup \{x_i\}$. Let $\mathcal{R}'$ be the subset of $S_{n,(c_1,c_2,\ldots,c_t)}$ such that every $\tau \in \mathcal{R}'$ satisfies the following: (i) for all $i \in \{j+1, j+2, \ldots, t+1\}$, $y \in A_i$, and $z \in (A_j \cup A_{j+1} \cup \cdots \cup A_{i-1})\backslash\{x_i\}$, we have $z \in \mathcal{A}_\tau(y)$; (ii) and for all $i \in \{j, j+1, \ldots, t+1\}$, τ_i avoids both 3412 and 3421.

Let $\tau \in \mathcal{R}'$. By Theorem 4.6.1, since $|A_j| = c_{(j)} - c_{(j-1)} - 1$, there are $\mathbb{S}_{c_{(j)}-c_{(j-1)}-2}$ possibilities for τ_j. Now let $i \in \{j+1, j+2, \ldots, t+1\}$. Inductively, we count the possibilities for τ_i when $\tau_j, \tau_{j+1}, \ldots, \tau_{i-1}$ are fixed. In this case, since $|A_i \cup \{x_i\}| = c_{(i)} - c_{(i-1)}$, by Theorem 4.6.1, there are $\mathbb{S}_{c_{(i)}-c_{(i-1)}-1}$ possibilities for τ_i. Hence, we have $|\mathcal{R}'| = \mathbb{S}_{c_{(j)}-c_{(j-1)}-2} \prod_{i=j+1}^{t+1} \mathbb{S}_{c_{(i)}-c_{(i-1)}-1}$.

It remains to show that $\mathcal{R}' = S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$.

Let $\tau \in S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$. Then Property (ii) is obviously satisfied. Now we show that τ satisfies Property (i). Suppose, by way of contradiction, that τ does not satisfy Property (i). Then there exist $i \in \{j+1, j+2, \ldots, t+1\}$, $y \in A_i$, and $z \in (A_j \cup A_{j+1} \cup \cdots \cup A_{i-1}) \backslash \{x_i\}$ such that $z \in D_\tau(y)$. Since x_i is the last term of the subpermutation of τ on $A_j \cup A_{j+1} \cup \cdots \cup A_{i-1}$, $c_{(i-1)}yzx_i$ is a subpermutation of τ. Since $y \in A_i$, we have $y > c_{(i-1)}$. Now since $z, x_i \in A_j \cup A_{j+1} \cup \cdots \cup A_{i-1}$, $c_{(i-1)}yzx_i$ is either a 3412 pattern or a 3421 pattern. This is a contradiction. Hence τ satisfies Property (i). It follows that $S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421) \subseteq \mathcal{R}'$.

We still need to prove that $\mathcal{R}' \subseteq S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$. Let $\tau \in \mathcal{R}'$. Suppose, by way of contradiction, that $abcd$ is a subpermutation of τ which is either a 3412 pattern or a 3421 pattern. Then we have $c < d < a < b$ or $d < c < a < b$. We have five cases:

Case 1: $a, b, c, d \notin \{c_1, c_2, \ldots, c_t\}$. Then $a \in A_k$ for some $k \in \{j, j+1, \ldots, t+1\}$. Since $c, d < a$, we have $c \in A_{i_1}$ and $d \in A_{i_2}$ for some $i_1, i_2 \leq k$. Since $b > a$, if $b \notin A_k$, then $b \in A_{i_3}$ with $i_3 > k$. Then Property (i) is violated because $c, d \in A_j \cup A_{j+1} \cup \cdots \cup A_k$. So we must have $b \in A_k$. By Property (i), at most one of c and d is in $A_j \cup A_{j+1} \cup \cdots \cup A_{k-1}$. In addition, if c or d is in $A_j \cup A_{j+1} \cup \cdots \cup A_{k-1}$, then it must be the last term of τ_{k-1}. So $abcd$ is a subpermutation on $A_k \cup \{x_k\}$. This contradicts Property (ii).

Case 2: $a \in \{c_1, c_2, \ldots, c_t\}$ but $b, c, d \notin \{c_1, c_2, \ldots, c_t\}$. Then $a = c_k$ for some $k \in [t]$. Since $b > a$, we have $b \in A_i$ for some $i > k$. Since $c, d < a$, we have $c, d \in$

$A_j \cup A_{j+1} \cup \cdots \cup A_k$. Since $c, d \in D_\tau(b)$, this violates Property (i). Hence we have a contradiction.

Case 3: $a, b \in \{c_1, c_2, \ldots, c_t\}$ but $c, d \notin \{c_1, c_2, \ldots, c_t\}$. Since $a < b$ and $c, d < a$, $V \neq \emptyset$ and $|[\max V] \backslash \{c_1, c_2, \ldots, c_t\}| \geq 2$ which is a contradiction.

Case 4: $a, b, c \in \{c_1, c_2, \ldots, c_t\}$ but $d \notin \{c_1, c_2, \ldots, c_t\}$. Then abc is a 231 pattern. Since $d < a$, $U \neq \emptyset$ and $|[\max U] \backslash \{c_1, c_2, \ldots, c_t\}| \geq 1$ which is a contradiction.

Case 5: $a, b, c, d \in \{c_1, c_2, \ldots, c_t\}$. This contradicts our convention that $(c_1, c_2, \ldots, c_t)$ avoids both 3412 and 3421.

Hence, τ avoids both 3412 and 3421. It follows that $\mathcal{R}' \subseteq S_{n,(c_1,c_2,\ldots,c_t)}(3412, 3421)$. This completes our proof. □

4.7 r-Wilf-Equivalence classes

In this section, we classify r-Wilf-equivalence classes for patterns of length three. Recall that for a fixed $r \in \mathbb{N}$, two patterns σ and σ' are called r-*Wilf equivalent* if $|S_{n,r}(\sigma)| = |S_{n,r}(\sigma')|$ for all $n \geq r$.

We start with some elementary results summarized in Table 4.2.

r	$\lvert S_{n,r}(123)\rvert$	$\lvert S_{n,r}(321)\rvert$	$\lvert S_{n,r}(132)\rvert$	$\lvert S_{n,r}(312)\rvert$	$\lvert S_{n,r}(213)\rvert$	$\lvert S_{n,r}(231)\rvert$
n	C_{n-1}	1	C_{n-1}	1	C_{n-1}	C_{n-1}
$n-1$	C_{n-1}	$n-1$	C_{n-1}	$n-1$	C_{n-2}	C_{n-2}
2	$n-1$	C_{n-1}	$n-1$	C_{n-1}	C_{n-2}	C_{n-2}
1	1	C_{n-1}	1	C_{n-1}	C_{n-1}	C_{n-1}

Table 4.2: Single Patterns of Length 3 for $n \geq 2$.

It is easy to check the correctness of the expressions in Table 4.2. As an example, we sketch the proof of the fact that $|S_{n,n-1}(123)| = C_{n-1}$ for all $n \geq 2$. For any $i, j \in \{2, 3, \ldots, n\}$ with $i < j$, either $\tau(i) < n-1$ or $\tau(j) < n-1$. It follows that $(n-1, \tau(i), \tau(j))$ will never form a 123 pattern for any $i, j \in \{2, 3, \ldots, n\}$ with $i < j$. Hence,

$\tau \in S_{n,n-1}$ avoids 123 if and only if $(\tau(2), \tau(3), \ldots, \tau(n))$ avoids 123. Therefore, by Theorem 1.3.1, we have $|S_{n,n-1}(123)| = |S_{n-1}(123)| = C_{n-1}$.

Next we classify the r-Wilf-equivalence classes for patterns of length three for all $r \in \mathbb{N}$.

Theorem 4.7.1. *There are two* 1*-Wilf-equivalence classes for patterns of length three:* $123 \overset{1}{\sim} 132$ *and* $321 \overset{1}{\sim} 312 \overset{1}{\sim} 213 \overset{1}{\sim} 231$*. For* $r \geq 2$*, there are three* r*-Wilf-equivalence classes for patterns of length three:* $213 \overset{r}{\sim} 231$*,* $123 \overset{r}{\sim} 132$*, and* $321 \overset{r}{\sim} 312$*.*

Proof. The fact that there are two 1-Wilf-equivalence classes for patterns of length three follows from the last row of Table 4.2.

Let $r \geq 2$. By Lemma 4.2.2, we have $213 \overset{r}{\sim} 231$, $123 \overset{r}{\sim} 132$, and $321 \overset{r}{\sim} 312$. We need to show that there exist $n_1, n_2, n_3 \geq r$ such that $|S_{n_1,r}(213)| \neq |S_{n_1,r}(123)|$, $|S_{n_2,r}(213)| \neq |S_{n_2,r}(321)|$, and $|S_{n_3,r}(123)| \neq |S_{n_3,r}(321)|$. There are three cases to consider.

Case 1: $r = 2$. Set $n_1 = n_2 = n_3 = 4$. By Tables 4.1 and 4.2, we have $|S_{4,2}(123)| = 4 - 1 = 3$, $|S_{4,2}(321)| = C_3 = 5$, and $|S_{4,2}(213)| = C_2 = 2$. Hence we have the desired result.

Case 2: $r = 3$. Set $n_1 = n_2 = n_3 = 4$. By Tables 4.1 and 4.2, we have $|S_{4,3}(123)| = C_4 = 5$, $|S_{4,3}(321)| = 4 - 1 = 3$, and $|S_{4,3}(213)| = C_2 = 2$. Hence we have the desired result.

Case 3: $r \geq 3$. Set $n_1 = n_2 = n_3 = r + 1$. By Table 4.2, we have $|S_{r+1,r}(123)| = C_r$, $|S_{r+1,r}(321)| = r$, and $|S_{r+1,r}(213)| = C_{r-1}$. By Table 4.1, we have $C_r > C_{r-1} > r$ and the theorem follows. □

In addition to the classical patterns we have studied so far in this paper, many papers studied consecutive patterns [38], bivincular patterns [20], and mesh patterns [23, 68]. Here we briefly describe, for all $r \geq 5$, the r-Wilf equivalence classes for vincular patterns of length three studied by Babson and Steingrímsson [8] and later, Claesson [29].

In vincular patterns [68, Section 2], some consecutive elements in a permutation pattern are required to be adjacent. We use overlines to indicate that the elements under the overlines are required to be adjacent. There are twelve vincular patterns of length three where one requires exactly two numbers to be adjacent. For example, a permutation $\tau \in S_n$ contains the pattern $1\overline{32}$ if there exist indices $i < j$ such that $\tau(i)\tau(j)\tau(j+1)$ is a 132 pattern. Other vincular patterns are defined similarly.

Example 4.7.2. In the permutation $\tau = 13542 \in S_5$, $\tau(2)\tau(3)\tau(5) = 352$ is a $\overline{23}1$ pattern and $\tau(1)\tau(4)\tau(5) = 142$ is a $1\overline{32}$ pattern, but τ avoids the pattern $\overline{21}3$.

Claesson [29] proved that there are two Wilf-equivalence classes for the twelve vincular patterns. They are counted either by the Catalan numbers or by the Bell numbers:

Theorem 4.7.3. *For all $n \geq 1$,*

$$\begin{aligned}|S_n(1\overline{23})| =& |S_n(3\overline{21})| = |S_n(\overline{12}3)| = |S_n(\overline{32}1)| = |S_n(1\overline{32})| \\ =& |S_n(3\overline{12})| = |S_n(\overline{21}3)| = |S_n(\overline{23}1)| = B_n,\end{aligned}$$

and

$$|S_n(2\overline{13})| = |S_n(2\overline{31})| = |S_n(\overline{13}2)| = |S_n(\overline{31}2)| = C_n,$$

where B_n is the nth Bell number and C_n is the nth Catalan number.

We first adapt some results in Claesson [29] to show r-Wilf equivalence for several vincular patterns.

Proposition 4.7.4. *For all* $r \in \mathbb{N}$, $2\overline{13} \overset{r}{\sim} 2\overline{31}$, $1\overline{23} \overset{r}{\sim} 1\overline{32}$, *and* $3\overline{21} \overset{r}{\sim} 3\overline{12}$.

Proof. Let $1 \leq r \leq n$. Using a short combinatorial argument, Claesson [29, Lemma 2] showed that for all $n \geq 1$, $\tau \in S_n$ avoids $2\overline{13}$ if and only if it avoids 213. Taking complements, for all $n \geq 1$, $\tau \in S_n$ avoids $2\overline{31}$ if and only if it avoids 231. Hence for all $\tau \in S_{n,r}$, τ avoids $2\overline{13}$ if and only if it avoids 213 and τ avoids $2\overline{31}$ if and only if it avoids 231. Then we have $|S_{n,r}(2\overline{13})| = |S_{n,r}(213)|$ and $|S_{n,r}(2\overline{31})| = |S_{n,r}(231)|$. Now by Lemma 4.2.2, we have $|S_{n,r}(2\overline{13})| = |S_{n,r}(213)| = |S_{n,r}(231)| = |S_{n,r}(2\overline{31})|$. Therefore, $2\overline{13} \overset{r}{\sim} 2\overline{31}$.

Claesson [29, Propositions 2 and 4] constructed bijections between $S_n(1\overline{23})$ and the partitions of $[n]$, and then between $S_n(1\overline{32})$ and the partitions of $[n]$. These bijections preserve the leading terms of permutations. So for all $1 \leq r \leq n$, we have $|S_{n,r}(1\overline{23})| = |S_{n,r}(1\overline{32})|$. Taking the complements, we also have $|S_{n,r}(3\overline{21})| = |S_{n,r}(3\overline{12})|$. Therefore, we have $1\overline{23} \overset{r}{\sim} 1\overline{32}$ and $3\overline{21} \overset{r}{\sim} 3\overline{12}$. □

By Proposition 4.7.4, there are at most nine r-Wilf-equivalence classes for vincular patterns. Table 4.3 lists the results we need to classify r-Wilf equivalence classes for all twelve vincular patterns.

Most of the expressions in Table 4.3 can be obtained by straightforward calculation using Theorem 4.7.3 and Table 4.2. We will only sketch the proofs of a few of them.

Lemma 4.7.5. *For all* $r \geq 3$,

$$|S_{r,r}(\overline{32}1)| = B_{r-2}.$$

Proof. Let $\tau \in S_{r,r}(\overline{32}1)$. If $\tau(2) \neq 1$, then $\tau(1)\tau(2)1$ is a $\overline{32}1$ pattern. So we must have $\tau(2) = 1$. At the same time, $(\tau(3), \tau(4), \ldots, \tau(n))$ avoids the pattern $\overline{32}1$. So by Theorem 4.7.3, we have $|S_{r,r}(\overline{32}1)| \leq |S_{r-2}(\overline{32}1)| = B_{r-2}$.

	$n=r$	$n=r+1$	$n=r+2$
$\lvert S_{n,r}(2\overline{13})\rvert = \lvert S_{n,r}(2\overline{31})\rvert$	C_{r-1}	C_{r-1}	
$\lvert S_{n,r}(\overline{13}2)\rvert$	C_{r-1}	C_r	
$\lvert S_{n,r}(3\overline{21})\rvert = \lvert S_{n,r}(3\overline{12})\rvert$	1	2^{r-1}	
$\lvert S_{n,r}(\overline{31}2)\rvert$	1	r	
$\lvert S_{n,r}(1\overline{23})\rvert = \lvert S_{n,r}(1\overline{32})\rvert$	B_{r-1}	B_r	$B_{r+1} - B_{r-1}$
$\lvert S_{n,r}(\overline{12}3)\rvert$	B_{r-1}	B_r	$B_{r+1} - B_r$
$\lvert S_{n,r}(\overline{21}3)\rvert$	B_{r-1}	B_{r-1}	
$\lvert S_{n,r}(\overline{23}1)\rvert$	B_{r-1}	$B_r - B_{r-1}$	
$\lvert S_{n,r}(\overline{32}1)\rvert$	B_{r-2}		

Table 4.3: Avoiding Vincular Patterns by Leading Terms for $r \geq 3$. (We leave some entries in the table blank and only include results that are needed to classify r-Wilf equivalence classes for the twelve vincular patterns.)

Now let $\tau \in S_{r,r}$ with $\tau(2) = 1$ and $(\tau(3), \tau(4), \ldots, \tau(n))$ avoiding the pattern $\overline{32}1$. Since $r1x$ and $1xy$ are never $\overline{32}1$ patterns, we must have $\tau \in S_{r,r}(\overline{32}1)$. Hence we have $B_{r-2} = |S_{r-2}(\overline{32}1)| \leq |S_{r,r}(\overline{32}1)|$. This completes the proof of the lemma. $\square$

Lemma 4.7.6. *For all* $r \geq 1$*,*

$$|S_{r+2,r}(1\overline{23})| = B_{r+1} - B_{r-1} \textit{ and } |S_{r+2,r}(\overline{12}3)| = B_{r+1} - B_r.$$

Proof. We first prove that $|S_{r+2,r}(1\overline{23})| = B_{r+1} - B_{r-1}$. Let $\tau \in S_{r+2,r}(1\overline{23})$. Then the subpermutation $(\tau(2), \tau(3), \ldots, \tau(r+2))$ avoids $1\overline{23}$. By Theorem 4.7.3, there are $|S_{r+1}(1\overline{23})| = B_{r+1}$ ways for $(\tau(2), \tau(3), \ldots, \tau(r+2))$ to avoid $1\overline{23}$. For these permutations on $\{1, 2, \ldots, r-1, r+1, r+2\}$, the only way that $r+1$ and $r+2$ are adjacent and the subpermutation on $\{r+1, r+2\}$ is $(r+1, r+2)$ is when $\tau(2) = r+1$ and $\tau(3) = r+2$ because otherwise $(\tau(2), \tau(3), \ldots, \tau(r+2))$ would contain a $1\overline{23}$ pattern. This is the only case that $(r, r+1, r+2)$ is a $1\overline{23}$ pattern. Since $(\tau(4), \tau(5), \ldots, \tau(r+2))$ also need to avoid $1\overline{23}$, by Theorem 4.7.3, the number of permutations $(\tau(2), \tau(3), \ldots, \tau(r+2))$ avoiding $1\overline{23}$, with $\tau(2) = r+1$ and $\tau(3) = r+2$, is $|S_{r-1}(1\overline{23})| = B_{r-1}$. Here it is easy to check that if $\tau(2) = r+1$, $\tau(3) = r+2$, and $(\tau(4), \tau(5), \dots, \tau(r+2))$ avoids $1\overline{23}$, then $(\tau(2), \tau(3), \dots, \tau(r+2))$ avoids $1\overline{23}$ as well. Therefore

$$|S_{r+2,r}(1\overline{23})| = |S_{r+1}(1\overline{23})| - |S_{r-1}(1\overline{23})| = B_{r+1} - B_{r-1}.$$

Next, we prove that $|S_{r+2,r}(\overline{12}3)| = B_{r+1} - B_r$. Let $\tau \in S_{r+2,r}(\overline{12}3)$. Then

$$(\tau(2), \tau(3), \dots, \tau(r+2))$$

avoids $\overline{12}3$. By Theorem 4.7.3, there are $|S_{r+1}(\overline{12}3)| = B_{r+1}$ ways for

$$(\tau(2), \tau(3), \dots, \tau(r+2))$$

to avoid $\overline{12}3$. For these permutations, the only way that we have a $\overline{12}3$ pattern starting with r is when $\tau(2) = r+1$, then $(\tau(1), \tau(2), r+2)$ is a $\overline{12}3$ pattern. Here, it is easy to see that if $\tau(2) = r+1$, then, for all $2 < i < j \leq r+2$, $(\tau(2), \tau(i), \tau(j))$ is never a $\overline{12}3$ pattern. Hence, by Theorem 4.7.3, the number of permutations $(\tau(2), \tau(3), \dots, \tau(r+2))$, with $\tau(2) = r+1$, avoiding $\overline{12}3$ is $|S_r(\overline{12}3)| = B_r$. Using subtraction, we have $|S_{r+2,r}(\overline{12}3)| = |S_{r+1}(\overline{12}3)| - |S_r(\overline{12}3)| = B_{r+1} - B_r$. □

Lemma 4.7.7. *For all* $r \geq 1$*,*

$$|S_{r+1,r}(3\overline{21})| = 2^{r-1}.$$

Proof. Let $\tau \in S_{r+1,r}(3\overline{21})$ and let $i > 1$ be such that $\tau(i) = r+1$. Then for all $j \in \{2, 3, \dots, i-2\}$, we must have $\tau(j) < \tau(j+1)$. To see this, suppose that $\tau(j) > \tau(j+1)$ for some $j \in \{2, 3, \dots, i-2\}$. Then $r\tau(j)\tau(j+1)$ is a $3\overline{21}$ pattern which is a contradiction. Similarly, for all $j \in \{i+1, i+2, \dots, n-1\}$, we must have $\tau(j) < \tau(j+1)$. Hence, we have $\tau(2) < \tau(3) < \dots < \tau(i-1)$ and $\tau(i+1) < \tau(i+2) < \dots < \tau(r+1)$.

On the other hand, it is easy to check that for all $\tau \in S_{r+1,r}$, if $\tau(i) = r+1$, $\tau(2) < \tau(3) < \cdots < \tau(i-1)$, and $\tau(i+1) < \tau(i+2) < \cdots < \tau(r+1)$ for some $i > 1$, then τ avoids $3\overline{21}$.

So $|S_{r+1,r}(3\overline{21})|$ is equal to the number of permutations $\tau \in S_{r+1,r}$ such that for some $i \in \{2, 3, \ldots, r+1\}$, we have $\tau(i) = r+1$, $\tau(2) < \tau(3) < \cdots < \tau(i-1)$, and $\tau(i+1) < \tau(i+2) < \cdots < \tau(r+1)$. Let τ be such a permutation and $i \in \{2, 3, \ldots, r+1\}$. Then there are $\binom{r-1}{i-2}$ ways to choose $i-2$ numbers from $\{1, 2, \ldots, r-1\}$ and assign them to $\tau(2), \tau(3), \ldots, \tau(i-1)$ so that $\tau(2) < \tau(3) < \cdots < \tau(i-1)$; once $\tau(2), \tau(3), \ldots, \tau(i-1)$ are determined, $\tau(i+1), \tau(i+2), \ldots, \tau(r+1)$ are uniquely determined as well. Hence we have

$$|S_{r+1,r}(3\overline{21})| = \sum_{i=2}^{r+1} \binom{r-1}{i-2} = \sum_{i=0}^{r-1} \binom{r-1}{i} = 2^{r-1}. \qquad \square$$

If $r \geq 5$, by Tables 4.1 and 4.3 and Lemmas 4.2.10 and 4.2.11, there are nine r-Wilf equivalence classes. To see this, it suffices to note that for each $r \geq 5$ and for any two distinct generalized patterns σ and σ' in different rows, either $|S_{r,r}(\sigma)| \neq |S_{r,r}(\sigma')|$, or $|S_{r+1,r}(\sigma)| \neq |S_{r+1,r}(\sigma')|$, or $|S_{r+2,r}(\sigma)| \neq |S_{r+2,r}(\sigma')|$. We briefly describe several of them as examples.

Example 4.7.8. By Table 4.3 and Lemma 4.2.10, for all $r \geq 5$, $|S_{r,r}(\overline{13}2)| = C_{r-1} < B_{r-1} = |S_{r,r}(\overline{12}3)|$. Hence, for all $r \geq 5$, $\overline{13}2$ and $\overline{12}3$ are not r-Wilf equivalent.

Example 4.7.9. By Table 4.3 and Lemma 4.2.11, for all $r \geq 5$, $|S_{r,r}(\overline{21}3)| = B_{r-1} = |S_{r,r}(\overline{23}1)|$, but $|S_{r+1,r}(\overline{21}3)| = B_{r-1} < B_r - B_{r-1} = |S_{r+1,r}(\overline{23}1)|$. Hence, for all $r \geq 5$, $\overline{21}3$ and $\overline{23}1$ are not r-Wilf equivalent.

Example 4.7.10. By Table 4.3, we have $|S_{r,r}(1\overline{23})| = B_{r-1} = |S_{r,r}(\overline{12}3)|$ and

$$|S_{r+1,r}(1\overline{23})| = B_r = |S_{r+1,r}(\overline{12}3)|,$$

but $|S_{r+2,r}(1\overline{23})| = B_{r+1} - B_{r-1} > B_{r+1} - B_r = |S_{r+2,r}(\overline{12}3)|$ for all $r \geq 5$. Hence $1\overline{23}$ and $\overline{12}3$ belong to two different equivalence classes when $r \geq 5$.

Example 4.7.11. By Table 4.3, we have $|S_{r,r}(2\overline{13})| = C_{r-1}$ and $|S_{r,r}(\overline{32}1)| = B_{r-2}$ for all $r \geq 5$. By Table 4.1 and the generating functions of the Catalan and Bell numbers [36, Sections 3.2 and 6.1], we have $B_{r-1} \neq C_r$ for all $r \geq 5$. Hence $2\overline{13}$ and $\overline{32}1$ belong to two different equivalence classes when $r \geq 5$.

The following theorem completely classifies, for all $r \geq 5$, the r-Wilf-equivalence classes for the twelve vincular patterns of length three.

Theorem 4.7.12. *For all $r \geq 5$, there are nine r-Wilf-equivalence classes for vincular patterns of length three: $2\overline{13} \overset{r}{\sim} 2\overline{31}$, $1\overline{23} \overset{r}{\sim} 1\overline{32}$, $3\overline{21} \overset{r}{\sim} 3\overline{12}$, and the other six classes each contains a single vincular pattern.*

4.8 Concluding remarks

Miner and Pak [88] used generalizations of Theorem 4.1.1 to study the limit shapes of random permutations avoiding a given pattern. In this section, we give some ideas about the limit shapes of random σ-avoiding, $\sigma \in S_3$, permutations with fixed prefix $(c_1, c_2, \ldots, c_t)$. Particularly, we are interested in exploring for large n, what a uniformly random permutation from $S_{(c_1,c_2,\ldots,c_t)}(\sigma)$ looks like. To do this, we follow Miner and Pak [88] and view permutations as matrices. That is, for each $\tau \in S_n$, we look at the $n \times n$ matrix $M(\tau)$ such that $(M(\tau))_{jk} = 1$ if $\tau(j) = k$ and $(M(\tau))_{jk} = 0$ if $\tau(j) \neq k$. By Lemma 4.2.2, complementary patterns may be studied in pairs and it suffices to examine permutations avoiding the patterns 123, 132, and 231.

In some situations, this question is easy to answer. If $1 \in \{c_1, c_2, \ldots, c_t\}$, then there is a unique permutation that avoids a 123 pattern, as the later $n - t$ digits need to be decreasing to avoid a 23 pattern in the second unfixed segment. So after asymptotic scaling, the limit of the unfixed segment is just the anti-diagonal $x + y = 1$. The situation becomes more complicated when $1 \notin \{c_1, c_2, \dots, c_t\}$. As shown in Theorem 4.4.4, we may project our permutation from S_n where the first t coordinates are fixed down to a permutation from S_{n-t+1} where only the first coordinate is fixed via 'standardization.' The limiting phenomenon of these generic 'reduced' 123-avoiding permutations were studied in Miner and Pak [88], where the anti-diagonal again shows up. As pointed out earlier in Section 4.4, the result for the 132 pattern is similar to the 123 pattern and the structure of the pattern-avoiding permutation is also preserved after projection via standardization. See Theorem 4.4.5 for more details. The limiting phenomenon of these reduced 132-avoiding permutations was also studied in Miner and Pak [88], where the anti-diagonal as well as the lower right corner show up in the asymptotic analysis. Unlike 123 and 132 patterns, fixing the prefix $(c_1, c_2, \dots, c_t)$, a uniformly random permutation avoiding a 231 pattern will instead display a block structure as hinted in our proof of Theorem 4.4.1. For the initial block which consists of $c_1, c_2, \dots, c_t$, the segment of the permutation will be a fixed curve that is asymptotically in correspondence to the prefix $(c_1, c_2, \dots, c_t)$; and for all the remaining blocks, the segment of the permutation will lie on the boundary of feasible 231-density asymptotically. See Kenyon et al. [73] and the references therein for a description of the limit shapes of these feasible regions.

We have only scratched the surface of enumerating pattern-avoiding permutations by fixed prefixes, mostly concentrating on patterns of length three. It would be interesting to study permutations with fixed prefixes that avoid other patterns; for instance, all single patterns of length greater than three are open. It would also be interesting to compute limits of pattern avoiding permutations chosen uniformly under certain constraints; fixing the prefix $(c_1, c_2, \dots, c_t)$ as we have done in this paper is only one of the many possibilities out there.

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Chapter 5: Pattern avoidance for permutations and their rotations

5.1 Introduction

Bóna and Smith [19] initiated the study of permutations p such that both p and p^2 avoid a given pattern. Here p^2 is the product of p and itself when one views S_n as a group. Burcroff and Defant [25] went further to study permutations p such that all powers of p avoid a given pattern. Pan [96] resolved a conjecture in [25] and there are still many open problems in this area. In this chapter, we consider a similar question but with more combinatorial flavor. Instead of looking at group-theoretic operations, we look at rotations of permutations.

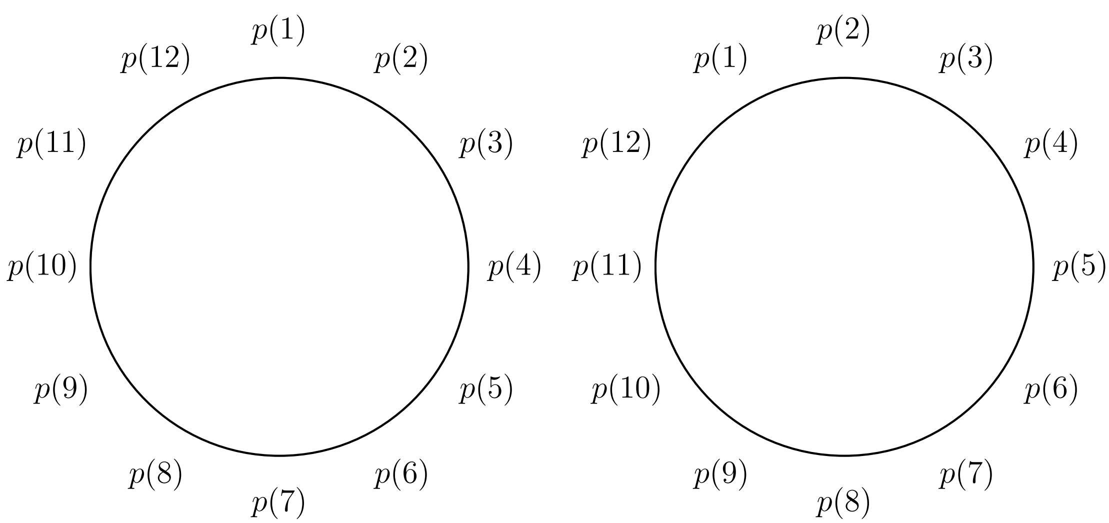

Figure 5.1: A permutation (left) of length 12 and its rotation (right).

For a permutation $p \in S_n$, we list the terms of p on a circle clockwise. We then look at pattern avoidance for p and the permutation obtained by rotating the circle counterclockwise so that each term is moved to its neighboring position on the circle. We call the latter a counterclockwise rotation (or simply, rotation) of p. Figure 5.1 shows an example of a permutation of length 12 and its rotation. For each permutation of length n, we can rotate it n times to get back to itself. Now we formally define the rotations of a permutation.

Definition 5.1.1. For all $p \in S_n$ and $k \in \{2, 3, \dots, n\}$, let

$$p^{(k)} := (p(k), p(k+1), \cdots, p(n), p(1), p(2), \cdots p(k-1))$$

be the *kth counterclockwise rotation* (or, simply, *rotation*) of p.

Example 5.1.2. Let $p = 2756431 \in S_7$. Then we have $p^{(2)} = 7564312$, $p^{(3)} = 5643127$, $p^{(4)} = 6431275$, $p^{(5)} = 4312756$, $p^{(6)} = 3127564$, and $p^{(7)} = 1275643$.

For consistency, we write $p^{(1)} = p$ and use them interchangeably.

Definition 5.1.3. For each $n \in \mathbb{N}$ and $k \in [n]$, let $S_n^{(k)}(\sigma)$ be the set of permutations $p \in S_n$ such that $p^{(1)}, p^{(2)}, \dots, p^{(k)}$ all avoid σ.

Notice that we have $S_n^{(1)}(\sigma) = S_n(\sigma)$ and

$$S_n^{(n)}(\sigma) \subseteq S_n^{(n-1)}(\sigma) \subseteq \cdots \subseteq S_n^{(2)}(\sigma) \subseteq S_n^{(1)}(\sigma) = S_n(\sigma).$$

Example 5.1.4. Let $p = 654231 \in S_6$. Then $p^{(1)} = p = 654231$ and $p^{(2)} = 542316$ both avoid the pattern 132. Hence $p \in S_6^{(2)}(132)$. However, in $p^{(3)} = 423165$, since 465 is a 132 pattern, $p \notin S_6^{(2)}(132)$

We first determine $|S_n^{(k)}(\sigma)|$ where $2 \leq k \leq n$ and $\sigma \in S_3$. For monotone patterns of length three, we have

$$|S_n^{(k)}(123)| = |S_n^{(k)}(321)| = 2^{n-k+1} + k - 2;$$

and for nonmonotone patterns of length three, we have

$$|S_n^{(k)}(213)| = |S_n^{(k)}(231)| = |S_n^{(k)}(312)| = |S_n^{(k)}(132)| = k - 2 + \sum_{\ell=0}^{n-k+1} C_\ell,$$

where C_ℓ is the ℓ-th Catalan number. Interestingly, for $k = 2$, the expression above becomes $\sum_{\ell=0}^{n-1} C_\ell$. That is, the number of permutations p on $\{1, 2, \dots, n\}$ such that both p and $p^{(2)}$ avoid a nonmonotone pattern of length three is the sum of first n Catalan numbers.

Archer and Geary [4] generalized the results of Bóna and Smith [19] by considering permutations and their power avoiding different permutations. They call them permutation chains. For example, if p avoids σ_1 and p^2 avoids σ_2, then they say p avoids the permutation chain $(\sigma_1; \sigma_2)$. Inspired by this, we next study permutations p such that $p^{(1)}$ and $p^{(2)}$ avoid two different patterns of length three. For two different patterns $\sigma_1, \sigma_2 \in S_3$, we use $S_n^{(2)}(\sigma_1; \sigma_2)$ to denote the set of permutations p in S_n such that $p^{(1)}$ avoids σ_1 but $p^{(2)}$ avoids σ_2. We call $(\sigma_1; \sigma_2)$ a 2-chain of length three. For example, $S_n^{(2)}(123; 231)$ is the set of permutations $p \in S_n$ such that $p^{(1)}$ avoids 123 and $p^{(2)}$ avoids 231. We show that, for many 2-chains (σ_1, σ_2) of length three, we have $|S_n^{(2)}(\sigma_1; \sigma_2)| = |S_n(\sigma_1, \sigma_2)|$ where $S_n(\sigma_1, \sigma_2)$ is the set of permutations $p \in S_n$ such that p avoids both σ_1 and σ_2. If $|S_n^{(2)}(\sigma_1; \sigma_2)| \neq |S_n(\sigma_1, \sigma_2)|$, and σ_1 and σ_2 are not both monotone, then there are three possible expressions for $|S_n^{(2)}(\sigma_1; \sigma_2)|$: $2n - 2$, $(n + 2) \cdot 2^{n-3}$, and $\frac{1}{3}n^3 - n^2 + \frac{5}{3}n$.

Given the results for patterns of length three, the next natural step would be to study patterns of length four. In general, patterns of length greater than three are more difficult to deal with. However, we mange to derive a recursive relation for $|S_n^{(2)}(1324)|$. Other patterns of length four will be studied in the future.

5.2 Preliminaries

We first define clockwise rotations and show a relationship between counterclockwise rotations and clockwise rotations.

Definition 5.2.1. For all $1 \leq k \leq n$ and $p \in S_n$, let

$$p^{(k)'} = (p(n-k+1), p(n-k+2), \ldots, p(n), p(1), p(2), \ldots, p(n-k))$$

be the *kth clockwise rotation* of p.

Let $S_n^{(k)'}(\sigma)$ be the set of permutations $p \in S_n$ such that $p^{(1)'} = p, p^{(2)'}, \ldots, p^{(k)'}$ all avoid σ.

Lemma 5.2.2. *For all n and for all $\sigma \in S_n$, $|S_n^{(k)}(\sigma)| = |S_n^{(k)'}(\sigma)|$.*

Proof. Consider $f : S_n^{(k)}(\sigma) \to S_n^{(k)'}(\sigma)$ such that $f(p) = p^{(k)}$. We will show that f is a bijection.

Let $p \in S_n^{(k)}(\sigma)$. Then $p, p^{(2)}, \ldots, p^{(k)}$ all avoid σ. Notice that by the definitions of clockwise and counterclock-wise rotations, for all $1 \leq \ell \leq k$, we have $(p^{(k)})^{(\ell)'} = p^{(k-\ell+1)}$. So $p^{(k)} \in S_n^{(k)'}(\sigma)$.

Let $p_1, p_2 \in S_n^{(k)}(\sigma)$ with $p_1 \neq p_2$. Then $p_1^{(k)} \neq p_2^{(k)}$. Hence f is one-to-tone.

Let $q \in S_n^{(k)'}(\sigma)$. Then $q^{(k)} \in S_n^{(k)}(\sigma)$ with $f(q^{(k)}) = q$. So f is onto. □

Now we define the complement and reverse of a permutation. We will use these to reduce the cases we need for enumeration.

Definition 5.2.3. For a permutation $p \in S_n$, the *complement* p^c of p is a permutation in S_n defined by setting $p^c(i) = n + 1 - p(i)$, and the *reverse* p^r of p is a permutation in S_n defined by setting $p^r(i) = p(n + 1 - i)$.

Lemma 5.2.4. *For all $p \in S_n$ and $k \leq n$, we have $(p^{(k)})^r = (p^r)^{(k)'}$.*

Proof. Let $p = (p(1), p(2), \ldots, p(n)) \in S_n$. Then

$$p^{(k)} = (p(k), p(k+1), \ldots, p(n), p(1), p(2), \ldots, p(k-1)),$$

and hence

$$(p^{(k)})^r = (p(k-1), p(k-2), \ldots, p(1), p(n), p(n-1), \ldots, p(k)).$$

At the same time, we have $p^r = (p(n), p(n-1), \ldots, p(1))$ and hence

$$(p^r)^{(k)'} = (p(k-1), p(k-2), \ldots, p(1), p(n), p(n-1), \ldots, p(k)).$$

Hence $(p^{(k)})^r = (p^r)^{(k)'}$. □

Lemma 5.2.5. *For all positive integers k and n with $k \leq n$, we have*

$$|S_n^{(k)}(\sigma)| = |S_n^{(k)}(\sigma^c)| = |S_n^{(k)}(\sigma^r)|.$$

Proof. It is easy to see that $\left(p^{(k)}\right)^c = (p^c)^{(k)}$ for all $p \in S_n$. So $p \in S_n^{(k)}(\sigma)$ if and only if $p^c \in S_n^{(k)}(\sigma^c)$. Hence $|S_n^{(k)}(\sigma)| = |S_n^{(k)}(\sigma^c)|$.

To show that $|S_n^{(k)}(\sigma)| = |S_n^{(k)}(\sigma^r)|$, it suffices to show that $|S_n^{(k)'}(\sigma)| = |S_n^{(k)}(\sigma^r)|$. Indeed, $g : p \in S_n^{(k)}(\sigma^r) \mapsto p^r \in S_n^{(k)'}(\sigma)$ is a bijection. To see this, let $p \in S_n^{(k)}(\sigma^r)$. Then $p, p^{(2)}, \ldots, p^{(k)}$ all avoid σ^r. So $(p^{(\ell)})^r$ avoids σ for all $\ell \leq k$. By Lemma 5.2.4, we have $(p^r)^{(\ell)'} = (p^{(\ell)})^r$ for all $\ell \leq k$. Hence, $(p^r)^{(\ell)'}$ avoids σ for all $\ell \leq k$. It follows that $p^r \in S_n^{(k)'}(\sigma)$. Hence g is a well-defined function. It is easy to check that g is both one-to-one and onto. □

Notice that even though $|S_n^{(k)}(\sigma)| = |S_n^{(k)}(\sigma^r)|$, reverses and rotations do not commute. For example, let $p = 1234$. Then $p^r = 4321$, $p^{(2)} = 2341$, $\left(p^{(2)}\right)^r = 1432$, and $(p^r)^{(2)} = 3214$.

The following observation will be used in Section 5.5. The proof is elementary, and hence we state it without proof.

Lemma 5.2.6. *Let $n \in \mathbb{N}$, $r \in \{1, 2, \ldots, n\}$, σ a pattern, and $S_{n,r}^{(n)}(\sigma)$ the set of permutations in $S_n^{(n)}(\sigma)$ with leading term r. Then $|S_n^{(n)}(\sigma)| = n|S_{n,r}^{(n)}(\sigma)|$.*

5.3 Single patterns of length three

In this section, we enumerate $S_n^{(k)}(\sigma)$ for $k \in \{2, 3, \ldots, n\}$ and $\sigma \in S_3$. Notice that by Theorem 1.3.1, for all $n \geq 1$ and $\sigma \in S_3$, we have

$$|S_n^{(1)}(\sigma)| = |S_n(\sigma)| = C_n.$$

Theorem 5.3.1. *For all $2 \leq k \leq n$, we have*

$$|S_n^{(k)}(123)| = |S_n^{(k)}(321)| = 2^{n-k+1} + k - 2.$$

Proof. Fix $n \geq 2$. By Lemma 5.2.5, it suffices to prove the result for $S_n^{(k)}(123)$.

We first prove this when $k = 2$. Let $\mathcal{A}$ be a subset of S_n such that for all $p \in \mathcal{A}$ with $p(1) = \ell$, the subpermutation of p on $\{\ell + 1, \ldots, n\}$ is $(n, n-1, \ldots, \ell + 1)$ and the subpermutation of p on $\{\ell, \ell - 1, \ldots, 1\}$ is $(\ell, \ell - 1, \ldots, 1)$. By this definition, for each fixed ℓ, there are $\binom{n-1}{\ell-1}$ permutations $p \in \mathcal{A}$ with $p(1) = \ell$. Adding the permutations in $\mathcal{A}$ by their leading terms, we have

$$|\mathcal{A}| = \sum_{\ell=1}^{n} \binom{n-1}{\ell-1} = 2^{n-1}.$$

Now we show that $\mathcal{A} \subseteq S_n^{(2)}(123)$. Let $p \in \mathcal{A}$ with $p(1) = \ell$. If a subpermutation abc of p is a 123 pattern, then either $a < b < \ell$ or $\ell < b < c$. Both cases contradict the construction of $\mathcal{A}$. So we must have $p \in S_n^{(1)}(123)$. Notice this automatically implies that $(p(2), p(3), \ldots, n)$ avoids the pattern 123. Now we show that $p^{(2)}$ avoids 123. Suppose, by way of contradiction, a subpermutation abc of $p^{(2)}$ is a 123 pattern. Since $p^{(2)} = (p(2), p(3), \ldots, p(n), \ell)$ and $(p(2), p(3), \ldots, p(n))$ avoids 123, we must have $c = \ell$.

So $a < b < \ell$ and hence ab is a subpermutation of $p^{(2)}$ on $\{\ell - 1, \dots, 1\}$ which is a contradiction. Therefore, $p^{(2)} \in S_n^{(2)}(123)$. Hence, we have $\mathcal{A} \subseteq S_n^{(2)}(123)$.

It remains to show that $S_n^{(2)}(123) \subseteq \mathcal{A}$. Let $p \in S_n^{(2)}(123)$. Suppose $p(1) = \ell$. Then the subpermutation of p on $\{\ell+1, \dots, n\}$ must be $(n, n-1, \dots, \ell+1)$ because otherwise p contains a 123 pattern. At the same time, the subpermutation of p on $\{\ell, \ell-1, \dots, 1\}$ is $(\ell, \ell-1, \dots, 1)$ because otherwise the subpermutation of $p^{(2)}$ on $\{\ell, \ell-1, \dots, 1\}$ would contain a 123 pattern. Hence, we have $S_n^{(2)}(123) \subseteq \mathcal{A}$. This completes the proof for $k = 2$.

Now let $k \geq 3$. Notice that $S_n^{(k)}(123) \subseteq S_n^{(2)}(123) = \mathcal{A}$. In particular, for all $p \in S_n^{(k)}(123)$ with $p(1) = \ell$, the subpermutation of p on $\{\ell+1, \dots, n\}$ is $(n, n-1, \dots, \ell+1)$ and the subpermutation of p on $\{\ell, \ell-1, \dots, 1\}$ is $(\ell, \ell-1, \dots, 1)$. Let $p \in S_n^{(k)}(123)$ with $p(1) = \ell$.

Case 1: $1 \leq \ell \leq k-2$. In this case, we will show that $p = (\ell, \ell-1, \dots, 1, n, n-1, \dots, \ell+1)$. Since $S_n^{(k)}(123) \subseteq \mathcal{A}$, this is true when $\ell = 1$. So we assume that $\ell \geq 2$. Suppose, by way of contradiction, $p \neq (\ell, \ell-1, \ell-2, \dots, 1, n, n-1, \dots, \ell+1)$. $S_n^{(k)}(123) \subseteq \mathcal{A}$, then there exists $i \leq \ell - 1$ such that n comes before $\ell - i$. Let $j \leq \ell - 1$ be the smallest such that n comes before $\ell - j$. Then $(\ell - j, \ell, n)$ is a subpermutation of $p^{(j+1)}$ and $(\ell - j, \ell, n)$ is a 123 patterns, we have a contradiction.

So for each $\ell \in \{1, 2, \dots, k-2\}$, there is a unique $p \in S_n^{(k)}(123)$ with $p(1) = \ell$.

Case 2: $\ell \geq k-1$. In this case, we have $(p(1), p(2), \dots, p(k-1)) = (\ell, \ell-1, \dots, \ell-k+2)$. The proof is similar to Case 1. Since the subpermutation of p on $\{1, 2, \dots, \ell-k+1\}$ is $(\ell-k+1, \ell-k, \dots, 1)$ and the subpermutation of p on $\{\ell+1, \ell+2, \dots, n\}$ is $(n, n-1, \dots, \ell+1)$, there are $\binom{n-\ell+\ell-k+1}{\ell-k+1} = \binom{n-k+1}{\ell-k+1}$ possibilities for p. It is easy to check that, for all these possibilities, $p, p^{(2)}, \dots, p^{(k)}$ all avoid 123.

Adding all the possibilities together, we have $|S_n^{(k)}(123)| = k - 2 + \sum_{\ell=k-1}^{n} \binom{n-k+1}{\ell-k+1} = 2^{n-k+1} + k - 2$. $\square$

Theorem 5.3.2. *For all* $2 \le k \le n$*, we have*

$$|S_n^{(k)}(213)| = |S_n^{(k)}(231)| = |S_n^{(k)}(312)| = |S_n^{(k)}(132)| = k - 2 + \sum_{\ell=0}^{n-k+1} C_\ell.$$

Proof. Since $(213)^c = 231$, $(213)^r = 312$, and $(312)^c = 132$, by Lemma 5.2.5, it suffices to prove the result for $S_n^{(k)}(213)$.

We first prove this for $k = 2$. Let $\mathcal{B} \subseteq S_n$ be the set of permutations such that for all $p \in \mathcal{B}$ with $p(1) = \ell$, $p = (\ell, p(2), p(3), \ldots, p(n-\ell+1), 1, 2, \ldots, \ell-1)$ where $(p(2), p(3), \ldots, p(n-\ell+1))$ avoids 213. For each ℓ, by Theorem 1.3.1, there are $C_{n-\ell}$ possibilities for $(p(2), p(3), \ldots, p(n-\ell+1))$. Hence, we have

$$|\mathcal{B}| = \sum_{\ell=1}^{n} C_{n-\ell} = \sum_{\ell=0}^{n-1} C_\ell.$$

We still need to show that $\mathcal{B} = S_n^{(2)}(213)$. Let $p \in \mathcal{B}$ with $p(1) = \ell$. We need to show that p and $p^{(2)}$ both avoid 213. Suppose, by way of contradiction, that abc is a subpermution of p and abc is a 213 pattern. Then $b < a < c$. If $a = \ell$, then since $b < a$ and $c > a$, c must come before b which is a contradiction. If $a > \ell$, then since $c > a > \ell$ and b comes before c, abc is a subpermutation of $(p(2), p(3), \ldots, p(n-\ell+1))$ which is a contradiction. If $a < \ell$, then since b comes after a by the definition of $\mathcal{A}$, we have $b < a$ which is a contradiction. Hence, we have $p \in S_n(213)$. We still need to show that $p \in S_n^{(2)}(123)$. Suppose, by way of contradiction, that abc is a subpermution of $p^{(2)}$ and abc is a 213 pattern. Then $b < a < c$. If $c < \ell$, by the definition of $\mathcal{B}$, b must come before a which is a contradiction. If $c > \ell$, then abc is a subpermutation of $(p(2), p(3), \ldots, p(n-\ell+1))$. Since $(p(2), p(3), \ldots, p(n-\ell+1))$ is 213 avoiding, we have a contradiction. If $c = \ell$, then $a < \ell$ and hence $b > a$ by the construction of $\mathcal{A}$ which is a contradiction. Hence $p^{(2)}$ avoids the pattern 213. This prove that $\mathcal{B} \subseteq S_n^{(2)}(213)$.

Now let $p \in S_n^{(2)}(213)$ with $p(1) = \ell$. Since $p^{(2)}$ avoids 213, the subpermutation of p on $\{1, 2, \ldots, \ell - 1\}$ must be $(1, 2, \ldots, \ell - 1)$. Otherwise, there would exist $a, b \in \{1, 2, \ldots, \ell - 1\}$ such that $a < b$ and $ba\ell$ is a subpermutation of $p^{(2)}$. Now let $x > \ell$ and $y < \ell$. Is y comes before x, then the subpermutation ℓyx of p would be a 213 pattern. Hence x comes before y. It follows that $p = (\ell, p(2), p(3), \ldots, p(n-\ell+1), 1, 2, \ldots, \ell-1)$. Since p avoids 213, we have that $(p(2), p(3), \ldots, p(n - \ell + 1))$ is 213-avoiding. Hence we have $S_n^{(2)}(213) \subseteq \mathcal{B}$.

So we have $|S_n^{(2)}(213)| = \sum_{\ell=0}^{n-1} C_\ell$.

Now suppose $k \geq 3$. Notice that $S_n^{(k)}(213) \subseteq S_n^{(2)}(213) = \mathcal{B}$. So for all $p \in S_n^{(k)}(213)$ with $p(1) = \ell$, $p = (\ell, p(2), p(3), \ldots, p(n - \ell + 1), 1, 2, \ldots, \ell - 1)$ where $(p(2), p(3), \ldots, p(n - \ell + 1))$ avoids 213. Let $p \in S_n^{(k)}(213)$ with $p(1) = \ell$.

Case 1: $n - \ell \leq k - 2$. In this case, we must have $(p(2), p(3), \ldots, p(n - \ell + 1)) = (\ell + 1, \ell + 2, \ldots, n)$. Suppose not. Then there exists $x > y > \ell$ such that x comes before y. Then $y\ell x$ is a 123 pattern and a subpermutation of $p^{(i)}$ for some $i \leq k$ which is a contradiction.

Hence, for each ℓ with $n - \ell \leq k - 2$, there is a unique $p \in S_n^{(k)}(213)$ with $p(1) = \ell$. Notice that if $n - \ell \leq k - 2$, then $k \geq n - k + 2$. So there are $k - 1$ such values for ℓ.

Case 2: $n - \ell \geq k - 1$. Then, similar to Case 1, we have $(p(2), p(3), \ldots, p(k - 1)) = (\ell + 1, \ell + 2, \ldots, \ell + k - 2)$. By Theorem 1.3.1, there are $C_{n-(\ell+k-2)}$ possibilities for $(p(k), p(k + 1), \ldots, p(n - \ell + 1))$. It is easy to check that, for all these possibilities, $p, p^{(2)}, \ldots, p^{(k)}$ all avoid 213.

Notice that if $n - \ell \geq k - 1$, then $k \leq n - k + 1$.

Adding all the possibilities together, we have

$|S_n^{(k)}(213)| = k-1+\sum_{\ell=1}^{n-k+1} C_{n-(\ell+k-2)} = k-1+\sum_{\ell=1}^{n-k+1} C_\ell = k-2+\sum_{\ell=0}^{n-k+1} C_\ell$. $\square$

Remark 5.3.3. If $\sigma \in S_3$ is not monotone, then

$$|S_n^{(2)}(\sigma)| = \sum_{\ell=0}^{n-1} C_\ell.$$

That is, $|S_n^{(2)}(\sigma)|$ is equal to the first n Catalan numbers. This is related to a result by Adeniran and Pudwell [1, Theorem 4.5].

Remark 5.3.4. By Theorems 5.3.1 and 5.3.2, we have $|S_n^{(n)}(\sigma)| = n$ for all $n \geq 1$ and for all $\sigma \in S_3$.

5.4 2-chains of patterns of length three

In this section, we enumerate $S_n^{(2)}(\sigma_1; \sigma_2)$ where $\sigma_1, \sigma_2 \in S_3$ and $\sigma_1 \neq \sigma_2$. Throughout this section, we assume that $n \geq 2$. Notice that, similar to Lemma 5.2.5, we have $|S_n^{(2)}(\sigma_1; \sigma_2)| = |S_n^{(2)}(\sigma_1^c; \sigma_2^c)|$. Hence, we restrict to the cases where $\sigma_1 \in \{123, 132, 213\}$. By the Erdős-Szekeres theorem [44, p. 467], we have $|S_n^{(2)}(123; 321)| = 0$ for all $n \geq 6$. Hence, we only need to consider the following 14 possibilities for $(\sigma_1; \sigma_2)$:

$$(123; 132), (123; 213), (123; 231), (123; 312), (132; 123), (132; 213), (132; 231),$$
$$(132; 321), (132; 312), (213; 123), (213; 132), (213; 231), (213; 312), (213; 321).$$

Our results are summarized in Table 5.1.

$(\sigma_1; \sigma_2)$	$\lvert S_n^{(2)}(\sigma_1; \sigma_2)\rvert$
$(123; 132)$, $(132; 231)$, $(132; 312)$, $(213; 123)$, $(213; 231)$, $(213; 312)$	2^{n-1}
$(123; 312)$, $(213; 321)$	$\binom{n}{2} + 1$
$(123; 213)$, $(132; 123)$, $(132; 213)$	$2n - 2$
$(123; 231)$, $(132; 321)$	$\frac{1}{3}n^3 - n^2 + \frac{5}{3}n$
$(213; 132)$	$(n + 2) \cdot 2^{n-3}$

Table 5.1: 2-chains of patterns of length three for $n \geq 2$.

To prove the results in Table 5.1, we need Theorem 4.5.1 by Simion and Schmidt [113] on the number of permutation avoiding two different patterns of length three. Similar to Chapter 4, we use $S_n(\sigma_1, \sigma_2)$ to denote the set of permutations $p \in S_n$ which avoids both σ_1 and σ_2. The key observation here is that if $p \in S_n^{(2)}(\sigma_1; \sigma_2)$, then $(p(2), p(3), \ldots, p(n))$ avoids both σ_1 and σ_2.

Proposition 5.4.1. *For all $n \geq 2$,*

$$\begin{aligned} |S_n^{(2)}(123; 132)| =& |S_n^{(2)}(132; 231)| = |S_n^{(2)}(132; 312)| = |S_n^{(2)}(213; 123)| \\ =& |S_n^{(2)}(213; 231)| = |S_n^{(2)}(213, 312)| = 2^{n-1}. \end{aligned}$$

Proof.

- $|S_n^{(2)}(123; 132)| = 2^{n-1}$. Let $p \in S_n^{(2)}(123, 132)$ with $p(1) = \ell$. Since p avoids 123, the subpermutation on $\{\ell+1, \ell+2, \ldots, n\}$ is decreasing. Since $p^{(2)}$ avoids 132, if $x > \ell$ and $y < \ell$, then x comes before y. So we have

  $$(p(2), p(3), \ldots, p(n-\ell+1)) = (n, n-1, \ldots, \ell+1).$$

  If $\ell = 1$, then there is one such permutation. If $\ell \geq 2$, by Theorem 4.5.1, there are $2^{\ell-2}$ choices for the subpermutation on $\{1, 2, \ldots, \ell-1\}$. Hence we have

  $$|S_n^{(2)}(123, 132)| = 1 + \sum_{\ell=2}^{n} 2^{\ell-2} = 2^{n-1}.$$

- $|S_n^{(2)}(132; 231)| = 2^{n-1}$. Let $p \in S_n^{(2)}(132; 231)$ with $p(1) = \ell$.

  We first suppose that $\ell \leq n-2$. Then either $(\ell, n, n-1)$ is a subpermutation of p or $(n-1, n, \ell)$ is a subpermutation of $p^{(2)}$.

Next suppose $\ell = n$ or $n-1$. Then $(p(2), p(3), \ldots, p(n))$ avoids both 132 and 231. By Theorem Theorem 4.5.1, there are $2^{(n-1)-1} = 2^{n-2}$ possibilities for each value of ℓ. So we have $2^{n-2} + 2^{n-2} = 2^{n-1}$ possibilities in total.

- $|S_n^{(2)}(132; 312)| = 2^{n-1}$. Let $p \in S_n^{(2)}(132; 312)$ with $p(1) = \ell$. Since p avoids 132, the subpermutation on $\{\ell+1, \ell+2, \ldots, n\}$ is increasing. Since $p^{(2)}$ avoids 312, for all $x < \ell$ and $y > \ell$, x comes before y. Otherwise, $yx\ell$ would be a 312 pattern. So $\{p(2), p(3), \ldots, p(\ell)\} = \{1, 2, \ldots, \ell-1\}$ and $(p(\ell+1), p(\ell+2), \ldots, p(n)) = (n, n-1, \ldots, \ell+1)$. It is easy to check that as long as $(p(2), p(3), \ldots, p(\ell))$ avoids both 132 and 312, we have $p \in S_n^{(2)}(132; 312)$. By Theorem 4.5.1, if $\ell \geq 2$, then there are $2^{(\ell-1)-1}$ possibilities for $(p(2), p(3), \ldots, p(\ell))$. Adding all the contributions, we have
$$|S_n^{(2)}(132; 312)| = 1 + \sum_{\ell=2}^{n} 2^{(\ell-1)-1} = 2^{n-1}.$$

- $|S_n^{(2)}(213; 123)| = 2^{n-1}$. Let $p \in S_n^{(2)}(213; 123)$. Since p avoids 213, for all $x < \ell$ and $y > \ell$, y comes before x. So $\{p(2), p(3), \ldots, p(n-\ell+1) = \{\ell+1, \ell+2, \ldots, n\}$ and $\{p(n-\ell+2), p(n-\ell+3), \ldots, p(n)\} = \{1, 2, \ldots, \ell-1\}$. Since $p^{(2)}$ avoids 123, the subpermutation on $\{1, 2, \ldots, \ell-1\}$ is decreasing. So there is only one possibility for p when $\ell = n$. If $\ell \leq n-1$, then by Theorem 4.5.1, there are $2^{(n-\ell)-1}$ possibilities for $(p(2), p(3), \ldots, p(n-\ell+1))$. Adding all the contributions, we have
$$|S_n^{(2)}(132; 312)| = 1 + \sum_{\ell=1}^{n-1} 2^{(n-\ell)-1} = 2^{n-1}.$$

- $|S_n^{(2)}(213; 231)| = 2^{n-1}$. This is similar to the previous case.

- $|S_n^{(2)}(213, 312)| = 2^{n-1}$. Let $p \in S_n^{(2)}(213, 312)$ with $p(1) = \ell$. We first suppose that $\ell \neq 1, n$. Then for some $x < \ell$ and $y > \ell$, either ℓxy is a subpermutation of p or $yx\ell$ is a subpermutation of $p^{(2)}$. So either p has a 213 pattern or $p^{(2)}$ has a 312 pattern which is a contradiction. Next, suppose $p(1) = 1$. Since p avoids 213, $(p(2), p(3), \ldots, p(n))$ avoids 213. Since $p^{(2)}$ avoids 312, $(p(2), p(3), \ldots, p(n))$ avoids 312. By Theorem 4.5.1, there are 2^{n-2} possibilities for $(p(2), p(3), \ldots, p(n))$. It is easy to check p avoids 213 and $p^{(2)}$ avoids 312 for all those possibilities. Similarly, there are 2^{n-2} possibilities when $p(1) = n$. Adding all the contributions, we have $|S_n^{(2)}(213, 312)| = 2^{n-2} + 2^{n-2} = 2^{n-1}$. □

Proposition 5.4.2. *For all $n \geq 2$, we have*

$$|S_n^{(2)}(123; 312)| = |S_n^{(2)}(213, 321)| = \binom{n}{2} + 1.$$

Proof. We will prove the result for $S_n^{(2)}(123; 312)$. The proof for $S_n^{(2)}(213, 321)$ is similar.

Let $p \in S_n^{(2)}(123; 312)$ with $p(1) = \ell < n$. Since p avoids 123, the subpermutation on $\{\ell+1, \ell+2, \ldots, n\}$ is decreasing. Since $p^{(2)}$ avoids 312, for all $x < \ell$ and $y > \ell$, x comes before y. Now since p avoids 123, the subpermutation on $\{1, 2, \ldots, \ell - 1\}$ is decreasing. So $p = (\ell, \ell - 1, \ldots, 1, n, n - 1, \ldots, \ell + 1)$.

Let $p \in S_n^{(2)}(123; 312)$ with $p(1) = n$. Then $(p(2), p(3), \ldots, p(n)) \in S_{n-1}(123, 312)$. By Theorem 4.5.1, there are $\binom{n-1}{2} + 1$ possibilities.

Adding all the contributions, we have $|S_n^{(2)}(123; 312)| = \binom{n-1}{2} + 1 + n - 1 = \binom{n}{2} + 1$. □

Proposition 5.4.3. *For all $n \geq 2$,*

$$|S_n^{(2)}(123; 213)| = |S_n^{(2)}(132; 123)| = |S_n^{(2)}(132; 213)| = 2n - 2.$$

Proof.

- $|S_n^{(2)}(123;213)| = 2n-2$. Let $p \in S_n^{(2)}(123,213)$ with $p(1) = \ell$. First suppose that $\ell \geq 4$. Since $p^{(2)}$ avoids 213 and $p^{(2)}(n) = \ell \geq 4$, the subpermutation on $\{1,2,3\}$ is 123 which contradicts that p avoids 123.

  Next, suppose $\ell = 1$. Since p avoids 123, the subpermutation on $\{2,3,\ldots,n\}$ is decreasing. So $p = (1,n,n-2,\ldots,2)$.

  Now, suppose $\ell = 2$. Since p avoids 123, the subpermutation on $\{3,\ldots,n\}$ is decreasing. Now there are $n-1$ possibilities for the location of 1.

  Finally, suppose $\ell = 3$. Since $p^{(2)}$ avoids 213 and $p^{(2)}(n) = 3$, the subpermutation on $\{1,2\}$ is 12. Since p avoids 123, the subpermutation on $\{4,5,\ldots,n\}$ is decreasing. For all $x > 3$, we must have that x comes before 2. Otherwise, $12x$ would be a 123 pattern. Hence we have $p(n) = 2$ and there are $n-2$ possibilities for the location of 1.

  Adding up all the contributions, we have

  $$|S_n^{(2)}(123,213)| = 1 + (n-1) + (n-2) = 2n-2.$$

- $|S_n^{(2)}(132;123)| = 2n-2$. This is similar to the previous case.

- $|S_n^{(2)}(132;213)| = 2n-2$. Let $p \in S_n^{(2)}(132;213)$ with $p(1) = \ell$. Since p avoids 132, the subpermutation on $\{\ell+1,\ell+2,\ldots,n\}$ is increasing. Since $p^{(2)}$ avoids 213 and $p^{(2)}(n) = \ell$, the subpermutation on $\{1,2,\ldots,\ell-1\}$ is also increasing. Since $(p(2),p(3),\ldots,p(n))$ avoids both 132 and 213, for all $x,y > \ell$ and $u,v < \ell$ with $x \neq y$ and $u \neq v$, u and v can be between x and y, and vice versa. Otherwise, we would have either a 132 pattern or 213 pattern. Hence either $(p(2),p(3),\ldots,p(\ell)) = (1,2,\ldots,\ell-1)$ or $(p(n-\ell+2),p(n-\ell+1),\ldots,p(n)) = (1,2,\ldots,\ell-1)$. Hence, if $\ell = 1$ or $\ell = n$, then there is one possibility; and if $\ell \neq 1,n$, then there are two possibilities. Adding up all the contributions, we have $|S_n^{(2)}(132;213)| = 1+1+2(n-2) = 2n-2$.

This completes the proof. $\square$

Proposition 5.4.4. *For all* $n \geq 2$*, we have*

$$|S_n^{(2)}(123;231)| = |S_n^{(2)}(132;321)| = \frac{1}{3}n^3 - n^2 + \frac{5}{3}n.$$

Proof. First notice that $|S_2^{(2)}(123;231)| = |S_2^{(2)}(132;321)| = 2$. Both of them satisfy the expression in the proposition. So for the rest of this proof, we assume that $n \geq 3$. We first prove the result for $S_n^{(2)}(123;231)$. We consider four cases based on the value of ℓ.

Let $p \in S_n^{(2)}$ with $p(1) = \ell$. First notice that $(p(2), p(3), \dots, p(n))$ avoids both 123 and 231. Moreover, the subpermutation of p on $\{\ell+1, \ell+2, \dots, n\}$ must be $(n, n-1, \dots, \ell+1)$.

Case 1: $\ell = 1$. Then $(p(2), \dots, p(n)) = (n, n-1, \dots, 2)$.

Case 2: $\ell = 2$. Then there are $n-1$ possibilities for the location of 1.

Case 3: $\ell = n$. Then $(p(2), \dots, p(n)) \in S_{n-1}(123, 231)$. By [113, Lemma 5], there are $\binom{n-1}{2} + 1$ possibilities.

Case 4: $3 \leq \ell \leq n-1$. Let p' be the subpermutation of p on $\{1, 2, \dots, \ell-1\}$.

Subcase 4.1: $p' = (\ell-1, \ell-2, \dots, 1)$. Then $\ell+1, \ell+2, \dots, n$ must be either before $p'(1)$ or after $p'(\ell-1)$. Otherwise, there would be a 231 pattern. Hence, there are $n-\ell+1$ possibilities.

Subcase 4.2: $p'(r) = \ell-1$ with $r \geq 2$. We will show that we must have $p' = (r-1, r-2, \dots, 1, \ell-1, \ell-2, \dots, r)$ in this case. Since p' avoids 123, $(p'(1), p'(2), \dots, p'(r-1))$ must be decreasing. Since p' avoids 231, for all $x \in \{p'(1), p'(2), \dots, p'(r-1)\}$ and $y \in \{p'(r+1), p'(r+2), \dots, p'(\ell-1)\}$, we must have $x < y$. It follows that $\{p'(1), p'(2), \dots, p'(r-1)\} = \{1, 2, \dots, r-1\}$. Hence $(p'(1), p'(2), \dots, p'(r-1)) = (r-1, r-2, \dots, 1)$. Since $r \geq 2$, 1 comes before $\ell-1$. Now if $(p'(r+1), p'(r+

$2), \ldots, p'(\ell-1))$ is not decreasing, then we would have a 123 pattern which is a contradiction. Hence $p' = (r-1, r-2, \ldots, 1, \ell-1, \ell-2, \ldots, r)$. Now the only possible locations for $\ell+1, \ell+2, \ldots, n$ are before $p'(1)$ or between $p'(r-1)$ and $p'(r)$. Otherwise, $(p(2), p(3), \ldots, p(n))$ either has a 231 pattern or a 123 pattern. In total, we have $(\ell-2)(n-\ell+1)$ possibilities for $(p(2), p(3), \ldots, p(n))$.

Subcase 4.3: $p'(1) = \ell - 1$ but $p' \neq (\ell-1, \ell-2, \ldots, 1)$. In this case, there exists $x < y < \ell-1$ such that x comes before y. Now let $z \in \{\ell+1, \ell+2, \ldots, n\}$. If z comes after y, then xyz is a 123 pattern; if z is located between $\ell-1$ and y, then $(\ell-1, z, y)$ is a 231 pattern. Hence the only possible locations for $\ell+1, \ell+2, \ldots, n$ are before $p'(1) = \ell-1$. To summarize, $p = (\ell, n, n-1, \ldots, \ell+1, \ell-1, p(n-\ell+3), p(n-\ell+4), \ldots, p(n))$. By Theorem 4.5.1, there are $\binom{\ell-2}{2} + 1 - 1 = \binom{\ell-2}{2}$ possibilities for $(p(n-\ell+3), p(n-\ell+4), \ldots, p(n)) \neq (\ell-2, \ell-3, \ldots, 1)$.

Adding them all together, we have

$$\begin{aligned}
&|S_n^{(2)}(123;231)| \\
=&1 + (n-1) + \binom{n-1}{2} + 1 + \sum_{\ell=3}^{n-1} \left[(n-\ell+1) + (\ell-2)(n-\ell+1) + \binom{\ell-2}{2} \right] \\
=&\frac{1}{3}n^3 - n^2 + \frac{5}{3}n.
\end{aligned}$$

Now we prove the result for $S_n^{(2)}(132;321)$. Consider the function $f : S_n^{(2)}(312;123) \to S_n^{(2)}(123;231)$ where

$$f(\tau) = (n+1-\tau_1, n+1-\tau_n, n+1-\tau_{n-1}, \ldots, n+1-\tau_2).$$

It is easy to check that f is a bijection and hence $|S_n^{(2)}(312;123)| = |S_n^{(2)}(123;231)|$.

Since $(132)^c = 312$ and $(321)^c = 123$, we have $|S_n^{(2)}(132;321)| = |S_n^{(2)}(312;123)| = |S_n^{(2)}(123;231)|$. □

Remark 5.4.5. The integer sequence in Proposition 5.4.4 is the sequence A116731 in OEIS [94] and it counts the number of permutations avoiding the patterns 321, 2143, and 3124, as well as permutations avoiding the patterns 132, 2314, and 4312.

Proposition 5.4.6. *For all $n \geq 2$, we have*

$$|S_n^{(2)}(213; 132)| = (n+2) \cdot 2^{n-3}.$$

Proof. Let $p \in S_n^{(2)}(213; 132)$ with $p(1) = \ell$. Since p avoids 213, if $x < \ell$ and $y > \ell$, then y comes before x. So $\{p(2), p(3), \ldots, p(n-\ell+1)\} = \{\ell+1, \ell+2, \ldots, n\}$ and $\{p(n-\ell+2), p(n-\ell+3), \ldots, p(n)\} = \{1, 2, \ldots, \ell-1\}$.

Since p avoids 213 and $p^{(2)}$ avoids 132, $(p(2), p(3), \ldots, p(n))$ avoids both 213 and 132. By Theorem 4.5.1, there are at most $2^{n-\ell-1}$ possibilities for $(p(2), p(3), \ldots, p(n-\ell+1))$ and at most $2^{\ell-1-1}$ possibilities for $(p(n-\ell+2), p(n-\ell+3), \ldots, p(n))$. So we have

$$|S_n^{(2)}(213; 132)| \leq 2^{n-1-1} + 2^{n-1-1} + \sum_{\ell=2}^{n-1} 2^{n-\ell-1} 2^{\ell-1-1} = (n+2) \cdot 2^{n-3}.$$

We still need to show that $|S_n^{(2)}(213; 132)| \geq (n+2) \cdot 2^{n-3}$. To do this, let $p \in S_n$ with $p(1) = \ell$, $\{p(2), p(3), \ldots, p(n-\ell+1)\} = \{\ell+1, \ell+2, \ldots, n\}$, $\{p(n-\ell+2), p(n-\ell+3), \ldots, p(n)\} = \{1, 2, \ldots, \ell-1\}$, and $(p(2), p(3), \ldots, p(n-\ell+1))$ and $(p(n-\ell+2), p(n-\ell+3), \ldots, p(n))$ both avoid 213 and 132. We need to show that p avoids 213 and $p^{(2)}$ avoids 132.

Suppose, by way of contradiction, that p has a subpermutation xyz which is a 213 pattern. Then $y < x < z$. If $x = \ell$, then z comes before y which is a contradiction; if $x > \ell$, then since $z > x > \ell$, xyz is a subpermutation of $(p(2), p(3), \ldots, p(n-\ell+1))$ which is a contradiction; and if $x < \ell$, then by the definition of p, xyz is a subpermutation of $(p(n-\ell+2), p(n-\ell+3), \ldots, p(n))$ which is again a contradiction. Hence, p avoids 213.

Now suppose, by way of contradiction, that $p^{(2)}$ has a subpermutation xyz which is a 132 pattern. Then $x < z < y$ Notice that $p^{(2)}(n) = \ell$. If $z = \ell$, then $x < z < y$ would imply that y comes before x which is a contradiction; if $z < \ell$, then $x < z < y$ would imply that $y < \ell$ and hence xyz is a subpermutation of $(p(n-\ell+2), p(n-\ell+3), \ldots, p(n))$ which is a contradiction; and if $z > \ell$, then xyz would be a subpermutation of $(p(2), p(3), \ldots, p(n-\ell+1))$ which again a contradiction. Hence, $p^{(2)}$ avoids 132.

Therefore, $|S_n^{(2)}(213; 132)| = (n+2) \cdot 2^{n-3}$. □

Remark 5.4.7. The integers sequence in Proposition 5.4.6 is consistent with A045623 in OEIS [94] which counts the number of 1's in all compositions of $n+1$.

5.5 The pattern 1324

In this section, we present a recursive relation on $|S_n^{(n)}(1324)|$. In the future, we plan to also study other patterns of length four.

Theorem 5.5.1. $|S_1^{(1)}(1324)| = 1$, $|S_2^{(2)}(1324)| = 2$*, and for all* $n \geq 3$*,*

$$|S_n^{(n)}(1324)| = \frac{3n}{n-1}|S_{n-1}^{(n-1)}(1324)| - \frac{n}{n-2}|S_{n-2}^{(n-2)}(1324)|.$$

Proof. Let $S_{n,1}^{(n)}(1324)$ be set consisting of $p \in S_n^{(n)}(1324)$ with $p(1) = 1$. By Lemma 5.2.6, we have $|S_{n,1}^{(n)}(1324)| = n|S_n^{(n)}(1324)|$. So the recursive relation in the theorem is equivalent to

$$|S_{n,1}^{(n)}(1324)| = 3|S_{n-1,1}^{(n-1)}(1324)| - |S_{n-2,1}^{(n-2)}(1324)|.$$

Also, $|S_{1,1}^{(1)}(1324)| = |S_{2,1}^{(2)}(1324)| = 1$. We classify $p \in S_{n,1}^{(n)}(1324)$ according to the relative location of n and $n-1$.

First suppose $n-1$ appears before n. Then p is of the form

$$(1, x_1, \ldots, x_r, n-1, y_1, \ldots, y_s, n, z_1, \ldots, z_t).$$

If $s > 0$, then $(1, n-1, y_1, n)$ is a 1324 pattern. So if $n-1$ appears before n, then n would immediately follow $n-1$. The number of such permutations is equal to $|S_{n-1,1}^{n-1}(1324)|$.

Now we assume that n appears before $n-1$. Then p is of the form

$$(1, x_1, \ldots, x_r, n, y_1, \ldots, y_s, n-1, z_1, \ldots, z_t).$$

The number of permutations where n appears immediately before $n-1$ (i.e., $r = 0$) is also $|S_{n-1,1}^{n-1}(1324)|$. So now we assume that $s > 0$. We will show that

$$(y_1, y_2, \ldots, y_s) = (n-s-1, n-s, \ldots, n-2).$$

To prove this, we first show that $\{y_1, y_2, \ldots, y_s\} = \{m-s-1, m-s, \ldots, m-2\}$. Since $p \in S_n^{(n)}(1324)$, $(y_1, \ldots, y_s, m-1, z_1, \ldots, z_t, 1, x_1, \ldots, x_r, m)$ avoids 1324. If $z_i > y_j$ for some i and j, then $(y_j, m-1, z_i, m)$ is a 1324 pattern. So $z_i < y_j$ for all i and j. Similarly, $x_i < y_j$ for all i and j. Hence $\{y_1, y_2, \ldots, y_s\} = \{m-s-1, m-s, \ldots, m-2\}$. Now if $y_i > y_j$ for some $i < j$, then $(1, y_i, y_j, m-1)$ would be a 1324 pattern. Therefore, we have $(y_1, y_2, \ldots, y_s) = (m-s-1, m-s, \ldots, m-2)$.

So the total number of such permutations is

$$\sum_{s=1}^{n-3} |S_{n-s-1,1}^{(n-s-1)}(1324)| =: d_n.$$

We'll show that $d_n = |S_{n-1,1}^{(n-1)}(1324)| - |S_{n-2,1}^{(n-2)}(1324)|$ for $n \geq 4$ with $d_3 = 0$. Writing

$$d_m = d_{m-1} + |S_{n-2,1}^{(n-2)}(1324)| \ ,$$

by induction

$$\begin{aligned} d_m &= \Big(|S_{n-2,1}^{(n-2)}(1324)| - |S_{n-3,1}^{(n-3)}(1324)|\Big) + |S_{n-2,1}^{(n-2)}(1324)| \\ &= \Big(3|S_{n-2,1}^{(n-2)}(1324)| - |S_{n-3,1}^{(n-3)}(1324)|\Big) - |S_{n-2,1}^{(n-2)}(1324)| \\ &= |S_{n-1,1}^{(n-1)}(1324)| - |S_{n-2,1}^{(n-2)}(1324)| \ . \end{aligned}$$

Adding up the contributions, we have $|S_{n,1}^{(n)}(1324)| = 2|S_{n-1,1}^{(n-1)}(1324)| + |S_{n-1,1}^{(n-1)}(1324)| - |S_{n-2,1}^{(n-2)}(1324)| = 3|S_{n-1,1}^{(n-1)}(1324)| - |S_{n-2,1}^{(n-2)}(1324)|$. □

5.6 Future work

While single patterns of length three are totally resolved in this paper, we have only studied one pattern of length four. In the future, we plan to study other patterns of length four as well. Another direction would be to study monotone patterns of arbitrary length. For a monotone pattern of length t, it is known [18, Theorem 4.21] that

$$|S_n(123\cdots t)| \leq (t-1)^{2n}.$$

Notice that, by Theorem 5.3.1, we have $|S_n^{(2)}(123)| = (3-1)^{n-1}$. It would be interesting to see whether one can get a strong upper bound for $|S_n^{(k)}(123\cdots t)|$.

On a more detailed level, Kitaev, Remmel, and Tiefenbruck [75] asked whether there is a combinatorial explanation of the fact that in S_n, the number of (132, 4321, 3421)-avoiding permutations is the same as the number of (321, 2143, 3142)-avoiding permutations. In

Remark 5.4.5, we noticed that

$$|S_n^{(2)}(123;231)| = |S_n(132,4321,3421)| = |S_n(321,2143,3142)|,$$

where $S_n(\sigma_1,\sigma_2,\sigma_3)$ is the set of permutations $p \in S_n$ such that p avoids σ_1, σ_2, and σ_3. We plan to build a bijection between $S_n^{(2)}(123;231)$ and $S_n(132,4321,3421)$, and a bijection between $S_n^{(2)}(123;231)$ and $S_n(321,2143,3142)$. This has the potential of resolving the open question asked by Kitaev, Remmel, and Tiefenbruck.

Chapter 6: Subsequence sums in permutations

6.1 Introduction

In 1973, Entringer and Jackson [42] asked whether every permutation of $[n]$ contains a monotone (i.e., increasing or decreasing) subsequence (x, y, z) which is also a three-term arithmetic progression, i.e., $x + z = 2y$. We call a monotone sequence $(x_1, x_2, \ldots, x_k)$ of length k a monotone k-AP if there exists an integer d such that for all $i \in \{2, \ldots, k\}$ $x_i = x_{i-1} + d$. Odda [93] provided a negative answer to Entringer and Jackson's question. The argument is as follows: if $(p_1, p_2, \ldots, p_m) \in S_m$ has no monotone 3-AP, then the permutation

$$(2p_1, 2p_2, \ldots, 2p_m, 2p_1 - 1, 2p_2 - 1, \ldots, 2p_m - 1) \in S_{2m}$$

also has no monotone 3-AP. Davis, Entringer, Graham, and Simmons [33] went one step further and showed that, for each positive integer n, there are at least 2^{n-1} permutations on $[n]$ which has no monotone 3-AP. An upper bound for the number of permutation without monotone 3-APs is also showned by Davis, Entringer, Graham, and Simmons, and improved by Sharma [111]. Davis, Entringer, Graham, and Simmons also considered similar questions on infinite permutations and showed that every permutation of the positive integers has no monotone 5-AP, but there exists permutation of the positive integers without monotone 3-AP. However, despite some recent effort [50, 82], we still do not know whether there exists a permutation of the positive integers wihout monotone 4-AP. We also note that Ardal, Brown, and Jungić [5] showed that there is a permutation of the real numbers which does not contain a monotone 3-AP.

The results on the permutations of the integers are different from the colorings of the integers such as the van der Waerden's theorem and Rado theorem as discussed in Chapter 1. Given the differences between permutations and colorings, one then might wonder, what kind of subsequences, with certain arithmetic properties, always exist in permutations. Thinking along this line, we notice that the equation $x + z = 2y$ for the 3-APs can be rewritten as $x + y + z = 3y$. It is easy to see that Odda's [93] construction can easily be adapted to show that for all $n \in \mathbb{N}$, there exists $p \in S_n$ which does not contain a monotone subsequence x, y, z such that $x + y + z = \ell y$ where ℓ is odd. What if we replace the right-hand-side of the last equation with x or z? A more general question then is the following:

Problem 6.1.1. Let $k \geq 3$ and $\ell \geq 2$ be positive integers. Does there exist $n \in \mathbb{N}$ such that every $p \in S_n$ has a subsequence $(x_1, x_2, \ldots, x_k)$ with

$$\sum_{i=1}^{k} x_i = \ell x_1 \text{ or } \sum_{i=1}^{k} x_i = \ell x_k? \tag{6.1}$$

If a subsequence satisfies Equation (6.1), then we say that it is *ℓ-additive*. In this paper, we provide an affirmative answer to Problem 6.1.1 when $\ell = 2$. For this case, we are also able to provide upper and lower bounds for the smallest n such that every $p \in S_n$ has an ℓ-additive subsequence of length k.

One might wonder if one could restrict subsequences to monotone subsequences. This leads the following question:

Problem 6.1.2. Let $k \geq 3$ and $\ell \geq 2$ be positive integers. Does there exist $n \in \mathbb{N}$ such that every $p \in S_n$ has a monotone subsequence $(x_1, x_2, \ldots, x_k)$ with

$$\sum_{i=1}^{k} x_i = \ell x_1 \text{ or } \sum_{i=1}^{k} x_i = \ell x_k? \tag{6.2}$$

If a monotone subsequence satisfies Equation (6.1), then we say that it is monotone ℓ*-additive*. Using elementary calculations, we show an affirmative answer to Problem 6.1.2 when $k = 3$ and $\ell = 2$. We also provide strong bounds on minimum number of monotone 2-additive subsequences of length 3.

6.2 Preliminaries

We first introduce some terminology which will be used in our proofs.

Definition 6.2.1. For all $p \in S_n$ and $s \in [n]$ with $p_i = s$, let $\mathcal{L}_p(s) = \{p_j : j < i\}$ and $\mathcal{R}_p(s) = \{p_j : j > i\}$.

That is, $\mathcal{L}_p(s)$ consists of all the terms before s and $\mathcal{R}_p(s)$ consists of all the terms after s when we arrange the terms of p on a horizontal line. For example, for the permutation $p = (5, 1, 4, 3, 2) \in S_5$, we have $\mathcal{L}_p(4) = \{1, 5\}$ and $\mathcal{R}_p(1) = \{2, 3, 4\}$.

Definition 6.2.2. Let $n \in \mathbb{N}$, $A \subseteq [n]$, and $p \in S_n$. The subpermutation of p on A is a sequence obtained by deleting all the terms of p which are not in A, but keeping the relative order of the terms that are in A.

For example, $(5, 1, 3)$ is a subpermutation of $(5, 1, 4, 3, 2)$ on $\{1, 3, 5\}$. If q is a subpermutation of p on A, then we will simply call q a subpermutation on A when there is no confusion.

Now we state two simple observations which will be used to simplify our proofs.

Lemma 6.2.3. *A sequence* $(x_1, x_2, \ldots, x_k)$ *is* ℓ*-additive if and only if* $\sum_{i=1}^{k-1} x_i = (\ell - 1)x_1$ *or* $\sum_{i=2}^{k} x_i = (\ell - 1)x_k$.

Lemma 6.2.4. *Let* $N, k, \ell \in \mathbb{N}$. *If every* $p \in S_N$ *has an* ℓ*-additive subsequence of length* k*, then for all* $n \geq N$*, every* $p \in S_n$ *has an* ℓ*-additive subsequence of length* k.

6.3 2-additive subsequences

We prove our main result in this section.

Theorem 6.3.1. *For all $k \geq 3$ and $n \geq (k-2)(k-1)^2(3k-4)^2/2$, every $p \in S_n$ has a 2-additive subsequence of length k.*

Proof. Let $k \geq 3$. For all $p = (p_1, p_2, \ldots, p_{k-2}, p_{k-1}, p_k, \ldots, p_{2k-3}) \in S_{2k-3}$, we define the following:

$$\alpha_p := \sum_{i=1}^{k-1} p_i,$$

$$\beta_p := \sum_{i=k-1}^{2k-3} p_i,$$

$$U_p := \{(k-2)\alpha_p, \alpha_p - p_1, \alpha_p - p_2, \ldots, \alpha_p - p_{k-1}\},$$

$$V_p := \{(k-2)\beta_p, \beta_p - p_{k-1}, \beta_p - p_k, \ldots, \beta_p - p_{2k-3}\},$$

and

$$L_p = \max\{\text{lcm}(a, b) : a \in U_p, b \in V_p\},$$

where lcm(a, b) is the least common multiple of a and b.

Let $N = \max_{p \in S_{2k-3}} 2L_p$. We will show that every $\sigma \in S_N$ has a 2-additive subsequence which would imply that for all $n \geq N$, every $\tau \in S_n$ has a 2-additive subsequence. Suppose, by way of contradiction, that $\sigma \in S_N$ does not have a 2-additive subsequence of length k. Then for all $s \in [N]$ and distinct integers $a_1, a_2, \ldots, a_{k-1} \in \mathcal{L}_p(s)$ or $a_1, a_2, \ldots, a_{k-1} \in \mathcal{R}_p(s)$ we have $\sum_{i=1}^{k-1} a_i \neq s$.

Let p be the subpermutation of σ on $[2k-3]$.

Claim 1. $U_p \cap \mathcal{L}_\sigma(p_{k-1}) \neq \emptyset$ and $V_p \cap \mathcal{R}_\sigma(p_{k-1}) \neq \emptyset$.

By symmetry, it suffices to prove that $U_p \cap \mathcal{L}_\sigma(p_{k-1}) \neq \emptyset$. Suppose, by way of contradiction, that $U_p \cap \mathcal{L}_\sigma(p_{k-1}) = \emptyset$. Then $U_p \subseteq \mathcal{R}_\sigma(p_{k-1})$. Since $\sum_{i=1}^{k-1}(\alpha_p - p_i) = (k-2)\alpha_p$, there exists $j \in [k-1]$ such that $\alpha_p - p_j \in \mathcal{L}_\sigma((k-2)\alpha_p)$. WLOG, we assume that $j = 1$. So we have $p_1, p_2, \ldots, p_{k-1}, \alpha_p - p_1 \in \mathcal{L}_\sigma((k-2)\alpha_p)$. Write $S_1 = \alpha_p - p_1$ and, for all $i \in \{2, 3, \ldots, k-1\}$, let $S_i := S_{i-1} + p_1 + \cdots + p_{i-1} + p_{i+1} + \cdots + p_{k-1}$.

We will show that $S_i \in \mathcal{L}_\sigma((k-2)\alpha_p)$ for all $i \in \{2, 3, \ldots, k-1\}$. Suppose not. Let $j \in \{2, 3 \ldots, k-1\}$ be the smallest index such that $S_j \in \mathcal{R}_\sigma((k-2)\alpha_p)$. Since $p_1, p_2 \ldots, p_{k-1}, S_{j-1} \in \mathcal{L}_\sigma((k-2)\alpha_p)$, we have $p_1, p_2, \ldots, p_{k-1}, S_{j-1} \in \mathcal{L}_\sigma(S_j)$ which is a contradiction because $S_j = S_{j-1} + p_1 + p_2 + \cdots + p_{j-1} + p_{j+1} + p_{j+2} + \cdots + p_{k-1}$.

Since $S_k = (k-2)\alpha_p$, we have $(k-2)\alpha_p \in \mathcal{L}_\sigma((k-2)\alpha_p)$ which is a contradiction.

Claim 2. For all $a \in U_p \cap \mathcal{L}_\sigma(p_{k-1})$ and for all $\ell \geq 2$, if $\ell a \leq N$, then $\ell a \in \mathcal{L}_\sigma(p_{k-1})$ and there exist $x_1, x_2, \ldots, x_{k-1} \in \mathcal{L}_\sigma(p_{k-1})$ such that $x_1 + x_2 + \cdots + x_{k-1} = \ell a$. Similarly, for all $b \in V_p \cap \mathcal{R}_\sigma(p_{k-1})$ and for all $\ell \geq 2$, if $\ell b \leq N$, then $\ell b \in \mathcal{R}_\sigma(p_{k-1})$ and there exist $x_1, x_2, \ldots, x_k \in \mathcal{R}_\sigma(p_k)$ such that $x_1 + x_2 + \cdots + x_{k-1} = \ell b$.

By symmetry, we only need to prove the former. We consider two cases:

Case 1: $(k-2)\alpha_p \in U_p \cap \mathcal{L}_\sigma(p_{k-1})$. We use induction to show that for all $\ell \geq 1$, if $\ell(k-2)\alpha_p \leq N$, then $\ell(k-2)\alpha_p \in U_p \cap \mathcal{L}_\sigma(p_{k-1})$. Write $T_0 := (k-2)\alpha_p$ and for all $j \in \{1, 2, \ldots, k-1\}$, let $T_j := T_{j-1} + p_1 + p_2 + \cdots + p_{j-1} + p_{j+1} + p_{j+2} + \cdots + p_{k-1}$. We will show that $T_j \in \mathcal{L}_\sigma(p_{k-1})$ for all $j = 1, 2, \ldots, k-1$. Suppose not. Let $j \in \{1, 2, \ldots, k-1\}$ be the smallest such that $T_j \notin \mathcal{L}_\sigma(p_{k-1})$. Then we have $T_j := T_{j-1} + p_1 + p_2 + \cdots + p_{j-1} + p_{j+1} + p_{j+2} + \cdots + p_{k-1}$ and $T_{j-1}, p_1, p_2, \ldots, p_{k-1} \in \mathcal{L}_\sigma(T_j)$ which is a contradiction. By our construction $T_{k-1} = 2(k-2)\alpha_p$. So $2(k-2)\alpha_p \in \mathcal{L}_\sigma(p_{k-1})$. Continue this inductively, we have that if $\ell(k-2) \leq N$, then $\ell(k-2)\alpha_p \in \mathcal{L}_\sigma(p_{k-1})$.

Case 2: $\alpha_p - p_i \in U_p \cap \mathcal{L}_\sigma(p_{k-1})$ for some $i \in [k-1]$. WLOG, we assume that $i = 1$. Let m be the largest integer such that $m(\alpha_p - p_1) \leq N$. Write $R_1 := \alpha_p - p_1$; and for all $\ell \in \{2, 3, \ldots, m\}$, let $R_\ell = R_{\ell-1} + p_2 + p_3 + \cdots + p_{k-1}$. Using the same arguments as in Case 1, we have $R_\ell \in \mathcal{L}_\sigma(p_{k-1})$ for all $\ell = 1, 2, \ldots, k-1$. By our construction, we have $R_\ell = \ell(\alpha_p - p_1)$ for all $\ell \leq m$. Hence if $\ell(\alpha_p - p_1) \leq N$, then $\ell(\alpha_p - p_1) \in \mathcal{L}_\sigma(p_{k-1})$.

By Claim 1, there exists integers a and b with $a \in U_p \cap \mathcal{L}_\sigma(p_{k-1})$ and $b \in V_p \cap \mathcal{R}_\sigma(p_{k-1})$. Then $aa' = 2L_p = bb'$ for some $a', b' \geq 2$. Since $2L_p \leq N$, by Claim 2, we have $aa' \in \mathcal{L}_\sigma(p_{k-1})$, $bb' \in \mathcal{R}_\sigma(p_{k-1})$, and there exist $x_1, x_2, \ldots, x_{k-1} \in \mathcal{L}_\sigma(p_{k-1})$ such that $x_1 + x_2 + \cdots + x_{k-1} = aa'$. Hence $x_1, x_2, \ldots, x_k \in \mathcal{L}_\sigma(bb')$ and $x_1 + x_2 + \cdots + x_{k-1} = bb'$ which is a contradiction.

Now we prove that $N \leq (k-2)(k-1)^2(3k-4)^2/2$. Let $p \in S_{2k-3}$, $a \in U_p$, and $b \in V_p$. By the definition of U_p and V_p, we have

$$\operatorname{lcm}(a, b) \leq (k-2)\alpha_p\beta_p < (k-2)\left(\sum_{i=k-1}^{2k-3} i\right)^2 = \frac{1}{4}(k-2)(k-1)^2(3k-4)^2.$$

Hence we have $N \leq (k-2)(k-1)^2(3k-4)^2/2$. □

The N in Theorem 6.3.1 could be improved. However, we do not attempt it here since our construction in the proof seems unlikely to be optimal.

For all $k \geq 3$, let $f(k)$ be the smallest n such that every $p \in S_n$ has a 2-additive subsequence of length k. By Theorem 6.3.1, we have $f(k) \leq (k-2)(k-1)^2(3k-4)^2/2$. Now we show that $f(k)$ is at least quadratic.

Theorem 6.3.2. *For all $k \geq 3$,*

$$f(k) \geq (k-1)(3k-4)/2.$$

Proof. Let $m = (k-1)(3k-4)/2$. Consider the permutation

$$p = (1, 2, \ldots, k-2, m-1, m-2, \ldots, k-1) \in S_{m-1}.$$

We will show that for all $s \in [m-1]$, neither $\mathcal{L}_p(s)$ nor $\mathcal{R}_p(s)$ contains a subset of size $k-1$ which sums to s. This is obviously true if $s < (k-1)k/2$. So we suppose that

$s \geq (k-1)k/2$. By the construction of p, $\mathcal{L}_p(s)$ only contains $k-2$ numbers that are smaller than s. So $\mathcal{L}_p(s)$ does not contain a subset of size $k-1$ which sums to s. As for $\mathcal{R}_\sigma(s)$, the sum of a subset of size $k-1$ is at least $\sum_{i=k-1}^{2k-3} i = \frac{1}{2}(k-1)(3k-4) > m-1$. □

It turns out that lower bound in Theorem 6.3.2 matches the exact value when $k = 3$. We have also checked that $f(4) \leq 28$. It is an open question to determine the correct order of $f(k)$.

6.4 Monotone 2-additive subsequences of length three

In this section, we first show that every $p \in S_{48}$ contains a monotone subsequence of length three which is 2-additive. Our proof is based on elementary calculation.

Lemma 6.4.1. *Let $p \in S_n$. If (x_1, x_2, x_3, x_4) is a subsequence of p with $x_1+x_2 = x_3+x_4 \leq n$, $x_1 < x_2$, and $x_3 > x_4$, then p contains a monotone subsequence (x, y, z) such that either $x + y = z$ or $x = y + z$.*

Proof. Let $a = x_1 + x_2 = x_3 + x_4$. Since $a \leq n$, either (a, x_3, x_4) or (x_1, x_2, a) is a monotone subsequence and we are done. □

Lemma 6.4.2. *Let $p \in S_n$. If there exists $a \leq n/12$ with the property that the subpermutation on $\{a, 2a, 3a\}$ is either $(a, 3a, 2a)$ or $(2a, 3a, a)$, then p contains a monotone subsequence (x, y, z) such that either $x + y = z$ or $x = y + z$.*

Proof. WLOG, we assume that the subpermutation on $\{a, 2a, 3a\}$ is $(a, 3a, 2a)$. Suppose, by way of contradiction, that p does not contain a monotone 2-additive subsequence (x, y, z).

Since $a + 3a = 4a$ and $3a + 2a = 5a$, we have $4a \in \mathcal{L}_p(3a)$ and $5a \in \mathcal{R}_p(3a)$. Since $a + 4a = 5a$, we have $4a \in \mathcal{L}_p(a)$.

Case 1: The subpermutation on $\{a, 2a, 3a, 4a, 5a\}$ is $(4a, a, 3a, 5a, 2a)$. Since $2a + 5a = 7a$, we have $7a \in \mathcal{R}_p(5a)$. Since $2a + 4a = 6a$, we have $6a \in \mathcal{R}_p(4a)$. Since

$a + 5a = 6a$, we have $6a \in \mathcal{L}_p(5a)$. Since $a + 6a = 7a$, we have $6a \in \mathcal{L}_p(a)$. Since $4a + 6a = 10a$, we have $10a \in \mathcal{L}_p(6a)$. Since $3a + 5a = 8a$, we have $8a \in \mathcal{L}_p(5a)$. Since $2a + 6a = 8a$, we have $8a \in \mathcal{R}_p(6a)$. Now $(10a, 8a, 2a)$ is a decreasing 2-additive subpermutation of p which is a contradiction.

Case 2: The subpermutation on $\{a, 2a, 3a, 4a, 5a\}$ is $(4a, a, 3a, 2a, 5a)$. Since $4a + 2a = 6a$ and $a + 5a = 6a$, we have $6a \in \mathcal{R}_p(4a) \cap \mathcal{L}_p(5a)$. Since $4a + 3a = 7a$ and $2a + 5a = 7a$, we have $7a \in \mathcal{R}_p(4a) \cap \mathcal{L}_p(5a)$. Since $4a + 7a = 11a$ and $6a + 5a = 11a$, by Lemma 6.4.1, the subpermutation on $\{4a, 5a, 6a, 7a\}$ is $(4a, 6a, 11a, 7a, 5a)$. Since $7a + 5a = 12a$, we have $12 \in \mathcal{R}_p(7a)$. Since $a + 11a = 12a$, we have $11a \in \mathcal{L}_p(a)$. So the subpermutation on $\{a, 2a, 3a, 4a, 5a, 6a, 11a\}$ is $(4a, 6a, 11a, a, 3a, 2a, 5a)$. Since $6a + 2a = 8a$ and $3a + 5a = 8a$, we have $8a \in \mathcal{R}_p(6a) \cap \mathcal{L}_p(5a)$. Since $4a + 8a = 12a$, we have $8a \in \mathcal{R}_p(12a)$. Since $4a + 6a = 10a$, we have $10a \in \mathcal{L}_p(6a)$. Now we have $10a \in \mathcal{L}_p(3a)$. Since $3a + 7a = 10a$, we have $7a \in \mathcal{R}_p(3a)$. Now $(a, 7a, 8a)$ is an increasing 2-additive subpermutation of p which is a contradiction. □

Theorem 6.4.3. *For all $n \geq 48$, every $p \in S_n$ has a monotone 2-additive subsequence of length three.*

Proof. By Lemmas 6.2.3 and 6.2.4, it suffices to show that all $p \in S_{48}$ has a monotone subsequence (x, y, z) such that either $x + y = z$ or $y + z = x$. Suppose, by way of contradiction, that $p \in S_{48}$ does not have a monotone subsequence (x, y, z) such that either $x + y = z$ or $y + z = x$. WLOG, we assume that the subpermutation on $\{1, 2\}$ is $(1, 2)$. Since $1 + 2 = 3$, the subpermutation of $\{1, 2, 3\}$ is either $(3, 1, 2)$ or $(1, 3, 2)$. Now by Lemma 6.4.2 with $a = 1$, the subpermutation of $\{1, 2, 3\}$ is $(3, 1, 2)$. Since $1 + 3 = 4$, we have $4 \in \mathcal{R}_p(3)$.

Case 1: The subpermutation on $\{1, 2, 3, 4\}$ is $(3, 4, 1, 2)$. By Lemma 6.4.2 with $a = 2$, we have $6 \in \mathcal{R}_p(2)$. By Lemma 6.4.2 with $a = 3$, we have $9 \in \mathcal{L}_p(3)$. So the subpermutation on $\{1, 2, 3, 4, 6, 9\}$ is $(9, 3, 4, 1, 2, 6)$. Since $4 + 1 = 5$, we have $5 \in \mathcal{R}_p(4)$.

Since $3+4=7=5+2$, by Lemma 6.4.1, we have $5 \in \mathcal{R}_p(2)$. Since $1+5=6$, we have $5 \in \mathcal{R}_p(6)$. Now the subpermutation on $\{1,2,3,4,5,6,9\}$ is $(9,3,4,1,2,6,5)$. Since $6+5=11$, we have $11 \in \mathcal{R}_p(6)$. Since $2+6=8$, we have $8 \in \mathcal{L}_p(6)$. Since $3+8=11$, we have $8 \in \mathcal{L}_p(3)$. Since $8+1=9$, we have $8 \in \mathcal{L}_p(9)$. Since $3+4=7$, we have $7 \in \mathcal{L}_p(4)$. Since $1+7=8$, we have $7 \in \mathcal{L}_p(8)$. Now $(7,8,9,6)$ is a subseqence with $7+8=9+6$. By Lemma 6.4.1, this is a contradiction.

Case 2: The subpermutation on $\{1,2,3,4\}$ is $(3,1,4,2)$. By Lemma 6.4.2 with $a=2$, we have $6 \in \mathcal{R}_p(2)$. By Lemma 6.4.2 with $a=3$, we have $9 \in \mathcal{L}_p(3)$. So the subpermutation on $\{1,2,3,4,6,9\}$ is $(9,3,1,4,2,6)$. Since $3+2=5$ and $1+4=5$, we have $5 \in \mathcal{R}_p(3) \cap \mathcal{L}_p(4)$. Since $1+5=6$, we have $5 \in \mathcal{L}_p(1)$. Since $3+5=8$, we have $8 \in \mathcal{L}_p(5)$. Since $1+8=9$, we have $8 \in \mathcal{L}_p(9)$. Now the subpermutation on $\{1,2,3,4,5,6,8,9\}$ is $(8,9,3,5,1,4,2,6)$. By Lemma 6.4.2 with $a=4$, we have $12 \in \mathcal{R}_p(4)$. Since $5+2=7$ and $1+6=7$, we have $7 \in \mathcal{R}_p(5) \cap \mathcal{L}_p(6)$. Since $5+7=12$, we have $7 \in \mathcal{R}_p(12)$. Hence $(1,12,7,6)$ is a subpermutation which is a contradiction by Lemma 6.4.1.

Case 3: The subpermutation on $\{1,2,3,4\}$ is $(3,1,2,4)$. By Lemma 6.4.2 with $a=2$, we have $6 \in \mathcal{L}_p(2)$. Since $1+4=5$ and $3+2=5$, we have $5 \in \mathcal{R}_p(3) \cap \mathcal{L}_p(4)$.

We now show that $6 \in \mathcal{L}_p(3)$. Suppose, by way of contradiction, that $6 \in \mathcal{R}_p(3)$. By Lemma 6.4.2 with $a=3$, we have $9 \in \mathcal{L}_p(3)$. Now $(9,5,4)$ is a decreasing 2-additive subsequence which is a contradiction.

Since $3+5=8$, we have $8 \in \mathcal{L}_p(5)$. By Lemma 6.4.2 with $a=4$, we have $12 \in \mathcal{R}_p(4)$. Since $5+4=9$, we have $9 \in \mathcal{R}_p(5)$. Since $3+9=12$, we have $9 \in \mathcal{R}_p(12)$. Since $6+2=8$, we have $8 \in \mathcal{R}_p(6)$. Since $1+8=9$ and $8 \in \mathcal{L}_p(9)$, we have $8 \in \mathcal{L}_p(1)$. Since $8+2=10$, we have $10 \in \mathcal{R}_p(8)$. Since $1+9=10$. we have $10 \in \mathcal{L}_p(9)$. Since $6+8=10+4$, by Lemma 6.4.1, $10 \in \mathcal{R}_p(4)$. Since $2+10=12$ and $10 \in \mathcal{R}_p(2)$, we have $10 \in \mathcal{R}_p(12)$. Now the subpermutation on $\{3,4,9,10,12\}$ is $(3,4,12,10,9)$. Since

$3+4=7$, we have $7 \in \mathcal{L}_p(4)$. So $(7, 12, 10, 9)$ is a subsequence of p. Since $7+12=10+9$, by Lemma 6.4.1, we have a contradiction. $\square$

For each $p \in S_n$, let $g_p(n)$ be the number of monotone 2-additive subsequences of p and let

$$g(n) = \min_{p \in S_n} g_p(n).$$

Theorem 6.4.4. *For all large enough n, we have*

$$\frac{n}{\log n} - 15 \le g(n) \le \frac{1}{18}n^2 + \frac{7}{6}n.$$

Proof. We first prove the lower bound. Let $p \in S_n$ and let a be a prime such that $a > 48$ and $48a \le n$. Consider the subpermutation p' of p on $\{a, 2a, \ldots, 48a\}$. By Theorem 6.4.3, p' contains a monotone 2-additive subsequence of length three. There are 15 primes less than 48 and, by the well-known prime number theorem [91, p. 274], there are at least $n/\log n$ primes in $\{1, 2, \ldots, n\}$ for large enough n. Here the explicit lower bound is recently proved by Dusart [35]. It follows that there are at least $n/\log n - 15$ primes a such that $a > 48$ and $48a \le n$. For each of these a, there is a monotone 2-additive subsequence of length three. Since each monotone 2-additive subsequence of length three has a distinct prime factor greater than 48, none of them are the same. Hence, we have $g(n) \ge n/\log n - 15$.

As for the upper bound, for $2 \le \ell \le \lceil n/2 \rceil$, let's consider the permutation

$$\left\lceil \frac{n}{\ell} \right\rceil, \left\lceil \frac{n}{\ell} \right\rceil - 1, \ldots, 1, n, n-1, \ldots, \left\lceil \frac{n}{\ell} \right\rceil + 1.$$

We note that the above permutation is inspired by the construction by Myers [89] who studied the minimum number of monotone subsequences of permutations without arithmetic properties. This permutation does not contain increasing subsequences of length three. All the decreasing subsequences come from either the first half, from $\lceil n/\ell \rceil$ to 1, or the second half, from n to $\lceil n/\ell \rceil + 1$. The number of decreasing 2-additive subsequences from the first half is at most

$$\sum_{i=1}^{\lfloor n/(2\ell)\rfloor} \left(\left\lceil \frac{n}{\ell} \right\rceil - 2i\right) \leq \sum_{i=1}^{\lfloor n/(2\ell)\rfloor} \left(\frac{n}{\ell} + 1 - 2i\right) \leq \left(\frac{n}{\ell} + 1\right)\frac{n}{2\ell} - \left(\frac{n}{2\ell} - 1\right)\frac{n}{2\ell} = \frac{1}{4\ell^2}n^2 + \frac{1}{\ell}n.$$

The number of decreasing 2-additive subsequences from the second half is at most

$$\begin{aligned}\sum_{i=\lceil n/\ell\rceil+1}^{\lfloor n/2\rfloor} (n-2i) =& n\left(\left\lfloor \frac{n}{2}\right\rfloor - \left\lceil \frac{n}{\ell}\right\rceil\right) - \left[\left\lfloor \frac{n}{2}\right\rfloor\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right) - \left\lceil \frac{n}{\ell}\right\rceil\left(\left\lceil \frac{n}{\ell}\right\rceil + 1\right)\right] \\ \leq& n\left(\frac{n}{2} - \frac{n}{\ell}\right) - \frac{n}{2}\left(\frac{n}{2} - 1\right) + \frac{n}{\ell}\left(\frac{n}{\ell} + 2\right) = \frac{\ell^2 + 4 - 4\ell}{4\ell^2}n^2 + \frac{\ell + 2}{2\ell}n.\end{aligned}$$

Adding them together, the total number of decreasing 2-additive subsequences is at most

$$\frac{\ell^2 + 5 - 4\ell}{4\ell^2}n^2 + \frac{\ell + 4}{2\ell}n.$$

For all large enough n, the above quantity reaches minimum when $\ell = 3$ which is $\frac{1}{18}n^2 + \frac{7}{6}n$. □

6.5 Future work

Let k, ℓ be positive integers. We have only answered Problem 6.1.1 for $\ell = 2$. In the future, we plan to study the cases when $\ell > 2$. We first notice that the answer to Problem 6.1.1 is negative if $k = \ell$.

Proposition 6.5.1. *If $k = \ell$, then $(1, 2, \ldots, n) \in S_n$ does not have an ℓ-additive subsequence.*

Proof. Let $(x_1, x_2, \ldots, x_k)$ be a subsequence of $(1, 2, \ldots, n)$. Then we have $x_1 + x_2 + \cdots + x_k > kx_1 = \ell x_1$ and $x_1 + x_2 + \cdots + x_k < kx_k = \ell x_k$. □

When $k \neq \ell$, some routine calculation suggests that we are likely to have an affirmative answer to Problem 6.1.1.

Proposition 6.5.2. *For all $n \geq 9$, every $p \in S_n$ has a 3-additive subsequence of length four.*

Proof. By Lemmas 6.2.3 and 6.2.4, it suffices to show that every $p \in S_9$ has a subsequence (x_1, x_2, x_3, x_4) such that either $x_1 + x_2 + x_3 = 2x_4$ or $x_2 + x_3 + x_4 = 2x_1$. Suppose, for a contradiction, that $p \in S_9$ is a permutation that does not have this property. WLOG, we assume that the subpermutation on $\{1, 2\}$ is $(1, 2)$. We split it into three cases based on the location of 5.

Case 1: The subpermutation on $\{1, 2, 5\}$ is $(5, 1, 2)$. Since $1 + 2 + 5 = 2 \cdot 4$, we have $4 \in \mathcal{R}_p(5) \cap \mathcal{L}_p(2)$. Since $1 + 2 + 7 = 2 \cdot 5$ and $1, 2 \in \mathcal{R}_p(5)$, we have $7 \in \mathcal{L}_p(5)$. Since $1+5+8 = 2 \cdot 7$ and $1, 5 \in \mathcal{R}_p(7)$, we have $8 \in \mathcal{L}_p(7)$. Hence $(8, 7, 5, 4)$ is a subpermutation of p with $4 + 5 + 7 = 2 \cdot 8$. This is a contradiction.

Case 2: The subpermutation on $\{1, 2, 5\}$ is $(1, 2, 5)$. Since $1 + 2 + 5 = 2 \cdot 4$, we have $4 \in \mathcal{R}_p(1) \cap \mathcal{L}_p(5)$. Since $1 + 2 + 7 = 2 \cdot 5$ and $1, 2 \in \mathcal{L}_p(5)$, we have $7 \in \mathcal{R}_p(5)$. Since $1+5+8 = 2 \cdot 7$ and $1, 5 \in \mathcal{L}_p(7)$, we have $8 \in \mathcal{R}_p(7)$. Hence $(4, 5, 7, 8)$ is a subpermutation of p with $4 + 5 + 7 = 2 \cdot 8$. This is a contradiction.

Case 3: The subpermutation on $\{1, 2, 5\}$ is $(1, 5, 2)$. Since $1 + 2 + 5 = 2 \cdot 4$, we have $4 \in \mathcal{R}_p(1) \cap \mathcal{L}_p(2)$. So the subpermutation on $\{1, 2, 4, 5\}$ is either $(1, 4, 5, 2)$ or $(1, 5, 4, 2)$. Now we split Case 3 into six subcases based on the subpermutation on $\{1, 2, 4, 5\}$ and the location of 8.

Subcase 3.1: The subpermutation on $\{1, 2, 4, 5\}$ is $(1, 4, 5, 2)$ and $8 \in \mathcal{R}_p(5)$. Since $1 + 5 + 8 = 2 \cdot 7$ and $1, 5 \in \mathcal{L}_p(8)$, we must have $7 \in \mathcal{L}_p(8)$. Now the subpermutation on $\{4, 5, 7, 8\}$ is $(4, 5, 7, 8)$, $(4, 7, 5, 8)$, or $(7, 4, 5, 8)$. Since $4 + 5 + 7 = 2 \cdot 8$, we have a contradiction.

Subcase 3.2: The subpermutation on $\{1, 2, 4, 5\}$ is $(1, 4, 5, 2)$ and $8 \in \mathcal{L}_p(4)$. Since $2 + 4 + 8 = 2 \cdot 7$ and $2, 4 \in \mathcal{R}_p(8)$, we have $7 \in \mathcal{R}_p(8)$. Now the subpermutation on $\{4, 5, 7, 8\}$ is $(8, 7, 4, 5)$, $(8, 4, 7, 5)$, or $(8, 4, 5, 7)$. Since $4 + 5 + 7 = 2 \cdot 8$, we have a contradiction.

Subcase 3.3: The subpermutation on $\{1, 2, 4, 5, 8\}$ is $(1, 4, 8, 5, 2)$. Since $1+5+8 = 2\cdot 7$ and $1, 8 \in \mathcal{L}_p(5)$, we have $7 \in \mathcal{L}_p(5)$. Since $2 + 4 + 8 = 2 \cdot 7$ and $2, 8 \in \mathcal{R}_p(4)$, we have $7 \in \mathcal{R}_p(4)$. Since $2 + 5 + 9 = 2 \cdot 8$ and $2, 5 \in \mathcal{R}_p(8)$, we have $9 \in \mathcal{L}_p(8)$. Since $1 + 4 + 9 = 2 \cdot 7$ and $1, 4 \in \mathcal{L}_p(7)$, we have $9 \in \mathcal{R}_p(7)$. So the subpermutation on $\{1, 2, 4, 5, 7, 8, 9\}$ is $(1, 4, 7, 9, 8, 5, 2)$. Since $1+4+7 = 2\cdot 6$, we have $6 \in \mathcal{R}_p(1)\cap\mathcal{L}_p(7)$. Now $(1, 6, 9, 8)$ is a subpermutation of p with $1 + 6 + 9 = 2 \cdot 8$. This is a contradiction.

Subcase 3.4: The subpermutation on $\{1, 2, 4, 5\}$ is $(1, 5, 4, 2)$ and $8 \in \mathcal{L}_p(5)$. Since $1 + 5 + 8 = 2 \cdot 7$ and $1, 8 \in \mathcal{L}_p(5)$, we have $7 \in \mathcal{L}_p(5)$. Since $2 + 4 + 8 = 2 \cdot 7$ and $2, 4 \in \mathcal{R}_p(8)$, we have $7 \in \mathcal{R}_p(8)$. Now $(8, 7, 5, 4)$ is a subpermutation of p with $4 + 5 + 7 = 2 \cdot 8$. This is a contradiction.

Case 3.5: The subpermutation on $\{1, 2, 4, 5\}$ is $(1, 5, 4, 2)$ and $8 \in \mathcal{R}_p(4)$. Since $2 + 4 + 8 = 2 \cdot 7$ and $2, 8 \in \mathcal{R}_p(4)$, we have $7 \in \mathcal{R}_p(4)$. Since $1 + 5 + 8 = 2 \cdot 7$ and $1, 5 \in \mathcal{L}_p(8)$, we have $7 \in \mathcal{L}_p(8)$. Now $(5, 4, 7, 8)$ is a subpermutation of p with $4 + 5 + 7 = 2 \cdot 8$. This is a contradiction.

Case 3.6: The subpermutation on $\{1, 2, 4, 5, 8\}$ is $(1, 5, 8, 4, 2)$. Since $2+4+8 = 2\cdot 7$, we have $7 \in \mathcal{R}_p(8)$. Now $(1, 5, 8, 7)$ is a subpermutation of p with $1 + 5 + 8 = 2 \cdot 7$. This is a contradiction. □

Proposition 6.5.3. *For all $n \geq 16$, every $p \in S_n$ has a 4-additive subsequence of length three.*

Proof. By Lemmas 6.2.3 and 6.2.4, it suffices to show that every $p \in S_{16}$ has a subsequence (x_1, x_2, x_3) such that either $x_1 + x_2 = 3x_3$ or $x_2 + x_3 = 3x_1$. Suppose, for a contradiction, that $p \in S_{16}$ is a permutation that does not have this property. WLOG, we assume that the subpermutation on $\{1, 5\}$ is $(1, 5)$.

Since $1 + 5 = 3 \cdot 2$, the subpermutation on $\{1, 2, 5\}$ is $(1, 2, 5)$. Since $2 + 13 = 3 \cdot 5$, we have $13 \in \mathcal{R}_p(5)$. Since $5 + 13 = 3 \cdot 6$, $6 \in \mathcal{R}_p(5) \cap \mathcal{L}_p(13)$. Now the subpermutation on $\{1, 2, 5, 6, 13\}$ is $(1, 2, 5, 6, 13)$.

Since $2 + 16 = 3 \cdot 6$ and $2 \in \mathcal{L}_p(3)$, we must have $16 \in \mathcal{R}_p(6)$. Since $5 + 16 = 3 \cdot 7$, we must have $7 \in \mathcal{R}_p(5) \cap \mathcal{L}_p(16)$. Since $2 + 7 = 3 \cdot 3$, we must have $3 \in \mathcal{R}_p(2) \cap \mathcal{L}_p(7)$. Since $1 + 8 = 3 \cdot 3$ and $1 \in \mathcal{L}_p(3)$, we have $8 \in \mathcal{R}_p(3)$. Since $7 + 8 = 3 \cdot 5$ and $7 \in \mathcal{R}_p(5)$, we must have $8 \in \mathcal{L}_p(5)$. Notice that now the subpermutation on $\{3, 5, 7\}$ must be $(3, 5, 7)$.

Since $4 + 5 = 3 \cdot 3$ and $5 \in \mathcal{R}_p(3)$, we have $4 \in \mathcal{L}_p(3)$. Now $(4, 5, 7)$ is a subpermutation of p with $5 + 7 = 3 \cdot 4$. This is a contradiction. □

Conjecture 6.5.4. If $k \geq 3$ and $k \neq \ell$, then for all large n, every $p \in S_n$ has an ℓ-additive subsequence of length k.

While it seems unlikely that Problem 6.1.1 could be easily resolved, the following question seems doable and we plan to tackle it in the future:

Problem 6.5.5. Is it true that for all $\ell \geq 4$, there exists n such that every $p \in S_n$ has an ℓ-additive subsequence of length three?

Problem 6.1.2 seems significantly harder than Problem 6.1.1 even for $\ell = 2$. The following problem seems more approachable:

Problem 6.5.6. Is it true that for all positive integers $\ell \geq 2$ and sufficiently large n, every $p \in S_n$ has a monotone ℓ-additive subsequences of length three?

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