6-15-2024

Building Blocks for W-Algebras of Classical Types

Vladimir Kovalchuk

Building Blocks for W-Algebras of Classical Types

Abstract

The universal 2-parameter vertex algebra W_∞ of type $W(2, 3, 4; \dots)$ serves as a classifying object for vertex algebras of type $W(2, 3, \dots, N)$ for some N in the sense that under mild hypothesis, all such vertex algebras arise as quotients of W_∞. There is an $\mathbb{N} \times \mathbb{N}$ family of such 1-parameter vertex algebras known as Y-algebras. They were introduced by Gaiotto and Rapčák are expected to be building blocks for all W-algebras in type A, i.e, every W-(super) algebra in type A is an extension of a tensor product of finitely many Y-algebras. Similarly, the orthosymplectic Y-algebras are 1-parameter quotients of a universal 2-parameter vertex algebra of type $W(2, 4, 6, \dots)$, which is a classifying object for vertex algebras of type $W(2, 4, \dots, 2N)$ for some N. Unlike type A, these algebras are not all the building blocks for W-algebras of types B, C, and D. In this thesis, we construct a new universal 2-parameter vertex algebra of type $W(1^3, 2, 3^3, 4, 5^3, 6, \dots)$ which we denote by $W_\infty{}^{\mathrm{sp}}$ since it contains a copy of the affine vertex algebra $V^k(\mathrm{sp}_2)$. We identify 8 infinite families of 1-parameter quotients $W_\infty{}^{\mathrm{sp}}$ which are analogues of the Y-algebras, and 4 infinite families with sp_2-level constant. We regard $W_\infty{}^{\mathrm{sp}}$ as a fundamental object on equal footing with W_∞ and $W_\infty{}^{\mathrm{ev}}$, and we give some heuristic reasons for why we expect the 1-parameter quotients of these three objects to be the building blocks for all W-algebras in classical types.

Document Type

Dissertation

Degree Name

Ph.D.

First Advisor

Andrew R. Linshaw

Second Advisor

Florencia Orosz

Third Advisor

Shashank Kanade

Keywords

Vertex algebras, Mathematics

Subject Categories

Algebra | Mathematics | Other Mathematics | Physical Sciences and Mathematics

Building Blocks for $\mathcal{W}$-algebras of Classical Types

---

A Dissertation

Presented to

the Faculty of the College of Natural Sciences and Mathematics

University of Denver

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Vladimir Kovalchuk

June 2024

Advisor: Dr. Andrew R. Linshaw

Author: Vladimir Kovalchuk
Title: Building Blocks for $\mathcal{W}$-agebras of Classical Types
Advisor: Dr. Andrew R. Linshaw
Degree Date: June 2024

ABSTRACT

The universal 2-parameter vertex algebra $\mathcal{W}_\infty$ of type $\mathcal{W}(2,3,4,\dots)$ serves as a classifying object for vertex algebras of type $\mathcal{W}(2,3,\dots,N)$ for some N in the sense that under mild hypothesis, all such vertex algebras arise as quotients of $\mathcal{W}_\infty$. There is an $\mathbb{N}\times\mathbb{N}$ family of such 1-parameter vertex algebras known as Y-algebras. They were introduced by Gaiotto and Rapčák are expected to be building blocks for all $\mathcal{W}$-algebras in type A, i.e, every $\mathcal{W}$-(super) algebra in type A is an extension of a tensor product of finitely many Y-algebras. Similarly, the orthosymplectic Y-algebras are 1-parameter quotients of a universal 2-parameter vertex algebra of type $\mathcal{W}(2,4,6,\dots)$, which is a classifying object for vertex algebras of type $\mathcal{W}(2,4,\dots,2N)$ for some N. Unlike type A, these algebras are not all the building blocks for $\mathcal{W}$-algebras of types B, C, and D. In this thesis, we construct a new universal 2-parameter vertex algebra of type $\mathcal{W}(1^3,2,3^3,4,5^3,6,\dots)$ which we denote by $\mathcal{W}_\infty^{\mathfrak{sp}}$ since it contains a copy of the affine vertex algebra $V^k(\mathfrak{sp}_2)$. We identify 8 infinite families of 1-parameter quotients $\mathcal{W}_\infty^{\mathfrak{sp}}$ which are analogues of the Y-algebras, and 4 infinite families with $\mathfrak{sp}_2$-level constant. We regard $\mathcal{W}_\infty^{\mathfrak{sp}}$ as a fundamental object on equal footing with $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{\mathrm{ev}}$, and we give some heuristic reasons for why we expect the 1-parameter quotients of these three objects to be the building blocks for all $\mathcal{W}$-algebras in classical types.

ACKNOWLEDGEMENTS

I am deeply indebted to the DU mathematics department, consisting of wonderful people who were apart of my long journey here. Firstly, I thank Susan Bolton and Gabbi Lucero for keeping this department thriving and functioning. It is a bargain to learn wisdom earned by years of experience and deep thought only by attending a lecture. For this gift, I am thankful to Nick Galatos, Shashank Kanade, Michael Kinyon, Paul Horn, Frederic Latremoliere, Andrew Linshaw, Ronnie Pavlov, and Petr Vojtechovksy. For much instruction on proof writing, good notation, and overall help, I am thankful to Alvaro Arias. I wish to express my gratitude to my advisor Andrew R. Linshaw - for the support, for the opportunity to work on a good problem, to travel, to meet great people, and the many conversations we have had; it has been an honor and pleasure to work with you. Finally, I want to thank Thomas Creutzig, Florencia Orosz Hunziker, and Fei Qi for many insightful conversations.

Before being mathematicians, we are human beings first; born of flesh, needing caffeine, we thrive off collaboration, free exchange of ideas and far off flights of thought. I am honored to have met many wonderful people while my time here. Thanks to Allison Achilles, Kempton (Kemptonius Rex) Albee, Mariah Barnes, Riley Britten, Terry Daniel, Ryan Demuse, Mark Doeller, Wesley Fussner, Isis Aymara Gallardo, Collier Gaiser, Dan Graybill, Devin Groh, Evans Hedges, Zion Hefty, Luke Hetzel, Joseph Ingenito, Paul Johnson, Eden Ketchum, Jordan Koler, Genevieve Maalouf, Dylan Magnani, Christian Naess, Matthew Napolitano, Lauren Nelson, Jesse (Jello) Parrish, Hwajin Park, Adam Percully, Di Qin, Casey Schlortt, JamieLynn Selman, Alexandra Sedlmayr, Alex Stevens, Gavin St. John, August (Noodles) Sangalli, and Xiao (Helios) Zhuang; I appreciate you all and the time we have shared.

TABLE OF CONTENTS

Chapter 1: Introduction

$\mathcal{W}$-algebras are an important class of vertex algebra that have been studied in both the mathematics and physics literature for nearly 40 years. For any Lie (super)algebra $\mathfrak{g}$ and nilpotent element f in the even part of $\mathfrak{g}$, the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ at level $k \in \mathbb{C}$, is defined via the generalized Drinfeld-Sokolov reduction [31]. They are a common generalization of affine vertex algebras and the Virasoro algebra, as well as the $\mathcal{N} = 1$, 2, and 4 superconformal algebras. When f is a principal nilpotent, $\mathcal{W}^k(\mathfrak{g}, f)$ is called a principal $\mathcal{W}$-algebra and is denoted by $\mathcal{W}^k(\mathfrak{g})$; they appear in many settings including integrable systems [9, 13, 36, 28], conformal field theory to higher spin gravity duality [35], Alday-Gaiotto-Tachikawa correspondence [1, 70, 12], and the quantum geometric Langlands program [37, 34, 2, 17, 39, 5]. In general, $\mathcal{W}^k(\mathfrak{g}, f)$ can be regarded as a chiralization of the finite $\mathcal{W}$-algebra $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$ defined by Premet [66], which is a quantization of the coordinate ring of the Slodowy slice $S_f \subseteq \mathfrak{g} \cong \mathfrak{g}^*$.

Principal $\mathcal{W}$-algebras satisfy Feigin-Frenkel duality, which is a vertex algebra isomorphism

$$\mathcal{W}^k(\mathfrak{g}) \cong \mathcal{W}^\ell({}^L\mathfrak{g}), \quad (k + h^\vee)(\ell +^L h^\vee) = r. \tag{1.1}$$

Here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, $h^\vee$, ${}^Lh^\vee$ are the dual Coxeter numbers of $\mathfrak{g}$, ${}^L\mathfrak{g}$, and r is the lacity [47]. For $\mathfrak{g}$ simply-laced, there is another duality called the Arakawa-Creutzig-Linshaw coset realization which was proven in [3]. For generic values of ℓ, we have a vertex algebras isomorphism

$$\mathcal{W}^\ell(\mathfrak{g}) \cong \text{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})), \qquad \ell + h^\vee = \frac{k + h^\vee}{k + h^\vee + 1}, \tag{1.2}$$

which descends to an isomorphism of simple vertex algebras

$$\mathcal{W}_\ell(\mathfrak{g}) \cong \mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$$

for all admissible levels k. This was a longstanding conjecture [10, 46, 40, 57], vastly generalizing the Goddard-Kent-Olive construction of the Virasoro algebra [33]. There is a different coset realization for types B and C [21], but no such result is known for types F and G.

Non-principal $\mathcal{W}$-algebras are not as well understood, but they have become increasingly important in physics in recent years; see for example [4, 17, 18, 30, 52, 67, 68, 69]. In [30], Gaiotto and Rapčák introduced a family of VOAs $Y_{N_1,N_2,N_3}[\psi]$ called Y-algebras, which are indexed by three integers $N_1, N_2, N_3 \geq 0$ and a complex parameter ψ. They are associated to interfaces of twisted $\mathcal{N} = 4$ supersymmetric gauge theories with gauge groups $U(N_1)$, $U(N_2)$, and $U(N_3)$. The interfaces satisfy a permutation symmetry which is expected to induce a corresponding symmetry on the VOAs. This led Gaiotto and Rapčák to conjecture a triality of isomorphisms of Y-algebras.

The Y-algebras with one label zero are (up to a Heisenberg algebra) the affine cosets of a family of non-principal $\mathcal{W}$-(super)algebras of type A which are known as hook-type. The triality conjecture when one label is zero was proved in [20], and is a vast generalization of both Feigin-Frenkel duality and the coset realization theorem in type A. The key idea is that the Y-algebras with one label zero can be realized explicitly as simple 1-parameter quotients of the universal 2-parameter vertex algebra $\mathcal{W}_\infty$ of type $\mathcal{W}(2, 3, \dots)$. This is a classifying object for vertex algebras of type $\mathcal{W}(2, 3, \dots, N)$ for some N satisfying some mild hypothesis. It was known to physicists since the early 1990s, and was constructed by Linshaw in [60]. Recently, it has been conjectured that the Y-algebras are the building blocks for all $\mathcal{W}$-algebras in type A in the sense that any such $\mathcal{W}$-algebra is an extension of a tensor product of finitely many Y-algebras; see [22, Conjecture B]. This is based on [22, Conjecture A], which says that the quantum Drinfeld-Solokov reduction can be carried out in stages; see [53] for a similar conjecture in the setting of finite $\mathcal{W}$-algebras. In [22], this conjecture was proven at the level of graded characters and was also verified by computer for all $\mathcal{W}$-algebras in type A of rank at most 4.

In [30], Gaiotto and Rapčák also introduced the orthosymplectic Y-algebras, which can be realized as affine cosets of $\mathcal{W}$-(super)algebras in types B, C, and D. They conjectured a similar triality of isomorphisms which were proven in [21] by realizing these algebras explicitly as simple 1-parameter quotients of the universal 2-parameter vertex algebra $\mathcal{W}^{\mathrm{ev}}_{\infty}$ of type $\mathcal{W}(2, 4, 6, \dots)$. This algebra was also known to physicists for many years and was constructed by Kanade and Linshaw in [55].

By analogy with the type A story, one might believe that the building blocks are the orthosymplectic Y-algebras. However it is readily seen that these are some, but not all, of the necessary building blocks. The reduction by stages conjecture says that if a nilpotent f can be decomposed as $f = f_1 + f_2$ in a sense we shall defined in Section (5), and $f_1 \geq f_2$ in a certain sense, that the reduction $H_f(V^k(\mathfrak{g}))$ coincides with $H_{f_2}(H_{f_1}(V^k(\mathfrak{g})))$. We call a nilpotent f indecomposable if it cannot be written in this way. The reason that only Y-algebras appear in [22, Conjecture A] is that the only indecomposable nilpotents in type A are the hook-type nilpotents. But in types B, C, and D, there are many more indecomposable nilpotents. This suggests that many more building blocks will be needed.

We will give some heuristic reasons why all the new building blocks should be 1-parameter quotients of a new universal vertex algebra $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ which is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, 5^3, 6, \dots). \tag{1.3}$$

The three fields in weight 1 generate a copy of the affine vertex algebra $V^k(\mathfrak{sp}_2)$, the fields in each even weight $2, 4, 6, \dots$ transform as the trivial $\mathfrak{sp}_2$ module, and the three fields in each odd weight $3, 5, 7, \dots$ each transform as the adjoint $\mathfrak{sp}_2$-module. There are 8 infinite families of 1-parameter vertex algebras with this generating type, which are either a $\mathcal{W}$-algebra (orbifolds of) or the affine cosets of $\mathcal{W}$-algebras, see (6.1). The list is quite parallel to the list of orthosymplectic Y-algebras; accordingly we call them the symplectic Y-algebras. In addition, there are 4 infinite families with $\mathfrak{sp}_2$-level is a half-integer, which have no analogues for $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{\text{ev}}$. These are the expected extra building blocks for $\mathcal{W}$-algebras of types B, C, and D.

The main result in this thesis is the construction of $\mathcal{W}_\infty^{\mathfrak{sp}}$; it is a 2-parameter VOA and serves as a classifying object for VOAs with this generating type. We regard it as a fundamental object on equal footing with $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{\text{ev}}$, but it has inexplicably never appeared before either in the mathematics or physics literature.

Chapter 2: Background

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional complex vector superspace with the parity function

$$|v| = \begin{cases} 0, & v \in V_{\bar{0}}, \\ 1, & v \in V_{\bar{1}}. \end{cases} .$$

In the category of vector superspaces, the morphisms are given by linear maps that respect parity,

$$f : V \to W, \quad f(V_{\bar{i}}) \subseteq W_{\bar{i}}, \quad i = 0, 1.$$

We say that a nondegenerate bilinear form g is supersymmetric if its restriction to the even part $g_{\bar{0}}$ is symmetric, its restriction to the odd part $g_{\bar{1}}$ is skew-symmetric, and $V_{\bar{0}}$ is orthogonal to $V_{\bar{1}}$. The superdimension of V is the difference $\mathrm{sdim}(V) = \dim(V_0) - \dim(V_1)$. Up to an isomorphism of vector superspaces, it is the superspace $\mathbb{C}^{\dim(V_{\bar{0}})|\dim(V_{\bar{1}})}$ of dimension $\dim(V_{\bar{0}})|\dim(V_{\bar{1}})$. For a linear map $A : V \to V$ its supertrace is defined as $\mathrm{sTr}(A) = \mathrm{Tr}(A|_{V_0}) - \mathrm{Tr}(A|_{V_1})$. Note that superdimension is the supertrace of the identity operator on a superspace.

2.1 Lie Superalgebras

Let $[\cdot, \cdot]$ be a homogeneous bilinear map, meaning $[V_{\bar{i}}, V_{\bar{j}}] \subset V_{\overline{i+j}}$. The pair $(V, [\cdot, \cdot])$ is a Lie superalgebra [56] if for all $a, b, c \in V$ it satisfies

$$\begin{aligned} 0 =& [a, b] + (-1)^{|a||b|}[b, a], \\ 0 =& (-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]]. \end{aligned} \tag{2.1}$$

As usual denote the adjoint map of the Lie superalgebra

$$\mathrm{ad}(a) : V \to V, \quad b \mapsto [a,b].$$

The Killing form on $\mathfrak{g}$ is a supersymmetric bilinear map defined by the supertrace

$$\mathrm{sTr}(\mathrm{ad}(a), \mathrm{ad}(b)).$$

For a simple Lie superalgebra $\mathfrak{g}$, we often work with the normalized Killing form

$$(a|b) := \frac{1}{2h^\vee}\mathrm{sTr}(\mathrm{ad}(a), \mathrm{ad}(b)),$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. A homomorphism of Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot])$ and $(\mathfrak{g}', [\cdot,\cdot]')$ is a map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ of vector superspaces that satisfies

$$\varphi([a,b]) = [\varphi(a), \varphi(b)]'.$$

The even part of the Lie superalgebra $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, and the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. The space of linear maps on a vector superspace $\mathbb{C}^{n|m}$, denoted $\mathfrak{gl}_{n|m}$, is naturally a Lie superalgebra. For homogeneous elements $a, b \in V$, the superbracket given by

$$[a,b] = ab - (-1)^{|a||b|}ba.$$

The even Lie subalgebra of $\mathfrak{gl}_{n|m}$ is the semisimple Lie algebra $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$, and the odd part transform as the $\mathbb{C}^n \otimes \mathbb{C}^{*m} \oplus \mathbb{C}^{*n} \otimes \mathbb{C}^m$ under $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$.

2.2 Orthosymplectic Lie (super)Algebras

In this thesis our main objects of study are orthosymplectic Lie (super)algebras. Let (V, g) be a vector superspace of dimension $n|2m$ and a supersymmetric bilinear form g.

Orthosymplectic Lie superalgebra $\mathfrak{osp}_{n|2m}$ is the Lie subsuperalgera of $\mathfrak{gl}_{n|2m}$ preserving g, specifically

$$\mathfrak{osp}_{n|2m} = \{A \in \mathfrak{gl}_{n|2m} |\ (Av|w) + (-1)^{|A||v|}(v|Aw) = 0,\ \forall v, w \in V\}.$$

The even subalgebra is the semisimple Lie algebra $\mathfrak{so}_n \oplus \mathfrak{sp}_{2m}$, where

- Orthogonal Lie algebra $\mathfrak{so}_n$ is the subalgebra of $\mathfrak{gl}_n$ preserving the symmetric form $g_{\bar{0}}$ on the even subspace $\mathbb{C}^{n|0}$.
- Symplectic Lie algebra $\mathfrak{sp}_{2m}$ is the subalgebra of $\mathfrak{gl}_{2m}$ preserving the symplectic form $g_{\bar{1}}$ on the odd subspace $\mathbb{C}^{0|2m}$.

The odd part of $\mathfrak{osp}_{n|2m}$ transforms as $\mathbb{C}^n \otimes \mathbb{C}^{2m}$ under $\mathfrak{so}_n \oplus \mathfrak{sp}_{2m}$. Note that setting $n = 0$ we have $\mathfrak{osp}_{0|2m} = \mathfrak{sp}_{2m}$, and setting $m = 0$ we recover $\mathfrak{osp}_{n|0} = \mathfrak{so}_n$. Therefore, orthosymplectic Lie algebra $\mathfrak{osp}_{n|2m}$ can be regarded as a common generalization of classical Lie algebras of type B, C and D. Now we describe some structure of the orthogonal, symplectic Lie subalgebras and orthosymplectic Lie superalgebras in greater detail.

Let $V \cong \mathbb{C}^{n|2m}$ be the standard representation of $\mathfrak{gl}_{n|2m}$. Choose an orthonormal basis $\{P_i | i = 1, \ldots, n\}$ of the even subspace $V_{\bar{0}}$, and a symplectic basis $\{Q_i, Q_{-i} | i = 1, \ldots, m\}$ for the odd subspace $V_{\bar{1}}$. With respect to this basis we have pairings

$$g(P_i, P_j) = \delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \quad g(Q_i, Q_j) = \omega_{i,j} := \begin{cases} \delta_{i,-j}, & i \geq 1, \\ -\delta_{i,-j}, & i \leq 1. \end{cases}$$

In particular above pairings identify $\mathbb{C}^{n|2m}$ with the dual space $(\mathbb{C}^{n|2m})^*$, equipped with the dual basis $\{P_i^* | i = 1, \ldots, n\} \cup \{Q_i^*, Q_{-i}^* | i = 1, \ldots, m\}$.

First, we consider the orthogonal Lie subalgebra $\mathfrak{so}_n$. Define maps $\{E_{i,j}|1 \leq i < j \leq n\}$ by their action on $\mathbb{C}^{n|0}$ as endomorphisms

$$\mathfrak{so}_n \to \mathfrak{gl}_{n|0}, \quad E_{i,j} \mapsto P_j^* \otimes P_i - P_i^* \otimes P_j. \tag{2.2}$$

Action (2.2) defines the standard representation of $\mathfrak{so}_n$. In this basis, the Lie bracket of $\mathfrak{so}_n$ can be computed from (2.2) to give

$$[E_{i,j}, E_{p,q}] = -\delta_{j,q}E_{i,p} + \delta_{j,p}E_{i,q} + \delta_{i,q}E_{j,p} - \delta_{i,p}E_{j,q}.$$

The normalized Killing form and the dual Coxeter number are

$$(E_{i,j}|E_{p,q}) = \delta_{i,p}\delta_{j,q} - \delta_{i,q}\delta_{j,p}, \quad h^\vee_{\mathfrak{so}_n} = n - 2. \tag{2.3}$$

In view of the above equation, we call this the orthonormal basis of $\mathfrak{so}_n$. Note that it is not the usual root basis of $\mathfrak{so}_n$.

Now, we consider the symplectic Lie subalgebra $\mathfrak{sp}_{2m}$. Define maps $\{G_{i,j}| - m \leq i, j \leq m, i, j \neq 0\}$ that act on $\mathbb{C}^{0|2m}$ as endomorphisms

$$\mathfrak{sp}_{2m} \to \mathfrak{gl}_{0|2m}, \quad G_{i,j} \mapsto Q_j^* \otimes Q_i + Q_i^* \otimes Q_j. \tag{2.4}$$

Action (2.4) defines the standard representation of $\mathfrak{sp}_{2n}$. In this basis, the Lie bracket of $\mathfrak{sp}_{2m}$ can be computed from (2.2) to give

$$[G_{i,j}, G_{p,q}] = \omega_{j,q}G_{i,p} + \omega_{j,p}G_{i,q} + \omega_{i,q}G_{j,p} + \omega_{i,p}G_{j,q}. \tag{2.5}$$

The normalized Killing form and the dual Coxeter number are

$$(G_{i,j}|G_{p,q}) = \omega_{i,p}\omega_{j,q} + \omega_{i,q}\omega_{j,q}, \quad h^\vee_{\mathfrak{sp}_{2m}} = m+1. \tag{2.6}$$

In view of the above equation, we call this the orthonormal basis of $\mathfrak{sp}_{2m}$. Note that it is not the usual root basis of $\mathfrak{sp}_{2m}$.

Lastly, we write down the basis for the odd part of $\mathfrak{osp}_{n|2m}$. Define maps $\{X_{i,j}|1 \leq i \leq n, -m \leq j \leq m, j \neq 0\}$ that act on $\mathbb{C}^{n|2m}$ as endomorphisms

$$X_{i,j} = P_i^* \otimes Q_j + Q_j^* \otimes P_i. \tag{2.7}$$

In this basis, the Lie bracket given by

$$[X_{i,j}, X_{p,q}] = \delta_{i,p} Q_{j,q} + \omega_{j,q} Q_{i,p}. \tag{2.8}$$

2.3 Invariant Theory

It is well known that the category of modules of semisimple Lie algebra is semisimple, i.e. every finite-dimensional module is a direct sum of simple finite-dimensional modules ([64], Sec. 6.3). This is not true for Lie superalgebras, with an exception of $\mathfrak{osp}_{1|2n}$. In this regard Lie superalgebra $\mathfrak{osp}_{1|2n}$ is similar to a semisimple Lie algebra.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Let M be a module, possibly infinite-dimensional that decomposes as a direct sum of finite-dimensional simple modules, with an action

$$\rho : \mathfrak{g} \to \text{End}(M).$$

The action of $\mathfrak{g}$ on M integrates to an action of a connected Lie group G on M, with Lie algebra $\mathfrak{g}$. The invariants M^G is the set of fixed points points under action of G

$$M^G = \{m \in M | g.m = m, \, \forall g \in G\}.$$

Invariants M^G can be identified with the space of vectors annihilated by $\mathfrak{g}$

$$M^{\mathfrak{g}} = \{m \in M | \rho(g).m = 0, \, \forall g \in \mathfrak{g}\}.$$

The same statement holds for Lie superalgebra $\mathfrak{g} = \mathfrak{osp}_{1|2n}$ and Lie supergroup $G = \mathrm{Osp}_{1|2n}$.

Let G be a Lie (super)group, and U a finite-dimensional representation. Consider the direct sum

$$V = \bigoplus_{k=1}^{\infty} U_k,$$

where $U_k \cong U$ for each k, and $\mathbb{C}[V]$ be the ring of polynomial functions on V. There is an induced action of G on functions

$$G \times \mathbb{C}[V] \to \mathbb{C}[V], \quad (g, f(x)) \mapsto f(g^{-1}x).$$

Denote by $\mathbb{C}[V]^G$ the set of G-invariant polynomial functions on V. The first fundamental theorem of invariant theory for the representation U of G is a description of the generators for the ring $\mathbb{C}[V]^G$. The second fundamental theorem of invariant theory for the representation U of G is a set of generators for the ideal of relations satisfied among the generators. Unfortunately, for general representations such results are not known. However, it is known in the case of U being the standard representation. In this case, for Lie algebras of classical types A, B, C, and D, the first and second fundamental theorems were obtained by Weyl in [74]. Later, Sergeev in [71] and [72] obtained the first and second fundamental theorems for the standard representations of Lie supergroups.

2.3.1 Orthogonal Invariants. For $k \geq 0$, let U_k be a copy of the standard O_n-module $\mathbb{C}^n$ with an orthonormal basis $\{x_{k,i} | i = 1, \dots, n\}$. The generators and relations for the ring $(\bigoplus_{k=1}^{\infty} U_k)^{\mathrm{O}_n}$ are given by Weyl's first and second fundamental theorems of invariant theory for the standard representation of O_n.

Theorem 1 ([74], Thm. 2.9.A and 2.17.A)**.** The ring of invariant polynomial functions

$$R = \mathbb{C}\Big[\bigoplus_{k=0}^{\infty} U_k\Big]^{\mathrm{O}_n}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n} x_{i,a}x_{i,b}, \quad a, b \geq 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$. The kernel I_n of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by determinants $p_{I,J}$ of degree $n+1$ in the variables $Q_{a,b}$, which are indexed by lists $I = (i_0, \dots i_n)$ and $J = (j_0, \cdots, j_n)$ of integers satisfying

$$0 \leq i_0 \leq \cdots \leq i_n, \quad 0 \leq j_0 \leq \cdots \leq j_n.$$

For $n = 1$ and $I = (i_0, i_1)$ and $J = (j_0, i_1)$, we have

$$d_{I,J} = q_{i_0,j_0}q_{i_1,j_1} - q_{i_1,j_0}q_{i_0,j_1},$$

and for $n > 1$ they satisfy the following recursion

$$d_{I,J} = \sum_{r=0}^{n} (-1)^{r+1} q_{i_r,j_0} d_{I_r,J'},$$

where $I_r = (i_0, \ldots, \hat{i_r}, \ldots, i_n)$ is obtained from I by omitting i_r, and $J' = (j_1, \ldots, j_n)$ is obtained from J by omitting j_0.

A similar statement holds for the exterior algebra $\bigwedge(\bigoplus_{i=1}^{\infty} U_k)^{\mathrm{O}_n}$.

Theorem 2. The ring of invariant polynomial functions

$$R = \bigwedge \Big[\bigoplus_{k=0}^{\infty} U_k\Big]^{\mathrm{O}_n}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n} x_{i,a} x_{i,b}, \quad a, b \geq 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = -Q_{b,a}$. The kernel I_n of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by polynomials $p_{I,J}$ of degree $n+1$ in the variables $Q_{a,b}$, which are indexed by lists $I = (i_0, \ldots i_n)$ and $J = (j_0, \cdots, j_n)$ of integers satisfying

$$0 \leq i_0 \leq \cdots \leq i_n, \quad 0 \leq j_0 \leq \cdots \leq j_n.$$

These relations are analogous to $(n+1) \times (n+1)$ determinants, but without the signs.

For $n = 1$ and $I = (i_0, i_1)$ and $J = (j_0, i_1)$, we have

$$d_{I,J} = q_{i_0,j_0} q_{i_1,j_1} + q_{i_1,j_0} q_{i_0,j_1},$$

and for $n > 1$ they satisfy the following recursion

$$d_{I,J} = \sum_{r=0}^{n} q_{i_r,j_0} d_{I_r,J'},$$

where $I_r = (i_0, \ldots, \hat{i_r}, \ldots, i_n)$ is obtained from I by omitting i_r, and $J' = (j_1, \ldots, j_n)$ is obtained from J by omitting j_0.

2.3.2 Symplectic Invariants. For $k \geq 0$, let U_k be a copy of the standard Sp_{2n}-module $\mathbb{C}^{2n}$ with a symplectic basis $\{x_{k,i} y_{k,i} | i = 1, \ldots, n\}$. The generators and relations for the ring $(\bigoplus_{k=1}^{\infty} U_k)^{\mathrm{Sp}_{2n}}$ are given by Weyl's first and second fundamental theorems of invariant theory for the standard representation of Sp_{2n}.

Theorem 3 ([74], Thm. 6.1.A and 6.1.B)**.** The ring of invariant polynomial functions

$$R = \mathbb{C}\left[\bigoplus_{k=0}^{\infty} U_k\right]^{\mathrm{Sp}_{2n}}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n}(x_{i,a} y_{i,b} - x_{i,b} y_{i,a}), \quad a, b \geq 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = -Q_{b,a}$. The kernel I_n of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by Pfaffians p_I of degree $n+1$ in the variables $Q_{a,b}$, which are indexed by lists $I = (i_0, \dots, i_{2n+1})$ satisfying

$$0 \le i_0 < \cdots < i_{2n+1}.$$

For $n = 1$ and $I = (i_0, i_1, i_2, i_3)$, we have

$$p_I = q_{i_0,i_1} q_{i_2,i_3} - q_{i_0,i_2} q_{i_1,i_3} + q_{i_0,i_3} q_{i_1,i_2},$$

and for $n > 1$ they satisfy the following recursion

$$p_I = \sum_{r=1}^{2n+1} (-1)^{r+1} q_{i_0,i_r} p_{I_r},$$

where $I_r = (i_1, \dots, \hat{i_r}, \dots, i_{2n+1})$ is obtained from I by omitting i_0 and i_r.

A similar statement holds for the exterior algebra $\bigwedge(\bigoplus_{i=1}^{\infty} U_k)^{\mathrm{Sp}_{2n}}$.

Theorem 4. The ring of invariant polynomial functions

$$R = \bigwedge \Big[\bigoplus_{k=0}^{\infty} U_k \Big]^{\mathrm{Sp}_{2n}}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2} \sum_{i=1}^{n} (x_{i,a} y_{i,b} + x_{i,b} y_{i,a}), \quad a, b \ge 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$. The kernel I_n of the homomorphism

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by polynomials p_I of degree $n+1$ in the variables $Q_{a,b}$, which are indexed by lists $I = (i_0, \dots, i_{2n+1})$, satisfying

$$0 \leq i_0 < \cdots < i_{2n+1}$$

For $n = 1$ and $I = (i_0, i_1, i_2, i_3)$, we have

$$p_I = q_{i_0,i_1} q_{i_2,i_3} + q_{i_0,i_2} q_{i_1,i_3} + q_{i_0,i_3} q_{i_1,i_2},$$

and for $n > 1$ they satisfy the following recursion

$$p_I = \sum_{r=1}^{2n+1} q_{i_0,i_r} p_{I_r},$$

where $I_r = (i_1, \dots, \hat{i_r}, \dots, i_{2n+1})$ is obtained from I by omitting i_0 and i_r.

2.3.3 Orthosymplectic Invariants. The generators and relations for $(\bigoplus_{k=1}^{\infty} U_k)^{\mathrm{Osp}_{1|2n}}$ are given by Sergeev's first and second fundamental theorems of invariant theory for the standard representation of $\mathrm{Osp}_{1|2n}$ (Theorems 1.3 of [71] and Theorem 4.5 of [72]). First, we state the theorem for $\mathbb{C}^{1|2n}$ representation.

Theorem 5. For $k \geq 0$, let U_k be a copy of the standard $\mathrm{Osp}_{1|2n}$-module $\mathbb{C}^{1|2n}$, which has odd subspace spanned by $\{x_{k,i}, y_{k,i} | i = 1, \dots, n\}$ and even subspace spanned by z_k. Then the ring of invariant polynomial functions

$$R = \mathbb{C}\Big[\bigoplus_{k=0}^{\infty} U_k\Big]^{\mathrm{Osp}_{1|2n}}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n}(x_{i,a}y_{i,b} + x_{i,b}y_{i,a}) + \frac{1}{2}z_a z_b, \quad a, b \geq 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$. The kernel of the map

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by polynomials of degree $2n+2$ in the variables $Q_{a,b}$ corresponding to rectangular Young tableau of size $2 \times (2n+2)$, filled by entries from a standard sequence I of length $4n+2$ from the set of indices $\{0, 1, 2, \dots\}$. The entries must weakly increase along rows and strictly increase along columns.

A similar statement holds for $\mathbb{C}^{2n|1}$ representation of $\mathrm{Osp}_{1|2n}$.

Theorem 6. For $k \geq 0$, let U_k be a copy of the standard $\mathrm{Osp}_{1|2n}$-module $\mathbb{C}^{2n|1}$, which has even subspace spanned by $\{x_{k,i}, y_{k,i} | i = 1, \dots, n\}$ and odd subspace spanned by z_k. Then ring of invariant polynomial functions

$$R = \mathbb{C}\Big[\bigoplus_{k=0}^{\infty} U_k\Big]^{\mathrm{Osp}_{1|2n}}$$

is generated by quadratics

$$q_{a,b} = \frac{1}{2}\sum_{i=1}^{n}(x_{i,a}y_{i,b} - x_{i,b}y_{i,a}) - \frac{1}{2}z_a z_b, \quad a, b \geq 0.$$

Let $Q_{a,b}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$. The kernel of the map

$$\mathbb{C}[Q_{a,b}] \to R, \quad Q_{a,b} \to q_{a,b}$$

is generated by polynomials of degree $2n+2$ in the variables $Q_{a,b}$ corresponding to rectangular Young tableau of size $2 \times (2n+2)$, filled by entries from a standard sequence I of length $4n+4$ from the set of indices $\{0, 1, 2, \dots\}$. The entries must strictly increase along rows and weakly increase along columns.

Chapter 3: Vertex Algebras

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. Denote by $\mathrm{End}(V)[[z, z^{-1}]]$ the set of all Laurent series in the formal variable z with coefficients in $\mathrm{End}(V)$. Introduce the vertex operator Y as the map

$$V \to \mathrm{End}(V)[[z, z^{-1}]], \quad a \mapsto Y(a, z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

Further, let there be a distinguished even element $\mathbf{1} \in V_{\bar{0}}$, and a linear map $\partial : V \to V$. Then vertex algebra is the data $(V, \mathbf{1}, \partial, Y)$, subject to the following axioms ([38], Sect. 1.3).

- Vacuum axiom:

$$Y(\mathbf{1}, z) = \mathrm{Id}_V, \quad Y(a, z)\mathbf{1} \in V[[z]], \quad \lim_{z \to 0} Y(a, z)\mathbf{1} = a, \quad \forall a \in V. \tag{3.1}$$

- Translation axiom:

$$[\partial, Y(a, z)] = \frac{d}{dz} Y(a, z), \quad \forall a \in V. \tag{3.2}$$

- Locality axiom:

$$(z - w)^N \left(Y(a, z)Y(b, w) - (-1)^{|a||b|} Y(b, w)Y(a, z) \right) = 0, \quad \forall a, b \in V, \tag{3.3}$$

  for some $N \geq 0$.

Homomorphisms, subalgebras, ideals and tensor products are defined in a natural way.

- A vertex algebra homomorphism φ between $(V, \mathbf{1}, \partial, Y)$ and $(V', \mathbf{1}', \partial', Y')$ is a linear map $V \to V'$ mapping $\mathbf{1}$ to $\mathbf{1}'$, intertwining the translation operators, and satisfying

$$\varphi(Y(a, z), b) = Y(\varphi(a), z)\varphi(b).$$

- A vertex subalgebra $V' \subset V$ is a ∂-invariant subspace containing the vacuum $\mathbf{1}$, and satisfying $a(z)b \in V'((z))$ for all $a, b \in V'$.

- A vertex algebra ideal $I \subset V$ is a ∂-invariant subspace satisfying $a(z)b \in I((z))$ and $b(z)a \in I((z))$ for all $a \in I$ and $b \in V$. It follows that for any ideal $I \subset V$, the quotient space V/I inherits a natural vertex algebra structure.

- For two vertex algebras $(V, \mathbf{1}, \partial, T)$ and $(V', \mathbf{1}', \partial', T')$, the data $(V \otimes V', \mathbf{1} \otimes \mathbf{1}', \partial \otimes \mathbf{1}' + \mathbf{1} \otimes \partial', Y \otimes Y')$ defines a tensor product of vertex algebras, where

$$(Y \otimes Y')(a \otimes a', z) = Y(a, z) \otimes Y'(a', z).$$

We often omit the Y and write $Y(a, z) = a(z)$, when no confusion arises. A subspace of fields $\mathcal{F}(V) \subset \text{End}(V)[[z, z^{-1}]]$ is spanned by linear maps

$$V \to V((z)) = \{\sum_{n \in \mathbb{Z}} v(n) z^{-n-1} | v(n) \in V, \ v(n) = 0 \text{ for } n >> 0\}.$$

For each $n \in \mathbb{Z}$, we have a bilinear operation on $\mathcal{F}(V)$, defined on homogeneous elements a and b by

$$a(w)_{(n)} b(w) = \text{Res}_z a(z) b(w) \iota_{|z|>|w|}(z - w)^n - (-1)^{|a||b|} \text{Res}_z b(w) a(z) \iota_{|w|>|z|}(z - w)^n. \tag{3.4}$$

Here $\iota_{|z|>|w|}f(z,w) \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$.

For $a, b \in \mathcal{F}(V)$, we have the following identity of power series known as the operator product expansion (OPE) formula.

$$a(z)b(w) = \sum_{n=0}^{\infty} a(w)_{(n)}b(w)(z-w)^{-n-1} + :a(z)b(w): . \tag{3.5}$$

Here $:a(z)b(z): = a(z)_{-}b(z) + (-1)^{|a||b|}b(z)a(z)_{+}$ denotes the normally ordered product, where

$$a(z)_{-} := \sum_{n=1}^{\infty} a_{(-n)}z^{n-1}, \quad a(z)_{+} := \sum_{n=0}^{\infty} a_{(n)}z^{-n-1}, \quad a(z) = a_{-}(z) + a_{+}(z).$$

Note that $:a(z)b(z):$ is a well-defined element of $\mathcal{F}(V)$, while the product $a(z)b(z)$ is in general not, see ([38], Sec. 2.2).

Often, (3.5) is written as

$$a(z)b(w) \sim \sum_{n=0}^{\infty} a(w)_{(n)}b(w)(z-w)^{-n-1}, \tag{3.6}$$

where $\sim$ means equal modulo the term $:a(z)b(w):$, which is regular at $z = w$.

Using identity (3.4), we see that $:a(z)b(z):$ coincides with $a(z)_{(-1)}b(z)$. More generally, we have

$$a(z)_{(-n-1)}b(z) = \frac{1}{n!} :(\partial^n a(z))\, b(z):, \quad \partial = \frac{d}{dz}, \quad n \geq 1. \tag{3.7}$$

For $a_1(z), a_2(z), \dots, a_n(z) \in \mathcal{F}(V)$, the n-fold iterated is defined inductively

$$:a_1(z)a_2(z)\cdots a_n(z): = :a_1(z)A(z):, \quad A(z) = :a_2(z)\cdots a_n(z): . \tag{3.8}$$

We often omit the formal variable z when no confusion arises.

A subspace $\mathcal{A} \subset \mathcal{F}(V)$ containing Id_V, which is closed under the products (3.4) is called a quantum operator algebra (QOA). Note that any VOA is a local QOA, i.e. satifies Locality axiom (3.3). An $\mathcal{A}$-module M is any vector superspace admitting a VOA homomorphism $\mathcal{A} \to \mathrm{End}(M)[[z, z^{-1}]]$.

3.1 Generation

The vertex algebra $\mathcal{A}$ is generated by a subset $S = \{a^i | i \in I\}$ if $\mathcal{A}$ is spanned by words formed as products built from the letters in S. We say that S strongly generates $\mathcal{A}$ if $\mathcal{A}$ is spanned by normally ordered products of elements in S and their derivatives. Equivalently, $\mathcal{A}$ is spanned by

$$\{:\partial^{k_1} a^{i_1} \cdots \partial^{k_m} a^{i_m}: | i_1, \dots, i_m \in I,\ k_1, \dots, k_m \geq 0\}.$$

Suppose that S is an ordered strong generating set $\{\alpha^i | i \in I\}$ for $\mathcal{A}$ which is at most countable. We say S freely generates $\mathcal{A}$, if it has a Poincaré-Birkhoff-Witt (PBW) basis

$$\begin{aligned} &:\partial^{k_1^1} a^{i_1} \cdots \partial^{k_{r_1}^1} a^{i_1} \partial^{k_1^2} a^{i_2} \cdots \partial^{k_{r_2}^2} a^{i_2} \cdots \partial^{k_1^n} a^{i_n} \cdots \partial^{k_{r_1}^n} a^{i_n}:, \quad 1 \leq i_1 < \cdots < i_n, \\ &k_1^1 \geq k_2^1 \geq \cdots k_{r_1}^1, \quad k_1^2 \geq k_2^2 \geq \cdots k_{r_2}^2, \cdots, k_1^n \geq k_2^n \geq \cdots k_{r_n}^n, \\ &k_1^t > k_2^t > \cdots > k_{r_t}^t \text{ if } a^{i_t} \text{ is odd.} \end{aligned} \tag{3.9}$$

3.2 Important Identities

For us it is convenient to work directly with modes of fields, rather than fields themselves. Using axioms (3.1-3.3) and OPE relation (3.6) we find that modes of the corresponding fields satisfy the Borcherds identities ([11], Sec. 3.3.10)

$$\begin{aligned}[][a_{(m)}, b_{(n)}] =& a_{(m)}b_{(n)} - (-1)^{|a||b|}b_{(n)}a_{(m)} = \sum_{i=0}^{\infty}\binom{m}{i}(c_i)_{(m+n-i)}, \\ (a_{(m)}b)_{(n)} =& \sum_{i=0}^{\infty}(-1)^i\binom{m}{i}\left(a_{(m-i)}b_{(n+i)} - (-1)^{|a||b|}(-1)^m b_{(n+m-i)}a_{(i)}\right).\end{aligned} \tag{3.10}$$

These identities together with (3.1) and (3.2) have the following useful consequences.

- Conformal identity:

$$(\partial a)_{(r)}b = -r a_{(r-1)}b, \quad r \in \mathbb{Z}. \tag{3.11}$$

- Skew-symmetry:

$$a_{(r)}b = (-1)^{|a||b|+r+1}b_{(r)}a + \sum_{i=1}^{\infty}\frac{(-1)^{|a||b|+r+i+1}}{i!}\partial^i\left(b_{(r+i)}a\right), \quad r \in \mathbb{Z}. \tag{3.12}$$

- Quasi-associativity:

$$:a:bc::=::ab:c: + \sum_{i=0}^{\infty}\frac{1}{(i+1)!}\left(:\partial^{i+1}(a)(b_{(i)}c): + (-1)^{|a||b|}:\partial^{i+1}(b)(a_{(i)}c):\right). \tag{3.13}$$

- Quasi-derivation:

$$a_{(r)}:bc:=:(a_{(r)}b)c: + (-1)^{|a||b|}:ba_{(r)}c: + \sum_{i=1}^{r}\binom{r}{i}(a_{(r-i)}b)_{(i-1)}c, \quad r \geq 0. \tag{3.14}$$

- Jacobi identity:

$$a_{(r)}(b_{(s)}c) = (-1)^{|a||b|}b_{(s)}(a_{(r)}c) + \sum_{i=0}^{r}\binom{r}{i}(a_{(i)}b)_{(r+s-i)}c, \quad r, s \geq 0. \tag{3.15}$$

We denote the above equation by $J_{r,s}(a,b,c)$, and use $J(a,b,c)$ to denote the set of all Jacobi identities $\{J_{r,s}(a,b,c)|r,s\geq 0\}$.

Once OPEs among the strong generators are defined, identities (3.11-3.14) determine the OPEs among the composite fields. We record this observation in the following proposition.

Proposition 7. Let $\mathcal{V}$ be a vertex algebra strongly generated by $S=\{a^i|i\in I\}$. Then the OPE algebra of $\mathcal{V}$ is determined by OPEs among the elements of S.

If we encounter a monomial that is not in the PBW form, we may use identities (3.11-3.14) to express it in PBW basis (3.9). This gives rise to a straightening algorithm, which is implemented in the Mathematica package `OPEdefs` [73]. In practice, the number of monomials arising in this process is at least exponential. It is an interesting problem to find optimal algorithms to efficiently run such computations.

3.3 Vertex Operator Algebras

3.3.1 Virasoro Algebra. Let Vir $=\mathbb{C}((t))\oplus\mathbb{C}C$ denote the Virasoro Lie algebra. It is which an infinite-dimensional Lie algebra with the following commutation relations

$$[L_n,L_m]=(n-m)L_{n+m}+n(n^2-1)\frac{C}{12}\delta_{n+m,0},\quad [C,L_n]=0,\quad n,m\in\mathbb{Z},\tag{3.16}$$

where $L_n=-t^{n+1}\frac{d}{dt}$. Consider the induced representation

$$\text{Vir}^c=U(\text{Vir})\otimes_{\text{Vir}_+}\mathbb{C}_c,$$

where $\mathbb{C}_c$ is the one-dimensional representation of the commutative Lie algebra $\text{Vir}_+=\mathbb{C}[[t]]\frac{d}{dt}\oplus\mathbb{C}C$ on which $\mathbb{C}[[t]]\frac{d}{dt}$ acts trivially and $\mathbb{C}C$ acts as the multiplication by c. The vector space Vir^c admits a PBW basis of the form

$$L_{-n_1} \cdots L_{-n_s} \mathbf{1}, \quad n_1 \geq \cdots \geq n_s \geq 2.$$

Let $\partial = L_{-1}$ and define a field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \sum_{n \in \mathbb{Z}} L_{(n)} z^{-n-1}$. This data define a unique vertex algebra structure on Vir^c called the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$. It is strongly generated by L satisfying the OPE relation

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}. \tag{3.17}$$

Commutation relations (3.16) imply that modes $\{L_{(0)}, L_{(1)}, L_{(2)}\}$ form a subalgebra isomorphic to $\mathfrak{sl}_2$.

3.3.2 Conformal Structure. A conformal structure with central charge c on a vertex algebra $\mathcal{A}$ is a Virasoro field L satisfying the OPE relation (3.17), such that $L_{(0)}a = \partial a$ and $L_{(1)}$ acts semisimply on $\mathcal{A}$. When a choice of conformal structure is made explicit, we follow tradition and call $(\mathcal{A}, L)$ a vertex operator algebra (VOA). A VOA homomorphism $\varphi : (\mathcal{V}, L) \to (\mathcal{V}', L')$ is then a homomorphism of vertex algebras that preserves the conformal vector

$$\varphi(L) = L'.$$

We say that a has conformal weight N if $L_{(1)}a = Na$, and denote the conformal weight N subspace by $\mathcal{A}[N]$, so that

$$V = \bigoplus_N V_N.$$

In all our examples, the conformal weight gradings will be either $\mathbb{Z}_{\geq 0}$ or $\frac{1}{2}\mathbb{Z}_{\geq 0}$. We say that a vertex algebra is of type

$$\mathcal{W}(a_1^{n_1}, \dots; b_1^{m_1}, \dots),$$

if it has a strong generating set consisting of n_1 even fields of weights a_1, and m_1 odd fields of weight b_1, etc. If $\mathcal{A}$ is freely generated of type $\mathcal{W}(a_1^{n_1},\dots;b_1^{m_1},\dots)$, then it has a graded character

$$\chi(\mathcal{A},q)=\sum_{N\geq 0}\dim(\mathcal{A}[N])q^N=\prod_{i,j\geq 0}\prod_{l=0}^{\infty}\frac{(1+q^{b_j+l})^{m_j}}{(1-q^{a_i+l})^{n_i}}.$$

A Virasoro primary vector P of conformal dimension N is any element of $\mathcal{A}$ satisfying the OPE relation

$$L(z)P(w)\sim NP(w)(z-w)^{-2}+\partial P(w)(z-w)^{-1}.$$

3.4 Coset Construction

Given a VOA $\mathcal{V}$ and a subVOA $\mathcal{A}\subset\mathcal{V}$, the coset or commutant of $\mathcal{A}$ in $\mathcal{V}$ is the vertex algebra

$$\text{Com}(\mathcal{A},\mathcal{V})=\{v\in\mathcal{V}|[a(z),v(w)]=0,\quad\forall a\in\mathcal{A}\}.$$

This was introduced by Frenkel and Zhu in [50], generalizing earlier constructions in [33]. Equivalently, $v\in\text{Com}(\mathcal{A},\mathcal{V})$ if and only if $a_{(n)}v=0$ for all $a\in\mathcal{A}$ and $n\geq 0$. Note that if $\mathcal{V}$ and $\mathcal{A}$ are conformal elements $L^{\mathcal{V}}$ and $L^{\mathcal{A}}$, then $\text{Com}(\mathcal{A},\mathcal{V})$ has conformal elements $L=L^{\mathcal{V}}-L^{\mathcal{A}}$, provided that $L^{\mathcal{A}}\neq L^{\mathcal{V}}$. Defining a conformal structure on the tensor product as the sum of conformal vectors, we may always realize the ambient VOA $\mathcal{V}$ as the following conformal extension

$$\mathcal{A}\otimes\text{Com}(\mathcal{A},\mathcal{V})\hookrightarrow\mathcal{V}.$$

3.5 Orbifold Construction

Let $\mathcal{A}$ be a VOA and $G\subset\text{Aut}(\mathcal{A})$ be a subgroup of VOA automorphisms. The set of elements fixed by G

$$\mathcal{A}^G=\{a\in\mathcal{A}|g(a)=a,\ \forall g\in G\}$$

inherits a VOA structure, and is called the G-orbifold of $\mathcal{A}$. This construction was originally introduced in physics; see for example [29] as well as [44] for the construction of the Moonshine VOA $V^\natural$ as an extension of the $\mathbb{Z}_2$-orbifold of the VOA associated to the Leech lattice. In all our examples, G will be either a finite group $\mathbb{Z}_2$, or one of the classical groups O_n, Sp_{2n}, or the supergroup $\mathrm{Osp}_{1|2n}$.

3.6 Free Field Algebras

A free field algebra is a vertex superalgebra $\mathcal{V}$ with weight grading

$$\mathcal{V} = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{V}[d], \quad \mathcal{V}[0] = \mathbb{C}\mathbf{1},$$

with strong generators $\{X^i | i \in I\}$ satisfying OPE relations

$$X^i(z)X^j(w) \sim a_{i,j}\mathbf{1}(z-w)^{-\Delta(X^i)-\Delta(X^j)}, \quad a_{i,j} \in \mathbb{C}, \quad a_{i,j} = 0 \text{ if } \Delta(X^i) + \Delta(X^j) \notin \mathbb{Z}.$$

Note $\mathcal{V}$ is not assumed to have a conformal structure. These generalize a well-known class of algebras such as the rank n Heisenberg algebra, rank n bc-system, and rank n $\beta\gamma$-system [51]. Here we recall the four families of standard free field algebras that were introduced by Creutzig and Linshaw in [20].

3.6.1 Even Algebras of Orthogonal Type. Let $n \geq 1$ and $k \geq 2$ be even. The even algebra of orthogonal type $\mathcal{O}_{\text{ev}}(n,k)$ is the vertex algebra with even generators $\{a^i | i = 1, \dots, n\}$ of weight $\frac{k}{2}$, satisfying OPE relations

$$a^i(z)a^j(w) \sim \delta_{i,j}\mathbf{1}(z-w)^{-k}.$$

In the case $k = 2$, $\mathcal{O}_{\text{ev}}(n, 2)$ is the rank n Heisenberg algebra $\mathcal{H}(n)$. Let $\{\pi^i | i = 1, \dots, n\}$ denote the standard generators for $\mathcal{H}(n)$ with OPEs

$$\pi^i(z)\pi^j(w) \sim \delta_{i,j}\mathbf{1}(z - w)^{-2}.$$

In this case, there is an n-dimensional family of conformal structures

$$L = \frac{1}{2}\sum_{i=1}^{n} : \pi^i \pi^i : + m_i \partial \pi^i, \quad c = n - 3\sum_{i=1}^{n} m_i^2.$$

For the choice of $m_i = 0$, generators $\{\pi^i | i = 1, \dots, n\}$ are primary of weight 1. The even algebra of orthogonal type $\mathcal{O}_{\text{ev}}(n, k)$ can be realized inside $\mathcal{H}(n)$ by setting

$$a^i = \frac{\epsilon}{\sqrt{(k-1)!}}\partial^{\frac{k}{2}}\pi^i, \quad i = 1, \dots, n,$$

where $\epsilon = \sqrt{-1}$ if $4|k$, and otherwise $\epsilon = 1$. Note that $\mathcal{O}_{\text{ev}}(n, k)$ has no conformal vector for $k > 2$, but for all k it is a simple and has full automorphism group the orthogonal group O_n.

3.6.2 Even Algebras of Symplectic Type. Let $n \geq 1$ and $k \geq 1$ be odd. The even algebra of symplectic type $\mathcal{S}_{\text{ev}}(n, k)$ is the vertex algebra with even generators $\{a^i, b^i | i = 1, \dots, n\}$ of weight $\frac{k}{2}$, satisfying OPE relations

$$\begin{aligned} &a^i(z)b^j(w) \sim \delta_{i,j}(z - w)^{-k}, \quad &&b^i(z)a^j(w) \sim -\delta_{i,j}(z - w)^{-k}, \\ &a^i(z)a^j(w) \sim 0, &&b^i(z)b^j(w) \sim 0. \end{aligned} \tag{3.18}$$

In the case $k = 1$, $\mathcal{O}_{\text{ev}}(n, 1)$ is the rank n $\beta\gamma$-system algebra $\mathcal{S}(n)$. Let $\{\beta^i, \gamma^i | i = 1, \dots, n\}$ be the standard generators of $\mathcal{S}(n)$ with OPEs

$$\beta^i(z)\gamma^j(w) \sim \delta_{i,j}\mathbf{1}(z - w)^{-1}. \tag{3.19}$$

In this case, there is an n-dimensional family of conformal structures

$$L = \sum_{i=1}^{n} (1-m_i) :\beta^i \partial \gamma^i: - m_i :\partial \beta^i \gamma^i:, \quad c = 2n + 12 \sum_{i=1}^{n} m_i(m_i - 1).$$

For the choice of $m_i = 1$, $c = -n$ and $\{\beta^i, \gamma^i | i = 1, \dots, n\}$ are primary of conformal weight $\frac{1}{2}$. The even algebra of symplectic type $\mathcal{S}_{\text{ev}}(n,k)$ can be realized as the subalgebra of $\mathcal{S}(n)$ with generators

$$a^i = \frac{\epsilon}{\sqrt{(k-1)!}} \partial^{\frac{k-1}{2}} \beta^i, \quad b^i = \frac{\epsilon}{\sqrt{(k-1)!}} \partial^{\frac{k-1}{2}} \beta^i, \quad i = 1, \dots, n.$$

Note that $\mathcal{O}_{\text{ev}}(n,k)$ has no conformal vector for $k > 1$, but for all k it is a simple and has full automorphism group the symplectic group Sp_{2n}.

3.6.3 Odd Algebras of Symplectic Type. Let $n \geq 1$ and $k \geq 2$ be even. The odd algebra of symplectic type $\mathcal{S}_{\text{odd}}(n,k)$ is the vertex superalgebra with odd generators $\{a^i, b^i | i = 1, \dots, n\}$ of weight $\frac{k}{2}$, satisfying OPE relations

$$\begin{aligned} &a^i(z)b^j(w) \sim \delta_{i,j}(z-w)^{-k}, \quad && b^i(z)a^j(w) \sim -\delta_{i,j}(z-w)^{-k}, \\ &a^i(z)a^j(w) \sim 0, && b^i(z)b^j(w) \sim 0. \end{aligned} \tag{3.20}$$

In the case $k = 2$, $\mathcal{S}_{\text{odd}}(n,2)$ is the rank n symplectic fermion algebra $\mathcal{A}(n)$. Let $\{e^i, f^i | i = 1, \dots, n\}$ be standard generators of $\mathcal{A}(n)$ with OPEs

$$e^i(z)f^j(w) \sim \delta_{i,j}\mathbf{1}(z-w)^{-2}. \tag{3.21}$$

In this case, there is only one conformal vector

$$L = \sum_{i=1}^{n} :e^i f^i:, \quad c = -2n,$$

and $\{e^i, f^i | i = 1, \dots, n\}$ are primary of conformal weight 1. The odd algebra of symplectic type $\mathcal{S}_{\text{odd}}(n, k)$ can be realized as the subalgebra of $\mathcal{A}(n)$ with generators

$$a^i = \frac{\epsilon}{\sqrt{(k-1)!}} \partial^{\frac{k}{2}-1} e^i, \quad b^i = \frac{\epsilon}{\sqrt{(k-1)!}} \partial^{\frac{k}{2}-1} f^i, \quad i = 1, \dots, n.$$

For $k > 2$, $\mathcal{S}_{\text{odd}}(n, k)$ has no conformal vector, but for all k it is a simple and has full automorphism group the symplectic group Sp_{2n}.

3.6.4 Odd Algebras of Orthogonal Type. Let $n \geq 1$ and $k \geq 1$ be odd. The odd algebra of orthogonal type $\mathcal{O}_{\text{odd}}(n, k)$ is the vertex superalgebra with odd generators $\{a^i | i = 1, \dots, n\}$ of weight $\frac{k}{2}$, satisfying OPE relations

$$a^i(z) a^j(w) \sim \delta_{i,j} (z - w)^{-k}.$$

In the case $k = 1$, $\mathcal{O}_{\text{odd}}(n, 1)$ is the rank n free fermion algebra $\mathcal{F}(n)$. Let $\{\phi^i | i = 1, \dots, n\}$ be standard generators of $\mathcal{F}(n)$ with OPEs

$$\phi^i(z) \phi^j(w) \sim \delta_{i,j} \mathbf{1} (z - w)^{-1}. \tag{3.22}$$

We remark that bc-system $\mathcal{E}(n)$ of rank n is isomorphic to $\mathcal{F}(2n)$; it has odd generators b^i, c^j and satisfying OPE relations

$$b^i(z) c^j(z) \sim \delta_{i,j} \mathbf{1} (z - w)^{-1}.$$

In this case, there is an n-dimensional family of conformal structures

$$L = \sum_{i=1}^{n} m_i :\partial b^i c^i: - (1 - m_i) :b^i \partial c^i:, \quad c = -2n - 12 \sum_{i=1}^{n} m_i (m_i - 1). \tag{3.23}$$

Generators $\{b^i | i = 1, \dots, n\}$ are primary of conformal weight $1-m_i$, and $\{c^i | i = 1, \dots, n\}$ are primary of conformal weight m_i. In the case there is an odd number of generators, we have

$$L = -\frac{1}{2} :\partial\phi^1\phi^1: + \sum_{i=1}^{n} m_i :\partial b^i c^i: -(1-m_i) :b^i\partial c^i:, \quad c = -2n - 1 - 12\sum_{i=1}^{n} m_i(m_i-1),$$

and ϕ_1 is primary of weight $\frac{1}{2}$. The odd algebra of orthogonal type $\mathcal{S}_{\text{odd}}(n,k)$ can be realized as the subalgebra of $\mathcal{F}(n)$ with generators

$$a^i = \frac{\epsilon}{\sqrt{(k-1)!}}\partial^{\frac{k-1}{2}}\phi^i, \quad i = 1, \dots, n.$$

Note that $\mathcal{O}_{\text{odd}}(n,k)$ has no conformal vector for $k > 1$, but for all k it is simple and has full automorphism group O_n.

Finally, we record some obvious isomorphisms among tensor products of free field algebras in the following.

Lemma 8. We have the following isomorphisms of vertex algebras.

$$\begin{aligned} \mathcal{O}_{\text{ev}}(m,k) \otimes \mathcal{O}_{\text{ev}}(n,k) &\cong \mathcal{O}_{\text{ev}}(n+m,k), \\ \mathcal{S}_{\text{ev}}(m,k) \otimes \mathcal{S}_{\text{ev}}(n,k) &\cong \mathcal{S}_{\text{ev}}(n+m,k), \\ \mathcal{O}_{\text{odd}}(m,k) \otimes \mathcal{O}_{\text{odd}}(n,k) &\cong \mathcal{O}_{\text{odd}}(n+m,k), \\ \mathcal{S}_{\text{odd}}(m,k) \otimes \mathcal{S}_{\text{odd}}(n,k) &\cong \mathcal{S}_{\text{odd}}(n+m,k). \end{aligned} \tag{3.24}$$

A particularly important class of free field algebras are those that decompose as a finite tensor product of the standard ones, discussed above. Specifically, affine $\mathcal{W}$-algebras admit a suitable limit which is a free field algebra of this form, see Theorem 12 which we discuss in Section 4. This feature provides a powerful tool for analyzing the structure $\mathcal{W}$-algebras, their orbifolds and cosets.

3.7 Affine Vertex Superalgebras

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie superalgebra with a normalized Killing form $(\cdot|\cdot)$. Let $\{q^\alpha|\alpha \in S\}$ be a basis of $\mathfrak{g}$ which is homogeneous with respect to parity. We define the corresponding structure constants $\{f_\gamma^{\alpha,\beta}|\alpha,\beta,\gamma \in S\}$ by

$$[q^\alpha, q^\beta] = \sum_{\gamma\in S} f_\gamma^{\alpha,\beta} q^\gamma.$$

The affine vertex algebra of $\mathfrak{g}$ associated to the bilinear form $(\cdot|\cdot)$ at level k is strongly generated by the fields $\{X^\alpha|\alpha \in S\}$ with operator products

$$X^\alpha(z)X^\beta(w) \sim k(q^\alpha|q^\beta)\mathbf{1}(z-w)^{-2} + \sum_{\gamma\in S} f_\gamma^{\alpha,\beta} X^\gamma(w)(z-w)^{-1}.$$

We define X_α to be the field corresponding to q_α where $\{q_\alpha|\alpha \in S\}$ is the dual basis of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$. The following is Sugawara conformal vector and the respective central charge

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_{\alpha\in S} (-1)^{|\alpha|} :X_\alpha X^\alpha:, \quad c^{\mathfrak{g}} = \frac{k \text{ sdim}\mathfrak{g}}{k+h^\vee}. \tag{3.25}$$

Fields $X^\alpha(z)$ and $X_\alpha(z)$ are primary with respect to $L^{\mathfrak{g}}$ and have conformal weight of 1.

Let $\mathcal{V}$ be a VOA equipped with a homomorphism $V^k(\mathfrak{g}) \to \mathcal{V}$. We continue using notation $\{X^\alpha|\alpha \in S\}$ to denote the image of the generators. In particular, the zero modes of affine fields afford an action of a classical Lie (super)algebra $\mathfrak{g}$. Let $P = \text{Span}\{P^i|i = 1,\dots,n\}$ denote some irreducible $\mathfrak{g}$-submodule arising in $\mathcal{V}$, and $\rho : \mathfrak{g} \to \text{End}(V)$ denote the corresponding action. We say that P is affine primary if it satisfies the OPE relations

$$X^\alpha(z)P^i(w) \sim (\rho(\alpha)P^i)(w)(z-w)^{-1}, \quad i = 1,\dots,n.$$

In all our examples these will be either trivial, standard or adjoint representations.

3.8 Action of Affine VOAs on Free Fields

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and let V be $\mathfrak{g}$-module of dimension d_1 with an action

$$\rho_1 : \mathfrak{g} \to \text{End}(V).$$

Equip $\mathfrak{g}$ with some basis $\{x_i | i = 1, \dots, n\}$, and let $\{x'_i | i = 1, \dots, n\}$ denote the dual basis with repect to the normalized Killing form $(\cdot|\cdot)$. There is an induced homomorphism of VOAs, again denoted ρ_1, defined as the following.

$$\rho_1 : V^{-k}(\mathfrak{g}) \to \mathcal{S}(d_1), \quad X^\alpha \mapsto -\sum_{i=1}^{d_1} :\gamma^{x'_i}\beta^{\rho_1(\alpha)(x_i)}: . \tag{3.26}$$

Here k is the scaling factor of the trace form to the normalized Killing form on $\mathfrak{g}$,

$$\text{Tr}(\rho(q^\alpha), \rho(q^\beta)) = k(q^\alpha | q^\beta). \tag{3.27}$$

Then $\{\beta^i | i = 1, \dots, n\}$ and $\{\gamma^i | i = 1, \dots, n\}$ transform under $\mathfrak{g}$ as V and V^*, respectively.

Specializing (3.26) for $\mathfrak{g} = \mathfrak{so}_n$ and $V = (\mathbb{C}^n)^{\oplus m}$, we have a homomorphism

$$V^{-2m}(\mathfrak{so}_n) \to \mathcal{S}(nm). \tag{3.28}$$

Note that for each $i = 1, \dots, n$, $U_i = \text{Span}\{\gamma^{i,1}, \dots, \gamma^{i,m}, \beta^{i,1}, \dots, \beta^{i,m}\}$ is a copy of the standard $\mathbb{C}^{2m}$ $\mathfrak{sp}_{2m}$-module. In addition, there is a homomorphism $V^{-\frac{n}{2}}(\mathfrak{sp}_{2m}) \to \mathcal{S}(nm)$

given by

$$\begin{aligned} G^{i,j} &\mapsto \sum_{l=1}^{n} :\beta^{l,i}\beta^{l,j}:, \quad 1 \le i \le j \le m, \\ G^{-i,-j} &\mapsto \sum_{l=1}^{n} :\gamma^{l,i}\gamma^{l,j}:, \quad 1 \le i \le j \le m, \\ G^{-i,j} &\mapsto \sum_{l=1}^{n} :\gamma^{l,i}\beta^{l,j}:, \quad 1 \le i, j \le m, \end{aligned} \tag{3.29}$$

where $G_{i,j}$ form an orthonormal basis (2.5). Moreover, the images of homomorphisms (3.28) and (3.29) commute in $\mathcal{S}(nm)$ [8]. Combining these two maps we have a homomorphism

$$V^{-\frac{n}{2}}(\mathfrak{sp}_{2m}) \otimes V^{-2m}(\mathfrak{so}_n) \to \mathcal{S}(nm). \tag{3.30}$$

In the section (7) we will need a special case of the above with $m = 1$.

Next, we consider affine algebra actions on the bc-systems and free fermion algebras. As before, let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and let W be $\mathfrak{g}$-module of dimension d_2 with an action

$$\rho_2 : \mathfrak{g} \to \text{End}(W).$$

There is an induced homomorphism of VOAs, again denoted ρ_2, defined as the following.

$$\rho_2 : V^{\ell}(\mathfrak{g}) \to \mathcal{E}(d_2), \quad X^{\alpha} \mapsto \sum_{i=1}^{d_2} :c^{x_i'} b^{\rho_2(\alpha)(x_i)}: . \tag{3.31}$$

Here, as in (3.27) ℓ is the scaling factor of the trace form to the normalized Killing form on $\mathfrak{g}$. Then $\{b^i | i = 1, \dots, n\}$ and $\{c^i | i = 1, \dots, n\}$ transform under $\mathfrak{g}$ as V and V^*, respectively.

Specializing (3.31) for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $V = (\mathbb{C}^{2n})^{\oplus m}$, we have a homomorphism

$$V^{n}(\mathfrak{sp}_{2m}) \to \mathcal{E}(2nm). \tag{3.32}$$

In addition, there is a homomorphism $V^m(\mathfrak{sp}_{2n}) \to \mathcal{E}(2nm)$ given by

$$\begin{aligned} G^{i,j} &\mapsto \sum_{l=1}^{n} :b^{l,-i}b^{l,j}: + :b^{l,-j}b^{l,i}:, \quad 1 \le i \le j \le m, \\ G^{-i,-j} &\mapsto \sum_{l=1}^{n} :c^{l,i}c^{l,-j}: + :c^{l,j}c^{l,-i}:, \quad 1 \le i \le j \le m, \\ G^{-i,j} &\mapsto \sum_{l=1}^{n} :c^{l,i}b^{l,j}: - :c^{l,-j}b^{l,-i}:, \quad 1 \le i, j \le m. \end{aligned} \tag{3.33}$$

Moreover, their images inside $\mathcal{E}(2nm)$ commute [65] and give rise to a homomorphism

$$L_n(\mathfrak{sp}_{2m}) \otimes L_m(\mathfrak{sp}_{2n}) \to \mathcal{E}(2nm). \tag{3.34}$$

In the section (7) we will need a special case of the above with $m = 1$.

Finally, we consider actions of affine vertex (super)algebras on tensor products of $\beta\gamma$-systems and bc-systems. Let $\mathfrak{g}$ be a simple Lie algebra and let $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V)$, and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(W)$ be finite-dimensional $\mathfrak{g}$-modules, as above. Then we have a homomorphism

$$\rho : V^{-k+\ell}(\mathfrak{g}) \to \mathcal{S}(d_1) \otimes \mathcal{E}(d_2), \quad X^\alpha \mapsto -\sum_{i=1}^{d_1} :\gamma^{x'_i}\beta^{\rho_1(\alpha)(x_i)}: + \sum_{i=1}^{d_2} :c^{x'_i}b^{\rho_2(\alpha)(x_i)}: . \tag{3.35}$$

Specializing (3.35) for $\mathfrak{g} = \mathfrak{osp}_{m|2r}$ and $V = (\mathbb{C}^{m|2r})^{\oplus n}$, we have a homomorphism

$$V^n(\mathfrak{osp}_{m|2r}) \to \mathcal{S}(nm) \otimes \mathcal{E}(2nm). \tag{3.36}$$

In addition, there is a homomorphism $V^{-\frac{m}{2}+r}(\mathfrak{sp}_{2n}) \to \mathcal{S}(nm) \otimes \mathcal{E}(2nr)$ given by

$$\begin{aligned} G^{i,j} &\mapsto \sum_{l=1}^{m} :\beta^{l,i}\beta^{l,j}: + \sum_{l=1}^{r} :b^{l,-i}b^{l,j}: + :b^{l,-j}b^{l,i}:, \quad 1 \leq i \leq j \leq n, \\ G^{-i,-j} &\mapsto \sum_{l=1}^{m} :\gamma^{l,i}\gamma^{l,j}: + \sum_{l=1}^{r} :c^{l,i}c^{l,-j}: + :c^{l,j}c^{l,-i}:, \quad 1 \leq i \leq j \leq n, \\ G^{-i,j} &\mapsto \sum_{l=1}^{m} :\gamma^{l,i}\beta^{l,j}: + \sum_{l=1}^{r} :c^{l,i}b^{l,j}: - :c^{l,-i}b^{l,-j}:, \quad 1 \leq i,j \leq n. \end{aligned} \tag{3.37}$$

Moreover, their images inside $\mathcal{S}(nm)\otimes\mathcal{E}(2nm)$ commute and give rise to a homomorphism

$$V^{-\frac{m}{2}+r}(\mathfrak{sp}_{2n}) \otimes V^{n}(\mathfrak{osp}_{m|2r}) \to \mathcal{S}(nm) \otimes \mathcal{E}(2nr). \tag{3.38}$$

In the section (7) we will need a special case of the above with $m = n = 1$.

Chapter 4: $\mathcal{W}$-superalgebras

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie (super)algebra equipped with normalized Killing form $(\cdot|\cdot)$, and f be a nilpotent element in the even part of $\mathfrak{g}$. Associated to the data $\mathfrak{g}$, f and a complex number k, is the $\mathcal{W}$-(super)algebra $\mathcal{W}^k(\mathfrak{g}, f)$ of level k. The definition is due to Kac, Roan, and Wakimoto [31], and it generalizes the definition for f a principal nilpotent and $\mathfrak{g}$ a Lie algebra given by Feigin and Frenkel [47].

4.1 BRST Reduction

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie (super)algebra with normalized Killing form $(\cdot|\cdot)$. Let $\{q^\alpha|\alpha \in S\}$ be a basis of $\mathfrak{g}$ which is homogeneous with respect to parity, and define the structure constants $\{f_\gamma^{\alpha,\beta}|\alpha, \beta, \gamma \in S\}$

$$[q^\alpha, q^\beta] = \sum_{\gamma \in S} f_\gamma^{\alpha,\beta} q^\gamma.$$

Recall the affine vertex algebra of $\mathfrak{g}$ associated to the bilinear form $(\cdot|\cdot)$ at level k is strongly generated by the fields $\{X^\alpha|\alpha \in S\}$ with operator products

$$X^\alpha(z)X^\beta(w) \sim k(q^\alpha|q^\beta)\mathbf{1}(z-w)^{-2} + \sum_{\gamma \in S} f_\gamma^{\alpha,\beta} X^\gamma(w)(z-w)^{-1}.$$

As before, let X_α to be the field corresponding to q_α where $\{q_\alpha|\alpha \in S\}$ is the dual basis of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$, and $L^{\mathfrak{g}}$ be the Sugawara vector (3.25).

Let f be a nilpotent in the even part of $\mathfrak{g}$, which we may complete to an $\mathfrak{sl}_2$-triple, thanks to the Jacobson-Morozov theorem [63], [54]. By abuse of notation we denote $\mathfrak{sl}_2 =$

$\{f, h, e\}$, which satisfies the well-known commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The semisimple element $x = \frac{1}{2}h$ induces a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ as follows.

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{a \in \mathfrak{g} | [x, a] = ja\}. \tag{4.1}$$

Let S_k be a basis of the weight space $\mathfrak{g}_k$ and extend to the basis of $\mathfrak{g}$, so that $S = \bigcup S_k$. Write $S_+ = \bigcup\{S_k | k < 0\}$ and $S_- = \bigcup\{S_k | k > 0\}$ for bases of respective subspaces

$$\mathfrak{g}_+ = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{>0}} \mathfrak{g}_j, \quad \mathfrak{g}_- = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{<0}} \mathfrak{g}_j.$$

Denote by $F(\mathfrak{g}_+)$ the charged fermions vertex algebra associated with the vector superspace $\mathfrak{g}_+ \oplus \mathfrak{g}_+^*$. It is strongly generated by fields $\{\varphi_\alpha, \varphi^\alpha | \alpha \in S_+\}$, where φ_α and φ^α have opposite parity to q^α. Their operator products are

$$\varphi_\alpha(z)\varphi^\beta(w) \sim \delta_{\alpha,\beta}(z-w)^{-1}, \quad \varphi_\alpha(z)\varphi_\beta(w) \sim 0 \sim \varphi^\alpha(z)\varphi^\beta(w).$$

The following is a conformal vector and the associated central charge, see (3.23).

$$L^{\text{ch}} = \sum_{\alpha \in S_+} (1 - m_\alpha) :\partial\varphi^\alpha\varphi_\alpha: - m_\alpha :\varphi^\alpha\partial\varphi_\alpha:, \quad c^{\text{ch}} = -\sum_{\alpha \in S_+} (-1)^{|\alpha|}(12m_\alpha^2 - 12m_\alpha + 2). \tag{4.2}$$

Fields $\varphi_\alpha(z)$ and $\varphi^\alpha(z)$ are primary with respect to L^{ch} and have conformal weights of $1 - m_\alpha$ and m_α, respectively.

Since nilpotent element f belongs to $\mathfrak{g}_{-1}$, it endows $\mathfrak{g}_{\frac{1}{2}}$ with a skew-symmetric bilinear form

$$\langle a, b\rangle = (f|[a,b]). \tag{4.3}$$

Denote by $F(\mathfrak{g}_{\frac{1}{2}})$ the neutral fermions vertex algebra associated to $\mathfrak{g}_{\frac{1}{2}}$. It is strongly generated by fields $\{\Phi_\alpha|\alpha \in S_{\frac{1}{2}}\}$ and Φ_α and q^α have the same parity. Their operator products are

$$\Phi_\alpha(z)\Phi_\beta(w) \sim \langle q^\alpha, q^\beta\rangle (z-w)^{-1}.$$

The following is a conformal vector and the associated central charge.

$$L^{\text{ne}} = \frac{1}{2}\sum_{\alpha\in S_{\frac{1}{2}}} :\partial\Phi^\alpha\Phi_\alpha:, \quad c^{\text{ne}} = -\frac{1}{2}\text{sdim}\mathfrak{g}_{\frac{1}{2}}.$$

Here vectors Φ^α are dual to Φ_α with respect bilinear form (4.3). Fields $\Phi^\alpha(z)$ and $\Phi_\alpha(z)$ are primary with respect to L^{ne} and have conformal weight of $\frac{1}{2}$.

As in [31], define a vertex algebra $C^k(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes F(\mathfrak{g}_+) \otimes F(\mathfrak{g}_{\frac{1}{2}})$. It admits a $\mathbb{Z}$-grading by charge

$$C^k(\mathfrak{g}, f) = \bigoplus_{j\in\mathbb{Z}} C_j,$$

by giving the φ_α charge -1, the φ^α charge 1, and all others 0. There is an odd field $d(z) = d_{\text{st}}(z) + d_{\text{tw}}(z)$ of charge -1, where

$$\begin{aligned} d_{\text{st}}(z) &= \sum_{\alpha\in S_+} (-1)^{|\alpha|} :X^\alpha\varphi^\alpha:(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in S_+} (-1)^{|\alpha||\gamma|} f_\gamma^{\alpha,\beta} :\varphi_\gamma\varphi^\alpha\varphi^\beta:(z), \\ d_{\text{tw}}(z) &= \sum_{\alpha\in S_+} (f|q^\alpha)\varphi^\alpha(z) + \sum_{\alpha\in S_{\frac{1}{2}}} :\varphi^\alpha\Phi_\alpha:(z). \end{aligned} \tag{4.4}$$

Theorem 9 ([31], Thm. 2.1)**.** The above defined field $d(z)$ satisfies

$$d(z)d(w) \sim 0.$$

In particular, since $d(z)$ is odd, the zero-mode $d_{(0)}$ is a square-zero map. In addition, charge considerations imply

$$[d_{(0)}, C_j] \subset C_{j+1}.$$

This endows the vertex algebra $C^k(\mathfrak{g}, f)$ with a structure of $\mathbb{Z}$-graded cohomology complex. It has a conformal vector $L = L^{\mathfrak{g}} + \partial x + L^{\text{ch}} + L^{\text{ne}}$ with central charge

$$c(\mathfrak{g}, f, k) = \frac{k\ \text{sdim}(\mathfrak{g})}{k + h^\vee} - 12k(x|x) - \sum_{\alpha \in S_+} (-1)^{|\alpha|}(12m_\alpha^2 - 12m_\alpha + 2) - \frac{1}{2}\text{sdim}(\mathfrak{g}_{\frac{1}{2}}), \quad (4.5)$$

where $m_\alpha = j$ if $\alpha \in S_j$. It is established in [31] that it is $d_{(0)}$-closed, that is $d_{(0)}L = 0$. Finally, the affine $\mathcal{W}$-superalgebra is defined to be its cohomology

$$\mathcal{W}^k(\mathfrak{g}, f) := H(C^k(\mathfrak{g}, f), d_{(0)}).$$

Remark 4.1.1. *$\mathcal{W}^k(\mathfrak{g}, f)$ can be endowed with other conformal structures, with respect to which conformal weights of generators may be shifted. Moreover, there can exist multiple commuting Virasoro subalgebras.*

The following fields feature prominantly in the description of $\mathcal{W}$-algebras.

$$J^\alpha(z) = X^\alpha(z) + \sum_{\beta,\gamma \in S_+} (-1)^{|\gamma|} f_\gamma^{\alpha,\beta} :\varphi_\gamma \varphi^\beta:(z) + \frac{(-1)^{|\alpha}}{2} \sum_{\beta,\gamma \in S_+} f_\gamma^{\beta,\alpha} :\Phi_\beta \Phi^\gamma:(z). \quad (4.6)$$

Let $\mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{g}^f$ denote the subspace of trivial $\mathfrak{sl}_2$-modules appearing in decomposition (4.1). The fields $\{J^\alpha | q^\alpha \in \mathfrak{a}\}$ close under OPEs, and generate to an affine vertex algebra of type $\mathfrak{a}$, with its level shifted.

Theorem 10 ([58], Thm. 2.1)**.**

$$J^\alpha(z)J^\beta(w) \sim (k(q^\alpha|q^\beta) + k^\Gamma(q^\alpha, q^\beta))\mathbf{1}(z-w)^{-2} + f^{\alpha,\beta}_\gamma J^\gamma(w)(z-w)^{-1}, \tag{4.7}$$

where

$$k^\Gamma(q^\alpha, q^\beta) = \frac{1}{2}\left(\kappa_{\mathfrak{g}}(q^\alpha, q^\beta) - \kappa_{\mathfrak{g}_0}(q^\alpha, q^\beta) - \kappa_{\frac{1}{2}}(q^\alpha, q^\beta)\right),$$

with $\kappa_{\mathfrak{g}}, \kappa_{\mathfrak{g}_0}$, and $\kappa_{\mathfrak{g}_{\frac{1}{2}}}$ the Killing forms, that is the supertrace of the adjoint representation of $\mathfrak{g}$,$\mathfrak{g}_0$, and $\mathfrak{g}_{\frac{1}{2}}$.

Define the following two subVOAs of C.

- C^- is strongly generated by the fields $\{J^\alpha | \alpha \in S_-\} \cup \{\varphi^\alpha | \alpha \in S_+\} \cup \{\Phi_\alpha | \alpha \in S_{\frac{1}{2}}\}$,
- C^+ is strongly generated by the fields $\{\varphi_\alpha | \alpha \in S_+\}$.

We have a decomposition of vector superspaces

$$C = C^+ \otimes C^-. \tag{4.8}$$

It is then shown that both C^- and C^+ are $d_{(0)}$-invariant, and the decomposition (4.8) holds at the level of complexes. Moreover, it is possible to establish that $H(\mathcal{C}^+, d_{(0)}) = \mathbb{C}\mathbf{1}$, and so by Künneth formula all nontrivial cohomology arises from C^-.

The key structure theorem is the following.

Theorem 11 ([31], Thm 4.1)**.** Let $\mathfrak{g}$ be a simple-finite dimensional Lie superalgebra with an invariant bilinear form $(\cdot|\cdot)$, and let x, f be a pair of even elements of $\mathfrak{g}$ such that $ad(x)$ is diagonalizable with half-integer eigenvalues and $[x, f] = -f$. Suppose that all eigenvalues of $ad(x)$ on $\mathfrak{g}^f$ are non-positive, so that $\mathfrak{g}^f = \bigoplus_{j \leq 0} \mathfrak{g}^f_j$. Then

1. For each $q^\alpha \in \mathfrak{g}^f_{-j}$, there exists a $d_{(0)}$-closed field $J^{\{\alpha\}}$ in C^- of conformal weight $1 + j$ (with respect to L) such that $J^{\{\alpha\}} - J^\alpha$ is a linear combination of normally ordered products of the fields J^β, where $\{q^\beta | 0 \leq s < j\}$, the fields $\{\Phi_\alpha | \alpha \in S_{\frac{1}{2}}\}$, and their derivatives.

2. The homology classes of the fields $J^{\{\alpha\}}$, where $\{q^\alpha\}$ is a basis of $\mathfrak{g}^f$, strongly and freely generate the vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$.

3. $H_0(C, d_{(0)}) = \mathcal{W}^k(\mathfrak{g}, f)$ and $H_j(C) = 0$ if $j \neq 0$.

This construction is a particular case of BRST reduction, often referred as the Drinfeld-Sokolov reduction of $V^k(\mathfrak{g})$. By part (3) of the above theorem, we may ease the notation and denote the reduction $H_f(\cdot)$, and in particular $\mathcal{W}^k(\mathfrak{g}, f) = H_f(V^k(\mathfrak{g}))$.

Finally, we note that this quantum Drinfeld-Sokolov reduction is a functor. Let M be a $V^k(\mathfrak{g})$-module, that is, a smooth $\hat{\mathfrak{g}}-$module of level k. It gives rise to a $\mathcal{W}^k(\mathfrak{g}, f)$-module as follows. Consider the C-module

$$C(M) = M \otimes F(\mathfrak{g}_+) \otimes F(\mathfrak{g}_{\frac{1}{2}}),$$

with M having zero charge. Then C-module $C(M)$ inherits a charge decomposition

$$C(M) = \bigoplus_{j \in \mathbb{Z}} C(M)_j.$$

Hence $(C(M), d_{(0)})$ is a C-module complex, and its cohomology $H_f(M) := \bigoplus_{i \in \mathbb{Z}} H^i_f(M)$ is a direct sum of $\mathcal{W}^k(\mathfrak{g}, f)$-modules. Thus, we have a functor

$$V^k(\mathfrak{g})\text{-Mod} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_f(M).$$

4.2 Classification of Nilpotent Orbits

In the above exposition of $\mathcal{W}$-algebras we have assumed an explicit choice of the nilpotent element f. Consider the adjoint action of G on $\mathfrak{g}$

$$G \times \mathfrak{g} \to \mathfrak{g}, \quad f \mapsto gfg^{-1}.$$

Then completing the conjugate gfg^{-1} to the conjugate $\mathfrak{sl}_2$-triple gives rise to an isomorphic $\mathcal{W}$-algebra. Consequently, the isomorphism type of the $\mathcal{W}^k(\mathfrak{g}, f)$ depends at most on the adjoint orbit $\mathcal{O}_f = G.f$. Nilpotent orbits admit the following partition characterizations.

1. The nilpotent orbits in $\mathfrak{sl}_n$ are in bijection with partitions of n ([16], Thm. 5.1.1).

2. The nilpotent orbits in $\mathfrak{so}_{2n+1}$ are in bijection with partitions of $2n+1$, where the even parts occurs with even multiplicity ([16], Thm. 5.1.2).

3. The nilpotent orbits in $\mathfrak{sp}_{2n}$ are in bijection with partitions of $2n$, where the odd parts occurs with even multiplicity ([16], Thm. 5.1.3).

4. The nilpotent orbits in $\mathfrak{so}_{2n}$ are in bijection with partitions of $2n$, where the even parts occurs with even multiplicity, except very even partitions (those with only even parts, each having even multiplicity) correspond to two orbits ([16], Thm. 5.1.4).

In types A, B and C the nilpotent orbits are in bijection with isomorphism classes $\mathcal{W}$-algebras. In type D, the two different nilpotent orbits arising from very even partitions give rise to isomorphic $\mathcal{W}$-algebras. Therefore, let $\mathfrak{g}$ be of fixed type, and let $f \in \mathfrak{g}$ be nilpotent; we denote the $\mathcal{W}$-algebra as $\mathcal{W}(\mathfrak{g}, f_\lambda)$, where λ is associated to f by the above procedure.

4.3 Large Level Limits of $\mathcal{W}$-algebras

Generally, vertex algebras defined over polynomial rings have nontrivial OPEs that are hard to work with directly. In this work we are interested in certain soft features of $\mathcal{W}$-algebras, such as the strong and weak generating types, which do not require knowledge of the full OPE algebra. To study such questions, we recall the technique of large level limits developed in ([20], Sec. 3). These are tensor products of free field algebras ([20], Cor. 3.4), and as heuristic, may be considered as the 0^{th}-order approximation to a $\mathcal{W}$-algebra.

First, we consider the case of affine vertex algebra. The integral form $V^{\mathbf{k}}(\mathfrak{g})$ over the polynomial ring $R = \mathbb{C}[\mathbf{k}]$ is a vertex algebra, so that

$$V^{\mathbf{k}}(\mathfrak{g})/(\mathbf{k} - k) \cong V^k(\mathfrak{g})$$

is the universal affine VOA over $\mathbb{C}$. Let $\hat{R} = \mathbb{C}[\mathbf{k}^{\pm\frac{1}{2}}]$, and consider the induced algebra $V^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g}) := V^{\mathbf{k}}(\mathfrak{g}) \otimes_R \hat{R}$. Introduce the integral form $V^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g})$ over $\hat{R}_\infty = \mathbb{C}[\mathbf{k}^{-\frac{1}{2}}]$, strongly generated by $\hat{a}_\infty = \mathbf{k}^{-\frac{1}{2}} a$ for $a \in \mathfrak{g}$. The large level limit $V^\infty(\mathfrak{g}) = \lim_{k\to\infty} V^k(\mathfrak{g})$ is by definition

$$V^\infty(\mathfrak{g}) := V^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g})/(\mathbf{k}^{\frac{1}{2}}).$$

Thanks to the above scaling, in the limit only the leading pole of OPEs is nonzero

$$X^\alpha(z)X^\beta(w) \sim (q^\alpha|q^\beta)\mathbf{1}(z-w)^{-2}.$$

Therefore, the large level limit $V^\infty(\mathfrak{g})$ is isomorphic to a free field $\mathcal{S}_{\text{ev}}(\text{dim}(\mathfrak{g}), 2)$. As for the general $\mathcal{W}$-algebra, we replace the BRST complex $C^k(\mathfrak{g}, f)$ with an integral form $C^{\mathbf{k}}(\mathfrak{g}, f)$ defined over $\hat{R}_\infty$, with appropriate modifications to the complex structure, see ([20], Sec. 3). As before, $\mathcal{W}^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g}, f)$ is the 0^{th} cohomology $H^0(C^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g}, f))$. Now, the large level limit of $\mathcal{W}$-algebra is defined as

$$\mathcal{W}^\infty(\mathfrak{g}, f) := \mathcal{W}^{\mathbf{k}}_{\hat{R}_\infty}(\mathfrak{g}, f)/(\mathbf{k}^{\frac{1}{2}}).$$

Theorem 12 ([20], Thm 3.5 and Cor. 3.4)**.** $\mathcal{W}^{\text{free}}(\mathfrak{g}, f)$ is a free field algebra with strong generators $\{X^\alpha | q^\alpha \in \mathfrak{g}^f\}$ and OPEs

$$X^\alpha(z)X^\beta(w) \sim (z-w)^{-2k}\delta_{j,k}B_k(q^\alpha, q^\beta) \tag{4.9}$$

for $q^\alpha \in \mathfrak{g}^f_{-k}$ and $q^\beta \in \mathfrak{g}^f_{-j}$, where

$$B_k : \mathfrak{g}^f_{-k} \times \mathfrak{g}^f_{-k} \to \mathbb{C}, \quad B_k(a,b) := (-1)^{2k}((\text{ad}(f))^{2k}b|a).$$

Moreover, $\mathcal{W}^\infty(\mathfrak{g}, f)$ decomposes as a tensor product of the standard free field algebras. Specifically, let us refer to $2k$ in (4.9) as pole order, and $X = \text{Span}\{X^\alpha | q^\alpha \in \mathfrak{g}^f_{-k}\}$. Then,

- if pole order is even and form $(\cdot|\cdot)$ is symmetric, then X generate an even algebra of orthogonal type.
- if pole order is odd and form $(\cdot|\cdot)$ is symmetric, then X generate an odd algebra of orthogonal type.
- if pole order is odd and form $(\cdot|\cdot)$ is skew-symmetric, then X generate an even algebra of symplectic type.
- if pole order is even and form $(\cdot|\cdot)$ is skew-symmetric, then X generate an odd algebra of symplectic type.

Chapter 5: Indecomposable Nilpotents andRreduction by Stages

In this section, we provide some motivation for our main result, which is the construction of a new universal 2-parameter vertex algebra that is freely generated of type (1.3). We begin by recalling some conjectures from the recent paper [22] on the structure of $\mathcal{W}$-algebras of type A. An arbitrary nilpotent corresponds to a Young diagram, and we can regard it as the sum of hook-type nilpotents which are indecomposable. If true, reduction in stages would imply that $\mathcal{W}$-algebras are extensions of tensor product of hook-type cosets. These hook-type cosets are quotients of $\mathcal{W}_\infty$, and we regard them as the building blocks of $\mathcal{W}$-superalgebras of type A. Moreover, these have Abelian Zhu algebra and we can regard this tensor product as analogue of the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_n)$ [3]. In the case of $L_k(\mathfrak{gl}_n)$ for $k \in \mathbb{N}$, this subVOA is a product of principal $\mathcal{W}$-algebras, but in general this is not the case.

One definition of indecomposable nilpotents that works in classical types is the following. A nilpotent f is decomposable if up to conjugacy it can be written in the form $f = f_1 + f_2$ where f_1, f_2 are both sums of consecutive simple roots, and these sets of simple roots are disjoint. Otherwise, f is indecomposable. Following Collingwood-MacGovern [16], we list the indecomposable nilpotents in the remaining classical types B, C, and D.

5.1 Indecomposable Nilpotents for Type C

1. For $1 \leq n < N$, let $f_{2n} \in \mathfrak{sp}_{2N}$ correspond to the partition $(2n, 1^{2N-2n})$. This is the principal nilpotent in $\mathfrak{sp}_{2n} \subseteq \mathfrak{sp}_{2N}$.

2. For $1 \leq n < \frac{N}{2}$, let $f_{2n+1,2n+1} \in \mathfrak{sp}_{2N}$ corresponding to the partition $(2n+1, 2n+1, 2N-2n-2)$.

5.2 Indecomposable Nilpotents for Type D

1. For $1 \leq n \leq N$, let $f_{2n-1,1} \in \mathfrak{so}_{2N}$ correspond to the partition $(2n-1, 1, 1^{2N-2n})$ of $2N$. This is the principal nilpotent in $\mathfrak{so}_{2n} \subseteq \mathfrak{so}_{2N}$.

2. For $1 \leq n \leq \frac{N}{2}$, let $f_{2n,2n} \in \mathfrak{so}_{2N}$ correspond to the partition $(2n, 2n, 1^{2N-4n})$ of $2N$.

3. For $0 \leq i < n \leq \frac{N}{2}$, let $f_{2n+2i+1,2n-2i-1} \in \mathfrak{so}_{2N}$ correspond to partition $(2n+2i+1, 2n-2i-1, 1^{2N-4n})$ of $2N$.

4. For $0 \leq i < n \leq \frac{N-1}{2}$, let $f_{2n+2i+1,2n-2i+1} \in \mathfrak{so}_{2N}$ correspond to partition $(2n+2i+1, 2n-2i+1, 1^{2N-4n-2}$ of $2N$.

5.3 Indecomposable Nilpotents for Type B

1. For $1 \leq n \leq N$, let $f_{2n+1} \in \mathfrak{so}_{2N+1}$ correspond to the partition $(2n+1, 1^{2N-2n})$ of $2N+1$. This is the principal nilpotent in $\mathfrak{so}_{2n+1} \subseteq \mathfrak{so}_{2N+1}$.

2. For $1 \leq n \leq \frac{N}{2}$, let $f_{2n,2n,1} \in \mathfrak{so}_{2N+1}$ correspond to the partition $(2n, 2n, 1^{1+2N-4n})$ of $2N+1$.

3. For $0 \leq i < n \leq \frac{N}{2}$, let $f_{2n+2i+1,2n-2i-1,1} \in \mathfrak{so}_{2N+1}$ correspond to the partition $(2n+2i+1, 2n-2i-1, 1, 1^{2N-4n})$ of $2N+1$.

4. For $0 \leq i < n \leq \frac{N-1}{2}$, let $f_{2n+2i+1,2n-2i+1,1} \in \mathfrak{so}_{2N+1}$ correspond to the partition $(2n+2i+1, 2n-2i+1, 1^{2N-4n-1})$ of $2N+1$.

5.4 Generating Types

We now consider the generating type of the corresponding $\mathcal{W}$-algebras in the cases where these nilpotents are distinguished, i.e., $\mathfrak{g}^\natural$ is trivial.

1. $\mathcal{W}^k(\mathfrak{sp}_{2N}, f_{2N})$ is the principal $\mathcal{W}$-algebras and has type $\mathcal{W}(2, 4, \dots, 2N)$.

2. $\mathcal{W}^k(\mathfrak{sp}_{2(2N+1)}, f_{2N+1,2N+1})$ has type $\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2N-1)^3, 2N, (2N+1)^3)$.

3. $\mathcal{W}^k(\mathfrak{so}_{2N}, f_{2N-1,1})$ is the principal $\mathcal{W}$-algebra and has type $\mathcal{W}(2, 4, \dots, 2N-2, N)$

4. $\mathcal{W}^k(\mathfrak{so}_{4N}, f_{2N,2N})$ has type $\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2N-1)^3, 2N)$.

5. $\mathcal{W}^k(\mathfrak{so}_{4N}, f_{2N+2i+1,2N-2i-1})$ for $0 \leq i < N$, has type

$$\begin{aligned}\mathcal{W}(2^2, 4^2, \dots, (2N-2i-2)^2, & 2N-2i, 2N-2i+2, \dots, 2N+2i, \\ & 2i+2, 2i+3, 2i+4, \dots, 2N).\end{aligned}$$

6. $\mathcal{W}^k(\mathfrak{so}_{4N+2}, f_{2N+2i+1,2N-2i+1})$ for $0 \leq i < N$ has type

$$\begin{aligned}\mathcal{W}(2^2, 4^2, \dots, (2N-2i)^2, & 2N-2i+2, 2N-2i+4, \dots, 2N+2i, \\ & 2i+1, 2i+2, 2i+3, \dots, 2N+1).\end{aligned}$$

7. $\mathcal{W}^k(\mathfrak{so}_{2N+1}, f_{2N+1})$ is the principal $\mathcal{W}$-algebra has type $\mathcal{W}(2, 4, \dots, 2N)$

8. $\mathcal{W}^k(\mathfrak{so}_{4N+1}, f_{2N,2N,1})$ has type

$$\mathcal{W}\big(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n, \left(\frac{2N+1}{2}\right)^2\big).$$

9. $\mathcal{W}^k(\mathfrak{so}_{4N+1}, f_{2N+2i+1,2N-2i-1,1})$ for $0 \leq i < N$ has type

$$\begin{aligned}\mathcal{W}(2^2, 4^2, \dots, (2N-2i-2)^2, & 2N-2i, 2N-2i+2, \dots, 2N+2i, \\ & 2i+2, 2i+3, 2i+4, \dots, 2N, N+i+1, N-i).\end{aligned}$$

10. $\mathcal{W}^k(\mathfrak{so}_{4N+3}, f_{2N+2i+1,2N-2i+1})$ for $0 \leq i < N$ has type

$$\mathcal{W}(2^2, 4^2, \dots, (2N-2i)^2, 2N-2i+2, 2N-2i+4, \dots, 2N+2i,$$
$$2i+1, 2i+2, 2i+3, \dots, 2N+1, N+i+1, N-i+1).$$

We already know the existence of a universal 2-parameter VOA of type $\mathcal{W}(2,4,6,\dots)$ that admits $\mathcal{W}^k(\mathfrak{sp}_{2n+1})$ and $\mathcal{W}^k(\mathfrak{so}_{2n})^{\mathbb{Z}_2}$ as 1-parameter quotients. Examples (2) and (4) above suggest existence of a universal VOA which is freely generated of type (1.3) admitting all these algebras as 1-parameter quotients. In fact, Example (8) has an action of $\mathbb{Z}_2$ and it is not difficult to see that the $\mathbb{Z}_2$-orbifold has this generating type.

For each $i \geq 0$, Example (6) suggests the existence of a universal VOA which is freely generated of type

$$\mathcal{W}(2^2, 4^2, 6^2, \dots, 2i+1, 2i+2, 2i+3, \dots), \tag{5.1}$$

admitting these algebras as 1-parameter quotients. Example (10) has an action of $\mathbb{Z}_2$ and the $\mathbb{Z}_2$-orbifold has this generating type of (5.1).

Similarly, for each $i \geq 0$, Example (5) suggests the existence of a universal VOA which is freely generated of type

$$\mathcal{W}(2^2, 4^2, 6^2, \dots, 2i+2, 2i+3, 2i+4, \dots), \tag{5.2}$$

admitting these algebras as 1-parameter quotients. As above, Example (9) has an action of $\mathbb{Z}_2$ and the $\mathbb{Z}_2$-orbifold has this generating type of (5.2).

Based on our previous experience with such universal objects, we expect that the universal objects of the form (1.3), (5.1), and (5.2) to be 2-parameter vertex algebras. In fact, we further expect the universal objects of the form (5.1), and (5.2) for all $i \geq 0$ to not be new in the sense that they are extensions of two commuting copies of $\mathcal{W}^{\text{ev}}_{\infty}$. The reason we expect this is as follows. Consider the decomposable nilpotent $f_{2N+2i,2N-2i} \in \mathfrak{sp}_{4N}$. Then $\mathcal{W}^k(\mathfrak{sp}_{4N}, f_{2N+2i,2N-2i})$ has type

$$\mathcal{W}(2^2, 4^2, \dots, (2N-2i)^2, 2N-2i+2, 2N-2i+4, \dots, 2N+2i, 2i+1, 2i+2, \dots, 2N).$$

If the universal VOA of type is indeed a 2-parameter VOA as we expect, then these VOAs are also 1-parameter quotient. Based on reduction by stages, $\mathcal{W}^k(\mathfrak{sp}_{4N}, f_{2N+2i,2N-2i})$ should be an extension of two quotients of $\mathcal{W}^{\text{ev}}_{\infty}$, so the universal VOA should have this property as well.

Similarly, for $0 \leq i \leq N$, $\mathcal{W}^k(\mathfrak{sp}_{2(2N+1)}, f_{2N+2i+2,2N-2i})$, which has type

$$\mathcal{W}(2^2, 4^2, \dots, (2N-2i)^2, 2N-2i+2, 2N-2i+4, \dots, 2N+2i+2,$$
$$2i+2, 2i+3, \dots, 2N+1).$$

Again, these are all expected to be quotients of universal VOA of type

$$\mathcal{W}(2^2, 4^2, 6^2, \dots, 2i+2, 2i+3, 2i+4, \dots).$$

If this universal VOA is indeed a 2-parameter VOA, this would imply that it is an extension of two commuting copies of $\mathcal{W}^{\text{ev}}_{\infty}$.

It can be seen easily that there is no family of decomposable nilpotents in any of these types whose unifying algebra is of the form (1.3). This suggests that the universal object of this type is genuinely new, and is not an extension of two commuting copies of $\mathcal{W}^{\text{ev}}_{\infty}$. Moreover, the above analysis suggests that the quotients of this algebra are the remaining building blocks.

In the next two sections, we enumerate 12 infinite families of 1-parameter VOAs with strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$. First, in section 6 we give 8 $\mathbb{N} \times \mathbb{N}$ families which contain the $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{so}_{4n}, f_{2n,2n})$ and $\mathcal{W}^k(\mathfrak{sp}_{2(2n+1)}, f_{2n+1,2n+1})$ as special cases. These families are all either $\mathcal{W}$-algebras or (orbifolds of) cosets of $\mathcal{W}$-algebras, and the list is quite parallel to the orthosymplectic Y-algebras as given in [21]. Therefore we call these the Y-algebras of type C. The remaining 4 families arise in a different way, and have the property that the level k of the affine subalgebra of $\mathfrak{sp}_2$ is a fixed integer or half integer, and also is simple.

Chapter 6: Y-algebras of Type C

Here we define 8 families of $\mathcal{W}$-(super)algebras that we need in a unified framework. First, let $\mathfrak{g}$ be a simple orthosymplectic Lie (super)algebra. We further assume that $\mathfrak{g}$ admits a decomposition as a $\mathfrak{b} \oplus \mathfrak{sp}_2 \oplus \mathfrak{a}$-module, where

$$\mathfrak{g} \cong \mathfrak{b} \oplus \mathfrak{sp}_2 \oplus \mathfrak{a} \oplus \rho_{\mathfrak{g}}^{\tau} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \rho_{\mathfrak{b}} \otimes \mathbb{C}^2 \otimes \rho_{\mathfrak{a}}, \tag{6.1}$$

where

$$\rho_{\mathfrak{g}}^{\tau} := \begin{cases} \rho_{\omega_2}, & \mathfrak{b} = \mathfrak{sp}_{2n}, \\ \rho_{2\omega_1}, & \mathfrak{b} = \mathfrak{so}_{2n+1}, \end{cases}$$

with the following properties.

1. $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$, and is either $\mathfrak{sp}_{2m}$ or $\mathfrak{so}_{2m+1}$.

2. $\mathfrak{a}$ is a Lie sub(super)algebra of $\mathfrak{g}$, and is either $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$,$\mathfrak{sp}_{2n}$, or $\mathfrak{osp}_{1|2n}$.

3. $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{b}}$ transform as the standard representation of $\mathfrak{a}$ and $\mathfrak{b}$, and have the same parity, which can be even or odd.

Note that if $\mathfrak{a} = \mathfrak{osp}_{1|2n}$, $\rho_{\mathfrak{a}}$ even means that $\rho_{\mathfrak{a}} \cong \mathbb{C}^{2n|1}$ as a vector superspace, whereas $\rho_{\mathfrak{a}}$ odd means that $\rho_{\mathfrak{a}} \cong \mathbb{C}^{1|2n}$. If $\mathfrak{g} = \mathfrak{osp}_{m|2n}$ we use the following convention for its dual Coxeter number $h^{\vee}$.

$$h^{\vee} = \begin{cases} m - 2n - 2 & \text{type } C \\ n + 1 - \frac{1}{2}m & \text{type } B \end{cases}, \quad \text{sdim}(\mathfrak{osp}_{m|2n}) = \frac{(m-2n)(m-2n-1)}{2}. \tag{6.2}$$

In this notation, type B (respectively C) means that the subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is of type B (respectively C), and the bilinear form on $\mathfrak{osp}_{m|2n}$ is normalized so that it coincides with the normalized Killing form on $\mathfrak{b}$. The cases that we need are recorded in the table (6.1).

Table 6.1: $\mathfrak{sp}_2$-rectangular $\mathcal{W}$-algebras with a tail.

Case	$\mathfrak{g}$	$\mathfrak{a}$	$\mathfrak{b}$	$\rho_{\mathfrak{a}} \otimes \mathbb{C}^2 \otimes \rho_{\mathfrak{b}}$	a
CD	$\mathfrak{so}_{2(2m)+2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{sp}_{2m}$	Even	$\psi - 2n$
CB	$\mathfrak{so}_{2(2m)+2n+1}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2m}$	Even	$\psi - 2n + 1$
CC	$\mathfrak{osp}_{2(2m)\|2n}$	$\mathfrak{sp}_{2n}$	$\mathfrak{sp}_{2m}$	Odd	$\frac{\psi}{2} - n$
CO	$\mathfrak{osp}_{2(2m)+1\|2n}$	$\mathfrak{osp}_{1\|2n}$	$\mathfrak{sp}_{2m}$	Odd	$\frac{\psi}{2} - n - \frac{1}{2}$
BD	$\mathfrak{osp}_{2n\|2(2m+1)}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2m+1}$	Odd	$-2\psi - 2n + 4$
BB	$\mathfrak{osp}_{2n+1\|2(2m+1)}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{so}_{2m+1}$	Odd	$-2\psi - 2n + 3$
BC	$\mathfrak{sp}_{2(2m+1)+2n}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2m+1}$	Even	$\psi - n - 2$
BO	$\mathfrak{osp}_{1\|2(2m+1)+2n}$	$\mathfrak{osp}_{1\|2n}$	$\mathfrak{so}_{2m+1}$	Even	$\psi + 2n$

Let $f_{\mathfrak{b}}$ be the nilpotent element which is principal in the first factor and trivial in $\mathfrak{sp}_2 \oplus \mathfrak{a}$. The corresponding $\mathcal{W}$-algebras $\mathcal{W}^{\ell}(\mathfrak{g}, f_{\mathfrak{b}})$ will be called $\mathfrak{sp}_2$-rectangular algebra with a tail. We also write

$$f_m = \begin{cases} f_{\mathfrak{so}_m}, & m \text{ is odd,} \\ f_{\mathfrak{sp}_m}, & m \text{ is even.} \end{cases}$$

Then, we have a uniform description of the nilpotent f_m and semisimple x_m elements.

$$x_m = \frac{1}{2}\sum_{i=1}^{m}(m+1-i)(e_{i,i} + e_{i-m-1,i-m-1}), \quad f_m = \sum_{i=1}^{m}(e_{i+1,i} - e_{-i,-i+1}), \tag{6.3}$$

where $e_{i,j}$ is the matrix with 1 in the i^{th} row and j^{th} column. These arise in accordance with the Dynkin diagram displayed in Figure 6.1.

Let ρ_d denotes the $(d+1)$-dimensional representation of the $\mathfrak{sl}_2$-triple $\{f_m, x_m, e_m\}$, and define two $\mathfrak{sl}_2$-modules

$$\text{Even}(m) = \bigoplus_{i=1}^{m} \rho_{4i-2}, \quad \text{Odd}(m) = \bigoplus_{i=1}^{m} \rho_{4i}. \tag{6.4}$$

1	2	$\dots$	$m-1$	m
$-m$	$1-m$	$\cdots$	-2	-1

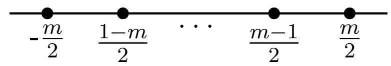

Figure 6.1: Dynkin pyramid associated to x_m and f_m. When n is odd (respectively even) the pyramid corresponds to type B (respectively C).

Recall the decomposition (6.1). Taking f to be principal in $\mathfrak{b}$ we have the following ismorphisms of $\mathfrak{sl}_2$-modules.

$$\rho_{\omega_2}^{\mathfrak{so}_{2n+1}} \cong \rho_{2\omega_1}^{\mathfrak{sp}_{2n}} \cong \text{Even}(n), \quad \rho_{2\omega_1}^{\mathfrak{so}_{2n+1}} \cong \text{Odd}(n), \quad \rho_{\omega_2}^{\mathfrak{sp}_{2n}} \cong \text{Odd}(n-1). \tag{6.5}$$

To compute the central charge (4.5) we have to evaluate contribution of each term in the expression

$$c = c_{\mathfrak{g}} + c_{\text{dilaton}} + c_{\text{ghost}}.$$

So we have evaluate c_{dilaton} and c_{ghost} from charged fermions arising from $\mathfrak{g}_+$.

First, we evaluate the dilaton contribution $-12\ell(x_m|x_m)$. Using representative (6.3), we evalute the Killing form

$$(x_m|x_m) = \text{Tr}(x_m^2).$$

In types B and C this contribution reduces to the sum of odd and even squares, respectively.

$$c_{\text{dilaton}} = -\,\ell \times \begin{cases} (m+2)(2m)(2m+1), & \mathfrak{b} = \mathfrak{so}_{2m+1}, \\ (2m-1)(2m)(2m+1), & \mathfrak{b} = \mathfrak{sp}_{2m}. \end{cases}$$

It remains to evaluate the contributions of c_{ghost}. Consider the decomposition $\mathfrak{g} = \bigoplus_d \rho_d$ intro irreducible $\mathfrak{sl}_2$-modules. Then each ρ_d gives rise a field of conformal weight $\frac{d+1}{2}$ in $\mathcal{W}^\ell(\mathfrak{g}, f_\mathfrak{b})$. Using (4.2), the corresponding ghosts give rise to a central charge contribution

$$c_d = -\frac{(d-1)(d^2-2d-1)}{2}. \tag{6.6}$$

Examining the decomposition (6.1), we find that charged fermions contribution consists of three terms

$$c_{\text{ghost}} = c_{\text{even}} + 3c_{\text{odd}} + 2\text{sdim}(\rho_\mathfrak{g})c_{d_\mathfrak{b}}. \tag{6.7}$$

To compute the contributions c_{Even} and c_{Odd} we use formula (6.6) and decomposition (6.4) evaluate the sums

$$c_{\text{Even}} = \sum_{i=0}^{n} c_{4i+3} = 6m^2 - 8m^4, \quad c_{\text{Odd}} = \sum_{i=0}^{n} c_{4i+1} = -2m(m+1)(4m(m+1)-1).$$

The third term in (6.7) is computed by applying formula (6.6).

It follows from above discussion that $\mathcal{W}^\ell(\mathfrak{g}, f_\mathfrak{b})$ is of type

$$\begin{aligned} &\mathcal{W}\left(1^{3+\dim(\mathfrak{a})}, 2, 3^3, 4, \dots, (2m-1)^3, 2m, (m+\frac{1}{2})^{2\dim(\rho_\mathfrak{a})}\right), \quad \mathfrak{b} = \mathfrak{sp}_{2m}, \\ &\mathcal{W}\left(1^{3+\dim(\mathfrak{a})}, 2, 3^3, 4, \dots, 2m, (2m+1)^3, (m+1)^{2\dim(\rho_\mathfrak{a})}\right), \quad \mathfrak{b} = \mathfrak{so}_{2m+1}. \end{aligned} \tag{6.8}$$

The affine subalgebra is $V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{a})$ for some level a, recorded in the table (6.1). We will always replace ℓ with the critically shifted level $\psi = \ell + h^\vee_\mathfrak{g}$, where $h^\vee_\mathfrak{g}$ is the dual Coxeter number of $\mathfrak{g}$. We now describe the examples we need in greater detail.

6.1 Case CD

For $\mathfrak{g} = \mathfrak{so}_{4m+2n}$, we have an isomorphism of $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{so}_{2n}$-modules

$$\mathfrak{so}_{4m+2n} \cong \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{so}_{2n} \oplus \rho_{\omega_2} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{2n},$$

and the critically shifted level $\psi = \frac{k+2m+n}{m} + 4m + 2n - 2$. We define

$$\mathcal{W}^{\psi}_{CD}(n,m) := \mathcal{W}^{\frac{k+2m+n}{m}}(\mathfrak{so}_{4m+2n}, f_{\mathfrak{sp}_{2m}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{\psi-2n}(\mathfrak{so}_{2n})$. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{CD}(n,m) := \mathrm{Com}(V^{\psi-2n}(\mathfrak{so}_{2n}), \mathcal{W}^{\psi}_{CD}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{so}_{2n}}$ has central charge

$$c_{CD} = -\frac{k(2k+1)(2km-k+2m-n)(2km+k+4m+n)}{(k+2)(k+n)(k+2m+n)}. \tag{6.9}$$

The free field limit of $\mathcal{W}^{\psi}_{CD}(n,m)$ is

$$\mathcal{O}_{\mathrm{ev}}(2n^2-n,2) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}_{\mathrm{ev}}(1,4i) \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{\mathrm{ev}}(3,4i-2)\right) \otimes \mathcal{S}_{\mathrm{ev}}(2n,2m+1).$$

Lemma 13. For $n+m \geq 1$, $C^{\psi}_{CD}(n,m)$ is of type $\mathcal{W}(1^3,2,3^3,4,\dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.2 Case CB

For $\mathfrak{g} = \mathfrak{so}_{4m+2n+1}$, we have an isomorphism of $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{so}_{2n+1}$-modules

$$\mathfrak{so}_{4m+2n+1} \cong \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{so}_{2n+1} \oplus \rho_{\omega_2} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{2n+1},$$

and the critically shifted level $\psi = \frac{k+2m+n}{m} + 4m + 2n - 1$. We define

$$\mathcal{W}^{\psi}_{CB}(n,m) := \mathcal{W}^{\frac{2k+4m+2n+1}{2m}}(\mathfrak{so}_{4m+2n+1}, f_{\mathfrak{sp}_{2m}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{\psi-2}(\mathfrak{so}_{2n+1})$. Next, we define its affine coset

$$\mathcal{C}^\psi_{CB}(n,m) := \text{Com}(V^{\psi-2n+1}(\mathfrak{so}_{2n+1}), \mathcal{W}^\psi_{CB}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{so}_{2n+1}}$ has central charge

$$c_{CB} = -\frac{k(2k+1)(4km-2k+4m-2n-1)(4km+2k+8m+2n+1)}{(k+2)(2k+2n+1)(2k+4m+2n+1)}. \quad (6.10)$$

The free field limit of $\mathcal{W}^\psi_{CB}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2) \otimes \left(\bigotimes_{i=1}^m \mathcal{O}_{\text{ev}}(1,4i) \otimes \bigotimes_{i=1}^m \mathcal{O}_{\text{ev}}(3,4i-2) \right) \otimes \mathcal{S}_{\text{ev}}(2n+1,2m+1).$$

Lemma 14. For $n+m \geq 1$, $C^\psi_{CB}(n,m)$ is of type $\mathcal{W}(1^3,2,3^3,4,\dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.3 Case CC

For $\mathfrak{g} = \mathfrak{osp}_{4m|2n}$, we have an isomorphism of $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n}$-modules

$$\mathfrak{osp}_{4m|2n} \cong \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n} \oplus \rho_{\omega_2} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{0|2n},$$

and the critically shifted level $\psi = \frac{k+2m-n}{m} + 4m - 2n - 2$. We define

$$\mathcal{W}^\psi_{CC}(n,m) := \mathcal{W}^{\frac{k+2m-n}{m}}(\mathfrak{osp}_{4m|2n}, f_{\mathfrak{sp}_{2m}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{-\frac{\psi}{2}-n}(\mathfrak{sp}_{2n})$. We consider the following extreme cases. Next, we define its affine coset

$$\mathcal{C}^\psi_{CC}(n,m) := \text{Com}(V^{-\frac{\psi}{2}-n}(\mathfrak{so}_{2n+1}), \mathcal{W}^\psi_{CC}(n,m))^{\mathbb{Z}_2},$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{sp}_{2n}}$ has central charge

$$c_{CC} = \frac{k(2k+1)(2km+k+4m-n)(2km-k+2m+n)}{(k+2)(n-k)(k+2m-n)}. \qquad (6.11)$$

The free field limit of $\mathcal{W}^{\psi}_{CC}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}_{\text{ev}}(1,4i) \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{\text{ev}}(3,4i-2) \right) \otimes \mathcal{O}_{\text{odd}}(4n,2m+1).$$

Lemma 15. For $n+m \geq 1$, $C^{\psi}_{CD}(n,m)$ is of type $\mathcal{W}(1^3,2,3^3,4,\dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.4 Case CO

For $\mathfrak{g} = \mathfrak{osp}_{4m+1|2n}$, we have an isomorphism of $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus \mathfrak{osp}_{1|2n}$-modules

$$\mathfrak{osp}_{4m+1|2n} \cong \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_2 \oplus \mathfrak{osp}_{1|2n} \oplus \rho_{\omega_2} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{1|2n},$$

and the critically shifted level $\psi = \frac{2k+4m-2n+1}{2m} + 4m - 2n - 1$. We define

$$\mathcal{W}^{\psi}_{CO}(n,m) := \mathcal{W}^{\frac{2k+4m-2n+1}{2m}}(\mathfrak{osp}_{4m+1|2n}, f_{\mathfrak{sp}_{2m}})),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{-\frac{\psi}{2}-n-\frac{1}{2}}(\mathfrak{osp}_{1|2n})$. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{CO}(n,m) := \text{Com}(V^{-\frac{\psi}{2}-n-\frac{1}{2}}(\mathfrak{osp}_{1|2n}), \mathcal{W}^{\psi}_{CO}(n,m))^{\mathbb{Z}_2},$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{osp}_{1|2n}}$ has central charge

$$c_{CO} = \frac{k(2k+1)(4km+2k+8m-2n+1)(4km-2k+4m+2n-1)}{(k+2)(-2k+2n-1)(2k+4m-2n+1)}. \qquad (6.12)$$

The free field limit of $\mathcal{W}^{\psi}_{CO}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2)\otimes\mathcal{S}_{\text{odd}}(n,2)\otimes\left(\bigotimes_{i=1}^{m}\mathcal{O}_{\text{ev}}(1,4i)\otimes\bigotimes_{i=1}^{m}\mathcal{O}_{\text{ev}}(3,4i-2)\right)\\ \otimes\mathcal{O}_{\text{odd}}(4n,2m+1)\otimes\mathcal{S}_{\text{ev}}(1,2m+1).$$

Lemma 16. For $n+m\geq 1$, $C^{\psi}_{CO}(n,m)$ is of type $\mathcal{W}(1^3,2,3^3,4,\dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.5 Case BD

For $\mathfrak{g}=\mathfrak{osp}_{2n|4m+2}$, we have an isomorphism of $\mathfrak{so}_{2m+1}\oplus\mathfrak{sp}_2\oplus\mathfrak{so}_{2n}$-modules

$$\mathfrak{osp}_{2n|4m+2}\cong\mathfrak{so}_{2m+1}\oplus\mathfrak{sp}_2\oplus\mathfrak{so}_{2n}\oplus\rho_{\omega_2}\otimes\mathbb{C}^3\otimes\mathbb{C}\oplus\mathbb{C}^{2m+1}\otimes\mathbb{C}^2\otimes\mathbb{C}^{0|2n},$$

and the critically shifted level $\psi=\frac{k+2m-n+2}{2m+1}+2m-r+2$. We define

$$\mathcal{W}^{\psi}_{BD}(n,m):=\mathcal{W}^{\frac{k+2m-n+2}{2m+1}}(\mathfrak{sp}_{4m+2n+2},f_{\mathfrak{so}_{2n+1}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2)\otimes V^{-2\psi-2m+4}(\mathfrak{so}_{2n})$. We consider the following extreme cases. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{BD}(n,m):=\text{Com}(V^{-2\psi-2n+4}(\mathfrak{so}_{2n}),\mathcal{W}^{\psi}_{BD}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L-L^{\mathfrak{sp}_2}-L^{\mathfrak{so}_{2n}}$ has central charge

$$c_{BD}=-\frac{k(2k+1)(2km+2k+4m-n+3)(2km+2m+n)}{(k+2)(k-n+1)(k+2m-n+2)}. \tag{6.13}$$

The free field limit of $\mathcal{W}^{\psi}_{BD}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2-n,2)\otimes\left(\bigotimes_{i=1}^{m}\mathcal{O}_{\text{ev}}(1,4i)\otimes\bigotimes_{i=1}^{m+1}\mathcal{O}_{\text{ev}}(3,4i-2)\right)\otimes\mathcal{S}_{\text{odd}}(2n,2m+2).$$

Lemma 17. For $n+m\geq 1$, $C^{\psi}_{BD}(n,m)$ is of type $\mathcal{W}(1^3,2,3^3,4,\dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.6 Case BB

For $\mathfrak{g}=\mathfrak{osp}_{2n+1|4m+2}$, we have an isomorphism of $\mathfrak{so}_{2m+1}\oplus\mathfrak{sp}_2\oplus\mathfrak{so}_{2n+1}$-modules

$$\mathfrak{osp}_{2n+1|4m+2}\cong\mathfrak{so}_{2m+1}\oplus\mathfrak{sp}_2\oplus\mathfrak{so}_{2n+1}\oplus\rho_{\omega_2}\otimes\mathbb{C}^3\otimes\mathbb{C}\oplus\mathbb{C}^{2n+1}\otimes\mathbb{C}^2\otimes\mathbb{C}^{2n+1},$$

and the critically shifted level $\psi=\frac{2k+4m-2n+3}{2(2m+1)}+2m-r+\frac{3}{2}$. We define

$$\mathcal{W}^{\psi}_{BB}(n,m):=\mathcal{W}^{\frac{2k+4m-2n+3}{2(2m+1)}}(\mathfrak{osp}_{2n+1|2(2m+1)},f_{\mathfrak{so}_{2m+1}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2)\otimes V^{-2\psi-2n+3}(\mathfrak{so}_{2n+1})$. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{BB}(n,m):=\text{Com}(V^{-2\psi-2n+3}(\mathfrak{so}_{2n+1}),\mathcal{W}^{\psi}_{BB}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L-L^{\mathfrak{sp}_2}-L^{\mathfrak{so}_{2n+1}}$ has central charge

$$c_{BB}=\frac{k(2k+1)(4km+4k+8m-2n+5)(4km+4m+2n+1)}{(k+2)(-2k+2n-1)(2k+4m-2n+3)}\tag{6.14}$$

The free field limit of $\mathcal{W}^{\psi}_{BB}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2)\otimes\left(\bigotimes_{i=1}^{m}\mathcal{O}_{\text{ev}}(1,4i)\otimes\bigotimes_{i=1}^{m+1}\mathcal{O}_{\text{ev}}(3,4i-2)\right)\otimes\mathcal{S}_{\text{odd}}(2n+1,2m+2).$$

Lemma 18. For $n+m \geq 1$, $C^{\psi}_{BB}(n,m)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.7 Case BC

For $\mathfrak{g} = \mathfrak{sp}_{2(2m+1)+2n}$, we have an isomorphism of $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n}$-modules

$$\mathfrak{sp}_{4m+2n+2} \cong \mathfrak{so}_{2m+1} \oplus \mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n} \oplus \rho_{\omega_2} \otimes \mathbb{C}^3 \otimes \mathbb{C} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{2n},$$

and the critically shifted level $\psi = \frac{k+2m+n+2}{2m+1} + 2m + n + 2$. We define

$$\mathcal{W}^{\psi}_{BC}(n,m) := \mathcal{W}^{\frac{k+2m+n+2}{2m+1}}(\mathfrak{sp}_{2(2m+1)+2n}, f_{\mathfrak{so}_{2m+1}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{\psi-n-2}(\mathfrak{sp}_{2n})$. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{BC}(n,m) := \text{Com}(V^{\psi-n-2}(\mathfrak{osp}_{1|2n})\mathcal{W}^{\psi}_{BC}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{sp}_{2n}}$ has central charge

$$c_{BC} = -\frac{k(2k+1)(2km+2m-n)(2km+2k+4m+n+3)}{(k+2)(k+n+1)(k+2m+n+2)}. \tag{6.15}$$

The free field limit of $\mathcal{W}^{\psi}_{BC}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2) \otimes \mathcal{S}_{\text{odd}}(n,2) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}_{\text{ev}}(1,4i) \otimes \bigotimes_{i=1}^{m+1} \mathcal{O}_{\text{ev}}(3,4i-2) \right) \otimes \mathcal{O}_{\text{ev}}(4n, 2m+2)$$

Lemma 19. For $n+m \geq 1$, $C^{\psi}_{BC}(n,m)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

6.8 Case BO

For $\mathfrak{g} = \mathfrak{osp}_{1|4m+2n+2}$, we have an isomorphism of $\mathfrak{so}_{2m+1} \oplus \mathfrak{sp}_2 \oplus \mathfrak{osp}_{1|2n}$-modules

$$\mathfrak{osp}_{1|4m+2n+2} \cong \mathfrak{so}_{2m+1} \oplus \mathfrak{sp}_2 \oplus \mathfrak{osp}_{1|2n} \oplus \rho_{2\omega_1} \otimes \mathbb{C}^3 \oplus \mathbb{C}^{2m+1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{2n|1},$$

and the critically shifted level $\psi = \frac{2k+4m+2n+3}{2(2m+1)} + 2n + r + \frac{3}{2}$. We define

$$\mathcal{W}^{\psi}_{BO}(n,m) := \mathcal{W}^k(\mathfrak{osp}_{1|2(2n+1)+2r}, f_{\mathfrak{so}_{2n+1}}),$$

which has affine subalgebra $V^k(\mathfrak{sp}_2) \otimes V^{\psi+2n}(\mathfrak{osp}_{1|2n})$. Next, we define its affine coset

$$\mathcal{C}^{\psi}_{BO}(n,m) := \text{Com}(V^{\psi+2n}(\mathfrak{osp}_{1|2n}), \mathcal{W}^{\psi}_{BO}(n,m))^{\mathbb{Z}_2}.$$

The conformal element $L - L^{\mathfrak{sp}_2} - L^{\mathfrak{osp}_{1|2n}}$ has central charge

$$c_{BO} = -\frac{k(2k+1)(4km+4m-2n+1)(4km+4k+8m+2n+5)}{(k+2)(2k+2n+1)(2k+4m+2n+3)} \tag{6.16}$$

The free field limit of $\mathcal{W}^{\psi}_{BO}(n,m)$ is

$$\mathcal{O}_{\text{ev}}(2n^2+n,2) \otimes \mathcal{S}_{\text{odd}}(n,2) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}_{\text{ev}}(1,4i) \otimes \bigotimes_{i=1}^{m+1} \mathcal{O}_{\text{ev}}(3,4i-2)\right)$$
$$\otimes \mathcal{O}_{\text{ev}}(2n,2m+2) \otimes \mathcal{S}_{\text{odd}}(1,2m+2).$$

Lemma 20. For $n+m \geq 1$, $C^{\psi}_{BO}(n,m)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$ as a 1-parameter vertex algebra. Equivalently, this holds for generic values of ψ.

Remark 6.8.1. *Let X be either B or C. Then we have the following relations among the central charges.*

$$c_{XD}(n,m) = c_{XC}(-n,m), \quad c_{XB}(n,m) = c_{XO}(-n,m).$$

This suggests a heuristic that type $\mathfrak{so}_{2n+1}$ is the negative of $\mathfrak{osp}_{1|2n}$, and $\mathfrak{so}_{2n}$ is the negative of $\mathfrak{sp}_{2n}$. In fact, similar feature continues to hold for diagonal cosets introduced in the following section (7).

Chapter 7: Families with $\mathfrak{sp}_2$-level Constant

Unlike $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{\mathrm{ev}}$ where the Y-algebras are expected to account for all the simple, strongly finitely generated 1-parameter quotients, $\mathcal{W}_\infty^{\mathfrak{sp}}$ admits at least 4 more infinite families of such 1-parameter quotients. These algebras all contain the simple affine vertex algebra $L_k(\mathfrak{sl}_2)$ for a fixed k.

7.1 Cases B and D

Specializing the homomorphism (3.28) to the case $m = 1$, we have an embedding of $V^{-2}(\mathfrak{so}_n)$ in the rank n $\beta\gamma$-system $\mathcal{S}(n)$

$$\varphi : V^{-2}(\mathfrak{so}_n) \hookrightarrow \mathcal{S}(n), \quad E_{i,j} \mapsto :\beta^i\gamma^j: - :\beta^j\gamma^i:, \quad 1 \leq i < j \leq n. \tag{7.1}$$

The commutant of $V^{-2}(\mathfrak{so}_n)$ inside $\mathcal{S}(n)$ is $L_{-\frac{n}{2}}(\mathfrak{sp}_2)$ ([62], Thm. 5.1), and is generated by the following $\mathfrak{so}_n$-invariants.

$$\begin{aligned} X &= \sum_{i=1}^{n} :\beta^i\beta^i:, \\ Y &= \sum_{i=1}^{n} :\gamma^i\gamma^i:, \\ H &= \sum_{i=1}^{n} :\beta^i\gamma^i: . \end{aligned} \tag{7.2}$$

The weight $\frac{1}{2}$ space is spanned by $\{\beta^i | i = 1, \dots, n\} \cup \{\gamma^i | i = 1, \dots, n\}$, and transforms as $\mathbb{C}^2 \otimes \mathbb{C}^n$ under $\mathfrak{sp}_2 \oplus \mathfrak{so}_n$. We have a diagonal action

$$V^{\ell}(\mathfrak{so}_n) \hookrightarrow V^{\ell-2}(\mathfrak{so}_n) \otimes \mathcal{S}(n), \quad E_{i,j} \mapsto E_{i,j} \otimes \text{Id} + \text{Id} \otimes \varphi(E_{i,j}). \tag{7.3}$$

Define the affine coset

$$\mathcal{C}^{\ell}\left(-\frac{n}{2}\right) = \text{Com}(V^{\ell}(\mathfrak{so}_n), V^{\ell-2}(\mathfrak{so}_n) \otimes \mathcal{S}(n))^{\mathbb{Z}_2}. \tag{7.4}$$

The free field limit of $\mathcal{C}^{\ell}\left(-\frac{n}{2}\right)$ is the invariant algebra $\mathcal{S}(n)^{\text{O}_n}$. By Weyl's first fundamental theorem for standard representation of O_n (1), the $\mathfrak{so}_n$-invariants are given by

$$\begin{aligned} X^{p,q} &= \sum_{i=1}^{n} :\partial^p \beta^i \partial^q \beta^i: \quad p \geq q \geq 0, \\ Y^{p,q} &= \sum_{i=1}^{n} :\partial^p \gamma^i \partial^q \gamma^i: \quad p \geq q \geq 0, \\ H^{p,q} &= \sum_{i=1}^{n} :\partial^p \beta^i \partial^q \gamma^i:, \quad p, q \geq 0. \end{aligned} \tag{7.5}$$

Removing the redundancy due to differential relations in the above generators, we find that orbifold $\mathcal{S}(n)^{\text{O}_{2n}}$ has a strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, which is not minimal. This continues to hold generically for the cosets (7.4), so we obtain the following.

Lemma 21. Let $n \in \mathbb{Z}_{\geq 0}$. Then the coset $\mathcal{C}^{\ell}\left(-\frac{n}{2}\right)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$.

7.2 Case C

Specializing the homomorphism (3.32) to the case $m = 1$, we have an embedding of $L_1(\mathfrak{sp}_{2n})$ inside the rank $2n$ bc-system $\mathcal{E}(2n)$

$$\varphi : L_1(\mathfrak{sp}_{2n}) \hookrightarrow \mathcal{E}(2n), \tag{7.6}$$

given by the mapping

$$G_{i,j} \mapsto :b^i c^{-j}: + :b^j c^{-i}:, \quad 1 \le i \le j \le n,$$

$$G_{-i,-j} \mapsto :b^{-i} c^j: + :b^{-j} c^i: \quad 1 \le i \le j \le n,$$

$$G_{-i,j} \mapsto :b^i c^j: - :b^{-j} c^{-i}:, \quad 1 \le i, j \le n.$$

The commutant of $L_1(\mathfrak{sp}_{2n})$ inside $\mathcal{E}(2n)$ is $L_n(\mathfrak{sp}_2)$, and it is generated by the following $\mathfrak{sp}_{2n}$-invariants.

$$\begin{aligned} X &= \sum_{i=1}^{n} :b^i b^{-i}:, \\ Y &= \sum_{i=1}^{n} :c^i c^{-i}:, \\ H &= \sum_{i=1}^{n} :b^i c^i: + :b^{-i} c^{-i}: . \end{aligned} \tag{7.7}$$

The weight $\frac{1}{2}$ space is spanned by $\{b^i, b^{-i} | i = 1, \dots, n\} \cup \{c^i, c^{-i} | i = 1, \dots, n\}$, and transforms under $\mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n}$ as $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$. Therefore, we have a diagonal action

$$V^{\ell}(\mathfrak{sp}_{2n}) \hookrightarrow V^{\ell-1}(\mathfrak{sp}_{2n}) \otimes \mathcal{E}(4n), \quad G_{i,j} \mapsto G_{i,j} \otimes \text{Id} + \text{Id} \otimes \varphi(G_{i,j}). \tag{7.8}$$

Define the affine coset

$$\mathcal{C}^{\ell}(n) = \text{Com}(V^{\ell}(\mathfrak{sp}_{2n}), V^{\ell-1}(\mathfrak{sp}_{2n}) \otimes \mathcal{E}(2n)). \tag{7.9}$$

The free field limit of $\mathcal{C}^{\ell}(n)$ is the invariant algebra $\mathcal{E}(2n)^{\text{Sp}_{2n}}$. By Weyl's first fundamental theorem for standard representation of Sp_{2n} with odd variables (4), the $\mathfrak{sp}_{2n}$-invariants are generated by

$$\begin{aligned}
X^{p,q} =& \frac{1}{2}\sum_{i=1}^{n} :\partial^p b^i \partial^q b^{-i}: + :\partial^q b^i \partial^p b^{-i}:, \quad p \geq q \geq 0,\\
Y^{p,q} =& \frac{1}{2}\sum_{i=1}^{n} :\partial^p c^i \partial^q c^{-i}: + :\partial^q c^i \partial^p c^{-i}:, \quad p \geq q \geq 0,\\
H^{p,q} =& \frac{1}{2}\sum_{i=1}^{n} :\partial^p b^i \partial^q c^{-i}: + :\partial^q b^i \partial^p c^{-i}:, \quad p, q \geq 0.
\end{aligned} \tag{7.10}$$

Removing the redundancy due to differential relations in the above generators, we find that orbifold $\mathcal{E}(2n)^{\mathrm{Sp}_{2n}}$ has a strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, which is not minimal. This continues to hold generically for the cosets (7.9).

Lemma 22. Let $n \in \mathbb{Z}_{\geq 0}$. Then the coset $\mathcal{C}^{\ell}(n)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$.

7.3 Case O

Specializing the homomorphism (3.37) to the case $m = 1$, we have an embedding of $L_{\frac{1}{2}}(\mathfrak{osp}_{1|2n})$ inside the tensor product of rank n bc-system with rank 1 $\beta\gamma$-system $\mathcal{E}(2n) \otimes \mathcal{S}(1)$

$$\varphi : V^k(\mathfrak{osp}_{1|2n}) \hookrightarrow \mathcal{S}(1) \otimes \mathcal{E}(2n) \tag{7.11}$$

extending the mapping (7.6) by

$$\begin{aligned}
X_{1,-j} \mapsto& :\gamma c^j: - :\beta b^{-j}:, \quad 1 \leq j \leq n,\\
X_{1,j} \mapsto& :\gamma c^{-j}: + :\beta b^j:, \quad 1 \leq j \leq n.
\end{aligned}$$

The commutant of $L_1(\mathfrak{osp}_{1|2n})$ in $\mathcal{E}(2n)\otimes\mathcal{S}(1)$ contains $L_{n-\frac{1}{2}}(\mathfrak{sp}_2)$, which it is generated by the following $\mathfrak{osp}_{1|2n}$-invariants.

$$\begin{aligned} X &= -:\beta\beta: - \sum_{i=1}^{n} :b^i b^{-i}:, \\ Y &=:\gamma\gamma: + \sum_{i=1}^{n} :c^i c^{-i}:, \\ H &=:\beta\gamma: + \sum_{i=1}^{n} :b^i c^i: + :b^{-i}c^{-i}: . \end{aligned} \tag{7.12}$$

The weight $\frac{1}{2}$ space is spanned by odd variables $\{b^i, b^{-i}, c^i, c^{-i} | i = 1, \dots, n\}$, and even variables $\{\beta, \gamma\}$, that transform under $\mathfrak{sp}_2 \oplus \mathfrak{osp}_{1|2n}$ as $\mathbb{C}^2 \otimes \mathbb{C}^{1|2n}$. Therefore, we have a diagonal action

$$V^{\ell}(\mathfrak{osp}_{1|2n}) \hookrightarrow V^{\ell-1}(\mathfrak{osp}_{2n}) \otimes \mathcal{E}(2n) \otimes \mathcal{S}(1), \quad G_{i,j} \mapsto G_{i,j} \otimes \mathrm{Id} + \mathrm{Id} \otimes \varphi(G_{i,j}). \tag{7.13}$$

Define the following $\mathbb{Z}_2$-orbifold of the affine coset

$$\mathcal{C}^{\ell}\left(n - \frac{1}{2}\right) = \mathrm{Com}(V^{\ell}(\mathfrak{osp}_{1|2n}), V^{\ell-1}(\mathfrak{osp}_{1|2n}) \otimes \mathcal{S}(1) \otimes \mathcal{E}(2n))^{\mathbb{Z}_2}. \tag{7.14}$$

The free field limit of $\mathcal{C}^{\ell}(n)$ is the invariant algebra $(\mathcal{S}(1) \otimes \mathcal{E}(2n))^{\mathrm{Osp}_{1|2n}}$. By Sergeev's first fundamental theorem for standard representation $\mathbb{C}^{1|2n}$ of $\mathrm{Osp}_{1|2n}$ (6), the $\mathfrak{osp}_{1|2n}$-

invariants are generated by

$$\begin{aligned}
X^{p,q} &= :\partial^n\beta\partial^m\beta: + \sum_{i=1}^{n} :\partial^n b^i\partial^m b^{-i}: + :\partial^m b^i\partial^m b^{-i}:, \quad p \geq q \geq 0, \\
Y^{p,q} &= :\partial^n\gamma\partial^m\gamma: + \sum_{i=1}^{n} :\partial^n c^i\partial^m c^{-i}: + :\partial^m c^i\partial^n c^{-i}:, \quad p \geq q \geq 0, \\
H^{p,q} &= :\partial^n\beta\partial^m\gamma: + \sum_{i=1}^{n} :\partial^n b^i\partial^m c^{-i}: + :\partial^m b^i\partial^n c^{-i}:, \quad p, q \geq 0.
\end{aligned} \tag{7.15}$$

Removing the redundancy due to differential relations in the above generators, we find that orbifold $(\mathcal{S}(1) \otimes \mathcal{E}(2n))^{\mathrm{Osp}_{1|2n}}$ has a strong generatying type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, which is not minimal. This continues to hold generically for the cosets (7.14).

Lemma 23. Let $n \in \mathbb{Z}_{\geq 0}$ Then the coset $\mathcal{C}^\ell\left(n - \frac{1}{2}\right)$ is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$.

Chapter 8: Universal 2-parameter Vertex Algebra $\mathcal{W}_\infty^{\mathfrak{sp}}$

In this section we will construct the universal 2-parameter algebra $\mathcal{W}_\infty^{\mathfrak{sp}}$ of type

$$\mathcal{W}(1^3, 2, 3^3, 4, 5^3, 6, \dots). \tag{8.1}$$

The three fields in weight 1 generate a copy of the affine vertex algebra $V^k(\mathfrak{sp}_2)$, the fields in even weights $\{L, W^{2i} | i \geq 2\}$ transform as the trivial $\mathfrak{sp}_2$-module, and the three fields in odd weights $\{X^{2i-1}, Y^{2i-1}, H^{2i-1} | i \geq 2\}$ transform as the adjoint $\mathfrak{sp}_2$-module. It is freely generated and defined over a localization of the polynomial ring $\mathbb{C}[c,k]$.

As in the case of universal 2-parameter algebras $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{\mathrm{ev}}$ constructed in [60] and [55], $\mathcal{W}_\infty^{\mathfrak{sp}}$ will be the universal enveloping vertex algebra of a nonlinear Lie conformal algebra $\mathcal{L}^{\mathfrak{sp}_2}$, defined over a localization of $\mathbb{C}[c,k]$ with generators $\{L, W^{2i} | i \geq 2\} \cup \{X^{2i-1}, Y^{2i-1}, H^{2-1} | i \geq 1\}$ and grading $\Delta(X^{2i-1}) = \Delta(Y^{2i-1}) = \Delta(H^{2i-1}) = 2i-1$, $\Delta(L) = 2$ and $\Delta(W^{2i}) = 2i$, in the sense of [26].

We shall work with the OPE rather than lambda-bracket formalism, so the sesquilinearity, skew-symmetry, and Jacobi identities from [26] are replaced with (3.11-3.15). As explained in ([60], Sec. 3), specifying a nonlinear Lie conformal algebra in the language of OPEs means specifying generators $\{a^1, a^2, \dots\}$ where field a^i has conformal weight $d_i > 0$, and pairwise product expansions

$$a^i(z)a^j(w) \sim \sum_{n=0}^{\infty} a^i(w)_{(n)}(w)a^j(w)(z-w)^{-n-1},$$

where each term $a^i(w)_{(n)}a^j(w)$ has conformal weight $d_i + d_j - n - 1$, and is a normally ordered polynomial in the generators and their derivatives. Additionally, for all $a, b, c \in \{a^1, a^2, \dots\}$, (3.11-3.14) hold, and all Jacobi identities (3.15) hold as a consequence of (3.11-3.14) alone. This data uniquely determines the universal enveloping vertex algebra which is freely generated by $\{a^1, a^2, \dots\}$.

In this notation, our goal will be to construct the OPE algebra with strong generators $\{L, W^{2i} | i \geq 2\} \cup \{X^{2i-1}, Y^{2i-1}, H^{2-1} | i \geq 1\}$ of $\mathcal{W}_\infty^{\mathfrak{sp}}$, such that identities (3.11-3.14) are imposed, and the Jacobi identities (3.15) hold as a consequence of (3.11-3.14) alone. We postulate that $\mathcal{W}_\infty^{\mathfrak{sp}}$ has the following features.

1. Weight 1 fields X^1, Y^1, H^1 generate the universal affine algebra $V^k(\mathfrak{sp}_2)$ of level k.

2. Weight 2 field L generates the universal Virasoro algebra Vir^c of central charge c.

3. Even weight strong generators transform as the trivial $\mathfrak{sp}_2$-module.

4. Odd weight strong generators transform as the adjoint $\mathfrak{sp}_2$-module.

5. Weight 3 strong generators X^3, Y^3, H^3 are primary for Vir^c.

6. Weight 4 strong generator W^4 is primary for Vir^c.

7. Subalgebras $V^k(\mathfrak{sp}_2)$ and Vir^c, together with W^4 weakly generate $\mathcal{W}_\infty^{\mathfrak{sp}}$. Specifically, W^4 satisfies the raising property:

$$\begin{gathered} W^4_{(1)}X^{2i-1} = X^{2i+1}, \quad W^4_{(1)}Y^{2i-1} = Y^{2i+1}, \quad W^4_{(1)}H^{2i-1} = H^{2i+1}, \\ W^4_{(1)}W^{2i} = W^{2i+2}. \end{gathered} \tag{8.2}$$

Since the strong generating type alternates by the parity of conformal weight, there are three types of interactions with some variation in their structure, e.g. see (8.40-8.42).

1. Even with even. For $n \geq 0$ and $0 \leq r \leq 2n-1$ we set

$$EE_r^{2n} = \{L_{(r)}W^{2n-2}, W^4_{(r)}W^{2n-4}, W^6_{(r)}W^{2n-6}, \dots\}.$$

2. Even with odd. Let us denote by $EO_r^{2i,2j-1}$ the set of following products

$(\cdot_{(r)}\cdot)$	X^{2j-1}	Y^{2j-1}	H^{2j-1}
W^{2i}	$W^{2i}_{(r)}X^{2j-1}$	$W^{2j}_{(r)}Y^{2j-1}$	$W^{2j}_{(r)}H^{2j-1}$

For $n \geq 0$ and $0 \leq r \leq 2n$ we set

$$EO_r^{2n+1} = EO_r^{2,2n-1} \cup EO_r^{4,2n-3} \cup EO_r^{6,2n-5} \cup \cdots .$$

3. Odd with odd. Let us denote by $OO_r^{2i-1,2j-1}$ the set of following products

$(\cdot_{(r)}\cdot)$	X^{2j-1}	Y^{2j-1}	H^{2j-1}
X^{2i-1}	$X^{2i-1}_{(r)}X^{2j-1}$	$X^{2i-1}_{(r)}Y^{2j-1}$	$X^{2i-1}_{(r)}H^{2j-1}$
Y^{2i-1}	$Y^{2i-1}_{(r)}X^{2j-1}$	$Y^{2i-1}_{(r)}Y^{2j-1}$	$Y^{2i-1}_{(r)}H^{2j-1}$
H^{2i-1}	$H^{2i-1}_{(r)}X^{2j-1}$	$H^{2i-1}_{(r)}Y^{2j-1}$	$H^{2i-1}_{(r)}H^{2j-1}$

For $n \geq 0$ and $0 \leq r \leq 2n-1$ we set

$$OO_r^{2n} = OO^{1,2n-1} \cup OO^{3,2n-3} \cup \cdots .$$

Let D_r^n denote the set of all r^{th} products among generators of total weight not exceeding n, specifically

$$D_r^{2m} = OO_r^{2m} \cup EE_r^{2m}, \quad D_r^{2m+1} = EO_r^{2m+1}.$$

Write $D^n = \bigcup_{r\leq n} D^n_r$ and $D_n = \bigcup_{m\leq n}^n D^m$ for the OPE data among fields of total weight n and not exceeding n, respectively. Lastly, let J^m denote the set of all Jacobi identities among fields with total weight exactly m, and $J_n = \bigcup_{m\leq n}^n J^m$.

Our strategy is similar to the one used in [60] and [55], and consists of four steps.

1. We begin by writing down the most general OPEs in D_9 compatible with conformal weight and $V^k(\mathfrak{sp}_2)$ symmetry. Next, we impose vertex algebra relations (3.11-3.14) along with Jacobi identities J_{11} to uniquely determine D_9 in terms of two parameters c and k, see Proposition (25).

2. Next, we use data D_9 obtained in the previous step, and raising property of W^4 field to explicitly evaluate an infinite set of structure constants, see Proposition (26).

3. Next, we proceed inductively; we assume $D^{2n} \cup D^{2n+1}$ to be known, and show that a subset of Jacobi identities in $J^{2n+4} \cup J^{2n+5}$ uniquely determines all OPEs in $D^{2n+2} \cup D^{2n+3}$, see Proposition (8.4). Note that we are not checking all the Jacobi identities $J^{2n+4} \cup J^{2n+5}$ at this stage, we leave open the possibility that some of them may not vanish but instead give rise to nontrivial null fields. At the end of induction, we obtain the existence of a possibly degenerate nonlinear Lie conformal algebra $\mathcal{L}^{\mathfrak{sp}_2}$ over a localization of $\mathbb{C}[c, k]$. We then invoke De Sole-Kac correspondence to conclude that universal enveloping vertex algebra $\mathcal{W}^{\mathfrak{sp}}_\infty$ indeed exists, see Theorem (35).

4. Lastly, we exhibit a family vertex algebras with known characters to prove free generation of $\mathcal{W}^{\mathfrak{sp}}_\infty$. Specifically, these are the generalized parafermions in type C, see Corollary (39).

8.1 Set-up

Before we begin with the base case computation, we first investigate some useful consequences of postulated features (1-3) in the list (8).

With foresight of the main result of this section, Theorem 35, we often mention the VOA $\mathcal{W}_\infty^{\mathfrak{sp}}$ during the process of its construction. At this stage it is not yet known if it is a freely generated 2-parameter VOA. However, by De Sole-Kac correspondence there always exists the universal enveloping VOA $\mathcal{W}_\infty^{\mathfrak{sp}}$ of the nonlinear Lie conformal algebra $\mathcal{L}^{\mathfrak{sp}_2}$, possibly degenerate or trivial. All our statements regarding this VOA continue to hold true before the construction is complete, even if it were to be degenerate or trivial.

8.1.1 PBW Monomials and Filtration. We begin by making the following choice of a lexicographic order on PBW monomials. First, order elements of lower conformal weight to be less than those of higher one, and when they are of the same conformal weight (necessarily odd), we order $X^{2i-1} < Y^{2i-1} < H^{2i-1}$ for each $i \geq 1$. Then we use the convention introduced in (3.9). Since $\mathcal{W}_\infty^{\mathfrak{sp}}$ is of type $\mathcal{W}(1^3, 2, \dots)$, its conformal weight N homogeneous component is spanned by PBW monomials of the form

$$\begin{aligned} &:\partial^{a_1^1} X^1 \cdots \partial^{a_{e_1}^1} X^1 \, \partial^{b_1^1} Y^1 \cdots \partial^{b_{f_1}^1} Y^1 \, \partial^{c_1^1} H^1 \cdots \partial^{c_{h_1}^1} H^1 \, \partial^{k_1^2} L \cdots \partial^{k_{r_2}^2} L \cdots :, \\ &a_1^1 \geq \cdots \geq a_{e_1}^1, \quad b_1^1 \geq \cdots \leq b_{f_1}^1, \quad c_1^1 \geq \cdots \geq c_{h_1}^1, \quad k_1^2 \geq \cdots \geq k_{r_1}^2, \dots, \\ &a_1 + \cdots + a_{e_1} + b_1 + \cdots + b_{f_1} + c_1 + \cdots + c_{h_1} + 2(k_1 + \cdots + k_{r_2}) + \cdots = N. \end{aligned} \tag{8.3}$$

Let ω be a sequence of partitions $(\omega_1, \omega_2, \dots)$. We say that a PBW monomial in (8.3) of type ω if

$$\omega_1 = (\{a_1^1, a_2^1, \dots, a_{e_1}^1\} \cup \{b_1^1, b_2^1, \dots, b_{f_1}^1\} \cup \{c_1^1, c_2^1, \dots, c_{h_1}^1\}), \quad \omega_2 = (k_1^2, k_2^2, \dots, k_{r_2}^2), \dots$$

where ω_1 is sorted in decreasing order, etc. Let $U^{(\omega)}$ be the subspace spanned by all PBW monomials of type ω. For example, $U^{(1^1,3)}$ is spanned by monomials $\{:\partial X^1 X^3:, :\partial H^1 X^3:, :\partial X^1 H^3:, :\partial X^1 Y^3:, :\partial Y^1 X^3:, :\partial Y^1 H^3:, :\partial H^1 Y^3:, :\partial Y^1 Y^3:\}$.

Remark 8.1.1. *Types of monomials are in bijection with basis of $\mathcal{W}_{1+\infty}$, which itself is bijective with plane partitions thanks to formula of MacMahon.*

Now we recall Li's canonical decreasing filtration [59]. Let F^p be spanned by elements of the form

$$:\partial^{n_1}\alpha^1\partial^{n_2}\alpha^2\dots\partial^{n_r}\alpha^r:,$$

where $\alpha^1,\dots,\alpha^r \in \mathcal{W}_\infty^{\mathfrak{sp}}$, $n_i \geq 0$, and $n_1+\cdots+n_r \geq p$. Then we have

$$\mathcal{W}_\infty^{\mathfrak{sp}} = F^0 \supseteq F^1 \supseteq \cdots. \tag{8.4}$$

Set

$$\mathrm{gr}^F(\mathcal{W}_\infty^{\mathfrak{sp}}) = \bigoplus_{p=0}^{\infty} F^p/F^{p+1},$$

and for $p \geq 0$ let

$$\pi_p : F^p \to F^p/F^{p+1} \subseteq \mathrm{gr}^F(\mathcal{W}_\infty^{\mathfrak{sp}}) \tag{8.5}$$

be a projection. Then $\mathrm{gr}^F(\mathcal{W}_\infty^{\mathfrak{sp}})$ is a graded commutative algebra with product

$$\pi_p(\alpha)\pi_q(\beta) = \pi_{p+q}(\alpha_{(-1)}\beta), \quad \alpha \in F^p, \beta \in F^q.$$

8.1.2 Symmetry. By our assumptions, weight 1 fields X^1,H^1 and Y^1 generate the affine VOA $V^k(\mathfrak{sp}_2)$. In particular, the zero-modes $\{X_0^1, H_0^1, Y_0^1\}$ generate a Lie algebra $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$, giving rise to an action of $\mathfrak{sp}_2$ on the VOA $\mathcal{W}_\infty^{\mathfrak{sp}}$. We proceed to investigate the consequences of this symmetry.

We begin by recalling the $\mathfrak{sp}_2$-commutation relation

$$X^1_{(0)}Y^1 = H^1, \quad X^1_{(0)}H^1 = -2X^1, \quad Y^1_{(0)}H^1 = 2Y^1.$$

In particular, we have the inner automorphism σ of $\mathfrak{sp}_2$ that maps

$$X^1 \mapsto Y^1, \quad Y^1 \to X^1, \quad H^1 \to -H^1, \quad L \mapsto L.$$

Thanks to the raising property of W^4, we extend σ to all strong generators

$$X^{2n-1} \mapsto Y^{2n-1}, \quad Y^{2n-1} \to X^{2n-1}, \quad H^{2n-1} \to -H^{2n-1}, \quad W^{2n} \mapsto W^{2n}, \quad n \geq 2.$$

Now we will use representation theory of $\mathfrak{sp}_2$ to organize the basis (8.3). First, decompose the weight N subspace $\mathcal{W}_\infty^{\mathfrak{sp}}[N]$ into irreducible $\mathfrak{sp}_2$-modules

$$\mathcal{W}_\infty^{\mathfrak{sp}}[N] = \bigoplus_{\mu=0}^{N} \mathbb{C}^{M(N,2\mu)} \otimes \rho_{2\mu}. \tag{8.6}$$

Here $M(N, 2\mu)$ is the multiplicity of a highest weight $\mathfrak{sp}_2$-module ρ_μ of highest weight μ.

We follow the standard choice of scaling for the basis elements of homogeneous weight spaces [64]. Specifically, we have the weight space decomposition

$$\rho_\mu = \rho_{\mu,-\mu} \oplus \rho_{\mu,-\mu+2} \oplus \cdots \oplus \rho_{\mu,\mu-2} \oplus \rho_\mu.$$

Here, the weight space $\rho_{\mu,\alpha}$ is spanned by the vector $v_{\mu,\mu-2\alpha} = \frac{1}{\alpha!}(Y^1_{(0)})^\alpha v_{\mu,\mu}$, with $v_{\mu,\mu}$ being the highest weight vector. With this choice of basis for weight spaces the $\mathfrak{sp}_2$-action is as follows.

$$X^1_{(0)} v_{\mu,\alpha} = (\mu - \alpha + 2) v_{\mu,\alpha-2}, \quad Y^1_{(0)} v_{\mu,\alpha} = (\alpha + 2) v_{\mu,\alpha+2}, \quad H^1_{(0)} v_{\mu,\alpha} = (\mu - 2\alpha) v_{\mu,\alpha}. \tag{8.7}$$

First, we set-up notation to help us organize the strong generators into $\mathfrak{sp}_2$-modules. Let us denote by U^n_μ the $\mathfrak{sp}_2$-module isomorphic to ρ_μ, spanned by generators of conformal weight n, and by $U^n_{\mu,\alpha}$ the subspace of Cartan weight α. By our assumptions, these are either trivial or adjoint $\mathfrak{sp}_2$-modules, specifically we have

$$U^{2i}_0 = \text{Span}\{W^{2i}\} \cong \rho_0, \quad U^{2i-1}_2 = \text{Span}\{W^{2i-1}_{2,-2}, W^{2i-1}_{2,-2}, W^{2i-1}_{2,0}\} \cong \rho_2, \tag{8.8}$$

where the weight spaces are spanned by the following basis vectors

$$W^2_{0,0} = L, \quad W^{2i}_{0,0} = W^{2i}, \quad W^{2i-1}_{2,2} = X^{2i-1}, \quad W^{2i-1}_{2,-2} = -Y^{2i-1}, \quad W^{2i-1}_{2,0} = -H^{2i-1}, \tag{8.9}$$

for $i \geq 1$. Note that in the case of strong generators, the conformal weight uniquely specifies the $\mathfrak{sp}_2$-module $\rho_{\bar{n}}$, where $\bar{n} = 2(n \bmod \mathbb{Z}/2\mathbb{Z})$. Similarly, we will write

$$\begin{aligned} W^{(2i-1,2j-1)}_{2,2} =: X^{2i-1}H^{2j-1} : - : H^{2i-1}X^{2j-1} :, \quad W^{(2i-1)^d}_{2,2} &= \partial^d X^{2i-1}, \\ W^{(2i)^d}_{0,0} &= \partial^d W^{2i}, \end{aligned} \tag{8.10}$$

for $j \geq i \geq 1$ and $d \geq 1$. The basis for the whole modules $U^{(2i-1,2j-1)}_2$ and U^{2i-1}_2 is generated in accordance to $\mathfrak{sp}_2$-action (8.7).

We proceed to organize the rest of the basis (8.3) into $\mathfrak{sp}_2$-modules. Due to nonassociativity and noncommutativity in $\mathcal{W}^{\mathfrak{sp}}_\infty$, the subspace U^ω is in general not an $\mathfrak{sp}_2$-module. However, its projection by the map (8.5) on the associated graded $\mathrm{gr}^F(\mathcal{W}^{\mathfrak{sp}}_\infty)$, again denoted U^ω, is an $\mathfrak{sp}_2$-module. Then we have an isomorphism of $\mathfrak{sp}_2$-modules

$$U^\omega \cong \mathcal{S}^{\omega_1}(V_1) \otimes \mathcal{S}^{\omega_2}(V_0) \otimes \cdots \otimes \mathcal{S}^{\omega_s}(V_{\bar{s}}) \cong \bigoplus_\mu U^\omega_\mu, \tag{8.11}$$

where functor $\mathcal{S}^\lambda$ defined as

$$\mathcal{S}^\lambda(V) = \mathcal{S}^{m_1}(V) \otimes \mathcal{S}^{m_2}(V) \otimes \cdots, \quad \lambda = (1^{m_1}, 2^{m_2}, \dots).$$

Let Ω^ω_μ be the index set for some basis of highest weight vectors in U^ω_μ of highest weight μ. By $\mathfrak{sp}_2$-action (8.7) the choice of a basis for $U^\omega_{\mu,\mu}$ determines the bases for each weight space $U^\omega_{\mu,\alpha}$, and thus the whole $\mathfrak{sp}_2$-module U^ω_μ. Thus chosen basis vectors for weight spaces $U^\omega_{\mu,\alpha}$ are denoted by $W^{\omega\chi}_{\mu,\alpha}$, where χ indexes the set Ω^ω_μ. If U^ω_μ has dimension 1, then we simply denote the basis for weight spaces by $W^{\omega}_{\mu,\alpha}$. For example, this happens if $\omega = n$, $\omega = (n)^1$ or $\omega = (2i-1, 2j-1)$, where we have made the choices of highest weight vectors in (8.9) and (8.10).

Remark 8.1.2. *In general, the decomposing (8.11) is not multiplicity free. It may be interesting to compute dimension dim*(Ω^{ω}_{μ}).

For $v \in U^{\omega}_{\mu}$ we conjecture that it is possible to correct for the effects of nonassociativity and noncommutativity in $\mathcal{W}^{\mathfrak{sp}}_{\infty}$, in a uniform manner. Specifically, we have a map

$$\iota : \mathrm{gr}^F(\mathcal{W}^{\mathfrak{sp}}_{\infty}) \to \mathcal{W}^{\mathfrak{sp}}_{\infty}, \tag{8.12}$$

given in the following.

Conjecture 8.1.1. *Let* $v \in U^{\omega}_{\mu}$ *be a highest weight vector of highest weight* μ *in the associated graded gr*$(\mathcal{W}^{\mathfrak{sp}}_{\infty})$*, of type* $\omega = (\omega_1, \dots, \omega_s)$*. Then the following expression is the highest weight vector of highest weight* μ *in the VOA* $\mathcal{W}^{\mathfrak{sp}}_{\infty}$*.*

$$\iota(v) = v + \sum_{n=1}^{\infty} e_n(\alpha_{1,\mu}, \alpha_{2,\mu}, \dots, \alpha_{l(\omega)-1,\mu})(X^1_{(0)}Y^1_{(0)})^n v, \quad \alpha_{n,\mu} = -\frac{1}{(n+2)(n+\mu+1)}, \tag{8.13}$$

Here $e(x_1, \dots, x_n)$ *is an elementary symmetric polynomial of degree* n*, and* $l(\omega)$ *is the sum of the lengths of partitions in* ω*.*

As an example, consider the Sugawara vector $L^{\mathfrak{sp}_2}$. In our notation, it is an element of $U^{(1,1)}_0$. The following is the highest weight vector in the associated graded $\mathrm{gr}^F(\mathcal{W}^{\mathfrak{sp}}_{\infty})$.

$$v = \frac{1}{2} :H^1H^1: + 2 :X^1Y^1: .$$

Since length of $(1,1)$ is 2, we only need to the $n=1$ correction in (8.13), and $\alpha_{1,0} = -\frac{1}{6}$. Evaluating $\iota(v)$ we find the well-known $\mathfrak{sp}_2$-invariant vector in VOA $\mathcal{W}_\infty^{\mathfrak{sp}}$

$$\iota(v) = \frac{1}{2} :H^1H^1: +2 :X^1Y^1: -\partial H^1.$$

Remark 8.1.3. *The deformation ι in (8.12) and (8.13) is not unique. For instance, any linear combination with other highest weight vector in the common subspace U_μ^ω gives a different deformation.*

We have verified conjecture (8.1.1) for $\mathcal{W}_\infty^{\mathfrak{sp}}[N]$ with $N \leq 9$, which is what we needed for the base computation (25). Note that in the equation (8.13) for $\iota(v)$, the correction terms to v has strictly greater degree in filtration (8.4) than that of v. Therefore, we can regard v as the leading term in (8.13). By abuse of notation we denote the image under the deformation map (8.12) $\iota(U_\mu^\omega)$ in $\mathcal{W}_\infty^{\mathfrak{sp}}$ as U_μ^ω, that transforms as ρ_μ $\mathfrak{sp}_2$-module. Finally, let $U_\mu(n)$ denote the subspace of conformal weight n transforming as the ρ_μ $\mathfrak{sp}_2$-module. For example, the conformal weight 2 space decomposes as the $\mathfrak{sp}_2$-modules

$$\mathcal{W}_\infty^{\mathfrak{sp}}[2] = U_0(2) \oplus U_2(2) \oplus U_4(2) \cong 2\rho_0 \oplus \rho_2 \oplus \rho_4,$$

where $U_0(2)$ is spanned by L and Sugawara vector $L^{\mathfrak{sp}_2}$, $U_2(2)$ is spanned by highest weight vector ∂X^1, and $U_4(2)$ is spanned by the highest weight vector $:X^1X^1:$.

Next, we explain how $\mathfrak{sp}_2$-symmetry imposes severe restrictions on the OPEs among strong generators. First, recall the following decompositions of $\mathfrak{sp}_2$-modules

$$\begin{aligned} \rho_2 \otimes \rho_2 \cong& \rho_0 \oplus \rho_2 \oplus \rho_4, \\ \rho_0 \otimes \rho_2 \cong& \rho_2, \\ \rho_0 \otimes \rho_0 \cong& \rho_0. \end{aligned} \tag{8.14}$$

From the above isomorphisms it follows that:

- only U_0^ω, U_2^ω, and U_4^ω can arise in the OPEs among the generators of odd conformal weights.
- only U_2^ω can arise in the OPEs among the generators of even and odd conformal weights.
- only U_0^ω can arise in the OPEs among the generators of even and even conformal weights.

Remark 8.1.4. *This constraint is analogous to $\mathbb{Z}_2$-symmetry of $\mathcal{W}_\infty$, which featured prominently in its construction [60].*

Further restrictions follow. Let $v_{\mu,\alpha} \otimes v_{\nu,\beta}$ denote the basis of $\mathfrak{sp}_2$-modules appearing on the left side of isomorphisms in (8.14), and denote the basis for $\mathfrak{sp}_2$-modules appearing on the right side by $u^{2,2}_{\mu,\alpha}$, $u^{0,2}_{2,\alpha}$, and $u^{0,0}_{0,0}$, respectively. In terms of this basis we have the following relations.

$$\begin{aligned} v_{2,\alpha} \otimes v_{2,\beta} =& \epsilon_0^{2,2}(\alpha,\beta) u_{0,\alpha+\beta} + \epsilon_2^{2,2}(\alpha,\beta) u_{2,\alpha+\beta} + \epsilon_4^{2,2}(\alpha,\beta) u_{4,\alpha+\beta}, \\ v_{0,0} \otimes v_{2,\alpha} =& \epsilon_2^{0,2}(0,\alpha) u_{2,\alpha}, \\ v_{0,0} \otimes v_{0,0} =& \epsilon_0^{0,0}(0,0) u_{0,0}. \end{aligned} \tag{8.15}$$

For each module arising on the right side of (8.14), we have a choice of the scaling for the highest weight vector. The constants $\epsilon_\gamma^{\mu,\nu}(\alpha,\beta)$ in (8.15) are uniquely determined by this choice. We can choose the scaling of the $u_{0,0}$ and $u_{2,2}$ so that $\epsilon_0^{0,0}$ and $\epsilon_2^{0,2}$ are identically equal to 1. We have chosen the scaling for $u^{2,2}_{0,0}$,$u^{2,2}_{2,2}$ and $u^{2,2}_{4,4}$ so that $\epsilon_0^{2,2}(2,-2) = \epsilon_2^{2,2}(2,-2) = \epsilon_4^{2,2}(2,-2) = 1$ in $v_{2,2} \otimes v_{2,-2}$. This determines the remaining displayed in the table (8.1).

$\rho_2 \otimes \rho_2$	$v_{2,-2}$	$v_{2,2}$	$v_{2,0}$
$v_{2,-2}$	$6v_{4,-4}$	$v_{0,0} - v_{2,0} + v_{4,0}$	$2v_{2,-2} + 3v_{4,-2}$
$v_{2,2}$	$v_{0,0} + v_{2,0} + v_{4,0}$	$6v_{4,4}$	$-2v_{2,2} + 3v_{4,2}$
$v_{2,0}$	$2v_{2,-2} + 3v_{4,-2}$	$-2v_{2,2} + 3v_{4,2}$	$-2v_{0,0} + 4v_{4,0}$

Figure 8.1: Relationships among structure constants by $\mathfrak{sp}_2$-symmetry in the tensor product $\rho_2 \otimes \rho_2$.

The following is the master OPE among any two strong generators of the VOA $\mathcal{W}^{\mathfrak{sp}}_\infty$.

$$W^n_{\mu,\alpha}(z)W^m_{\nu,\beta}(w) \sim \sum_{r=0}^{n+m-1} \sum_{\gamma=0,2,4} \sum_{\omega\in U_\gamma(r)} \sum_{\chi\in\Omega^\omega_\gamma} w^{n,m}_{\omega_\chi,\gamma}(\alpha,\beta) W^{\omega_\chi}_{\gamma,\alpha+\beta}(w)(z-w)^{-n-m+r}. \tag{8.16}$$

Thanks to $\mathfrak{sp}_2$-symmetry we can have the following factorization of structure constants

$$w^{n,m}_{\omega_\chi,\gamma}(\alpha,\beta) = \epsilon^{\mu,\nu}_\gamma(\alpha,\beta) w^{n,m}_{\omega_\chi,\gamma}, \quad \chi \in \Omega^\omega_{\gamma,\alpha+\beta} \tag{8.17}$$

From now on we will write (8.16) in the following shorthand notation.

$$W^n(z)W^m(w) \sim \sum_{r=0}^{n+m-1} \sum_{\gamma=0,2,4} \sum_{\omega\in U_\gamma(r)} \sum_{\chi\in\Omega^\omega_\gamma} \epsilon^{\bar{n},\bar{m}}_\gamma w^{n,m}_{\omega_\chi,\mu} W^{\omega_\chi}_\mu(w)(z-w)^{-n-m+r}. \tag{8.18}$$

To extract the OPEs among fields in (8.16) from (8.18), we proceed as follows:

1. Determine that W^n transforms as $\rho_{\bar{n}}$, and W^m as $\rho_{\bar{m}}$, and recall the decomposition of $\rho_{\bar{n}} \otimes \rho_{\bar{m}}$ in (8.14).

2. Given a choice of Cartan weights α and β appearing in modules $U^n_{\bar{n}}$ and $U^m_{\bar{m}}$, select the basis of vectors $W^n_{\bar{n},\alpha}$ and $W^m_{\bar{m},\beta}$ of weight spaces $U^n_{\bar{n},\alpha}$ and $U^n_{\bar{m},\beta}$, determined by our choice of the highest weight vector (8.9) and $\mathfrak{sp}_2$-action (8.7).

3. Given that same choice of Cartan weights α and β, use the table (8.1) to evaluate $\epsilon^{\bar{n},\bar{m}}_\gamma(\alpha,\beta)$, for γ a highest weight of the highest weight module appearing in the

decomposition of $\rho_{\bar{n}} \otimes \rho_{\bar{m}}$ in (8.14). Note that only in the case of interaction of odd with odd weights this is nontrivial.

4. Select the basis of vectors $W^{\omega_\chi}_{\mu,\alpha+\beta}$ and of the weight spaces $U^{\omega}_{\mu,\alpha+\beta}$, determined a choice of highest weight vectors in U^{ω}_{μ} and the $\mathfrak{sp}_2$-action (8.7). Finally, we recover OPE relations (8.16).

We will be imposing Jacobi identities $J_{r,s}(W^u, W^n, W^m)$. However, using the shorthand notation (8.18) they still evaluate to expressions involving ϵ symbols introduced in (8.15). Similarly as in the above discussion regarding how to extract (8.16) from shorthand (8.18), we can extract from expressions $J_{r,s}(W^u, W^n, W^m)$ the Jacobi identities $J(W^u_{\bar{u},\gamma}, W^n_{\bar{n},\mu}, W^m_{\bar{m},\beta})$.

Finally, we note that structure constants $w^{n,m}_{\omega_\chi,\mu}$ arising in (8.18) depend on the choice of bases indexed by sets Ω^{ω}_{μ}. Given such a choice, the structure constants are uniquely determined as rational functions in terms of c and k, i.e. the universal algebra $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ is a 2-parameter VOA. This feature is independent of the choice of basis.

8.2 Step 1: Base Computation

The aim of this subsection is to describe the computation in sufficient detail, that the reader can reproduce our results. The computation was done using a Mathematica package `OPEdefs` [73]. We begin by setting up the most general OPEs in D_9 compatibile with assumptions (8). Next, we impose conformal and affine symmetry constraints which yield linear relations reducing the number of undetermined constants to just 24. Finally, we impose the remaining Jacobi equations in J_{11}, which allow us to express all structure constants as rational functions of 2 parameters c and k, see Proposition (25).

8.2.1 New Generators. We begin by introducing a strong generating set $\bigcup\{\tilde{W}^i | i \geq 1\}$, that is adapted for the purposes of base case evaluation. First, the affine fields W^1 cannot be corrected and remain fixed, thus $\tilde{W}^1 = W^1$. Next, the conformal vector W^2 must be corrected to the $\mathfrak{sp}_2$-coset Virasoro field $\tilde{W}^2 = W^2 - L^{\mathfrak{sp}_2}$, where we recall

$$L^{\mathfrak{sp}_2} = \frac{1}{2(k+2)}\Big(\frac{1}{2}:H^1H^1: +2:X^1Y^1: -\partial H^1\Big).$$

Weight three triple W^3 all remain the same, since they are primary for $V^k(\mathfrak{sp}_2)$, so $\tilde{W}^3 = W^3$. Finally, we deform weight four singlet $\tilde{W}^4$ so it commutes with affine fields.

Lemma 24. Up to a scaling parameter, there is unique generator of weight four field that commutes with weight one affine, and is primary for the Virasoro algebra. Moreover, it admits an explicit form

$$\tilde{W}^4 = W^4 - \frac{1}{k+4}\Big(\frac{1}{2}:H^1H^3: + :X^1Y^3: + :Y^1X^3:\Big) \tag{8.19}$$

Proof. Consider a defomation of $\tilde{W}^4 = W^4 + \cdots$, where the omitted terms are normally ordered products of monomials in $\{W^j | j = 1, 2, 3\}$ and their derivatives. Imposing desired constraints, namely that

$$W^1(z)\tilde{W}^4(w) \sim 0, \quad \tilde{W}^2(z)\tilde{W}^4(w) \sim 4\tilde{W}^4(w)(z-w)^{-2} + \partial\tilde{W}^4(w)(z-w)^{-1}.$$

evaluates to a unique deformation (8.19). □

Lastly, we postulate that the raising property is respected by fields of higher conformal weights

$$\tilde{W}^4_{(1)}\tilde{W}^n = \tilde{W}^{n+2}, \quad n \geq 3. \tag{8.20}$$

8.2.2 Set-up. First, we cast our assumptions on algebra $\mathcal{W}^{\mathfrak{sp}}_\infty$ into OPE form.

1. Weight one fields W^1 generate the universal affine algebra $V^k(\mathfrak{sp}_2)$ of level k:

$$W^1(z)W^1(w) \sim \epsilon_0^{2,2}k\mathbf{1}(z-w)^{-2} + \epsilon_2^{2,2}W^1(w)(z-w)^{-1}.$$

2. Weight two field W^2 generates the universal Virasoro algebra of central charge c. So the coset field $\tilde{W}^2 = W^2 - L^{\mathfrak{sp}_2}$ generates Virasoro with $\tilde{c} = c - \frac{3k}{k+2}$:

$$\tilde{W}^2(z)\tilde{W}^2_{0,0}(w) \sim \frac{\tilde{c}}{2}\mathbf{1}(z-w)^{-4} + 2\tilde{W}^2(w)(z-w)^{-2}(w) + \partial\tilde{W}^2(w)(z-w)^{-1}. \tag{8.21}$$

In particular, the action on higher weight generator is as follows

$$\begin{aligned} \tilde{W}^2(z)\tilde{W}^{2n}(w) &\sim \cdots + 2n\tilde{W}^{2n}(w)(z-w)^{-2}(w) + \partial\tilde{W}^{2n}(w)(z-w)^{-1},\ n \geq 3, \\ \tilde{W}^2(z)\tilde{W}^{2n-1}(w) &\sim \cdots + (W^2_{(1)}\tilde{W}^{2n-1} - L^{\mathfrak{sp}_2}_{(1)}\tilde{W}^{(2n-1)})(w)(z-w)^{-2} \\ &\qquad + (W^2_{(1)}\tilde{W}^{2n-1} - L^{\mathfrak{sp}_2}_{(1)}\tilde{W}^{2n-1}(w)(z-w)^{-1}, \quad n \geq 3. \end{aligned} \tag{8.22}$$

3. Using $\tilde{W}^2 = W^2 - L^{\mathfrak{sp}_2}$ we compute its action on $W^{(3)}$:

$$\tilde{W}^2(z)W^3(w) \sim \frac{3k+4}{k+2}W^3(w)(z-w)^{-2} + \left(\partial W^3 - \frac{1}{k+2}W_2^{(1,3)}\right)(w)(z-w)^{-1}.$$

4. By Lemma (24), weight four field $\tilde{W}^4$ is primary for $\tilde{W}^2$:

$$\tilde{W}^2(z)\tilde{W}^4(w) \sim 4\tilde{W}^4(w)(z-w)^{-2} + \partial\tilde{W}^4(w)(z-w)^{-1}.$$

5. By (8.20), fields W^{2n} and W^{2n+1} are primary for $V^k(\mathfrak{sp}_2)$:

$$W^1(z)\tilde{W}^{2n}(w) \sim 0, \quad W^1(z)\tilde{W}^{2n+1}(w) \sim \epsilon_2^{2,2}\tilde{W}^{2n+1}(w)(z-w)^{-1}, \quad n \geq 1. \tag{8.23}$$

6. $\tilde{W}^4$ satisfies the raising property (8.20):

$$\tilde{W}^4(z)\tilde{W}^n(w) \sim \cdots + \tilde{W}^{n+2}(w)(z-w)^{-2} + \cdots, \quad n \geq 2.$$

For the remaining OPEs, we posit they have the most general form compatible with the conformal weight gradation and $\mathfrak{sp}_2$-symmetry, as in (8.18). For clarity, we specialize 8.18 for each type of interaction

$$\begin{aligned}
\tilde{W}^{2i}(z)\tilde{W}^{2j}(w) &\sim \sum_{r=0}^{2i+2j-1} \left(V_0^{2i,2j}(r)\right)(w)(z-w)^{-r-1}, \\
\tilde{W}^{2i}(z)\tilde{W}^{2j-1}(w) &\sim \sum_{r=0}^{2i+2j-2} \left(V_2^{2i,2j-1}(r)\right)(w)(z-w)^{-r-1}, \\
\tilde{W}^{2i-1}(z)\tilde{W}^{2j-1}(w) &\sim \sum_{r=0}^{2i+2j-3} \sum_{\gamma=0,2,4} \left(V_\gamma^{2i-1,2j-1}(r)\right)(w)(z-w)^{-r-1}.
\end{aligned} \tag{8.24}$$

In above (8.24), $V_\mu^{n,m}(r)$ represents the general linear combination of all PBW monomials of conformal weight $n+m-r$ transforming as the ρ_μ $\mathfrak{sp}_2$-module. As an example, we have used the table (8.1) to write the first order poles among weights 3 and 5 fields.

$W^3_{(0)}W^5$	Y^5	X^5	H^5
Y^3	$6V^{3,5}_{-4,4}(0)$	$V^{3,5}_{0,0}(0) - V^{3,5}_{2,0}(0) + V^{3,5}_{4,0}(0)$	$2V^{3,5}_{2,-2}(0) + 3V^{3,5}_{4,-2}(0)$
X^3	$V^{3,5}_{0,0}(0) + V^{3,5}_{2,0}(0) + V^{3,5}_{4,0}(0)$	$6V^{3,5}_{4,4}(0)$	$-2V^{3,5}_{2,2}(0) + 3V^{3,5}_{4,2}(0)$
H^3	$2V^{3,5}_{2,-2}(0) + 3V^{3,5}_{4,-2}(0)$	$-2V^{3,5}_{2,2}(0) + 3V^{3,5}_{4,2}(0)$	$-2V^{3,5}_{2,0}(0) + 4V^{3,5}_{4,0}(0)$

8.2.3 Affine and Conformal Symmetry. Let $\tilde{W}^n$ and $\tilde{W}^m$ be two fields of the VOA $\mathcal{W}^{\mathfrak{sp}}_\infty$. In our ansatz (8.18) we have already imposed classical $\mathfrak{sp}_2$-symmetry and the conformal weight grading conditions. Now we impose the full affine and conformal symmetry constraints on OPEs $\tilde{W}^n(z)\tilde{W}^m(w)$ for all OPEs in D_9. This is equivalent to the vanishing of Jacobi identities $J(W^1, \tilde{W}^n, \tilde{W}^m)$ and $J(\tilde{W}^2, \tilde{W}^n, \tilde{W}^m)$. These give rise to linear constraints among the undetermined structure constants. Doing so, we find the following.

$$\tilde{W}^n(z)\tilde{W}^m(w) \sim \sum_{\gamma=0,2,4} \epsilon_{\gamma}^{\bar{n},\bar{m}} \sum_{\omega\in U_\gamma(r)} \sum_{\chi\in\Omega_\gamma^\omega} \tilde{w}_{\omega_\chi,\gamma}^{n,m} \sum_{\Lambda} \Big(\beta_\gamma^{n,m}(\Lambda) :\Lambda\tilde{W}_\gamma^{\omega_\chi}: (w)(z-w)^{\Delta(\Lambda)+\Delta(\omega)-n-m}\Big). \tag{8.25}$$

Here $\tilde{W}_\gamma^{\omega_\chi}$ are certain fields which we define later in (8.27) and (8.28). Fields Λ are contained in $V^k(\mathfrak{sp}_2)\otimes \text{Vir}^c$, and $\beta_\gamma^{n,m}(\Lambda)$ are rational functions of c and k. In particular, we have $\beta_\gamma^{n,m}(\mathbf{1}) = 1$. Accordingly, if any additional parameters in the OPE algebra of $\mathcal{W}_\infty^{\mathfrak{sp}}$ exist, they must arise as structure constants $\tilde{w}_{\omega_\chi,\gamma}^{n,m}$. Since our aim to show that $\mathcal{W}_\infty^{\mathfrak{sp}}$ is a 2-parameter VOA, all the relevant data is contained in these constants. Therefore we extract the terms in equation (8.25) with $\Lambda = \mathbf{1}$, and use the following shorthand notation, as in [15].

$$\tilde{W}^n \times \tilde{W}^m = \sum_{\gamma=0,2,4} \sum_{\omega\in U_\gamma} \sum_{\chi\in\Omega_\gamma^\omega} \epsilon_\gamma^{\bar{n},\bar{m}} w_{\omega_\chi,\gamma}^{n,m} W_\gamma^{\omega_\chi}. \tag{8.26}$$

The above shorthand (8.26) represents the notation (8.25), which itself stands for the OPEs (8.16) among fields which are the basis of Cartan weight spaces in $U_{\bar{n}}^n$ and $U_{\bar{m}}^m$ as in (8.9), and to extract fields on the right side of (8.26) we follow the same procedure as explained in (8.1.2). Using above notation, the imposition of affine and conformal symmetry for OPEs in D_9 leaves us with following undetermined constants.

$$
\begin{aligned}
\tilde{W}^4 \times \tilde{W}^4 =& \tilde{w}_0^{3,3}\mathbf{1} + \tilde{w}_4^{4,4}\tilde{W}^4 + \tilde{W}^6, \\
\tilde{W}^4 \times \tilde{W}^3 =& \tilde{w}_3^{3,3}W^3 + \tilde{W}^5, \\
W^3 \times W^3 =& \epsilon_0^{2,2}\tilde{w}_0^{3,3}\mathbf{1} + \epsilon_2^{2,2}\tilde{w}_3^{3,3}W^3 + \epsilon_0^{2,2}\tilde{w}_4^{3,3}\tilde{W}^4 + \epsilon_2^{2,2}\tilde{w}_5^{3,3}\tilde{W}^5, \\
\tilde{W}^4 \times \tilde{W}^5 =& \tilde{w}_3^{3,5}W^3 + \tilde{w}_5^{4,5}\tilde{W}^5 + \tilde{W}^7, \\
W^3 \times \tilde{W}^6 =& \tilde{w}_3^{3,6}W^3 + \tilde{w}_5^{3,6}\tilde{W}^5 + \tilde{w}_7^{3,6}\tilde{W}^7 + \tilde{w}_{(3,4),2}^{3,6}\tilde{W}_2^{(3,4)} \\
& + \tilde{w}_{(3,5),2}^{3,6}\tilde{W}_2^{(3,5)} + \tilde{w}_{(1,3,4),2}^{3,6}\tilde{W}_2^{(1,3,4)}, \\
W^3 \times \tilde{W}^5 =& \epsilon_0^{2,2}\tilde{w}_0^{3,5}\mathbf{1} + \epsilon_2^{2,2}\tilde{w}_3^{3,5}W^3 + \epsilon_0^{2,2}\tilde{w}_4^{3,5}\tilde{W}^4 + \epsilon_0^{2,2}\tilde{w}_6^{3,5}\tilde{W}^6 + \epsilon_0^{2,2}\tilde{w}_{(3,3),0}^{3,5}\tilde{W}_0^{(3,3)} \\
& + \epsilon_4^{2,2}\tilde{w}_{(3,3),4}^{3,5}\tilde{W}_4^{(3,3)} + \epsilon_2^{2,2}\tilde{w}_7^{3,5}\tilde{W}^7 + \epsilon_2^{2,2}\tilde{w}_{(3,4),2}^{3,5}\tilde{W}_2^{(3,4)} + \epsilon_4^{2,2}\tilde{w}_{(1,3,3),4}^{3,5}\tilde{W}_4^{(1,3,3)}.
\end{aligned}
\tag{8.27}
$$

Here, we have the $\mathfrak{sp}_2$-modules generated by the highest weight vectors

$$
\begin{aligned}
\tilde{W}_{4,4}^{(3,3)} &= :\tilde{X}^3\tilde{X}^3:, \\
\tilde{W}_0^{(3,3)} &= 2:\tilde{X}^3\tilde{Y}^3: + \frac{1}{2}:\tilde{H}^3\tilde{H}^3: + \cdots, \\
\tilde{W}_{2,2}^{(3,4)} &= :\tilde{X}^3\tilde{W}^4:, \\
\tilde{W}_{4,4}^{(1,3,3)} &= :X^1\tilde{X}^3\tilde{H}^3: - :H^1\tilde{X}^3\tilde{X}^3:, \\
\tilde{W}_{2,2}^{(1,3,4)} &= :X^1\tilde{H}^3\tilde{W}^4: - :H^1\tilde{X}^3\tilde{W}^4:,
\end{aligned}
\tag{8.28}
$$

where the omitted terms in $\tilde{W}_0^{(3,3)}$ represent normally ordered monomials in $\{\tilde{W}^i | i \leq 5\}$ and their derivatives, which are necessary to make $\tilde{W}_0^{(3,3)}$ be $\mathfrak{sp}_2$-invariant.

Remark 8.2.1. *It is a well-known fact [11] that conformal symmetry fixes the coefficients of all fields appearing in the OPEs $W^n(z)W^m(w)$ in terms of the coefficients of Virasoro primaries only. Our computations suggests that a similar fact holds when symmetry is enlarged to $V^k(\mathfrak{sp}_2) \otimes Vir^c$.*

8.2.4 Nonlinear constraints. The imposition of Jacobi identities

$$J(\tilde{W}^3, \tilde{W}^3, \tilde{W}^3), J(\tilde{W}^4, W^3, W^3), J(\tilde{W}^4, \tilde{W}^4, W^3)$$

allows us to express all of the structure constants in (8.27) as rational functions in the central charge c and $\mathfrak{sp}_2$-level k. Specifically, we obtain a system of quadratic equations. The solution breaks up in two parts, which is explained by the automorphism σ. We choose our scaling to eliminate any square roots in the OPE algebra, so that it is defined over a localization of polynomial ring, see (8.30) and (8.31). Finally, deform generating set $\{\tilde{W}^i | i \geq 1\}$ back to $\{W^i | i \geq 1\}$. Further, renormalize the fields W^3 and W^4 to have the leading poles

$$\begin{aligned}
w_0^{3,3} =& \frac{c(k-1)(k+2)\left(ck+2c+6k^2+11k+4\right)}{576k(k+1)(2k+1)(3k+4)}, \\
w_0^{4,4} =& \frac{c(k-1)\left(ck+2c+6k^2+11k+4\right)\alpha(c,k)}{20736(5c+22)k^2(k+1)^2(k+2)^2(2k+1)^2(3k+4)}, \\
\alpha(c,k) =& c^2(k+2)^2(2k-1)(2k+3)(3k+4) + ck(k+2)\Big(\\
& \left(192k^4+1216k^3+2510k^2+1961k+478\right) \\
& + 2k(2k+1)\left(96k^4+1172k^3+4014k^2+5311k+2376\right)\Big).
\end{aligned} \tag{8.29}$$

Remark 8.2.2. *Thanks to weight 3 fields W^3 the difficulty of this computation is comparable to that of $\mathcal{W}_\infty$ in [60]. It is also similar to $\mathcal{W}_\infty^{ev}$ in [55], in that some quadratic equations must be solved.*

Although the structure constants in the OPE algebra are not polynomials in c and k, they have only finitely many poles. Specifically, these are contained in the set

$$\{(5c+22), k, (2k+1), (k+1), (k+2), (k+4), (3k+4), (5k+8), (7k+16)\}. \tag{8.30}$$

Let D be the multiplicatively closed set containing (8.30) and define the following localization of the polynomial ring $\mathbb{C}[c,k]$

$$R = D^{-1}\mathbb{C}[c,k]. \tag{8.31}$$

We conclude this subsection with the following.

Proposition 25.

1. Data D_9 is fully determined in terms of the central charge c and $\mathfrak{sp}_2$-level k. Moreover, all Jacobi identities J_{11} vanish.

2. Under the scaling (8.29) all structure constants appearing in D_9 are elements of the ring R.

3. We have $w^{4,3}_{(15),2} = w^{4,5}_{(17),2} = w^{4,5}_{(35),2} = w^{3,6}_{(17),2} = w^{3,6}_{(35),2} = 0$, and moreover

$$\begin{aligned}
W^4_{(1)}W^3 =& W^{(5)}, \quad W^4_{(0)}W^3 = \frac{3}{5}\partial W^5 + \cdots, \\
W^4_{(1)}W^4 =& W^6, \quad W^4_{(0)}W^4 = \frac{1}{2}W^6 + \cdots, \\
W^3_{(1)}W^3 =& \epsilon^{2,2}_0 W^4 + \cdots, \quad W^3_{(0)}W^3 = \epsilon^{2,2}_2 \frac{3}{5}W^5 + \epsilon^{2,2}_0\frac{1}{2}\partial W^4 + \cdots \\
W^3_{(1)}W^5 =& \epsilon^{2,2}_0 \frac{5}{4}W^6 + \cdots, \quad W^3_{(0)}W^5 = \epsilon^{2,2}_2 \frac{3}{7}W^5 + \epsilon^{2,2}_0\frac{5}{12}\partial W^6 + \cdots, \\
W^4_{(1)}W^5 =& W^7 + \cdots, \quad W^4_{(0)}W^5 = \frac{3}{7}\partial W^7 + \cdots, \\
W^6_{(1)}W^3 =& \frac{6}{5}W^7 + \cdots, \quad W^6_{(0)}W^3 = \frac{6}{7}\partial W^7 + \cdots, \\
W^8_{(1)}W^1 =& \frac{16}{5}W^7 + \cdots, \quad W^8_{(0)}W^1 = \frac{16}{5}\partial W^7 + \cdots,
\end{aligned} \tag{8.32}$$

From now on we will be working with the original strong generating set $\{W^i | i \geq 1\}$, unless specified otherwise.

8.3 Step 2: Constant Structure Constants

Here, we consider consequences of the weak generation property (8.2). In Proposition (26), we find explicit forms for infinitely many structure constants specified in the following products.

$$W^n_{(1)}W^m = \epsilon^{\bar{n},\bar{m}}_{\overline{n+m-2}} w^{n,m}_{n+m-2} W^{n+m-2} + \cdots . \tag{8.33}$$

In general, for the structure constants $w^{n,m}_{\omega_\chi,\mu}$ in (8.18) to be well-defined, we must make explicit the choice of basis indexed by the set Ω^ω_μ for the subspace U^ω_μ. Since we only need to solve for some structure constants, and not all, we make only a partial choice of basis. We must verify that the structure constants we define are independent of the choice of basis for the complementary subspace. To resolve this ambiguity, we introduce a decreasing degree filtration (8.34), so that the complementary subspace belongs to a higher filtration degree.

Now we introduce a decreasing $\mathfrak{sp}_2$-invariant filtration on the VOA $\mathcal{W}^{\mathfrak{sp}}_\infty$. Define degree on the PBW monomials as follows.

$$\deg(:\partial^{d_1}a_1\cdots\partial^{d_n}a_n:) = n + \sum_{i=1}^{n} d_i.$$

Let G_n denote the span of all PBW monomials of degree greater or equal to n. Then G_n is a decreasing filtration

$$\mathcal{W}^{\mathfrak{sp}}_\infty = G_0 \supset G_1 \supset G_2 \supset \cdots . \tag{8.34}$$

Denote the canonical surjections

$$\sigma_n : G_n \to G_n/G_{n+1}, \tag{8.35}$$

and form the associated graded of VOA $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ with respect to the above decreasing degree filtration

$$\mathrm{gr}^G(\mathcal{W}^{\mathfrak{sp}}_{\infty}) := \bigoplus_{n=0}^{\infty} G_n/G_{n+1}. \tag{8.36}$$

The graded algebra $\mathrm{gr}^G(\mathcal{W}^{\mathfrak{sp}}_{\infty})$ is neither commutative nor associative, and is not good in the sense of [59]. However, the quasi-derivation identity (3.14) and existence of a conformal weight grading structure implies that

$$:a_{(-1)}b:\in G_{n+m}, \quad a \in G_n, b \in G_m, \tag{8.37}$$

and for all nonnegative r^{th}-products we have

$$a_{(r)}G_n \subset G_{n-r}. \tag{8.38}$$

Note that G_1/G_2 is the span of all strong generators of the algebra, and G_2/G_3 is spanned by all strong generators with one derivative and normally ordered quadratics with no derivatives.

Recall the subspace U^{ω}_{μ} discussed in (8.1.1), and consider $U^{\omega}_{\mu}\cap G_i$. The map σ_i restricts to the intersection

$$\sigma_i : U^{\omega}_{\mu} \cap G_i \to U^{\omega}_{\mu} \cap G_i/U \cap G_{i+1} \subseteq G_i/G_{i+1}.$$

Fix some basis of $\sigma_i(U^{\omega}_{\mu} \cap G_i)$ and lift it to a set of linearly independent vectors $\bigcup_{\alpha}\{W^{\omega_\chi}_{\mu,\alpha}|\chi \in \Omega^{\omega}_{\mu}\}$ in $U^{\omega}_{\mu} \cap G_i$ so that we have splitting of vector space

$$U^{\omega}_{\mu} = \mathrm{Span}(\bigcup_{\alpha}\{W^{\omega_\chi}_{\mu,\alpha}|\chi \in \Omega^{\omega}_{\mu}\}) \oplus C.$$

Here the complement C lies $G_{i+1} \cap U^{\omega}_{\gamma,\gamma}$. Without the loss of generality, since degree filtration (8.34) is $\mathfrak{sp}_2$-invariant, we can assume that this basis is compatible with the $\mathfrak{sp}_2$-action (8.7). Now the structure constants $w^{n,m}_{\omega_\chi,\gamma}$ are well-defined, i.e. independent of the choice of basis for C. In what follows we apply this procedure to the cases of $i = 1$ and $i = 2$, and $U^{\omega}_{\mu,\mu}$ is of dimension 1. Specifically, our partial basis will be given by (8.9) and (8.10).

We will need to impose Jacobi identities $J_{2,0}(W^{2i}, W^n, W^m)$ of the form

$$J_{2,0}(W^{2i}, W^n, W^m) = \varepsilon^{2i,n,m}_{2,0} W^{2i+n+m-4} + \cdots = 0, \tag{8.39}$$

where expression $\varepsilon^{2i,n,m}_{2,0}$ will give rise to an equation (8.44). Property (8.38) together with the structure of conformal gradation implies that only linear monomial with at most one derivative or quadratics term with no derivatives may contribute to the coefficient of $W^{2i+n+m-4}$ in (8.39). Recall that we have already made this choice in (8.9) and (8.10). We proceed to analyze the 3 types of interactions - even with even, even with odd, and odd with odd weight fields.

- Even with even. The first order pole of W^{2i} and W^{2j} has odd conformal weight $2i + 2j - 1$ and transforms as the trivial $\mathfrak{sp}_2$-module. So only $\partial W^{2i+2j-2}$ contributes in (8.39), and we write

$$\begin{aligned} W^{2i}_{(0)} W^{2j} &= v^{2i,2j}_0 \partial W^{2i+2j-2} + W^{2i,2j}_{0,0}(0), \\ W^{2i}_{(1)} W^{2j} &= v^{2i,2j}_1 W^{2i+2j-2} + W^{2i,2j}_{0,0}(1), \end{aligned} \tag{8.40}$$

  where $W^{2i,2j}_{0,0}(-)$ are some normally ordered polynomial in the generators $\{W^n | n \leq 2i + 2j - 3\}$ and their derivatives. Here, we used a new variable for the structure constant $v^{2i,2j}_0 = w^{2i,2j}_{(2i+2j-2)^1,0}$ and $v^{2i,2j}_1 = w^{2i,2j}_{2i+2j-2,0}$. Note that the $\mathfrak{sp}_2$-label is uniquely determined.

- Even with odd. The first order pole of W^{2i} and W^{2j-1} has even conformal weight $2i+2j-2$ and transforms as the adjoint $\mathfrak{sp}_2$-module. So $\partial W^{(2i+2j-3)}$ and quadratics $W_2^{(2l-1,2i+2j-2l-3)}$ contribute in (8.39), and we write

$$\begin{aligned} W^{2i}_{(0)}W^{2j-1} &= v_0^{2i,2j-1}\partial W^{2j+2i-3} + \sum_{l=0}^{\lfloor\frac{i+j}{2}\rfloor} q_l^{2i,2j-1} W_2^{(2l-1,2i+2j-2l-3)} \\ &\quad + W_2^{2i,2j-1}(0), \\ W^{2i}_{(1)}W^{2j-1} &= v_1^{2i,2j-1}W^{2j+2i-3} + W_2^{2i,2j-1}(1), \end{aligned} \tag{8.41}$$

where $W_{2,-}^{2i,2j-1}(-)$ are some normally ordered polynomial in generators $\{W^n | n \leq 2i+2j-4\}$. Here, we used a new variable for the structure constant $v_0^{2i,2j-1} = w^{2i,2j-1}_{(2i+2j-1)^1,2}$, $v_1^{2i,2j-1} = w^{2i,2j}_{2i+2j-1,2}$ and $q_l^{2i,2j-1} = w^{2i,2j-1}_{(2l-1,2i+2j-2l-3),2}$. Note that the $\mathfrak{sp}_2$-label is uniquely determined.

- Odd with odd. The first order pole of W^{2i-1} and W^{2j-1} has odd conformal weight $2i+2j-3$ and transforms under $\mathfrak{sp}_2$ as the $\rho_0 \oplus \rho_2 \oplus \rho_4$. So only $\partial W^{(2i+2j-3)}$ contribute in (8.39), and we write

$$\begin{aligned} W^{2i-1}_{(0)}W^{2j-1} =& \epsilon_2^{2,2} a^{2i-1,2j-1} W^{2j+2i-3} + \epsilon_0^{2,2} v_0^{2i-1,2j-1}\partial W^{2i+2j-4} \\ & + W_0^{2i-1,2j-1}(0) + W_2^{2i-1,2j-1}(0) + W_4^{2i-1,2j-1}(0), \\ W^{2i-1}_{(1)}W^{2j-1} =& \epsilon_0^{2,2} v_1^{2i-1,2j-1} W^{2i+2j-4} + W_0^{2i-1,2j-1}(1) + W_2^{2i-1,2j-1}(1) \\ & + W_4^{2i-1,2j-1}(1), \end{aligned} \tag{8.42}$$

where terms $W_{-,-}^{2i-1,2j-1}(-)$ are some normally ordered polynomial in the generators $\{W^n | n \leq 2i+2j-4\}$, and their derivatives. Here, we used a new variable for the structure constant $v_0^{2i-1,2j-1} = w^{2i-1,2j-1}_{(2i+2j-4)^1,0}$, $v_1^{2i-1,2j-1} = w^{2i-1,2j-1}_{2i+2j-4,0}$ and $a^{2i-1,2j-1} = w^{2i-1,2j-1}_{2i+2j-3,2}$. Note that the $\mathfrak{sp}_2$-label is uniquely determined.

Remark 8.3.1. *Ansatz (8.40) is formally the same as the one used in the construction of $\mathcal{W}^{ev}_{\infty}$, see ([55], Sect. 3, Eq. 3.2 and Eq. 3.3). Unlike in the constructions of $\mathcal{W}_{\infty}$ and $\mathcal{W}^{ev}_{\infty}$, we see a quadratic $W_2^{(2l-1,2i+2j-2l-3)}$ arising in (8.41) and a derivative-free monomial $W^{2j+2i-3}$ in (8.42). This is a consequence of the nontrivial C_2-algebra, see Subsection 8.6.*

Our notation above extends the one used in structure constants (8.17). Specifically if we denote by $W^{n,m}_{\gamma,\alpha+\beta}(r)(\alpha,\beta)$ the normally ordered differential polynomial, arising as above in $W^n_{\bar{n},\alpha}(z)W^m_{\bar{m},\beta}(w)$, then affine symmetry affords a factorization

$$W^{n,m}_{\gamma,\alpha+\beta}(r)(\alpha,\beta) = \epsilon^{\bar{n},\bar{m}}_{\gamma}(\alpha,\beta)W^{n,m}_{\gamma,\alpha+\beta}(r), \quad r = 0, 1.$$

Therefore, up to $\mathfrak{sp}_2$-symmetry, all relevant structure is contained in the constants with the form $v^{n,m}_0$, $v^{n,m}_1$ and $a^{2i-1,2j-1}$, and normally ordered differential polynomials $W^{n,m}_{\gamma,\alpha+\beta}(r)$. Denote the double factorial by

$$a!! = \begin{cases} (2n-1)(2n-3)\cdots 3, & a = 2n-1, \\ (2n)(2n-2)\cdots 2, & a = 2n. \end{cases}$$

Proposition 26. Let the notation be fixed as in (8.40-8.42) We have the following expressions for the first and second order poles among the strong generators.

1. Even and even weight fields.

$$\begin{aligned} W^{2i}_{(0)}W^{2j} =& \frac{(2i-1)(2i)!!(2j)!!}{8(2i+2j-2)(2i+2j-4)!!}\partial W^{2i+2j-2} + W^{2i,2j}_0(0), \\ W^{2i}_{(1)}W^{2j} =& \frac{(2i)!!(2j)!!}{8(2i+2j-4)!!}W^{2i+2j-2} + W^{2i,2j}_0(1), \quad i \geq 2. \end{aligned}$$

2. Even and odd weight fields.

$$W^{2i}_{(0)}W^{2j-1} = \frac{(2i-1)(2i)!!(2j-1)!!}{8(2i+2j-3)(2i+2j-5)!!}\partial W^{2j+2i-3} + W_2^{2i,2j-1}(0),$$
$$W^{2i}_{(1)}W^{2j-1} = \frac{(2i)!!(2j-1)!!}{8(2i+2j-5)!!}W^{2j+2i-3} + W_2^{2i,2j-1}(1), \quad i \geq 2.$$

3. Odd and odd weight fields:

$$\begin{aligned} W^{2i-1}_{(0)}W^{2j-1} =& \epsilon_2^{2,2}\frac{(2i-1)!!(2j-1)!!}{(2i+2j-3)!!}W^{2j+2i-3} \\ &+ \epsilon_0^{2,2}\frac{2(2i-2)(2i-1)!!(2j-1)!!}{9(2i+2j-4)(2i+2j-6)!!}\partial W^{2i+2j-4} \\ &+ W_0^{2i-1,2j-1}(0) + W_2^{2i-1,2j-1}(0) + V_4^{2i-1,2j-1}(0), \\ W^{2i-1}_{(1)}W^{2j-1} =& \epsilon_0^{2,2}\frac{2(2i-1)!!(2j-1)!!}{9(2i+2j-6)!!}W^{2i+2j-4} + W_0^{2i-1,2j-1}(1) \\ &+ W_2^{2i-1,2j-1}(1) + W_4^{2i-1,2j-1}(1). \end{aligned}$$

Proof. We will proceed by induction on N. Our base case is the Proposition (25), with $N = 3$. Inductively, assume that all structure constants defined in (8.40-8.42) have the form in as in the Proposition (26) for products in $D_0^{2N+1} \cup D_0^{2N+2}$. In particular, it means that all $\{q_l^{2i,2j-1} | 2i+2j-1 = 2N+1, l \geq 1\}$ vanish. First, we will show that constants $w_l^{2i,2j+1}$ arising in D_0^{2N+3} vanish.

To this end, let i, j, l be integers such that $2i + 2j + 2l = 2N + 6$ and $l \geq 2$, and consider the Jacobi identity $J_{0,0}(W^{2l}, W^{2i-1}, W^{2j-1})$, which has the form

$$W^{2l}_{(0)}(W^{2i-1}_{(0)}W^{2j-1}) = W^{2i-1}_{(0)}(W^{2l}_{(0)}W^{2j-1}) + (W^{2i}_{(0)}W^{2i-1})_{(0)}W^{2j-1}.$$

Since $l \geq 2$, by induction only the left side contributes a nonzero quadratic expression (8.42) and $w^{2i-1,2j-1}_{2i+2j-3} \neq 0$, thus no quadratics arise in the product $W^{2l}_{(0)}W^{2i+2j-3}$. It remains to show that product $W^{2n}_{(0)}W^3$ has no nonzero quadratics. This can be done by imposition of Jacobi identity$J_{1,0}(W^4, W^{2N-2}, W^3)$ which has the form

$$\begin{aligned} W^4_{(1)}(W^{2N-2}_{(0)}W^3) =& W^{2N-2}_{(0)}W^5 + W^{2N-2}_{(0)}W^3 + (W^4_{(0)}W^{2N-2})_{(1)}W^3 \\ =& W^{2n-2}_{(0)}W^5 + (1 - v_0^{4,2N-2})W^{2N}_{(0)}W^3 + (W_0^{4,2N-2}(0))_{(1)}W^3. \end{aligned} \tag{8.43}$$

Since by induction $v_0^{4,2N-2} \neq 1$, product $W^{2N}_{(0)}W^{(3)}$ has no nonzero quadratics. Finally, the term $W^{2N+2}_{(0)}W^{(1)}$ has no quadratics since W^{2N+2} transforms as the trivial $\mathfrak{sp}_2$-module.

Now, assuming that no quadratics arise in D_0^{2N+3}, we can show that structure constants have the form in a uniform manner. Let $m \neq 1$ and $n \neq 2$, or $m = 1$ and n is even, see Remark (8.3.2), and so that $2l + n + m \leq 2N + 6$. Extracting coefficient of fields $W^{2l+n+m-4}$ in identities $J_{2,0}(W^{2l}, W^n, W^m)$ gives a relation

$$v_1^{2l,n+m-2}v_0^{n,m} + (v_0^{2l,n} - v_1^{2l,n})v_1^{2l+n-2,m} = 0. \tag{8.44}$$

Set $l = 1$ in(8.44), and recall that Virasoro action implies that $v_0^{2,n} = 1$ and $v_1^{2,n} = n$ for $n \geq 1$. Thus we find

$$v_0^{n,m} = \frac{n-1}{n+m-2}v_1^{n,m}.$$

Next, set $l = 2$ in (8.44) and using part above relation we obtain a recurrence

$$v_1^{n+2,m} = -\frac{(n-1)v_1^{4,n+m-2}}{(n+m-2)(\frac{3}{n+2}v_1^{4,n} - v_1^{4,n})}v_1^{n,m} = \frac{n+2}{n+m-2}v_1^{n,m}, \tag{8.45}$$

where we have used raising the property $v_1^{4,n} = v_1^{4,n+m-2} = 1$. Thanks to skew-symmetry we exchange indices n and m to obtain

$$v_1^{n,m+2} = \frac{m+2}{n+m-2}v_1^{n,m}. \tag{8.46}$$

This proves inductive hypothesis for $v_1^{n,m}$ and $v_0^{n,m}$. Lastly, we evaluate $a^{2i-1,2l-1}$. Let i, l be integers such that $2l + 2i - 2 = 2N$. We extract the coefficient of X^{2N-1} in Jacobi identity $J_{0,1}(W^{2l}, X^1, H^{2i-1})$, and obtain a relation

$$v_0^{2l,1} a^{2l-1,2i-1} = -2v_1^{2l,2i-1}. \tag{8.47}$$

Since we have already determined $v_0^{2l,1}$ and $v_0^{2l,2i-1}$, the above allows to solve for $a^{2i-1,2l-1}$.

□

Remark 8.3.2. *When $n = 2j - 1$ and $m = 1$, identity (8.44) develops a contribution from structure constant of the monomial $:\partial H^1 W^{2i+2j-4}:$ arising in $W^{2i}_{(0)} W^{2j-1}$. Though it can computed exactly, we do not require it for this proof.*

8.4 Step 3: Induction

The main result of this subsection is Theorem (35). It is proved by induction, and the process is similar to that of [60] and [55]. Our base case is the Proposition (25). By inductive data we mean the set of OPEs in $D_{2n} \cup D_{2n+1}$, and that they are fully expressed in terms of parameters c and k. At this stage, the OPEs in $D^{2n+2} \cup D^{2n+3}$ are yet undetermined. We will use a subset of Jacobi identities in $J^{2n+4} \cup J^{2n+5}$ to express $D^{2n+2} \cup D^{2n+3}$ in terms of inductive data. We write $A \equiv 0$ to denote that A is computable from inductive data $D_{2n} \cup D_{2n+1}$.

Lemma 27. Let n be a positive integer. Then we have the following.

1. OPEs $W^4(z)W^n(w)$ and $L(z)W^n(w)$ together determine $L(z)W^{n+2}(w)$.

2. OPEs $W^3(z)W^n(w)$ and $W^1(z)W^n(w)$ together determine $W^1(z)W^{n+2}(w)$.

Proof. The Jacobi identity $J_{1,r}(W^4, W^1, W^n)$ gives rise to the following relation

$$W^1_{(r)} W^{n+2} = W^4_1 W^1_{(r)} W^n - r W^3_{(r)} W^n.$$

Note it also reproduces the affine action when restricted to $r = 0$. Similarly, the Jacobi identity $J_{1,r}(W^4, L, W^n)$ gives rise to a relation

$$L_{(r)}W^{n+2} = W^4_{(1)}L_{(r)}W^n - (3r-1)W^4_{(r)}W^n,$$

which also reproduces the Virasoro action when restricted to $r = 1$ and $r = 0$. $\square$

In the Lemmas (29-31) we use the raising property of W^4 to establish the following.

Proposition 28.

1. $W^4(z)W^{2n-2}(w)$ with inductive data D_{2n} together determine EE^{2n+2}.
2. $W^4(z)W^{2n-2}(w)$ and $W^3(z)W^{2n-1}(w)$ with inductive data D_{2n} together determine OO^{2n+2}.
3. $W^4(z)W^{2n-1}(w)$ with inductive data D_{2n+1} together determine EO^{2n+3}.

First, we consider the first order poles. In the Proposition (26) we have already determined some structure constants arising in the first order poles. Therefore, to determine first order poles it is sufficient to analyze the normally ordered differential monomials $W^{i,j}_{0,0}(0)$, $W^{i,j}_{2,0}(0)$, and $W^{i,j}_{4,0}(0)$, defined in (8.40-8.42). Note that by Proposition (26), the coefficients of ∂W^{i+j} arising in Jacobi identities $J_{1,0}(W^4, W^i, W^j)$.

Lemma 29. Modulo the inductive data, we have that

1. $W^4_{(0)}W^{2n-2}$ determine $\{W^{2i+4}_{(0)}W^{2n-2i-2} | i \geq 1\}$.
2. $W^4_{(0)}W^{2n-1}$ determine $\{W^{2i+4}_{(0)}W^{2n-2i-1} | i \geq 1\}$.
3. $W^3_{(0)}W^{2n-1}$ and $W^4_{(0)}W^{2n-2}$ determine $\{W^{2i+3}_{(0)}W^{2n-2i-1} | i \geq 1\}$.

Proof. Consider a general Jacobi identity $J_{1,0}(W^4, W^i, W^j)$ which reads

$$(v_1^{4,i} - v_0^{4,i})W_\mu^{i+2,j}(0) = \epsilon_\mu^{\bar{i},\bar{j}} v_0^{i,j} W_\mu^{4,i+j-2}(0) - W_\mu^{i,j+2}(0) + R_{1,0}^{4,i,j}, \quad \mu = 0, 2, 4, \quad (8.48)$$

where

$$R_{1,0}^{4,i,j} = \epsilon_\mu^{\bar{i},\bar{j}} W_{(1)}^4 \left(W_\mu^{i,j}(0)\right) - \left(W_{\bar{i}}^{4,i}(0)\right)_{(1)} W^j, \quad \mu = 0, 2, 4. \quad (8.49)$$

Note that $R_{1,0}$ is known from inductive data, so we may write

$$W_\mu^{i+2,j}(0) \equiv \epsilon_\mu^{\bar{i},\bar{j}} \frac{v_1^{i,j}}{1 - v_0^{4,i}} W_\mu^{4,i+j-2}(0) - \frac{1}{1 - v_0^{4,i}} W_\mu^{i,j+2}(0), \quad \mu = 0, 2, 4. \quad (8.50)$$

Note that $1 - v_0^{4,1} = 0$, and hence recursion (8.50) is not valid for $i = 1$. However, this is not an issue since products $W_{(0)}^1 W^n$ are known by our assumptions. Finally, we iterate recursion (8.50) and instantiate to the case of gives rise to linear relations

$$\begin{aligned}
W_{0,0}^{2l+4,2n-2l-2}(0) \equiv& p_l(n) W_{0,0}^{4,2n-2}(0),\\
W_{2,0}^{2l+4,2n-2l-1}(0) \equiv& q_l(n) W_{2,0}^{4,(2n-1)}(0),\\
W_{0,0}^{2l+3,2n-2l-1}(0) \equiv& h_l(n) W_{0,0}^{4,2n-2}(0) + d^l W_{0,0}^{3,2n-1}(0),\\
W_{2,0}^{2l+3,2n-2l-1}(0) \equiv& d^l W_{2,0}^{(3),(2n-1)}(0),\\
W_{4,0}^{2l+3,2n-2l-1}(0) \equiv& d^l W_{4,0}^{(3),(2n-1)}(0),
\end{aligned} \quad (8.51)$$

where $p_l(n), q_l(n), h_l(n)$ are some nonzero rational functions in n, and d^l are constants. □

Next, we consider the second order poles. As before, by Proposition (26) we have already determined some structure constants arising in the second order poles. Therefore, to determine first order poles it is sufficient to analyze the normally ordered differential monomials $W_0^{i,j}(1)$, $W_2^{i,j}(1)$, and $W_4^{i,j}(1)$. Moreover by Proposition (26), the coefficients of W^{i+j} arising in Jacobi identities $J_{1,1}(W^4, W^i, W^j)$ vanish.

Lemma 30. Modulo the inductive data, we have that

1. $W^6_{(1)}W^{2n-4}$ determine $\{W^{2i+4}_{(1)}W^{2n-2i-2}|i \geq 2\}$.

2. $W^6_{(1)}W^{2n-3}$ determine $\{W^{2i+4}_{(1)}W^{2n-2i-3})|i \geq 2\}$.

3. $W^3_{(1)}H^{2n-1}$ determine $\{W^{2i+3}_{(1)}W^{2n-2i-1})|i \geq 1\}$.

Proof. Consider a general Jacobi identity $J_{1,1}(W^4, W^i, W^j)$ which reads

$$(1 - 2v_0^{4,i})W_\mu^{i+2,j}(1) = W_\mu^{i,j+2}(1) + R_{1,1}^{4,i,j}, \quad \mu = 0, 2, 4, \tag{8.52}$$

where

$$R_{1,1}^{4,i,j} = W^4_{(1)}\left(W_\mu^{i,j}(1)\right) - \left(W_{\bar{i}}^{4,i}(0)\right)_{(2)} W^j.$$

Note that $R_{1,1}^{4,(i),(j)}$ is known from inductive data, so we may write

$$W_\mu^{i+2,j}(1) \equiv \frac{1}{1 - 2v_0^{4,i}} W_\mu^{i,j+2}(1), \quad \mu = 0, 2, 4. \tag{8.53}$$

Note that $1 - 2v_0^{4,4} = 0$ and thus (8.53) is not valid for $i = 2$. However, this is not an issue since we have assumed that raising property holds. Iterating recursion (8.50) gives rise to the desired linear relations. □

Lastly, we consider the higher order products.

Lemma 31. Let $r > 1$. Modulo the inductive data, we have that

1. $W^4_{(r)}W^{2n-2}$ and determine $\{W^{2i+2}_{(r)}W^{2n-2i}|i \geq 2\}$.

2. $W^3_{(r)}W^{2n}$ determine $\{W^{2i+3}_{(r)}W^{2n-2i-1}|i \geq 1\}$.

3. $W^3_{(r)}W^{2n-1}$ determine $\{W^{2i+3}_{(r)}W^{2n-2i-1}|i \geq 1\}$.

Proof. Consider Jacobi identity $J_{r,1}(W^4, W^i, W^j)$ which reads

$$(r - (r+1)v_0^{4,i})W^{i+2}_{(r)}W^j = v_1^{i,j}W^4_{(r)}W^{i+j-2} + R^{4,i,j}_{r,1}, \tag{8.54}$$

where

$$R^{4,i,j}_{r,1} = W^4_{(r)}W^{i,j}_\mu(1) - \left(W^{4,i}_{\bar{i}}(0)\right)_{(r+1)} W^j - \sum_{j=2}^{r}\binom{r}{j}(W^4_j W^i)_{(r+1-j)}W^j.$$

Note that $R^{4,i,j}_{r,1}$ is known from inductive data, so we may write

$$W^{i+2}_{(r)}W^j \equiv \frac{v_1^{i,j}}{r-(r+1)v_0^{4,i}}W^4_{(r)}W^{i+j-2}. \tag{8.55}$$

Iterating recursions (8.55) gives rise to the desired linear relations. □

From now on we assume that Jacobi identities used in the proof of Proposition (28) have been imposed. In the next series of Lemmas (32,33,34), we write down a small set of Jacobi identities to obtain linear relations among the desired products. This reduces our problem to solving a linear system of equations. Their proofs are similar, so we only provide an account for the most complicated case, which is part (1) of the following.

Lemma 32. D_0^{2n+2} and D_0^{2n+3} is determined from inductive data with D_1^{2n+2} and D_1^{2n+3}. Specifically, assuming Lemma (29) we have the following.

1. $J_{0,1}(W^4, H^3, H^{2n-3})$ and $J_{0,0}(X^3, Y^3, H^{2n-3})$ express $W^4_{(0)}W^{2n-2}$ and $H^3_{(0)}H^{2n-1}$ in terms of inductive data and OO_1^{2n+2}.

2. $J_{0,1}(W^4, X^3, H^{2n-3})$ expresses $X^3_{(0)}H^{2n-1}$ in terms of inductive data and OO_1^{2n+2}.

3. $J_{0,1}(W^4, W^6, H^{2n-5})$ expresses $W^4_{(0)}H^{2n-3}$ in terms of inductive data together with EE_1^{2n+2} and EO_1^{2n+3}.

Proof. Here we only prove part (1), and the rest is similar. To determine all products in D_0^{2n+2}, it suffices to determine $W_{0,0}^{4,2n-2}(0)$, $W_{0,0}^{3,2n-1}(0)$, $W_{2,0}^{3,2n-1}(0)$ and $W_{4,0}^{3,2n-1}(0)$. Expanding $J_{0,1}(W^4, H^3, H^{2n-3})$ and $J_{0,0}(X^3, Y^3, H^{2n-3})$, projecting onto ρ_0 component, and omitting the inductively known data we obtain two linear relations

$$\begin{aligned} 0 \equiv& v_1^{3,2n-3} W_{0,0}^{4,2n-2}(0) - v_0^{4,2n-3} W_{0,0}^{3,2n-1}(0) + v_0^{4,3} W_{0,0}^{5,2n-3}(0) - v_0^{4,2n-3} \partial W_{0,0}^{3,2n-1}(1) \\ 0 \equiv& a^{3,3} W_{0,0}^{5,2n-3}(0) - 2v_1^{3,2n-3} W_{0,0}^{3,2n-1}(0). \end{aligned} \tag{8.56}$$

Using recurrences obtained in Lemma (8.51), we can express $W_{0,0}^{5,2n-3}(0)$ in terms of $W_{0,0}^{3,2n-1}(0)$, modulo inductive data. Finally, we observe that two linear relations (8.56) are linearly independent, and provide solutions

$$\begin{aligned} W_{0,0}^{4,2n-2}(0) \equiv& \frac{9(2n+3)(n-1)!}{4n(2n-1)(2n+1)\left(\frac{1}{2}\right)_{n-1}} \partial W_{0,0}^{3,2n-1}(1), \\ W_{0,0}^{3,2n-1}(0) \equiv& \frac{3}{n(2n+1)} \partial W_{0,0}^{3,2n-1}(1). \end{aligned}$$

Thus products $W^4_{(0)} W^{2n-2}$ and $W^3_{(0)} W^{2n-3}$ are determined by inductive data together with $\partial W_{0,0}^{3,2n-1}(1) \in OO_1^{2n+2}$. Similarly, the ρ_2 and ρ_4 components of the Jacobi identity $J_{0,0}(X^3, Y^3, H^{2n-3})$ express $W_{2,0}^{3,2n-1}(0)$ and $W_{4,0}^{3,2n-1}(0)$ in terms of inductive data. This completes the proof of part (1). □

Lemma 33. Data D_1^{2n+2} and D_1^{2n+3} is determined from inductive data with D_2^{2n+2} and D_2^{2n+3}. Specifcially, we have the following.

1. $J_{0,2}(W^4, H^3, H^{2n-3})$ expresses $H^3_{(1)} H^{2n-1}$ in terms of inductive data and OO_2^{2n+2}.
2. $J_{0,2}(W^4, W^4, W^{2n-4})$ expresses $W^6_{(1)} W^{2n-4}$ in terms of inductive data and EE_2^{2n+2}.

3. $J_{0,2}(W^4, W^6, H^{2n-5})$ expresses $W^6_{(1)}H^{2n-3}$ in terms of inductive data and EO_2^{2n+3} and EE_2^{2n+2}.

Lemma 34. Data D_r^{2n+2} and D_r^{2n+3} is determined from inductive data with D_{r+1}^{2n+2} and D_{r+1}^{2n+3}. Specifically, we have the following.

1. Let $r > 1$. $J_{1,r}(W^{2n-1}, H^3, H^3)$ and $J_{1,r}(W^4, H^{2n-3}, W^4)$ express $H^3_{(r)}H^{2n-1}$ in terms of inductive data and OO_{r+1}^{2n+2}.
2. Let $r > 0$. $J_{r+1,0}(W^4, X^3, H^{2n-3})$ expresses $X^3_{(r)}H^{2n-1}$ in terms of inductive data.
3. Let $r > 1$. $J_{1,r}(W^{2n-4}, W^4, W^4)$ and $J_{1,r}(W^4, W^{2n-4}, W^4)$ express $W^4_{(r)}W^{2n-2}$ in terms of inductive data and EE_{r+1}^{2n+2}.
4. Let $r > 1$. $J_{1,r}(W^4, W^6, H^{2n-5})$ expresses $W^4_{(r)}H^{2n-1}$ in terms of inductive data.

This process terminates after finitely many steps, since all elements of $D_r^{2n+3} \cup D_r^{2n+4}$ vanish for $r \geq 2n+2$. Therefore, we have proven the following.

Theorem 35. There exists a nonlinear conformal algebra $\mathcal{L}^{\mathfrak{sp}_2}$ over the localized ring $D^{-1}\mathbb{C}[c, k]$ with D being the multipliticatively closed set containing

$$\{(5c+22), k, (2k+1), (k+1), (k+2), (k+4), (3k+4), (5k+8), (7k+16)\},$$

satisfying features in list (8) whose universal enveloping vertex algebra $\mathcal{W}^{\mathfrak{sp}}_\infty$ has the following properties.

1. It has conformal weight grading

$$\mathcal{W}^{\mathfrak{sp}}_\infty = \bigoplus_{N=0}^{\infty} \mathcal{W}^{\mathfrak{sp}}_\infty[N], \quad \mathcal{W}^{\mathfrak{sp}}_\infty[0] = D^{-1}\mathbb{C}[c, k].$$

2. It is strongly generated by fields $\{X^{2i-1}, Y^{2i-1}, H^{2-1} | i \geq 1\} \cup \{L, W^{2i} | i \geq 2\}$ and satisfies the OPE relations in Proposition (25), Jacobi identities in J_{11} and those which appear in Lemmas (29 -34).

3. It is the unique initial object in the category of vertex algebras with the above properties.

Remark 8.4.1. *Note that we have $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ under the weak generation hypothesis. Thus to show that an algebra arises as a quotient of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$, in addition to having strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, it must have the weak generation property (8.20).*

8.5 Step 4. Free Generation

There are more Jacobi identities than those imposed in Lemmas (29-34). So it is not yet clear that all Jacobi identities among the strong generators hold as a consequence of (3.11-3.14) alone, or equivalently, that $\mathcal{L}^{\mathfrak{sp}_2}$ is a nonlinear Lie conformal algebra and $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ is freely generated. In order to prove it, we will consider certain simple quotients of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$. First, recall the localized ring R (8.31), and let

$$I \subseteq R \cong \mathcal{W}^{\mathfrak{sp}}_{\infty}[0]$$

be an ideal, and let $I \cdot \mathcal{W}^{\mathfrak{sp}}_{\infty}$ denote the vertex algebra ideal generated by I. The quotient

$$\mathcal{W}^{\mathfrak{sp},I} = \mathcal{W}^{\mathfrak{sp}}_{\infty} / I \cdot \mathcal{W}^{\mathfrak{sp}}_{\infty} \tag{8.57}$$

has strong generators $\{W^i | i \geq 1\}$ satisfying the same OPE relations as the corresponding generators of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ where all structure constants in R are replaced by their images in R/I.

We now consider a localization of $\mathcal{W}^{\mathfrak{sp},I}$. Let $D \subseteq R/I$ be a multiplicatively closed set, and let $S = D^{-1}(R/I)$ denote the localization of R/I along S.

Thus we have localization of R/I-modules

$$\mathcal{W}_S^{\mathfrak{sp},I} = S \otimes_{R/I} \mathcal{W}_\infty^{\mathfrak{sp},I},$$

which is a vertex algebra over S.

Theorem 36. Let I, D and S be as above, and let $\mathcal{W}$ be a simple vertex algebra over S with the following properties.

1. $\mathcal{W}$ is generated by affine fields $\bar{X}^1, \bar{H}^1, \bar{Y}^1$, Virasoro field $\bar{L}$ of central charge c and a weight 4 primary field $\bar{W}^4$.

2. Setting $\bar{W}^{n+2} = \bar{W}^4_{(1)}\bar{W}^n$ for all $i \geq 2$, the OPE relations for $\bar{W}^n(z)\bar{W}^m(z)$ for $n + m \leq 9$ are the same as in $\mathcal{W}_\infty^{\mathfrak{sp}}$ if the structure constants are replaced with their images in S.

Then $\mathcal{W}$ is the simple quotient of $\mathcal{W}_S^{\mathfrak{sp},I}$ by its maximal graded ideal $\mathcal{I}$.

Proof. The assumption that generators $\{W^i | i \geq 1\}$ satisfy the above OPE relations is equivalent to the statement that the Jacobi identities in J_{11} hold in the corresponding nonlinear Lie conformal algebra, which is possibly degenerate. Then all OPE relations among the generators $\{\bar{W}^i | i \geq 1\}$ of $\mathcal{W}_S^{\mathfrak{sp},I}$ must also hold among the fields $\{\bar{W}^i | i \geq 1\}$, since they are formal consequences of these OPE relations together with Jacobi identities, which hold in $\mathcal{W}$. It follows that $\{\bar{W}^i | i \geq 1\}$ close under OPE and strongly generate a vertex subalgebra $\mathcal{W}' \subset \mathcal{W}$, which must coincide with $\mathcal{W}$ since $\mathcal{W}$ is assumed to be generated by $\{\bar{X}^1, \bar{Y}^1, \bar{H}^1, \bar{L}, \bar{W}^4\}$. So $\mathcal{W}$ has the same strong generating set and OPE algebra as $\mathcal{W}_S^{\mathfrak{sp},I}$. Since $\mathcal{W}$ is simple and the category of vertex algebras over R with this strong generating set and OPE algebra has a unique simple graded object, $\mathcal{W}$ must be the simple quotient of $\mathcal{W}_S^{\mathfrak{sp},I}$ by its maximal graded ideal. $\square$

8.5.1 Generalized parafermions of type C. Let $n \geq 1$, and consider a family of vertex algebras $\mathcal{C}^l(n)$ that arise as quotients of $\mathcal{W}_\infty^{\mathfrak{sp}}$, which we call generalized parafermions of type C, by analogy with other types A [60] and B, D [24]. The embedding of Lie algebras $\mathfrak{sp}_{2n} \to \mathfrak{sp}_{2n+2}$ lifts to the embedding of vertex algebras

$$V^k(\mathfrak{sp}_{2n}) \to V^k(\mathfrak{sp}_{2n+2}).$$

We define

$$\mathcal{C}^k(n) = \text{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2})). \tag{8.58}$$

These are a special case of the Y-algebras of type C, specifically $\mathcal{W}_{BC}^{\psi}(0,m)$. As a $\mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n}$-module, $\mathfrak{sp}_{2n+2}$ decomposes as $\mathfrak{sp}_2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2n} \oplus \mathfrak{sp}_{2n}$, where $\mathbb{C}^{2n}$ transforms as standard $\mathfrak{sp}_{2n}$-module. From this decomposition it is clear that $\mathcal{C}^k(n)$ has an affine subalgebra $V^k(\mathfrak{sp}_2)$. We regard $\mathcal{C}^k(n)$ as a vertex algebra over localization $D^{-1}\mathbb{C}[k]$, where D is a multiplicatively closed set containing

$$\{k, (2k+1), (k+1), (k+2), (k+4), (3k+4), (5k+8), (7k+16), (k+n+1), (k+n+2)\}.$$

It has central charge

$$c(k) = \frac{kn(2k+1)(2k+n+3)}{(k+2)(k+n+1)(k+n+2)}. \tag{8.59}$$

for the conformal vector $L^{\mathfrak{sp}_{2n+2}} - L^{\mathfrak{sp}_2} - L^{\mathfrak{sp}_{2n}}$, so it commutes with $V^k(\mathfrak{sp}_2)$. By proper rescaling of generators in $\mathcal{C}^k(n)$, algebra $\mathcal{C}^k(n)$ has a well-defined free field limit

$$\lim_{k\to\infty} \mathcal{C}^k(n) \simeq \mathcal{H}(3) \oplus \mathcal{H}(4n)^{\text{Sp}_{2n}}.$$

Only the second summand in the above decomposition requires further analysis. Recall that rank $4n$ Heisenber algebra $\mathcal{H}(4n)$ has full automorphism O_{4n} (3.6.1). There is an embedding of $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{O}_{4n}$, such that the standard representation $\mathbb{C}^{4n}$ of O_{4n} decomposes as 2 standard representations $\mathbb{C}^{2n}$ of Sp_{2n}. We choose generators $\{u^i, u^{-i} | i = 1, \dots n\}$ and $\{v^i, v^{-i} | i = 1, \dots n\}$ for $\mathcal{H}(4n)$ which transform as the standard modules in the usual symplectic basis. They satisfy the following OPE relations

$$u^i(z)v^{-j}(w) \sim \delta_{i,j}\mathbf{1}(z-w)^{-2}, \quad u^{-i}(z)v^j(w) \sim -\delta_{i,j}\mathbf{1}(z-w)^{-2}. \tag{8.60}$$

For a power series $f(q) = \sum_{n=0}^{\infty} f_n q^n$, let us denote by $f(q)[q^m]$ the coefficient f_n. We have the following.

Proposition 37. Orbifold $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$ has the following properties.

1. $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$ is of type $\mathcal{W}(2, 3^3, 4, \dots, N)$ for some $N \geq n^2+3n+1$ and whose graded satisfies

$$\chi(\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}, q)[q^a] = \prod_{i=1}^{\infty}\prod_{j=2}^{\infty}\prod_{l=1}^{\infty} \frac{1}{(1-q^{2i+2j+1})^3(1-q^{2i+2l+2})}[q^a],$$

for $a \leq n^2+3n+1$. Here n^2+3n+2 is the conformal weight of the first non-trivial relation among its strong generators.

2. $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$ is generated by conformal weights 2 and 3 fields.

3. $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$ is a simple vertex algebra.

Proof. First, $\mathcal{H}(4n)$ has a good increasing filtration [59], such that the associated graded $\mathrm{gr}^F(\mathcal{H}(4n)^{\mathrm{Sp}_{2n}})$ is isomorphic to a classical invariant ring

$$\Big(\mathrm{Sym}\Big(\bigoplus_{i=0}^{\infty}(V_i \oplus U_i)\Big)\Big)^{\mathrm{Sp}_{2n}},$$

where $U_i \cong V_i \cong \mathbb{C}^{2n}$ as Sp_{2n}-module. Generators for this ring are given by Weyl's first fundamental theorems of invariant theory for the standard representation of Sp_{2n} (3). These are quadratics, each corresponding to a pair of distinct modules in $\{U_i, V_i | i \geq 1\}$. Let $\{\partial^d u^i, \partial^d u^{-i} | i = 1, \dots, n\}$ and $\{\partial^d v^i, \partial^d v^{-i} | i = 1, \dots, n\}$ be the symplectic bases for each U_d and V_d, respectively. We have 3 kinds of $\mathfrak{sp}_{2n}$-invariants

$$\begin{aligned} X^{a,b} &= \sum_{i=1}^{n} :\partial^a u^i \partial^b u^{-i}: - :\partial^b u^i \partial^a u^{-i}:, \quad a > b \geq 0, \\ Y^{a,b} &= \sum_{i=1}^{n} :\partial^a v^i \partial^b v^{-i}: - :\partial^b v^i \partial^a v^{-i}:, \quad a > b \geq 0, \\ H^{a,b} &= \sum_{i=1}^{n} :\partial^a u^i \partial^b v^{-i}: - :\partial^b u^i \partial^a v^{-i}:, \quad a, b \geq 0, \end{aligned} \tag{8.61}$$

each of conformal weight $a + b + 2$. The strong generating sets $\{X^{a,b} | a > b \geq 0\}$, $\{Y^{a,b} | a > b \geq 0\}$, and $\{H^{a,b} | a, b \geq 0\}$ are not independent as elements of a ∂-ring. It is easy to see that a smaller set

$$\{X^{2a+1,0} | a \geq 1\} \cup \{Y^{2a+1,0} | a \geq 1\} \cup \{H^{a,0} | a \geq 2\} \tag{8.62}$$

suffices. It follows that $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$ has a strong generating type $\mathcal{W}(2, 3^3, 4, 5^3, \dots)$, which is not minimal. By the theorem ([61], Thm. 6.6), it is of type $\mathcal{W}(1^3, 2^3, 3^3, 4, \dots, N)$ for some N. Thus a finite subset of generators (8.62) is sufficient to generate the orbifold $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$.

Next, we write a lower bound on N, which will be quadratic in n. By Weyl's second fundamental theorem of invariant theory for the standard representation of Sp_{2n} (3), the relations are Pfaffians. These are polynomials of degree $n + 1$ in the quadratic generators and correspond to a list of $2n + 2$ distinct modules in $\{U_i, V_i | i \geq 1\}$. Since U_i and V_i have conformal weight $i + 1$, the weight of the first relation occurs in weight $n^2 + 3n + 2$, corresponding to the list $\{U_0, V_0, U_1, V_1, \dots, U_n, V_n\}$. Thus $N \geq n^2 + 3n + 1$ and we have proven (1). Next, we prove the weak generation. We have the following relations in $\mathcal{H}(4n)^{\mathrm{Sp}_{2n}}$.

$$\begin{aligned} H^{1,0}_{(0)} X^{2a+1,0} =& 2X^{2a+3,0} + \cdots , \\ H^{1,0}_{(0)} Y^{2a+1,0} =& -2Y^{2a+3,0} + \cdots , \\ H^{1,0}_{(1)} H^{a,0} =& (a+3)H^{a+1,0} + \cdots , \end{aligned} \tag{8.63}$$

where the omitted terms are in the span of $\{\partial^{2k} X_{2a-3k+3}, \partial^{2k} Y_{2a-3k+3}, \partial^k X_{a-k+1} | k \geq 1\}$. By induction we can get all the generators in (8.62), thus proving (2). Finally, part (3) follows [27]. □

Lemma 38. As a one-parameter vertex algebra, $\mathcal{C}^k(n)$ inherits these properties; it is generated by weights 2 and 4 fields, is of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots,)$, and simple. Equivalently, $\mathcal{C}^k(n)$ inherits these properties for a generic value of k.

Proof. Strong generation is established by ([19], Lem. 3.2), weak generation by ([7], Lem. 2.5), and simplicity is similar to the proof in ([20], Thm. 3.6). □

Conjecture 8.5.1. *Often, it is the case that the relation occurring in the minimal weight is decoupling. If so, we conjecture that the affine coset $\mathcal{C}^k(n)$ if has a minimal strong generating set of type*

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (n^2+3n-1)^3, (n^2+3n), (n^2+3n+1)^3).$$

Therefore, we have the following.

Corollary 39. All Jacobi identities among generators $\{W^i | i \geq 1\}$ holds as consequences of (3.11-3.14) alone in $\mathcal{L}^{\mathfrak{sp}_2}$, so $\mathcal{L}^{\mathfrak{sp}_2}$ is a nonlinear Lie conformal algebra with generators $\{W^i | i \geq 1\}$. Equivalently, $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ is freely generated by $\{W^i | i \geq 1\}$ and has graded character

$$\chi(\mathcal{W}^{\mathfrak{sp}}_{\infty}, q) = \sum_{n=0}^{\infty} rank_R(\mathcal{W}^{\mathfrak{sp}}_{\infty}[n]) q^n = \prod_{i,j,l=1}^{\infty} \frac{1}{(1-q^{2i+2j+1})^3(1-q^{2i+2l+2})}. \tag{8.64}$$

For any prime ideal $I \subseteq R$, $\mathcal{W}^{\mathfrak{sp},I}$ is freely generated by $\{W^i | i \geq 1\}$ as a vertex algebra over R/I and

$$\chi(\mathcal{W}^{\mathfrak{sp},I}, q) = \sum_{n=0}^{\infty} rank_{R/I}(\mathcal{W}^{\mathfrak{sp},I}[n]) q^n = \prod_{i,j,l=1}^{\infty} \frac{1}{(1-q^{2i+2j+1})^3(1-q^{2i+2l+2})}.$$

For any localization $S = (D^{-1}R)/I$ along a multiplicatively closed set $S \subseteq R/I$, $\mathcal{W}^{\mathfrak{sp},I}_R$ is freely generated by $\{W^i | i \geq 1\}$ and

$$\chi(\mathcal{W}^{\mathfrak{sp},I}_R, q) = \sum_{n=0}^{\infty} rank_S(\mathcal{W}^{\mathfrak{sp},I}_R[n]) q^n = \prod_{i,j,l=1}^{\infty} \frac{1}{(1-q^{2i+2j+1})^3(1-q^{2i+2l+2})}.$$

Proof. If some Jacobi identity among generators $\{W^i | i \geq 1\}$ does not hold as a consequence of $(3.11 - 3.14)$, there would be a null vector of weight N in $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ for some N. Then $\text{rank}_R(\mathcal{W}^{\mathfrak{sp}}_{\infty}[N])$ would be smaller than that given by (8.64), and the same would hold in any quotient of $\mathcal{W}^{\mathfrak{sp}}_{\infty}[N]$, as well as any localization of such a quotient. But since $\mathcal{C}^k(N)$ is a localization of such a quotient and generated of type $\mathcal{W}(1^3, 2, 3^3, 4, \dots, M)$ for $M \geq 2N^2 + 3N + 1$, this is impossible. $\square$

Corollary 40. The vertex algebra $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ is simple.

Proof. If $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ is not simple, it would have a singular vector in some weight N. Let $p \in R$ be an irreducible polynomial and let ideal $I = (p) \subset R$. By rescaling if necessary, we can assume without loss of generality that ω is not divisible by p, and hence descends to a nontrivial singular vector in $\mathcal{W}^{\mathfrak{sp},I}$. Then for any localization R of R/I, the simple quotient of $\mathcal{W}^{\mathfrak{sp},I}_R$ would have a smaller weight N submodule than $\mathcal{W}^{\mathfrak{sp},I}_R$ for all such I. This contradicts that $\mathcal{C}^k(N)$ is quotient. □

Corollary 41. The vertex algebra $\mathcal{W}^{\mathfrak{sp}}_\infty$ has full automorphism group $SL_2(\mathbb{C})$.

Proof. Let g be any automorphism of $\mathcal{W}^{\mathfrak{sp}}_\infty$ that fixes weight one fields. By definition, g preserves the Virasoro generator L, and by our assumption it acts as identity on weight one space. Let us consider a general deformation $g(W^4) = a_4 W^4 + \dots$ and $g(X^3) = a_3 X^3 + \dots$, where the omitted terms are normally ordered products in $X^1, Y^1, H^1, X^3, H^3, Y^3$ and that transform as trivial and adjoint $\mathfrak{sp}_2$-module. Imposing conditions that fields $g(X^3)$ and $g(W^4)$ be primary with respect to Virasoro element L, and that raising property holds

$$g(W^4)(z)X^1(w) \sim g(X^3)(z-w)^{-2} + \partial g(X^3)(z-w)^{-1},$$

we find the solution $g(W^4) = a_4 W^4$ and $g(X^3) = a_{1,3} X^3$. Finally, the following vertex algebra products must be respected

$$\begin{aligned} g(W^4_{(7)}W^4) =& g(W^4)_{(7)}g(W^4),\\ g(H^3_{(5)}H^3) =& g(H^3)_{(5)}g(H^3),\\ g(W^4_{(3)}W^4) =& g(W^4)_{(3)}g(W^4). \end{aligned} \tag{8.65}$$

Solving above constraints fixes automorphism g to be the identity map. □

8.6 Poisson Vertex Algebra Structure

Recall the Li filtration (8.4). There is a differential ∂ on $\mathrm{gr}^F(\mathcal{V})$ defined by

$$\partial(\sigma_p(\alpha)) = \sigma_{p+1}(\partial\alpha), \quad \alpha \in F^p.$$

Finally, $\mathrm{gr}^F(\mathcal{W}^{\mathfrak{sp}}_\infty)$ has a Poisson vertex algebra structure [59]; for $n \geq 0$, $\alpha \in F^p$ and $\beta \in F^p$, define

$$\sigma_p(\alpha)_{(n)}\sigma_q(\beta) = \sigma_{p+q-n}(\alpha_{(n)}\beta).$$

The subalgebra $F^0(\mathcal{W}^{\mathfrak{sp}}_\infty)/F^1(\mathcal{W}^{\mathfrak{sp}}_\infty)$ is isomorphic to Zhu's commutative algebra $C(\mathcal{W}^{\mathfrak{sp}}_\infty)$, and is known to generate $\mathrm{gr}^F(\mathcal{W}^{\mathfrak{sp}}_\infty)$ as a differential graded algebra. We change our notation slightly and denote by $\overline{\alpha}$ the image of α in $C(\mathcal{V})$. It is a Poisson algebra with product $\overline{\alpha} \cdot \overline{\beta} = \overline{\alpha_{(-1)}\beta}$ and Poisson bracket $\{\overline{\alpha}, \overline{\beta}\} = \overline{\alpha_{(0)}\beta}$.

Unlike the universal objects $\mathcal{W}_\infty$ and $\mathcal{W}^{\mathrm{ev}}_\infty$, the Poisson structure is nontrivial.

Theorem 42. The Poisson algebra $C(\mathcal{W}^{\mathfrak{sp}}_\infty)$ is nontrivial. Specifically, we have

$$\{W^{2i-1}, W^{2j-1}\} = \epsilon_2^{2,2}\frac{(2i-1)!!(2j-1)!!}{(2i+2j-3)!!}W^{2j+2i-3} + \cdots . \tag{8.66}$$

Moreover it is defined over the ring R (8.31).

Proof. Note that passing from the universal 2-parameter VOA $\mathcal{W}^{\mathfrak{sp}}_\infty$ to its C_2 algebra allows us to conclude that there at most two parameters. To see that the C_2 algebra is defined over R, consider a truncation of $\mathcal{W}^{\mathfrak{sp}}_\infty$ along the curve defined by relation (8.59). Examining OPEs in this quotient, it is readily seen that level parameter k arises nontrvially in the Poisson bracket, and that the Poisson algebras are not isomorphic for generic values of k. $\square$

8.7 Quotients by Maximal Ideals of $\mathcal{W}^{\mathfrak{sp}}_\infty$

So far, we have considered quotients of the form $\mathcal{W}^{\mathfrak{sp},I}_R$ which are 1-parameter vertex algebras in the sense that R has Krull dimension 1. Here, we consider simple quotients of $\mathcal{W}^{\mathfrak{sp},I}$ where $I \subseteq R$ is a maximal ideal. Such an ideal always has the form $I = (c - c_0, k - k_0)$ for $c_0, k_0 \in \mathbb{C}$, and $\mathcal{W}^{\mathfrak{sp},I}$ is a vertex algebra over $\mathbb{C}$. We first need a criterion for when the simple quotients of two such vertex algebras are isomorphic.

Theorem 43. Let c_0, c_1, k_0, k_1 be complex numbers in the complement of D, see (8.30), and let

$$I_0 = (c - c_0, k - k_0), \quad I_1 = (c - c_1, k - k_1)$$

be the corresponding maximal ideals in R. Let $\mathcal{W}_0$ and $\mathcal{W}_1$ be the simple quotients of $\mathcal{W}^{\mathfrak{sp}, I_0}$ and $\mathcal{W}_R^{\mathfrak{sp}, I_1}$, respectively. Then $\mathcal{W}_0 \simeq \mathcal{W}_1$ are isomorphic if and only if $c_0 = c_1$ and $k_0 = k_1$.

Proof. Since parameters c and k arise as the central charge of the Virasoro field L and level of $V^k(\mathfrak{sp}_2)$, it follows that if $\mathcal{W}_0 \simeq \mathcal{W}_1$, then necessarily $c_0 = c_1$ and $k_0 = k_1$. $\square$

Corollary 44. Let $I = (p)$ and $J = (q)$ be prime ideals in R which lie in the Shapovalov spectrum of $\mathcal{W}_\infty^{\mathfrak{sp}}$. Then any pointwise coincidences between the simple quotients of $\mathcal{W}^{\mathfrak{sp}, I}$ and $\mathcal{W}^{\mathfrak{sp}, J}$ must correspond to intersection points of the truncation curves $V(I) \cap V(J)$.

Corollary 45. Suppose that $\mathcal{A}$ is a simple, 1-parameter vertex algebra which is isomorphic to the simple quotient of $\mathcal{W}^{\mathfrak{sp}, I}$ for some prime ideal $I \subseteq R$, possibly after localization. Then if $\mathcal{A}$ is the quotient of $\mathcal{W}_\infty^{\mathfrak{sp}, I}$ for some prime ideal J, possibly localized, we must have $I = J$.

Proof. This is immediate from Theorem (43) and Corollary (44), since if I and J are distinct prime ideals, their truncation curves $V(I)$ and $V(J)$ can intersect in at most finitely many points. The simple quotients of $\mathcal{W}_\infty^{\mathfrak{sp}, I}$ and $\mathcal{W}_\infty^{\mathfrak{sp}, J}$ therefore cannot coincide as one-parameter families. $\square$

Chapter 9: Reconstruction

Recall that we have constructed $\mathcal{W}_{\infty}^{\mathfrak{sp}}$ under the hypothesis that it is weakly generated by the fields in weights at most 4. In Section 6, we introduced 8 families of 1-parameter vertex algebras of this generating which are analogues of Y-algebras, and in Section 7 we introduced 4 additional families for which the level k is constant. We expect that these 12 families all arise as 1-parameter quotients of $\mathcal{W}_{\infty}^{\mathfrak{sp}}$, but in order to prove this it is necessary to prove that they satisfy the above weak generation hypothesis. In the setting of $\mathcal{W}_{\infty}$ and $\mathcal{W}_{\infty}^{\mathrm{ev}}$, the same problem arose, and the proof involved finding infinitely many intersection points between the truncation curves for these algebras and the truncation curves for $\mathcal{W}^k(\mathfrak{sl}_m)$ and $\mathcal{W}^k(\mathfrak{sp}_{2m})$ where the weak generation property was known for all noncritical levels. In our case, we do not have such a family where we know the weak generation property for all noncritical levels, so the problem is more difficult. However, to prove the weak generation property for a 1-parameter vertex algebra of this type, it suffices to show that it holds at just one point, and this could be the point at infinity.

For example, Proposition 37 shows that $\mathcal{H}(4m)^{\mathrm{Sp}_{4m}}$ is weakly generated by the fields in weights 2 and 3, so $\mathcal{H}(3) \otimes \mathcal{H}(4m)^{\mathrm{Sp}_{2m}}$, which is the large level limit of the coset $\mathrm{Com}(V^k(\mathfrak{sp}_{2m}), V^k(\mathfrak{sp}_{2m+2}))$, is weakly generated by the fields in weights $1, 2, 3$. It follows that for generic values of k, $\mathrm{Com}(V^k(\mathfrak{sp}_{2m}), V^k(\mathfrak{sp}_{2m+2}))$ has the weak generation property ([7], Lem. 2.5), or equivalently, it holds as a 1-parameter vertex algebra. By a similar argument involving a free field limit which we omit, the same can be shown for the following examples.

1. $\mathrm{Com}(V^k(\mathfrak{so}_{2m}), V^k(\mathfrak{osp}_{2m|2}))^{\mathbb{Z}_2}$ (Case BB),

2. $\mathrm{Com}(V^k(\mathfrak{so}_{2m+1}), V^k(\mathfrak{osp}_{2m+1|2}))^{\mathbb{Z}_2}$ (Case BD),

3. $\mathrm{Com}(V^\ell(\mathfrak{sp}_{2n}), V^{\ell-1}(\mathfrak{sp}_{2n}) \otimes \mathcal{E}(2n))$,

4. $\mathrm{Com}(V^\ell(\mathfrak{so}_n), V^{\ell-2}(\mathfrak{sp}_n) \otimes \mathcal{S}(n))^{\mathbb{Z}_2}$,

5. $\mathrm{Com}(V^\ell(\mathfrak{osp}_{1|2n}), V^{\ell-1}(\mathfrak{osp}_{1|2n}) \otimes \mathcal{S}(1) \otimes \mathcal{E}(2n))^{\mathbb{Z}_2}$.

We would like to also explain why we expect the weak generation property to holds for many simple quotients. Consider Y-algebra $\mathcal{C}^\psi_{XY}(n,m)$, and fix some integer $k \geq m+3$. As we have seen, the affine coset $\mathcal{C}^\ell(k)$ has the weak generation generically. From this we can deduce that it holds for all but finitely many levels ℓ. At the point where $c = c_{XY}(k)$, we have an isomorphism between the simple quotients of the subalgebras of $\mathcal{C}^\psi_{XY}(n,m)$ and $\mathcal{C}^\ell(k)$, generated by the fields of weight at most 4. It follows from Weyl's theorem that the first relation among the generators of $\mathcal{C}^\ell(k)$ for generic ℓ occurs at weight $k+1$, and therefore this holds for all but finitely many values of ℓ. We conclude that for all but finitely many values of n, $\mathcal{C}^\psi_{XY}(n,m)$ has the property that the subagebra generated by the fields in weight at most 4 contains all the generators up to weight $m+2$. Finally, in the large level limit, the generators of weight up to $m+2$ suffice to weakly generate the algebra. Therefore it follows that for all but finitely many values of n, $\mathcal{C}_{XY}(n,m)$ has the weak generation property.

Finally, we mention the last method that can be used to prove this conjecture in full generality; it consists of two steps. First, we prove the weak generation property of Y-algebras associated with minimal $\mathcal{W}$-algebras of types B (Case $\mathcal{W}_{CB}(n,1)$) and D (Case $\mathcal{W}_{CD}(n,1)$), for all levels above the critical level. Next, using intersection points of the truncations curves arguments, as above, we deduce weak generation property of all Y-algebras of type C. If we are successful, it will appear in our follow-up paper [25].

Conjecture 9.0.1. *Y-algebras of type C are weakly generated by fields of weight at most 4.*

Assume this conjecture; alternatively, we may assume that we are only working with Y-algebras for which this property holds. Under this hypothesis, by Theorem (36) all Y-algebras of type C and the 4 $\mathfrak{sp}_2$-level k constant families arise as 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$. So we use the same notation for the generators of $\mathcal{W}$. Specifically, we have fields of odd weight $\{X^{2i-1}, Y^{2i-1}, H^{2i-1} | i \geq 1\}$ and fields of even weight $\{W^{2i} | i \geq 1\}$, which we refer to uniformly as W^n, see (8.9).

A consequence of Lemmas (13-20, 21-23) and Theorem (12) is the following.

Proposition 46. Let X stand for B or C, and Y for B, C, D or O.

1. The Y-algebra $\mathcal{C}^{\psi}_{XY}(n,m)$ is simple for generic values of ψ.

2. We may replace fields $W^1, W^2, W^3, \dots, W^{2m}$ in strong generating set of the $\mathfrak{sp}_2$-rectangular algebra $\mathcal{W}^{\psi}_{CY}(n,m)$ with fields $\omega^1, \omega^2, \omega^3, \dots, \omega^{2m} \in \mathcal{C}^{\psi}_{CY}(n,m)$. Similarly, we may replace fields $W^1, W^2, W^3, \dots, W^{2m+1}$ in strong generating set of $\mathcal{W}^{\psi}_{BY}(n,m)$ with fields $\omega^1, \omega^2, \omega^3, \dots, \omega^{2m+1} \in \mathcal{C}^{\psi}_{BY}(n,m)$.

3. Let $U \cong \mathbb{C}^2 \otimes \rho_{\mathfrak{a}}$, where $\rho_{\mathfrak{a}}$ is the standard representation of $\mathfrak{a}$. It is spanned by $P^{1,j}, P^{-1,j}$, where j runs over a basis of $\rho_{\mathfrak{a}}$. Then U has a supersymmetric bilinear form

$$\langle\ ,\ \rangle : U \to \mathbb{C}, \qquad \begin{cases} \langle a,b\rangle = a_{(2m)}b, & X = C, \\ \langle a,b\rangle = a_{(2m+1)}b & X = B. \end{cases}$$

This form is nondegenerate and coincides with the standard pairing on $\mathbb{C}^2 \otimes \rho_{\mathfrak{a}}$. Hence, without loss of generality, we may normalize the fields in U as

$$P^{\mu,i}(z)P^{\nu,j}(w) \sim \begin{cases} \delta_{i,j}\delta_{\mu+\nu,0}\mathbf{1}(z-w)^{-2m-1} + \cdots, & X = C, \\ \delta_{i,j}\delta_{\mu+\nu,0}\mathbf{1}(z-w)^{-2m-2} + \cdots, & X = B. \end{cases} \tag{9.1}$$

Here, in addition to strong generators of $V^a(\mathfrak{a})$, the remaining terms depend only on $\omega^1, \omega^2, \omega^3, \dots, \omega^{2m}$ if $X = C$, and on $\omega^1, \omega^2, \omega^3, \dots, \omega^{2m+1}$ if $X = B$.

Recall the $\mathfrak{sp}_2$-rectangular $\mathcal{W}$-algebra $\mathcal{W}^{\psi}_{XY}(n,m)$ from Section 6 whose coset by $V^a(\mathfrak{a})$ is the Y-algebra $\mathcal{C}^{\psi}_{XY}(n,m)$, where $\mathfrak{a}$-level a is determined in terms of $\mathfrak{sp}_2$-level k, as in the table (6.1). The main result of this section is the reconstruction theorem (47), which states that the full OPE algebra of $\mathcal{W}^{\psi}_{XY}(n,m)$ is determined by the action of $\mathfrak{sp}_2 \oplus \mathfrak{a}$ on the generating fields.

We would like to handle cases when the nilpotent is of type B and C in a uniform way. Above, m is the rank of the Lie algebra in which the nilpotent element is principal, and it enters our procedure as a degree of the leading pole in (9.1). Therefore we adopt the following convention. In what follows, let m be the conformal weight of the extension fields, which is either an integer or half-integer.

- If $m \geq 1$ is an integer, then nilpotent is principal in $\mathfrak{so}_{2m-1}$.
- If $m \geq \frac{3}{2}$ is a half-integer, then nilpotent is principal in $\mathfrak{sp}_{2m-2}$. If $m = \frac{1}{2}$ then we are in the case of diagonal cosets discussed in Section 7.

Theorem 47. Let $\mathcal{A}^{\psi}_{XY}(n,m)$ be a simple 1-parameter vertex (super)algebra with the following properties, which are shared with $\mathcal{W}^{\psi}_{XY}(n,m)$.

1. $\mathcal{A}^{\psi}_{XY}(n,m)$ is a conformal extension of $\mathcal{W} \otimes V^a(\mathfrak{a})$, where $\mathcal{W}$ is some 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$.

2. The extension is generated by fields $P^{\mu,j}$ in weight m which are primary for the conformal vector $L + L^{\mathfrak{a}}$ of $\mathcal{A}^{\psi}_{XY}(n,m)$, primary for the affine subVOA $V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{a})$, and transform as $\mathbb{C}^2 \otimes \rho_{\mathfrak{a}}$ under $\mathfrak{sp}_2 \oplus \mathfrak{a}$.

3. The extension fields have the same parity as the corresponding fields of $\mathcal{W}^{\psi}_{XY}(n,m)$ appearing in Table (6.1). Algebra $\mathcal{A}^{\psi}_{XY}(n,m)$ is strongly generated by these fields, together with the generators of $\mathcal{W} \otimes V^a(\mathfrak{a})$.

4. The restriction of the Shapovalov form to the extension fields is non-degenerate.

Then $\mathcal{A}^{\psi}_{XY}(n,m)$ is isomorphic to $\mathcal{W}^{\psi}_{XY}(n,m)$ as 1-parameter vertex (super)algebras.

Our strategy is similar to the one used in [20] and [21], and consists of 3 steps.

1. We first show that the existence of an extension $\mathcal{A}^{\psi}_{XY}(n,m)$ satisfying (47) uniquely determines the truncation curve expressing $\mathcal{W}$ as a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$. It can be uniformly expressed in the form

$$c\,(4\lambda(k+2)+(k+1)(2k+1)) = 4k\lambda\,(4\lambda(k+1)(k+2)-(2k+1)(2k+3))\,, \tag{9.2}$$

where λ is given by (9.3)

2. We express the OPEs $W^3(z)P^{\mu,j}(w)$ for all $P^{\mu,j} \in U$, in terms of $\mathfrak{sp}_2$-level k.

3. We argue that the all OPEs of $\mathcal{A}^{\psi}_{XY}(n,m)$ are determined from OPEs of $\mathcal{W}^{\mathfrak{sp}}_{\infty}$ together with $W^3(z)P^{\mu,j}(w)$ for all $P^{\mu,j} \in U$.

9.1 Set-up

Let $\{q^{\alpha}|\alpha \in S\}$ denote the basis of $\mathfrak{a}$, and $X^{\alpha}(z)$ be the corresponding fields. We use generators $\{\tilde{W}^i|i \geq 1\}$ as in (8.2), to cast our assumptions (47) in the OPE form.

1. $V^a(\mathfrak{a})$ is affine subalgebra:

$$X^\alpha(z)X^\beta(w) \sim a(q^\alpha|q^\beta)\mathbf{1}(z-w)^{-2} + (\sum_{\gamma\in S} f_\gamma^{\alpha,\beta}X^\gamma)(w)(z-w)^{-1}.$$

2. $V^a(\mathfrak{a})$ commutes with $\mathcal{W}$:

$$\tilde{W}^n(z)X^\alpha(w) \sim 0, \quad q^\alpha \in \mathfrak{a}, \quad n \geq 1.$$

3. Fields $P^{\mu,j}$ are primary for $V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{a})$ and transform as the $\mathbb{C}^2 \otimes \rho_\mathfrak{a}$:

$$X^\alpha(z)P^{\mu,j}(w) \sim (\rho_{\mathfrak{sp}_2\oplus\mathfrak{a}}(q^\alpha)P^{\mu,j})(w)(z-w)^{-1}, \quad q^\alpha \in \mathfrak{sp}_2 \oplus \mathfrak{a}.$$

4. In $\mathcal{W} \otimes V^a(\mathfrak{a})$ the total Virasoro field is $T = \tilde{L} + L^{\mathfrak{sp}_2} + L^\mathfrak{a}$, so the OPEs of extension fields L are

$$\tilde{L}(z)P^{\mu,j}(w) \sim \lambda P^{\mu,j}(w)(z-w)^{-2} + (\partial P^{\mu,j} - L^{\mathfrak{sp}_2}_{(0)}P^{\mu,j} - L^{\mathfrak{a}}_{(0)}P^{\mu,j})(w)(z-w)^{-1}$$

where constant

$$\lambda = m - \frac{3}{4(k+2)} - \frac{\mathrm{Cas}}{a + h^\vee_\mathfrak{a}}. \tag{9.3}$$

Here Cas is the eigenvalue of the Casimir in $U(\mathfrak{a})$ of the standard representation $\rho_\mathfrak{a}$.

5. Since $\mathcal{A}$ is nondegenerate, so we can renormalize fields $P^{\mu,j}$ so that (9.1) holds.

Next, we proceed to set-up OPEs among the generators of $\mathcal{W}$ and extension fields $P^{\mu,j}$.

For our computation we need only a few structure constants, defined in the following OPEs.

$$\begin{aligned}\tilde{W}^{2n}(z)P^{-1,1}(w) \sim & p_0^{2n}P^{-1,1}(w)(z-w)^{-2m} + (p_1^{2n}\partial P^{-1,1} + \cdots)(w)(z-w)^{-2n+1} \\ & + (p_{1,1}^{2n}\partial^2 P^{-1,1} + p_2^{2n} :LP^{-1,1}: + \cdots)(w)(z-w)^{-2m+2} + \cdots , \\ \tilde{X}^{2n-1}(z)P^{-1,1}(w) \sim & p_0^{2n-1}P^{1,1}(w)(z-w)^{-2m+1} \\ & + (p_1^{2n-1}\partial P^{1,1} + a^{2n-1}L_{(0)}^{\mathfrak{sp}_2}P^{1,1} + b^{2n-1}L_{(0)}^{\mathfrak{a}}P^{1,1})(w)(z-w)^{-2m+2} \\ & + (p_{1,1}^{2n-1}\partial^2 P^{1,1} + p_2^{2n} :LP^{1,1}: + \dots)(w)(z-w)^{-2m+3} + \cdots .\end{aligned} \tag{9.4}$$

From (9.4), using Jacobi identities $J_{0,r}(X^\alpha, W^n, P^{-1,1})$ for $X^\alpha \in V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{a})$ one can determine all OPEs $W^n(z)P^{\mu,i}(w)$ for $P^{\mu,j} \in U$. Finally, we posit that OPEs among the extension fields of weight m have the most general form that is compatible with the conformal weight grading and $\mathfrak{sp}_2 \oplus \mathfrak{a}$-symmetry.

9.2 Step 1: Truncation

Computation of truncation curve amounts to the imposistion of several Jacobi identities. We will arrive at system consisting of a quadratic, cubic and quartic equations. Solving it, we will obtain the formula (9.2).

First, we impose conformal symmetry thanks to the Jacobi identities $J(L, W^3, P^{-1,1})$. For the purposes of evaluating the truncation curve, only the structure constants displayed in (9.4) are relevant. We find the following expressions.

$$\begin{aligned}
p^3_1 =&\frac{(3k+4)p^3_0}{2(k+2)\lambda},\\
p^3_{1,1} =&\frac{(3k+4)(2ck+3c+4\lambda k)p^3_0}{2(k+2)^2\left(2c\lambda+c+16\lambda^2-10\lambda\right)\lambda},\\
p^3_2 =&\frac{2(3k+4)(4\lambda k+8\lambda-4k-5)p^3_0}{(k+2)^2\left(2c\lambda+c+16\lambda^2-10\lambda\right)},\\
a^3_1 =&\frac{\left(4\lambda k^2-3k^2+8\lambda k-k+4\right)p^3_0}{4(k-1)(k+2)^2\lambda},\\
b^3_1 =&\frac{\left(2\lambda k^2-3k^2-k-8\lambda+4\right)p^3_0}{2(k-1)(k+2)^2\lambda}.
\end{aligned} \tag{9.5}$$

Next, we impose the Jacobi identities $J_{2,1}(X^3, H^3, P^{-1,1})$, $J_{1,3}(H^3, H^3, P^{-1,1})$ and $J_{2,4}(W^4, W^4, P^{-1,1})$ to determine the relevant structure constants (9.4) arising in the OPEs $W^4(z)P^{-1,1}(w)$, $W^5(z)P^{-1,1}(w)$ and $W^6(z)P^{-1,1}(w)$ in terms of variables p^3_0 and c, k. Extracting the coefficients of $:X^1P^{1,1}:$, $P^{1,1}$ and $\partial^2P^{1,1}$ in identities $J_{2,1}(X^3, X^3, P^{-1,1})$, $J_{3,2}(X^3, W^4, P^{-1,1})$ and $J_{1,3}(W^4, W^4, P^{-1,1})$ we obtain a quadratic, cubic and quartic equations in variables c, k and p^3_0 , respectively. Finally, assuming $p^3_0 \neq 0$, and solving this system we uniquely determines c and p^3_0 as functions of k.

$$\begin{aligned}
c =&4k\lambda\frac{4\lambda(k+1)(k+2)-(2k+1)(2k+3)}{4\lambda(k+2)+(k+1)(2k+1)},\\
p^3_0 =&\frac{2\lambda(k-1)(k+1)(k+2)^2(2k+1)(4\lambda(k+2)+2k+1)}{(3k+4)\left(4\lambda(k+2)+(k+1)(2k+1)\right)}.
\end{aligned} \tag{9.6}$$

In particular, this proves part (1) of Theorem (47). Moreover, we have the following.

Lemma 48. The OPEs $W^3(z)P^{\mu,j}(w)$ for $P^{\mu,j} \in U$ are expressed as rational functions of the $\mathfrak{sp}_2$-level k.

Remark 9.2.1. *Substituting the values of λ as in (9.3), we recover the formulae for the central charges obtained in (6.9-6.12).*

Remark 9.2.2. *The truncation curve depends on the extension data only via the parameter λ. Similar feature is exhibited by $\mathcal{W}_\infty$ and $\mathcal{W}_\infty^{ev}$ algebras.*

9.3 Step 2: Reconstruction

By Lemma 48, OPEs $W^3(z)P^{\mu,j}(w)$ for $P^{\mu,j} \in U$ are fully determined in terms of the $\mathfrak{sp}_2$-level k. In the following we show that this property propagates to all OPEs $W^n(z)P(w)$ for $n \geq 4$.

Lemma 49. Let $n \geq 3$. Then OPEs $W^n(z)P^{\mu,j}(w)$ for $P^{\mu,j} \in U$, are uniquely determined by $W^3(z)P^{\nu,i}(w)$ for $P^{\nu,i} \in U$ and OPEs of $\mathcal{W}$.

Proof. We proceed by induction, with our base case the Lemma (48). Inductively, assume that OPEs $W^i(z)P^{\mu,j}(w)$ for $P^{\mu,j} \in U$ have been expressed in terms of k. Imposing the Jacobi identity $J_{r,1}(W^3, W^n, P^{\mu,j})$ we obtain a relation

$$W^{n+1}_{(r)}P^{\mu,j} = \frac{1}{\epsilon^{2,\bar{n}}_{\bar{n}}} \frac{1}{rv^{3,n}_1 - (r+1)v^{3,n}_0} \left(W^4_{(r)}W^n_{(1)}P^{\mu,j} - W^n_{(1)}W^4_{(r)}P^{\mu,j} - R^{3,n}_{r,1}\right), \quad (9.7)$$

where

$$R^{3,n}_{r,1} = (W^{4,n}_{\bar{n}}(0))_{(r+1)}P^{\mu,j} - \sum_{j=2}^{r} \binom{r}{j} (W^4_{(j)}W^i)_{(r+1-j)}P^{\mu,j}. \quad (9.8)$$

By induction, the right side of (9.7) is known, and this completes the proof. □

Let $i, j \in \{1, \ldots, \dim(\rho_\mathfrak{a})\}$ be fixed. We have the following.

Lemma 50. OPEs $P^{\mu,i}(z)P^{\nu,j}(w)$ are fully determined from $W^n(z)P^{\alpha,l}(w)$ for $P^{\alpha,l} \in U$, and OPEs of $\mathcal{W}$.

Proof. We proceed inductively to determine $P^{\mu,i}_{(r)}P^{\nu,j}$ for all $1 \leq r \leq 2m-1$. Our base case is $r = 2m-1$, which is known by assumption (9.1). Inductively, assume that all products $\{P^{\alpha,a}_{(s)}P^{\beta,b} | \alpha, \beta \in \{-1,1\}, a, b \in \{1, \ldots, \dim(\rho_\mathfrak{a})\}, s > r\}$ are expressed in terms the

$\mathfrak{sp}_2$-level k. Consider the Jacobi identity $J_{1,r}(X^3, P^{-1,i}, P^{1,j})$, which reads as follows.

$$X^3_{(1)} P^{-1,i}_{(r)} P^{1,j} = (X^3_{(1)} P^{-1,i})_{(r)} P^{1,j} + (X^3_{(0)} P^{-1,i})_{(r+1)} P^{1,j} + P^{-1,i}_{(r)} X^3_{(1)} P^{1,j}. \tag{9.9}$$

Note that the left side of relation (9.9) is known by induction, and only the right side gives rise to a contribution of the desired product $P^{-1,i}_{(r-1)} P^{-1,j}$. Each such contribution arises from monomials in the products $X^3_{(1)} P^{-1,i}$, $X^3_{(0)} P^{-1,i}$ and $X^3_{(1)} P^{1,i}$. Upon collecting every contribution of the desired product $P^i_{(r-1)} P^j$, we may express it in terms of inductively data. □

Assuming the Conjecture (9.0.1), this completes the proof of Theorem (47).

Remark 9.3.1. *The Jacobi identities $J_{1,0}(W^3, P^{\mu,i}, P^{\nu,j})$ gives rise to relations expressing strong generators W^n for $n \geq m$, in terms of extension fields.*

Remark 9.3.2. *Note that for weight 1 and 2 fields, the Jacobi identities $J(W^1, P^{\mu,i}, P^{\nu,j})$ and $J(L, P^{\mu,i}, P^{\nu,j})$ express the conformal and affine symmetries. More generally, each generator W^n of $\mathcal{W}^{\mathfrak{sp}}_\infty$ the Jacobi identity $J(W^n, P^{\mu,i}, P^{\nu,j})$ gives rise to an family of recursions among the products $P^{\mu,i}_{(r)} P^{\nu,j}$. In this sense, $\mathfrak{sp}_2$-rectangular $\mathcal{W}$-algebras with a tail possesses a larger symmetry, namely the universal 2-parameter VOA $\mathcal{W}^{\mathfrak{sp}}_\infty$.*

BIBLIOGRAPHY

[1] L. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys., 91(2):167-197, 2010.

[2] M. Aganagic, E. Frenkel, and A. Okounkov, *Quantum q-Langlands correspondence*, Trans. Moscow Math. Soc. 79 (2018).

[3] T. Arakawa, T. Creutzig, and A. Linshaw, *$\mathcal{W}$-algebras as coset vertex algebras*, Invent. Math. 218, no. 1 (2019), 145-195.

[4] T. Arakawa, *Chiral algebras of class $\mathcal{S}$ and Moore-Tachikawa symplectic varieties*, arXiv:1811.01577.

[5] T. Arakawa and E. Frenkel, *Quantum Langlands duality of representations of $\mathcal{W}$-algebras*, Compos. Math. 155, no. 12 (2019), 2235-2262.

[6] T. Arakawa, T. Creutzig, K. Kawasetsu and A. R. Linshaw, *Orbifolds and cosets of minimal $\mathcal{W}$-algebras*, Communications in Mathematical Physics, 355, 339-372, 2017.

[7] M. Al-Ali and A. R. Linshaw, *Orbifolds of Gaiotto-Rapcak Y-algebras*, Journal of Algebra, 625, 1-27, 2023.

[8] A. R. Linshaw and B. Song, *Cosets of free field algebras via arc spaces*. International Mathematics Research Notices, 2024(1), 47-114.

[9] A. A. Belavin, *KdV-Type Equations and W-Algebras*, Integrable Systems in Quantum Field Theory. Adv. Stud. Pure Math. 19, Academic Press, San Diego, 1989, Pages 117-125.

[10] F. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, *Coset construction for extended Virasoro algebras*, Nuclear Phys. B , 304(2) (1988), 371-391.

[11] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Physics B, 241(2) (1984), 333-380.

[12] A. Braverman, M. Finkelberg, and H. Nakajima, *Instanton moduli spaces and $\mathcal{W}$-algebras*, Astérisque, (385):vii+128, 2016.

[13] B. Bakalov and T. Milanov, *$\mathcal{W}$-constraints for the total descendant potential of a simple singularity*, Compos. Math. 149 (2013), no. 5, 840-888.

[14] R. Borcherds, *Vertex operator algebras, Kac-Moody algebras and the Monster*, PNAS 83 (1986) 3068.

[15] C. Candu and C. Vollenweider, The $\mathcal{N} = 1$ algebra $\mathcal{W}_\infty[\mu]$ and its truncations, JHEP 11 (2013), 032.

[16] Collinwood and D. H., MacGovern, W. M., *Nilpotent orbits in semisimple Lie algebra*, Van Nostrand Reinhold Mathematics Series, 1992.

[17] T. Creutzig and D. Gaiotto, *Vertex Algebras for S-duality*, Comm. Math. Phys.379(2020), no.3, 785-845.

[18] T. Creutzig and Y. Hikida, *Rectangular $\mathcal{W}$-algebras, extended higher spin gravity and dual coset CFTs*, JHEP 147 (2019).

[19] T. Creutzig and A. R. Linshaw, *Cosets of affine vertex algebras inside larger structures*, Journal of Algebra, 517, 396-438 (2019).

[20] T. Creutzig and A. R. Linshaw, *Trialities of $\mathcal{W}$-algebras*, Camb. J. Math. vol. 10, no. 1 (2022), 69-194.

[21] T. Creutzig and A. R. Linshaw, *Trialities of orthosymplectic $\mathcal{W}$-algebras*, Adv. Math. 409 (2022), 108678, 79pp.

[22] T. Creutzig, J. Fasquel, A. R. Linshaw and S. Nakatsuka, *On the structure of $\mathcal{W}$-algebras in type A*, arXiv:2403.08212.

[23] T. Creutzig, N. Genra, and A. R. Linshaw, *Category $\mathcal{O}$ for vertex algebras of $\mathfrak{osp}_{1|2n}$*, arXiv:2203.08188.

[24] T. Creutzig, V. Kovalchuk, and A. R. Linshaw, *Generalized parafermions of orthogonal type*, Journal of Algebra 593 (2022), 178-192.

[25] T. Creutzig, V. Kovalchuk, and A. R. Linshaw, *Building blocks of orthosymplectic $\mathcal{W}$-algebras*, in prepration.

[26] A. De Sole and V. Kac, *Freely generated vertex algebras and non-linear Lie conformal algebras*, Comm. Math. Phys. 254 (2005), no. 3, 659-694.

[27] C. Dong, H. Li, and G. Mason, *Compact automorphism groups of vertex operator algebras*, Internat. Math. Res. Notices (1996), no. 18, 913-921.

[28] V. Drinfeld and V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Current problems in mathematics, Vol. 24, 81-180, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[29] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models*, Comm. Math. Phys. 123 (1989), 485-528.

[30] D. Gaiotto and M. Rapčák, *Vertex Algebras at the Corner*, J. High Energy Phys. 2019, no. 1, 160, front matter+85 pp.

[31] V. Kac, S. Roan, and M. Wakimoto, *Quantum reduction for affine superalgebras*, Comm. Math. Phys. 241 (2003), no. 2-3, 307–342.

[32] V. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10. American Math. Soc., 1998.

[33] P. Goddard, A. Kent, and D. Olive, *Virasoro algebras and coset space models*, Phys. Lett B 152 (1985) 88-93.

[34] D. Gaitsgory, *Quantum Langlands Correspondence*, `arXiv:1601.05279`.

[35] M. Gaberdiel and R. Gopakumar, *An* AdS_3 *Dual for Minimal Model CFTs*, Phys. Rev. D 83 (2011) 066007.

[36] I. Gelfand and L. Dickey, *A family of Hamiltonian structures connected with integrable nonlinear differential equations*, Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No. 136 (1978).

[37] E. Frenkel, *Langlands correspondence for loop groups*, volume 103 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007.

[38] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys and Monographs, Vol. 88, American Math. Soc., 2001.

[39] E. Frenkel and D. Gaiotto, *Quantum Langlands Dualities of Boundary Conditions, D-modules, and conformal blocks*, Commun. Number Theory Phys. 14 (2020), no. 2, 199-313.

[40] E. Frenkel, V. Kac, and M. Wakimoto, *Characters and fusion rules for $\mathcal{W}$-algebras via quantized Drinfeld-Sokolov reduction*, Comm. Math. Phys. 147 (1992), 295–328.

[41] I. B. Frenkel, *Representations of Kac-Moody algebras and dual resonance models*, Applications of group theory in physics and mathematical physics (Chicago, 1982), 325-353, Lectures in Appl. Math., 21, Amer. Math. Soc., Providence, RI, 1985.

[42] I. B. Frenkel, Y-Z Huang, and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. 104 (1993), no. 494, viii+64 pp.

[43] I. B. Frenkel and V. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. 62 (1980), 23–66.

[44] I. B. Frenkel, J. Lepowsky, and A. Meurmann, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.

[45] A. Elashvili, V. Kac, and E. Vinberg, *On exceptional nilpotents in semisimple Lie algebras*, J. Lie Theory 19 (2009), no. 2, 371-390.

[46] V. Fateev and S. Lykyanov, *The models of two-dimensional conformal quantum field theory with $\mathbb{Z}_n$ symmetry*, Internat. J. Modern Phys. A, 3(2) (1988), 507-520.

[47] B. Feigin and E. Frenkel, *Duality in $\mathcal{W}$-algebras*, Int. Math. Res. Notices 1991 (1991), no. 6, 75-82.

[48] B. Feigin and E. Frenkel, *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. B 246 (1990), 75–81.

[49] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, In: “Integrable systems and quantum groups” (Montecatini Terme, 1993), 349–418, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[50] I. B. Frenkel and Y. C. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J, Vol. 66, No. 1, (1992), 123-168.

[51] D. Friedan, E. Martinec, and S. Shenker, *Conformal invariance, supersymmetry and string theory*, Nucl. Phys. B271 (1986) 93-165.

[52] D. Gaiotto, M. Rapčák, and T. Zhou, *Deformed Double Current Algebras, Matrix Extended $\mathcal{W}_\infty$ Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection*, arXiv:2309.16929.

[53] N. Genra and T. Juillard, *Reduction by stages for finite W-algebras*, arXiv:2212.06022.

[54] N. Jacobson, *Completely reducible Lie algebras of linear transformations*, Proc. Amer. Math. Soc., vol. 2, 105–113, 1951.

[55] S. Kanade and A. R. Linshaw, *Universal two-parameter even spin $\mathcal{W}_\infty$-algebra*, Adv. Math. 355 (2019), 106774.

[56] V. Kac, *Lie superalgebras*. Advances in mathematics 26, no. 1, (1977), 8-96.

[57] V. Kac and M. Wakimoto, *Classification of modular invariant representations of affine algebras*, In Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), volume 7 of Adv. Ser. Math. Phys. pages 138–177. World Sci. Publ., Teaneck, NJ, 1989.

[58] V. Kac and M. Wakimoto. *Quantum reduction and representation theory of superconformal algebras*, Advances in Mathematics 185, no. 2 (2004): 400-458.

[59] H. Li, *Vertex algebras and vertex Poisson algebras*, Communications in Contemporary Mathematics, 6(01), 61-110, 2004.

[60] A. R. Linshaw, *Universal two-parameter $\mathcal{W}_\infty$-algebra and vertex algebras of type* $\mathcal{W}(2,3,\dots,N)$, Compos. Math.157, no. 1 (2021), 12-82.

[61] A. R. Linshaw, *Invariant subalgebras of affine vertex algebras*, Advances in Mathematics, 234, pp. 61-84, 2013.

[62] A. R. Linshaw, G. W. Schwarz, and B. Song. *Arc spaces and the vertex algebra commutant problem*, Advances in Mathematics 277 (2015): 338-364.

[63] V. V. Morozov, *On a nilpotent element in a semi-simple Lie algebra*, In CR (Doklady) Acad. Sci. URSS (NS), vol. 36, pg. 83-86, 1942.

[64] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 9. Springer Science, Business Media, 2012.

[65] V. Ostrik, E. Rowell, and M. Sun, *Symplectic level-rank duality via tensor categories*, J. Lie Theory 30 (2020), no. 4, 909-924.

[66] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. 170 (2002), no. 1, 1-55.

[67] T. Procházka, *Exploring $\mathcal{W}_\infty$ in the quadratic basis*, J. High Energy Phys. 2015, no. 9, 116, front matter+62 pp.

[68] T. Procházka, *$\mathcal{W}$-symmetry, topological vertex, and affine Yangian*, J. High Energy Phys. 2016, no. 10, 077, front matter+72 pp.

[69] T. Procházka and Rapčák, *Webs of $\mathcal{W}$-algebras*, J. High Energy Phys. 2018, no. 11, 109, front matter+87 pp.

[70] O. Schiffmann and E. Vasserot, *Cherednik algebras, $\mathcal{W}$-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$*, Publ. Math. Inst. Hautes Etudes Sci. 118 (2013), 213–342.

[71] A. Sergeev, *An analog of the classical invariant theory for Lie superalgebras*, Michigan Math. J. 49 (2001), no. 1, 113-146.

[72] A. Sergeev, *An analog of the classical invariant theory for Lie superalgebras*, Michigan Math. J. 49 (2001), no. 1, 147-168.

[73] K. Thielemans, *A Mathematica package for computeing operator product expansions*, Int. Jour. Mod. Phys. C2 (1991) p.787.

[74] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1946.