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ABSTRACT. We prove a compactness theorem for the dual Gromov-Hausdorff propinquity as a noncommutative analogue of the Gromov compactness theorem for the Gromov-Hausdorff distance. Our theorem is valid for subclasses of quasi-Leibniz quantum compact metric spaces of the closure of finite dimensional quasi-Leibniz quantum compact metric spaces for the dual propinquity. While finding characterizations of this class proves delicate, we show that all nuclear, quasi-diagonal quasi-Leibniz quantum compact metric spaces are limits of finite dimensional quasi-Leibniz quantum compact metric spaces. This result involves a mild extension of the definition of the dual propinquity to quasi-Leibniz quantum compact metric spaces, which is presented in the first part of this paper.

CONTENTS

1. INTRODUCTION

The dual Gromov-Hausdorff propinquity [19, 16, 18] is an analogue of the Gromov-Hausdorff distance [6] defined on a class of noncommutative algebras, called Leibniz quantum compact metric spaces, and seen as a generalization of the algebras of continuous functions over metric spaces. The dual propinquity is designed to provide a framework to extend techniques from metric geometry [7] to noncommutative geometry [4]. In this paper, we prove a generalization of Gromov's compactness theorem to the dual propinquity, and study the related issue of finite dimensional approximations for Leibniz quantum compact metric spaces.

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An important property of the Gromov-Hausdorff distance is a characterization of compact classes of compact metric spaces [6]:

Theorem 1.1 (Gromov's Compactness Theorem). *A class $\mathcal{S}$ of compact metric spaces is totally bounded for the Gromov-Hausdorff distance if, and only if the following two assertions hold:*

1. *there exists $D \geqslant 0$ such that for all $(X, \mathrm{m}) \in \mathcal{S}$, the diameter of (X, m) is less or equal to D,*
2. *there exists a function $G : (0, \infty) \to \mathbb{N}$ such that for every $(X, \mathrm{m}) \in \mathcal{S}$, and for every $\varepsilon > 0$, the smallest number $\mathrm{Cov}_{(X,\mathrm{m})}(\varepsilon)$ of balls of radius ε needed to cover (X, m) is no more than $G(\varepsilon)$.*

Since the Gromov-Hausdorff distance is complete, a class of compact metric spaces is compact for the Gromov-Hausdorff distance if and only if it is closed and totally bounded.

Condition (2) in Theorem (1.1) is meaningful since any compact metric space is totally bounded, which is also equivalent to the statement that compact metric spaces can be approximated by their finite subsets for the Hausdorff distance. Thus, intimately related to the question of extending Theorem (1.1) to the dual propinquity, is the question of finite dimensional approximations of Leibniz quantum compact metric spaces. However, this latter question proves delicate. In order to explore this issue, we are led to work within larger classes of generalized Lipschitz algebras, which we name the quasi-Leibniz quantum compact metric spaces. The advantage of these classes is that it is possible to extend to them the dual propinquity, and then prove that a large class of Leibniz quantum compact metric spaces are limits of finite dimensional quasi-Leibniz quantum compact metric spaces for the dual propinquity. In particular, the closure of many classes of finite dimensional quasi-Leibniz quantum compact metric spaces contain all *nuclear quasi-diagonal* quasi-Leibniz quantum compact metric spaces. The various possible classes of quasi-Leibniz quantum compact metric spaces we introduce represent various degrees of departure from the original Leibniz inequality.

The dual propinquity is our answer to the challenge raised by the quest for a well-behaved analogue of the Gromov-Hausdorff distance designed to work within the C*-algebraic framework in noncommutative metric geometry [3, 4, 24, 25, 37, 27, 26, 28, 29, 30, 31, 33, 32, 34, 35, 36]. Recent research in noncommutative metric geometry suggests, in particular, that one needs a strong tie between the quantum topological structure, provided by a C*-algebra, and the quantum metric structure, if one wishes to study the behavior of C*-algebraic related structures such as projective modules under metric convergence. The quantum metric structure over a C*-algebra $\mathfrak{A}$ is given by a seminorm L defined on a dense subspace $\mathrm{dom}(\mathsf{L})$ of the self-adjoint part of $\mathfrak{A}$ such that:

1. $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
2. the distance on the state space $\mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ defined, for any two $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}) \text{ and } \mathsf{L}(a) \leqslant 1\} \tag{1.1}$$

   metrizes the weak* topology on $\mathscr{S}(\mathfrak{A})$.

Such a seminorm is called a Lip-norm, and the metric defined by Expression (1.1) is called the Monge-Kantorovich metric, by analogy with the classical picture [8, 9]. In particular, it appears that Lip-norms should satisfy a form of the Leibniz inequality. In [14] and onward, we thus connected quantum topology and quantum metric by adding to Lip-norms L on unital C*-algebras $\mathfrak{A}$ the requirement that:

$$\mathsf{L}\left(\frac{ab+ba}{2}\right) \leqslant \mathsf{L}(a)\|b\|_{\mathfrak{A}} + \mathsf{L}(b)\|a\|_{\mathfrak{A}} \tag{1.2}$$

and

$$\mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant \mathsf{L}(a)\|b\|_{\mathfrak{A}} + \mathsf{L}(b)\|a\|_{\mathfrak{A}}, \tag{1.3}$$

where $\|\cdot\|_{\mathfrak{A}}$ is the norm of the underlying C*-algebra $\mathfrak{A}$.

Yet, the quantum Gromov-Hausdorff distance, introduced by Rieffel [37] as the first noncommutative analogue of the Gromov-Hausdorff distance, did not capture the C*-algebraic structure, as illustrated with the fact that the distance between two non-isomorphic C*-algebras could be null. In order to strengthen Rieffel's distance to make *-isomorphism necessary, one may consider one of at least two general approaches. A first idea is to modify the quantum Gromov-Hausdorff distance so that it captures some quantum topological aspects of the underlying quantum metric spaces, while not tying the metric and topological structure together. For instance, Kerr's distance [10] replaces the state space in Expression (1.1) with spaces of unital completely positive matrix valued linear maps, which capture some additional topological information, while the notion of Lip-norm, i.e. metric structure, is essentially unchanged and in particular, does not involve the Leibniz properties (1.2), (1.3). This first approach is shared, in some fashion or another, by all early attempts to fix the weakness of the coincidence axiom [10, 20, 22, 11].

A second approach is to tie together the metric and topological structure of quantum metric spaces before attempting to define a Gromov-Hausdorff distance. This approach involves working on a more restrictive category of quantum compact metric spaces. One then realizes that quite a few challenges arise when trying to define an analogue of the Gromov-Hausdorff distance. These challenges owe to the fact that the definition of an analogue of the Gromov-Hausdorff distance involves a form of embedding of two compact quantum metric spaces into some other space, and various properties of this analogue, such as the triangle inequality, become harder to establish when one puts strong constraints on the possible embedding. The Leibniz property of Expressions (1.2) and (1.3) are examples of such strong constraints. A first step in this direction can be found in Rieffel's quantum proximity [32], which is not known to even be a pseudo-metric, as the proof of the triangle inequality is elusive.

The dual propinquity follows the second approach, and owes its name to Rieffel's proximity. Its construction answered a rather long-standing challenge to employ the second approach described above to a successful conclusion. The dual propinquity is defined on Leibniz quantum compact metric spaces, which are compact quantum metric spaces whose Lip-norms are defined on dense Jordan-Lie sub-algebras of the self-adjoint part of the underlying C*-algebras, and which satisfy the Leibniz inequalities (1.2) and (1.3). All the main examples of compact quantum metric spaces are in fact Leibniz quantum compact metric spaces [37, 27, 26, 23, 21]. Now, the dual propinquity is a complete metric on the class of Leibniz quantum compact metric spaces: in particular, distance zero implies *-isomorphism, in addition to isometry of quantum metric structures, and the triangle inequality is satisfied. Several examples of convergence for the dual propinquity are known [12, 19, 34].

Moreover, several stronger ties between quantum metric and quantum topology have been proposed, most notably Rieffel's strong Leibniz property and Rieffel's compact C*-metric spaces [32], both of which are special cases of Leibniz quantum compact metric spaces. The dual propinquity can be specialized to these classes, in the sense that its construction may, if desired, only involve quantum metric spaces in these classes. We also note that the notion of Leibniz quantum compact metric space can be extended to the framework of quantum locally compact metric spaces [13, 15, 17].

The problem of determining which Leibniz quantum compact metric spaces is a limit of finite dimensional Leibniz quantum compact metric spaces for the dual propinquity, however, challenges us in this paper to explore a somewhat relaxed form of the Leibniz inequality, while keeping, informally, the same tie between quantum topology and quantum metric structure. Indeed, while any compact quantum metric space is within an arbitrarily small quantum Gromov-Hausdorff distance to some finite dimensional quantum metric space [37, Theorem 13.1], Rieffel's construction of these finite dimensional approximations does not produce finite dimensional C*-algebras nor Leibniz Lip-norms. In fact, the construction of finite dimensional approximations for Leibniz quantum compact metric spaces using the dual propinquity remains elusive in general. If $(\mathfrak{A}, \mathsf{L})$ is a Leibniz quantum compact metric spaces, we seek a sequence $(\mathfrak{B}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ of finite dimensional Leibniz quantum compact metric spaces which converge to $(\mathfrak{A}, \mathsf{L})$ for the dual propinquity. The first question is: what is the source of the C*-algebras $\mathfrak{B}_n$?

A natural approach to the study of this problem is to first seek C*-algebras which naturally come with finite dimensional C*-algebra approximations in a quantum topological sense. Nuclearity and quasi-diagonality, for instance, provide such approximations. The next natural question becomes: how to equip the finite dimensional topological approximations of some C*-algebra $\mathfrak{A}$ with quantum metric structures, given a Lip-norm $\mathfrak{A}$? Our effort led us to an answer in this paper, if we allow a bit of flexibility. When working with a nuclear quasi-diagonal Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$, then we can equip finite dimensional approximations with Lip-norms which are not necessarily Leibniz, but satisfy a slight generalized form of the Leibniz identity, in such a way as to obtain a metric approximation for the dual propinquity. This weakened form of the Leibniz property is referred to the quasi-Leibniz property. Informally, one may require that the deficiency in the Leibniz property for approximations of nuclear quasi-diagonal Leibniz quantum compact metric spaces be arbitrarily small, though not null.

To make sense of this statements, we first must check that the dual propinquity extends to classes of quasi-Leibniz quantum compact metric spaces while retaining all of its basic properties. This is achieved in the first section of this paper. We then prove our compactness theorem for quasi-Leibniz quantum compact metric spaces. This theorem includes a statement about the original class of Leibniz quantum compact metric spaces. We then prove that nuclear-quasi-diagonal quasi-Leibniz quantum compact metric spaces are within the closure of many classes of quasi-Leibniz quantum compact metric spaces, to which our compactness theorem would thus apply.

2. Quasi-Leibniz quantum compact metric spaces and the Dual Gromov-Hausdorff Propinquity

The framework for our paper is a class of compact quantum metric spaces constructed over C*-algebras and whose Lip-norms are well-behaved with respect to the multiplication. The desirable setup is to require the Leibniz property [31, 32, 14, 16, 18]. Yet, as we shall see in the second half of this work, the Leibniz property is difficult to obtain for certain finite-dimensional approximations. We are thus led to a more flexible framework, although we purposefully wish to stay, informally, close to the original Leibniz property, while accommodating the constructions of finite dimensional approximations and potential future examples. It could also be noted that certain constructions in noncommutative geometry, such as twisted spectral triples [5], would lead to seminorms which are not satisfying the Leibniz inequality, yet would fit within our new framework. With this in mind, we propose the following as the basic objects of our study:

Notation 2.1. The norm of any normed vector space X is denoted by $\|\cdot\|_X$.

Notation 2.2. Let $\mathfrak{A}$ be a unital C*-algebra. The unit of $\mathfrak{A}$ is denoted by $1_{\mathfrak{A}}$. The subspace of the self-adjoint elements in $\mathfrak{A}$ is denoted by $\mathfrak{sa}(\mathfrak{A})$. The state space of $\mathfrak{A}$ is denoted by $\mathscr{S}(\mathfrak{A})$.

Notation 2.3. Let $\mathfrak{A}$ be a C*-algebra. The *Jordan product* of $a, b \in \mathfrak{sa}(\mathfrak{A})$ is the element $a \circ b = \frac{1}{2}(ab + ba)$ and the *Lie product* of $a, b \in \mathfrak{sa}(\mathfrak{A})$ is the element $\{a, b\} = \frac{1}{2i}(ab - ba)$.

Definition 2.4. A pair $(\mathfrak{A}, \mathsf{L})$ of a unital C*-algebra $\mathfrak{A}$ and a seminorm L defined on a dense subspace $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ is a (C, D)*-quasi-Leibniz quantum compact metric space* for some $C \geqslant 1$ and $D \geqslant 0$ if:

(1) $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
(2) L is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$,
(3) the *Monge-Kantorovich metric* on $\mathscr{S}(\mathfrak{A})$, defined for any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}) \text{ and } \mathsf{L}(a) \leqslant 1\right\}, \tag{2.1}$$

induces the weak* topology on $\mathscr{S}(\mathfrak{A})$,
(4) the following inequalities hold:

$$\mathsf{L}(a \circ b) \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b) \tag{2.2}$$

and

$$\mathsf{L}(\{a, b\}) \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b). \tag{2.3}$$

A $(1,0)$-quasi-Leibniz quantum compact metric space is called a *Leibniz quantum compact metric space* [19]. The seminorm L of a (C,D)-quasi-Leibniz quantum compact metric space is called a *(C,D)-quasi-Leibniz Lip-norm*, and a $(1,0)$-quasi-Leibniz Lip-norm is called a *Leibniz Lip-norm*.

We note that a pair $(\mathfrak{A},\mathsf{L})$ satisfying Assertion (1) of Definition (2.4) is a Lipschitz pair, and a *Lipschitz pair* which satisfies Assertion (3) of Definition (2.4) is a *compact quantum metric space*, while L is then known as a Lip-norm. When a seminorm satisfies Assertion (4), then it is called a *(C,D)-quasi-Leibniz seminorm*.

We use the following convention in this paper:

Convention 2.5. If L is a seminorm defined on a dense subset $\mathrm{dom}(\mathsf{L})$ of some vector space X then we extend L to X by setting $\mathsf{L}(a) = \infty$ for all $a \in X \setminus \mathrm{dom}(\mathsf{L})$, and we check that if L is lower semi-continuous with respect to a norm on X, then so is its extension. We adopt the convention that $0 \cdot \infty = 0$ when working with seminorms.

We remark that if $(\mathfrak{A},\mathsf{L})$ is a (C,D)-quasi-Leibniz quantum compact metric space then, since $\mathsf{L}(1_{\mathfrak{A}}) = 0$ and thus $\mathsf{L}(a) = \mathsf{L}(a1_{\mathfrak{A}}) \leqslant C\mathsf{L}(a)$ for all $a \in \mathfrak{sa}(\mathfrak{A})$, we must have $C \geqslant 1$, as given in Definition (2.4).

Remark 2.6. Of course, if $(\mathfrak{A},\mathsf{L})$ is a (C,D)-quasi-Leibniz quantum compact metric space, then it is a (C',D')-quasi-Leibniz quantum compact metric space for any $C' \geqslant C$ and $D' \geqslant D$.

Remark 2.7. If L_1 and L_2 are respectively (C_1,D_1) and (C_2,D_2)-quasi-Leibniz Lip-norms on some C*-algebra $\mathfrak{A}$, then $\max\{\mathsf{L}_1,\mathsf{L}_2\}$ is a $(\max\{C_1,C_2\},\max\{D_1,D_2\})$-quasi-Leibniz Lip-norm on $\mathfrak{A}$.

A morphism $\pi : (\mathfrak{A},\mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ is easily defined as a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ whose dual map, restricted to the state space $\mathscr{S}(\mathfrak{B})$, is a Lipschitz map from $(\mathscr{S}(\mathfrak{B}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$. In this manner, quasi-Leibniz quantum compact metric spaces form a category. The notion of isomorphism in this category is the noncommutative generalization of bi-Lipschitz maps. However, as is usual with metric spaces, a more constrained notion of isomorphism is given by isometries.

Definition 2.8. An isometric isomorphsism $\pi : (\mathfrak{A},\mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ between two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ is a unital *-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ whose dual map is an isometry from $(\mathscr{S}(\mathfrak{B}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ onto $(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

The assumption that Lip-norms are lower semi-continuous in Definition (2.4) allows for the following useful characterizations of isometric isomorphisms:

Theorem 2.9 (Theorem 6.2, [37])**.** *Let $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is an isometric isomorphism if and only if $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.*

The fundamental characterization of quantum compact metric spaces, due to Rieffel [24, 25, 23], will be used repeatedly in this work, so we include it here:

Theorem 2.10 (Rieffel's characterization of compact quantum metric spaces)**.** *Let $(\mathfrak{A}, \mathsf{L})$ be a pair of a unital C*-algebra $\mathfrak{A}$ and a seminorm L defined on a dense subspace of $\mathfrak{sa}(\mathfrak{A})$ such that $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$. The following assertions are equivalent:*

1. *The Monge-Kantorovich metric mk_{L} metrizes the weak* topology of $\mathscr{S}(\mathfrak{A})$,*
2. $\operatorname{diam}(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}) < \infty$ *and $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\}$ is totally bounded for $\|\cdot\|_{\mathfrak{A}}$,*
3. *for some state $\mu \in \mathscr{S}(\mathfrak{A})$, the set $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mu(a) = 0$ and $\mathsf{L}(a) \leqslant 1\}$ is totally bounded for $\|\cdot\|_{\mathfrak{A}}$,*
4. *for all states $\mu \in \mathscr{S}(\mathfrak{A})$, the set $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mu(a) = 0$ and $\mathsf{L}(a) \leqslant 1\}$ is totally bounded for $\|\cdot\|_{\mathfrak{A}}$.*

One may replace total boundedness by actual compactness in Theorem (2.10) when working with quasi-Leibniz quantum compact metric spaces, since their Lip-norms are lower semi-continuous.

We extend the theory of the dual Gromov-Hausdorff propinquity to the class of quasi-Leibniz quantum compact metric spaces. Our original construction [16] was developed for Leibniz quantum compact metric spaces. Much of our work carry naturally to quasi-Leibniz quantum compact metric spaces. We follow the improvements to our original construction made in [18] for our general construction, and we refer to these two works heavily, focusing here only on the changes which are needed.

We begin by extending the notion of a tunnel between quasi-Leibniz quantum compact metric spaces, which is our noncommutative analogue of a pair of isometric embeddings of two quasi-Leibniz quantum compact metric spaces into a third quasi-Leibniz quantum compact metric space.

Definition 2.11. Let $C \geqslant 1$ and $D \geqslant 0$. Let $(\mathfrak{A}_1, \mathsf{L}_1)$, $(\mathfrak{A}_2, \mathsf{L}_2)$ be two (C, D)-quasi-Leibniz quantum compact metric spaces. A (C, D)-*tunnel* $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$ is a (C, D)-quasi-Leibniz quantum compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ and two *-epimorphisms $\pi_1 : \mathfrak{D} \twoheadrightarrow \mathfrak{A}_1$ and $\pi_2 : \mathfrak{D} \twoheadrightarrow \mathfrak{A}_2$ such that, for all $j \in \{1, 2\}$, and for all $a \in \mathfrak{sa}(\mathfrak{A}_j)$, we have:

$$\mathsf{L}_j(a) = \inf\{\mathsf{L}_{\mathfrak{D}}(d) : d \in \mathfrak{sa}(\mathfrak{D}) \text{ and } \pi_j(d) = a\}.$$

We assign a numerical value to a tunnel designed to measure how far two quasi-Leibniz quantum compact metric spaces are, or rather how long the given tunnel is. As a matter of notation:

Notation 2.12. The Hausdorff distance on compact subsets of a compact metric space (X, m) is denoted by $\mathsf{Haus}_{\mathrm{m}}$. If N is a norm on a vector space X, the Hausdorff distance on the class of closed subsets of any compact subset of X is denoted by Haus_N.

Notation 2.13. For any positive unital linear map $\pi : \mathfrak{A} \to \mathfrak{B}$ between two unital C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, the map $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is denote by π^*.

Definition 2.14. Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two (C, D)-quasi-Leibniz quantum compact metric spaces for some $C \geqslant 1$ and $D \geqslant 0$. The *extent* of a (C, D)-tunnel

$(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$ is the real number:

$$\max\left\{ \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}(\mathscr{S}(\mathfrak{D}), \pi_j^*(\mathscr{S}(\mathfrak{A}_j))) : j \in \{1,2\} \right\}.$$

The dual propinquity may be specialized to various subclasses of Leibniz quantum compact metric spaces. Somewhat different requirements can be made on the class of allowable tunnels; for our purpose we shall follow the notion of an appropriate collection of tunnels, as in [18]. We begin with the following observation:

Theorem 2.15. *Let $C \geqslant 1$ and $D \geqslant 0$. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and $(\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})$ be three (C,D)-quasi-Leibniz quantum compact metric spaces. Let $\tau_1 = (\mathfrak{D}_1, \mathsf{L}_1, \pi_1, \pi_2)$ be a (C,D)-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and $\tau_2 = (\mathfrak{D}_2, \mathsf{L}_2, \rho_1, \rho_2)$ be a (C,D)-tunnel from $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ to $(\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})$. Let $\varepsilon > 0$.*

If, for all $(d_1, d_2) \in \mathfrak{sa}(\mathfrak{D}_1 \oplus \mathfrak{D}_2)$, we set:

$$\mathsf{L}(d_1, d_2) = \max\left\{ \mathsf{L}_1(d_1), \mathsf{L}_2(d_2), \frac{1}{\varepsilon} \|\pi_2(d_1) - \rho_1(d_2)\|_{\mathfrak{B}} \right\},$$

and if $\eta_1 : (d_1, d_2) \in \mathfrak{sa}(\mathfrak{D}_1 \oplus \mathfrak{D}_2) \mapsto d_1$ and $\eta_2 : (d_1, d_2) \in \mathfrak{sa}(\mathfrak{D}_1 \oplus \mathfrak{D}_2) \mapsto d_2$, then $\tau_3 = (\mathfrak{D}_1 \oplus \mathfrak{D}_2, \mathsf{L}, \eta_1, \eta_2)$ is a (C,D)-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})$, whose extent satisfies:

$$\chi(\tau_3) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon.$$

Proof. Let $N(d_1, d_2) = \frac{1}{\varepsilon}\|\pi_2(d_1) - \rho_1(d_2)\|_{\mathfrak{B}}$ for all $(d_1, d_2) \in \mathfrak{sa}(\mathfrak{D}_1 \oplus \mathfrak{D}_2)$. For all $d_1, d_1' \in \mathfrak{sa}(\mathfrak{D}_1)$ and $d_2, d_2' \in \mathfrak{sa}(\mathfrak{D}_2)$, we have:

$$\begin{aligned}
N(d_1 d_1', d_2 d_2') &\leqslant \|d_1\|_{\mathfrak{D}_1} N(d_1', d_2') + \|d_2'\|_{\mathfrak{D}_2} N(d_1, d_2) \\
&\leqslant \|(d_1, d_2)\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} N(d_1', d_2') + \|(d_1', d_2')\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} N(d_1, d_2) \\
&\leqslant C\left(\|(d_1, d_2)\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} N(d_1', d_2') + \|(d_1', d_2')\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} N(d_1, d_2)\right) \\
&\leqslant C\left(\|(d_1, d_2)\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} \mathsf{L}(d_1', d_2') + \|(d_1', d_2')\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} \mathsf{L}(d_1, d_2)\right).
\end{aligned}$$

It thus follows that:

$$\begin{aligned}
N\left(d_1 \circ d_1', d_2 \circ d_2'\right) \leqslant \\
C\left(\|(d_1, d_2)\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} \mathsf{L}(d_1', d_2') + \|(d_1', d_2')\|_{\mathfrak{D}_1 \oplus \mathfrak{D}_2} \mathsf{L}(d_1, d_2)\right) \\
+ D\mathsf{L}(d_1, d_2)\mathsf{L}(d_1', d_2').
\end{aligned}$$

and similarly for the Lie product. Thus L is a (C,D)-quasi-Leibniz seminorm.

The proof of [18, Theorem 3.1] now applies to reach the conclusion of this theorem. □

We thus define, following [18, Definition 3.5]:

Definition 2.16. Let $C \geqslant 1$ and $D \geqslant 0$. Let $\mathcal{C}$ be a nonempty class of (C,D)-quasi-Leibniz quantum compact metric spaces. A class $\mathcal{T}$ of (C,D)-tunnels is $(\mathcal{C}, C, D)$-*appropriate* when:

1. for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ in $\mathcal{C}$, there exists $\tau \in \mathcal{T}$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,
2. if there exists an isometric isomorpism $h : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ for any two $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ in $\mathcal{C}$, then both $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}, h)$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, h^{-1}, \mathrm{id}_{\mathfrak{B}})$ are elements of $\mathcal{T}$, where id_E is the identity map on any set E,
3. if $(\mathfrak{D}, \mathsf{L}, \pi, \rho) \in \mathcal{T}$ then $(\mathfrak{D}, \mathsf{L}, \rho, \pi) \in \mathcal{T}$,

(4) if $(\mathfrak{D}, \mathsf{L}, \pi, \rho) \in \mathcal{T}$ then the co-domains of π and ρ lie in $\mathcal{C}$,
(5) if $\tau_1, \tau_2 \in \mathcal{T}$ and $\varepsilon > 0$, then the tunnel τ_3 defined by Theorem (2.15) also lies in $\mathcal{T}$.

Let $C \geqslant 1$, $D \geqslant 0$ and let $\mathcal{C}$ be a nonempty class of (C, D)-quasi-Leibniz quantum compact metric spaces. Let $\mathcal{T}$ be a $(\mathcal{C}, C, D)$-appropriate class of (C, D)-tunnels. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be in $\mathcal{C}$. The set of all (C, D)-tunnels in $\mathcal{T}$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is denoted by:

$$\mathscr{T}\!\mathit{unnels}\left[(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xrightarrow{\mathcal{T}, C, D} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right].$$

The main tool for our work is a distance constructed on the class of quasi-Leibniz quantum compact metric spaces. This distance is a form of our dual propinquity adapted to quasi-Leibniz quantum compact metric spaces, and its construction is modeled after the Gromov-Hausdorff distance [7, 6]:

Definition 2.17. Let $C \geqslant 1$ and $D \geqslant 0$. Let $\mathcal{C}$ be some nonempty class of (C, D)-quasi-Leibniz quantum compact metric spaces and let $\mathcal{T}$ be a class of $\mathcal{C}$-appropriate tunnels. The $(\mathcal{T}, C, D)$-*dual Gromov-Hausdorff propinquity* $\Lambda_{\mathcal{T}, C, D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ between two (C, D)-quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is the number:

$$\inf\left\{\chi(\tau) : \tau \in \mathscr{T}\!\mathit{unnels}\left[(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xrightarrow{\mathcal{T}, C, D} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right]\right\}.$$

We simply shall write $\Lambda_{C,D}$ for the dual propinquity defines on the class of all (C, D)-quasi-Leibniz quantum compact metric spaces, defined using the class of all (C, D)-tunnels.

In [16, 18], we proved that the dual propinquity is a complete metric on the class of Leibniz quantum compact metric spaces; in particular it satisfies the triangle inequality and distance zero between two Leibniz quantum compact metric spaces implies that they are isometrically isomorphic. The same holds in our present context. We simply explain in the proof of our theorem how to modify some estimates in [16, 18] to obtain the desired result.

Notation 2.18. The diameter of any metric space (X, m) is denoted by $\operatorname{diam}(X, \mathrm{m})$.

Theorem 2.19. *Let $C \geqslant 1$ and $D \geqslant 0$. Let $\mathcal{C}$ be some nonempty class of (C, D)-quasi-Leibniz quantum compact metric spaces and let $\mathcal{T}$ be a $(\mathcal{C}, C, D)$-appropriate class of (C, D)-tunnels.*

The $(\mathcal{T}, C, D)$-dual propinquity $\Lambda_{\mathcal{T}, C, D}$ is a metric on $\mathcal{C}$, i.e. for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ in $\mathcal{C}$:

(1) $\Lambda_{\mathcal{T}, C, D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \Lambda_{\mathcal{T}, C, D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}))$,
(2) *We have:*

$$\Lambda_{\mathcal{T}, C, D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})) \leqslant \Lambda_{\mathcal{T}, C, D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \Lambda_{\mathcal{T}, C, D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})),$$

(3) $\Lambda_{\mathcal{T}, C, D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ *if and only if there exists an isometric isomorphism between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,*

(4) *We have:*

$$\Lambda_{\mathcal{T},C,D}((\mathfrak{A},\mathsf{L}_{\mathfrak{A}}),(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})) \leqslant \max\left\{\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right),\right. \\ \left.\mathrm{diam}\left(\mathscr{S}(\mathfrak{B}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}\right)\right\}.$$

Moreover, $\Lambda_{C,D}$ is a complete metric on the class of all (C,D)-quasi-Leibniz quantum compact metric spaces.

Proof. The proofs are given in [19, 18], for the case when $C = 1$ and $D = 0$. The more general setting of this paper is however an immediate consequence of the work in [19, 18], once we make the following simple observations.

(1) [19, Lemma 4.2, Proposition 4.4, Corollary 4.5, Proposition 4.6] remain unchanged in our setting.
(2) [18, Proposition 2.12] remains unchanged as well.
(3) The estimate in [19, Proposition 4.8] should be modified very slightly. Let $\tau = (\mathfrak{D},\mathsf{L}_{\mathfrak{D}},\pi_{\mathfrak{A}},\pi_{\mathfrak{B}})$ be a (C,D)-tunnel from $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$. We recall from [19, Definition 4.1] that for $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) < \infty$ and for all $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the *target set* $\mathfrak{t}_{\tau}(a|l)$ is the set:

$$\mathfrak{t}_{\tau}(a|l) = \{\pi_{\mathfrak{B}}(d) : d \in \mathfrak{sa}(\mathfrak{D}), \pi_{\mathfrak{A}}(d) = a, \mathsf{L}_{\mathfrak{D}}(d) \leqslant l\}.$$

If $a, a' \in \mathfrak{sa}(\mathfrak{A})$ with $\max\{\mathsf{L}_{\mathfrak{A}}(a),\mathsf{L}_{\mathfrak{A}}(a')\} \leqslant l$ for some $l > 0$, and if $d, d' \in \mathfrak{sa}(\mathfrak{D})$ with $\pi_{\mathfrak{A}}(d) = a$, $\pi_{\mathfrak{A}}(d') = a'$ and $\max\{\mathsf{L}_{\mathfrak{D}}(d),\mathsf{L}_{\mathfrak{D}}(d')\} \leqslant l$, then by [19, Proposition 4.6]:

$$\|d\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}} + 4l\chi(\tau) \text{ and } \|d'\|_{\mathfrak{D}} \leqslant \|a'\|_{\mathfrak{A}} + 4l\chi(\tau).$$

Consequently, using the (C,D)-Leibniz property of $\mathsf{L}_{\mathfrak{D}}$:

$$\begin{aligned}\mathsf{L}_{\mathfrak{D}}\left(d \circ d'\right) &\leqslant C\left(\|d\|_{\mathfrak{D}}\mathsf{L}_{\mathfrak{D}}(d') + \|d'\|_{\mathfrak{D}}\mathsf{L}_{\mathfrak{D}}(d)\right) + D\mathsf{L}_{\mathfrak{D}}(d)\mathsf{L}_{\mathfrak{D}}(d') \\ &\leqslant Cl\left(\|a\|_{\mathfrak{A}} + \|a'\|_{\mathfrak{A}} + 8l\chi(\tau)\right) + 8l^2.\end{aligned} \tag{2.4}$$

Thus one may conclude that, if $b \in \mathfrak{t}_{\tau}(a|l)$ and $b' \in \mathfrak{t}_{\tau}(a'|l)$ then:

$$b \circ b' \in \mathfrak{t}_{\tau}\left(a \circ a' \middle| Cl(\|a\|_{\mathfrak{A}} + \|a'\|_{\mathfrak{A}} + 8l\chi(\tau)) + Dl^2\right). \tag{2.5}$$

A similar modification of [19, Proposition 4.8] holds for the Lie product. Moreover, we observe that, up to replacing the factor 8 by a factor 4 in Inequality (2.5), we could substitute the tunnel length to its extent.
(4) One may now check that the proof of [19, Theorem 4.16] carries unchanged, except with the use of Expression (2.5) in lieu of [19, Corollary 4.14, Proposition 4.8]. It is easy to verify that the change in the constant involved in the target set does not impact the proof at all. Thus [19, Theorem 4.16] holds.
(5) [19, Section 6] can be entirely rewritten with (C,D)-quasi-Lip-norms. Thus, a Cauchy sequence of (C,D)-quasi-Leibniz quantum compact metric spaces converge, for the dual propinquity, to a (C,D)-quasi-Leibniz quantum compact metric space.
(6) Using Theorem (2.15) in place of [18, Theorem 3.1], we can deduce that the dual propinquity satisfies the triangle inequality on the class of (C,D)-quasi-Leibniz quantum compact metric spaces as in [18, Theorem 3.7].

□

Remark 2.20. We emphasize that Theorem (2.19) requires that we first fix a $C \geqslant 1$ and $D \geqslant 0$; if we were working on the class of all quasi-Leibniz quantum compact metric spaces, then the estimates needed for Theorem (2.19) would not be valid any longer, since we would essentially lose control of the quasi-Leibniz property.

We conclude with a few useful definitions for our later work. The original construction of the dual propinquity [16] involved the *length* of a tunnel, rather than its extent. The extent [18] was useful to obtain the triangle inequality more easily. However, estimates on the dual propinquity may be easier to derive using our notion of length. The length and extent are equivalent, in the sense described below.

Definition 2.21. Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a (C, D)-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

(1) The *reach* of τ is:
$$\rho(\tau) = \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{A}), \mathscr{S}(\mathfrak{B})\right).$$
(2) The *depth* of τ is:
$$\delta(\tau) = \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \overline{\mathrm{co}}\left(\mathscr{S}(\mathfrak{A}) \cup \mathscr{S}(\mathfrak{B})\right)\right),$$
where $\overline{\mathrm{co}}(X)$ is the closed convex hull of X.

(3) The *length* of τ is:
$$\lambda(\tau) = \max\left\{\delta(\tau), \rho(\tau)\right\}.$$

Proposition 2.22 (Proposition 2.12, [18])**.** *For any tunnel τ, we have:*
$$\lambda(\tau) \leqslant \chi(\tau) \leqslant 2\lambda(\tau).$$

We conclude this section with a brief remark on extending the quantum propinquity to quasi-Leibniz quantum compact metric spaces. The quantum propinquity [14] is a specialized version of the dual propinquity, where for any two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, all the tunnels from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are of the form:
$$(\mathfrak{A} \oplus \mathfrak{B}, \mathsf{L}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$$
where $\pi_{\mathfrak{A}}$ and $\pi_{\mathfrak{B}}$ are the canonical surjections onto $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and for all $(a, b) \in \mathfrak{A} \oplus \mathfrak{B}$, the Lip-norm L is of the form:
$$\mathsf{L}(a, b) = \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda}\|\rho_{\mathfrak{A}}(a)\omega - \omega\rho_{\mathfrak{B}}(b)\|_{\mathfrak{D}}\right\} \tag{2.6}$$
where $\mathfrak{D}$ is some C*-algebra, $\rho_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\rho_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ are unital *-monomorphisms, and $\omega \in \mathfrak{D}$ is an element of $\mathfrak{D}$ such that there exists at least one state $\varphi \in \mathscr{S}(\mathfrak{D})$ such that:
$$\varphi((1_{\mathfrak{D}} - \omega)(1_{\mathfrak{D}} - \omega)^*) = \varphi((1_{\mathfrak{D}} - \omega)^*(1_{\mathfrak{D}} - \omega)) = 0.$$
The quadruples $(\mathfrak{D}, \omega, \rho_{\mathfrak{A}}, \rho_{\mathfrak{B}})$ are called bridges in [14]. The quantum propinquity was our first construction of a metric adapted to Leibniz quantum compact metric spaces, and asking tunnels of the form Expression (2.6) presents some potential algebraic advantages, such as the construction of matricial versions of these Lip-norms. As any specialization, the quantum propinquity is stronger than the

dual propinquity in general. Moreover, the quantum propinquity provides a useful technique to compute bounds for the dual propinquity (see for instance [19]).

The observation in Theorem (2.15) proves that Expression (2.6) leads to a (C,D)-quasi-Lip-norm if $\mathsf{L}_{\mathfrak{A}}$, $\mathsf{L}_{\mathfrak{B}}$ are (C,D)-quasi-Lip-norms. One can then verify that the quantum propinquity extends to a metric to the class of (C,D)-quasi-Leibniz quantum compact metric spaces for any $C \geqslant 1$ and $D \geqslant 0$, with the natural notion of a (C,D)-bridge.

3. Compactness for some classes of Finite Dimensional quasi-Leibniz quantum compact metric spaces

We begin this section with a characterization of Leibniz seminorms based upon their unit balls. We recall that a convex subset $S \subseteq \mathfrak{A}$ is *balanced* if $\{\lambda x : x \in S\} \subseteq S$ for all $\lambda \in [-1,1]$. If L is a seminorm on some space A then $\{x \in A : \mathsf{L}(x) \leqslant 1\}$ is convex and balanced. Conversely, if S is a convex and balanced subset of A, then the *gauge seminorm* or *Minkowsky functional* L associated with S is the seminorm on A defined by $\mathsf{L}(x) = \inf\{\lambda > 0 : \lambda x \in S\}$ for all $x \in A$. In particular, if A is a normed vector space, then these constructions establish a bijection between the set of closed, convex balanced subsets of A and the seminorms on A which are lower semi-continuous with respect to the norm of A.

Lemma 3.1. *Let $\mathfrak{A}$ be a C*-algebra. Let $\mathfrak{S}$ be a closed, balanced convex subset of $\mathfrak{A}$. Let $C \geqslant 1$ and $D \geqslant 0$. The Minkowsky functional L of $\mathfrak{S}$ is (C,D)-quasi-Leibniz if and only if for all $a,b \in \mathfrak{S}$:*

$$a \circ b \in \left[C\left(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}\right) + D\right]\mathfrak{S}$$

and

$$\{a,b\} \in \left[C\left(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}\right) + D\right]\mathfrak{S}.$$

Proof. Assume that L is a (C,D)-quasi-Leibniz seminorm for some $C \geqslant 1$, $D \geqslant 0$. Let $a,b \in \mathfrak{S}$. If $a,b \neq 0$ then:

$$\mathsf{L}(a\circ b) \leqslant C\left(\|a\| + \|b\|\right) + D \text{ i.e. } \mathsf{L}\left(\frac{1}{C\left(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}\right) + D} a \circ b\right) \leqslant 1$$

as $C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + D > 0$, and thus $\frac{1}{C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + D} a \circ b \in \mathfrak{S}$ so:

$$a \circ b \in \left(C\left(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}\right) + D\right)\mathfrak{S}.$$

If $a = 0$ or $b = 0$, then $\mathsf{L}(ab) = 0$, and thus:

$$0 = a \circ b \in \left(C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + D\right)\mathfrak{S}$$

as $0 \in \mathfrak{S}$.

We obtain a similar result for the Lie product, and thus our condition is necessary.

Assume conversely, that for all $a,b \in \mathfrak{S}$ we have $a \circ b \in \left(C\left(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}\right) + D\right)\mathfrak{S}$ for some $C \geqslant 1$ and $D \geqslant 0$. Let $a,b \in \mathrm{dom}(\mathsf{L})$. Assume first that $\mathsf{L}(a)\mathsf{L}(b) > 0$. We then have $\mathsf{L}(a)^{-1}a, \mathsf{L}(b)^{-1}b \in \mathfrak{S}$ and thus:

$$\frac{1}{\mathsf{L}(a)\mathsf{L}(b)} a \circ b \in \left(C\left(\left\|\frac{1}{\mathsf{L}(a)}a\right\|_{\mathfrak{A}} + \left\|\frac{1}{\mathsf{L}(b)}b\right\|_{\mathfrak{A}}\right) + D\right)\mathfrak{S}$$

i.e.

$$\mathsf{L}\left(\frac{1}{\mathsf{L}(a)\mathsf{L}(b)}a\circ b\right)\leqslant C\left(\left\|\frac{1}{\mathsf{L}(a)}a\right\|_{\mathfrak{A}}+\left\|\frac{1}{\mathsf{L}(b)}b\right\|_{\mathfrak{A}}\right)+D.$$

Consequently:

$$\begin{aligned}\mathsf{L}(a\circ b)&=\mathsf{L}(a)\mathsf{L}(b)\mathsf{L}\left(\frac{1}{\mathsf{L}(a)\mathsf{L}(b)}a\circ b\right)\\&\leqslant\mathsf{L}(a)\mathsf{L}(b)\left(C\left(\frac{1}{\mathsf{L}(a)}\|a\|_{\mathfrak{A}}+\frac{1}{\mathsf{L}(b)}\|b\|_{\mathfrak{A}}\right)+D\right)\\&=C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b)+\|b\|_{\mathfrak{A}}\mathsf{L}(a)\right)+D\mathsf{L}(a)\mathsf{L}(b).\end{aligned}$$

Now, assume $\mathsf{L}(a)=0$ and $\mathsf{L}(b)>0$. Let $\lambda>0$. Since $\mathsf{L}(\lambda^{-1}a)=0$ we have $\lambda^{-1}a\in\mathfrak{S}$. Of course, $\mathsf{L}(b)^{-1}b\in\mathfrak{S}$. Hence:

$$\left(\lambda^{-1}a\right)\circ\left(\mathsf{L}^{-1}(b)b\right)\in\left(C\left(\|\lambda^{-1}a\|_{\mathfrak{A}}+\|\mathsf{L}(b)^{-1}b\|_{\mathfrak{A}}\right)+D\right)\mathfrak{S}$$

so

$$\mathsf{L}\left(\left(\lambda^{-1}a\right)\circ\left(\mathsf{L}^{-1}(b)b\right)\right)\leqslant C\left(\lambda^{-1}\|a\|_{\mathfrak{A}}+\mathsf{L}(b)^{-1}\|b\|_{\mathfrak{A}}\right)+D$$

and thus:

$$\mathsf{L}(a\circ b)\leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b)+\lambda\|b\|_{\mathfrak{A}}\right)+D\lambda\mathsf{L}(b).$$

As $\lambda>0$ is arbitrary, we conclude that $\mathsf{L}(ab)\leqslant C\|a\|_{\mathfrak{A}}\mathsf{L}(b)$ as desired.

The case $\mathsf{L}(a)>0,\mathsf{L}(b)=0$ is dealt with symmetrically. Last, if $\mathsf{L}(a)=\mathsf{L}(b)=0$, then again, for all $\mu,\lambda>0$, we have:

$$\left(\lambda^{-1}a\right)\circ\left(\mu^{-1}b\right)\in\left(C\left(\|\lambda^{-1}a\|_{\mathfrak{A}}+\|\mu^{-1}b\|_{\mathfrak{A}}\right)+D\right)\mathfrak{S}$$

and thus $\mathsf{L}(a\circ b)\leqslant\mu\|a\|_{\mathfrak{A}}+\lambda\|b\|_{\mathfrak{A}}+D\mu\lambda$. Again, since λ,μ are arbitrary positive numbers, we conclude that $\mathsf{L}(a\circ b)=0=\|a\|_{\mathfrak{A}}\mathsf{L}(b)+\|b\|_{\mathfrak{A}}\mathsf{L}(a)+D\mathsf{L}(a)\mathsf{L}(b)$.

The proof is analogue for the Lie product and the product. This concludes our lemma. ☐

One important use for Lemma (3.1) is to help determine whether a seminorm whose unit ball is constructed as the Hausdorff limit of the unit balls of quasi-Leibniz Lip-norms possesses a quasi-Leibniz property. The following result will help with these considerations.

Lemma 3.2. *Let $\mathfrak{A}$ be a C*-algebra, and let $C\geqslant 1$ and $D\geqslant 0$. For all $n\in\mathbb{N}$, let $\mathfrak{L}_n$ be a closed, balanced convex subset of $\mathfrak{sa}(\mathfrak{A})$ such that, for all $a,b\in\mathfrak{L}_n$ we have:*

$$a\circ a',\left\{a,a'\right\}\in\left[C\left(\|a\|_{\mathfrak{A}}+\|b\|_{\mathfrak{A}}\right)+D\right]\mathfrak{L}_n.$$

If $(\mathfrak{L}_n)_{n\in\mathbb{N}}$ converges to some closed set $\mathfrak{L}$ for the Hausdorff distance associated with the norm $\|\cdot\|_{\mathfrak{A}}$, then $\mathfrak{L}$ is a balanced convex subset of $\mathfrak{sa}(\mathfrak{A})$ such that, for all $a,a'\in\mathfrak{L}$, we also have:

$$a\circ a',\left\{a,a'\right\}\in\left[C\left(\|a\|_{\mathfrak{A}}+\|b\|_{\mathfrak{A}}\right)+D\right]\mathfrak{L}.$$

Proof. Let Haus be the Hausdorff distance on closed subsets of $\mathfrak{A}$, associated with $\|\cdot\|_{\mathfrak{A}}$.

Let $a, b \in \mathfrak{L}$ and let $\varepsilon > 0$. If either a or b is 0 then $ab = 0 \in [C(\|a\| + \|b\|) + D]\,\mathfrak{L}$ trivially, so we henceforth assume that $0 \notin \{a, b\}$ and we further impose that $\varepsilon < \frac{1}{2}\min\{\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}\}$.

There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\mathsf{Haus}(\mathfrak{L}_n, \mathfrak{L}) \leqslant \varepsilon$. Thus there exists $c, d \in \mathfrak{L}_n$ such that:

$$\|a - c\|_{\mathfrak{A}} \leqslant \varepsilon \text{ and } \|b - d\|_{\mathfrak{A}} \leqslant \varepsilon.$$

Note that thanks to our choice of ε, we can assert that $c, d \neq 0$.

By assumption on L_n, we have:

$$c \circ d \in [C(\|c\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}) + D]\,\mathfrak{L}_n$$

On the other hand, we note that since a, b, c, d are self-adjoint:

$$\begin{aligned}\|a \circ b - c \circ d\|_{\mathfrak{A}} &\leqslant \frac{1}{2}\left(\|ab - cd\|_{\mathfrak{A}} + \|ba - dc\|_{\mathfrak{A}}\right)\\ &= \frac{1}{2}\left(\|ab - cd\|_{\mathfrak{A}} + \|(ab - cd)^*\|_{\mathfrak{A}}\right) = \|ab - cd\|_{\mathfrak{A}}\\ &\leqslant \|a\|_{\mathfrak{A}}\|b - d\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}\|a - c\|_{\mathfrak{A}}.\end{aligned}$$

Therefore:

$$\|a \circ b - c \circ d\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\|b - d\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}\|a - c\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\varepsilon + (\|b\|_{\mathfrak{A}} + \varepsilon)\varepsilon.$$

To ease notation, let $\lambda = C(\|c\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}) + D > 0$.

Let $e_\varepsilon \in \mathfrak{L}$ such that $\|c \circ d - \lambda e_\varepsilon\|_{\mathfrak{A}} \leqslant \lambda\varepsilon$. We note that:

$$\begin{aligned}\|e_\varepsilon\|_{\mathfrak{A}} &\leqslant \varepsilon + \lambda^{-1}\|c \circ d\|_{\mathfrak{A}}\\ &\leqslant \varepsilon + \frac{\|c\|_{\mathfrak{A}}\|d\|_{\mathfrak{A}}}{C(\|c\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}) + D}\\ &\leqslant \varepsilon + \frac{(\|a\|_{\mathfrak{A}} + \varepsilon)(\|b\|_{\mathfrak{A}} + \varepsilon)}{C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}} - 2\varepsilon) + D}\\ &\leqslant \varepsilon + \frac{9\|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}}}{2C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + 4D}.\end{aligned}$$

Consequently:

$$\begin{aligned}&\|a \circ b - [C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + D]\,e_\varepsilon\|_{\mathfrak{A}}\\ &\quad\leqslant \|a \circ b - c \circ d\|_{\mathfrak{A}} + \|c \circ d - [C(\|c\|_{\mathfrak{A}} + \|d\|_{\mathfrak{A}}) + D]\,e_\varepsilon\|_{\mathfrak{A}}\\ &\qquad + \|\,[C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}} - \|c\|_{\mathfrak{A}} - \|d\|_{\mathfrak{A}})]\,e_\varepsilon\|_{\mathfrak{A}}\\ &\quad\leqslant \varepsilon(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}} + \varepsilon) + [C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}} + 2\varepsilon) + D]\,\varepsilon\\ &\qquad + 2C\varepsilon\left(\varepsilon + \frac{9\|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}}}{2C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + 4D}\right)\\ &\quad= \mathscr{O}(\varepsilon).\end{aligned}$$

Hence, as $\varepsilon \in \left(0, \frac{1}{2}\min\{\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}\}\right)$ is arbitrary, and $\mathfrak{L}$ is closed, we conclude that:

$$a \circ b \in [C(\|a\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}}) + D]\,\mathfrak{L}.$$

The same proof holds for the Lie product. This proves the key fact of our lemma.

As the Hausdorff limit of a convex balanced set, it is easy to see that $\mathfrak{L}$ is balanced and convex. Indeed, if $a, b \in \mathfrak{L}$, then for all $n \in \mathbb{N}$ there exists $a_n, b_n \in \mathfrak{L}_n$ such that $\|a - a_n\|_{\mathfrak{A}} \leqslant \mathsf{Haus}(\mathfrak{L}_n, \mathfrak{L}) + \frac{1}{n+1}$ and $\|b - b_n\|_{\mathfrak{A}} \leqslant \mathsf{Haus}(\mathfrak{L}_n, \mathfrak{L}) + \frac{1}{n+1}$.

Now, if $t \in [0,1]$, then $ta_n + (1-t)b_n \in \mathfrak{L}_n$ since $\mathfrak{L}_n$ is convex. Therefore, there exists $c_n \in \mathfrak{L}$ such that $\|c_n - (ta_n + (1-t)b_n)\|_{\mathfrak{A}} \leqslant \mathsf{Haus}(\mathfrak{L}_n, \mathfrak{L}) + \frac{1}{n+1}$. Now:

$$\begin{aligned}\|ta + (1-t)b - c_n\|_{\mathfrak{A}} &\leqslant \|t(a - a_n) + (1-t)(b - b_n)\|_{\mathfrak{A}} \\ &\quad + \|ta_n + (1-t)b_n - c_n\|_{\mathfrak{A}} \leqslant 2\mathsf{Haus}(\mathfrak{L}_n, \mathfrak{L}) + \frac{2}{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty} 2\mathsf{Haus}(\mathfrak{L}, \mathfrak{L}_n) + \frac{2}{n+1} = 0$, we conclude that $(c_n)_{n\in\mathbb{N}}$ converges to $ta + (1-t)b$, and since $\mathfrak{L}$ is closed, we have $ta + (1-t)b \in \mathfrak{L}$. The same reasoning applies to show that $\mathfrak{L}$ is balanced as well. $\square$

We now establish a first compactness result, which serves as the basis for our main compactness theorem on quasi-Leibniz quantum compact metric spaces in the next section.

Notation 3.3. If $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space, we denote the diameter of $\mathscr{S}(\mathfrak{A})$ for the Monge-Kantorovich metric mk_{L} is denoted by $\mathrm{diam}\,(\mathfrak{A}, \mathsf{L})$.

Theorem 3.4. *Let $C \geqslant 1$ and $D \geqslant 0$ and let $\mathcal{QQCMS}_{C,D}$ be the class of all (C,D)-quasi-Leibniz quantum compact metric spaces. Let $d \in \mathbb{N} \setminus \{0\}$ and $K > 0$. The class:*

$$\mathcal{C}_{C,D,K,d} = \{(\mathfrak{A}, \mathsf{L}) \in \mathcal{QQCMS}_{C,D} : \dim_{\mathbb{C}} \mathfrak{A} \leqslant d \text{ and } \mathrm{diam}\,(\mathfrak{A}, \mathsf{L}) \leqslant K\}$$

is compact for the dual propinquity $\Lambda_{C,D}$.

Proof. Let $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{C,D,K,d}$. Let $\mathfrak{M}_d$ be the C*-algebra of $d \times d$ matrices.

First, fix $n \in \mathbb{N}$. We identify the C*-algebra $\mathfrak{A}_n$ with a C*-subalgebra of $\mathfrak{M}_d$ as follows. Up to $*$-isomorphism, we can write $\mathfrak{A}_n = \oplus_{j\in J} \mathfrak{M}_{t(j)}$ where $J = \{1, \ldots, d\}$ and $t(1) \geqslant t(2) \geqslant \cdots \geqslant t(d)$. We note that t may be zero for some $j \in \{1, \ldots, d\}$, and the zero set of t is a tail of J.

Let $j \in \{1, \ldots, d\}$. Let $s(j) = \sum_{k=1}^{j} t(j)$ and set $s(0) = 0$. We now let Q_j be the projection given as the diagonal matrix whose only nonzero entries are 1 on the diagonal, from row $s(j-1)+1$ to $s(j)$, i.e. in block form:

$$Q_j = \begin{pmatrix} 0_{s(j-1)} & & \\ & 1_{s(j)-s(j-1)} & \\ & & 0_{d-s(j)} \end{pmatrix}.$$

Of course the projections Q_j are orthogonal and sum to the identity of $\mathfrak{M}_d$.

It then follows trivially that $\mathfrak{A}_n$ is isomorphic to $\mathfrak{A}'_n = \sum_{j\in J} Q_j \mathfrak{M}_d Q_j$. Let $\pi_n : \mathfrak{A}_n \rightarrow \mathfrak{A}'_n$ be the *-isomorpism thus constructed. Note that π_n is *not* a unital map from $\mathfrak{A}_n$ into $\mathfrak{M}_d$.

If 1_n is the unit of $\mathfrak{A}_n$ for all $n \in \mathbb{N}$, then $(\pi_n(1_n))_{n\in\mathbb{N}}$ is a sequence of diagonal projections, i.e. diagonal $d \times d$-matrices with entries in $\{0,1\}$. Thus, there exists a constant subsequence $(\pi_{g(n)}(1_{g(n)}))_{n\in\mathbb{N}}$, with value denoted by p, of $(\pi_n(1_n))_{n\in\mathbb{N}}$.

Let $\mathfrak{B} = p\mathfrak{M}_d p$. Note that for all $n \in \mathbb{N}$, the map $p\pi_{g(n)}p : \mathfrak{A}_n \to \mathfrak{B}$ is now a unital *-monomorphism. We shall henceforth omit the notation $\mathrm{ad}_p \pi_{g(n)}$ and simply identify $\mathfrak{A}_{g(n)}$ with $p\pi_{g(n)}(\mathfrak{A}_{g(n)})p$. We emphasize that with this identification, $1_{g(n)} = 1_{\mathfrak{B}}$.

Let $\mathfrak{R} = \{b \in \mathfrak{sa}(\mathfrak{B}) : \|b\| \leqslant K\}$ be the closed ball of center 0 and radius K in $\mathfrak{sa}(\mathfrak{B})$. Since $\mathfrak{sa}(\mathfrak{B})$ is finite dimensional, the set $\mathfrak{R}$ is compact in norm. We shall denote by Haus the Hausdorff distance defined by the norm of $\mathfrak{sa}(\mathfrak{B})$ on the compact subsets of $\mathfrak{R}$. Since $\mathfrak{R}$ is compact in norm, Haus induces a compact topology on the set of compact subsets of $\mathfrak{R}$ as well [2, Theorem 7.3.8].

We fix a state $\varphi \in \mathscr{S}(\mathfrak{B})$ and identify φ with its restriction to $\mathfrak{A}_{g(n)}$, which is a state of $\mathfrak{A}_{g(n)}$, for all $n \in \mathbb{N}$.

Now, for all $n \in \mathbb{N}$, let:

$$\mathfrak{L}_n = \left\{ a \in \mathfrak{sa}\left(\mathfrak{A}_{g(n)}\right) : \mathsf{L}_{g(n)}(a) \leqslant 1 \right\}$$

and

$$\mathfrak{D}_n = \{a \in \mathfrak{L}_n : \varphi(a) = 0\}.$$

Fix $n \in \mathbb{N}$. By construction, we check that $\mathfrak{L}_n = \mathfrak{D}_n + \mathbb{R}1_{\mathfrak{B}}$, since for all $a \in \mathfrak{sa}\left(\mathfrak{A}_{g(n)}\right)$ we check easily that $\mathsf{L}_{g(n)}(a \pm \varphi(a)1_n) = \mathsf{L}_{g(n)}(a)$. On the other other hand, we note that $\mathfrak{D}_n$ is a compact subset of $\mathfrak{R}$ since $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}_n), \mathsf{mk}_{\mathsf{L}_n}\right) \leqslant K$. Indeed, if $a \in \mathfrak{D}_n$ then, for all $\psi \in \mathscr{S}(\mathfrak{A}_{g(n)})$, we have:

$$|\psi(a)| = |\psi(a) - \varphi(a)| \leqslant \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) \leqslant K.$$

Moreover, compactness of $\mathfrak{D}_n$ follows from Theorem (2.10) since $(\mathfrak{A}_{g(n)}, \mathsf{L}_{g(n)})$ is a Leibniz quantum compact metric space for all $n \in \mathbb{N}$.

Thus, there exists a convergent subsequence $(\mathfrak{D}_{f(n)})_{n\in\mathbb{N}}$ of $(\mathfrak{D}_n)_{n\in\mathbb{N}}$ for Haus, whose limit we denote by $\mathfrak{D}$.

We now define $\mathfrak{L} = \mathfrak{D} + \mathbb{R}1_{\mathfrak{B}}$. Let us first check that $(\mathfrak{L}_{f(n)})_{n\in\mathbb{N}}$ converges to $\mathfrak{L}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{B}}}(\mathfrak{D}_{f(n)}, \mathfrak{D}) \leqslant \varepsilon.$$

Let $n \geqslant N$. We observe that, for any $a \in \mathfrak{L}_{f(n)}$, there exists $a' \in \mathfrak{D}_{f(n)}$ and $t \in \mathbb{R}$ such that $a = a' + t1_{\mathfrak{B}}$. Now, there exists $b' \in \mathfrak{D}$ such that $\|a' - b'\|_{\mathfrak{B}} \leqslant \varepsilon$. Let $b = b' + t1_{\mathfrak{B}} \in \mathfrak{L}$. Then $\|a - b\|_{\mathfrak{B}} = \|a' - b'\|_{\mathfrak{B}} \leqslant \varepsilon$, so $\mathfrak{L}_{f(n)}$ is included in an ε-neighborhood of $\mathfrak{L}$. Using a symmetric argument, we conclude:

$$\mathsf{Haus}\left(\mathfrak{L}_{f(n)}, \mathfrak{L}\right) \leqslant \varepsilon$$

and thus $\left(\mathfrak{L}_{f(n)}\right)_{n\in\mathbb{N}}$ converges to $\mathfrak{L}$ for the Hausdorff distance Haus.

Moreover, $\mathfrak{D} = \{a \in \mathfrak{L} : \varphi(a) = 0\}$ by construction and continuity of φ. Last, as $\mathfrak{D}$ is compact, hence closed, the set $\mathfrak{L} = \mathfrak{D} + \mathbb{R}1_{\mathfrak{B}}$ is closed as well: if $(l_n)_{n\in\mathbb{N}} \in \mathfrak{L}$ converges to some l in $\mathfrak{B}$ then $(l_n - \varphi(l_n)1_{\mathfrak{B}})_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{D}$, and thus by continuity of φ and since $\mathfrak{D}$ is closed, $l - \varphi(l)1_{\mathfrak{B}} \in \mathfrak{D}$. Thus $l \in \mathfrak{L}$.

For all $b \in \mathfrak{sa}(\mathfrak{B})$ we define:

$$\mathsf{L}(b) = \inf\{\lambda > 0 : b \in \lambda\mathfrak{L}\}.$$

By Lemma (3.2), the set $\mathfrak{L}$ satisfies Lemma (3.1), and thus L is a (C, D)-quasi-Leibniz seminorm.

Certainly L may assume the value ∞. Let $\mathfrak{J} = \mathrm{dom}(\mathsf{L})$ be the set of self-adjoint elements in $\mathfrak{B}$ for which L is finite.

If $a, b \in \mathfrak{J}$ then:

$$\mathsf{L}(a \circ b), \mathsf{L}(\{a, b\}) \leqslant C\left(\|a\|\mathsf{L}(b) + \|b\|\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b) < \infty$$

so $\mathfrak{J}$ is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{B})$. We define:

$$\mathfrak{A} = \{b \in \mathfrak{B} : \Re(b), \Im(b) \in \mathfrak{J}\}$$

and we check that $\mathfrak{A}$ is a C*-subalgebra of $\mathfrak{B}$ with the same unit as $\mathfrak{B}$ and such that $\mathfrak{sa}(\mathfrak{A}) = \mathfrak{J}$.

If $\mathsf{L}(a) = 0$ for some $a \in \mathfrak{J}$, then we have $\mathsf{L}(a - \varphi(a)1_{\mathfrak{A}}) = 0$ as well since $\mathsf{L}(1_{\mathfrak{A}}) = 0$ and L is a seminorm by construction. Thus $a - \varphi(a)1_{\mathfrak{A}} \in \mathfrak{D}$. Now, for any $t \in \mathbb{R}$, we have $\varphi(t(a - \varphi(a)1_{\mathfrak{A}})) = 0$ and $\mathsf{L}(t(a - \varphi(a)1_{\mathfrak{A}})) = 0$, so $t(a - \varphi(a)1_{\mathfrak{A}}) \in \mathfrak{D}$ for all $t \in \mathbb{R}$. Since $\mathfrak{D}$ is norm bounded, we conclude that $a = \varphi(a)1_{\mathfrak{A}}$ as desired.

Since $\mathfrak{D} \subseteq \mathfrak{K}$, for any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ and for all $a \in \mathfrak{L}$ we have $|\varphi(a) - \psi(a)| \leqslant K$ and thus $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}\right) \leqslant K$.

Moreover, since $\mathfrak{D}$ is compact, we conclude that L is a Leibniz Lip-norm and $(\mathfrak{A}, \mathsf{L})$ is a Leibniz quantum compact metric space by Theorem (2.10).

We now prove that $\dim \mathfrak{A} \leqslant d$. First, for all $n \in \mathbb{N}$, let $d_n = \dim_{\mathbb{R}} \mathfrak{sa}\left(\mathfrak{A}_{g(f(n))}\right)$. By assumption, since $\dim_{\mathbb{R}} \mathfrak{sa}\left(\mathfrak{A}_{g(f(n))}\right) = \dim_{\mathbb{C}} \mathfrak{A}_{g(f(n))}$, we conclude that $d_n \in \{1, \ldots, d\}$. Thus, there exists a constant subsequence $(d_{f_1(n)})_{n\in\mathbb{N}}$ of $(d_n)_{n\in\mathbb{N}}$. Set $g_1 = g \circ f \circ f_1$ and $\delta = d_{f_1(0)} = d_{f_1(1)} = \ldots$.

Now, for all $n \in \mathbb{N}$, there exists a basis $(c_1^n, \ldots, c_\delta^n)$ of $\mathfrak{sa}\left(\mathfrak{A}_{g_1(n)}\right)$ such that $c_1^n = 1_{\mathfrak{B}}$. In particular, $\mathsf{L}_{g_1(n)}(c_j^n) \in (0, \infty)$ for all $j \in \{2, \ldots, \delta\}$ since $\mathsf{L}_{g_1(n)}$ is a Lip-norm on a finite dimensional space.

Now we set $d_1^n = 1_{\mathfrak{B}}$ and $d_j^n = \mathsf{L}_{g_1(n)}(c_j^n)^{-1} d_j^n$, then we have constructed a basis $(d_1^n, \ldots, d_\delta^n)$ of $\mathfrak{sa}\left(\mathfrak{A}_{g_1(n)}\right)$ consisting of elements in $\mathfrak{L}_{g_1(n)}$. We can improve somewhat on this construction. Indeed, for $j \in \{2, \ldots, \delta\}$, we have $\varphi(d_j^n - \varphi(d_j^n)d_1^n) = 0$. Thus, if $b_1^n = d_1^n$ and $b_j^n = d_j^n - \varphi(d_j^n)d_1^n$, then we have constructed a basis $\{b_1^n, \ldots, b_\delta^n\}$ of $\mathfrak{A}_{g_1(n)}$ with $b_1^n = 1_{\mathfrak{B}}$ and $b_j^n \in \mathfrak{D}_{g_1(n)}$.

Since $\mathfrak{K}$ is compact and, for any j in the *finite* set $\{1, \ldots, \delta\}$, the sequence $(b_j^n)_{n\in\mathbb{N}}$ lies in $\mathfrak{K}$, there exists a strictly increasing function $f_2 : \mathbb{N} \to \mathbb{N}$ such that for all $j \in \{1, \ldots, \delta\}$, the sequence $(b_j^{f_2(n)})_{n\in\mathbb{N}}$ converges in norm to some $b_j \in \mathfrak{K}$. Let $g_2 = g_1 \circ f_2$.

Since $(\mathfrak{D}_n)_{n\in\mathbb{N}}$ converges to $\mathfrak{D}$ for the Hausdorff distance on $\mathfrak{K}$ associated with the norm of $\mathfrak{B}$, we conclude that $b_j \in \mathfrak{D}$ for all $j \in \{2, \ldots, \delta\}$. Of course, $b_1 = 1_{\mathfrak{A}}$.

Now, let $b \in \mathfrak{sa}(\mathfrak{A})$ be arbitrary. By construction, $a = rb$ for some $r \in \mathbb{R}$ and $a \in \mathfrak{L}$. Since $\mathfrak{L}_{g_2(n)}$ converges to $\mathfrak{L}$, there exists $a_n \in \mathfrak{L}_{g_2(n)}$ such that $\lim_{n\to\infty} a_n = a$. For each $n \in \mathbb{N}$ we write $a_n = \sum_{j=1}^{\delta} \lambda_j^n b_j^n$.

There exists $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$, we have $\|a_n\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{B}} + 1$. Thus $\left(\lambda_j^n\right)_{n\in\mathbb{N}}$ is a bounded sequence for all j in the finite set $\{1,\ldots,\delta\}$. Consequently, there exists a strictly increasing $f_3 : \mathbb{N} \to \mathbb{N}$ with $(\lambda_j^{f_3(n)})_{n\in\mathbb{N}}$ converging to some limit $\lambda_j \in \mathbb{R}$ for all $j \in \{1,\ldots,\delta\}$.

Let $\varepsilon > 0$. Summarizing our construction thus far, there exists $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ we have $\|a - a_{f_3(n)}\| \leqslant \frac{\varepsilon}{3}$. There exists $N_3 \in \mathbb{N}$ such that for all $n \geqslant N_3$ and all $j \in \{1,\ldots,\delta\}$ we have $|\lambda_j^{f_3(n)} - \lambda_j| \leqslant \frac{\varepsilon}{3\max\{\|b_1\|_{\mathfrak{B}},\ldots,\|b_\delta\|_{\mathfrak{B}}\}}$. Last there exists $N_4 \in \mathbb{N}$ such that for all $n \geqslant N_4$ we have $\|b_j^{f_2(f_3(n))} - b_j\|_{\mathfrak{B}} \leqslant \frac{\varepsilon}{3\delta\max\{\lambda_1,\ldots,\lambda_\delta,1\}}$.

Now for $n \geqslant \max\{N_2, N_3, N_4\}$:

$$\begin{aligned}\left\|a - \sum_{j=1}^{\delta}\lambda_j b_j\right\|_{\mathfrak{B}} &\leqslant \left\|a - a_{f_3(n)}\right\|_{\mathfrak{B}} + \left\|a_{f_3(n)} - \sum_{j=1}^{\delta}\lambda_j b_j\right\|_{\mathfrak{B}} \\ &\leqslant \frac{\varepsilon}{3} + \left\|a_{f_3(n)} - \sum_{j=1}^{\delta}\lambda_j b_j^{f_2(f_3(n))}\right\|_{\mathfrak{B}} + \left\|\sum_{j=1}^{\delta}\lambda_j\left(b_j^{f_2(f_3(n))} - b_j\right)\right\|_{\mathfrak{B}} \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus a lies in the closure of the span of $\{b_1,\ldots,b_\delta\}$, which is closed since finite dimensional. Hence $\{b_1,\ldots,b_\delta\}$ spans $\mathfrak{A}$ and thus $\dim\mathfrak{A} \leqslant \delta \leqslant d$.

Now, we wish to conclude by showing that $(\mathfrak{A}_{g(f(n))}, \mathsf{L}_{g(f(n))})_{n\in\mathbb{N}}$ converges to $(\mathfrak{A}, \mathsf{L})$ for the quantum propinquity. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\mathsf{Haus}(\mathfrak{L}_{g\circ f(n)}, \mathfrak{L}) \leqslant \varepsilon$. Let now $n \geqslant N$.

For all $a \in \mathfrak{A}_{g\circ f(n)}$ and $b \in \mathfrak{A}$, we set:

$$N_n(a,b) = \frac{1}{\varepsilon}\|a - b\|_{\mathfrak{B}},$$

and

$$\mathsf{L}^n(a,b) = \max\left\{\mathsf{L}_{g\circ f(n)}(a), \mathsf{L}(b), N_n(a,b)\right\}.$$

It is easily checked that N_n is a bridge in the sense of [37, Definition 5.1]: in particular, if $a \in \mathfrak{A}_{g\circ f(n)}$ with $\mathsf{L}_{g\circ f(n)}(a) \leqslant 1$ then there exists $b \in \mathfrak{L}$ with $\|a - b\|_{\mathfrak{B}} \leqslant \varepsilon$, which implies $\mathsf{L}^n(a,b) = 1$; similarly if $b \in \mathfrak{A}$ with $\mathsf{L}(b) \leqslant 1$, i.e. $b \in \mathfrak{L}$, then there exists $a \in \mathfrak{L}_{f(n)}$ with $\|a - b\|_{\mathfrak{B}} \leqslant \varepsilon$ and thus $\mathsf{L}^n(a,b) = 1$.

Hence by [37, Theorem 5.2], the seminorm L^n is a Lip-norm. It is lower semi-continuous by construction, and it is easily checked that L^n is (C,D)-quasi-Leibniz, as in our proof of Theorem (2.15).

Let $\tau_n = (\mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A}, \mathsf{L}, \rho_n, \rho)$ with $\rho_n : \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A} \twoheadrightarrow \mathfrak{A}_{g\circ f(n)}$ and $\rho : \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A} \twoheadrightarrow \mathfrak{A}$ the two canonical surjections. By construction, τ_n is a (C,D)-tunnel.

The depth of τ_3 (Definition (2.21)) is null, and thus the length of τ_3 is its spread. If $\mu \in \mathscr{S}(\mathfrak{A}_{g\circ f(n)})$ and $\nu \in \mathscr{S}(\mathfrak{A})$ and if $(a,b) \in \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A}$ with $\mathsf{L}^n(a,b) \leqslant 1$, then:

$$|\mu(a) - \nu(b)| \leqslant \|a - b\|_{\mathfrak{B}} \leqslant \varepsilon$$

and thus $\lambda(\tau_3) \leqslant \varepsilon$. Thus $\chi(\tau_3) \leqslant 2\varepsilon$.

Consequently: for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\Lambda_{C,D}((\mathfrak{A}_{g\circ f(n)}, \mathsf{L}_{g\circ f(n)}), (\mathfrak{A}, \mathsf{L})) \leqslant 2\varepsilon.$$

Moreover, $(\mathfrak{A}, \mathsf{L})$ is a (C, D)-quasi-Leibniz quantum compact metric space of diameter at most K and dimension at most d. This completes our proof. □

We conclude by observing that, using the notations of the proof of Theorem (3.4), the quadruple $(\mathfrak{B}, 1_{\mathfrak{B}}, \iota_n, \iota)$ where $\iota_n : \mathfrak{A}_{g\circ f(n)} \hookrightarrow \mathfrak{B}$ and $\iota : \mathfrak{A} \hookrightarrow \mathfrak{B}$ are the inclusion maps, is a bridge for the extension of the quantum propinquity to (C, D)-quasi-Leibniz quantum compact metric spaces. Thus, Theorem (3.4) is also valid for the quantum propinquity.

4. A compactness Theorem

This section establishes the core result of our paper, which characterizes compact classes of quasi-Leibniz quantum compact metric spaces for the dual propinquity among all subclasses of the closure of finite dimensional quasi-Leibniz quantum compact metric spaces. Our reason for working within this closure is that our main theorem employs the following key notion, motivated by Gromov's Theorem (1.1):

Definition 4.1. Let $C \geqslant 1$, $D \geqslant 0$ and let $\mathcal{QQCMS}_{C,D}$ be the class of all (C, D)-quasi-Leibniz quantum compact metric spaces. Let $(\mathfrak{A}, \mathsf{L})$ be a (C, D)-quasi-Leibniz quantum compact metric space and let $\varepsilon > 0$. The *covering number* $\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}|\varepsilon)$ is:

$$\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}|\varepsilon) = \inf \left\{ n \in \mathbb{N} : \begin{array}{l} \exists (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQMS}_{C,D} \text{ such that} \\ \dim_{\mathbb{C}} \mathfrak{B} \leqslant n \text{ and} \\ \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon \end{array} \right\}.$$

Our generalization of Gromov's Compactness Theorem (1.1) is given by:

Theorem 4.2. *Let $C \geqslant 1$, $D \geqslant 0$, and let $\mathcal{A}$ be a class of (C, D)-quasi-Leibniz quantum compact metric spaces in the closure of the finite dimensional (C, D)-quasi-Leibniz quantum compact metric spaces for the dual propinquity. The following assertions are equivalent:*

1. *$\mathcal{A}$ is totally bounded for $\Lambda_{C,D}$,*
2. *there exists a function $F : (0, \infty) \to \mathbb{N}$ and $K > 0$ such that for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A}$, we have:*
   - $\mathrm{diam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \leqslant K$,
   - *for all $\varepsilon > 0$ we have* $\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}|\varepsilon) \leqslant F(\varepsilon)$.

Proof. Assume (2), i.e. assume that there exists $F : (0, \infty) \to \mathbb{N}$ and $K > 0$ such that for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A}$ and $\varepsilon > 0$, we have $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right) \leqslant K$ and:

$$\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}|\varepsilon) \leqslant F(\varepsilon).$$

Let $\varepsilon > 0$.

First, we note that if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A}$, then there exists a (C, D)-quasi-Leibniz quantum compact metric space $(\mathfrak{A}_f, \mathsf{L}_f)$ such that:

- $\dim_{\mathbb{C}} \mathfrak{A}_f \leqslant F\left(\frac{\varepsilon}{3}\right)$,
- $\Lambda_{C,D}((\mathfrak{A}_f, \mathsf{L}_f), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})) \leqslant \frac{\varepsilon}{3}$.

Consequently, we note that $\mathrm{diam}\left(\mathfrak{A}_f, \mathsf{L}_f\right) \leqslant K + \frac{2\varepsilon}{3}$, since the function $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A} \mapsto \mathrm{diam}\,(\mathfrak{A}, \mathsf{L})$ is 2-Lipschitz for the quantum Gromov-Hausdorff distance, and thus for the dual propinquity, by [37, Lemma 13.6].

Now, by Theorem (3.4), the class:

$$\mathcal{F}_\varepsilon = \left\{ (\mathfrak{B}, \mathsf{L}) \in \mathcal{QQCMS}_{C,D} : \dim_{\mathbb{C}} \mathfrak{B} \leqslant F\left(\frac{\varepsilon}{3}\right) \text{ and } \mathrm{diam}\,(\mathfrak{B}, \mathsf{L}) \leqslant K + \frac{2\varepsilon}{3} \right\}$$

is compact for $\Lambda_{C,D}$.

Let:

$$\mathcal{G}_\varepsilon = \left\{ (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{F}_\varepsilon : \exists (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A} \quad \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \frac{\varepsilon}{3} \right\}.$$

Since $\mathcal{G}_\varepsilon \subseteq \mathcal{F}_\varepsilon$, we conclude that $\mathcal{G}_\varepsilon$ is totally bounded for the dual propinquity. Thus, there exists a finite subset $\mathcal{J}_\varepsilon$ of $\mathcal{G}_\varepsilon$ which is $\frac{\varepsilon}{3}$ dense in $\mathcal{G}_\varepsilon$.

Therefore, up to invoking choice, there exists a finite subset $\mathcal{A}_\varepsilon$ of $\mathcal{A}$ such that for all $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{J}_\varepsilon$ there exists $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A}_\varepsilon$ such that $\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \frac{\varepsilon}{3}$.

Now, let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathcal{A}$. There exists $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{G}_\varepsilon$ such that:

$$\Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})) \leqslant \frac{\varepsilon}{3}.$$

Now, there exists $(\mathfrak{C}, \mathsf{L}_{\mathfrak{C}}) \in \mathcal{J}_\varepsilon$ such that:

$$\Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})) \leqslant \frac{\varepsilon}{3}.$$

Last, by our choice, there exists $(\mathfrak{A}_f, \mathsf{L}_f) \in \mathcal{A}_\varepsilon$ with:

$$\Lambda_{C,D}((\mathfrak{A}_f, \mathsf{L}_f), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})) \leqslant \frac{\varepsilon}{3}.$$

Consequently:

$$\begin{aligned}
&\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{A}_f, \mathsf{L}_f)) \\
&\quad \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})) + \Lambda_{C,D}((\mathfrak{C}, \mathsf{L}_{\mathfrak{C}}), (\mathfrak{A}_f, \mathsf{L}_f)) \\
&\quad \leqslant \varepsilon.
\end{aligned}$$

Thus $\mathcal{A}_\varepsilon$ is ε-dense in $\mathcal{A}$ for the dual propinquity, and is a finite set. Thus, $\mathcal{A}$ is totally bounded for $\Lambda_{C,D}$.

Assume (1) now, i.e. assume that $\mathfrak{A}$ is totally bounded. Since the function:

$$(\mathfrak{A}, \mathsf{L}) \in \mathcal{A} \mapsto \mathrm{diam}\,(\mathfrak{A}, \mathsf{L})$$

is 2-Lipschitz for the quantum Gromov-Hausdorff distance by [37, Lemma 13.6], it is continuous, and thus it is bounded above since $\mathcal{A}$ is totally bounded.

Now, let $\varepsilon > 0$. Since $\mathfrak{A}$ is totally bounded, there exists a finite subset $\mathfrak{A}_\varepsilon$ of $\mathfrak{A}$ which is $\frac{\varepsilon}{2}$-dense in $\mathcal{A}$ for $\Lambda_{C,D}$. For each $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}_\varepsilon$, there exists a finite dimensional (C, D)-quasi-Leibniz quantum compact metric space $F(\mathfrak{A}, \mathsf{L})$ such that:

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}), F(\mathfrak{A}, \mathsf{L})) \leqslant \frac{\varepsilon}{2}$$

by assumption on $\mathcal{A}$.

Let:

$$F(\varepsilon) = \max \left\{ \dim_{\mathbb{C}} F(\mathfrak{A}, \mathsf{L}) : (\mathfrak{A}, \mathsf{L}) \in \mathcal{A}_\varepsilon \right\}.$$

If $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$, then there exists $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{A}_\varepsilon$ such that $\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \frac{\varepsilon}{2}$. Thus:

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}), (F(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))) \leqslant \varepsilon.$$

Thus $\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant F(\varepsilon)$ by definition.

This completes our proof. □

We thus conclude:

Corollary 4.3. *Let $C \geqslant 1$, $D \geqslant 0$, and let $\mathcal{A}$ be a class of (C, D)-quasi-Leibniz quantum compact metric spaces which lies within the closure of the finite dimensional (C, D)-quasi-Leibniz quantum compact metric spaces for the dual propinquity.*

The class $\mathcal{A}$ is compact for the dual propinquity $\Lambda_{C,D}$ if and only if there exists $F : (0, \infty) \rightarrow \mathbb{N}$ and $K > 0$ such that:

1. *for all $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$ we have* $\mathrm{diam}(\mathfrak{A}, \mathsf{L}) \leqslant K$,
2. *for all $(\mathfrak{A}, \mathsf{L}) \in \mathcal{A}$ and for all $\varepsilon > 0$, we have:*
$$\mathrm{cov}_{(C,D)}(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant F(\varepsilon)$$
3. *$\mathcal{A}$ is closed for $\Lambda_{C,D}$.*

Proof. By Theorem (2.19), the dual propinquity is complete. The result then follows from Theorem (4.2). □

Thus, we are led to study the closure of finite dimensional quasi-Leibniz quantum compact metric spaces for the dual propinquity. This question proves tricky. However, our use of quasi-Leibniz Lip-norms, instead of Leibniz Lip-norms, allows us to establish that a large class of (C, D)-quasi-Leibniz quantum compact metric spaces lie within the closure of the class of finite dimensional (C', D')-quasi-Leibniz quantum compact metric spaces for arbitrary $C' > C \geqslant 1$ and $D' > D \geqslant 0$. This will be the subject of the next section of this paper.

5. Finite Dimensional Approximations for pseudo-diagonal quasi-Leibniz quantum compact metric spaces

For an arbitrary Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ and $\varepsilon > 0$, it is not immediately clear how to find a finite dimensional C*-algebra $\mathfrak{B}$ and a Leibniz Lip-norm $\mathsf{L}_{\mathfrak{B}}$ on $\mathfrak{B}$ such that $\Lambda((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon$. In this section, we propose to work with Leibniz quantum compact metric spaces whose underlying C*-algebra provides natural topological, finite dimensional approximations, and then attempt to construct Lip-norms on these approximations. As we shall see, however, these Lip-norms will be quasi-Leibniz. Yet, for any $C > 1$ and $D > 0$, and for any ε, we will show that $(\mathfrak{A}, \mathsf{L})$ is within ε-distance of some (C, D)-quasi-Leibniz quantum compact metric space of finite dimension, for the dual propinquity, as long as $\mathfrak{A}$ satisfies the following condition:

Definition 5.1. A unital C*-algebra $\mathfrak{A}$ is *pseudo-diagonal*, when for all $\varepsilon > 0$ and for all finite subset $\mathfrak{F}$ of $\mathfrak{A}$, there exist a finite dimensional C*-algebra $\mathfrak{B}$ and two positive unital linear maps $\varphi : \mathfrak{B} \rightarrow \mathfrak{A}$ and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that:

1. for all $a \in \mathfrak{F}$, we have $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,
2. for all $a, b \in \mathfrak{F}$, we have $\|\psi(a \circ b) - \psi(a) \circ \psi(b)\|_{\mathfrak{B}} \leqslant \varepsilon$,

(3) for all $a, b \in \mathfrak{F}$, we have $\|\psi(\{a,b\}) - \{\psi(a), \psi(b)\}\|_{\mathfrak{B}} \leqslant \varepsilon$.

Note that unital positive linear maps have norm 1, and thus are contractions.

Definition (5.1) is motivated by a characterization of unital nuclear, quasi-diagonal C*-algebras, due to Blackadar and Kirchberg [1, Theorem 5.2.2]:

Theorem 5.2 (Theorem 5.2.2, [1]). *A unital C*-algebra $\mathfrak{A}$ is* nuclear Quasi-Diagonal *if and only if, for all $\varepsilon > 0$ and all finite subset $\mathfrak{F} \subseteq \mathfrak{A}$, there exist a finite dimensional C*-algebra $\mathfrak{B}$ and two unital contractive completely positive map $\psi : \mathfrak{A} \to \mathfrak{B}$ and $\varphi : \mathfrak{B} \to \mathfrak{A}$ such that:*

(1) *for all $a, b \in \mathfrak{F}$ we have $\|\psi(ab) - \psi(a)\psi(b)\|_{\mathfrak{B}} \leqslant \varepsilon$,*
(2) *for all $a \in \mathfrak{F}$ we have $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,*

Remark 5.3. With the notations of Theorem (5.2), if $a \in \mathfrak{F}$ then $\|a\|_{\mathfrak{A}} = \|a - \varphi(\psi(a))\|_{\mathfrak{A}} + \|\varphi(\psi(a))\|_{\mathfrak{A}} \leqslant \varepsilon + \|\psi(a)\|_{\mathfrak{B}}$, i.e.

$$\forall a \in \mathfrak{F} \quad \|a\|_{\mathfrak{A}} \leqslant \|\psi(a)\|_{\mathfrak{B}} + \varepsilon.$$

This property, together with the first assertion of Theorem (5.2), characterizes quasi-diagonal C*-algebras, as proved by D. Voiculescu [38]. The existence of the map φ, in turn, adds the nuclearity property.

While Theorem (5.2) involves completely positive maps, Definition (5.1) only requires positive maps, as this suffices for our proof of finite approximation in the sense of the dual propinquity. On the other hand, Theorem (5.2) does not provide unital maps, while Definition (5.1) requires them, as we will find this useful — for instance, to map states to states. Nonetheless, unital nuclear quasi-diagonal C*-algebras are unital pseudo-diagonal. To this end, we show that we can, in fact, require the maps in Theorem (5.2) to be unital. We begin with:

Lemma 5.4. *Let $\mathfrak{A}$ be a unital C*-algebra such that, for every nonempty finite subset $\mathfrak{F}$ of $\mathfrak{A}$ and for every $\varepsilon > 0$, there exists a finite dimensional C*-algebra $\mathfrak{B}_\varepsilon$ and two positive contractions $\varphi_\varepsilon : \mathfrak{B}_\varepsilon \to \mathfrak{A}$ and $\psi_\varepsilon : \mathfrak{A} \to \mathfrak{B}_\varepsilon$ such that for all $x, y \in \mathfrak{F}$:*

$$\|x - \varphi_\varepsilon \circ \psi_\varepsilon(x)\|_{\mathfrak{A}} \leqslant \varepsilon \tag{5.1}$$

and

$$\|\psi_\varepsilon(x)\psi_\varepsilon(y) - \psi_\varepsilon(xy)\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon. \tag{5.2}$$

Then for all $\varepsilon > 0$, there exists a C-subalgebra $\mathfrak{D}_\varepsilon$ of $\mathfrak{B}_\varepsilon$ with unit $1'_\varepsilon$ and a positive contractive map $\varsigma_\varepsilon : \mathfrak{A} \to \mathfrak{D}_\varepsilon$ such that, for all $x, y \in \mathfrak{F}$:*

$$\lim_{\varepsilon \to 0} \|x - \varphi_\varepsilon \circ \varsigma_\varepsilon(x)\|_{\mathfrak{A}} = 0$$

and

$$\lim_{\varepsilon \to 0} \|\varsigma_\varepsilon(x)\varsigma_\varepsilon(y) - \varsigma_\varepsilon(xy)\|_{\mathfrak{D}_\varepsilon} = 0$$

while:

$$\lim_{\varepsilon \to 0} \|1'_\varepsilon - \varsigma_\varepsilon(1_{\mathfrak{A}})\|_{\mathfrak{D}_\varepsilon} = 0.$$

Proof. Let $\mathfrak{F}$ be finite subset of $\mathfrak{A}$. Let $\mathfrak{F}_1 = \mathfrak{F} \cup \{1_{\mathfrak{A}}\}$. For all $\varepsilon > 0$, by assumption, there exist a finite dimensional C*-algebra $\mathfrak{B}_\varepsilon$ and two positive contractions $\varphi_\varepsilon : \mathfrak{B}_\varepsilon \to \mathfrak{A}$ and $\psi_\varepsilon : \mathfrak{A} \to \mathfrak{B}_\varepsilon$ such that, for all $a, b \in \mathfrak{F}_1$:

(1) $\|\psi_\varepsilon(ab) - \psi_\varepsilon(a)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon$,
(2) $\|a - \varphi_\varepsilon \circ \psi_\varepsilon(a)\|_{\mathfrak{A}} \leqslant \varepsilon$.

We shall henceforth tacitly identify $\mathfrak{B}_\varepsilon$ with some subalgebra of a full matrix algebra $\mathfrak{M}_\varepsilon$.

Let $\varepsilon \in \left(0, \frac{1}{4}\right)$. Our first observation is that $\psi_\varepsilon(1_{\mathfrak{A}})$ is a positive operator of norm at most 1 since ψ_ε is a positive contraction. Thus there exists a unitary u in $\mathfrak{M}_\varepsilon$ such that $u\psi_\varepsilon(1_{\mathfrak{A}})u^*$ is a positive diagonal matrix less than the identity. Now, if we replace ψ_ε with $u\psi_\varepsilon(\cdot)u^*$ and φ_ε with $\varphi_\varepsilon(u^* \cdot u)$, then we obtain a pair of maps which also satisfy Assertions (5.1) and (5.2) (up to replacing $\mathfrak{B}_\varepsilon$ with $u\mathfrak{B}_\varepsilon u^*$). We shall henceforth assume we have made this change.

Thus, $\psi_\varepsilon(1_{\mathfrak{A}})$ is a diagonal matrix which we denote by D such that $0 \leqslant D \leqslant 1$. Now, by construction, $\|\psi_\varepsilon(1_{\mathfrak{A}}) - \psi_\varepsilon(1)^2\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon < \frac{1}{4}$. Hence we conclude that the spectrum $\sigma(D)$ of D is a compact subset of $[0,\varepsilon] \cup [1-\varepsilon, 1]$; moreover $[0,\varepsilon] \cap [1-\varepsilon,1] = \emptyset$.

Let P be the projection on the sum of the spectral subspaces of D associated with the eigenvalues in $[1-\varepsilon, 1]$. Thus $1 - P$ is the projection on the sum of the spectral subspaces of D associated with eigenvalues in $[0,\varepsilon]$.

Let now $x \in \mathfrak{F}_1$. Then:

$$\begin{aligned}
&\|\psi_\varepsilon(x) - D\psi_\varepsilon(x)D\|_{\mathfrak{B}_\varepsilon} \\
&\qquad \leqslant \|\psi(x) - \psi(x)D\|_{\mathfrak{B}_\varepsilon} + \|\psi(x)D - D\psi(x)D\|_{\mathfrak{B}_\varepsilon} \\
&\qquad = \|\psi(x1_{\mathfrak{A}}) - \psi(x)\psi(1_{\mathfrak{A}})\|_{\mathfrak{B}_\varepsilon} + \|\left(\psi(1_{\mathfrak{A}}x) - \psi(1_{\mathfrak{A}})\psi(x)\right)D\|_{\mathfrak{B}_\varepsilon} \\
&\qquad \leqslant \varepsilon + \varepsilon\|D\|_{\mathfrak{B}_\varepsilon} \leqslant 2\varepsilon.
\end{aligned}$$

Moreover:

$$\begin{aligned}
\|D\psi(x)D - P\psi(x)P\|_{\mathfrak{B}_\varepsilon} &= \|(D-P)\psi(x)D\|_{\mathfrak{B}_\varepsilon} + \|P\psi(x)(D-P)\|_{\mathfrak{B}_\varepsilon} \\
&\leqslant 2\|D-P\|_{\mathfrak{B}_\varepsilon}\|x\|_{\mathfrak{A}}.
\end{aligned}$$

Now by construction, $\|D - P\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon$. Thus:

$$\begin{aligned}
\|\psi(x) - P\psi(x)P\|_{\mathfrak{B}_\varepsilon} &\leqslant \|\psi(x) - D\psi(x)D\|_{\mathfrak{B}_\varepsilon} + \|D\psi(x)D - P\psi(x)P\|_{\mathfrak{B}_\varepsilon} \\
&\leqslant 2\varepsilon(1 + \|x\|_{\mathfrak{A}}).
\end{aligned} \tag{5.3}$$

The map $\varsigma_\varepsilon = P\psi_\varepsilon(\cdot)P$ is a positive contraction by construction. We now check that our construction leads to the desired conclusion, with $\mathfrak{D}_\varepsilon = P\mathfrak{B}_\varepsilon P$. We rename P as $1'_\varepsilon$. We have:

$$\|1'_\varepsilon - \varsigma_\varepsilon(1_{\mathfrak{A}})\|_{\mathfrak{B}_\varepsilon} = \|P - PDP\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon.$$

Thus, as $\varepsilon > 0$ is arbitrary, we have established:

$$\lim_{t \to 0} \|1'_t - \varsigma_t(1_{\mathfrak{A}})\|_{\mathfrak{D}_t} = 0.$$

Now let $x \in \mathfrak{F}_1$. Using Inequality (5.3):

$$\begin{aligned}
\|x - \varphi_\varepsilon(\varsigma_\varepsilon(x))\|_{\mathfrak{A}} &= \|x - \varphi_\varepsilon(P\psi_\varepsilon(x)P)\|_{\mathfrak{A}} \\
&\leqslant \|x - \varphi_\varepsilon(\psi_\varepsilon(x))\|_{\mathfrak{A}} + \|\varphi_\varepsilon(\psi_\varepsilon(x)) - \varphi_\varepsilon(P\psi_\varepsilon(x)P)\|_{\mathfrak{A}} \\
&\leqslant \varepsilon + \|\psi_\varepsilon(x) - P\psi_\varepsilon(x)P\|_{\mathfrak{B}_\varepsilon} \\
&\leqslant \varepsilon(3 + 2\|x\|_{\mathfrak{A}}).
\end{aligned}$$

Consequently, we have for all $x \in \mathfrak{F}_1$:

$$\lim_{t\to 0} \|x - \varphi_t \circ \varsigma_t(x)\|_{\mathfrak{A}} = 0$$

Last, let $x, y \in \mathfrak{F}_1$. Since $1_{\mathfrak{A}} \in \mathfrak{F}_1$ as well, we obtain:

$$\begin{aligned}
&\|\varsigma_\varepsilon(x)\varsigma_\varepsilon(y) - \varsigma_\varepsilon(xy)\|_{\mathfrak{B}_\varepsilon} \\
&\quad = \|P\psi_\varepsilon(x)P\psi_\varepsilon(y)P - P\psi_\varepsilon(xy)P\|_{\mathfrak{B}_\varepsilon} \\
&\quad \leqslant \|\psi_\varepsilon(x)P\psi_\varepsilon(y) - \psi_\varepsilon(xy)\|_{\mathfrak{B}_\varepsilon} \\
&\quad \leqslant \|\psi_\varepsilon(x)P\psi_\varepsilon(y) - \psi_\varepsilon(x)\psi_\varepsilon(y)\|_{\mathfrak{B}_\varepsilon} + \|\psi_\varepsilon(x)\psi_\varepsilon(y) - \psi_\varepsilon(xy)\|_{\mathfrak{B}_\varepsilon} \\
&\quad \leqslant \|\psi_\varepsilon(x)\psi_\varepsilon(1_{\mathfrak{A}})\psi_\varepsilon(y) - \psi_\varepsilon(x1_{\mathfrak{A}})\psi_\varepsilon(y)\| + \varepsilon \\
&\quad \leqslant \varepsilon\|y\|_{\mathfrak{A}} + \varepsilon = \varepsilon(1 + \|y\|_{\mathfrak{A}}).
\end{aligned}$$

Note (although not needed for our proof) that by symmetry, grouping $1_{\mathfrak{A}}$ with y instead of x above, we get:

$$\|\psi_\varepsilon(x)\psi_\varepsilon(y) - \psi_\varepsilon(xy)\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon \min\{\|x\|, \|y\|\}.$$

Thus, for all $x, y \in \mathfrak{F}_1$:

$$\lim_{t\to 0} \|\varsigma_t(xy) - \varsigma_t(x)\varsigma_t(y)\|_{\mathfrak{D}_t} = 0.$$

This concludes our proof. □

Corollary 5.5. *A unital nuclear quasi-diagonal C*-algebra $\mathfrak{A}$ is a unital pseudo-diagonal C*-algebra.*

Proof. Let $\mathfrak{F}$ be a finite subset of $\mathfrak{A}$, which we assume without loss of generality, contains $1_{\mathfrak{A}}$. Let $\varepsilon < \frac{1}{4}$. By Lemma (5.4), there exist a finite dimensional C*-algebra $\mathfrak{B}_\varepsilon$ and two positive contractions $\varphi_\varepsilon : \mathfrak{B}_\varepsilon \to \mathfrak{A}$ and $\psi_\varepsilon : \mathfrak{A} \to \mathfrak{B}_\varepsilon$ such that:

(1) for all $x \in \mathfrak{F}$ we have $\|x - \varphi_\varepsilon \circ \psi_\varepsilon(x)\|_{\mathfrak{A}} \leqslant \varepsilon$,
(2) for all $x, y \in \mathfrak{F}$ we have $\|\psi_\varepsilon(xy) - \psi_\varepsilon(x)\psi_\varepsilon(y)\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon$,
(3) we have $\|1_\varepsilon - \psi_\varepsilon(1_\varepsilon)\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon$, where 1_ε is the unit of $\mathfrak{B}_\varepsilon$.

Since $\|\psi_\varepsilon(1_{\mathfrak{A}}) - 1_\varepsilon\|_{\mathfrak{B}_\varepsilon} < 1$, the matrix $\psi_\varepsilon(1_{\mathfrak{A}})$ is invertible, while it is also positive, of course.

We also note that:

$$\begin{aligned}
\|1_{\mathfrak{A}} - \varphi_\varepsilon(1_\varepsilon)\|_{\mathfrak{A}} &= \|1_{\mathfrak{A}} - \varphi_\varepsilon \circ \psi_\varepsilon(1_{\mathfrak{A}})\|_{\mathfrak{B}_\varepsilon} + \|\varphi_\varepsilon \circ \psi_\varepsilon(1_{\mathfrak{A}}) - \varphi_\varepsilon(1_\varepsilon)\|_{\mathfrak{A}} \\
&\leqslant \varepsilon + \|\psi_\varepsilon(1_{\mathfrak{A}}) - 1_\varepsilon\|_{\mathfrak{B}_\varepsilon} \\
&\leqslant 2\varepsilon < 1.
\end{aligned}$$

Thus $\varphi(1_{\mathfrak{A}})$ is also invertible in $\mathfrak{A}$ — and of course also positive.

Define:

$$\theta_\varepsilon = \varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\varphi_\varepsilon(\cdot)\varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}} \text{ and } \varsigma_\varepsilon = \psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\psi_\varepsilon(\cdot)\psi(1_{\mathfrak{A}})^{-\frac{1}{2}}.$$

We first note that θ_ε and ς_ε are both positive linear maps. Moreover, θ_ε and ς_ε are both unital by construction, so they are contractions as well.

Now, let $a, b \in \mathfrak{F}$. Then:

$$\begin{aligned}
&\|\varsigma_\varepsilon(ab) - \varsigma_\varepsilon(a)\varsigma_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} \\
&\quad\leqslant \|\psi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\|^2_{\mathfrak{B}_\varepsilon}\|\psi_\varepsilon(ab) - \psi_\varepsilon(a)\psi_\varepsilon^{-1}(1_\varepsilon)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} \\
&\quad\leqslant \|\psi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\|^2_{\mathfrak{B}_\varepsilon} \\
&\quad\left(\|\psi_\varepsilon(a)\left(1_\varepsilon - \psi_\varepsilon^{-1}(1_\varepsilon)\right)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} + \|\psi_\varepsilon(ab) - \psi_\varepsilon(a)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon}\right) \\
&\quad\leqslant \|\psi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\|^2_{\mathfrak{B}_\varepsilon}\left(\left\|1_\varepsilon - \psi_\varepsilon^{-1}(1_\varepsilon)\right\|_{\mathfrak{B}_\varepsilon}\|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}} + \|\psi_\varepsilon(ab) - \psi_\varepsilon(a)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon}\right)
\end{aligned}$$

since ψ_ε is a contraction. Now, the spectrum of the self-adjoint element $\psi_\varepsilon(1_{\mathfrak{A}})$ is a subset of $[1-\varepsilon, 1]$ since $\|\psi_\varepsilon(1_{\mathfrak{A}}) - 1_\varepsilon\|_{\mathfrak{B}_\varepsilon} \leqslant \varepsilon$. Consequently, as $\varepsilon < 1$, by the functional mapping theorem, the spectrum of $\psi_\varepsilon(1_{\mathfrak{A}})^{-1} - 1_\varepsilon$ is included in $[0, \frac{1}{1-\varepsilon} - 1]$. Thus, as $\psi_\varepsilon(1_{\mathfrak{A}})^{-1} - 1_\varepsilon$ is a self-adjoint element:

$$\left\|\psi_\varepsilon(1_{\mathfrak{A}})^{-1} - 1_\varepsilon\right\|_{\mathfrak{B}_\varepsilon} \leqslant \frac{1}{1-\varepsilon} - 1,$$

and thus $\lim_{\varepsilon\to 0}\|\psi_\varepsilon(1_{\mathfrak{A}})^{-1} - 1_\varepsilon\|_{\mathfrak{B}_\varepsilon} = 0$. Similarly, $\lim_{\varepsilon\to 0}\|1_\varepsilon - \psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\|_{\mathfrak{B}_\varepsilon} = 0$, and thus in particular, $\lim_{\varepsilon\to 0}\|\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\|_{\mathfrak{B}_\varepsilon} = 1$.

Since $\lim_{\varepsilon\to 0}\|\psi_\varepsilon(ab) - \psi_\varepsilon(a)\psi_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} = 0$, we obtain:

$$\lim_{\varepsilon\to 0}\|\varsigma_\varepsilon(ab) - \varsigma_\varepsilon(a)\varsigma_\varepsilon(b)\|_{\mathfrak{B}_\varepsilon} = 0.$$

Similarly, for all $a \in \mathfrak{F}$, we have:

$$\begin{aligned}
&\|a - \theta_\varepsilon \circ \varsigma_\varepsilon(a)\|_{\mathfrak{A}} \\
&\quad= \|a - \varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\varphi_\varepsilon\left(\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\psi_\varepsilon(a)\psi_\varepsilon^{-\frac{1}{2}}(1_{\mathfrak{A}})\right)\varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\|_{\mathfrak{A}} \\
&\quad\leqslant \|a - \varphi_\varepsilon \circ \psi_\varepsilon(a)\|_{\mathfrak{A}} \\
&\qquad+ \left\|\varphi_\varepsilon\left(\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\psi_\varepsilon(a)\left(1_\varepsilon - \psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\right)\right)\right\|_{\mathfrak{A}} \\
&\qquad+ \left\|\varphi_\varepsilon\left(\left(1_\varepsilon - \psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\right)\psi_\varepsilon(a)\right)\right\|_{\mathfrak{A}} \\
&\qquad+ \left\|\left(1_{\mathfrak{A}} - \varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\right)\varphi_\varepsilon\left(\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\psi_\varepsilon(a)\psi_\varepsilon^{-\frac{1}{2}}(1_{\mathfrak{A}})\right)\right\|_{\mathfrak{A}} \\
&\qquad+ \left\|\varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\varphi_\varepsilon\left(\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\psi_\varepsilon(a)\psi_\varepsilon^{-\frac{1}{2}}(1_{\mathfrak{A}})\right)\left(1_{\mathfrak{A}} - \varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\right)\right\|_{\mathfrak{A}} \\
&\quad\leqslant \|a - \varphi_\varepsilon \circ \psi_\varepsilon(a)\|_{\mathfrak{A}} \\
&\qquad+ \left\|1_\varepsilon - \psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\right\|_{\mathfrak{B}_\varepsilon}\|a\|_{\mathfrak{A}}\left(1 + \|\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}\|_{\mathfrak{B}_\varepsilon}\right) \\
&\qquad+ \left\|1_{\mathfrak{A}} - \varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\right\|_{\mathfrak{A}}\|\psi_\varepsilon(1_{\mathfrak{A}})^{-\frac{1}{2}}(1_{\mathfrak{A}})\|^2_{\mathfrak{B}_\varepsilon}\|a\|_{\mathfrak{A}}\left(1 + \|\varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}}\|_{\mathfrak{A}}\right).
\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\|\varphi_\varepsilon(1_\varepsilon) - 1_{\mathfrak{A}}\|_{\mathfrak{A}} = 0$, we check again easily that $\lim_{\varepsilon\to 0}\|\varphi_\varepsilon(1_\varepsilon)^{-\frac{1}{2}} - 1_{\mathfrak{A}}\|_{\mathfrak{A}} = 0$, and thus, since $\lim_{\varepsilon\to 0}\|a - \varphi_\varepsilon \circ \psi_\varepsilon(a)\|_{\mathfrak{A}} = 0$, we conclude:

$$\lim_{\varepsilon\to 0}\|a - \theta_\varepsilon \circ \varsigma_\varepsilon(a)\|_{\mathfrak{A}} = 0.$$

Now, if $c, c' \in \mathfrak{F}$, then we have:

$$\begin{aligned}0 &\leqslant \|\varsigma_\varepsilon\left(c \circ c'\right) - \varsigma_\varepsilon(c) \circ \varsigma_\varepsilon(c')\| \\ &\leqslant \frac{1}{2}\left(\|\varsigma_\varepsilon(c)\varsigma_\varepsilon(c') - \varsigma_\varepsilon(cc')\|_{\mathfrak{B}_\varepsilon} + \|\varsigma_\varepsilon(c')\varsigma_\varepsilon(c) - \varsigma_\varepsilon(c'c)\|_{\mathfrak{B}_\varepsilon}\right)\end{aligned}$$

and thus, for all $c, c' \in \mathfrak{F}$:

$$\lim_{\varepsilon \to 0} \|\varsigma_\varepsilon\left(c \circ c'\right) - \varsigma_\varepsilon(c) \circ \varsigma_\varepsilon(c')\|_{\mathfrak{B}_\varepsilon} = 0.$$

The same holds for the Lie product.

This concludes our proof. □

If $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space, with $\mathfrak{A}$ a pseudo-diagonal C*-algebra, then Definition (5.1) provides some finite dimensional C*-algebraic approximations. The question, of course, is how to construct Lip-norms on these finite dimensional approximations. The main difficulty is to obtain Lip-norms with the Leibniz property. We are, in fact, only able to construct Lip-norms with the quasi-Leibniz property, albeit arbitrarily close to the Leibniz property.

This construction is given in the following lemma:

Lemma 5.6. *Let $C \geqslant 1$ and $D \geqslant 0$. Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ be a (C, D)-quasi-Leibniz quantum compact metric space, and let $\varepsilon > 0$. Assume that we are given a ε^2-dense finite subset $\mathfrak{F}$ of:*

$$\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_\mathfrak{A}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$$

for some $\mu \in \mathscr{S}(\mathfrak{A})$, as well as a finite dimensional C-algebra $\mathfrak{B}$ and two unital, positive linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that:*

1. *for all $a \in \mathfrak{F}$, we have $\|a - \varphi \circ \psi(a)\|_\mathfrak{A} \leqslant \varepsilon^2$,*
2. *for all $a, b \in \mathfrak{F}$, we have $\|\psi(a) \circ \psi(b) - \psi\left(a \circ b\right)\|_\mathfrak{B} \leqslant \varepsilon^2$,*
3. *for all $a, b \in \mathfrak{F}$, we have $\|\{\psi(a), \psi(b)\} - \psi\left(\{a, b\}\right)\|_\mathfrak{B} \leqslant \varepsilon^2$.*

We then define:

$$\mathfrak{L} = \left\{b \in \mathfrak{sa}(\mathfrak{B}) : \exists a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{L}_\mathfrak{A}(a) \leqslant 1 \text{ and } \|b - \psi(a)\|_\mathfrak{B} \leqslant \varepsilon\right\}.$$

Then $\mathfrak{L}$ is a balanced convex set and the associated Minkowsky functional L is a:

$$\left[C(1 + 2\varepsilon), 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D\right]$$

quasi-Leibniz Lip-norm on $\mathfrak{B}$.

Proof. We split our proof in a few steps for clarity.

Step 1. *We first check that we may replace $\mathfrak{F}$ with $\mathfrak{F} + \mathbb{R}1_\mathfrak{A}$ in our lemma, at no cost.*

Our first observation is that if $t, u \in \mathbb{R}$ then, and for all $c, c' \in \mathfrak{F}$, we have:

$$\begin{aligned}&\|\psi(c + t1_\mathfrak{A})\psi(c' + u1_\mathfrak{A}) - \psi((c + t1_\mathfrak{A})(c' + u1_\mathfrak{A}))\|_\mathfrak{A} \\ &\qquad \leqslant \|\psi(c)\psi(c') - \psi(cc')\|_\mathfrak{A} + \|t\psi(c') - t\psi(c')\|_\mathfrak{A} \\ &\qquad\quad + \|u\psi(c) - u\psi(c)\|_\mathfrak{A} + \|(tu - tu)1_\mathfrak{A}\|_\mathfrak{A} \\ &\qquad = \|\psi(c)\psi(c') - \psi(cc')\|_\mathfrak{A}.\end{aligned}$$

Consequently, since $c, c' \in \mathfrak{sa}(\mathfrak{A})$, $t, u \in \mathbb{R}$ and ψ is positive (hence star-preserving), we have:

$$\|\psi(c + t1_{\mathfrak{A}}) \circ \psi(c' + u1_{\mathfrak{A}}) - \psi\left((c + t1_{\mathfrak{A}}) \circ \left(c' + u1_{\mathfrak{A}}\right)\right)\|_{\mathfrak{A}} \leqslant \varepsilon^2$$

and

$$\|\left\{\psi(c + t1_{\mathfrak{A}}), \psi(c' + u1_{\mathfrak{A}})\right\} - \psi\left(\left\{c + t1_{\mathfrak{A}}, c' + u1_{\mathfrak{A}}\right\}\right)\|_{\mathfrak{A}} \leqslant \varepsilon^2.$$

We also have:

$$\|c + t1_{\mathfrak{A}} - \varphi \circ \psi(c + t1_{\mathfrak{A}})\|_{\mathfrak{A}} = \|c - \varphi \circ \psi(c)\|_{\mathfrak{A}} \leqslant \varepsilon^2.$$

Hence, we may replace $\mathfrak{F}$ with $\mathfrak{G} = (\mathfrak{F} \cup \{0\}) + \mathbb{R}1_{\mathfrak{A}}$, while Assumptions (1), (2) and (3) of our Lemma remain satisfied.

Now, let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$. Then $\mathsf{L}_{\mathfrak{A}}(a - \mu(a)1_{\mathfrak{A}}) \leqslant 1$ and $\mu(a - \mu(a)1_{\mathfrak{A}}) = 0$. Thus there exists $c \in \mathfrak{F}$ such that $\|(a - \mu(a)1_{\mathfrak{A}}) - c\|_{\mathfrak{A}} \leqslant \varepsilon^2$. Hence, if $d = c + \mu(a)1_{\mathfrak{A}}$ then $d \in \mathfrak{G}$ and of course $\|a - d\|_{\mathfrak{A}} \leqslant \varepsilon^2$.

Step 2. *With these observations in mind, let us study the geometry of the set $\mathfrak{L}$. We begin by showing that $\mathfrak{L}$ is a balanced, convex set, and that L is finite on $\mathfrak{B}$.*

If $b, b' \in \mathfrak{L}$ then there exists $a, a' \in \mathfrak{sa}(\mathfrak{A})$ with $\|b - \psi(a)\|_{\mathfrak{B}} \leqslant \varepsilon$ and $\|b' - \psi(a')\|_{\mathfrak{B}} \leqslant \varepsilon$ while $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$ and $\mathsf{L}_{\mathfrak{A}}(a') \leqslant 1$. Thus, for any $t \in [0,1]$, we have $\mathsf{L}_{\mathfrak{A}}(ta + (1-t)a) \leqslant 1$ and $\|(tb + (1-t)b') - (ta + (1-t)a')\|_{\mathfrak{B}} \leqslant \varepsilon$. Thus $tb + (1-t)b' \in \mathfrak{L}$, i.e $\mathfrak{L}$ is convex.

Moreover, for all $t \in [-1,1]$, we have $\mathsf{L}_{\mathfrak{A}}(ta) \leqslant t \leqslant 1$ and $\|tb - \psi(ta)\|_{\mathfrak{B}} \leqslant t\varepsilon \leqslant \varepsilon$, so $\mathfrak{L}$ is balanced.

Since $\mathsf{L}_{\mathfrak{A}}(0) = 0$, we have $0 \in \mathfrak{L}$, and thus by construction of $\mathfrak{L}$, the closed ball of center 0 and radius $\varepsilon > 0$ lies in $\mathfrak{L}$. This implies in turn that the Minkowsky functional of $\mathfrak{L}$ is finite on $\mathfrak{B}$.

Step 3. *We now check that $\mathsf{L}(b) = 0$ if and only if b is a scalar multiple of the unit of $\mathfrak{B}$.*

First, let $t \in \mathbb{R}$. Then $\|t1_{\mathfrak{B}} - \psi(t1_{\mathfrak{A}})\|_{\mathfrak{B}} = 0$ and $\mathsf{L}_{\mathfrak{A}}(t1_{\mathfrak{A}}) = 0$ so $t1_{\mathfrak{B}} \in \mathfrak{L}$. Thus $\mathsf{L}(t1_{\mathfrak{B}}) \leqslant 1$ and thus, as $t \in \mathbb{R}$ is arbitrary, then $\mathsf{L}(1_{\mathfrak{B}}) = 0$.

Assume that $\mathsf{L}(b) = 0$ for some $b \in \mathfrak{sa}(\mathfrak{B})$. Then $\mathsf{L}(tb) = 0$ for all $t \in \mathbb{R}$, so $tb \in \mathfrak{L}$ for all $t \in \mathbb{R}$. Thus for all $t \in \mathbb{R}$ there exists $a_t \in \mathfrak{sa}(\mathfrak{A})$ such that $\mathsf{L}_{\mathfrak{A}}(a_t) \leqslant 1$ and:

$$\|tb - \psi(a_t)\|_{\mathfrak{B}} \leqslant \varepsilon.$$

Thus, if $t > 0$, then $\left\|b - \psi\left(\frac{1}{t}a_t\right)\right\|_{\mathfrak{B}} \leqslant \frac{1}{t}\varepsilon$. Let $a'_t = \frac{1}{t}a_t$ for all $t > 0$. Now $\mathsf{L}_{\mathfrak{A}}(a'_t) \leqslant t^{-1}$.

Now, for all $n \in \mathbb{N} \setminus \{0\}$, there exists $c_n \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(c_n) \leqslant 1$ and $\|a'_n - c_n\|_{\mathfrak{A}} \leqslant \varepsilon^2$. Thus:

$$\begin{aligned}\|b - \psi(c_n)\|_{\mathfrak{B}} &\leqslant \|b - \psi(a'_n)\|_{\mathfrak{B}} + \|\psi(a'_n - c_n)\|_{\mathfrak{B}} \\ &\leqslant \frac{\varepsilon}{n} + \varepsilon^2\end{aligned}$$

and thus, we have:

$$\|a'_n\|_{\mathfrak{A}} \leqslant \|c_n\| + \varepsilon^2 \leqslant \|\psi(c_n)\|_{\mathfrak{B}} + 2\varepsilon^2 \leqslant \varepsilon + 3\varepsilon^2 + \|b\|_{\mathfrak{B}}$$

by Remark (5.3), while $\mathsf{L}(a'_n) \leqslant \frac{1}{n}$. Therefore, the sequence $(a'_n)_{n \in \mathbb{N}, n > 0}$ lies in the compact set $\{c \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(c) \leqslant 1, \|c\|_{\mathfrak{A}} \leqslant \|b\|_{\mathfrak{B}} + \varepsilon + 3\varepsilon^2\}$ (using Theorem

(2.10)). So we can extract a subsequence of $(a'_n)_{n\in\mathbb{N},n>0}$ converging to some a. Now by continuity, $\|b-\psi(a)\|_{\mathfrak{B}}=0$ while, by lower semi-continuity, $\mathsf{L}_{\mathfrak{A}}(a)=0$. Thus $a=k1_{\mathfrak{A}}$ for some $k\in\mathbb{R}$ since $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ is a Lipschitz pair. Thus as ψ is unital, we conclude that $b=k1_{\mathfrak{B}}$.

Step 4. *We prove that for any $\nu\in\mathscr{S}(\mathfrak{B})$, we have $\mathfrak{L}=\{b\in\mathfrak{L}:\nu(b)=0\}+\mathbb{R}1_{\mathfrak{A}}$.*

If $b\in\mathfrak{L}$ then there exists $a\in\mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1$ and $\|\psi(a)-b\|_{\mathfrak{B}}\leqslant\varepsilon$. Thus for all $t\in\mathbb{R}$, $\mathsf{L}_{\mathfrak{A}}(a-t1_{\mathfrak{A}})=\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1$ since $\mathsf{L}_{\mathfrak{A}}(1_{\mathfrak{A}})=0$ and $\mathsf{L}_{\mathfrak{A}}$ is a seminorm, and then:

$$\|(b-t1_{\mathfrak{B}})-\psi(a-t1_{\mathfrak{A}})\|_{\mathfrak{B}}=\|b-\psi(a)\|_{\mathfrak{B}}\leqslant\varepsilon.$$

Consequently, $b-t1_{\mathfrak{B}}\in\mathfrak{L}$. From this, we conclude that:

- if $b\in\mathfrak{L}$ then $b-\nu(b)1_{\mathfrak{B}}\in\mathfrak{L}$. Since $\nu(b-\nu(b)1_{\mathfrak{B}})=0$, and $b=(b-\nu(b)1_{\mathfrak{B}})+\nu(b)1_{\mathfrak{B}}$, we have $b\in\{d\in\mathfrak{L}:\nu(d)=0\}+\mathbb{R}1_{\mathfrak{B}}$.
- if $b=d+t1_{\mathfrak{B}}$ with $d\in\mathfrak{L}$ and $\nu(d)=0$, then $b\in\mathfrak{L}$.

This proves our step.

Step 5. *Fix $\nu\in\mathscr{S}(\mathfrak{B})$. We now prove that $\mathfrak{L}$ is closed and $\{b\in\mathfrak{L}:\nu(b)=0\}$ is compact.*

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{L}$ with $\nu(b_n)=0$. Then for all $n\in\mathbb{N}$, let $a_n\in\mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a_n)\leqslant 1$ and $\|\psi(a_n)-b\|_{\mathfrak{B}}\leqslant\varepsilon$. Let $\xi=\nu\circ\psi\in\mathscr{S}(\mathfrak{A})$ (note that ψ is unital, positive, linear).

Let $n\in\mathbb{N}$. We first note that:

$$|\xi(a_n)|\leqslant|\xi(a_n)-\nu(b_n)|+|\nu(b_n)|\leqslant|\nu(\psi(a_n)-b_n)|\leqslant\varepsilon.$$

Hence, for all $\eta\in\mathscr{S}(\mathfrak{A})$, we have:

$$|\eta(a_n)|\leqslant|\eta(a_n)-\xi(a_n)|+|\xi(a_n)|\leqslant\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}(\eta,\xi)+\varepsilon$$

since $\mathsf{L}_{\mathfrak{A}}(a_n)\leqslant 1$; since $a_n\in\mathfrak{sa}(\mathfrak{A})$ we conclude:

$$\|a_n\|_{\mathfrak{A}}\leqslant\operatorname{diam}\left(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right)+\varepsilon.$$

We thus observe that:

$$\|b\|_{\mathfrak{B}}\leqslant\|a\|_{\mathfrak{A}}+\varepsilon\leqslant\operatorname{diam}\left(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right)+2\varepsilon.$$

As $\mathfrak{B}$ is finite dimensional, the bounded sequence $(b_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(b_{g(n)})_{n\in\mathbb{N}}$ with limit denoted by $b\in\mathfrak{sa}(\mathfrak{B})$. By continuity of ν we have $\nu(b)=0$.

We note, as a digression, that we also can conclude that $\operatorname{diam}(\mathscr{S}(\mathfrak{B}),\mathsf{mk}_{\mathsf{L}})<\infty$, though we will conclude a stronger fact shortly.

Indeed, we have now proven that for all $n\in\mathbb{N}$, we have $\mathsf{L}_{\mathfrak{A}}(a_{g(n)})\leqslant 1$ and $\|a_{g(n)}\|_{\mathfrak{A}}\leqslant K$ for some fixed $K>0$. The set $\{a\in\mathfrak{sa}(\mathfrak{A}):\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1,\|a\|_{\mathfrak{A}}\leqslant K\}$ is norm compact since $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ is a quasi-Leibniz quantum compact metric space by Theorem (2.10). Thus, there exists a convergent subsequence $(a_{g\circ f(n)})_{n\in\mathbb{N}}$ of $(a_{g(n)})_{n\in\mathbb{N}}$ which converges in norm to some $a\in\mathfrak{sa}(\mathfrak{A})$, with $\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1$, as $\mathsf{L}_{\mathfrak{A}}$ is a lower semi-continuous Lip-norm.

Thus by continuity, $\|b-\psi(a)\|_{\mathfrak{B}}\leqslant\varepsilon$ and thus $b\in\mathfrak{L}$. Hence, we have shown that any sequence of elements in $\{b\in\mathfrak{L}:\nu(b)=0\}$ has a convergent subsequence with limit in the same set. Therefore, $\{b\in\mathfrak{L}:\nu(b)=0\}$ is norm compact.

Thus, $\mathfrak{L} = \{b \in \mathfrak{L} : \nu(b) = 0\} + \mathbb{R}1_{\mathfrak{B}}$ is closed. Thus L is a lower semi-continuous Lip-norm on $\mathfrak{B}$ by Theorem (2.10).

Step 6. *It remains to study the quasi-Leibniz property of* L.

Let $b, b' \in \mathfrak{L}$. There exist $a, a' \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$, $\mathsf{L}_{\mathfrak{A}}(a') \leqslant 1$ while $\|\psi(a) - b\|_{\mathfrak{B}} \leqslant \varepsilon$ and $\|\psi(a') - b'\|_{\mathfrak{B}} \leqslant \varepsilon$. Then there exists $c, c' \in \mathfrak{G}$ with $\|a - c\|_{\mathfrak{A}} \leqslant \varepsilon^2$ and $\|a' - c'\|_{\mathfrak{A}} \leqslant \varepsilon^2$, as explained in Step 1.

Now:

$$\|\psi(a)\|_{\mathfrak{B}} \leqslant \|b\|_{\mathfrak{B}} + \varepsilon \text{ and } \|\psi(c)\|_{\mathfrak{B}} \leqslant \|\psi(c) - \psi(a)\|_{\mathfrak{B}} + \|\psi(a)\|_{\mathfrak{B}} \leqslant \|b\|_{\mathfrak{B}} + \varepsilon + \varepsilon^2.$$

Our goal is now to find an upper bound for $\|b \circ b' - \psi(a \circ a')\|_{\mathfrak{B}}$. Thus, as a, a', b, b' are all self-adjoint and ψ preserves the star operation:

$$
\begin{aligned}
&\|b \circ b' - \psi\left(a \circ a'\right)\|_{\mathfrak{B}} && (5.4)\\
&\leqslant \|b \circ b' - \psi(a) \circ \psi(a')\|_{\mathfrak{B}} + \|\psi(a) \circ \psi(a') - \psi\left(a \circ a'\right)\|_{\mathfrak{B}} && (5.5)\\
&\leqslant \|b\|_{\mathfrak{B}}\|b' - \psi(a')\|_{\mathfrak{B}} && (5.6)\\
&\quad + \|\psi(a')\|_{\mathfrak{B}}\|b - \psi(a)\|_{\mathfrak{B}} && (5.7)\\
&\quad + \|\psi(a) \circ \psi(a') - \psi(c) \circ \psi(a')\|_{\mathfrak{B}} && (5.8)\\
&\quad + \|\psi(c) \circ \psi(a') - \psi(c) \circ \psi(c')\|_{\mathfrak{B}} && (5.9)\\
&\quad + \|\psi(c) \circ \psi(c') - \psi\left(c \circ c'\right)\|_{\mathfrak{B}} && (5.10)\\
&\quad + \|\psi\left(c \circ c'\right) - \psi\left(a \circ a'\right)\|_{\mathfrak{B}}. && (5.11)
\end{aligned}
$$

We now provide a bound for each of the Terms (5.6–5.11). By construction, $\|b' - \psi(a')\|_{\mathfrak{B}} \leqslant \varepsilon$, so Term (5.6) is dominated by $\|b\|_{\mathfrak{B}}\varepsilon$.

Similarly, since $\|\psi(a')\|_{\mathfrak{B}} \leqslant \|b'\|_{\mathfrak{B}} + \varepsilon$ and $\|\psi(a) - b\|_{\mathfrak{B}} \leqslant \varepsilon$, we conclude that Terms (5.7) is dominated by $(\|b'\|_{\mathfrak{B}} + \varepsilon)\,\varepsilon$.

Now:

$$\|\psi(a) \circ \psi(a') - \psi(c) \circ \psi(a')\|_{\mathfrak{B}} \leqslant \|\psi(a) - \psi(c)\|_{\mathfrak{B}}\|\psi(a')\|_{\mathfrak{A}} \leqslant \varepsilon^2\left(\|b'\|_{\mathfrak{B}} + \varepsilon\right)$$

which gives us our bound for Term (5.8).

We derive a similar upper bound on Term (5.9):

$$
\begin{aligned}
\|\psi(c) \circ \psi(a') - \psi(c) \circ \psi(c')\|_{\mathfrak{B}} &\leqslant \|\psi(a') - \psi(c')\|_{\mathfrak{B}}\|\psi(c)\|_{\mathfrak{A}}\\
&\leqslant \varepsilon^2\left(\|b\|_{\mathfrak{B}} + \varepsilon + \varepsilon^2\right).
\end{aligned}
$$

By our choice of ψ, Term (5.10) is bounded above by ε^2. It is the appearance of this term, and the quasi-multiplicative property of ψ, which motivates our computation.

Last, concerning of Term (5.11), we first note that $\|a\|_{\mathfrak{A}} \leqslant \|c\|_{\mathfrak{A}} + \varepsilon^2$ while $\|c\|_{\mathfrak{A}} \leqslant \|\psi(c)\|_{\mathfrak{B}} + \varepsilon^2$ by Remark (5.3). Thus:

$$\|a\|_{\mathfrak{A}} \leqslant \|b\|_{\mathfrak{B}} + \varepsilon + 3\varepsilon^2,$$

while similarly, $\|c'\|_{\mathfrak{A}} \leqslant \|\psi(c')\| + \varepsilon^2 \leqslant \|b'\| + \varepsilon + 2\varepsilon^2$. Thus:

$$\|aa' - cc'\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\|a' - c'\|_{\mathfrak{A}} + \|c'\|_{\mathfrak{A}}\|a - c\|_{\mathfrak{A}} \leqslant \varepsilon^2\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}} + 2\varepsilon + 5\varepsilon^2\right).$$

Consequently:

$$\|\psi(a \circ a' - b \circ b')\|_{\mathfrak{B}} \leqslant \|a \circ a' - c \circ c'\|_{\mathfrak{A}} \leqslant \varepsilon^2 \left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}} + 2\varepsilon + 5\varepsilon^2 \right).$$

We thus obtain the following estimate:

$$\begin{aligned} \|b \circ b' - \psi(a \circ a')\|_{\mathfrak{B}} \leqslant \|b\|_{\mathfrak{B}}\varepsilon + \left(\|b'\|_{\mathfrak{B}} + \varepsilon\right)\varepsilon \\ + \varepsilon^2 \left(\|b'\| + \varepsilon\right) + \varepsilon^2 \left(\|b\| + \varepsilon + \varepsilon^2\right) \\ + \varepsilon^2 + \varepsilon^2 \left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}} + 2\varepsilon + 5\varepsilon^2\right), \end{aligned} \tag{5.12}$$

which simplifies to:

$$\|b \circ b' - \psi\left(a \circ a'\right)\|_{\mathfrak{B}} \leqslant \varepsilon \left((1+2\varepsilon)\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}}\right) + 2\varepsilon + 4\varepsilon^2 + 6\varepsilon^3 \right). \tag{5.13}$$

We now compute an upper bound on the Lip-norm of the Jordan product $a \circ a'$, using the (C, D)-quasi-Leibniz property of $\mathsf{L}_{\mathfrak{A}}$, and using $\|b\|_{\mathfrak{B}}$ and $\|b'\|_{\mathfrak{B}}$:

$$\begin{aligned} \mathsf{L}_{\mathfrak{A}}\left(a \circ a'\right) &\leqslant C\left(\|a\|_{\mathfrak{A}} + \|a'\|_{\mathfrak{A}}\right) + D \\ &\leqslant C\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}} + 2\varepsilon + 6\varepsilon^2\right) + D. \end{aligned} \tag{5.14}$$

Let:

$$\begin{aligned} \omega &= C(1+2\varepsilon)\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}} + 2\varepsilon + 6\varepsilon^2\right) + D \\ &= C(1+2\varepsilon)\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}}\right) + 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3. \end{aligned}$$

Trivially, from Expression (5.13), we have:

$$\|b \circ b' - \psi\left(a \circ a'\right)\|_{\mathfrak{B}} \leqslant \varepsilon\left(C(1+2\varepsilon)\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}}\right) + 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D\right) = \varepsilon\omega,$$

since $C \geqslant 1$, $D \geqslant 0$, so:

$$\left\| \frac{1}{\omega} b \circ b' - \psi\left(\frac{1}{\omega} a \circ a'\right) \right\|_{\mathfrak{B}} \leqslant \varepsilon. \tag{5.15}$$

Similarly, from Expression (5.14), we have:

$$\mathsf{L}_{\mathfrak{A}}\left(\frac{1}{\omega} a \circ a'\right) \leqslant 1. \tag{5.16}$$

Inequalities (5.15) and (5.16) together prove that $\frac{1}{\omega} b \circ b' \in \mathfrak{L}$. Thus, to summarize the heavy computation above, we conclude that:

$$b \circ b' \in \left[C\left(1+2\varepsilon\right)\left(\|b\|_{\mathfrak{B}} + \|b'\|_{\mathfrak{B}}\right) + \left(2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D\right) \right] \mathfrak{L}.$$

The same computations hold for the Lie product as well.

Thus Lemma (3.1) proves that L satisfies the $\left(C(1+2\varepsilon), 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D\right)$-quasi-Leibniz identity. $\square$

We thus arrive at the following finite dimensional approximation result which applies to a large class of quasi-Leibniz quantum compact metric spaces, including in particular nuclear, quasi-diagonal Leibniz quantum compact metric spaces. This result employs the generalized dual propinquity adapted to quasi-Leibniz quantum compact metric spaces, as Lemma (5.6) only provides us with such finite dimensional approximations, although the departure from the Leibniz property can be controlled to be made arbitrarily small.

Theorem 5.7. *Let $C \geqslant 1$, $D \geqslant 0$ and Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a (C, D)-quasi-Leibniz quantum compact metric space where $\mathfrak{A}$ is a unital pseudo-diagonal C*-algebra. Then $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is the limit of finite dimensional $(C(1+\varepsilon), D+\varepsilon)$-quasi-Leibniz quantum compact metric spaces for the $(C(1+\varepsilon), D+\varepsilon)$-dual propinquity, for any $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$. Let $\mu \in \mathscr{S}(\mathfrak{A})$ be arbitrary. The set:

$$\mathfrak{l} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1 \text{ and } \mu(a) = 0\}$$

is norm-compact, since $\mathsf{L}_{\mathfrak{A}}$ is a lower semi-continuous Lip-norm, by Theorem (2.10). Thus there exists a ε^2-dense finite subset $\mathfrak{F}$ of $\mathfrak{l}$.

Since $\mathfrak{A}$ is pseudo-diagonal, there exists a finite dimensional C*-algebra $\mathfrak{B}$ and two positive linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that:

(1) for all $a \in \mathfrak{F}$ we have $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon^2$,
(2) for all $a, b \in \mathfrak{F}$ we have $\|\psi(a) \circ \psi(b) - \psi(a \circ b)\|_{\mathfrak{B}} \leqslant \varepsilon^2$,
(3) for all $a, b \in \mathfrak{F}$ we have $\|\{\psi(a), \psi(b)\} - \psi(\{a, b\})\|_{\mathfrak{B}} \leqslant \varepsilon^2$.

Now, as with step 1 of the proof of Lemma (5.6), Assertions (1),(2) and (3) above remain valid if $\mathfrak{F}$ is replaced by $\mathfrak{F}_1 = \mathfrak{F} + \mathbb{R}1_{\mathfrak{A}}$. Moreover for any $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$ then there exists $c \in \mathfrak{F}_1$ such that $\|a - c\|_{\mathfrak{A}} \leqslant \varepsilon^2$. We shall use this henceforth.

By Lemma (5.6), if we set:

$$\mathfrak{L} = \{b \in \mathfrak{sa}(\mathfrak{B}) : \exists a \in \mathfrak{sa}(\mathfrak{A}) \quad \|b - \psi(a)\|_{\mathfrak{B}} \leqslant \varepsilon \text{ and } \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}$$

and if $\mathsf{L}_{\mathfrak{B}}$ is the associated Minkowsky functional, then L is a:

$$\left[C(1+2\varepsilon), 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D\right]$$

quasi-Leibniz Lip-norm on $\mathfrak{B}$. We now wish to compute the dual propinquity between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

For all $a \in \mathfrak{sa}(\mathfrak{A})$ and $b \in \mathfrak{sa}(\mathfrak{B})$ we define:

$$N(a, b) = \frac{1}{\varepsilon}\|\psi(a) - b\|_{\mathfrak{B}},$$

and

$$\mathsf{L}(a, b) = \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), N(a, b)\}.$$

We note that N is norm continuous on $\mathfrak{sa}(\mathfrak{A} \oplus \mathfrak{B})$, while $N(1_{\mathfrak{A}}, 1_{\mathfrak{B}}) = 0$ yet $N(1_{\mathfrak{A}}, 0) \neq 0$ since ψ is unital.

Now, let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) = 1$. Then since $\|\psi(a) - \psi(a)\|_{\mathfrak{B}} = 0$, we conclude that $\mathsf{L}_{\mathfrak{B}}(\psi(a)) \leqslant 1$ and $N(a, \psi(a)) = 0$. Note that $\mathsf{L}(a, \psi(a)) = \mathsf{L}_{\mathfrak{A}}(a) = 1$ in this case.

Second, let $b \in \mathfrak{sa}(\mathfrak{B})$ with $\mathsf{L}_{\mathfrak{B}}(b) = 1$. Then there exists $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$ and $N(a,b) \leqslant 1$ by definition of N and $\mathsf{L}_{\mathfrak{B}}$. Again note that $\mathsf{L}(a,b) = \mathsf{L}_{\mathfrak{B}}(b) = 1$.

Consequently, N is a bridge in the sense of [37]. We conclude immediately from the above computation that in fact, the quotient of L on $\mathfrak{sa}(\mathfrak{A})$ is $\mathsf{L}_{\mathfrak{A}}$ and the quotient of L on $\mathfrak{sa}(\mathfrak{B})$ is $\mathsf{L}_{\mathfrak{B}}$.

However, to conclude that we have constructed a tunnel for the propinquity, we must also study the quasi-Leibniz property for L.

Let $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $b, b' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$. Let $t = \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b)\}$ and $u = \max\{\mathsf{L}_{\mathfrak{A}}(a'), \mathsf{L}_{\mathfrak{B}}(b')\}$.

Assume first $t \leqslant 1$ and $u \leqslant 1$. Thus, there exists $c, c' \in \mathfrak{F}_1$ such that $\|a - c\|_{\mathfrak{A}} \leqslant \varepsilon^2$ and $\|a' - c'\|_{\mathfrak{A}} \leqslant \varepsilon^2$ again. Thus, similarly to our computation in the proof of Lemma (5.6), we have:

$$
\begin{aligned}
&\|bb' - \psi(aa')\|_{\mathfrak{B}} \\
&\quad \leqslant \|b\|_{\mathfrak{B}} \|b' - \psi(a')\|_{\mathfrak{B}} && (5.17)\\
&\quad + \|\psi(a')\|_{\mathfrak{B}} \|b - \psi(a)\|_{\mathfrak{B}} && (5.18)\\
&\quad + \|\psi(a)\psi(a') - \psi(c)\psi(a')\|_{\mathfrak{B}} && (5.19)\\
&\quad + \|\psi(c)\psi(a') - \psi(c)\psi(c')\|_{\mathfrak{B}} && (5.20)\\
&\quad + \|\psi(c)\psi(c') - \psi(cc')\|_{\mathfrak{B}} && (5.21)\\
&\quad + \|\psi(cc') - \psi(aa')\|_{\mathfrak{B}}. && (5.22)
\end{aligned}
$$

- Term (5.17) is bounded above by $\|b\|_{\mathfrak{B}}(\varepsilon N(a', b'))$.
- Term (5.18) is bounded above by $\|a'\|_{\mathfrak{A}}(\varepsilon N(a, b))$.
- Term (5.19) is bounded above by $\|a'\|\varepsilon^2$.
- Term (5.20) is bounded above by $\left(\|a\| + \varepsilon^2\right)\varepsilon^2$.
- Term (5.21) is bounded above by ε^2.
- Term (5.22) is bounded above by $\|cc' - aa'\|_{\mathfrak{A}}$ which in turn, is bounded by $\left(\|a\|_{\mathfrak{A}} + \|a'\|_{\mathfrak{A}} + \varepsilon^2\right)\varepsilon^2$.

Let us now assume simply that $t, u > 0$. Thus, we get from our estimates above:

$$
\begin{aligned}
\frac{1}{tu} N\left(aa', bb'\right) &= N\left(\frac{1}{t}a\frac{1}{u}a', \frac{1}{t}b\frac{1}{u}b'\right)\\
&\leqslant \left\|\frac{1}{u}b'\right\|_{\mathfrak{B}} N\left(\frac{1}{t}a, \frac{1}{t}b\right) + \left\|\frac{1}{t}a\right\|_{\mathfrak{A}} N\left(\frac{1}{u}a', \frac{1}{u}b'\right)\\
&\quad + 2\left(\left\|\frac{1}{t}a\right\|_{\mathfrak{A}} + \left\|\frac{1}{u}a'\right\|_{\mathfrak{A}}\right)\varepsilon\\
&\quad + \varepsilon + 2\varepsilon^3.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
N(aa', bb') \leqslant \|b'\|_{\mathfrak{B}} N(a,b) &+ \|a\|_{\mathfrak{A}} N(a', b')\\
&+ 2\|a\|_{\mathfrak{A}} u\varepsilon + 2\|a'\|_{\mathfrak{A}} t\varepsilon + \varepsilon tu + 2\varepsilon^3 tu.
\end{aligned}
$$

Since $\|(a,b)\|_{\mathfrak{A}\oplus\mathfrak{B}} = \max\{\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{B}}\}$, $\|(a',b')\|_{\mathfrak{A}\oplus\mathfrak{B}} = \max\{\|a'\|_{\mathfrak{A}}, \|b'\|_{\mathfrak{B}}\}$ while $\max\{t, N(a,b)\} = \mathsf{L}(a,b)$ and $\max\{u, N(a',b')\} = \mathsf{L}(a',b')$, we get:

$$\begin{aligned} N(aa',bb') \leqslant (&\|(a',b')\|_{\mathfrak{A}\oplus\mathfrak{B}}(1+2\varepsilon))\mathsf{L}(a,b) \\ &+ (1+2\varepsilon)\|(a,b)\|_{\mathfrak{A}\oplus\mathfrak{B}}\mathsf{L}(a',b') + \left(\varepsilon+2\varepsilon^3\right)\mathsf{L}(a,b)\mathsf{L}(a',b'). \end{aligned} \tag{5.23}$$

If, instead, $t = 0$, then a and b are scalar multiple of the identities in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and Inequality (5.23) trivially holds. The same is true if $u = 0$. In conclusion, Inequality (5.23) holds for all $a, a' \in \mathfrak{sa}(\mathfrak{A})$ and $b, b' \in \mathfrak{sa}(\mathfrak{B})$.

Since we must use the dual propinquity adapted, in particular, to the $(C(1+2\varepsilon), 2\varepsilon+10\varepsilon^2+12\varepsilon^3+D)$-quasi-Leibniz quantum compact metric space $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, we simply note the weaker estimate:

$$\begin{aligned} N(aa',bb') \leqslant C(1+2\varepsilon)&\left(\|(a,b)\|_{\mathfrak{A}\oplus\mathfrak{B}}\mathsf{L}(a',b') + \|(a',b')\|_{\mathfrak{A}\oplus\mathfrak{B}}\mathsf{L}(a,b)\right) \\ &+ \left(\varepsilon+10\varepsilon^2+12\varepsilon^3+D\right)\mathsf{L}(a',b')\mathsf{L}(a,b). \end{aligned}$$

It is then straightforward that for a, a', b, b' are self-adjoint, we have:

$$\begin{aligned} &\max\left\{N\left(a\circ a', b\circ b'\right), N\left(\{a,a'\},\{b,b'\}\right)\right\} \leqslant \\ &\quad \mathbb{C}(1+2\varepsilon)\left(\|(a,b)\|_{\mathfrak{A}\oplus\mathfrak{B}}\mathsf{L}(a',b') + \|(a',b')\|_{\mathfrak{A}\oplus\mathfrak{B}}\mathsf{L}(a,b)\right) \\ &\qquad + \left(\varepsilon+10\varepsilon^2+12\varepsilon^3+D\right)\mathsf{L}(a',b')\mathsf{L}(a,b). \end{aligned}$$

From this we conclude that, if $\pi_{\mathfrak{A}} : \mathfrak{A}\oplus\mathfrak{B} \twoheadrightarrow \mathfrak{A}$ and $\pi_{\mathfrak{B}} : \mathfrak{A}\oplus\mathfrak{B} \twoheadrightarrow \mathfrak{B}$ are the canonical surjections, then $\tau = (\mathfrak{A}\oplus\mathfrak{B}, \mathsf{L}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a $(C(1+2\varepsilon), 2\varepsilon+10\varepsilon^2+12\varepsilon^3+D)$-tunnel in the sense of Definition (2.11).

We can thus compute the length of the tunnel τ, as defined in Definition (2.21), to obtain an upper bound for the dual propinquity between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

The depth of τ is trivially zero, so we simply compute its reach.

If $\theta \in \mathscr{S}(\mathfrak{B})$ then we set $\varsigma = \theta\circ\psi \in \mathscr{S}(\mathfrak{A})$ (as ψ is unital, positive linear) and observe that:

$$\begin{aligned} \mathsf{mk}_{\mathsf{L}}(\theta,\varsigma) &= \sup\{|\theta(b) - \theta\circ\psi(a)| : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1, \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1, \|b-\psi(a)\|_{\mathfrak{B}} \leqslant \varepsilon\} \\ &\leqslant \varepsilon. \end{aligned}$$

If $\varsigma \in \mathscr{S}(\mathfrak{A})$, then let $\theta = \varsigma\circ\varphi \in \mathscr{S}(\mathfrak{B})$ (since φ is unital, positive linear). If $N(a,b) \leqslant 1$ for some $(a,b) \in \mathfrak{sa}(\mathfrak{A}\oplus\mathfrak{B})$ then $\|b-\psi(a)\|_{\mathfrak{B}} \leqslant \varepsilon$ and, since $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$, there exists $c \in \mathfrak{F}_1$ such that $\|a-c\|_{\mathfrak{A}} \leqslant \varepsilon^2$, so:

$$\begin{aligned} \|\varphi(b) - a\|_{\mathfrak{A}} &\leqslant \|\varphi(b) - \varphi\circ\psi(a)\|_{\mathfrak{A}} + \|\varphi\circ\psi(a) - a\|_{\mathfrak{A}} \\ &\leqslant \|b-\psi(a)\|_{\mathfrak{B}} + \|\varphi\circ\psi(a) - \varphi\circ\psi(c)\|_{\mathfrak{A}} \\ &\quad + \|\varphi\circ\psi(c) - c\|_{\mathfrak{A}} + \|c-a\|_{\mathfrak{A}} \\ &\leqslant \varepsilon + 3\varepsilon^2. \end{aligned}$$

Consequently, $\mathsf{mk}_{\mathsf{L}}(\theta,\varsigma) \leqslant \varepsilon + 3\varepsilon^2$.

Thus the length of our tunnel is bounded above by $\varepsilon + 3\varepsilon^2$ (and is extent is bounded above by $2(\varepsilon+3\varepsilon^2)$).

In summary, for all $\varepsilon > 0$, there exists a finite dimensional $(C(1+2\varepsilon), 2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 + D)$-quasi-Leibniz quantum compact metric space $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ such that:

$$\Lambda_{C(1+2\varepsilon),2\varepsilon+10\varepsilon^2+12\varepsilon^3+D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2\left(\varepsilon + 3\varepsilon^2\right).$$

This proves our theorem. Indeed: fix $\delta > 0$ and let $\varepsilon > 0$ be chosen so that $1 + 2\varepsilon \leqslant 1 + \delta$ and $2\varepsilon + 10\varepsilon^2 + 12\varepsilon^3 \leqslant \delta$. Then, we have shown that there exists a finite dimensional $(C(1+\delta), D+\delta)$-quasi-Leibniz quantum compact metric space $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ such that $\Lambda_{C(1+\delta),D+\delta}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2(\varepsilon + 3\varepsilon^2)$. From this, we conclude that $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is the limit, for $\Lambda_{C(1+\delta,D+\delta)}$, of finite dimensional $(C(1+\delta), D+\delta)$-quasi-Leibniz quantum compact metric spaces. $\square$

Now, in particular, we conclude that for any $C > 1$ and $D > 0$, if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a pseudo-diagonal Leibniz quantum compact metric space, then $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a limit of finite dimensional (C, D)-quasi-Leibniz quantum compact metric spaces for $\Lambda_{C,D}$. Our justification for working with quasi-Leibniz quantum compact metric spaces stems from the fact that our proof, in general, does not allow $C = 1$ and $D = 0$. Of course, [19], [32] provide examples of Leibniz quantum compact metric spaces which are limits of finite dimensional Leibniz quantum compact metric spaces for the dual propinquity, so there are known nontrivial examples when this desirable result holds.

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