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AN ISOMETRIC DYNAMICS FOR A CAUSAL SET APPROACH TO DISCRETE QUANTUM GRAVITY

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Abstract

We consider a covariant causal set approach to discrete quantum gravity. We first review the microscopic picture of this approach. In this picture a universe grows one element at a time and its geometry is determined by a sequence of integers called the shell sequence. We next present the macroscopic picture which is described by a sequential growth process. We introduce a model in which the dynamics is governed by a quantum transition amplitude. The amplitude satisfies a stochastic and unitary condition and the resulting dynamics becomes isometric. We show that the dynamics preserves stochastic states. By “doubling down” on the dynamics we obtain a unitary group representation and a natural energy operator. These unitary operators are employed to define canonical position and momentum operators.

1 Microscopic Picture

We call a finite poset $(x, <)$ a *causet* and interpret $a < b$ in x to mean that b is in the causal future of a . If x and y are causets with cardinality $|y| = |x| + 1$, then x *produces* y (denoted $x \mapsto y$) if y is obtained from x by adjoining a

single maximal element to x . If $x \rightarrow y$ we call y an *offspring* of x . A *labeling* for a causet x is a bijection

$$\ell: x \rightarrow \{1, 2, \dots, |x|\}$$

such that $a, b \in x$ with $a < b$ implies $\ell(a) < \ell(b)$. We then call $x = (x, \ell)$ a *labeled causet*. A labeling of x corresponds to a “birth order” for the elements of x . Two labeled causets x, y are *isomorphic* if there is a bijection $\phi: x \rightarrow y$ such that $a < b$ in x if and only if $\phi(a) < \phi(b)$ in y and $\ell[\phi(a)] = \ell(a)$ for all $a \in x$. A causet is *covariant* if it has a unique labeling (up to isomorphisms) and we call a covariant causet a *c-causet*. Covariance corresponds to the properties of a manifold being independent of the coordinate system used to describe it. Denote the set of *c-causets* with cardinality n by \mathcal{P}_n and the set of all *c-causets* by \mathcal{P} . It is shown in [3] that any $x \in \mathcal{P}$ with $x \neq \emptyset$ has a unique producer in \mathcal{P} and precisely two offspring in \mathcal{P} . It follows that $|\mathcal{P}_n| = 2^{n-1}$, $n = 1, 2, \dots$. For more background concerning the causet approach to discrete quantum gravity we refer the reader to [4, 5, 7]. For more information about *c-causets* the reader can refer to [1, 2, 3].

Two elements $a, b \in x$ are *comparable* if $a < b$ or $b < a$. We say that a is a *parent* of b and b is a *child* of a if $a < b$ and there is no $c \in x$ with $a < c < b$. A *path* from a to b in x is a sequence $a_1 = a, a_2, \dots, a_{n-1}, a_n = b$ where a_i is a parent of a_{i+1} , $i = 1, \dots, n-1$. The *height* $h(a)$ of $a \in x$ is the cardinality minus one of a longest path in x that ends with a . If there is no such path, we set $h(a) = 0$. It is shown in [3] that a causet x is covariant if and only if $a, b \in x$ are comparable whenever $h(a) \neq h(b)$.

If $x \in \mathcal{P}$, we call the sets

$$S_j(x) = \{a \in x: h(a) = j\}, j = 0, 1, 2, \dots$$

shells and the sequence of integers $s_j(x) = |S_j(x)|$, $j = 0, 1, 2, \dots$ is the *shell sequence* for x [1]. A *c-causet* is uniquely determined by its shell sequence and we think of $\{s_j(x)\}$ as describing the “shape” or geometry of x . The tree $(\mathcal{P}, \rightarrow)$ can be thought of as a growth model and an $x \in \mathcal{P}_n$ is a possible universe at step (time) n . An instantaneous universe $x \in \mathcal{P}_n$ grows one element at a time in one of two ways. If $x \in \mathcal{P}_n$ has shell sequence $(s_0(x), s_1(x), \dots, s_m(x))$, then $x \rightarrow x_0$ or $x \rightarrow x_1$ where x_0, x_1 have shell sequence $(s_0(x), s_1(x), \dots, s_m(x) + 1)$ and $(s_0(x), s_1(x), \dots, s_m(x), 1)$, respectively. In this way, we recursively order the *c-causets* in \mathcal{P} using the notation

$x_{n,j}$, $n = 1, 2, \dots$, $j = 0, 1, 2, \dots, 2^{n-1} - 1$, where $n = |x_{n,j}|$. For example, in terms of their shell sequences we have:

$$\begin{aligned} x_{1,0} &= (1), x_{2,0} = (2), x_{2,1} = (1, 1), x_{3,0} = (3), x_{3,1} = (2, 1), x_{3,2} = (1, 2), x_{3,3} = (1, 1, 1) \\ x_{4,0} &= (4), x_{4,1} = (3, 1), x_{4,2} = (2, 2), x_{4,3} = (2, 1, 1), x_{4,4} = (1, 3), x_{4,5} = (1, 2, 1) \\ x_{4,6} &= (1, 1, 2), x_{4,7} = (1, 1, 1, 1) \end{aligned}$$

In the microscopic picture, we view a c -causet as a framework or scaffolding for a possible universe. The vertices of x represent small cells that can be empty or occupied by a particle. The shell sequence for x gives the geometry of the framework. In [1] we have shown how to construct a metric or distance function on x . This metric has simple and useful properties. However, the present paper is mainly devoted to the macroscopic picture and the quantum dynamics that can be developed in that picture. Figure 1 illustrates the first four steps of the sequential growth process $(\mathcal{P}, \rightarrow)$. Notice that this is a multiverse model in which infinite paths represent the histories of “completed” universes [4].

2 Macroscopic Picture

We now study the macroscopic picture which describes the evolution of a universe as a quantum sequential growth process. In such a process, the probabilities and propensities of competing geometries are determined by quantum amplitudes. These amplitudes provide interferences that are characteristic of quantum systems. A *transition amplitude* is a map $\tilde{a}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$ satisfying $\tilde{a}(x, y) = 0$ if $x \not\rightarrow y$ and $\sum_{y \in \mathcal{P}} \tilde{a}(x, y) = 1$ for every $x \in \mathcal{P}$. Since $x_{n,j}$ only has the offspring $x_{n+1,2j}$ and $x_{n+1,2j+1}$ we have that

$$\sum_{k=0}^1 \tilde{a}(x_{n,j}, x_{n+1,2j+k}) = 1 \quad (2.1)$$

for all $n = 1, 2, \dots$, $j = 0, 1, 2, \dots, 2^{n-1} - 1$. We call \tilde{a} a *unitary transition amplitude* (uta) if \tilde{a} also satisfies $\sum_{y \in \mathcal{P}} |\tilde{a}(x, y)|^2 = 1$ or as in (2.1) we have

$$\sum_{k=0}^1 |\tilde{a}(x_{n,j}, x_{n+1,2j+k})|^2 = 1 \quad (2.2)$$

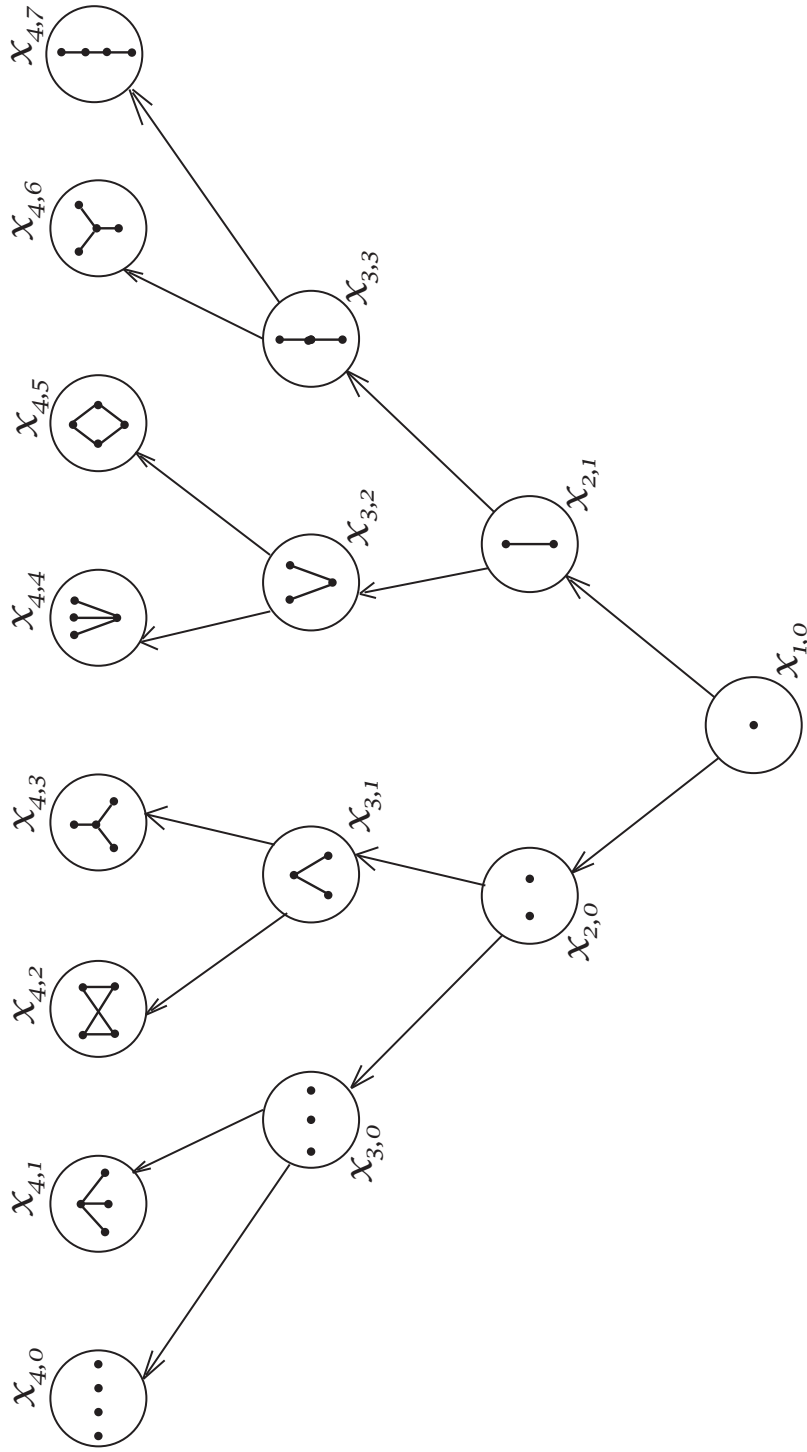


Figure 1 (Ordering for first four steps.)

One might suspect that these restrictions on a uta are so strong that the possibilities are very limited. This would be true if \tilde{a} were real valued. In this case, $\tilde{a}(x, y) = 1$ for one y with $x \rightarrow y$ and $\tilde{a}(x, y) = 0$, otherwise. However, in the complex case, the next result shows that there are a continuum of possibilities.

Theorem 2.1. *Two complex numbers a, b satisfy $a + b = |a|^2 + |b|^2 = 1$ if and only if there exists a $\theta \in [0, \pi)$ such that $a = \cos \theta e^{i\theta}$ and $b = -i \sin \theta e^{i\theta}$. Moreover, θ is unique.*

Proof. Necessity is clear. For sufficiency, suppose the conditions $a + b = |a|^2 + |b|^2 = 1$ hold. Then

$$1 = |a|^2 + |b|^2 = |a|^2 + |1 - a|^2 = |a|^2 + (1 - a)(1 - \bar{a}) = 1 - 2 \operatorname{Re} a + 2 |a|^2$$

Hence, $|a|^2 = \operatorname{Re} a$. Letting $a = |a| e^{i\theta}$ we have that $|a|^2 = |a| \cos \theta$. If $a = 0$, the result holds with $\theta = \pi/2$. If $a \neq 0$, we have that $|a| = \cos \theta$ and $\operatorname{Re} a = |a| \cos \theta$. Hence, $a = \cos \theta e^{i\theta}$ and

$$\begin{aligned} b &= 1 - \cos \theta e^{i\theta} = 1 - \cos^2 \theta - i \cos \theta \sin \theta = \sin \theta (\sin \theta - i \cos \theta) \\ &= -i \sin \theta e^{i\theta} \end{aligned}$$

Uniqueness follows from the fact that $\cos \theta$ is injective on $[0, \pi)$. □

If $\tilde{a}: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$ is a uta, we call

$$c_{n,j}^k = \tilde{a}(x_{n,j}, x_{n+1,2j+k}), \quad k = 0, 1$$

the coupling constants for \tilde{a} . It follows from Theorem 2.1 that there exist $\theta_{n,j} \in [0, \pi)$ such that

$$c_{n,j}^0 = \cos \theta_{n,j} e^{i\theta_{n,j}}, \quad c_{n,j}^1 = -i \sin \theta_{n,j} e^{i\theta_{n,j}}$$

It follows that $c_{n,j}^0 + c_{n,j}^1 = |c_{n,j}^0|^2 + |c_{n,j}^1|^2 = 1$ for all $n = 1, 2, \dots, j = 0, 1, 2, \dots, 2^{n-1} - 1$. Let H_n be the Hilbert space

$$H_n = L_2(\mathcal{P}_n) = \{f: \mathcal{P}_n \rightarrow \mathbb{C}\}$$

with the standard inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}_n} \overline{f(x)} g(x)$$

A *path* in \mathcal{P} is a sequence $\omega = \omega_1\omega_2\cdots$ where $\omega_i \in \mathcal{P}_i$ and $\omega_i \rightarrow \omega_{i+1}$. Similarly, an *n-path* has the form $\omega = \omega_1\omega_2\cdots\omega_n$ where again $\omega_i \in \mathcal{P}_i$ and $\omega_i \rightarrow \omega_{i+1}$. We denote the set of paths by Ω and the set of *n*-paths by Ω_n . Since every $x \in \mathcal{P}_n$ has a unique *n*-path terminating at x , we can identify \mathcal{P}_n with Ω_n and we write $\mathcal{P} \approx \Omega_n$. Similarly, we identify H_n with $L_2(\Omega_n)$. If \tilde{a} is a uta and $\omega = \omega_1\omega_2\cdots\omega_n \in \Omega_n$, we define the *amplitude* of ω to be

$$a(\omega) = \tilde{a}(\omega_1, \omega_2)\tilde{a}(\omega_2, \omega_3)\cdots\tilde{a}(\omega_{n-1}, \omega_n)$$

Moreover, we define the *amplitude* of $x \in \mathcal{P}_n$ to be $a(\omega)$ where $\omega \in \Omega_n$ terminates at x .

Let \hat{x}_n be the unit vector in H_n given by the characteristic function $\chi_{x_{n,j}}$. Then clearly, $\{\hat{x}_{n,j}: j = 0, 1, \dots, 2^{n-1} - 1\}$ forms an orthonormal basis for H_n . Define the operators $U_n: H_n \rightarrow H_{n+1}$ by

$$U_n\hat{x}_{n,j} = \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k}$$

and extend U_n to H_n by linearity.

Theorem 2.2. (i) *The adjoint of U_n is given by $U_n^*: H_{n+1} \rightarrow H_n$ where*

$$U_n^*\hat{x}_{n+1,2j+k} = \bar{c}_{n,j}^k \hat{x}_{n,j}, \quad k = 0, 1 \quad (2.3)$$

(ii) *U_n is a partial isometry with $U_n^*U_n = I_n$ and*

$$U_nU_n^* = \sum_{j=0}^{2^{n-1}-1} \left| \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k} \right\rangle \left\langle \sum_{k=0}^1 \bar{c}_{n,j}^k \hat{x}_{n+1,2j+k} \right| \quad (2.4)$$

Proof. (i) To show that (2.3) holds, we have

$$\begin{aligned} \langle U_n^*\hat{x}_{n+1,2j'+k'}, \hat{x}_{n,j} \rangle &= \langle \hat{x}_{n+1,2j'+k'}, U_n\hat{x}_{n,j} \rangle \\ &= \left\langle \hat{x}_{n+1,2j'+k'}, \sum_{k=0}^1 c_{n,j}^k \hat{x}_{n+1,2j+k} \right\rangle \\ &= c_{n,j}^k \delta_{jj'} \delta_{kk'} = \left\langle \bar{c}_{n,j}^{k'} \hat{x}_{n,j'}, \hat{x}_{n,j} \right\rangle \end{aligned}$$

(ii) To show that $U_n^*U_n = I_n$ we have by (i) that

$$U_n^*U_n\hat{x}_{n,j} = \sum_{k=0}^1 c_{n,j}^k U_n^*\hat{x}_{n+1,2j+k} = \sum_{k=0}^1 |c_{n,j}^k|^2 \hat{x}_{n,j} = \hat{x}_{n,j}$$

Since $\{\widehat{x}_{n,j}: j = 0, 1, \dots, 2^{n-1} - 1\}$ forms an orthonormal basis for H_n , the result follows. Equation (2.4) holds because it is well-known that $U_n U_n^*$ is the projection onto the range of $\mathcal{R}(U_n)$. We can also show this directly as follows

$$\begin{aligned}
& \sum_{j=0}^{2^{n-1}-1} \left| \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \right\rangle \left\langle \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \right| \widehat{x}_{n+1,2j'+k'} \\
&= \sum_{j=0}^{2^{n-1}-1} \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j+k} \overline{c_{n,j'}^{k'}} \delta_{jj'} = \overline{c_{n,j'}^{k'}} \sum_{k=0}^1 c_{n,j}^k \widehat{x}_{n+1,2j'+k} \\
&= \overline{c_{n,j'}^{k'}} U_n \widehat{x}_{n,j'} = U_n U_n^* \widehat{x}_{n+1,2j'+k'} \quad \square
\end{aligned}$$

It follows from Theorem 2.2 that the dynamics $U_n: H_n \rightarrow H_{n+1}$ for a utau \widetilde{a} is an isometric operator. As usual a *state* on H_n is a positive operator ρ on H_n with $\text{tr}(\rho) = 1$. A *stochastic state* on H_n is a state ρ that satisfies $\langle \rho 1_n, 1_n \rangle = 1$ where $1_n = \chi_{\mathcal{P}_n}$; that is, $1_n(x) = 1$ for every $x \in \mathcal{P}_n$. Notice that $U_n^* 1_{n+1} = 1_n$.

Lemma 2.3. (i) *If ρ is a state on H_n , then $U_n \rho U_n^*$ is a state on H_{n+1} .* (ii) *If ρ is a stochastic state on H_n , then $U_n \rho U_n^*$ is a stochastic state on H_{n+1} .*

Proof. (i) To show that $U_n \rho U_n^*$ is positive, we have

$$\langle U_n \rho U_n^* \phi, \phi \rangle = \langle \rho U_n^* \phi, U_n^* \phi \rangle \geq 0$$

for all $\phi \in H_{n+1}$. Moreover, by Theorem 2.2(ii) we have

$$\text{tr}(U_n \rho U_n^*) = \text{tr}(U_n^* U_n \rho) = \text{tr}(\rho) = 1$$

(ii) Since $U_n^* 1_{n+1} = 1_n$ we have

$$\langle U_n \rho U_n^* 1_{n+1}, 1_{n+1} \rangle = \langle \rho U_n^* 1_{n+1}, U_n^* 1_{n+1} \rangle = \langle \rho 1_n, 1_n \rangle = 1 \quad \square$$

Denoting the time evolution of states by $\rho_n \rightarrow \rho_{n+1}$, Lemma 2.3 shows that $\rho \rightarrow U_n \rho U_n^*$ gives a quantum dynamics for states. We now show this explicitly for the transition amplitude. Since

$$\langle \widehat{x}_{n+1,2j+k}, U_n \widehat{x}_{n,j} \rangle = c_{n,j}^k = \widetilde{a}(x_{n,j}, x_{n+1,2j+k})$$

we have for every $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$ that

$$a(\omega) = \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle$$

Define the operator ρ_n on H_n by $\langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle = \overline{a(\omega)} a(\omega')$ where $\widehat{\omega} = \chi_{\{\omega\}} \in H_n$ for any $\omega \in \Omega_n$.

Theorem 2.4. *The operator ρ_n is a stochastic state on H_n .*

Proof. To show that ρ_n is positive we have

$$\begin{aligned}
\langle f, \rho_n f \rangle &= \left\langle \sum \langle \widehat{\gamma}_i, f \rangle \widehat{\gamma}_i, \rho_n \sum \langle \widehat{\gamma}_j, f \rangle \widehat{\gamma}_j \right\rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f \rangle} \sum \langle \widehat{\gamma}_j, f \rangle \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j \rangle \\
&= \sum \overline{\langle \widehat{\gamma}_i, f \rangle} \sum \langle \widehat{\gamma}_j, f \rangle \overline{a(\gamma_i)} a(\gamma_j) \\
&= \left| \sum a(\gamma_i) \langle \widehat{\gamma}_i, f \rangle \right|^2 \geq 0
\end{aligned}$$

To show that ρ_n is a state on H_n we have that

$$\begin{aligned}
\text{tr}(\rho_n) &= \sum \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_i \rangle = \sum \overline{a(\gamma_i)} a(\gamma_i) = \sum |a(\gamma_i)|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_n} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 |\langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle|^2 \cdots |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 \\
&\quad \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2} \rangle|^2 \sum_{\omega_n} |\langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle|^2 \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 \cdots |\langle \widehat{\omega}_{n-1}, U_{n-2} \widehat{\omega}_{n-2} \rangle|^2 \\
&\quad \vdots \\
&= \sum_{\omega_2} |\langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle|^2 = 1
\end{aligned}$$

Finally, ρ_n is stochastic on H_n because

$$\begin{aligned}
\langle 1_n, \rho_n 1_n \rangle &= \left\langle \sum \widehat{\gamma}_i, \rho_n \sum \widehat{\gamma}_j \right\rangle = \sum_{i,j} \langle \widehat{\gamma}_i, \rho_n \widehat{\gamma}_j \rangle \\
&= \sum_{i,j} \overline{a(\gamma_i)} a(\gamma_j) = \left| \sum a(\gamma_i) \right|^2
\end{aligned}$$

As before, we obtain

$$\begin{aligned}
\sum_{\omega \in \Omega_n} a(\omega) &= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_n} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2 \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_n, U_{n-1} \widehat{\omega}_{n-1} \rangle \\
&= \sum_{\omega_2} \sum_{\omega_3} \cdots \sum_{\omega_{n-1}} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle \langle \widehat{\omega}_3, U_2, \widehat{\omega}_2 \rangle \cdots \langle \widehat{\omega}_{n-1} U_{n-2} \widehat{\omega}_{n-2} \rangle \\
&\quad \vdots \\
&= \sum_{\omega_2} \langle \widehat{\omega}_2, U_1 \widehat{\omega}_1 \rangle = 1 \quad \square
\end{aligned}$$

If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$, we have seen that ω_n produces two offspring $\omega_{n,0}, \omega_{n,1} \in \mathcal{P}_{n+1}$. We call the set

$$(\omega \rightarrow) = \{\omega_1 \omega_2 \cdots \omega_n \omega_{n,0}, \omega_1 \omega_2 \cdots \omega_n \omega_{n,1}\} \subseteq \Omega_{n+1}$$

the *one-step causal future* of ω . We say that the sequence ρ_n is *consistent* if

$$\langle (\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge \rangle = \langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle$$

for every $\omega, \omega' \in \Omega_n$ where $(\omega \rightarrow)^\wedge = \chi_{(\omega \rightarrow)}$. Consistency is important because it follows that the probabilities and propensities given by the dynamics ρ_n are conserved in time [2, 3].

Theorem 2.5. *The sequence ρ_n is consistent.*

Proof. Let $\omega = \omega_1 \omega_2 \cdots \omega_n$, $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n \in \Omega_n$ and suppose that $\omega_n \rightarrow \omega_{n,0}, \omega_{n,1}$ and $\omega'_n \rightarrow \omega'_{n,0}, \omega'_{n,1}$. We then have

$$\begin{aligned}
&\langle (\omega \rightarrow)^\wedge, \rho_{n+1}(\omega' \rightarrow)^\wedge \rangle \\
&= \langle (\omega \omega_{n,0})^\wedge + (\omega \omega_{n,1})^\wedge, \rho_{n+1} [(\omega' \omega'_{n,0})^\wedge + (\omega' \omega'_{n,1})^\wedge] \rangle \\
&= \langle (\omega \omega_{n,0})^\wedge, \rho_{n+1}(\omega' \omega'_{n,0})^\wedge \rangle + \langle (\omega \omega_{n,0})^\wedge, \rho_{n+1}(\omega' \omega'_{n,1})^\wedge \rangle \\
&\quad + \langle (\omega \omega_{n,1})^\wedge, \rho_{n+1}(\omega' \omega'_{n,0})^\wedge \rangle + \langle (\omega \omega_{n,1})^\wedge, \rho_{n+1}(\omega' \omega'_{n,1})^\wedge \rangle \\
&= \overline{a(\omega) \widetilde{a}(\omega_n \omega_{n,0})} a(\omega') [\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})] \\
&\quad + \overline{a(\omega) \widetilde{a}(\omega_n, \omega_{n,1})} a(\omega') [\widetilde{a}(\omega'_n, \omega'_{n,0}) + \widetilde{a}(\omega'_n, \omega'_{n,1})] \\
&= \overline{a(\omega)} a(\omega') = \langle \widehat{\omega}, \rho_n \widehat{\omega}' \rangle \quad \square
\end{aligned}$$

The *n-decoherence functional* is the map $D_n: 2^{\Omega_n} \times 2^{\Omega_n} \rightarrow \mathbb{C}$ defined by [4, 5, 7]

$$D_n(A, B) = \langle \chi_B, \rho_n \chi_A \rangle$$

The functional $D_n(A, B)$ gives a measure of the interference between A and B when the system is in the state ρ_n . Clearly $D_n(\Omega_n, \Omega_n) = 1$, $D_n(A, B) = \overline{D_n(A, B)}$ and $A \mapsto D_n(A, B)$ is a complex measure for every $B \in 2^{\Omega_n}$. It is also well-known that if $A_1, \dots, A_n \in 2^{\Omega_n}$, then the matrix with entries $D_n(A_j, A_k)$ is positive semidefinite [5]. Notice that

$$D_n(\{\omega\}, \{\omega'\}) = \overline{a(\omega)}a(\omega')$$

for every $\omega, \omega' \in \Omega_n$ and

$$D(A, B) = \sum \left\{ \overline{a(\omega)}a(\omega') : \omega \in A, \omega' \in B \right\}$$

Since ρ_n is consistent, we have that

$$D_{n+1}((A \rightarrow), (B \rightarrow)) = D_n(A, B)$$

for every $A, B \in 2^{\Omega_n}$ where $(A \rightarrow) = \cup \{(\omega \rightarrow) : \omega \in A\}$. The corresponding *q-measure* [2, 5, 6] is the map $\mu_n: 2^\Omega \rightarrow \mathbb{R}^+$ defined by

$$\mu_n(A) = D_n(A, A) = \langle \chi_A, \rho_n \chi_A \rangle$$

It follows that $\mu_n(\Omega_n) = 1$ and $\mu_{n+1}((A \rightarrow)) = \mu_n(A)$ for all $A \in 2^{\Omega_n}$. Although μ_n is not additive, it satisfies the *grade 2-additive condition*: if $A, B, C \in 2^{\Omega_n}$ are mutually disjoint then [4, 5, 6, 7]

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

Since μ_n is not a measure we do call it a probability but we interpret $\mu_n(A)$ as the quantum propensity for the occurrence of A . We have discussed in [2, 3] ways of extending the μ_n s to a *q-measure* μ on suitable subsets of Ω .

A uta is *completely stationary* (cs) with parameter $\theta \in [0, \pi)$ if $\theta_{n,j} = \theta$ for all n, j . For example, let \tilde{a} be cs with parameter 0. Then the path $x_{1,0}x_{2,0}x_{3,0} \dots$ has *q-measure* 1 and all other paths have *q-measure* 0. Now consider a general cs uta \tilde{a} with parameter $\theta \in (0, \pi)$. When a path “turns left” \tilde{a} has the value $\cos \theta e^{i\theta}$ and when it “turns right” \tilde{a} has the value $-i \sin \theta e^{i\theta}$. Hence if $\omega \in \Omega_n$ turns left ℓ times and right r times we have

$$a(\omega) = (\cos \theta)^\ell (-i)^r (\sin \theta)^r e^{in\theta}$$

We then have

$$\mu_n(\{\omega\}) = |a(\omega)|^2 = |\cos \theta|^{2\ell} |\sin \theta|^{2r}$$

Hence, $\lim_{n \rightarrow \infty} \mu_n(\{\omega\}) = 0$ and it is natural to define $\mu(\{\omega\}) = 0$.

A vector

$$v = \sum_{j=0}^{2^n-1} v_j \hat{x}_{n,j} = (v_0, v_1, \dots, v_{2^n-1}) \in H_n$$

is called a *stochastic state vector* if $\|v\| = 1$ and $\langle v, 1_n \rangle = 1$. We call the vector

$$\hat{a}_n = (a(x_{n,0}), a(x_{n,1}), \dots, a(x_{n,2^n-1})) \in H_n$$

an *amplitude vector*. Of course, \hat{a}_n is a stochastic state vector.

Theorem 2.6. (i) If $v \in H_n$ is a stochastic state vector, then $U_n v \in H_{n+1}$ is also. (ii) $U_n \hat{a}_n = \hat{a}_{n+1}$. (iii) $U_n^* \hat{a}_{n+1} = \hat{a}_n$.

Proof. (i) This follows from the fact that U_n is isometric and $U_n^* 1_{n+1} = 1_n$.

(ii) This holds because

$$\begin{aligned} U_n \hat{a}_n &= U_n \sum_{j=0}^{2^n-1} a(x_{n,j}) \hat{x}_{n,j} = \sum_{j=0}^{2^n-1} [a(x_{n,j}) c_{n,j}^0 \hat{x}_{n+1,2j} + a(x_{n,j}) c_{n,j}^1 \hat{x}_{n+1,2j+1}] \\ &= \sum_{j=0}^{2^n-1} a(x_{n+1,j}) \hat{x}_{n+1,j} = \hat{a}_{n+1} \end{aligned}$$

(iii) This is obtained from

$$\begin{aligned} U_n^* \hat{a}_{n+1} &= \sum [a(x_{n+1,2j}) \hat{x}_{n+1,2j} + a(x_{n+1,2j+1}) \hat{x}_{n+1,2j+1}] \\ &= \sum [a(x_{n+1,2j}) \bar{c}_{n,j}^0 \hat{x}_{n,j} + a(x_{n+1,2j+1}) \bar{c}_{n,j}^1 \hat{x}_{n,j}] \\ &= \sum [c_{n,j}^0 a(x_{n,j}) \bar{c}_{n,j}^0 + c_{n,j}^1 a(x_{n,j}) \bar{c}_{n,j}^1] \hat{x}_{n,j} \\ &= \sum a(x_{n,j}) \hat{x}_{n,j} = \hat{a}_n \quad \square \end{aligned}$$

Actually, (iii) follows from (ii) in Theorem 2.6 because $\hat{a}_{n+1} = U_n \hat{a}_n \in \mathcal{R}(U_n)$ so $U_n^* \hat{a}_{n+1} = U_n^* U_n \hat{a}_n = \hat{a}_n$. Interference in \mathcal{P}_n or Ω_n can be described by the nonadditivity of the q -measure μ_n . We say that $x, y \in \mathcal{P}_n$ do not interfere if

$$\mu_n(\{x, y\}) = \mu_n(\{x\}) + \mu_n(\{y\})$$

The next result gives an application of this concept.

Theorem 2.7. *If $x, y \in \mathcal{P}$ have the same producer, then x and y do not interfere.*

Proof. Suppose $x = x_{n+1,2j}, y = x_{n+1,2j+1}$ so x, y have the same producer $x_{n,j}$. Then

$$\begin{aligned}
\mu_{n+1}(\{x, y\}) &= |a(x) + a(y)|^2 = |a(x_{n,j})c_{n,j}^0 + a(x_{n,j})c_{n,j}^1|^2 \\
&= |a(x_{n,j})|^2 |c_{n,j}^0 + c_{n,j}^1|^2 = |a(x_{n,j})|^2 \\
&= |a(x_{n,j})|^2 \left[|c_{n,j}^0|^2 + |c_{n,j}^1|^2 \right] \\
&= |a(x_{n,j})c_{n,j}^0|^2 + |a(x_{n,j})c_{n,j}^1|^2 \\
&= |a(x_{n+1,2j})|^2 + |a(x_{n+1,2j+1})|^2 = \mu_{n+1}(\{x\}) + \mu_{n+1}(\{y\})
\end{aligned}$$

Hence, x and y do not interfere. \square

In general, the noninterference result in Theorem 2.7 does not hold if x and y have different producers. This is shown in the next two examples.

Example 1. For simplicity, suppose the uta is cs so we have just two coupling constants c^0, c^1 . We have seen in Theorem 2.7 that $x_{3,0}$ and $x_{3,1}$ do not interfere. In a similar way, we see that $x_{3,0}$ and $x_{3,2}$ do not interfere. We also have that $x_{3,1}$ does not interfere with $x_{3,j}, j = 0, 1, 2$ and $x_{3,2}$ does not interfere with $x_{3,j}, j = 0, 1, 3$. Let us now consider $x_{3,0}$ and $x_{3,3}$. We have that

$$\begin{aligned}
\mu_3(\{x_{3,0}, x_{3,3}\}) &= |a(x_{3,0}) + a(x_{3,3})|^2 = |(c^0)^2 + (c^1)^2|^2 \\
&= |\cos^2 \theta - \sin^2 \theta| = \cos^2 2\theta
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu_3(\{x_{3,0}\}) + \mu_2(\{x_{3,3}\}) &= |a(x_{3,0})|^2 + |a(x_{3,3})|^2 \\
&= \cos^4 \theta + \sin^4 \theta = \frac{1}{2}(1 + \cos^2 2\theta)
\end{aligned}$$

so $x_{3,0}$ and $x_{3,3}$ interfere, in general.

Example 2. If the uta is not cs, the situation is more complicated and we incur more interference. In the cs case, we saw in Example 1 that $x_{3,0}$ and $x_{3,2}$ do not interfere. However, in this more general case we have

$$\mu_3(\{x_{3,0}, x_{3,2}\}) = |a(x_{3,0}) + a(x_{3,2})| = |c_{1,0}^0 c_{2,0}^0 + c_{1,0}^1 c_{2,1}^0|^2$$

On the other hand

$$\mu_3(\{x_{3,0}\}) + \mu_3(\{x_{3,2}\}) = |a(x_{3,0})|^2 + |a(x_{3,2})|^2 = |c_{1,0}^0|^2 |c_{2,0}^0|^2 + |c_{1,0}^1|^2 |c_{2,1}^0|^2$$

But these two quantities do not agree unless

$$\operatorname{Re}(c_{1,0}^0 c_{2,0}^0 \bar{c}_{1,0}^1 \bar{c}_{2,1}^0) = 0$$

so $x_{3,0}$ and $x_{3,2}$ interfere, in general.

3 Double-Down To Unitary

We have seen that corresponding to a uta with coupling constants $c_{n,j}^k$, there are isometries $U_n: H_n \rightarrow H_{n+1}$ that describe the dynamics for a quantum sequential growth process on $(\mathcal{P}, \rightarrow)$. The operators U_n cannot be unitary because H_n and H_{n+1} are different dimensional Hilbert spaces. However, we can “double-down” the U_n to form operators $V_{n+1}: H_{n+1} \rightarrow H_{n+1}$ by

$$\begin{aligned} V_{n+1} \hat{x}_{n+1,2j} &= c_{n,j}^0 \hat{x}_{n+1,2j} + c_{n,j}^1 \hat{x}_{n+1,2j+1} \\ V_{n+1} \hat{x}_{n+1,2j+1} &= c_{n,j}^1 \hat{x}_{n+1,2j} + c_{n,j}^0 \hat{x}_{n+1,2j+1} \end{aligned}$$

Theorem 3.1. *The operators V_{n+1} are unitary and $V_{n+1} 1_{n+1} = 1_{n+1}$, $n = 1, 2, \dots$*

Proof. Since $\|V_{n+1} \hat{x}_{n+1,2j}\| = \|V_{n+1} \hat{x}_{n+1,2j+1}\| = 1$ and

$$\langle V_{n+1} \hat{x}_{n+1,2j}, V_{n+1} \hat{x}_{n+1,2j+1} \rangle = \bar{c}_{n,j}^0 c_{n,j}^1 + \bar{c}_{n,j}^1 c_{n,j}^0 = 0$$

we conclude that V_{n+1} sends an orthonormal basis to an orthonormal basis. Hence, V_{n+1} is unitary. To show that $V_{n+1} 1_{n+1} = 1_{n+1}$ we have

$$\begin{aligned} V_{n+1} 1_{n+1} &= \sum_j (V_{n+1} \hat{x}_{n+1,2j} + V_{n+1} \hat{x}_{n+1,2j+1}) \\ &= \sum_j [(c_{n,j}^0 + c_{n,j}^1) \hat{x}_{n+1,2j} + (c_{n,j}^1 + c_{n,j}^0) \hat{x}_{n+1,2j+1}] \\ &= \sum_j (\hat{x}_{n+1,2j} + \hat{x}_{n+1,2j+1}) = 1_{n+1} \quad \square \end{aligned}$$

The unitary operator V_2 corresponds to the coupling constants $c_{1,0}^0, c_{1,0}^1$ and relative to the basis $\{\widehat{x}_{2,0}, \widehat{x}_{2,1}\}$ has the form

$$V_2 = \begin{bmatrix} c_{1,0}^0 & c_{1,0}^1 \\ c_{1,0}^1 & c_{1,0}^0 \end{bmatrix}$$

Besides being unitary, V_2 is doubly stochastic (row and column sums are one). Of course, this is also true of V_n . By Theorem 2.1, there exists a unique $\theta \in [0, \pi)$ such that $c_{1,0}^0 = \cos \theta e^{i\theta}$, $c_{1,0}^1 = -i \sin \theta e^{i\theta}$. To make θ explicit, we write $V_2 = V_2(\theta)$.

Lemma 3.2. *The operator $V_2(\theta)$ has eigenvalues $1, e^{2i\theta}$ with corresponding unit eigenvectors $2^{-1/2}(1, 1)$, $2^{-1/2}(1, -1)$.*

Proof. By direct verification we have

$$\begin{aligned} V_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} c_{1,0}^0 + c_{1,0}^1 \\ c_{1,0}^1 + c_{1,0}^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ V_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} c_{1,0}^0 - c_{1,0}^1 \\ c_{1,0}^1 - c_{1,0}^0 \end{bmatrix} = (c_{1,0}^0 - c_{1,0}^1) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} c_{1,0}^0 - c_{1,0}^1 &= c_{1,0}^0 - (1 - c_{1,0}^0) = 2c_{1,0}^0 - 1 = 2 \cos \theta e^{i\theta} - 1 \\ &= 2 \cos^2 \theta + 2i \cos \theta \sin \theta - 1 = \cos^2 \theta - \sin^2 \theta + i \sin 2\theta \\ &= \cos 2\theta + i \sin 2\theta = e^{2i\theta} \end{aligned} \quad \square$$

We can write the 2^n -dimensional Hilbert space H_{n+1} as

$$H_{n+1} = H_2 \oplus H_2 \oplus \cdots \oplus H_2$$

where there are 2^{n-1} summands and the j th summand has the basis $\{\widehat{x}_{n+1,2j}, \widehat{x}_{n+1,2j+1}\}$. In general, V_{n+1} has the form

$$V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}}) = V_2(\theta_1) \oplus V_2(\theta_2) \oplus \cdots \oplus V_2(\theta_{2^{n-1}})$$

It follows from Lemma 3.2 that $V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}})$ has eigenvalues 1 (with multiplicity 2^{n-1}) and $e^{2i\theta_1}, e^{2i\theta_2}, \dots, e^{2i\theta_{2^{n-1}}}$. The unit eigenvectors corresponding to 1 are

$$2^{-1/2}(\widehat{x}_{n+1,2j} + \widehat{x}_{n+1,2j+1}), \quad j = 0, 1, \dots, 2^{n-1} - 1$$

and the unit eigenvector corresponding to $e^{2i\theta_j}$ is

$$2^{-1/2}(\widehat{x}_{n+1,2j} - \widehat{x}_{n+1,2j+1})$$

Let $\mathcal{S}(H_{n+1})$ be the set of operators on H_{n+1} of the form

$$\mathcal{S}(H_{n+1}) = \{V_{n+1}(\theta_1, \theta_2, \dots, \theta_{2^{n-1}}): \theta_n \in [0, \pi)\}$$

Now $[0, \pi)$ forms an abelian group with operations $a \oplus b = a + b \pmod{\pi}$.

Lemma 3.3. *For $\theta_1, \theta_2 \in [0, \pi)$ we have $V_2(\theta_1)V_2(\theta_2) = V_2(\theta_1 + \theta_2)$.*

Proof. Since $V_2(\theta_1)$ and $V_2(\theta_2)$ have the same eigenvectors, they commute and can be simultaneously diagonalized as

$$V_2(\theta_1) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_1} \end{bmatrix} \quad V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta_2} \end{bmatrix}$$

Hence, if $\theta_1 + \theta_2 < \pi$ then

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2)} \end{bmatrix} = V_2(\theta_1 \oplus \theta_2)$$

and if $\theta_1 + \theta_2 \geq \pi$ then $\theta_1 \oplus \theta_2 = \theta_1 + \theta_2 - \pi$ and we have

$$V_2(\theta_1)V_2(\theta_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(\theta_1+\theta_2-\pi)} \end{bmatrix} = V_2(\theta_1 + \theta_2 - \pi) = V_2(\theta_1 \oplus \theta_2) \quad \square$$

We now form the product group $[0, \pi)^{2^{n-1}} = [0, \pi) \times \dots \times [0, \pi)$ to obtain the following result.

Corollary 3.4. *Under operator multiplication, $\mathcal{S}(H_{n+1})$ is an abelian group and $(\theta_1, \dots, \theta_{2^{n-1}}) \mapsto V_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})$ is a unitary representation of the group $[0, \pi)^{2^{n-1}}$.*

Since V_{n+1} is unitary, there exists a unique self-adjoint operator K_{n+1} on H_{n+1} such that $V_{n+1} = e^{iK_{n+1}}$. We call K_{n+1} a *Hamiltonian operator*. For $V_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})$ the eigenvalues of K_{n+1} are 0 (with multiplicity 2^{n-1}) and $2\theta_1, \dots, 2\theta_{2^{n-1}}$. Hence, $\theta_j = 2^{-1}\lambda_j$ where λ_j is the j th energy value,

$j = 1, \dots, 2^{n-1}$. This gives a physical significance for the angles θ_j . The corresponding eigenvectors are the same as those given for V_{n+1} .

It is natural to define the *position operator* Q_{n+1} on H_{n+1} by $Q_{n+1}f(\hat{x}_{n+1,k}) = k$. Thus, $Q_{n+1}\hat{x}_{n+1,2j} = 2j$ and $Q_{n+1}\hat{x}_{n+1,2j+1} = 2j + 1$. Since Q_{n+1} is diagonal, we immediately see that its eigenvalues are $0, 1, \dots, 2^n - 1$ with corresponding eigenvector $\hat{x}_{n+1,k}$. It also seems natural to define the *canonical momentum operator* P_{n+1} on the subspace generated by $\{\hat{x}_{n+1,2j}, \hat{x}_{n+1,2j+1}\}$ as

$$\begin{aligned} P_2(\theta_j) &= V_2(\theta_j)^* Q_2(\theta_j) V_2(\theta_j) \\ &= \begin{bmatrix} \bar{c}_{n,j}^0 & \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^1 & \bar{c}_{n,j}^0 \end{bmatrix} \begin{bmatrix} 2j & 0 \\ 0 & 2j+1 \end{bmatrix} \begin{bmatrix} c_{n,j}^0 & c_{n,j}^1 \\ c_{n,j}^1 & c_{n,j}^0 \end{bmatrix} \\ &= \begin{bmatrix} 2j + |c_{n,j}^1|^2 & c_{n,j}^0 \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^0 c_{n,j}^1 & 2j + |c_{n,j}^0|^2 \end{bmatrix} = \begin{bmatrix} 2j + \sin^2 \theta_{n,j} & \frac{i}{2} \sin 2\theta_{n,j} \\ -\frac{i}{2} \sin 2\theta_{n,j} & 2j + \cos^2 \theta_{n,j} \end{bmatrix} \end{aligned}$$

The eigenvalues of $P_2(\theta_j)$ are $2j$ and $2j + 1$ with corresponding unit eigenvectors

$$\begin{aligned} V_2(\theta_j)^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \bar{c}_{n,j}^0 \\ \bar{c}_{n,j}^1 \end{bmatrix} \\ V_2(\theta_j)^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} \bar{c}_{n,j}^1 \\ \bar{c}_{n,j}^0 \end{bmatrix} \end{aligned}$$

The complete momentum operator P_{n+1} is given by

$$P_{n+1}(\theta_1, \dots, \theta_{2^{n-1}}) = P_2(\theta_1) \oplus P_2(\theta_2) \oplus \dots \oplus P_2(\theta_{2^{n-1}})$$

We now compute the commutator

$$\begin{aligned} [P_2(\theta_j), Q_2(\theta_j)] &= P_2(\theta_j)Q_2(\theta_j) - Q_2(\theta_j)P_2(\theta_j) = c_{n,j}^0 \bar{c}_{n,j}^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{i}{2} \sin 2\theta_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The complete commutation relation is

$$\begin{aligned} & [P_{n+1}(\theta_1, \dots, \theta_{2^{n-1}}), Q_{n+1}(\theta_1, \dots, \theta_{2^{n-1}})] \\ &= [P_2(\theta_1), Q_2(\theta_1)] \oplus \dots \oplus [P_2(\theta_{2^{n-1}}), Q_2(\theta_{2^{n-1}})] \end{aligned}$$

As in the Heisenberg uncertainty relation, the number $|\langle \phi, [P_{n+1}, Q_{n+1}] \phi \rangle|$ gives a lower bound for the product of the variances of P_{n+1} and Q_{n+1} . We now compute this number for an amplitude state \hat{a}_{n+1} . We have that

$$\begin{aligned} & \langle \hat{a}_{n+1}, [P_{n+1}, Q_{n+1}] \hat{a}_{n+1} \rangle \\ &= \sum_j \left\langle \begin{bmatrix} a(x_{n+1,2j}) \\ a(x_{n+1,2j+1}) \end{bmatrix}, c_{n,j}^0 c_{n,j}^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a(x_{n+1,2j}) \\ a(x_{n+1,2j+1}) \end{bmatrix} \right\rangle \\ &= \sum_j c_{n,j}^0 \bar{c}_{n,j}^1 [\bar{a}(x_{n+1,2j})a(x_{n+1,2j+1}) + \bar{a}(x_{n+1,2j+1})a(x_{n+1,2j})] \\ &= \sum_j c_{n,j}^0 \bar{c}_{n,j}^1 |a(x_{n,j})|^2 [\bar{c}_{n,j}^0 c_{n,j}^1 + \bar{c}_{n,j}^1 c_{n,j}^0] = 0 \end{aligned}$$

This shows that even though P_{n+1} and Q_{n+1} do not commute, there is no lower bound for the product of their variances when the system is in an amplitude state.

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