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EXTENDER SETS AND MULTIDIMENSIONAL SUBSHIFTS

NIC ORMES AND RONNIE PAVLOV

Abstract. In this paper, we consider a \( \mathbb{Z}^d \) extension of the well-known fact that subshifts with only finitely many follower sets are sofic. As in [4], we adopt a natural \( \mathbb{Z}^d \) analogue of a follower set called an extender set. The extender set of a finite word \( w \) in a \( \mathbb{Z}^d \) subshift \( X \) is the set of all configurations of symbols on the rest of \( \mathbb{Z}^d \) which form a point of \( X \) when concatenated with \( w \). As our main result, we show that for any \( d \geq 1 \) and any \( \mathbb{Z}^d \) subshift \( X \), if there exists \( n \) so that the number of extender sets of words on a \( d \)-dimensional hypercube of side length \( n \) is less than or equal to \( n \), then \( X \) is sofic. We also give an example of a non-sofic system for which this number of extender sets is \( n + 1 \) for every \( n \).

We prove this theorem in two parts. First we show that if the number of extender sets of words on a \( d \)-dimensional hypercube of side length \( n \) is less than or equal to \( n \) for some \( n \), then there is a uniform bound on the number of extender sets for words on any sufficiently large rectangular prisms; to our knowledge, this result is new even for \( d = 1 \). We then show that such a uniform bound implies soficity.

Our main result is reminiscent of the classical Morse-Hedlund theorem, which says that if \( X \) is a \( \mathbb{Z} \) subshift and there exists an \( n \) such that the number of words of length \( n \) is less than or equal to \( n \), then \( X \) consists entirely of periodic points. However, most proofs of that result use the fact that the number of words of length \( n \) in a \( \mathbb{Z} \) subshift is nondecreasing in \( n \), and we present an example (due to Martin Delacourt) which shows that this monotonicity does not hold for numbers of extender sets (or follower sets) of words of length \( n \).

1. Introduction

For any \( \mathbb{Z} \) subshift \( X \) and finite word \( w \) appearing in some point of \( X \), the follower set of \( w \), written \( F_X(w) \), is defined as the set of all one-sided infinite sequences \( s \) such that the infinite word \( ws \) occurs in some point of \( X \). (In some sources, the follower set is defined as the set of all finite words which can legally follow \( w \), but the former definition may be obtained by taking limits of the latter.) It is well-known that for a \( \mathbb{Z} \) subshift \( X \), finiteness of \( \{ F_X(w) : w \text{ in the language of } X \} \) is equivalent to \( X \) being sofic, i.e. the image of a shift of finite type under a continuous shift-commuting map. (For instance, see [5].)

In [4], extender sets were defined and introduced as a natural extension of follower sets to \( \mathbb{Z}^d \) subshifts with \( d > 1 \). The extender set of any finite word \( w \) in the language of \( X \) with shape \( S \subset \mathbb{Z}^d \), written \( E_X(w) \), is the set of all configurations on \( \mathbb{Z}^d \setminus S \) which, when concatenated with \( w \), form a legal point of \( X \). We can no longer speak of a subshift having only finitely many extender sets, since extender sets of patterns with different shapes cannot be compared as in the one-dimensional case. One way to deal with this is examine the growth rate of the number of distinct

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extender sets for words in $X$ with a given shape $S$ (which we denote by $N_S(X)$), as the size of $S$ approaches infinity. This works nicely in the one-dimensional case; our Lemma 3.4 (which is routine) shows that soficity of a one-dimensional subshift is equivalent to boundedness of the number of extender sets of $n$-letter words as $n \to \infty$. Interestingly, this sequence need not stabilize; Example 3.5, due to Martin Delacourt ([1]), demonstrates a Z sofic shift $X$ where $N_{[1,n]}(X)$ oscillates between two values as $n$ increases. In this paper, $[a, b]$ for $a, b \in \mathbb{Z}$ will always represent the set $\{a, a+1, \ldots, b\}$.

There are many relations between properties of $X$ and the behavior of $N_S(X)$. For instance, it is easy to see that when $X$ is a nearest-neighbor shift of finite type, the extender set of a word with shape $[1, n]^d$ is determined entirely by the letters on the boundary. This implies that for such $X$, $N_{[1,n]^d}(X)$ is bounded from above by $|A_X|^{d} |(1, n)^d| \leq |A_X|^{2dn(d-1)}$, where $A_X$ denotes the alphabet of $X$. It was conjectured in [4] that $X$ sofic implies that $\frac{\log N_{[1,n]^d}(X)}{n^d} \to 0$, but this remains open.

A partial answer was proven in [4], using an argument basically present in [7]. A finite sequence $S_n$ of sets, $1 \leq n \leq N$, was defined to be a union increasing chain if $S_n \not\subseteq \bigcup_{j=1}^{n-1} S_j$ for all $1 \leq n \leq N$. Theorem 2.3 of [4] states that if there exist union increasing chains of size $e^{c(n^d-1)}$ of extender sets of words with shape $[1, n]^d$, then $X$ is not sofic. These results can, broadly speaking, be thought of as showing that a very fast growth rate for extender sets implies that a subshift is not an SFT or sofic. Our main result is in the opposite direction, namely it demonstrates that a slow enough growth rate implies soficity.

**Theorem 1.1.** For any $d \geq 1$ and any $\mathbb{Z}^d$ subshift $X$, if there exists $n$ so that $N_{[1,n]^d}(X) \leq n$, then $X$ is sofic.

The proof of Theorem 1.1 is broken into two mostly disjoint parts:

**Theorem 1.2.** For any $d \geq 1$ and any $\mathbb{Z}^d$ subshift $X$, if there exists $n$ so that $N_{[1,n]^d}(X) \leq n$, then there exist $K, N$ such that for any rectangular prism $R$ with dimensions at least $K$, $N_R(X) \leq N$.

**Theorem 1.3.** For any $d \geq 1$ and any $\mathbb{Z}^d$ subshift $X$, if there exist $K, N$ so that $N_R(X) \leq N$ for all rectangular prisms $R$ with all dimensions at least $K$, then $X$ is sofic.

Theorem 1.3 can be thought of as a generalization of the previously mentioned fact that one-dimensional shifts with only finitely many follower sets are sofic. We also show that the upper bound in Theorem 1.1 cannot be improved.

**Theorem 1.4.** For any $d \geq 1$, there exists a nonsofic $\mathbb{Z}^d$ subshift $X$ for which $N_{[1,n]^d}(X) = n + 1$ for all $n$.

Our results have similarities to the famous Morse-Hedlund theorem.

**Theorem 1.5.** ([6]) If $X$ is a $\mathbb{Z}$ subshift and there exists an $n$ such that the number of words of length $n$ is less than or equal to $n$, then $X$ consists entirely of periodic points. Equivalently, there is a uniform upper bound on the number of words of length $n$.

It is well-known that the bound in Theorem 1.5 is also tight. Sturmian subshifts have no periodic points and have so-called minimal complexity, i.e. any Sturmian
subshift has $n+1$ words of length $n$ for all $n$. (For an introduction to Sturmian subshifts, see [2].)

There are similarities between Theorems 1.1 and 1.5; in fact Theorem 1.5 is used in our proof of Theorem 1.2. However, there are also some interesting differences. In the usual proof of Theorem 1.5, a key component is that the number of $n$-letter words is nondecreasing in $n$. However, Example 3.5 shows that $N_{[1,n]}(X)$ is not necessarily nondecreasing.

2. Definitions and preliminaries

Let $A$ denote a finite set, which we will refer to as our alphabet.

**Definition 2.1.** A pattern over $A$ is a member of $A^S$ for some $S \subset \mathbb{Z}^d$, which is said to have shape $S$. For $d = 1$, patterns are generally called words, especially in the case where $S$ is an interval.

For any patterns $v \in A^S$ and $w \in A^T$ with $S \cap T = \emptyset$, we define the concatenation $vw$ to be the pattern in $A^{S \cup T}$ defined by $(vw)|_S = v$ and $(vw)|_T = w$.

**Definition 2.2.** For any finite alphabet $A$, the $\mathbb{Z}^d$-shift action on $A^{\mathbb{Z}^d}$, denoted by $\{\sigma_t\}_{t \in \mathbb{Z}^d}$, is defined by $(\sigma_t x)(s) = x(s + t)$ for $s, t \in \mathbb{Z}^d$.

We always think of $A^{\mathbb{Z}^d}$ as being endowed with the product discrete topology, with respect to which it is obviously compact.

**Definition 2.3.** A $\mathbb{Z}^d$ subshift is a closed subset of $A^{\mathbb{Z}^d}$ which is invariant under the $\mathbb{Z}^d$-shift action.

**Definition 2.4.** The language of a $\mathbb{Z}^d$ subshift $X$, denoted by $L(X)$, is the set of all patterns which appear in points of $X$. For any finite $S \subset \mathbb{Z}^d$, $L_S(X) := L(X) \cap A^S$, the set of patterns in the language of $X$ with shape $S$.

Any subshift inherits a topology from $A^{\mathbb{Z}^d}$, and is compact. Each $\sigma_t$ is a homeomorphism on any $\mathbb{Z}^d$ subshift, and so any $\mathbb{Z}^d$ subshift, when paired with the $\mathbb{Z}^d$-shift action, is a topological dynamical system. Any $\mathbb{Z}^d$ subshift can also be defined in terms of disallowed patterns: for any set $F$ of patterns over $A$, one can define the set $X(F) := \{x \in A^{\mathbb{Z}^d} : x|_S \notin F \text{ for all finite } S \subset \mathbb{Z}^d\}$. It is well known that any $X(F)$ is a $\mathbb{Z}^d$ subshift, and all $\mathbb{Z}^d$ subshifts are representable in this way. All $\mathbb{Z}^d$ subshifts are assumed to be nonempty in this paper.

**Definition 2.5.** A $\mathbb{Z}^d$ shift of finite type (SFT) is a $\mathbb{Z}^d$ subshift equal to $X(F)$ for some finite $F$. If $F$ consists only of patterns consisting of pairs of adjacent letters, then $X(F)$ is called nearest-neighbor.

**Definition 2.6.** A (topological) factor map is any continuous shift-commuting map $\phi$ from a $\mathbb{Z}^d$ subshift $X$ onto a $\mathbb{Z}^d$ subshift $Y$. A factor map $\phi$ is 1-block if $(\phi x)(v)$ depends only on $x(v)$ for $v \in \mathbb{Z}^d$.

**Definition 2.7.** A $\mathbb{Z}^d$ sofic shift is the image of a $\mathbb{Z}^d$ SFT under a factor map. It is well-known that for any $\mathbb{Z}^d$ sofic shift $Y$, there exists a nearest-neighbor $\mathbb{Z}^d$ SFT $X$ and 1-block factor map $\phi$ so that $Y = \phi(X)$.

For $d = 1$, any $\mathbb{Z}$ sofic shift can also be defined using graphs; a $\mathbb{Z}$ subshift is sofic if and only if it is the set of labels of biinfinite paths for some (edge-)labeled graph $G$ (see [5] for a proof.)
Definition 2.8. For any $\mathbb{Z}^d$ subshift $X$ and rectangular prism $R = \prod_{i=1}^{d} [0, n_i - 1]$, the $R$-higher power shift of $X$, denoted $X^R$, is a $\mathbb{Z}^d$ subshift with alphabet $L_R(X)$ defined by the following rule: $x \in (L_R(X))^{\mathbb{Z}^d} \subseteq X^R$ if and only if the point $y$ defined by concatenating the $x(v)$, viewed themselves as patterns with shape $X$, is in $X$. Formally,

$$\forall v \in \mathbb{Z}^d, y(v) := (v([v_1 n_1^{-1}], \ldots, [v_d n_d^{-1}])) (v_1 \mod n_1, \ldots, v_d \mod n_d).$$

Definition 2.9. For any $\mathbb{Z}$-subshift $X$ and word $w \in L_{[1,n]}(X)$, the follower set of $w$ is $F_X(w) = \{x \in A^{(n+1,n+2,\ldots)}: wx \in L(X)\}$. For any $n$, we use $M_{[1,n]}(X)$ to denote $|\{F_X(w): w \in L_{[1,n]}(X)\}|$, the number of distinct follower sets of words of length $n$.

Definition 2.10. For any $\mathbb{Z}^d$-subshift $X$ and pattern $w \in L_S(X)$, the extender set of $w$ is $E_X(w) = \{x \in A^{\mathbb{Z}^d \setminus S}: wx \in X\}$. For any $S$, we use $N_S(X)$ to denote $|\{E_X(w): w \in L_S(X)\}|$, the number of distinct extender sets of patterns with shape $S$.

3. Proofs

For the proof of Theorem 1.2 we need the following finite version of the Morse-Hedlund theorem. We include a proof for completeness, though it is essentially the same proof as that of the original theorem.

Lemma 3.1. For any word $w \in A^N$ and $n \leq \frac{N}{2}$ so that the number of $n$-letter subwords of $w$ is less than or equal to $n$, we can write $w = twu$ where $|t| = |w| = n$ and $u$ is periodic with some period less than or equal to $n$.

Proof. Since there are less than or equal to $n$ subwords of $w$ of length $n$ and there are $N-n+1 > n$ values of $i$ for which $w(i)w(i+1)\ldots w(i+n-1)$ is a subword of $w$, there exists an $n$-letter subword of $w$ which appears twice. In fact, by the pigeonhole principle we may fix indices $i < k \in [1,n+1]$ such that $w(i)w(i+1)\ldots w(i+n-1) = w(k)w(k+1)\ldots w(k+n-1)$. Similarly, we may fix indices $\ell < j \in [N-n,N]$ such that $w(\ell-n+1)\ldots w(\ell-1)w(\ell) = w(j-n+1)\ldots w(j-1)w(j)$. Set $w' = w(i)w(i+1)\ldots w(j-1)w(j)$.

It suffices to show that $w'$ is periodic of period less than or equal to $n$; if this is true, then taking $t = w(1)\ldots w(n)$, $u = w(n+1)\ldots w(N-n)$, and $v = w(N-n+1)\ldots w(N)$ completes the proof since $u$ is a subword of $w'$.

Let us now consider the number of $m$-letter subwords of $w'$ for values of $m \in [1,n]$. If the number of one-letter subwords of $w'$ is equal to 1, then $w'$ is of the form $ss\ldots s$ for some symbol $s$ and we are done. If not, then the number of one-letter subwords of $w'$ is greater than 1, whereas the number of $n$-letter subwords of $w'$ is less than or equal to $n$. Therefore, there must be an $m \in [1,n-1]$ for which the number of $(m+1)$-letter subwords of $w'$ is less than or equal to the number of $m$-letter subwords of $w'$. Fix $m$ to be this number for the remainder of the proof.

We now claim that for every $m$-letter subword $t$ of $w'$, there exists $a \in A$ so that $ta$ is a subword of $w'$ as well. For any choice of $t$ aside from the $m$-letter suffix of $w'$, this is obvious. But it is true for the suffix as well, since by construction of $w'$, if $t$ is a suffix of $w'$ then $t$ is also the suffix of $w(\ell-n+1)\ldots w(\ell-1)w(\ell)$ which means $tw(\ell+1)$ is a subword of $w'$. A similar argument shows that for every $m$-letter subword $t$ of $w'$, there exists a $b \in A$ so that $bt$ is a subword of $w'$ as well.
Note that because the number of $m$-letter subwords is less than or equal to the number of $(m+1)$-letter subwords of $w'$, the $a$ and $b$ constructed in the previous paragraph are always unique.

Let $p = k - i$, and note that $w(i)w(i+1)\ldots w(i+m-1) = w(i+p)w(i+1+p)\ldots w(i+m+p)$. Since there is a unique $a$ which extends the word $w(i)w(i+1)\ldots w(i+m-1)$ as a subword of $w'$, we get that $w(i+1)w(i+2)\ldots w(i+m) = w(i+1+p)w(i+2+p)\ldots w(i+m+p)$. Using the same argument and working inductively, we see that $w(i+r)w(i+r)\ldots w(i+r) = w(i+r+p)w(i+r+p)\ldots w(i+r+p)$ for any $0 \leq r \leq j - i - p$. In other words, $w'$ is periodic with period $p \leq n$. □

We remark that since the word $u$ in the previous lemma is periodic with period less than or equal to $n$, this clearly implies that $u$ is periodic with period $n!$ (though this may be a meaningless statement if $|u| \leq n$)

**Proof of Theorem 1.2.** Consider a $\mathbb{Z}^d$ subshift $X$ and $n$ so that $|N_{[1,n]}(X)| \leq n$. Define an equivalence relation on $L_{[1,n]}(X)$ by $w \sim w'$ if $E_X(w) = E_X(w')$. For each of the $k \leq n$ equivalence classes, choose a lexicographically maximal element, and denote the collection of these words by $E$. Let $M$ be the entire word on $R$. Then for every $w \in L_{[1,n]}(X)$, there exists $w' \in M$ so that $E_X(w) = E_X(w')$. Equivalently, in any $x \in X$ containing $w$, $w$ can be replaced by $w'$ to make a new point $x' \in X$.

Now, consider any rectangular prism $R = \prod_{i=1}^d [1,n_i]$ with $n_i > max 4n, 2n + n!$ for all $i$, and any finite word $v \in L_R(X)$. Iterate the following procedure: if $v$ contains a subword with shape $[1,n]^d$ which is not in $M$, then replace it by the element of $M$ in its equivalence class. Since each of these replacements increases the entire word on $R$ in the lexicographic ordering, the procedure will eventually terminate, yielding a word $v'$ in which every subword with shape $[1,n]^d$ is in $M$. (These replacements could possibly be done in many different ways or orders; simply choose a particular one and call the result $v'$.) In particular, $v'$ contains less than or equal to $n$ distinct subwords with shape $[1,n]^d$. Since $v'$ is obtained from $v$ by a sequence of replacements with identical extender sets, $E_X(v) = E_X(v')$.

We wish to bound the number of such possible $v'$ for a given $R$. For any translate of the $(d-1)$-dimensional hypercube $t + [1,n]^{d-1} \subset \prod_{i=2}^d [1,n_i]$, consider the subpattern $v'|_{[1,n_1] \times (t+[1,n]^{d-1})}$. This can be viewed as an $n_1$-letter word in the $x_1$-direction, where each “letter” is a cross-section with shape $t + [1,n]^{d-1}$. When viewed in this way, each $n$-letter subword of $v'|_{[1,n_1] \times (t+[1,n]^{d-1})}$ is a subpattern of $v'$ with shape $[1,n]^d$, and there are less than or equal to $n$ such subpatterns. Therefore, by Lemma 3.1, $v'|_{[n+1,n_1-1] \times (t+[1,n]^{d-1})}$ is periodic with period $n!e_1$. Since $t + [1,n]^{d-1}$ was arbitrary, in fact $v'|_{[n+1,n_1-1] \times \prod_{i=2}^d [1,n_i]}$ is periodic with period $n!e_1$ as well. In other words, if $t$ and $t + n!e_1$ both have first coordinate between $n + 1$ and $n_1 - n$ inclusive, then $v'(t) = v'(t + n!e_1)$. A similar proof shows that if $t$ and $t + n!e_i$ both have $i$th coordinate between $n + 1$ and $n_i - n$ inclusive, then $v'(t) = v'(t + n!e_i)$.

The above shows that except for sites within $n$ of the boundary of $R$, $v'$ is determined by the subpattern occurring within a $d$-dimensional hypercube of side length $n!$. More specifically, the values of $v'$ on the sites in $\prod_{i=1}^d ([1,n] \cup [n_1 - n + 1, n_1] \cup [n + 1, n + n!])$ uniquely determine $v'$, and there are $(2n+n!)^d$ such sites. So, regardless of our choice for $R$, there are less than or equal to $|A_X|(2n+n!)^d$ possible $v'$. Since $E_X(v) = E_X(v')$ for every $v$, this shows that $|N_R(X)| \leq |A_X|(2n+n!)^d$ for
every $R$ with all dimensions at least $n$, completing the proof for $K = 2n + n!$ and $N = |A_X|^{(2n+n)!}$. \hfill \Box

We now need a few lemmas for the proof of Theorem 1.3. The first shows that for the purposes of proving $X$ sofic, we may always without loss of generality pass to a higher power shift.

**Lemma 3.2.** For any $d$, for any $\Z^d$ subshift $X$ and rectangular prism $R \subseteq \Z^d$, $X$ is sofic if and only if the higher power shift $X^{[R]}$ is sofic.

**Proof.** $\implies$: Suppose that $X$ is sofic. Then there is a 1-block factor map $\phi$ and $\Z^d$ nearest-neighbor SFT $Y$ so that $X = \phi(Y)$. But then it is easy to check that $X^{[R]} = \phi^{[R]}(Y^{[R]})$, where $\phi^{[R]}$ acts on patterns in $A^R_Y$ via coordinatewise action of $\phi$. Since $Y^{[R]}$ is a $\Z^d$ SFT and $\phi^{[R]}$ is a factor map, clearly $X^{[R]}$ is sofic.

$\impliedby$: Suppose that $X^{[R]}$ is sofic, and without loss of generality, write $R = \prod_{i=1}^d [0, n_i]$. Then there is a 1-block factor map $\psi$ and $\Z^d$ nearest-neighbor SFT $Z$ so that $X^{[R]} = \psi(Z)$. Define a $\Z^d$ nearest-neighbor SFT $Z'$ with alphabet $A_Z \times R$ by the following rules:

1. In the $x_i$-direction, a letter of the form $(a, (v_1, \ldots, v_d))$ must be followed by a letter of the form $(b, (v_1, \ldots, v_i, v_i + 1 (\text{mod } n_i), v_{i+1}, \ldots, v_d))$.
2. In rule (1), if $v_i \neq n_i - 1$, then $b = a$.
3. In rule (1), if $v_i = n_i - 1$, then $b$ must be a legal follower of $a$ in the $x_i$-direction in the nearest-neighbor SFT $Z$.

The effect of these rules is that in any point of $Z'$, $\Z^d$ is partitioned into translates of $R$, each translate of $R$ has a constant “label” from $A_Z$, and the “labels” of these translates comprise a legal point of $Z$. We now define a 1-block factor map $\phi'$ on $Z'$ by the rule $\phi'(a, v) = (\phi(a))(v)$, i.e. the letter of $A_Z$ appearing at location $v$ in $\phi(a)$, which was by definition a pattern in $A^R_Z$. This has the effect of, in each point of $Z'$, filling every translate of $R$ with the image under $\phi$ of the letter of $A_Z$ which was its label. Since these labels comprise a point of $Z$ and since $\phi(Z) = X^{[R]}$, the reader may check that $\phi'(Z') = X$, and so $X$ is sofic. \hfill \Box

Our next lemma shows that an upper bound for $N_R(X)$ over all large finite rectangular prisms $R$ must also be an upper bound for $N_R(X)$ even when we allow $R$ to have some infinite dimensions.

**Lemma 3.3.** For any $d$ and any $\Z^d$ subshift $X$, if there exist $K, N$ so that $N_R(X) \leq N$ for any rectangular prism $R$ with dimensions at least $K$, then it is also the case that $N_{R'}(X) \leq N$ for any “infinite rectangular prism” of the form $R' = \prod_{i=1}^d I_i$, where each of the $I_i$ is either an interval of integers with length at least $K$ or $\Z$.

**Proof.** Consider any $K, N, X$ satisfying the hypotheses of the theorem, and any “infinite rectangular prism” $R'$ with all dimensions either finite and greater than $K$ or infinite. Suppose for a contradiction that there exist $N + 1$ distinct configurations $w_1, \ldots, w_{N+1}$ in $L_{R'}(X)$ and that their extender sets $E_X(w_i)$ are distinct. Then, for each pair $i < j \in [1, N + 1]$, there exists a pattern $v_{ij} \in L_{R'}(X)$ s.t. $v_{ij}w_i \in X$ and $v_{ij}w_j \notin X$ or vice versa. By compactness, for each $v_{ij}$, there exists $n_{ij}$ so
that \(v_{ij}w_i|_{[-n_{ij},n_{ij}]} \cap R^c \notin L(X)\) and \(v_{ij}w_i|_{[-n_{ij},n_{ij}]} \cap R^c \notin L(X)\), or vice versa. This property is clearly preserved by increasing \(n_{ij}\). Therefore, if we define \(M = \max(K, \{n_{ij}\})\), then for every \(i < j \in [1, N + 1]\), either \(v_{ij}w_i|_{[-M,M]} \cap R^c \notin L(X)\) and \(v_{ij}w_i|_{[-M,M]} \cap R^c \notin L(X)\) or vice versa. Put another way, \(E_X(w_i|_{[-M,M]} \cap R^c)\) contains a pattern which equals \(v_{ij}\) on \(R^c\), and \(E_X(w_j|_{[-M,M]} \cap R^c)\) contains no such pattern, or vice versa. Either way, this shows that \(E_X(w_i|_{[-M,M]} \cap R^c) \neq E_X(w_j|_{[-M,M]} \cap R^c)\) and, since \(i,j\) were arbitrary, that all \(N + 1\) of the extender sets \(E_X(w_i|_{[-M,M]} \cap R^c), i \in [1, N + 1]\), are distinct. Since \([-M,M] \cap R^c\) is a finite rectangular prism with all dimensions at least \(K\), this contradicts the hypotheses of the theorem. Our original assumption was therefore wrong, and \(N_{R^c}(X) \leq N\).

Our final preliminary lemma shows that for \(d = 1\), boundedness of \(N_{[1,n]}(X)\) is equivalent to soficity of \(X\).

**Lemma 3.4.** For a \(\mathbb{Z}\) subshift \(X\), \(X\) is sofic if and only if \(N_{[1,n]}(X)\) is a bounded sequence.

**Proof.** \(\implies\): If \(X\) is sofic, then there is a 1-block map \(\phi\) and nearest-neighbor SFT \(Y\), with alphabet \(A_Y\), so that \(X = \phi(Y)\). Then, for any finite word \(w \in L_{[1,n]}(Y)\), clearly \(E_X(w) = \bigcup_{y \in \phi^{-1}(w)} \phi(E_Y(y))\). Since \(Y\) is a nearest-neighbor SFT, this clearly depends only on the set of pairs of first and last letters of \(\phi\)-preimages of \(w\), and there are fewer than \(2^{|A_X|^2}\) such sets. Therefore, \(N_{[1,n]}(X) \leq 2^{|A_X|^2}\) for all \(n\), and so the sequence \(N_{[1,n]}(X)\) is bounded.

\(\impliedby\): We prove the contrapositive, the proof will be similar to Lemma 3.3. Suppose that \(X\) is not sofic. Then there are infinitely many follower sets \(F(p)\) for infinite pasts \(p \in A_X^\mathbb{Z}\). For any \(N\), choose \(N\) pasts \(p_1, \ldots, p_N\) with distinct follower sets. This means that for every \(i < j \in [1, N]\), there exists a future \(f_{ij} \in A_X^\mathbb{Z}\) so that either \(p_if_{ij} \in X\) and \(p_jf_{ij} \notin X\), or vice versa. By compactness, there exists \(N_{ij}\) so that for any \(n > N_{ij}\), either \(p_i|_{[-n,0]}f_{ij} \in X\) and \(p_j|_{[-n,0]}f_{ij} \notin X\), or vice versa. But then if we take \(M = \max(N_{ij})\), then for every \(i < j \in [1, N]\), either \(p_i|_{[-M,0]}f_{ij} \in X\) and \(p_j|_{[-M,0]}f_{ij} \notin X\), or vice versa, meaning that the \(N\) extender sets \(E_X(p_i|_{[-M,0]}), i \in [1, N]\), are distinct. Therefore, \(N_{[1,M]}(X) \geq N\). Since\(N\) was arbitrary, \(N_{[1,n]}(X)\) is not bounded.

As an aside, before proving Theorem 1.3 we present the example mentioned in the introduction, of a \(\mathbb{Z}\) sofic shift \(X\) where \(N_{[1,n]}(X)\) is bounded, but does not stabilize. In fact, the number of distinct follower sets of words of length \(n\) also fails to stabilize for this shift, which may be of independent interest.

**Example 3.5.** ([1]) Define \(X\) to be the sofic shift consisting of all labels of biinfinite paths on the labeled graph \(G\) below. Then for all \(n > 1\), \(M_{[1,2n]}(X) = 14, M_{[1,2n+1]}(X) = 13, N_{[1,2n]}(X) = 46,\) and \(N_{[1,2n+1]}(X) = 44\).
Proof. The reader may check that $\mathcal{G}$ is follower-separated (see [5] for a definition), and so for any $w \in L(X)$, the follower set $F_X(w)$ is determined by the set of terminal vertices for paths in $\mathcal{G}$ with label $w$, which we’ll denote by $T(w)$. We now simply describe the possible sets $T(w)$ for words of even and odd length, with examples of words realizing each set. We use the notation * to indicate that any word may replace the *, and $n$ to represent any nonnegative integer.

<table>
<thead>
<tr>
<th>$T(w)$</th>
<th>$w$</th>
<th>$T(w)$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0}$</td>
<td>*cc</td>
<td>${0}$</td>
<td>*cc</td>
</tr>
<tr>
<td>${1}$</td>
<td>*ccb</td>
<td>${1}$</td>
<td>*cccb</td>
</tr>
<tr>
<td>${2}$</td>
<td>*cbb</td>
<td>${2}$</td>
<td>*cbb</td>
</tr>
<tr>
<td>${3}$</td>
<td>*bbba</td>
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<td>*bbba</td>
</tr>
<tr>
<td>${4}$</td>
<td>*bbcb</td>
<td>${4}$</td>
<td>*bbcb</td>
</tr>
<tr>
<td>${5}$</td>
<td>*bcbb</td>
<td>${5}$</td>
<td>*bcbb</td>
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<td>*cba</td>
</tr>
<tr>
<td>${7}$</td>
<td>*bca</td>
<td>${7}$</td>
<td>*bca</td>
</tr>
<tr>
<td>${8}$</td>
<td>*aca</td>
<td>${8}$</td>
<td>*aca</td>
</tr>
<tr>
<td>${1,5}$</td>
<td>$a^n\cdot cb$</td>
<td>${1,5}$</td>
<td>$a^n\cdot cb$</td>
</tr>
<tr>
<td>${3,6}$</td>
<td>$ba^{2n+1}$</td>
<td>${2,5}$</td>
<td>$ba^{2(n+1)}$</td>
</tr>
<tr>
<td>${4,7}$</td>
<td>$ba^{2n+1}\cdot c$</td>
<td>${0,4,7}$</td>
<td>$a^n\cdot c$</td>
</tr>
<tr>
<td>${2,3,5,6,7,8}$</td>
<td>$a^{2(n+1)}$</td>
<td>${2,3,5,6,7,8}$</td>
<td>$a^{2(n+1)}$</td>
</tr>
</tbody>
</table>

We leave it to the reader to check that there are no follower sets aside from the ones described here, and so $M_{[1,2n]}(X) = 14$ and $M_{[1,2n+1]}(X) = 13$ for all $n > 1$. Informally, the reason that words of even length have an additional follower set is that the word $ba^{2n}\cdot c$ has a follower set (given by the set $\{3,6\}$ of terminating states) which can not be recreated by odd length: every cycle has even length, knowledge of at least one letter on each side of the cycle is required to create a new follower set, and knowledge of two letters on either side makes the word synchronizing (meaning there is only a single terminating state.)
Since listing 46 and 44 extender sets similarly (for even and odd lengths respectively) would be quite cumbersome, we will not give a complete list of these, but will give a sketch of how they appear. First, note that $G$ is also predecessor separated, and so the extender set of a word $w \in L(X)$ is determined entirely by the set $\{v \to v'\}$ of possible pairs of initial and terminal vertices of paths in $G$ with label $w$, which we denote by $S(w)$. Note that partitioning the vertices into \{0, 2, 5, 7\} and \{1, 3, 4, 6, 8\} shows that $G$ is bipartite. The reader may check that every possible singleton $\{v \to v'\}$ for pairs $v, v'$ in the same vertex class occurs as $S(w)$ for a word $w$ of even length, and every possible singleton $\{v \to v'\}$ for pairs $v, v'$ in opposite vertex classes occurs as $S(w)$ for a word $w$ of odd length. This contributes $5^2 + 4^2 = 41$ extender sets to $N_{[1,2n]}(X)$ and $2 \cdot 5 \cdot 4 = 40$ extender sets to $N_{[1,2n+1]}(X)$ for every $n > 1$. The remaining sets $S(w)$, along with $w$ presenting them, appear in the table below. Note that though we can informally pair up the first through fourth types in each case, again the word $ba^{2n-2}c$ creates an extender set with no analogous extender set for a word of odd length.

\[
\begin{array}{c|c}
\text{(even length)} & \text{w} \\
\hline
\{2 \to 2, 3 \to 3, 5 \to 5, 6 \to 6, 7 \to 7, 8 \to 8\} & a^{2n} \\
\{1 \to 3, 4 \to 6\} & ba^{2n-1} \\
\{2 \to 5, 7 \to 1\} & a^{2n-2}cb \\
\{3 \to 4, 6 \to 7, 8 \to 0\} & a^{2n-1}c \\
\{1 \to 4, 4 \to 7\} & ba^{2n-2}c \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{(odd length)} & \text{w} \\
\hline
\{2 \to 3, 3 \to 2, 5 \to 6, 6 \to 5, 7 \to 8, 8 \to 7\} & a^{2n-1} \\
\{1 \to 2, 4 \to 5\} & ba^{2n} \\
\{3 \to 5, 8 \to 1\} & a^{2n-1}cb \\
\{2 \to 4, 5 \to 7, 7 \to 0\} & a^{2n}c \\
\end{array}
\]

\[\square\]

**Proof of Theorem 1.3.** Our proof proceeds by induction on $d$. The base case $d = 1$ is precisely Lemma 3.4. We now assume that the result holds for $\mathbb{Z}^{d-1}$ subshifts, and will prove it for $\mathbb{Z}^d$ subshifts. To that end, assume that $X$ is a $\mathbb{Z}^d$ subshift and that there exist $K, N$ so that for any rectangular prism $R$ with dimensions at least $K, N_R(X) \leq N$. By Lemma 3.3, the same is true even if $R$ has some infinite dimensions.

Note that by Lemma 3.2, we may without loss of generality replace $X$ by the higher power shift $X^{[1,K]^d}$. Since $N_R(X^{[1,K]^d}) \leq N$ for all rectangular prisms, with no restrictions on the dimension, we will assume this property for $X$ in the remainder of the proof.

Define $X' = \{ x_{|z|^{d-1} \times \{0\}} : x \in X \}$, the set of restrictions of points of $X$ to hyperplanes spanned by the first $d-1$ cardinal directions. By the assumption above, there are fewer than $N$ distinct extender sets for $x \in X'$, and so we define equivalence classes $C_i, i \in [1, M], M \leq N$, for the equivalence relation defined by
$x \sim y$ if $E_X(x) = E_X(y)$. In a slight abuse of notation, we denote by $E_X(C_i)$ the common extender set shared by all $x \in C_i$.

Now, consider any $x \in X'$ and $k \in [1, d - 1]$. By the pigeonhole principle, there exist $i < j \in [1, M + 1]$ so that $\sigma_{i\epsilon_k}x \sim \sigma_{j\epsilon_k}x$. But then for $y \in L_{Z^{d-1}}(\{0\}^c)(X)$, $y \in E_X(x) \iff \sigma_{i\epsilon_k}y \in E_X(\sigma_{i\epsilon_k}x) \iff \sigma_{j\epsilon_k}y \in E_X(\sigma_{j\epsilon_k}x) \iff y \in E_X(\sigma_{(j-i)\epsilon_k}x)$. Therefore, $x \sim \sigma_{(j-i)\epsilon_k}x$, and the same logic shows that $\sigma_{(j-i)m_{\epsilon_k}}x \sim x$ for any $m \in Z$. Since $j - i \leq M \leq N$, this shows that the $C_i$ containing $x$ is invariant under shifts by $N!e_k$ for $k \in [1, d - 1]$. Since $x \in X'$ was arbitrary, this means that every $C_i$ is invariant under shifts by each $N!e_k$. We may then, again by Lemma 3.2, replace $X$ by its higher power shift $X[[1,N]^d_{-1} \times \{0\}]$, which allows us to assume without loss of generality that all of the $C_i$ are shift-invariant subsets of $A_X^{Z_{d-1}}$. The classes $C_i$ need not, however, be closed. Their closures are $Z^{d-1}$ subshifts though, and we will show that they in fact must be sofic.

Claim 1: $\overline{C_i}$ is sofic for each $i$.

It suffices to show that for any rectangular prism $R \subseteq Z^{d-1}$ and $w, w' \in L_R(\overline{C_i})$, $E_X(w) = E_X(w') \implies E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$, since then $N_{R \times \{0\}}(X) \leq N \implies N_R(\overline{C_i}) \leq N$ for all rectangular prisms $R$, which will imply the desired conclusion by our inductive hypothesis.

So, assume that $E_X(w) = E_X(w')$ for $w, w' \in L_R(\overline{C_i})$. Suppose also that $vw \in \overline{C_i}$ for some $v \in A_X^{Z_{d-1}\setminus R}$. Then there exists $v_n \in A_X^{Z_{d-1}\setminus R}$ so that $v_n \rightarrow v$ and $v_nw \in C_i$ for all $n$. Then for any $y \in E_X(C_i)$, $yv_nw \in X$, since all $v_nw$ share the same class $C_i$. Since $E_X(w) = E_X(w')$, $yv_nw' \in X$ as well. Similarly, for any $y \notin E_X(C_i)$, $yv_nw \notin X$, and so $yv_nw' \notin X$. But then $E_X(v_nw') = E_X(C_i)$, and so $v_nw' \in C_i$. By taking limits, $vw' \in \overline{C_i}$. We’ve then shown that $vw \in \overline{C_i} \implies vw' \in \overline{C_i}$. The converse is true by the same proof, and so $E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$, completing the proof of soficity of $\overline{C_i}$ as described above.

Since all elements of any class $C_i$ are interchangeable in points of $X$, we can define $V \subseteq [1, M]^Z$ which lists legal sequences of classes (in the $e_d$-direction) within points in $X$:

$$V := \{(i_n) \in [1, M]^Z : \exists x \in X \text{ such that } \forall n \in Z, x|_{Z^{d-1} \times \{n\}} \in C_{i_n}\}.$$ 

It is obvious that $V$ is shift-invariant since $X$ is shift-invariant. However, it is not immediately clear that $V$ is closed since the $C_i$ are not necessarily closed. We will show that $V$ is closed by proving the following claim.

Claim 2: $X = \{x \in A_X^{Z_d} : \exists (i_n) \in V \text{ such that } \forall n \in Z, x|_{Z^{d-1} \times \{n\}} \in \overline{C_{i_n}}\}$. 

In other words, given \( v \in V \), not only can you make points of \( X \) by substituting in configurations from the classes given by the letters in \( v \), but you may also substitute configurations from the closures of these classes.

\[
X \subseteq \{ x \in A_{X}^{d} \mid \exists (i_{n}) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_{n}}} \}:
\]

First, we note that for any \( x \in X \), by definition \( x|_{\mathbb{Z}^{d-1} \times \{n\}} \in X' \) for all \( n \in \mathbb{Z} \), and so each of these is in some class \( C_{i} \). Define \( v = (i_{n}) \in [1, M]^{\mathbb{Z}} \) by saying that the \( x|_{\mathbb{Z} \times \{n\}} \) of \( x \) is in \( C_{i_{n}} \). Then by definition of \( V \), \( v \in V \). This clearly shows the desired containment.

\[
X \supseteq \{ x \in A_{X}^{d} \mid \exists (i_{n}) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z} \times \{n\}} \in \overline{C_{i_{n}}} \}:
\]

Choose any \( x \in A_{X}^{d} \) so that there is \( v = (i_{n}) \in V \) with the property that \( \forall n \in \mathbb{Z} \), \( x|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_{n}} \). Then, for each \( n \in \mathbb{Z} \), there exists a sequence \( x^{(k,n)} \in C_{i_{n}} \) so that \( x^{(k,n)} \rightarrow x|_{\mathbb{Z}^{d-1} \times \{n\}} \) for all \( n \). Also, since \( v \in V \), there exists \( x' \in X \) so that \( \forall n \in \mathbb{Z} \), \( x'|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_{n}} \).

We now define, for every \( k \), the point \( x^{(k)} \in A_{X}^{d} \) by

\[
x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = \begin{cases} x^{(k,n)} \text{ if } |n| \leq k \\ x'|_{\mathbb{Z}^{d-1} \times \{n\}} \text{ if } k \leq |n| \end{cases}
\]

The central \( 2k+1 \) \((d-1)\)-dimensional hyperplanes of \( x^{(k)} \) are given by the \( x^{(k,n)} \), and the remaining \((d-1)\)-dimensional hyperplanes are unchanged from \( x' \). We note that \( x^{(k)} \) can be obtained from \( x' \in X \) by making \( 2k+1 \) consecutive replacements of \( x'|_{\mathbb{Z}^{d-1} \times \{n\}} \) by \( x^{(k,n)} \). Since these replacements involve configurations in the same class \( C_{i_{n}} \), each of these replacements preserves being in \( X \), and so \( x^{(k)} \in X \) for all \( k \). Finally, we note that \( x^{(k)} \rightarrow x \), so \( x \in X \) as well, showing the desired containment.

Claim 3: \( V \) is a sofic subshift.

We first show that \( V \) is closed and therefore a subshift. Let \( v^{(k)} \in V \) and \( v^{(k)} \rightarrow v = (i_{n}) \). By passing to a subsequence, we may assume that for all \( k \geq |n| \), \( v^{(k)} = i_{n} \). By definition of \( V \), for every \( k \in \mathbb{N} \) there exists \( x^{(k)} \in X \) where \( x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v^{(k)}} \) for every \( n \). For all \( n \leq k \), since \( C_{v^{(k)}} = C_{i_{n}} = C_{v^{(n)}} \) we may replace the pattern \( x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \) in \( x^{(k)} \) with \( x^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}} \) to form a legal point in \( X \). In such a way we obtain a new sequence of points \( y^{(k)} \in X \) where \( y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v^{(k)}} \), but with the additional property that \( y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}} \) for \( k \geq |n| \). The sequence \( y^{(k)} \) clearly converges to a point \( y \in A_{X}^{d} \), and \( y \in X \) since \( X \) is closed. Since \( y|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_{n}} \), we have \( y \in V \).

We claim that \( N_{1,m}(V) \leq N \) for all \( m \in \mathbb{N} \), which will prove the claim by Lemma 3.4. Suppose for a contradiction that there are \( N+1 \) words \( v^{(1)}, \ldots, v^{(N+1)} \in L_{m}(V) \) s.t. the extender sets \( E_{V}(v^{(i)}) \) are all distinct. Then, for every \( i < j \in \mathbb{N} \).
[1, N + 1], there exists \( w^{(ij)} \in L_{[1,m]}\) (V) s.t. either \( v^{(i)}w^{(ij)} \in V \) and \( v^{(i)}w^{(ij)} \notin V \) or vice versa.

For each \( v^{(i)} = v_1^{(i)} \ldots v_m^{(i)} \), define \( S^{(i)} \in A_{X_{[1,m]}^d} \) by choosing \( S^{(i)}|_{[Z^{d-1}]^{(n)}} \) to be any row in \( C_{w^{(i)}} \). Similarly, for each \( w^{(ij)} \), define a pattern \( O^{(ij)} \in A_{X_{[1,m]}^{d-1}} \) by choosing \( O^{(ij)}|_{[Z^{d-1}]^{(n)}} \) to be any row in \( C_{w^{(i)}} \). Then, by Claim 2, \( S^{(i)}O^{(ij)} \in X \) and \( S^{(j)}O^{(ij)} \notin X \) or vice versa, meaning that all extender sets \( E_X(S^{(i)}), i \in [1, N + 1] \), are distinct. This is a contradiction to Lemma 3.3, and so our original assumption was wrong, \( N_{[1,m]}(V) \) is a bounded sequence, and \( V \) is sofic.

We may now finally construct an SFT cover of \( X \) to show that it is sofic. Since \( V \) is sofic by Claim 3, we may define a 1-block factor \( \psi \) and nearest-neighbor SFT \( W \) so that \( \psi(W) = V \). For each \( a \in A_V \), since \( C_a \) is sofic by Claim 1, there is a 1-block factor \( \phi_a \) and nearest-neighbor \( \mathbb{Z}^{d-1} \) SFT \( Y_a \) whose alphabet we denote by \( A_a \) so that \( \phi_a(Y_a) = C_a \). Now, define a nearest-neighbor \( \mathbb{Z}^d \) SFT \( Y \) with alphabet \( A_Y := \bigcup_{a \in A_W} (\{a\} \times A_Y(a)) \) by the following adjacency rules:

1. Any pair of letters \((a, t)\) which are adjacent in one of the first \( d - 1 \) cardinal directions must share the same first coordinate, i.e. \( a = b \).
2. \((a, s)\) may legally precede \((a, t)\) in the \( e_i\)-direction for \( i \in [1, d - 1] \) if and only if \( s \) may legally precede \( t \) in the same direction in \( Y(a) \).
3. \((a, s)\) may legally precede \((b, t)\) in the \( e_d\)-direction if and only if \( a \) may legally precede \( b \) in \( W \). (There is no restriction on the second coordinates \( s, t \).)

Clearly for any \( y \in Y \), these rules force the \((d - 1)\)-dimensional hyperplanes \( y|_{[Z^{d-1}]^{(n)}} \) to have constant first coordinate (say \( a_n \)), force the second coordinates to form a point in \( Y(a_n) \), and force the sequence \((a_n)\) to be in \( W \). We now define the 1-block factor map \( \phi \) on \( Y \) by \( \phi(a, s) = \phi_{\psi(a)}(s) \).

**Claim 4:** \( \phi(Y) = X \).

\( \phi(Y) \subseteq X \): Take any \( y \in Y \), and define \( a_n \) to be the first coordinate shared by all letters in \( y|_{[Z^{d-1}]^{(n)}} \). Then by definition of \( Y \), \((a_n) \in W \). Also by definition of \( Y \), the second coordinates of the letters in \( y|_{[Z^{d-1}]^{(n)}} \) form a point of \( Y_{\psi(a_n)} \), call it \( b^{(n)} \). Then, \((\psi(a_n)) \in V \), and for every \( n \in \mathbb{Z} \), \((\phi(y))|_{[Z^{d-1}]^{(n)}} = \phi_{\psi(a_n)}(b^{(n)}) \) is the \( \phi_{\psi(a_n)} \)-image of a point of \( Y_{\psi(a_n)} \), and so is in \( C_{a_n} \). But then, by Claim 2, \( \phi(y) \in X \), and since \( y \in Y \) was arbitrary, \( \phi(Y) \subseteq X \).

\( \phi(Y) \supseteq X \): Choose any \( x \in X \). For every \( n \in \mathbb{Z} \), \( x|_{[Z^{d-1}]^{(n)}} \) is in one of the \( C_i \), and if we define a sequence \( i_n \) by \( x|_{[Z^{d-1}]^{(n)}} \in C_{i_n} \), then \((i_n) \in \mathcal{V} \). Choose any \((a_n) \in W \) s.t. \((\psi(a_n)) = (i_n) \). For each \( n \in \mathbb{Z} \), since \( x|_{[Z^{d-1}]^{(n)}} \in C_{i_n} = C_{\psi(a_n)} \), there exists \( b^{(n)} \in Y_{\psi(a_n)} \) s.t. \( \phi_{\psi(a_n)}(b^{(n)}) = x|_{[Z^{d-1}]^{(n)}} \). Define a point \( y \in A_Y \) by setting, for all \( t = (t_1, \ldots, t_d) \in \mathbb{Z}^d \), \( y(t) = (a_n, b^{(n)}((t_1, \ldots, t_{d-1})) \). Then \( y \in Y \) since \((a_n) \) is in \( W \), and for all \( n \in \mathbb{Z} \), the second coordinates in each hyperplane
$y|_{Z^{d-1} \times \{n\}}$ create a point of $Y_{\psi(a_n)}$. It is also clear that $\phi(y) = x$, and since $X$ was arbitrary, $\phi(Y) \supseteq X$.

Since $Y$ was an SFT and $\phi$ a factor map, this shows that $X$ is sofic, completing the proof of Theorem 1.3.

We would like to say a bit more about shifts satisfying the hypotheses of Theorem 1.3, i.e. those with eventually bounded numbers of extender sets, because in fact they satisfy a much stronger (though technical) condition than just being sofic.

**Definition 3.6.** We say that a $Z^d$ nearest-neighbor SFT $X$ is **decouplable** if either $d = 1$ (in which case $X$ is automatically called decouplable) or there exist $i \in [1, d]$, a $Z$ nearest-neighbor SFT $W$, and decouplable $Z^{d-1}$ nearest-neighbor SFTs $Y_a$ for each $a \in A_W$, with disjoint alphabets, so that

$$X = \{ x \in A_X^{Z^d} : \exists w = (w_n) \in W \text{ s.t. } \forall n \in Z, x|_{Z^{d-1} \times \{n\} \times Z^{d-i}} \in Y_{w_n} \}. $$

(Here, we have made the obvious identification between $Z^{i-1} \times \{n\} \times Z^{d-i}$ and $Z^{d-1}$.)

In other words, $X$ is decouplable if one can construct it by starting from a one-dimensional nearest neighbor SFT and then arbitrarily replacing occurrences of each letter in its alphabet by points from a $Z^{d-1}$ decouplable nearest-neighbor SFT associated to that letter. This definition is obviously recursive; $X$ is decouplable if its $(d-1)$-dimensional hyperplanes are given by decouplable SFTs, whose $(d-2)$-dimensional hyperplanes are given by decouplable SFTs, and so on. This means that though $X$ is a $Z^d$ SFT, its behavior is in some sense one-dimensional.

**Remark 3.7.** In fact the SFT cover in the proof of Theorem 1.3 can always be chosen to be decouplable. By the inductive hypothesis, the covers $Y_a$ can each be chosen to be decouplable $Z^{d-1}$ SFTs, and then the construction of $X$ from $W$ and all $Y_a$ clearly yields a decouplable $Z^d$ SFT.

In order to give an application of Theorem 1.3 and to elucidate the idea of decouplable SFTs, we will present a brief example.

**Example 3.8.** Define $X$ to be the $Z^2$ subshift on $\{0, 1\}$ consisting of all $x \in \{0, 1\}^{Z^2}$ with either no 1s, a single 1, or two 1s.

Then it is not hard to see that for any $S \subseteq Z^2$ with $|S| > 1$, $N_S(X) = 3$. It is easily checked that the three possible extender sets for $w \in L_S(X)$ are:

- If $w$ contains no 1s, then $E_X(w)$ consists of all patterns on $S^c$ with either no 1s, a single 1, or two 1s.
- If $w$ contains a single 1, then $E_X(w)$ consists of all patterns on $S^c$ with either no 1 or a single 1.
- If $w$ contains two 1s, then $E_X(w)$ consists of the single pattern on $S^c$ with no 1s, namely $0^{S^c}$.

We will now describe how the proof of Theorem 1.3 yields an SFT cover for $X$. Using the language of the proof of Theorem 1.3, $X'$ consists of all biinfinite 0-1
sequences with either no 1s, a single 1, or two 1s. $X'$ is broken into three classes of rows with the same extender sets in $X$, which are again classified by number of 1s contained:

- $C_0 = \{x \in \{0, 1\}^\mathbb{Z} : x$ contains no 1s $\} = \{0^\mathbb{Z}\}$.
- $C_1 = \{x \in \{0, 1\}^\mathbb{Z} : x$ contains exactly one 1 $\}$.
- $C_2 = \{x \in \{0, 1\}^\mathbb{Z} : x$ contains exactly two 1s $\}$.

(We’ve written the $C_i$s with subscripts starting from 0 rather than 1 so that $V$ can be more easily described; clearly this has no effect on the proof.) Note that each of the $C_i$ is shift-invariant, but $C_1$ and $C_2$ are not closed. However, each closure $\overline{C_i}$ is sofic. This is easily checked, but we will momentarily explicitly describe SFT covers of the $\overline{C_i}$ anyway.

We now wish to find $V$, the $\mathbb{Z}$ subshift with alphabet $\{0, 1, 2\}$ which describes how the rows in various classes can fit together to make points of $X$. This is not so hard to see: since points of $X$ must have at most two 1s, and since the classes $C_i$ are partitioned by number of 1s,

$$V = \{v \in \{0, 1, 2\}^\mathbb{Z} : v$ has only finitely many nonzero digits, and $\sum v_n \leq 2\}.$$ 

Points of $X$ are then constructed by beginning with a sequence in $V$, writing it vertically, and replacing each letter $v_n$ with an arbitrary element of $C_{v_n}$. So, for instance, one could start with $\ldots 0002000 \ldots \in V$, replace all 0s by the single sequence $\ldots 000000 \ldots \in C_0$, and replace 2 by any sequence in $C_2$, for instance $\ldots 0001001000 \ldots$. Clearly every point obtained in this way will have at most two 1s and so will be in $X$. In addition though, as described in Claim 2, one can also replace each $v_n$ by an arbitrary element of the closure $\overline{C_{v_n}}$. For instance, if we chose to replace the 2 in our earlier sequence by $\ldots 0001000 \ldots$, which is not in $C_2$ (in fact it’s in $C_1$), but is in $\overline{C_2}$, we would still arrive at a legal point of $X$.

We now note that $V$ is a sofic shift, with nearest-neighbor SFT cover $W$ defined as follows: $A_W = \{A, B, C, D, E, F\}$, and legal adjacent pairs in $W$ are $AA$, $AB$, $BC$, $CC$, $CD$, $DE$, $EE$, $AF$, and $FE$. So,

$$W = \{A^\infty, C^\infty, E^\infty, A^\infty BC^\infty, C^\infty DE^\infty, A^\infty BC^n DE^\infty, A^\infty FE^\infty\}.$$ 

The factor $\psi$ is defined by $\psi(A) = \psi(C) = \psi(E) = 0$, $\psi(B) = \psi(D) = 1$, and $\psi(F) = 2$, and it is easily checked that $\psi(W) = V$.

Following our proof of Theorem 1.3, the next step is to construct SFT covers of each $\overline{C_i}$, which is straightforward. Define $Y_0$ to consist of the single fixed point $\{a^\infty\}$, and $\phi_0$ by $\phi_0(a) = 0$. Define $Y_1$ to have alphabet $\{a, b, c\}$ with legal adjacent pairs $aa$, $ab$, $bc$, and $cc$: then $Y_1 = \{a^\infty, c^\infty, a^\infty bc^\infty\}$. Define $\phi_1$ by $\phi_1(a) = \phi_1(c) = 0$ and $\phi_1(b) = 1$. Finally, define $Y_2$ to have alphabet $\{a, b, c, d, e\}$ with legal adjacent pairs $aa$, $ab$, $bc$, $cc$, $cd$, $de$, and $ee$; then $Y_2 = \{a^\infty, c^\infty, e^\infty, a^\infty bc^\infty, c^\infty de^\infty, a^\infty bc^\infty de^\infty\}$. Define $\phi_2$ by $\phi_2(a) = \phi_2(c) = \phi_2(e) = 0$ and $\phi_2(b) = \phi_2(d) = 1$. The reader may check that $\phi_i(Y_i) = \overline{C_i}$ for each $i$.

We may now construct an SFT cover $Y$ for $X$ following our proof. The alphabet $A_Y := \bigcup_{a \in A_W} \{(a) \times A_{\psi(a)}\}$ = \{(A, a), (B, a), (B, b), (B, c), (C, a), (D, a), (D, b), (D, c), (E, a), (F, a), (F, b), (F, c), (F, d), (F, e)\}. The adjacency rules are that horizontally adjacent letters have the same first (capital) coordinate $\Pi$ and second (lowercase) coordinates satisfying the adjacency rules given by $Y_{\psi(\Pi)}$, and that vertically adjacent letters have first (capital) coordinates satisfying the adjacency rules of $W$. So,
for instance, \((D, a)\) cannot appear immediately to the left of \((D, c)\), since \(ac\) is not a legal pair in \(Y_{D(D)} = Y_1\). On the other hand, \((A, a)\) can appear below \((F, d)\), since \(AF\) is a legal pair in \(W\). The map \(\phi\) is defined, as before by \(\phi(a, b) = \phi_s(a), \cdots, (b)\), meaning that \(\phi(B, b) = \phi(D, b) = \phi(F, b) = \phi(F, d) = 1\), and all other letters of \(A_Y\) have \(\phi\)-image 0. It’s easy to see that \(\phi(Y) = X\); the rules defining \(Y\) mean that there are at most two letters with second coordinate \(b\) or \(d\), and these are the only letters of \(A_Y\) which have \(\phi\)-image 1.

Finally, we will prove Theorem 1.4, but first need the following definition and theorem from [3].

**Definition 3.9.** ([3]) A \(\mathbb{Z}^d\) subshift \(X\) is effective if there exists a forbidden list \((w_n)\) for \(X\) and a Turing machine which, on input \(n\), outputs \(w_n\).

It is easy to see that not every subshift is effective; there are only countably many Turing machines, and so only countably many effective subshifts.

**Theorem 3.10.** ([3]) Any \(\mathbb{Z}^d\) sofic shift is effective.

**Proof of Theorem 1.4.** For any Sturmian \(\mathbb{Z}\) subshift \(S\), we can extend \(S\) to a \(\mathbb{Z}^d\) subshift \(\tilde{S}\) by enforcing constancy along all cardinal directions \(e_i\), \(i \in [2, d]\). There are uncountably many Sturmian subshifts, and so there exists one, call it \(\tilde{S}'\), s.t. \(\tilde{S}'\) is not effective. (In fact, effectiveness of Sturmian \(S\) and/or the shift \(\tilde{S}\) is equivalent to computability of the rotation number defining \(S\), but we will not need this fact.) By Theorem 3.10, \(\tilde{S}'\) is not sofic. In addition, by the minimal complexity definition of Sturmian subshifts, for every \(n \in \mathbb{N}\), \(|L_{[1,n]}(\tilde{S}')| = |L_{[1,n]}(S')| = n + 1\), and so trivially \(N_{[1,n]}(\tilde{S}') \leq n + 1\). Since \(\tilde{S}'\) is not sofic, by Theorem 1.1, in fact \(N_{[1,n]}(\tilde{S}') = n + 1\) for all \(n\).

□

**References**


Nic Ormes, Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208
E-mail address: normes@du.edu
URL: www.math.du.edu/~ormes/

Ronnie Pavlov, Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208
E-mail address: rpavlov@du.edu
URL: www.math.du.edu/~rpavlov/