

2013

Compatibility for Probabilistic Theories

Stan Gudder

Follow this and additional works at: https://digitalcommons.du.edu/math_preprints



Part of the [Mathematics Commons](#)

Recommended Citation

Gudder, Stan, "Compatibility for Probabilistic Theories" (2013). *Mathematics Preprint Series*. 23.
https://digitalcommons.du.edu/math_preprints/23

This Article is brought to you for free and open access by the Department of Mathematics at Digital Commons @ DU. It has been accepted for inclusion in Mathematics Preprint Series by an authorized administrator of Digital Commons @ DU. For more information, please contact jennifer.cox@du.edu.

COMPATIBILITY FOR PROBABILISTIC THEORIES

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@du.edu

Abstract

We define an index of compatibility for a probabilistic theory (PT). Quantum mechanics with index 0 and classical probability theory with index 1 are at the two extremes. In this way, quantum mechanics is at least as incompatible as any PT. We consider a PT called a concrete quantum logic that may have compatibility index strictly between 0 and 1, but we have not been able to show this yet. Finally, we show that observables in a PT can be represented by positive, vector-valued measures.

1 Observables in Probabilistic Theories

This paper is based on the stimulating article [1] by Busch, Heinosaari and Schultz. The authors should be congratulated for introducing a useful new tool for measuring the compatibility of a probabilistic theory (PT). In this paper, we present a simpler, but coarser, measure of compatibility that we believe will also be useful.

A *probabilistic theory* is a σ -convex subset \mathcal{K} of a real Banach space \mathcal{V} . That is, if $0 \leq \lambda_i \leq 1$ with $\sum \lambda_i = 1$ and $v_i \in \mathcal{K}$, $i = 1, 2, \dots$, then $\sum \lambda_i v_i$ converges in norm to an element of \mathcal{K} . We call the elements of \mathcal{K} *states*. There is no loss of generality in assuming that \mathcal{K} generates \mathcal{V} in the sense that the closed linear hull of \mathcal{K} equals \mathcal{V} . Denote the collection of Borel

subsets of \mathbb{R}^n by $\mathcal{B}(\mathbb{R}^n)$ and the set of probability measures on $\mathcal{B}(\mathbb{R}^n)$ by $\mathcal{M}(\mathbb{R}^n)$. If \mathcal{K} is a PT, an n -dimensional observable on \mathcal{K} is a σ -affine map $M: \mathcal{K} \rightarrow \mathcal{M}(\mathbb{R}^n)$. We denote the set of n -dimensional observables by $\mathcal{O}_n(\mathcal{K})$ and write $\mathcal{O}(\mathcal{K}) = \mathcal{O}_1(\mathcal{K})$. We call the elements of $\mathcal{O}(\mathcal{K})$ *observables*. For $M \in \mathcal{O}(\mathcal{K})$, $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$, we interpret $M(s)(A)$ as the probability that M has a value in A when the system is in state s .

A set of observables $\{M_1, \dots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ is *compatible* or *jointly measurable* if there exists an $M \in \mathcal{O}_n(\mathcal{K})$ such that for every $A \in \mathcal{B}(\mathbb{R})$ and every $s \in \mathcal{K}$ we have

$$\begin{aligned} M(s)(A \times \mathbb{R} \times \dots \times \mathbb{R}) &= M_1(s)(A) \\ M(s)(\mathbb{R} \times A \times \mathbb{R} \times \dots \times \mathbb{R}) &= M_2(s)(A) \\ &\vdots \\ M(s)(\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times A) &= M_n(s)(A) \end{aligned}$$

In this case, we call M a *joint observable* for $\{M_1, \dots, M_n\}$ and we call $\{M_1, \dots, M_n\}$ the *marginals* for M . It is clear that if $\{M_1, \dots, M_n\}$ is compatible, then any proper subset is compatible. However, we suspect that the converse is not true. If a set of observables is not compatible we say it is *incompatible*.

It is clear that convex combinations of observables give an observable so $\mathcal{O}(\mathcal{K})$ forms a convex set. In the same way, $\mathcal{O}_n(\mathcal{K})$ is a convex set. Another way of forming new observables is by taking functions of an observable. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function and $M \in \mathcal{O}(\mathcal{K})$, the observable $f(M): \mathcal{K} \rightarrow \mathcal{M}(\mathbb{R})$ is defined by $f(M)(s)(A) = M(s)(f^{-1}(A))$ for all $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$.

Theorem 1.1. *If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are functions of a single observable M , then M_1, M_2 are compatible.*

Proof. Suppose $M_1 = f(M)$, $M_2 = g(M)$ where f and g are Borel functions. For $A, B \in \mathcal{B}(\mathbb{R})$, $s \in \mathcal{K}$ define $\widetilde{M}(s)$ on $A \times B$ by

$$\widetilde{M}(s)(A \times B) = M(s) [f^{-1}(A) \cap g^{-1}(B)]$$

By the Hahn extension theorem, $\widetilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\widetilde{M} \in \mathcal{O}_2(\mathcal{K})$ and the marginals of \widetilde{M} are $f(M)$ and $g(M)$. We conclude that $M_1 = f(M)$ and $M_2 = g(M)$ are compatible \square

It follows from Theorem 1.1 that an observable is compatible with any Borel function of itself and in particular with itself. In a similar way we obtain the next result.

Theorem 1.2. *If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are compatible and f, g are Borel functions, then $f(M_1)$ and $g(M_2)$ are compatible.*

Proof. Since M_1, M_2 are compatible, they have a joint observable $M \in \mathcal{O}_2(\mathcal{K})$. For $A, B \in \mathcal{B}(\mathbb{R})$, $s \in \mathcal{K}$ define $\widetilde{M}(s)$ on $A \times B$ by

$$\widetilde{M}(s)(A \times B) = M(s) [f^{-1}(A) \times g^{-1}(B)]$$

As in the proof of Theorem 1.1, $\widetilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\widetilde{M} \in \mathcal{O}(\mathcal{K})$ and the marginals of \widetilde{M} are

$$\widetilde{M}(s)(A \times \mathbb{R}) = M(s) [f^{-1}(A) \times \mathbb{R}] = M_1(s) [f^{-1}(A)] = f(M_1)(s)(A)$$

$$\widetilde{M}(s)(\mathbb{R} \times A) = M(s) [\mathbb{R} \times g^{-1}(A)] = M_2(s) [g^{-1}(A)] = g(M_2)(s)(A)$$

We conclude that $f(M_1)$ and $g(M_2)$ are compatible. \square

The next result is quite useful and somewhat surprising.

Theorem 1.3. *Let $M_i^j \in \mathcal{O}(\mathcal{K})$ for $i = 1, \dots, n$, $j = 1, \dots, m$ and suppose $\{M_i^1, \dots, M_i^m\}$ is compatible, $i = 1, \dots, n$. If $\lambda_i \in [0, 1]$ with $\sum \lambda_i = 1$, $i = 1, \dots, n$, then*

$$\left\{ \sum_{i=1}^n \lambda_i M_i^1, \sum_{i=1}^n \lambda_i M_i^2, \dots, \sum_{i=1}^n \lambda_i M_i^m \right\}$$

is compatible.

Proof. Let $\widetilde{M}_i \in \mathcal{O}_m(\mathcal{K})$ be the joint observable for $\{M_i^1, \dots, M_i^m\}$, $i = 1, \dots, n$. Then $\widetilde{M} = \sum_{i=1}^n \lambda_i \widetilde{M}_i$ is an m -dimensional observable with marginals

$$\begin{aligned}
\widetilde{M}(s)(A \times \mathbb{R} \times \cdots \times \mathbb{R}) &= \sum_{i=1}^n \lambda_i \widetilde{M}_i(s)(A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \sum_{i=1}^n \lambda_i M_i^1(s)(A) \\
\widetilde{M}(s)(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}) &= \sum_{i=1}^n \lambda_i \widetilde{M}_i(s)(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}) \\
&= \sum_{i=1}^n \lambda_i M_i^2(s)(A) \\
&\vdots \\
\widetilde{M}(s)(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) &= \sum_{i=1}^n \lambda_i \widetilde{M}_i(s)(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) \\
&= \sum_{i=1}^n \lambda_i M_i^m(s)(A)
\end{aligned}$$

The result now follows \square

Corollary 1.4. *Let $M, N, P \in \mathcal{O}(\mathcal{K})$ and $\lambda \in [0, 1]$. If M is compatible with N and P , then M is compatible with $\lambda N + (1 - \lambda)P$.*

Proof. Since $\{M, N\}$ and $\{M, P\}$ are compatible sets, by Theorem 1.3, we have that $M = \lambda M + (1 - \lambda)M$ is compatible with $\lambda N + (1 - \lambda)P$. \square

2 Noisy Observables

If $p \in \mathcal{M}(\mathbb{R})$, we define the *trivial observable* $T_p \in \mathcal{O}(\mathcal{K})$ by $T_p(s) = p$ for every $s \in \mathcal{K}$. A trivial observable represents noise in the system. We denote the set of trivial observables on \mathcal{K} by $\mathcal{T}(\mathcal{K})$. The set $\mathcal{T}(\mathcal{K})$ is convex with

$$\lambda T_p + (1 - \lambda)T_q = T_{\lambda p + (1 - \lambda)q}$$

for every $\lambda \in [0, 1]$ and $p, q \in \mathcal{M}(\mathbb{R})$. An observable $M \in \mathcal{O}(\mathcal{K})$ is compatible with any $T_p \in \mathcal{T}(\mathcal{K})$ and a joint observable $\widetilde{M} \in \mathcal{O}_2(\mathcal{K})$ is given by

$$\widetilde{M}(s)(A \times B) = p(A)M(s)(B)$$

If $M \in \mathcal{O}(\mathcal{K})$, $T \in \mathcal{T}(\mathcal{K})$ and $\lambda \in [0, 1]$ we consider $\lambda M + (1 - \lambda)T$ as the observable M together with noise. Stated differently, we consider $\lambda M +$

$(1 - \lambda)T$ to be a noisy version of M . The parameter $1 - \lambda$ gives a measure of the proportion of noise and is called the *noise index*. Smaller λ gives a larger proportion of noise. As we shall see, incompatible observables may have compatible noisy versions.

The next lemma follows directly from Corollary 1.4. It shows that if M is compatible with N , then M is compatible with any noisy version of N .

Lemma 2.1. *If $M \in \mathcal{O}(\mathcal{K})$ is compatible with $N \in \mathcal{O}(\mathcal{K})$, then M is compatible with $\lambda N + (1 - \lambda)T$ for any $\lambda \in [0, 1]$ and $T \in \mathcal{T}(\mathcal{K})$.*

The following lemma shows that for any $M, N \in \mathcal{O}(\mathcal{K})$ a noisy version of N with noise index λ is compatible with any noisy version of M with noise index $1 - \lambda$. The lemma also shows that if M is compatible with a noisy version of N , then M is compatible with a still noisier version of N .

Lemma 2.2. *Let $M, N \in \mathcal{O}(\mathcal{K})$ and $S, T \in \mathcal{T}(\mathcal{K})$. (a) If $\lambda \in [0, 1]$, then $\lambda M + (1 - \lambda)T$ and $(1 - \lambda)N + \lambda S$ are compatible. (b) If M is compatible with $\lambda N + (1 - \lambda)T$, then M is compatible with $\mu N + (1 - \mu)T$ where $0 \leq \mu \leq \lambda \leq 1$.*

Proof. (a) Since $\{M, S\}$ and $\{T, N\}$ are compatible sets, by Theorem 1.3 $\lambda M + (1 - \lambda)T$ is compatible with $\lambda S + (1 - \lambda)N$. (b) We can assume that $\lambda > 0$ and we let $\alpha = \mu/\lambda$ so $0 \leq \alpha \leq 1$. Since $\{M, \lambda N + (1 - \lambda)T\}$ and $\{M, T\}$ are compatible sets, by Theorem 1.3, $M = \alpha M + (1 - \alpha)M$ is compatible with

$$\begin{aligned} \alpha [\lambda N + (1 - \lambda)T] + (1 - \alpha)T &= \alpha \lambda N + [\alpha(1 - \lambda) + (1 - \alpha)]T \\ &= \mu N + (1 - \mu)T \end{aligned} \quad \square$$

The *compatibility region* $J(M_1, M_2, \dots, M_n)$ of observables $M_i \in \mathcal{O}(\mathcal{K})$, $i = 1, \dots, n$, is the set of points $(\lambda_1, \lambda_2, \dots, \lambda_n) \in [0, 1]^n$ for which there exist $T_i \in \mathcal{T}(\mathcal{K})$, $i = 1, 2, \dots, n$, such that

$$\{\lambda_i M_i + (1 - \lambda_i)T_i\}_{i=1}^n$$

form a compatible set. Thus, $J(M_1, M_2, \dots, M_n)$ gives parameters for which there exist compatible noisy versions of M_1, M_2, \dots, M_n . It is clear that $0 = (0, \dots, 0) \in J(M_1, M_2, \dots, M_n)$ and we shall show that $J(M_1, M_2, \dots, M_n)$ contains many points. We do not know whether $J(M_1, M_2, \dots, M_n)$ is symmetric under permutations of the M_i . For example, is $J(M_1, M_2) = J(M_2, M_1)$?

Theorem 2.3. $J(M_1, M_2, \dots, M_n)$ is a convex subset of $[0, 1]^n$.

Proof. Suppose $(\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n) \in J(M_1, \dots, M_n)$. We must show that

$$\begin{aligned} & \lambda(\lambda_1, \dots, \lambda_n) + (1 - \lambda)(\mu_1, \dots, \mu_n) \\ &= (\lambda\lambda_1 + (1 - \lambda)\mu_1, \dots, \lambda\lambda_n + (1 - \lambda)\mu_n) \in J(M_1, \dots, M_n) \end{aligned}$$

for all $\lambda \in [0, 1]$. Now there exist $S_1, \dots, S_n, T_1, \dots, T_n \in \mathcal{T}(\mathcal{K})$ such that $\{\lambda_i M_i + (1 - \lambda_i) S_i\}_{i=1}^n$ and $\{\mu_i M_i + (1 - \mu_i) T_i\}_{i=1}^n$ are compatible. By Theorem 1.3 the set of observables

$$\begin{aligned} & \{\lambda[\lambda_i M_i + (1 - \lambda_i) S_i] + (1 - \lambda)[\mu_i M_i + (1 - \mu_i) T_i]\} \\ &= \{(\lambda\lambda_i + (1 - \lambda)\mu_i) M_i + \lambda(1 - \lambda_i) S_i + (1 - \lambda)(1 - \mu_i) T_i\} \end{aligned}$$

is compatible. Since

$$\begin{aligned} \lambda(1 - \lambda_i) + (1 - \lambda)(1 - \mu_i) &= 1 - \lambda\lambda_i - \mu_i + \lambda\mu_i \\ &= 1 - [\lambda\lambda_i + (1 - \lambda)\mu_i] \end{aligned}$$

letting $\alpha_i = \lambda\lambda_i + (1 - \lambda)\mu_i$ we have that

$$U_i = \frac{1}{1 - \alpha_i} [\lambda(1 - \lambda_i) S_i + (1 - \lambda)(1 - \mu_i) T_i] \in \mathcal{T}(\mathcal{K})$$

Since $\{\alpha_i M_i + (1 - \alpha_i) U_i\}_{i=1}^n$ forms a compatible set, we conclude that $(\alpha_1, \dots, \alpha_n) \in J(M_1, \dots, M_n)$. \square

Let $\Delta_n = \{(\lambda_1, \dots, \lambda_n) \in [0, 1]^n : \sum \lambda_i \leq 1\}$. To show that Δ_n forms a convex subset of $[0, 1]^n \subseteq \mathbb{R}^n$, let $(\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n) \in \Delta_n$ and $\lambda \in [0, 1]$. Then $\lambda(\lambda_1, \dots, \lambda_n) + (1 - \lambda)(\mu_1, \dots, \mu_n) \in [0, 1]^n$ and

$$\sum_{i=1}^n [\lambda\lambda_i + (1 - \lambda)\mu_i] = \lambda \sum \lambda_i + (1 - \lambda) \sum \mu_i \leq \lambda + (1 - \lambda) = 1$$

Theorem 2.4. If $\{M_1, \dots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$, then $\Delta_n \subseteq J(M_1, \dots, M_n)$.

Proof. Let $\delta_0 = (0, 0, \dots, 0) \in \mathbb{R}^n$, $\delta_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, $i = 1, \dots, n$ where 1 is in the i th coordinate. It is clear that

$$\delta_i \in J(M_1, \dots, M_n) \cap \Delta_n, \quad i = 0, 1, \dots, n$$

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, letting $\mu = \sum \lambda_i$ we have that $0 \leq \mu \leq 1$, $\sum \lambda_i + (1 - \mu) = 1$ and

$$\lambda = \sum_{i=1}^n \lambda_i \delta_i + (1 - \mu) \delta_0$$

It follows that Δ_n is the convex hull of $\{\delta_0, \delta_1, \dots, \delta_n\}$. Since

$$\{\delta_0, \delta_1, \dots, \delta_n\} \subseteq J(M_1, \dots, M_n)$$

and $J(M_1, \dots, M_n)$ is convex, it follows that $\Delta_n \in J(M_1, \dots, M_n)$. \square

The *n-dimensional compatibility region* for PT \mathcal{K} is defined by

$$J_n(\mathcal{K}) = \cap \{J(M_1, \dots, M_n) : M_i \in \mathcal{O}(\mathcal{K}), i = 1, \dots, n\}$$

We have that $\Delta_n \subseteq J_n(\mathcal{K}) \subseteq [0, 1]^n$ and $J_n(\mathcal{K})$ is a convex set that gives a measure of the incompatibility of observables on \mathcal{K} . As $J_n(\mathcal{K})$ gets smaller, \mathcal{K} gets more incompatible and the maximal incompatibility is when $J_n(\mathcal{K}) = \Delta_n$. For the case of quantum states \mathcal{K} , the set $J_2(\mathcal{K})$ has been considered in detail in [1].

We now introduce a measure of compatibility that we believe is simpler and easier to investigate than $J_2(M, N)$. For $M, N \in \mathcal{O}(\mathcal{K})$, the *compatibility interval* $I(M, N)$ is the set of $\lambda \in [0, 1]$ for which there exists a $T \in \mathcal{T}(\mathcal{K})$ such that M is compatible with $\lambda N + (1 - \lambda)T$. Of course, $0 \in I(M, N)$ and M and N are compatible if and only if $1 \in I(M, N)$. We do not know whether $I(M, N) = I(N, M)$. It follows from Lemma 2.2(b) that if $\lambda \in I(M, N)$ and $0 \leq \mu \leq \lambda$, then $\mu \in I(M, N)$. Thus, $I(M, N)$ is an interval with left endpoint 0. The *index of compatibility* of M and N is $\lambda(M, N) = \sup \{\lambda : \lambda \in I(M, N)\}$. We do not know whether $\lambda(M, N) \in I(M, N)$ but in any case $I(M, N) = [0, \lambda(M, N)]$ or $I(M, N) = [0, \lambda(M, N))$. For a PT \mathcal{K} , we define the *interval of compatibility* for \mathcal{K} to be

$$I(\mathcal{K}) = \cap \{I(M, N) : M, N \in \mathcal{O}(\mathcal{K})\}$$

The *index of compatibility* of \mathcal{K} is

$$\lambda(\mathcal{K}) = \inf \{\lambda(M, N) : M, N \in \mathcal{O}(\mathcal{K})\}$$

and $I(\mathcal{K}) = [0, \lambda(\mathcal{K})]$ or $I(\mathcal{K}) = [0, \lambda(\mathcal{K}))$. Again, $\lambda(\mathcal{K}) = 0$ gives a measure of incompatibility of the observables in $\mathcal{O}(\mathcal{K})$.

Example 1. (Classical Probability Theory) Let (Ω, \mathcal{A}) be a measurable space and let \mathcal{V} be the Banach space of real-valued measures on \mathcal{A} with the total variation norm. If \mathcal{K} is the σ -convex set of probability measures on \mathcal{A} , then \mathcal{K} generates \mathcal{V} . There are two types of observables on \mathcal{K} , the *sharp* and *fuzzy* observables. The sharp observables have the form M_f where f is a measurable function $f: \Omega \rightarrow \mathbb{R}$ and $M_f(s)(A) = s[f^{-1}(A)]$. If M_f, M_g are sharp observables, form the unique 2-dimensional observable \widetilde{M} satisfying

$$\widetilde{M}(s)(A \times B) = s[f^{-1}(A) \cap g^{-1}(B)]$$

Then \widetilde{M} is a joint observable for M_f, M_g so M_f and M_g are compatible. The unsharp observables are obtained as follows. Let $\mathcal{F}(\Omega)$ be the set of measurable functions $f: \Omega \rightarrow [0, 1]$. Let $\widehat{M}: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\Omega)$ satisfy $\widehat{M}(\mathbb{R}) = 1$, $\widehat{M}(\cup A_i) = \sum \widehat{M}(A_i)$. An unsharp observable has the form

$$M(s)(A) = \int \widehat{M}(A) ds$$

Two unsharp observables M, N are also compatible because we can form the joint observable \widehat{M} given by

$$\widehat{M}(S)(A \times B) = \int \widehat{M}(A) \widehat{N}(B) ds$$

We conclude that $J(\mathcal{K}) = [0, 1] \times [0, 1]$ and $I(\mathcal{K}) = [0, 1]$ so \mathcal{K} has the maximal amount of compatibility.

Example 2. (Quantum Theory) Let H be a separable complex Hilbert space and let \mathcal{K} be the σ -convex set of all trace 1 positive operators on H . Then \mathcal{K} generates the Banach space of self-adjoint trace-class operators with the trace norm. It is well known that $M \in \mathcal{O}(\mathcal{K})$ if and only if there exists a positive operator-valued measure (POVM) P such that $M(s)(A) = \text{tr}[sP(A)]$ for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$. It is shown in [1] that if $\dim H = \infty$, then there exist $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ such that $J_2(M_1, M_2) = \Delta_2$ and hence $J(\mathcal{K}) = \Delta_2$. If $\dim H < \infty$, then $J(\mathcal{K})$ is not known, although partial results have been obtained and it is known that $J(\mathcal{K}) \rightarrow \Delta_2$ as $\dim H \rightarrow \infty$

Now let H be an arbitrary complex Hilbert space with $\dim H \geq 2$. Although the Pauli matrices σ_x, σ_y are 2-dimensional, we can extend them from a 2-dimensional subspace H_0 of H to all of H by defining $\sigma_x \psi = 0$ for

all $\psi \in H_0^\perp$. Define the POVMs M_x, M_y on H by $M_x(\pm 1) = \frac{1}{2}(I \pm \sigma_x)$, $M_y(\pm 1) = \frac{1}{2}(I \pm \sigma_y)$. It is shown in [1] that

$$J(M_x, M_y) = \{(\lambda, \mu) \in [0, 1] \times [0, 1] : \lambda^2 + \mu^2 \leq 1\}$$

Thus, $J(M_x, M_y)$ is a quadrant of the unit disk. We conclude that M_x is compatible with $\mu M_y + (1 - \mu)T$ for $T \in \mathcal{T}(\mathcal{K})$ if and only if $1 + \mu^2 \leq 1$. Therefore, $\mu = 0$, so $I(M_x, M_y) = \{0\}$ and $\lambda(M_x, M_y) = 0$. Thus, $I(\mathcal{K}) = \{0\}$ and $\lambda(\mathcal{K}) = 0$. We conclude that quantum mechanics has the smallest index of compatibility possible for a PT. The index of compatibility for a classical system is 1, so we have the two extremes. It would be interesting to find $\lambda(\mathcal{K})$ for other PTs.

3 Concrete Quantum Logics

We now consider a PT that seems to be between the classical and quantum PTs of Examples 1 and 2. A collection of subsets \mathcal{A} of a set Ω is a σ -class if $\emptyset \in \mathcal{A}$, $A^c \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and if A_i are mutually disjoint, $i = 1, 2, \dots$, then $\cup A_i \in \mathcal{A}$. If \mathcal{A} is a σ -class on Ω , we call (Ω, \mathcal{A}) a *concrete quantum logic*. A σ -state on \mathcal{A} is a map $s: \mathcal{A} \rightarrow [0, 1]$ such that $s(\Omega) = 1$ and if $A_i \in \mathcal{A}$ are mutually disjoint, then $s(\cup A_i) = \sum s(A_i)$. If \mathcal{K} is the set of σ -states on (Ω, \mathcal{A}) , we call \mathcal{K} a *concrete quantum logic* PT. Let \mathcal{A}_σ be the σ -algebra generated by \mathcal{A} . A σ -state s is *classical* if there exists a probability measure μ on \mathcal{A}_σ such that $s = \mu \upharpoonright \mathcal{A}$. As in the classical case, an observable is *sharp* if it has the form $M_f(s)(A) = s[f^{-1}(A)]$ for an \mathcal{A} -measurable function $f: \Omega \rightarrow \mathbb{R}$. If f and g are \mathcal{A} -measurable functions satisfying $f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A}$ for all $A, B \in \mathcal{B}(\mathbb{R})$, then M_f and M_g are compatible because they have a joint observable M satisfying $M(s)(A \times B) = s[f^{-1}(A) \cap g^{-1}(B)]$ for all $s \in \mathcal{K}$, $A, B \in \mathcal{B}(\mathbb{R})$. We do not know whether M_f and M_g compatible implies that $f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A}$ holds for every $A, B \in \mathcal{B}(\mathbb{R})$, although we suspect it does not.

Example 3. This is a simple example of a concrete quantum logic. Let $\Omega = \{1, 2, 3, 4\}$ and let \mathcal{A} be the collection of subsets of Ω with even cardinality. Then

$$\mathcal{A} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}$$

Let \mathcal{K} be the sets of all states on \mathcal{A} . Letting $a = \{1, 2\}$, $a' = \{3, 4\}$, $b = \{1, 3\}$, $b' = \{3, 4\}$, $c = \{1, 4\}$, $c' = \{2, 3\}$ we can represent an $s \in \mathcal{K}$ by

$$\begin{aligned}\widehat{s} &= (s(a), s(a'), s(b), s(b'), s(c), s(c')) \\ &= (s(a), 1 - s(a), s(b), 1 - s(b), s(c), 1 - s(c))\end{aligned}$$

Thus, every $s \in \mathcal{K}$ has the form

$$s = (\lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2, \lambda_3, 1 - \lambda_3)$$

for $0 \leq \lambda_i \leq 1$, $i = 1, 2, 3$. The pure (extremal) classical states are the 0-1 states: $\delta_1 = (1, 0, 1, 0, 1, 0)$, $\delta_2 = (1, 0, 0, 1, 0, 1)$, $\delta_3 = (0, 1, 1, 0, 0, 1)$, $\delta_4 = (0, 1, 0, 1, 1, 0)$. The pure nonclassical states are the 0-1 states: $\gamma_1 = 1 - \delta_1$, $\gamma_2 = 1 - \delta_2$, $\gamma_3 = 1 - \delta_3$, $\gamma_4 = 1 - \delta_4$ where $1 = (1, 1, 1, 1, 1, 1)$. For example, to see that γ_1 is not classical, we have that $\gamma_1 = (0, 1, 0, 1, 0, 1)$. Hence, $\gamma_1(\{3, 4\}) = \gamma_1(\{2, 4\}) = \gamma_1(\{2, 3\}) = 1$. If there exists a probability measure μ such that $\gamma_1 = \mu \upharpoonright \mathcal{A}$ we would have $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 0$ which is a contradiction. The collection of sharp observable is very limited because a measurable function $f: \Omega \rightarrow \mathbb{R}$ can have at most two values. Thus, if M_f is a sharp observable there exists $a, b \in \mathbb{R}$ such that $M_f(s)(\{a, b\}) = 1$ for every $s \in \mathcal{K}$. There are many observables with more than two values (non-binary observables) and these are not sharp. Even for this simple example, it appears to be challenging to investigate the region and interval of compatibility.

4 Vector-Valued Measures

Let \mathcal{K} be a PT with generated Banach space \mathcal{V} and \mathcal{V}^* be the Banach space dual of \mathcal{V} . A *normalized vector-valued measure* (NVM) for \mathcal{K} is a map $\Gamma: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R})$ for every $s \in \mathcal{K}$. Thus, Γ satisfies the conditions:

- (1) $\Gamma(\mathbb{R})(s) = 1$ for every $s \in \mathcal{K}$,
- (2) $0 \leq \Gamma(A)(s) \leq 1$ for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$,
- (3) If $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, $i = 1, 2, \dots$, then

$$\Gamma(\cup A_i)(s) = \sum \Gamma(A_i)(s)$$

for every $s \in \mathcal{K}$.

This section shows that there is a close connection between observables on \mathcal{K} and NVMs for \mathcal{K} .

Theorem 4.1. *If Γ is a NVM for \mathcal{K} , then $M: \mathcal{K} \rightarrow \mathcal{M}(\mathbb{R})$ given by $M(s)(A) = \Gamma(A)(s)$, $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$, is an observable on \mathcal{K} .*

Proof. Since $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R})$ we have that $A \mapsto M(s)(A) \in \mathcal{M}(\mathbb{R})$. Let $\lambda_i \in [0, 1]$ with $\sum \lambda_i = 1$, $s_i \in \mathcal{K}$, $i = 1, 2, \dots$, and suppose that $s = \sum \lambda_i s_i$. Then $\lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i s_i = s$ in norm and since $s \mapsto \Gamma(A)(s) \in \mathcal{V}^*$, for every $A \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned} M(s)(A) &= M\left(\sum \lambda_i s_i\right)(A) = \Gamma(A)\left(\sum \lambda_i s_i\right) = \Gamma(A)\left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i s_i\right) \\ &= \lim_{n \rightarrow \infty} \Gamma(A)\left(\sum_{i=1}^n \lambda_i s_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i \Gamma(A)(s_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i M(s_i)(A) = \sum_{i=1}^{\infty} \lambda_i M(s_i)(A) \end{aligned}$$

It follows that $M(\sum \lambda_i s_i) = \sum \lambda_i M(s_i)$ so $M \in \mathcal{O}(\mathcal{K})$. \square

The converse of Theorem 4.1 holds if some mild conditions are satisfied. To avoid some topological and measure-theoretic technicalities, we consider the special case where \mathcal{V} is finite-dimensional. Assuming that \mathcal{K} is the base of a generating positive cone \mathcal{V}^+ , we have that every element $v \in \mathcal{V}^+$ has a unique form $v = \alpha s$, $\alpha \geq 0$, $s \in \mathcal{K}$ and that $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ where $\mathcal{V}^- = -\mathcal{V}^+$ and $\mathcal{V}^+ \cap \mathcal{V}^- = \{0\}$. If $M \in \mathcal{O}(\mathcal{K})$, then for every $A \in \mathcal{B}(\mathbb{R})$, $s \mapsto M(s)(A)$ is a convex, real-valued function on \mathcal{K} . A standard argument shows that this function has a unique linear extension $\widehat{M}(A) = \mathcal{V}^*$ for every $A \in \mathcal{B}(\mathbb{R})$. Hence

$$\widehat{M}(A)(s) = M(s)(A) \tag{4.1}$$

for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$. Since $A \mapsto \widehat{M}(A)(s) = M(s)(A) \in \mathcal{M}(\mathbb{R})$ we conclude that $A \mapsto \widehat{M}(A)$ is a NVM and \widehat{M} is the unique NVM satisfying (4.1). It follows that the converse of Theorem 4.1 holds in this case.

Example 1'. (Classical Probability Theory) In this example \mathcal{V}^* is the Banach space of bounded measurable functions $f: \Omega \rightarrow \mathbb{R}$ with norm $\|f\| =$

$\sup |f(\omega)| < \infty$ and duality given by

$$\langle \mu, f \rangle = f(\mu) = \int f d\mu$$

The function $1(\omega) = 1$ for every $\omega \in \Omega$ is the natural unit satisfying $1(\mu) = 1$ for every $\mu \in \mathcal{K}$. In this case, \mathcal{K} is a base for the generating positive cone \mathcal{V}^+ of bounded measures and the converse of Theorem 4.1 holds. Then a NVM Γ has the form $0 \leq \Gamma(A)(\omega) \leq 1$ for every $A \in \mathcal{B}(\mathbb{R})$, $\omega \in \Omega$ and $\Gamma(\mathbb{R}) = 1$. Thus $\Gamma(A) \in \mathcal{F}(\Omega)$ and if M is the corresponding observable, then

$$M(\mu)(A) = \Gamma(A)(\mu) = \int \Gamma(A) d\mu$$

In particular, if $T_p \in \mathcal{T}(\mathcal{K})$ then the corresponding NVM Γ_p has the form

$$\Gamma_p(A)(\mu) = T_p(\mu)(A) = p(A)$$

so $\Gamma_p(A)$ is the constant function $p(A)$. Moreover, if $M_p \in \mathcal{O}(\mathcal{K})$ is sharp, then the corresponding NVM Γ_f satisfies

$$\int \Gamma_f(A) d\mu = \Gamma_f(A)(\mu) = M_f(\mu)(A) = \mu[f^{-1}(A)] = \int \chi_{f^{-1}(A)} d\mu$$

Hence, $\Gamma_f(A) = \chi_{f^{-1}(A)}$ for every $A \in \mathcal{B}(\mathbb{R})$.

Example 2'. (Quantum Theory) In this example \mathcal{V}^* is the Banach space $\mathcal{B}(H)$ of bounded linear operators on H with norm

$$\|L\| = \sup \{\|L\psi\| : \|\psi\| = 1\}$$

and duality given by

$$\langle s, L \rangle = L(a) = \text{tr}(sL)$$

The identity operator I is the natural unit satisfying $I(s) = 1$ for all $s \in \mathcal{K}$. In this case, \mathcal{K} is a base for the generating cone \mathcal{V}^+ of positive trace class operators and the converse of Theorem 4.1 holds. If Γ is a NVM, then $\Gamma(A)$ is a positive operator satisfying $0 \leq \Gamma(A) \leq I$ called an *effect* and $\Gamma(\mathbb{R}) = I$. According to the converse of Theorem 4.1, if M is an observable, then there exists a POVM Γ such that

$$M(s)(A) = \text{tr}[s\Gamma(A)]$$

for every $s \in \mathcal{K}$ and $A \in \mathcal{B}(\mathbb{R})$. In particular, if $T_p \in \mathcal{T}(\mathcal{K})$, then the corresponding NVM Γ_p has the form

$$\text{tr}[s\Gamma_p(A)] = \Gamma_p(A)(s) = T_p(s)(A) = p(A) = \text{tr}[sp(A)I]$$

so $\Gamma_p(A) = p(A)I$ for all $A \in \mathcal{B}(\mathbb{R})$.

Similar to a NVM, we define an n -dimensional NVM to be a map $\Gamma: \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R}^b)$ for every $s \in \mathcal{K}$. Moreover, a set $\{\Gamma_1, \dots, \Gamma_n\}$ of NVMs for \mathcal{K} is compatible if there exists an n -dimensional NVM Γ such that

$$\begin{aligned} \Gamma(A \times \mathbb{R} \times \dots \times \mathbb{R}) &= \Gamma_1(A) \\ &\vdots \\ \Gamma(\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times A) &= \Gamma_n(A) \end{aligned}$$

for every $A \in \mathcal{B}(\mathbb{R})$. The proof of the following theorem is straightforward.

Theorem 4.2. *If $\{M_1, \dots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ and $\{\Gamma_1, \dots, \Gamma_n\}$ are the corresponding NVM for \mathcal{K} , then $\{M_1, \dots, M_n\}$ are compatible if and only if $\{\Gamma_1, \dots, \Gamma_n\}$ are compatible.*

References

- [1] P. Busch, T. Heinosaari and J. Schultz, Quantum theory contains maximally incompatible observables, arXiv: 1210.4142 v1 [quant-ph], Oct. 15, 2012.