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An Einstein Equation for Discrete Quantum Gravity

S. Gudder

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S. Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208, U.S.A.
sgudder@du.edu

Abstract

The basic framework for this article is the causal set approach to discrete quantum gravity (DQG). Let Q_n be the collection of causal sets with cardinality not greater than n and let K_n be the standard Hilbert space of complex-valued functions on Q_n. The formalism of DQG presents us with a decoherence matrix $D_n(x,y)$, $x,y \in Q_n$. There is a growth order in Q_n and a path in Q_n is a maximal chain relative to this order. We denote the set of paths in Q_n by Ω_n. For $\omega, \omega' \in \Omega_n$ we define a bidifference operator $\bigtriangledown^n_{\omega,\omega'}$ on $K_n \otimes K_n$ that is covariant in the sense that $\bigtriangledown^n_{\omega,\omega'}$ leaves D_n stationary. We then define the curvature operator $\mathcal{R}^n_{\omega,\omega'} = \bigtriangledown^n_{\omega,\omega'} - \bigtriangledown^n_{\omega',\omega}$. It turns out that $\mathcal{R}^n_{\omega,\omega'}$ naturally decomposes into two parts $\mathcal{R}^n_{\omega,\omega'} = \mathcal{D}^n_{\omega,\omega'} + \mathcal{T}^n_{\omega,\omega'}$ where $\mathcal{D}^n_{\omega,\omega'}$ is closely associated with D_n and is called the metric operator while $\mathcal{T}^n_{\omega,\omega'}$ is called the mass-energy operator. This decomposition is a discrete analogue of Einstein's equation of general relativity. Our analogue may be useful in determining whether general relativity theory is a close approximation to DQG.

1 Causet Approach to DQG

A *causal set* (*causet*) is a finite partially ordered set x. Thus, x is endowed with an irreflexive, transitive relation $<$ [1, 8, 10]. That is, $a \not< a$ for all $a \in x$

and $a < b$, $b < c$ imply that $a < c$ for $a, b, c \in x$. The relation $a < b$ indicates that b is in the causal future of a. Let $\mathcal{P}_n$ be the collection of all causets with cardinality n, $n = 1, 2, \ldots$, and let $\mathcal{P} = \cup \mathcal{P}_n$. For $x \in \mathcal{P}$, an element $a \in x$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x *produces* y if y is obtained from x by adjoining a single new element a to x that is maximal in y. In this way, there is no element of y in the causal future of a. If x produces y, we say that y is an *offspring* of x and write $x \to y$.

A *path* in $\mathcal{P}$ is a string (sequence) $x_1 x_2 \cdots$ where $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$, $i = 1, 2, \ldots$. An *n-path* in $\mathcal{P}$ is a finite string $x_1 x_2 \cdots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \to x_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n. We think of $\omega \in \Omega$ as a possible universe (or universe history). The set of paths whose initial n-path is $\omega_0 \in \Omega_n$ is called an *elementary cylinder set* and is denoted by $\mathrm{cyl}(\omega_0)$. Thus, if $\omega_0 = x_1 x_2 \cdots x_n$, then

$$\mathrm{cyl}(\omega_0) = \{\omega \in \Omega \colon \omega = x_1 x_2 \cdots x_n y_{n+1} y_{n+2} \cdots\}$$

The *cylinder set generated* by $A \subseteq \Omega_n$ is defined by

$$\mathrm{cyl}(A) = \bigcup_{\omega \in A} \mathrm{cyl}(\omega)$$

The collection $\mathcal{A}_n = \{\mathrm{cyl}(A) \colon A \subseteq \Omega_n\}$ forms an increasing sequence of algebras $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ on Ω and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of Ω. We denote the σ-algebra generated by $\mathcal{C}(\Omega)$ as $\mathcal{A}$.

It is shown [6, 11] that a classical sequential growth process (CSGP) on $\mathcal{P}$ that satisfies natural causality and covariance conditions is determined by a sequence of nonnegative numbers $c = (c_0, c_1, \ldots)$ called *coupling constants*. The coupling constants specify a unique probability measure ν_c on $\mathcal{A}$ making $(\Omega, \mathcal{A}, \nu_c)$ a probability space. The *path Hilbert space* is given by $H = L_2(\Omega, \mathcal{A}, \nu_c)$. If $\nu_c^n = \nu_c \mid \mathcal{A}_n$ is the restriction of ν_c to $\mathcal{A}_n$, then $H_n = L_2(\Omega, \mathcal{A}_n, \nu_c^n)$ is an increasing sequence of closed subspaces of H.

A bounded operator T on H_n will also be considered as a bounded operator on H by defining $Tf = 0$ for all $f \in H_n^\perp$. We denote the characteristic function of a set $A \in \mathcal{A}$ by χ_A and use the notation $\chi_\Omega = 1$. A *q-probability operator* is a bounded positive operator ρ_n on H_n that satisfies $\langle \rho_n 1, 1 \rangle = 1$. Denote the set of q-probability operators on H_n by $\mathcal{Q}(H_n)$. For $\rho_n \in \mathcal{Q}(H_n)$ we define the *n-decoherence functional* [3, 5, 8] $D_n \colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

The functional $D_n(A,B)$ gives a measure of the interference between the events A and B when the system is described by ρ_n. It is clear that $D_n(\Omega_n,\Omega_n)=1$, $D_n(A,B)=\overline{D_n(B,A)}$ and $A\mapsto D_n(A,B)$ is a complex measure for all $B\in\mathcal{A}_n$. It is also well known that if $A_1,\ldots,A_n\in\mathcal{A}_n$, then the matrix with entries $D_n(A_j,A_k)$ is positive semidefinite. In particular the positive semidefinite matrix with entries

$$D_n(\omega_i,\omega_j)=D_n\left(\mathrm{cyl}(\omega_i),\mathrm{cyl}(\omega_j)\right),\quad \omega_i,\omega_j\in\Omega_n$$

is called the n-*decoherence matrix*.

We define the map $\mu_n\colon\mathcal{A}_n\to\mathbb{R}^+$ given by

$$\mu_n(A)=D_n(A,A)=\langle\rho_n\chi_A,\chi_A\rangle$$

Notice that $\mu_n(\Omega)=1$. Although μ_n is not additive in general, it satisfies the *grade*-2 *additivity condition* [2, 3, 5, 7, 9]: if $A,B,C\in\mathcal{A}_n$ are mutually disjoint then

$$\mu_n(A\cup B\cup C)=\mu_n(A\cup B)+\mu_n(A\cup C)+\mu_n(B\cup C)-\mu_n(A)-\mu_n(B)-\mu_n(C)$$

We call μ_n the *q-measure* corresponding to ρ_n and interpret $\mu_n(A)$ as the quantum propensity for the occurrence of the event $A\in\mathcal{A}_n$. A simple example is to let $\rho_n=I$, $n=1,2,\ldots$. Then

$$D_n(A,B)=\langle\chi_B,\chi_A\rangle=\nu_c^n(A\cap B)$$

and $\mu_n(A)=\nu_c^n(A)$, the classical measure. A more interesting example is to let $\rho_n=|1\rangle\langle 1|$, $n=1,2,\ldots$. Then

$$D_n(A,B)=\langle|1\rangle\langle 1|\chi_B,\chi_A\rangle=\nu_c^n(A)\nu_c^n(B)$$

and $\mu_n(A)=\left[\nu_c^n(A)\right]^2$, the classical measure square.

We call a sequence $\rho_n\in\mathcal{Q}(H_n)$, $n=1,2,\ldots$, *consistent* if

$$D_{n+1}(A,B)=D_n(A,B)$$

for all $A,B\in\mathcal{A}_n$. A *quantum sequential growth process* (QSGP) is a consistent sequence $\rho_n\in\mathcal{Q}(H_n)$ [3, 4] . We consider a QSGP as a model for discrete quantum gravity. It is hoped that additional theoretical principles or experimental data will help determine the coupling constants and hence ν_c, which is the classical part of the process, and also the $\rho_n\in\mathcal{Q}(H_n)$, which is the quantum part. Moreover, it is believed that general relativity will eventually be shown to be a close approximation to this discrete model. Until now it has not been clear how this can be accomplished. However, in Section 3 we shall derive a discrete Einstein equation which might be useful in performing these tasks.

2 Difference Operators

Let $Q_n=\bigcup\limits_{j=1}^n\mathcal{P}_j$ be the collection of causets with cardinality not greater than n and let K_n be the (finite-dimensional) Hilbert space $\mathbb{C}^{Q_n}$ with the standard inner product

$$\langle f,g\rangle=\sum_{x\in Q_n}\overline{f(x)}g(x)$$

Let $L_n=K_n\otimes K_n$ which we identify with $\mathbb{C}^{Q_n\times Q_n}$ having the standard inner product. We shall also have need to consider the Hilbert space

$$K=\left\{f\in\mathbb{C}^\mathcal{P}\colon\sum_{x\in\mathcal{P}}\left|f(x)\right|^2<\infty\right\}$$

with the standard inner product and we define $L=K\otimes K$. Notice that $K_1\subseteq K_2\subseteq\cdots K$ form an increasing sequence of subspaces of K that generate K in the natural way.

Let $\rho_n\in\mathcal{Q}(H_n)$ be a QSGP with corresponding decoherence matrix $D_n(\omega,\omega')$, $\omega,\omega'\in\Omega_n$. If $\omega=\omega_1\omega_2\cdots\omega_n\in\Omega_n$ and $\omega_j=x$ for some j, we say that ω *goes through* x. For $x,y\in Q_n$ we define

$$D_n(x,y)=\sum\left\{D_n(\omega,\omega')\colon\omega\text{ goes through }x,\ \omega'\text{ goes through }y\right\}$$

Due to the consistency of ρ_n, $D_n(x,y)$ is independent of n. That is, $D_n(x,y)=D_m(x,y)$ if $x,y\in Q_n\cap Q_m$. Moreover, $D_n(x,y)$ are the components of a positive semidefinite matrix. We view Q_n as the analogue of a differentiable manifold and $D_n(x,y)$ as the analogue of a metric tensor. One might think that the elements of causets should be analogous to points of a differential manifold and not the causets themselves. However, if $x\in Q_n$, then x is intimately related to its producers, each of which determines a unique $a\in x$.

Moreover, if $y \to x$ we view (y, x) as a tangent vector at x. In this way, there are as many tangent vectors at x as there are producers of x. Finally, the elements of Ω_n are analogues of curves and the elements of K_n are analogues of smooth functions on a manifold.

For $x \in Q_n$, $|x|$ denotes the cardinality of x. Notice if $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$ and $\omega_j = x$, then $j = |x|$ and ω goes through x if and only if $\omega_{|x|} = x$. We see that a path ω through x determines a tangent vector $(\omega_{|x|-1}, \omega_{|x|})$ at x (assuming that $|x| \geq 2$). For $\omega \in \Omega_n$ we define the *difference operator* $\triangle_\omega^n$ on K_n by

$$\triangle_\omega^n f(x) = \left[f(x) - f(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}}$$

for all $f \in K_n$, where $\delta_{x,\omega_{|x|}}$ is the Kronecker delta. Thus, $\triangle_\omega^n f(x)$ gives the change of f along the tangent vector $(\omega_{|x|-1}, \omega_{|x|})$ if ω goes through x. It is clear that $\triangle_\omega^n$ is a linear operator on K_n. We now show that $\triangle_\omega^n$ satisfies a discrete form of Leibnitz's rule. For $f, g \in K_n$ we have

$$\begin{aligned}\triangle_\omega^n fg(x) &= \left[f(x)g(x) - f(\omega_{|x|-1})g(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} \\ &= \left\{\left[f(x)g(x) - f(x)g(\omega_{|x|-1})\right] + \left[f(x)g(\omega_{|x|-1}) - f(\omega_{|x|-1})g(\omega_{|x|-1})\right]\right\} \\ &\qquad \cdot \delta_{x,\omega_{|x|}} \\ &= f(x) \triangle_\omega^n g(x) + g(\omega_{|x|-1}) \triangle_\omega^n f(x) \qquad (2.1)\end{aligned}$$

Of course, it also follows that

$$\triangle_\omega^n fg(x) = \triangle_\omega^n gf(x) = g(x) \triangle_\omega^n f(x) + f(\omega_{|x|-1}) \triangle_\omega^n g(x) \qquad (2.2)$$

Given a function of two variables $f \in \mathbb{C}^{Q_n \times Q_n} = L_n$ we have a function $\widetilde{f} \in K_n$ of one variable where $\widetilde{f}(x) = f(x, x)$ and given a function $f \in K_n$ we have the functions of two variables $f_1, f_2 \in L_n$ where $f_1(x, y) = f(x)$ and $f_2(x, y) = f(y)$ for all $x, y \in Q_n$. For $\omega, \omega' \in \Omega_n$ we want an operator $\triangle_{\omega,\omega'}^n \colon L_n \to L_n$ that extends $\triangle_\omega^n$ and satisfies a discrete Leibnitz's rule. That is,

$$\triangle_{\omega,\omega'}^n f_1(x, y) = \triangle_\omega^n f(x)\delta_{y,\omega'_{|x|}}, \triangle_{\omega,\omega'}^n f_2(x, y) = \triangle_{\omega'}^n f(y)\delta_{x,\omega_{|x|}} \qquad (2.3)$$

and

$$\triangle_{\omega,\omega'}^n fg(x, y) = f(x, y) \triangle_{\omega,\omega'} g(x, y) + g(\omega_{|x|-1}, \omega'_{|x|-1}) \triangle_{\omega,\omega'}^n f(x, y) \quad (2.4)$$

Theorem 2.1. *A linear operator* $\triangle^n_{\omega,\omega'} \colon L_n \to L_n$ *satisfies* (2.3) *and* (2.4) *if and only if* $\triangle^n_{\omega,\omega'}$ *has the form*

$$\triangle^n_{\omega,\omega'} f(x,y) = \left[f(x,y) - f(\omega_{|x|-1}, \omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \tag{2.5}$$

Proof. If $\triangle^n_{\omega,\omega'}$ is defined by (2.5), then for $f \in K_n$ we have

$$\begin{aligned}
\triangle^n_{\omega,\omega'} f_1(x,y) &= \left[f_1(x,y) - f_1(\omega_{|x|-1}, \omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \\
&= \left[f(x) - f(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \\
&= \triangle^n_{\omega} f(x) \delta_{y,\omega'_{|y|}}
\end{aligned}$$

In a similar way, $\triangle^n_{\omega,\omega'}$ satisfies the second equation in (2.3). Moreover, we have

$$\begin{aligned}
\triangle^n_{\omega,\omega'} f(x,y) &= \left[f(x,y)g(x,y) - f(\omega_{|x|-1}\omega'_{|y|-1}) g(\omega_{|x|-1}\omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \\
&= \left[f(x,y)g(x,y) - f(x,y) g(\omega_{|x|-1}, \omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \\
&\quad + \left[f(x,y) g(\omega_{|x|-1}, \omega'_{|y|-1}) - f(\omega_{|x|-1}\omega'_{|y|-1}) g(\omega_{|x|-1}\omega'_{|y|-1})\right] \\
&\qquad \cdot \delta_{x,\omega_{|x|}} \delta_{y,\omega'|y|} \\
&= f(x,y) \triangle^n_{\omega,\omega'} g(x,y) + g(\omega_{|x|-1}, \omega'_{|y|-1}) \triangle^n_{\omega,\omega'} f(x,y)
\end{aligned}$$

Conversely, suppose the linear operator $\triangle^n_{\omega,\omega'} \colon L_n \to L_n$ satisfies (2.3) and (2.4). If $f \in L_n$ has the form $f(x,y) = g(x)h(y)$, then

$$\begin{aligned}
\triangle^n_{\omega,\omega'} f(x,y) &= \triangle^n_{\omega,\omega'} gh(x,y) = \triangle^n_{\omega,\omega'} g_1 h_2(x,y) \\
&= g_1(x,y) \triangle^n_{\omega,\omega'} h_2(x,y) + h_2(\omega_{|x|-1}\omega'_{|y|-1}) \triangle^n_{\omega,\omega'} g_1(x,y) \\
&= g(x) \triangle^n_{\omega'} h(y) \delta_{x,\omega_{|x|}} + h(\omega'_{|y|-1}) \triangle^n_{\omega} g(x) \delta_{y,\omega'_{|y|}} \\
&= \left\{g(x) \left[h(y) - h(\omega'_{|y|-1})\right] + h(\omega'_{|y|-1}) \left[g(x) - g(\omega_{|y|-1})\right]\right\} \\
&\qquad \cdot \delta_{x,\omega_{|x|}} \delta_{y,\omega'|y|} \\
&= \left[g(x)h(y) - g(\omega_{|x|-1}) h(\omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \\
&= \left[f(x,y) - f(\omega_{|x|-1}\omega'_{|y|-1})\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}}
\end{aligned}$$

Since $\triangle^n_{\omega,\omega'}$, is linear and every element of L_n is a linear combination of product functions, the result follows. □

Of course, Theorem 2.1 is not surprising because (2.5) is the natural extension of $\triangle_\omega^n$ to L_n. Also $\triangle_{\omega,\omega'}^n$ extends $\triangle_\omega^n$ in the sense that for any $f \in L_n$ we have

$$\begin{aligned}\triangle_{\omega,\omega}^n f(x,x) &= \left[f(x,y) - f(\omega_{|x|-1},\omega_{|x|-1})\right]\delta_{x,\omega_{|x|}}\\ &= \left[\widetilde{f}(x) - \widetilde{f}(\omega_{|x|-1})\right]\delta_{x,\omega_{|x|}} = \triangle_\omega^n \widetilde{f}(x)\end{aligned}$$

As before, $\triangle_{\omega,\omega'}^n$ satisfies

$$\begin{aligned}\triangle_{\omega,\omega'}^n fg(x,y) &= \triangle_{\omega,\omega'}^n gf(x,y)\\ &= g(x,y)\,\triangle_{\omega,\omega'}^n\,f(x,y) + f(\omega_{|x|-1},\omega'_{|y|-1})\,\triangle_{\omega,\omega'}^n\,g(x,y)\end{aligned}$$

The next result characterizes $\triangle_\omega^n$ and $\triangle_{\omega,\omega'}^n$ up to a multiplicative constant.

Theorem 2.2. (a) *An operator* $T_\omega \colon K_n \to K_n$ *satisfies* (2.1) *and* $T_\omega f(x) = 0$ *if* $\omega_{|x|} \ne x$ *if and only if there exists a function* $\beta_\omega \in K_n$ *such that* $T_\omega = \beta_\omega \triangle_\omega^n$.
(b) *An operator* $T_{\omega,\omega'} \colon L_n \to L_n$ *satisfies* (2.4) *and* $T_{\omega,\omega'} f(x,y) = 0$ *if* $\omega_{|x|} \ne x$ *or* $\omega'_{|y|} \ne y$ *if and only if there exists a function* $\beta_{\omega,\omega'} \in L_n$ *such that* $T_{\omega,\omega'} = \beta_{\omega,\omega'}\triangle_{\omega,\omega'}^n$.

Proof. If T_ω satisfies (2.1), it follows from (2.2) that

$$f(x)T_\omega g(x) + g(\omega_{|x|-1})T_\omega f(x) = g(x)T_\omega f(\omega) + f(\omega_{|x|-1})T_\omega g(x)$$

Hence,

$$\left[g(x) - g(\omega_{|x|-1})\right]T_\omega f(x) = T_\omega g(x)\left[f(x) - f(\omega_{|x|-1})\right]$$

Therefore, if $g(x) - g(\omega_{|x|-1}) \ne 0$, we have

$$T_\omega f(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})}\left[f(x) - f(\omega_{|x|-1})\right]$$

Letting

$$\beta_\omega(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})}$$

gives the result. The converse is straightforward. The proof of (b) is similar. □

It is clear that $\mu_n(x) = D_n(x,x)$ is not stationary; that is $\triangle_\omega^n \mu_n(x) \neq 0$ for all $x \in Q_n$ in general. Is there a function $\alpha_\omega \in K_n$ such that $(\triangle_\omega^n + \alpha_\omega)\mu_n(x) = 0$ for all $x \in Q_n$? If α_ω exists, we obtain

$$\left[\mu_n(x) - \mu_n(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} + \alpha_\omega(x)\mu_n(x) = 0$$

If $\omega_{|x|} = x$ and $\mu_n(x) = 0$, this would imply that $\mu_n(\omega_{|x|-1}) = 0$. Continuing this process would give

$$\mu_n(\omega_{|x|-2}) = \mu(\omega_{|x|-3}) = \cdots = 0$$

which leads to a contradiction. It is entirely possible for $\mu_n(x)$ to be zero for some $x \in Q_n$ so we abandon this attempt. How about functions $\alpha_\omega, \beta_\omega \in K_n$ such that $(\beta_\omega \triangle_\omega^n + \alpha_\omega)\mu_n(x) = 0$ for all $x \in Q_n$? We then obtain

$$\beta_\omega(x) \left[\mu_n(x) - \mu_n(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} + \alpha_\omega(x)\mu_n(x) = 0 \tag{2.6}$$

If $\omega_{|x|} = x$ and $\mu_n(x) = 0$ but $\beta_\omega(x) \neq 0$ we obtain the same contradiction as before. We conclude that $\beta_\omega(x) = 0$ whenever $\mu_n(x) = 0$. The simplest choice of such a β_ω is $\beta_\omega(x) = \mu_n(x)$. This choice also has the advantage of being independent of ω. Equation (2.6) becomes

$$\mu_n(x) \left[\mu_n(x) - \mu_n(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} + \alpha_\omega(x)\mu_n(x) = 0 \tag{2.7}$$

If $\mu_n(x) = 0$, then (2.7) holds. If $\mu_n(x) \neq 0$, then we obtain

$$\alpha_\omega(x) = \left[\mu_n(\omega_{|x|-1}) - \mu_n(x)\right] \delta_{x,\omega_{|x|}} = -\triangle_\omega^n \mu_n(x)$$

The numbers $\alpha_\omega(x)$ are an analogue of the Christoffel symbols. We call $\nabla_\omega^n = \mu_n \triangle_\omega^n + \alpha_\omega$ the *covariant difference operator*. The operator ∇_ω^n is not a difference operator in the usual sense because $\nabla_\omega^n 1 \neq 0$. Instead, we have $\nabla_\omega^n 1 = \alpha_\omega$.

Following the previous steps for $\triangle_{\omega,\omega'}^n$ we define the *covariant bidifference operator* $\nabla_{\omega,\omega'}^n = D_n \triangle_{\omega,\omega'}^n + \alpha_{\omega,\omega'}$ where

$$\alpha_{\omega,\omega'}(x,y) = \left[D_n(\omega_{|x|-1}, \omega'_{|y|-1}) - D_n(x,y)\right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}}$$

and again, $\alpha_{\omega,\omega'}(x,y)$ are analogous to Christoffel symbols. Notice that $\nabla_{\omega,\omega'}^n D_n(x,y) = 0$ for all $x, y \in Q_n$ and $\nabla_{\omega,\omega}^n f(x,x) = \nabla_\omega^n \widetilde{f}(x)$. Complete expressions for ∇_ω^n and $\nabla_{\omega,\omega'}^n$ are

$$\nabla_\omega^n f(x) = \left[\mu_n(\omega_{|x|-1}) f(x) - \mu_n(x) f(\omega_{|x|-1})\right] \delta_{x,\omega_{|x|}} \tag{2.8}$$

and

$$\bigtriangledown^n_{\omega,\omega'} f(x,y) = \left[D_n(\omega_{|x|-1},\omega'_{|y|-1})f(x,y) - D_n(x,y)f(\omega_{|x|-1},\omega'_{|y|-1})\right] \cdot \delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}} \tag{2.9}$$

The form of (2.8) and (2.9) shows that $\bigtriangledown^n_\omega$ and $\bigtriangledown^n_{\omega,\omega'}$ are "weighted" difference operators.

3 Curvature Operators

The linear operator $\mathcal{R}^n_{\omega,\omega'} \colon L_n \to L_n$ defined by

$$\mathcal{R}^n_{\omega,\omega'} = \bigtriangledown^n_{\omega,\omega'} - \bigtriangledown^n_{\omega',\omega}$$

is called the *curvature operator*. Applying (2.9) we have

$$\begin{aligned}
&\mathcal{R}^n_{\omega,\omega'} f(x,y) \\
&\quad = D_n(x,y)\left[f(\omega'_{|x|-1},\omega_{|y|-1})\delta_{x,\omega'_{|x|}}\delta_{y,\omega_{|y|}} - f(\omega_{|x|-1},\omega'_{|y|-1})\delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}}\right] \\
&\qquad + \left[D_n(\omega_{|x|-1},\omega'_{|y|-1})\delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}} - D_n(\omega'_{|x|-1},\omega_{|y|-1})\delta_{x,\omega'_{|x|}}\delta_{y,\omega_{|y|}}\right] f(x,y)
\end{aligned} \tag{3.1}$$

If $x = y$, then (3.1) reduces to

$$\begin{aligned}
\mathcal{R}^n_{\omega,\omega'} f(x,x) &= \mu_n(x)\left[f(\omega'_{|x|-1},\omega_{|x|-1}) - f(\omega_{|x|-1},\omega'_{|x|-1})\right]\delta_{x,\omega_{|x|}} \\
&\quad + 2i\,\mathrm{Im} D_n(\omega_{|x|-1},\omega'_{|x|-1})f(x,x)\delta_{x,\omega_{|x|}}
\end{aligned}$$

We call the operator $\mathcal{D}^n_{\omega,\omega'} \colon L_n \to L_n$ given by

$$\begin{aligned}
&\mathcal{D}^n_{\omega,\omega'} f(x,y) \\
&\quad = D_n(x,y)\left[f(\omega'_{|x|-1},\omega_{|y|-1})\delta_{x,\omega'_{|x|}}\delta_{y,\omega_{|y|}} - f(\omega_{|x|-1},\omega'_{|y|-1})\delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}}\right]
\end{aligned}$$

the *metric operator* and the operator $\mathcal{T}^n_{\omega,\omega'} \colon L_n \to L_n$ given by

$$\begin{aligned}
&\mathcal{T}^n_{\omega,\omega'} f(x,y) \\
&\quad = \left[D_n(\omega_{|x|-1},\omega'_{|y|-1})\delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}} - D_n(\omega'_{|x|-1},\omega_{|y|-1})\delta_{x,\omega_{|x|}}\delta_{y,\omega'_{|y|}}\right] f(x,y)
\end{aligned}$$

the *mass-energy operator*. Then (3.1) gives

$$\mathcal{R}^n_{\omega,\omega'} = \mathcal{D}^n_{\omega,\omega'} + \mathcal{T}^n_{\omega,\omega'} \tag{3.2}$$

Equation (3.2) is a discrete analogue of Einstein's equation [12]. In this sense, Einstein's equation always holds in the present framework no matter what we have for the quantum dynamics ρ_n. One might argue that we obtained this discrete analogue of Einstein's equation just by definition. However, $\mathcal{R}^n_{\omega,\omega'}$ is a reasonable counterpart of the curvature tensor in general relativity [12] and $\mathcal{D}^n_{\omega,\omega'}$ is certainly a counterpart of the metric tensor.

Equation (3.2) does not give direct information about $D_n(x,y)$ and $D_n(\omega,\omega')$ (which are, after all, what we want to find), but it may give useful indirect information. If we can find $D_n(\omega,\omega')$ such that the classical Einstein equation is an approximation to (3.2), then this would give information about $D_n(\omega,\omega')$. Moreover, an important problem in discrete quantum gravity theory is how to test whether general relativity is a close approximation to the theory. Whether Einstein's equation is an approximation to (3.2) would provide such a test. Another variant of a discrete Einstein equation can be obtained by defining the operator $\mathcal{R}^n_{x,y}$ for $x,y\in Q_n$ by

$$\mathcal{R}^n_{x,y} = \sum\left\{\mathcal{R}^n_{\omega,\omega'}\colon \omega_{|x|}=x, \omega'_{|y|}=y\right\}$$

With similar definitions for $\mathcal{D}^n_{x,y}$ and $\mathcal{T}^n_{x,y}$ we obtain

$$\mathcal{R}^n_{x,y} = \mathcal{D}^n_{x,y} + \mathcal{T}^n_{x,y}$$

In order to consider approximations by Einstein's equation, it may be necessary to let $n\to\infty$ in (3.2). However, the convergence of the operators depends on D_n and will be left to a later paper. In a similar vein, it may be possible that limit operators $\mathcal{R}_{\omega,\omega'}$, $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{T}_{\omega,\omega'}$ can be defined as (possibly unbounded) operators directly on the Hilbert space L.

4 Matrix Elements

We have introduced several operators on K_n and L_n in Sections 2 and 3. In order to understand such operators more directly, it is frequently useful to write them in terms of their matrix elements. First we denote the standard basis on K_n by $e^n_x, x\in Q_n$. The matrix that is zero except for a one in the xy entry is denoted by $|e_x^n\rangle\langle e_y^n|$ and we call this the *xy matrix element*, $x, y \in Q_n$. Of course, in Dirac notation, $|e_x^n\rangle\langle e_y^n|$ can be considered directly as a linear operator without referring to a matrix. In any case, every linear operator T on K_n can be represented uniquely as

$$T = \sum_{x,y \in Q_n} t_{x,y} |e_x^n\rangle\langle e_y^n|$$

for $t_{x,y} \in \mathbb{C}$. In a similar way, $e_x^n \otimes e_y^n$, $x, y \in Q_n$ form an orthonormal basis for $L_n = K_n \otimes K_n$ and every linear operator T on L_n has a unique representation

$$T = \sum \left\{ t_{x,y;x',y'} \left| e_x^n \otimes e_y^n \right\rangle \left\langle e_{x'}^n \otimes e_{y'}^n \right| : x, y, x', y' \in Q_n \right\}$$

Theorem 4.1. *If* $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$, *then*

$$\triangle_\omega^n = \sum_{j=1}^n \left| e_{\omega_j}^n \right\rangle \left(\left\langle e_{\omega_j}^n \right| - \left\langle e_{\omega_{j-1}}^n \right| \right)$$

and

$$\nabla_\omega^n = \sum_{j=1}^n \left| e_{\omega_j}^n \right\rangle \left[\mu_n(\omega_{j-1}) \left\langle e_{\omega_j}^n \right| - \mu_n(\omega_j) \left\langle e_{\omega_{j-1}}^n \right| \right]$$

where we use the conventions $\mu_n(\omega_0) = e_{\omega_0}^n = e_{\omega_{n+1}}^n = 0$.

Proof. We first observe that

$$\sum_{j=1}^n \left| e_{\omega_j}^n \right\rangle \left(\langle e_{\omega_n}^n | - \left\langle e_{\omega_{j-1}}^n \right| \right) e_{\omega_k}^n = e_{\omega_k}^n - e_{\omega_{k+1}}^n$$

On the other hand

$$\triangle_\omega^n e_{\omega_k}^n(x) = \left[e_{\omega_k}^n(x) - e_{\omega_k}^n(\omega_{|x|-1}) \right] \delta_{x,\omega_{|x|}} \tag{4.1}$$

Now the right side of (4.1) is zero if $\omega_{|x|} \neq x$, 1 if $\omega_k = x$ and -1 if $\omega_{k+1} = x$. The first result now follows. The second result is similar. $\square$

The proof of the next theorem is similar to that of Theorem 4.1.

Theorem 4.2. *If* $\omega = \omega_1\omega_2\cdots\omega_n$, $\omega' = \omega'_1\omega'_2\cdots\omega'_n \in \Omega_n$, *then*

$$\triangle^n_{\omega,\omega'} = \sum_{j,k=1}^n \left| e^n_{\omega_j} \otimes e^n_{\omega'_k} \right\rangle \left[\left\langle e^n_{\omega_j} \otimes e^n_{\omega'_k} \right| - \left\langle e^n_{\omega_{j-1}} \otimes e^n_{\omega'_{k-1}} \right| \right]$$

and

$$\begin{aligned}&\nabla^n_{\omega,\omega'}\\ &\quad = \sum_{j,k=1}^n \left| e^n_{\omega_j} \otimes e^n_{\omega'_k} \right\rangle \left[D_n(\omega_{j-1},\omega'_{k-1}) \left\langle e^n_{\omega_j} \otimes e^n_{\omega'_k} \right| - D_n(\omega_j,\omega'_k) \left\langle e^n_{\omega_{j-1}} \otimes e^n_{\omega'_{k-1}} \right| \right]\end{aligned}$$

It follows from Theorem 4.2 that

$$\begin{aligned}\mathcal{R}^n_{\omega,\omega'} &= \nabla^n_{\omega,\omega'} - \nabla^n_{\omega',\omega}\\ &= \sum_{j,k=1}^n \Big[D_n(\omega_{j-1},\omega'_{k-1}) \left| e^n_{\omega_j} \otimes e^n_{\omega'_{k-1}} \right\rangle \left\langle e^n_{\omega_j} \otimes e^n_{\omega'_k} \right| \\ &\qquad - D(\omega'_{j-1},\omega_{k-1}) \left| e^n_{\omega'_j} \otimes e^n_{\omega_k} \right\rangle \left\langle e^n_{\omega'_j} \otimes e^n_{\omega_k} \right| \Big] \\ &\qquad + \sum_{j,k=1}^n \Big[D_n(\omega'_k,\omega_j) \left| e^n_{\omega'_j} \otimes e^n_{\omega_k} \right\rangle \left\langle e^n_{\omega'_{j-1}} \otimes e^n_{\omega_{k-1}} \right| \\ &\qquad - D(\omega_j,\omega'_k) \left| e^n_{\omega_j} \otimes e^n_{\omega'_k} \right\rangle \left\langle e^n_{\omega_{j-1}} \otimes e^n_{\omega'_{k-1}} \right| \Big] \end{aligned} \tag{4.2}$$

The matrix element representations of $\mathcal{D}^n_{\omega,\omega'}$ and $\mathcal{T}^n_{\omega,\omega'}$ can now be obtained from (4.2)

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