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AN EINSTEIN EQUATION FOR DISCRETE QUANTUM GRAVITY

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Abstract

The basic framework for this article is the causal set approach to discrete quantum gravity (DQG). Let Q_n be the collection of causal sets with cardinality not greater than n and let K_n be the standard Hilbert space of complex-valued functions on Q_n . The formalism of DQG presents us with a decoherence matrix $D_n(x, y)$, $x, y \in Q_n$. There is a growth order in Q_n and a path in Q_n is a maximal chain relative to this order. We denote the set of paths in Q_n by Ω_n . For $\omega, \omega' \in \Omega_n$ we define a bidifference operator $\nabla_{\omega, \omega'}^n$ on $K_n \otimes K_n$ that is covariant in the sense that $\nabla_{\omega, \omega'}^n$ leaves D_n stationary. We then define the curvature operator $\mathcal{R}_{\omega, \omega'}^n = \nabla_{\omega, \omega'}^n - \nabla_{\omega', \omega}^n$. It turns out that $\mathcal{R}_{\omega, \omega'}^n$ naturally decomposes into two parts $\mathcal{R}_{\omega, \omega'}^n = \mathcal{D}_{\omega, \omega'}^n + \mathcal{T}_{\omega, \omega'}^n$ where $\mathcal{D}_{\omega, \omega'}^n$ is closely associated with D_n and is called the metric operator while $\mathcal{T}_{\omega, \omega'}^n$ is called the mass-energy operator. This decomposition is a discrete analogue of Einstein's equation of general relativity. Our analogue may be useful in determining whether general relativity theory is a close approximation to DQG.

1 Causet Approach to DQG

A *causal set* (*causet*) is a finite partially ordered set x . Thus, x is endowed with an irreflexive, transitive relation $<$ [1, 8, 10]. That is, $a \not< a$ for all $a \in x$

and $a < b$, $b < c$ imply that $a < c$ for $a, b, c \in x$. The relation $a < b$ indicates that b is in the causal future of a . Let \mathcal{P}_n be the collection of all causets with cardinality n , $n = 1, 2, \dots$, and let $\mathcal{P} = \cup \mathcal{P}_n$. For $x \in \mathcal{P}$, an element $a \in x$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x produces y if y is obtained from x by adjoining a single new element a to x that is maximal in y . In this way, there is no element of y in the causal future of a . If x produces y , we say that y is an *offspring* of x and write $x \rightarrow y$.

A *path* in \mathcal{P} is a string (sequence) $x_1 x_2 \dots$ where $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$, $i = 1, 2, \dots$. An *n-path* in \mathcal{P} is a finite string $x_1 x_2 \dots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$. We denote the set of paths by Ω and the set of n -paths by Ω_n . We think of $\omega \in \Omega$ as a possible universe (or universe history). The set of paths whose initial n -path is $\omega_0 \in \Omega_n$ is called an *elementary cylinder set* and is denoted by $\text{cyl}(\omega_0)$. Thus, if $\omega_0 = x_1 x_2 \dots x_n$, then

$$\text{cyl}(\omega_0) = \{\omega \in \Omega: \omega = x_1 x_2 \dots x_n y_{n+1} y_{n+2} \dots\}$$

The *cylinder set generated* by $A \subseteq \Omega_n$ is defined by

$$\text{cyl}(A) = \bigcup_{\omega \in A} \text{cyl}(\omega)$$

The collection $\mathcal{A}_n = \{\text{cyl}(A): A \subseteq \Omega_n\}$ forms an increasing sequence of algebras $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ on Ω and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of Ω . We denote the σ -algebra generated by $\mathcal{C}(\Omega)$ as \mathcal{A} .

It is shown [6, 11] that a classical sequential growth process (CSGP) on \mathcal{P} that satisfies natural causality and covariance conditions is determined by a sequence of nonnegative numbers $c = (c_0, c_1, \dots)$ called *coupling constants*. The coupling constants specify a unique probability measure ν_c on \mathcal{A} making $(\Omega, \mathcal{A}, \nu_c)$ a probability space. The *path Hilbert space* is given by $H = L_2(\Omega, \mathcal{A}, \nu_c)$. If $\nu_c^n = \nu_c \upharpoonright \mathcal{A}_n$ is the restriction of ν_c to \mathcal{A}_n , then $H_n = L_2(\Omega, \mathcal{A}_n, \nu_c^n)$ is an increasing sequence of closed subspaces of H .

A bounded operator T on H_n will also be considered as a bounded operator on H by defining $Tf = 0$ for all $f \in H_n^\perp$. We denote the characteristic function of a set $A \in \mathcal{A}$ by χ_A and use the notation $\chi_\Omega = 1$. A *q-probability operator* is a bounded positive operator ρ_n on H_n that satisfies $\langle \rho_n 1, 1 \rangle = 1$. Denote the set of q -probability operators on H_n by $\mathcal{Q}(H_n)$. For $\rho_n \in \mathcal{Q}(H_n)$ we define the *n-decoherence functional* [3, 5, 8] $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$ by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

The functional $D_n(A, B)$ gives a measure of the interference between the events A and B when the system is described by ρ_n . It is clear that $D_n(\Omega_n, \Omega_n) = 1$, $D_n(A, B) = \overline{D_n(B, A)}$ and $A \mapsto D_n(A, B)$ is a complex measure for all $B \in \mathcal{A}_n$. It is also well known that if $A_1, \dots, A_n \in \mathcal{A}_n$, then the matrix with entries $D_n(A_j, A_k)$ is positive semidefinite. In particular the positive semidefinite matrix with entries

$$D_n(\omega_i, \omega_j) = D_n(\text{cyl}(\omega_i), \text{cyl}(\omega_j)), \quad \omega_i, \omega_j \in \Omega_n$$

is called the *n-decoherence matrix*.

We define the map $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$ given by

$$\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle$$

Notice that $\mu_n(\Omega) = 1$. Although μ_n is not additive in general, it satisfies the *grade-2 additivity condition* [2, 3, 5, 7, 9]: if $A, B, C \in \mathcal{A}_n$ are mutually disjoint then

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

We call μ_n the *q-measure* corresponding to ρ_n and interpret $\mu_n(A)$ as the quantum propensity for the occurrence of the event $A \in \mathcal{A}_n$. A simple example is to let $\rho_n = I$, $n = 1, 2, \dots$. Then

$$D_n(A, B) = \langle \chi_B, \chi_A \rangle = \nu_c^n(A \cap B)$$

and $\mu_n(A) = \nu_c^n(A)$, the classical measure. A more interesting example is to let $\rho_n = |1\rangle\langle 1|$, $n = 1, 2, \dots$. Then

$$D_n(A, B) = \langle |1\rangle\langle 1| \chi_B, \chi_A \rangle = \nu_c^n(A) \nu_c^n(B)$$

and $\mu_n(A) = [\nu_c^n(A)]^2$, the classical measure square.

We call a sequence $\rho_n \in \mathcal{Q}(H_n)$, $n = 1, 2, \dots$, *consistent* if

$$D_{n+1}(A, B) = D_n(A, B)$$

for all $A, B \in \mathcal{A}_n$. A *quantum sequential growth process* (QSGP) is a consistent sequence $\rho_n \in \mathcal{Q}(H_n)$ [3, 4]. We consider a QSGP as a model for discrete quantum gravity. It is hoped that additional theoretical principles or experimental data will help determine the coupling constants and hence ν_c , which is the classical part of the process, and also the $\rho_n \in \mathcal{Q}(H_n)$, which is

the quantum part. Moreover, it is believed that general relativity will eventually be shown to be a close approximation to this discrete model. Until now it has not been clear how this can be accomplished. However, in Section 3 we shall derive a discrete Einstein equation which might be useful in performing these tasks.

2 Difference Operators

Let $Q_n = \bigcup_{j=1}^n \mathcal{P}_j$ be the collection of causets with cardinality not greater than n and let K_n be the (finite-dimensional) Hilbert space \mathbb{C}^{Q_n} with the standard inner product

$$\langle f, g \rangle = \sum_{x \in Q_n} \overline{f(x)} g(x)$$

Let $L_n = K_n \otimes K_n$ which we identify with $\mathbb{C}^{Q_n \times Q_n}$ having the standard inner product. We shall also have need to consider the Hilbert space

$$K = \left\{ f \in \mathbb{C}^{\mathcal{P}} : \sum_{x \in \mathcal{P}} |f(x)|^2 < \infty \right\}$$

with the standard inner product and we define $L = K \otimes K$. Notice that $K_1 \subseteq K_2 \subseteq \dots \subseteq K$ form an increasing sequence of subspaces of K that generate K in the natural way.

Let $\rho_n \in \mathcal{Q}(H_n)$ be a QSGP with corresponding decoherence matrix $D_n(\omega, \omega')$, $\omega, \omega' \in \Omega_n$. If $\omega = \omega_1 \omega_2 \dots \omega_n \in \Omega_n$ and $\omega_j = x$ for some j , we say that ω goes through x . For $x, y \in Q_n$ we define

$$D_n(x, y) = \sum \{ D_n(\omega, \omega') : \omega \text{ goes through } x, \omega' \text{ goes through } y \}$$

Due to the consistency of ρ_n , $D_n(x, y)$ is independent of n . That is, $D_n(x, y) = D_m(x, y)$ if $x, y \in Q_n \cap Q_m$. Moreover, $D_n(x, y)$ are the components of a positive semidefinite matrix. We view Q_n as the analogue of a differentiable manifold and $D_n(x, y)$ as the analogue of a metric tensor. One might think that the elements of causets should be analogous to points of a differential manifold and not the causets themselves. However, if $x \in Q_n$, then x is intimately related to its producers, each of which determines a unique $a \in x$.

Moreover, if $y \rightarrow x$ we view (y, x) as a tangent vector at x . In this way, there are as many tangent vectors at x as there are producers of x . Finally, the elements of Ω_n are analogues of curves and the elements of K_n are analogues of smooth functions on a manifold.

For $x \in Q_n$, $|x|$ denotes the cardinality of x . Notice if $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$ and $\omega_j = x$, then $j = |x|$ and ω goes through x if and only if $\omega_{|x|} = x$. We see that a path ω through x determines a tangent vector $(\omega_{|x|-1}, \omega_{|x|})$ at x (assuming that $|x| \geq 2$). For $\omega \in \Omega_n$ we define the *difference operator* Δ_ω^n on K_n by

$$\Delta_\omega^n f(x) = [f(x) - f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}}$$

for all $f \in K_n$, where $\delta_{x, \omega_{|x|}}$ is the Kronecker delta. Thus, $\Delta_\omega^n f(x)$ gives the change of f along the tangent vector $(\omega_{|x|-1}, \omega_{|x|})$ if ω goes through x . It is clear that Δ_ω^n is a linear operator on K_n . We now show that Δ_ω^n satisfies a discrete form of Leibnitz's rule. For $f, g \in K_n$ we have

$$\begin{aligned} \Delta_\omega^n fg(x) &= [f(x)g(x) - f(\omega_{|x|-1})g(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &= \{ [f(x)g(x) - f(x)g(\omega_{|x|-1})] + [f(x)g(\omega_{|x|-1}) - f(\omega_{|x|-1})g(\omega_{|x|-1})] \} \\ &\quad \cdot \delta_{x, \omega_{|x|}} \\ &= f(x) \Delta_\omega^n g(x) + g(\omega_{|x|-1}) \Delta_\omega^n f(x) \end{aligned} \quad (2.1)$$

Of course, it also follows that

$$\Delta_\omega^n fg(x) = \Delta_\omega^n gf(x) = g(x) \Delta_\omega^n f(x) + f(\omega_{|x|-1}) \Delta_\omega^n g(x) \quad (2.2)$$

Given a function of two variables $f \in \mathbb{C}^{Q_n \times Q_n} = L_n$ we have a function $\tilde{f} \in K_n$ of one variable where $\tilde{f}(x) = f(x, x)$ and given a function $f \in K_n$ we have the functions of two variables $f_1, f_2 \in L_n$ where $f_1(x, y) = f(x)$ and $f_2(x, y) = f(y)$ for all $x, y \in Q_n$. For $\omega, \omega' \in \Omega_n$ we want an operator $\Delta_{\omega, \omega'}^n: L_n \rightarrow L_n$ that extends Δ_ω^n and satisfies a discrete Leibnitz's rule. That is,

$$\Delta_{\omega, \omega'}^n f_1(x, y) = \Delta_\omega^n f(x) \delta_{y, \omega'_{|x|}}, \Delta_{\omega, \omega'}^n f_2(x, y) = \Delta_{\omega'}^n f(y) \delta_{x, \omega_{|x|}} \quad (2.3)$$

and

$$\Delta_{\omega, \omega'}^n fg(x, y) = f(x, y) \Delta_{\omega, \omega'}^n g(x, y) + g(\omega_{|x|-1}, \omega'_{|x|-1}) \Delta_{\omega, \omega'}^n f(x, y) \quad (2.4)$$

Theorem 2.1. *A linear operator $\Delta_{\omega, \omega'}^n : L_n \rightarrow L_n$ satisfies (2.3) and (2.4) if and only if $\Delta_{\omega, \omega'}^n$ has the form*

$$\Delta_{\omega, \omega'}^n f(x, y) = [f(x, y) - f(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \quad (2.5)$$

Proof. If $\Delta_{\omega, \omega'}^n$ is defined by (2.5), then for $f \in K_n$ we have

$$\begin{aligned} \Delta_{\omega, \omega'}^n f_1(x, y) &= [f_1(x, y) - f_1(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x) - f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= \Delta_{\omega}^n f(x) \delta_{y, \omega'_{|y|}} \end{aligned}$$

In a similar way, $\Delta_{\omega, \omega'}^n$ satisfies the second equation in (2.3). Moreover, we have

$$\begin{aligned} \Delta_{\omega, \omega'}^n f(x, y) &= [f(x, y)g(x, y) - f(\omega_{|x|-1}\omega'_{|y|-1})g(\omega_{|x|-1}\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x, y)g(x, y) - f(x, y)g(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &\quad + [f(x, y)g(\omega_{|x|-1}, \omega'_{|y|-1}) - f(\omega_{|x|-1}\omega'_{|y|-1})g(\omega_{|x|-1}\omega'_{|y|-1})] \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= f(x, y) \Delta_{\omega, \omega'}^n g(x, y) + g(\omega_{|x|-1}, \omega'_{|y|-1}) \Delta_{\omega, \omega'}^n f(x, y) \end{aligned}$$

Conversely, suppose the linear operator $\Delta_{\omega, \omega'}^n : L_n \rightarrow L_n$ satisfies (2.3) and (2.4). If $f \in L_n$ has the form $f(x, y) = g(x)h(y)$, then

$$\begin{aligned} \Delta_{\omega, \omega'}^n f(x, y) &= \Delta_{\omega, \omega'}^n gh(x, y) = \Delta_{\omega, \omega'}^n g_1 h_2(x, y) \\ &= g_1(x, y) \Delta_{\omega, \omega'}^n h_2(x, y) + h_2(\omega_{|x|-1}\omega'_{|y|-1}) \Delta_{\omega, \omega'}^n g_1(x, y) \\ &= g(x) \Delta_{\omega}^n h(y) \delta_{x, \omega_{|x|}} + h(\omega'_{|y|-1}) \Delta_{\omega}^n g(x) \delta_{y, \omega'_{|y|}} \\ &= \{g(x) [h(y) - h(\omega'_{|y|-1})] + h(\omega'_{|y|-1}) [g(x) - g(\omega_{|y|-1})]\} \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [g(x)h(y) - g(\omega_{|x|-1})h(\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x, y) - f(\omega_{|x|-1}\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \end{aligned}$$

Since $\Delta_{\omega, \omega'}^n$ is linear and every element of L_n is a linear combination of product functions, the result follows. \square

Of course, Theorem 2.1 is not surprising because (2.5) is the natural extension of Δ_ω^n to L_n . Also $\Delta_{\omega,\omega'}^n$ extends Δ_ω^n in the sense that for any $f \in L_n$ we have

$$\begin{aligned}\Delta_{\omega,\omega}^n f(x, x) &= [f(x, y) - f(\omega_{|x|-1}, \omega_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &= [\tilde{f}(x) - \tilde{f}(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} = \Delta_\omega^n \tilde{f}(x)\end{aligned}$$

As before, $\Delta_{\omega,\omega'}^n$ satisfies

$$\begin{aligned}\Delta_{\omega,\omega'}^n f g(x, y) &= \Delta_{\omega,\omega'}^n g f(x, y) \\ &= g(x, y) \Delta_{\omega,\omega'}^n f(x, y) + f(\omega_{|x|-1}, \omega'_{|y|-1}) \Delta_{\omega,\omega'}^n g(x, y)\end{aligned}$$

The next result characterizes Δ_ω^n and $\Delta_{\omega,\omega'}^n$ up to a multiplicative constant.

Theorem 2.2. (a) *An operator $T_\omega: K_n \rightarrow K_n$ satisfies (2.1) and $T_\omega f(x) = 0$ if $\omega_{|x|} \neq x$ if and only if there exists a function $\beta_\omega \in K_n$ such that $T_\omega = \beta_\omega \Delta_\omega^n$.*
(b) *An operator $T_{\omega,\omega'}: L_n \rightarrow L_n$ satisfies (2.4) and $T_{\omega,\omega'} f(x, y) = 0$ if $\omega_{|x|} \neq x$ or $\omega'_{|y|} \neq y$ if and only if there exists a function $\beta_{\omega,\omega'} \in L_n$ such that $T_{\omega,\omega'} = \beta_{\omega,\omega'} \Delta_{\omega,\omega'}^n$.*

Proof. If T_ω satisfies (2.1), it follows from (2.2) that

$$f(x)T_\omega g(x) + g(\omega_{|x|-1})T_\omega f(x) = g(x)T_\omega f(\omega) + f(\omega_{|x|-1})T_\omega g(x)$$

Hence,

$$[g(x) - g(\omega_{|x|-1})] T_\omega f(x) = T_\omega g(x) [f(x) - f(\omega_{|x|-1})]$$

Therefore, if $g(x) - g(\omega_{|x|-1}) \neq 0$, we have

$$T_\omega f(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})} [f(x) - f(\omega_{|x|-1})]$$

Letting

$$\beta_\omega(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})}$$

gives the result. The converse is straightforward. The proof of (b) is similar. \square

It is clear that $\mu_n(x) = D_n(x, x)$ is not stationary; that is $\Delta_\omega^n \mu_n(x) \neq 0$ for all $x \in Q_n$ in general. Is there a function $\alpha_\omega \in K_n$ such that $(\Delta_\omega^n + \alpha_\omega) \mu_n(x) = 0$ for all $x \in Q_n$? If α_ω exists, we obtain

$$[\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0$$

If $\omega_{|x|} = x$ and $\mu_n(x) = 0$, this would imply that $\mu_n(\omega_{|x|-1}) = 0$. Continuing this process would give

$$\mu_n(\omega_{|x|-2}) = \mu_n(\omega_{|x|-3}) = \cdots = 0$$

which leads to a contradiction. It is entirely possible for $\mu_n(x)$ to be zero for some $x \in Q_n$ so we abandon this attempt. How about functions $\alpha_\omega, \beta_\omega \in K_n$ such that $(\beta_\omega \Delta_\omega^n + \alpha_\omega) \mu_n(x) = 0$ for all $x \in Q_n$? We then obtain

$$\beta_\omega(x) [\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0 \quad (2.6)$$

If $\omega_{|x|} = x$ and $\mu_n(x) = 0$ but $\beta_\omega(x) \neq 0$ we obtain the same contradiction as before. We conclude that $\beta_\omega(x) = 0$ whenever $\mu_n(x) = 0$. The simplest choice of such a β_ω is $\beta_\omega(x) = \mu_n(x)$. This choice also has the advantage of being independent of ω . Equation (2.6) becomes

$$\mu_n(x) [\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0 \quad (2.7)$$

If $\mu_n(x) = 0$, then (2.7) holds. If $\mu_n(x) \neq 0$, then we obtain

$$\alpha_\omega(x) = [\mu_n(\omega_{|x|-1}) - \mu_n(x)] \delta_{x, \omega_{|x|}} = -\Delta_\omega^n \mu_n(x)$$

The numbers $\alpha_\omega(x)$ are an analogue of the Christoffel symbols. We call $\nabla_\omega^n = \mu_n \Delta_\omega^n + \alpha_\omega$ the *covariant difference operator*. The operator ∇_ω^n is not a difference operator in the usual sense because $\nabla_\omega^n 1 \neq 0$. Instead, we have $\nabla_\omega^n 1 = \alpha_\omega$.

Following the previous steps for $\Delta_{\omega, \omega'}^n$, we define the *covariant bidifference operator* $\nabla_{\omega, \omega'}^n = D_n \Delta_{\omega, \omega'}^n + \alpha_{\omega, \omega'}$ where

$$\alpha_{\omega, \omega'}(x, y) = [D_n(\omega_{|x|-1}, \omega'_{|y|-1}) - D_n(x, y)] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}}$$

and again, $\alpha_{\omega, \omega'}(x, y)$ are analogous to Christoffel symbols. Notice that $\nabla_{\omega, \omega'}^n D_n(x, y) = 0$ for all $x, y \in Q_n$ and $\nabla_{\omega, \omega'}^n f(x, x) = \nabla_\omega^n \tilde{f}(x)$. Complete expressions for ∇_ω^n and $\nabla_{\omega, \omega'}^n$ are

$$\nabla_\omega^n f(x) = [\mu_n(\omega_{|x|-1}) f(x) - \mu_n(x) f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \quad (2.8)$$

and

$$\begin{aligned} \nabla_{\omega, \omega'}^n f(x, y) &= [D_n(\omega_{|x|-1}, \omega'_{|y|-1})f(x, y) - D_n(x, y)f(\omega_{|x|-1}, \omega'_{|y|-1})] \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \end{aligned} \quad (2.9)$$

The form of (2.8) and (2.9) shows that ∇_{ω}^n and $\nabla_{\omega, \omega'}^n$ are “weighted” difference operators.

3 Curvature Operators

The linear operator $\mathcal{R}_{\omega, \omega'}^n: L_n \rightarrow L_n$ defined by

$$\mathcal{R}_{\omega, \omega'}^n = \nabla_{\omega, \omega'}^n - \nabla_{\omega', \omega}^n$$

is called the *curvature operator*. Applying (2.9) we have

$$\begin{aligned} \mathcal{R}_{\omega, \omega'}^n f(x, y) &= D_n(x, y) \left[f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right] \\ &\quad + \left[D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y) \end{aligned} \quad (3.1)$$

If $x = y$, then (3.1) reduces to

$$\begin{aligned} \mathcal{R}_{\omega, \omega'}^n f(x, x) &= \mu_n(x) [f(\omega'_{|x|-1}, \omega_{|x|-1}) - f(\omega_{|x|-1}, \omega'_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &\quad + 2i \operatorname{Im} D_n(\omega_{|x|-1}, \omega'_{|x|-1}) f(x, x) \delta_{x, \omega_{|x|}} \end{aligned}$$

We call the operator $\mathcal{D}_{\omega, \omega'}^n: L_n \rightarrow L_n$ given by

$$\begin{aligned} \mathcal{D}_{\omega, \omega'}^n f(x, y) &= D_n(x, y) \left[f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right] \end{aligned}$$

the *metric operator* and the operator $\mathcal{T}_{\omega, \omega'}^n: L_n \rightarrow L_n$ given by

$$\begin{aligned} \mathcal{T}_{\omega, \omega'}^n f(x, y) &= \left[D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y) \end{aligned}$$

the *mass-energy operator*. Then (3.1) gives

$$\mathcal{R}_{\omega,\omega'}^n = \mathcal{D}_{\omega,\omega'}^n + \mathcal{T}_{\omega,\omega'}^n \quad (3.2)$$

Equation (3.2) is a discrete analogue of Einstein's equation [12]. In this sense, Einstein's equation always holds in the present framework no matter what we have for the quantum dynamics ρ_n . One might argue that we obtained this discrete analogue of Einstein's equation just by definition. However, $\mathcal{R}_{\omega,\omega'}^n$ is a reasonable counterpart of the curvature tensor in general relativity [12] and $\mathcal{D}_{\omega,\omega'}^n$ is certainly a counterpart of the metric tensor.

Equation (3.2) does not give direct information about $D_n(x, y)$ and $D_n(\omega, \omega')$ (which are, after all, what we want to find), but it may give useful indirect information. If we can find $D_n(\omega, \omega')$ such that the classical Einstein equation is an approximation to (3.2), then this would give information about $D_n(\omega, \omega')$. Moreover, an important problem in discrete quantum gravity theory is how to test whether general relativity is a close approximation to the theory. Whether Einstein's equation is an approximation to (3.2) would provide such a test. Another variant of a discrete Einstein equation can be obtained by defining the operator $\mathcal{R}_{x,y}^n$ for $x, y \in Q_n$ by

$$\mathcal{R}_{x,y}^n = \sum \{ \mathcal{R}_{\omega,\omega'}^n : \omega|_x = x, \omega'|_y = y \}$$

With similar definitions for $\mathcal{D}_{x,y}^n$ and $\mathcal{T}_{x,y}^n$ we obtain

$$\mathcal{R}_{x,y}^n = \mathcal{D}_{x,y}^n + \mathcal{T}_{x,y}^n$$

In order to consider approximations by Einstein's equation, it may be necessary to let $n \rightarrow \infty$ in (3.2). However, the convergence of the operators depends on D_n and will be left to a later paper. In a similar vein, it may be possible that limit operators $\mathcal{R}_{\omega,\omega'}$, $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{T}_{\omega,\omega'}$ can be defined as (possibly unbounded) operators directly on the Hilbert space L .

4 Matrix Elements

We have introduced several operators on K_n and L_n in Sections 2 and 3. In order to understand such operators more directly, it is frequently useful to write them in terms of their matrix elements. First we denote the standard basis on K_n by $e_x^n, x \in Q_n$. The matrix that is zero except for a one in

the xy entry is denoted by $|e_x^n\rangle\langle e_y^n|$ and we call this the xy matrix element, $x, y \in Q_n$. Of course, in Dirac notation, $|e_x^n\rangle\langle e_y^n|$ can be considered directly as a linear operator without referring to a matrix. In any case, every linear operator T on K_n can be represented uniquely as

$$T = \sum_{x, y \in Q_n} t_{x, y} |e_x^n\rangle\langle e_y^n|$$

for $t_{x, y} \in \mathbb{C}$. In a similar way, $e_x^n \otimes e_y^n$, $x, y \in Q_n$ form an orthonormal basis for $L_n = K_n \otimes K_n$ and every linear operator T on L_n has a unique representation

$$T = \sum \{t_{x, y; x', y'} |e_x^n \otimes e_y^n\rangle\langle e_{x'}^n \otimes e_{y'}^n| : x, y, x', y' \in Q_n\}$$

Theorem 4.1. *If $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$, then*

$$\Delta_\omega^n = \sum_{j=1}^n |e_{\omega_j}^n\rangle \left(\langle e_{\omega_j}^n| - \langle e_{\omega_{j-1}}^n| \right)$$

and

$$\nabla_\omega^n = \sum_{j=1}^n |e_{\omega_j}^n\rangle \left[\mu_n(\omega_{j-1}) \langle e_{\omega_j}^n| - \mu_n(\omega_j) \langle e_{\omega_{j-1}}^n| \right]$$

where we use the conventions $\mu_n(\omega_0) = e_{\omega_0}^n = e_{\omega_{n+1}}^n = 0$.

Proof. We first observe that

$$\sum_{j=1}^n |e_{\omega_j}^n\rangle \left(\langle e_{\omega_n}^n| - \langle e_{\omega_{j-1}}^n| \right) e_{\omega_k}^n = e_{\omega_k}^n - e_{\omega_{k+1}}^n$$

On the other hand

$$\Delta_\omega^n e_{\omega_k}^n(x) = [e_{\omega_k}^n(x) - e_{\omega_k}^n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \quad (4.1)$$

Now the right side of (4.1) is zero if $\omega_{|x|} \neq x$, 1 if $\omega_k = x$ and -1 if $\omega_{k+1} = x$. The first result now follows. The second result is similar. \square

The proof of the next theorem is similar to that of Theorem 4.1.

Theorem 4.2. *If $\omega = \omega_1\omega_2\cdots\omega_n$, $\omega' = \omega'_1\omega'_2\cdots\omega'_n \in \Omega_n$, then*

$$\Delta_{\omega,\omega'}^n = \sum_{j,k=1}^n \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left[\left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| - \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right]$$

and

$$\begin{aligned} \nabla_{\omega,\omega'}^n &= \sum_{j,k=1}^n \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left[D_n(\omega_{j-1}, \omega'_{k-1}) \left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| - D_n(\omega_j, \omega'_k) \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right] \end{aligned}$$

It follows from Theorem 4.2 that

$$\begin{aligned} \mathcal{R}_{\omega,\omega'}^n &= \nabla_{\omega,\omega'}^n - \nabla_{\omega',\omega}^n \\ &= \sum_{j,k=1}^n \left[D_n(\omega_{j-1}, \omega'_{k-1}) \left| e_{\omega_j}^n \otimes e_{\omega'_{k-1}}^n \right\rangle \left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| \right. \\ &\quad \left. - D(\omega'_{j-1}, \omega_{k-1}) \left| e_{\omega'_j}^n \otimes e_{\omega_k}^n \right\rangle \left\langle e_{\omega'_j}^n \otimes e_{\omega_k}^n \right| \right] \\ &\quad + \sum_{j,k=1}^n \left[D_n(\omega'_k, \omega_j) \left| e_{\omega'_j}^n \otimes e_{\omega_k}^n \right\rangle \left\langle e_{\omega'_{j-1}}^n \otimes e_{\omega_{k-1}}^n \right| \right. \\ &\quad \left. - D(\omega_j, \omega'_k) \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right] \end{aligned} \quad (4.2)$$

The matrix element representations of $\mathcal{D}_{\omega,\omega'}^n$ and $\mathcal{T}_{\omega,\omega'}^n$ can now be obtained from (4.2)

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