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Discrete Quantum Gravity is not Isometric

S. Gudder

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We show that if a discrete quantum gravity is not classical, then it cannot be generated by an isometric dynamics. In particular, we show that if the quantum measure $\mu$ (or equivalently the decoherence functional) is generated by an isometric dynamics, then there is no interference between events so the system describing evolving universes is classical. The result follows from a forbidden configuration in the path space of causal sets.

1 Introduction

This introduction presents an overview of the article and precise definitions will be given in Section 2. We denote the collection of causal sets of cardinality $i$ by $\mathcal{P}_i$, $i = 1, 2, \ldots$. If $x_i \in \mathcal{P}_i$, $x_{i+1} \in \mathcal{P}_{i+1}$ satisfy a certain growth relationship, we write $x_i \rightarrow x_{i+1}$. A path is a sequence $x_1 x_2 \cdots, x_i \in \mathcal{P}_i$ with $x_i \rightarrow x_{i+1}$ and an $n$-path is a sequence of length $n$, $x_1 x_2 \cdots x_n$, $x_i \in \mathcal{P}_i$ with $x_i \rightarrow x_{i+1}$. We denote the set of paths by $\Omega$ and the set of $n$-paths by $\Omega_n$. For $\omega \in \Omega_n$, cyl$(\omega)$ is the collection of all paths whose initial $n$-path is $\omega$ and $\mathcal{A}_n$ is the algebra generated by cyl$(\omega)$ for all $\omega \in \Omega_n$. Letting $\mathcal{A}$ be the $\sigma$-algebra generated by $\mathcal{A}_n$, $n = 1, 2, \ldots$, we have that $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$. The theory of classical sequential growth process [3, 4, 7] provides us with a
probability measure \( \nu \) on \( \mathcal{A} \) so \( (\Omega, \mathcal{A}, \nu) \) becomes a probability space. Letting \( \nu_n = \nu | \mathcal{A}_n \) be the restriction of \( \nu \) to \( \mathcal{A}_n \) we obtain the Hilbert space \( H = L^2(\Omega, \mathcal{A}, \nu) \) together with the increasing sequence of closed subspaces \( H_n = L^2(\Omega, \mathcal{A}_n, \nu_n) \). The dynamics of a discrete quantum gravity is described by a sequence of positive operators \( \rho_n \) on \( H_n \), \( n = 1, 2, \ldots \), satisfying a normalization and consistency condition [2].

We call \( \mathcal{P}_n \) the \( n \)-site space and the associated Hilbert space \( K_n \) is the \( n \)-site Hilbert space. Let \( U_n : K_n \to K_{n+1} \) be an isometric operator (isometry), \( n = 1, 2, \ldots \), that is compatible with the growth relation \( x \to y \). When \( U_n \) describes the evolution of the system, there is a standard prescription [5, 6] for defining the amplitude \( a_n(\omega) \) in terms of \( U_n \) for every \( \omega \in \Omega_n \). Also, for \( \omega = x_1 x_2 \cdots x_n, \omega' = x'_1 x'_2 \cdots x'_n \) one defines the decoherence

\[
D_n(\omega, \omega') = \overline{a(\omega)} a(\omega') \delta_{x_n, x'_n}
\]

Moreover, the decoherence functional \( D_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C} \) is given by

\[
D_n(A, B) = \sum \{ D_n(\omega, \omega') : \omega \in A, \omega' \in B \}
\]

and the quantum measure \( \mu_n : \mathcal{A}_n \to \mathbb{R}^+ \) is defined as \( \mu_n(A) = D_n(A, A) \). It can be shown that the matrix with components \( D_n(\omega, \omega') \) defines a positive operator \( \rho_n \) on \( H_n \) satisfying the conditions of the previous paragraph. In this case, we say that \( \rho_n \) is generated by the isometry \( U_n \).

Our main result follows from a forbidden configuration (FC) theorem for the path space \( \Omega \). If \( x \in \mathcal{P}_n, y \in \mathcal{P}_{n+1} \) with \( x \to y \) we say that \( x \) produces \( y \) and \( y \) is an offspring of \( x \). The FC theorem states that two different producers cannot have two distinct offspring in common. The FC theorem greatly restricts the allowed isometries \( U_n \) which in turn restricts the possible generated operators \( \rho_n \). In fact, if \( \rho_n \) is generated by an isometry, then its matrix representation \( D_n(\omega, \omega') \) is diagonal. This implies that there is no interference between paths and that \( \mu_n \) is a classical probability measure. We conclude that if a discrete quantum gravity is not classical, then it cannot be generated by an isometric dynamics. Of course, almost by definition, a discrete quantum gravity is not classical, hence the title of this paper. Since \( \rho_n \) is not generated by an isometry, we must obtain \( \rho_n \) in other ways. We refer the reader to [2] for a study of this problem.
2 Discrete Quantum Gravity

A partially ordered set (poset) is a set \( x \) together with an irreflexive, transitive relation \(<\) on \( x \). In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let \( \mathcal{P}_n \) be the collection of all posets with cardinality \( n, n = 1, 2, \ldots \), and let \( \mathcal{P} = \bigcup \mathcal{P}_n \). An element of \( \mathcal{P} \) is called a causal set and if \( a < b \) for \( a, b \in x \) where \( x \in \mathcal{P}_n \), then \( b \) is in the causal future of \( a \). If \( x \in \mathcal{P}_n, y \in \mathcal{P}_{n+1} \), then \( x \) produces \( y \) if \( y \) is obtained from \( x \) by adjoining a single new element \( a \) to \( x \) that is maximal in \( y \). Thus, \( a \in y \) and there is no \( b \in y \) such that \( a < b \). In this case, we write \( y = x \uparrow a \). We also say that \( x \) is a producer of \( y \) and \( y \) is an offspring of \( x \). If \( x \) produces \( y \) we write \( x \rightarrow y \). We denote the set of offspring of \( x \) by \( x \rightarrow \) and for \( A \in \mathcal{P}_n \) we use the notation \( A \rightarrow = \{ y \in \mathcal{P}_{n+1} : x \rightarrow y, x \in A \} \).

The transitive closure of \( \rightarrow \) makes \( \mathcal{P} \) itself a poset [1, 3, 5].

A path in \( \mathcal{P} \) is a string (sequence) \( x_1 x_2 \cdots \) where \( x_i \in \mathcal{P}_i \) and \( x_i \rightarrow x_{i+1} \), \( i = 1, 2, \ldots \). An \( n \)-path in \( \mathcal{P} \) is a finite string \( x_1 x_2 \cdots x_n \) where again \( x_i \in \mathcal{P}_i \) and \( x_i \rightarrow x_{i+1} \). We denote the set of paths by \( \Omega \) and the set of \( n \)-paths by \( \Omega_n \). The set of paths whose initial \( n \)-path is \( \omega_n \in \Omega_n \) is denoted by \( \omega_n \Rightarrow \).

Thus, if \( \omega_n = x_1 x_2 \cdots x_n \) then

\[ \omega_n \Rightarrow = \{ \omega \in \Omega : x_1 x_2 \cdots x_n y_{n+1} y_{n+2} \cdots \} \]

For \( A \subseteq \Omega_n \) we use the notation

\[ A \Rightarrow = \cup \{ \omega \Rightarrow : \omega \in A \} \]

Thus, \( A \Rightarrow \) is the set of paths whose initial \( n \)-paths are elements of \( A \). We call \( A \Rightarrow \) a cylinder set and define

\[ \mathcal{A}_n = \{ A \Rightarrow : A \subseteq \Omega_n \} \]

In particular, if \( \omega_n \in \Omega_n \) then the elementary cylinder set \( \text{cyl}(\omega_n) \) is given by \( \text{cyl}(\omega_n) = \omega_n \Rightarrow \). It is easy to check that \( \mathcal{A}_n \) forms a increasing sequence \( \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \) of algebras on \( \Omega \) and hence \( C(\Omega) = \cup \mathcal{A}_n \) is an algebra of subsets of \( \Omega \). We denote the \( \sigma \)-algebra generated by \( C(\Omega) \) by \( \mathcal{A} \).

It is shown in [4, 7] that a classical sequential growth process (CSGP) on \( \mathcal{P} \) that satisfies natural causality and covariance conditions is determined by
a sequence of nonnegative numbers $c = (c_1, c_2, \ldots)$ called \textit{coupling constants}. The coupling constants determine a unique probability measure $\nu_c$ on $\mathcal{A}$ making $(\Omega, \mathcal{A}, \nu_c)$ a probability space. The \textit{path Hilbert space} is given by $H = L_2(\Omega, \mathcal{A}, \nu_c)$. If $\nu^n_c = \nu_c \mid \mathcal{A}_n$ is the restriction of $\nu_c$ to $\mathcal{A}_n$, then $H_n = L_2(\Omega, \mathcal{A}, \nu^n_c)$ is an increasing sequence of closed subspaces of $H$. Assuming that $\nu^n_c (\text{cyl}(\omega)) \neq 0$, an orthonormal basis for $H_n$ is

$$e^n_\omega = \nu^n_c (\text{cyl}(\omega))^{-1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n$$

where $\chi_A$ denotes the characteristic function of a set $A$.

A bounded operator $T$ on $H_n$ will also be considered as a bounded operator on $H$ by defining $Tf = 0$ for all $f \in H_n^\perp$. We employ the notation $\chi_{\Omega} = 1$. A \textit{q-probability operator} is a positive operator $\rho_n$ on $H_n$ that satisfies $\langle \rho_n 1, 1 \rangle = 1$. Denote the set of q-probability operators on $H_n$ by $\mathcal{Q}(H_n)$. For $\rho_n \in \mathcal{Q}(H_n)$ we define the \textit{n-decoherence functional} [1, 2, 3] $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$ by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

The functional $D_n(A, B)$ gives a measure of the interference between $A$ and $B$ when the system is described by $\rho_n$. It is clear that $D_n(\Omega_n, \Omega_n) = 1$, $D_n(A, B) = D_n(B, A)$ and $A \mapsto D_n(A, B)$ is a complex measure for every $B \in \mathcal{A}_n$. It is also well known that if $A_1, \ldots, A_n \in \mathcal{A}_n$ then the matrix with entries $D_n(A_j, A_k)$ is positive semidefinite. We define the map $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$ by

$$\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle$$

Notice that $\mu_n(\Omega_n) = 1$. Although $\mu_n$ is not additive, it satisfies the \textit{grade 2-additive condition}: if $A, B, C \in \mathcal{A}_n$ are mutually disjoint, then

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

We call $\mu_n$ the \textit{q-measure} corresponding to $\rho_n$ [1, 5, 6].

We call a sequence $\rho_n \in \mathcal{Q}(H_n)$, $n = 1, 2, \ldots$, \textit{consistent} if $D_{n+1}(A, B) = D_n(A, B)$ for all $A, B \in \mathcal{A}_n$. Of course, if the sequence $\rho_n$, $n = 1, 2, \ldots$, is consistent, then $\mu_{n+1}(A) = \mu_n(A)$ for every $A \in \mathcal{A}_n$. In the present context, a \textit{quantum sequential growth process} (QSGP) is a consistent sequence $\rho_n \in \mathcal{Q}(H_n)$. We consider a QSGP as a model for discrete quantum gravity. It is hoped that additional theoretical principles or experimental data will help determine the coupling constants and the $\rho_n \in \mathcal{Q}(H_n)$. We will then know $\nu_c$ which is the classical part of the process and $\rho_n$, $n = 1, 2, \ldots$, which is the quantum part.
3 Isometric Generation

Let $K_n$ be the Hilbert space of complex-valued functions on $\mathcal{P}_n$ with the usual inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}_n} f(x)g(x)$$

We call $K_n$ the \textit{n-site Hilbert space} and we denote the standard basis $\chi_{\{x\}}$ of $K_n$ by $e^n_x$, $x \in \mathcal{P}_n$. The projection operators $P_n(x) = |e^n_x\rangle\langle e^n_x|$, $x \in \mathcal{P}_n$, describe the site at step $n$ of the process. Let $U_n: K_n \to K_{n+1}$ be an operator satisfying the two conditions

1. $U_n^*U_n = I_n$ (isometry).
2. If $x_n \not\to x_{n+1}$, then $\langle e_{x_{n+1}}^{n+1}, U_n e^n_x \rangle = 0$ (compatibility).

Condition (1) implies that $U_n$ is an isometry; that is,

$$\langle U_n f, U_n g \rangle = \langle f, g \rangle$$

for all $f, g \in K_n$. The compatibility condition (2) ensures that $U_n$ preserves the growth relation $x_n \to x_{n+1}$; that is, when $e^n_x$ corresponds to site $x_n$, then $U_n e^n_x$ corresponds to sites in $x_n \to$. Notice that $Q_n = U_n U_n^*$ is the projection from $K_{n+1}$ onto Range ($U_n$). We call

$$a(x_n \to x_{n+1}) = \langle e_{x_{n+1}}^{n+1}, U_n e^n_x \rangle$$

the transition amplitude from $x_n$ to $x_{n+1}$. Of course, by (2) $a(x_n \to x_{n+1}) = 0$ if $x_n \not\to x_{n+1}$. The corresponding transition probability is $|a(x_n \to x_{n+1})|^2$.

Since

$$U_n e^n_x = \sum_{x_{n+1} \in \mathcal{P}_{n+1}} a(x_n \to x_{n+1}) e^{n+1}_{x_{n+1}}$$

we conclude that $|a(x_n \to x_{n+1})|^2$ can be interpreted as a probability because

$$\sum_{x_{n+1} \in \mathcal{P}_{n+1}} |a(x_n \to x_{n+1})|^2 = \|U_n e^n_x\|^2 = 1 \quad (3.1)$$

For $r \leq s \in \mathbb{N}$, define $U(s, r): K_r \to K_s$ by $U(r, r) = I_r$ if $r = s$ and if $r < s$, then

$$U(s, r) = U_r U_{r+1} \cdots U_{s-1}$$
Then $U(s, r)$ is an isometry and $U(t, r) = U(t, s)U(s, r)$ for all $r \leq s \leq t \in \mathbb{N}$. We call $U(s, r)$ for $r \leq s \in \mathbb{N}$ a discrete isometric system. Such systems frequently describe the dynamics (evolution) in quantum mechanics [1, 5, 6].

We can assume that all paths or $n$-paths begin at the poset $x_1$ that has one element. We describe the $n$-path $\omega = x_1 x_2 \cdots x_n$ quantum mechanically by the operator $C_n(\omega) : K_1 \to K_n$ given as

$$C_n(\omega) = P_n(x_n)U_{n-1}(x_{n-1})U_{n-2} \cdots P_2(x_2)U_1$$  \hspace{1cm} (3.2)

Defining the amplitude $a(\omega)$ of $\omega$ by

$$a(\omega) = a(x_{n-1} \to x_n)a(x_{n-2} \to x_{n-1}) \cdots a(x_1 \to x_2)$$  \hspace{1cm} (3.3)

we can write (3.2) as

$$C_n(\omega) = a(\omega)\langle e_{x_n}^n | e_{x_1}^1 \rangle$$  \hspace{1cm} (3.4)

We interpret $|a(\omega)|^2$ as the probability of the $n$-path $\omega$ according to the dynamics $U(s, r)$. It follows from (3.1) that

$$\sum_{\omega \in \Omega_n} |a(\omega)|^2 = 1$$

so $|a(\omega)|^2$ is indeed a probability distribution on $\Omega_n$.

The operator $C_n(\omega')^*C_n(\omega)$ describes the interference between the two $n$-paths $\omega, \omega' \in \Omega_n$. Applying (3.4) we conclude that

$$C_n(\omega')^*C_n(\omega) = \overline{a(\omega')}a(\omega)\delta_{x_n, x_n'}I_1$$

which we can identify with the complex number

$$D_n(\omega, \omega') = \overline{a(\omega')}a(\omega)\delta_{x_n, x_n'}$$  \hspace{1cm} (3.5)

The matrix $D_n$ with entries $D_n(\omega, \omega')$ is called the decoherence matrix. We say that a QSGP $\rho_n$, $n = 1, 2, \ldots$, is isometrically generated if there exists a discrete isometric system given by $U_n : K_n \to K_{n+1}$ such that $\rho_n$ is the operator corresponding to the matrix $D_n$; that is,

$$\langle \rho_n e^n_\omega, e^n_{\omega'} \rangle = D_n(\omega', \omega)$$  \hspace{1cm} (3.6)

for every $\omega, \omega' \in \Omega_n$. At first sight, isometric generation appears to be a natural way to construct a QSGP. However, the next section shows that this does not work unless the QSGP is classical. For methods of constructing such processes that are truly quantum, we refer the reader to [2].
Various configurations can occur in the poset \((\mathcal{P}, \rightarrow)\). For instance, it is quite common for two different producers to share a common offspring. The next example discusses the case in which more than two producers share a common offspring.

**Example 1.** Figure 1 illustrates a case in which three producers share a common offspring. In this figure, a rising line called a *link* from vertex \(a\) to vertex \(b\) designates that \(a < b\) and there is no \(c\) such that \(a < c < b\). In this figure, \(y_1\), \(y_2\) and \(y_3\) produce the offspring \(y\). This is the smallest cardinality example of this configuration. Indeed, if \(y\) has four elements then \(y\) would need three nonisomorphic maximal elements to have three producers and this is impossible. Figure 2 illustrates a poset that is the offspring of \(n\) producers.

4 Forbidden Configurations

We call the next result the *forbidden configuration* (FC) theorem. The proof of the theorem is illustrated in Figure 3.

**Theorem 4.1.** Two different producers cannot have two distinct offspring in common.

**Proof.** Suppose \(x_1 \neq x_2\) both produce \(y_1 \neq y_2\) where \(\Rightarrow\) means isomorphic. Then there exist \(a_1, a_2, b_1, b_2\) such that \(y_1 = x_1 \uparrow a_1, y_2 = x_1 \uparrow a_2, y_1 = x_2 \uparrow b_1, y_2 = x_2 \uparrow b_2\).
Figure 3

\[ b_1, y_2 = x_2 \uparrow b_2. \] Since \( y_1 \neq y_2, a_1 \neq a_2 \) in the sense that the links of \( a_1 \) are not the same as the links of \( a_2 \). Similarly, \( b_1 \neq b_2 \). Since \( x_1 \neq x_2 \), we have that \( a_1 \neq b_1 \). Similarly, \( a_2 \neq b_2 \). Since \( b_1 \in y_1 \) we have that \( b_1 \in x_1 \). Since \( b_2 \in y_2 \) we have that \( b_2 \in x_1 \). Hence, \( \{b_1, b_2\} \subseteq x_1 \) and similarly \( \{a_1, a_2\} \subseteq x_2 \). Since \( a_1 \notin x_1 \) we conclude that \( a_1 \neq b_1 \). Hence, \( \{a_1, b_1, b_2\} \subseteq y_1 \) and similarly \( \{a_1, a_2, b_2\} \subseteq y_2 \). Let \( z_1 = y_1 \setminus \{a_1, b_1\} \) and \( z_2 = y_2 \setminus \{a_2, b_2\} \). Then \( z_1 \neq z_2 \) because \( b_2 \in z_1 \) and \( b_2 \notin z_2 \). Now \( x_1 = z_1 \uparrow b_1 \) and \( x_1 = z_2 \uparrow b_2 \). Similarly, \( x_2 = z_1 \uparrow a_1 \) and \( x_2 = z_2 \uparrow a_2 \). We conclude that \( x_1 \) and \( x_2 \) are common offspring of distinct producers \( z_1 \) and \( z_2 \). Of course,

\[
\text{card} (z_1) = \text{card} (z_2) = \text{card} (x_1) - 1
\]

We can continue this process until we obtain distinct producers of cardinality 2 at which point we have a contradiction.

We now present our main result.

**Theorem 4.2.** If a QSGP \( \rho_n \) is generated by isometries \( U_n : K_n \to K_{n+1} \) then the corresponding q-measures \( \mu_n \) are classical probability measures, \( n = 1, 2, \ldots \).

**Proof.** Suppose \( \rho_n \) is generated by isometries \( U_n : K_n \to K_{n+1} \). Letting \( \omega = x_1x_2 \cdots x_n, \omega' = x'_1x'_2 \cdots x'_n \) be \( n \)-paths in \( \Omega_n \) with \( \omega \neq \omega' \) we shall show that \( D_n(\omega, \omega') = 0 \). If \( x_n \neq x'_n \), then by (3.5) we have \( D_n(\omega, \omega') = 0 \) so suppose that \( x_n = x'_n \). Assume that \( x_{n-1} \neq x'_{n-1} \) so \( x_n \) is a common offspring of
the distinct producers \(x_{n-1}, x'_{n-1}\). By Theorem 4.1, \(x_n\) is the only common offspring of \(x_{n-1}, x'_{n-1}\) so by the compatibility condition we have

\[
\overline{a(x_{n-1} \rightarrow x_n)} a(x'_{n-1} \rightarrow x_n) = \langle U_{n-1} e_{x_{n-1}}^{n-1}, e_{x_n}^n \rangle \langle e_{x_n}^n, U_{n-1} e_{x_{n-1}}^{n-1} \rangle = \sum_{y \in P_n} \langle U_{n-1} e_{x_{n-1}}^{n-1}, e_y^n \rangle \langle e_y^n, U_{n-1} e_{x'_{n-1}}^{n-1} \rangle = \langle U_{n-1} e_{x_{n-1}}^{n-1}, U_{n-1} e_{x'_{n-1}}^{n-1} \rangle = 0
\]

It follows that \(a(x_{n-1} \rightarrow x_n) = 0\) or \(a(x'_{n-1} \rightarrow x_n) = 0\). Applying (3.3) we conclude that \(a(\omega) = 0\) or \(a(\omega') = 0\) and hence, by (3.5) \(D_n(\omega, \omega') = 0\). If \(x_{n-1} = x'_{n-1}\), since \(\omega \neq \omega'\) we will eventually have a largest \(r \in \mathbb{N}\) such that \(x_r \neq x'_r, 2 \leq r \leq n - 2\). We now proceed as before to obtain \(D_n(\omega, \omega') = 0\).

It follows from (3.6) that \(\langle \rho_{n e_{\omega}^n}, e_{\omega'}^n \rangle = 0\). Hence, if \(A \in A_n\) we have

\[
\mu_n(A) = \sum \left\{ \langle \rho_{n e_{\omega}^n}, e_{\omega'}^n \rangle : \omega \Rightarrow, \omega' \Rightarrow \subseteq A \right\} = \sum \{ \rho \omega_{e_{\omega}^n}, e_{\omega'}^n \} : \omega \Rightarrow \subseteq A \}
\]

We conclude that \(\mu_n\) is a classical probability measure, \(n = 1, 2, \ldots\) \(\square\)

If \(\mu_n\) is a classical probability measure, then there is no interference between events and the QSGP is classical. We conclude that if a QSGP is isometrically generated, then it is classical.

References


