Models for Discrete Quantum Gravity

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MODELS FOR
DISCRETE QUANTUM GRAVITY

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Abstract
We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of \( q \)-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is “quantized” to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

1 Introduction

In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) \( \rho_n, n = 1, 2, \ldots \). Instead of requiring that \( \rho_n \) be a state on a Hilbert space \( H_n \), we require that \( \rho_n \) be a \( q \)-probability operator on \( H_n \). This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring \( \text{tr}(\rho_n) = 1 \), this condition normalizes the
corresponding quantum measure. By superimposing a concrete DQP on a
classical sequential growth process we obtain a model for discrete quantum
gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of
$q$-probability operators is a convex set and find its set of extreme elements.
We discuss the property of consistency for a DQP and introduce the so-
called quadratic algebra of suitable sets. The suitable sets are those on
which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process
(CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets
are discussed. In Section 4 we show how to “quantize” a CSGP to obtain
a quantum sequential growth process (QSGP). Some results concerning the
consistency of a DQP are given. Finally, Section 5 provides two methods for
constructing examples of QSGP.

2 Discrete Quantum Processes

Let $(\Omega, A, \nu)$ be a probability space and let

$$H = L_2(\Omega, A, \nu) = \left\{ f : \Omega \to \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\}$$

be the corresponding Hilbert space. Let $A_1 \subset A_2 \subset \cdots \subset A$ be an increasing
sequence of sub $\sigma$-algebras of $A$ that generate $A$ and let $\nu_n = \nu | A_n$ be the
restriction of $\nu$ to $A_n$, $n = 1, 2, \ldots$. Then $H_n = L_2(\Omega, A_n, \nu_n)$ forms an
increasing sequence of closed subspaces of $H$ called a filtration of $H$. A
bounded operator $T$ on $H_n$ will also be considered as a bounded operator
on $H$ by defining $Tf = 0$ for all $f \in H_n$. We denote the characteristic
function $\chi_{\Omega}$ of $\Omega$ by $1$. Of course, $\|1\| = 1$ and $\langle 1, f \rangle = \int f \, d\nu$ for every
$f \in H$. A $q$-probability operator is a bounded positive operator $\rho$ on $H$ that
satisfies $\langle \rho 1, 1 \rangle = 1$. Denote the set of $q$-probability operators on $H$
and $H_n$ by $Q(H)$ and $Q(H_n)$, respectively. Since $1 \in H_n$, if $\rho \in Q(H_n)$ by our
previous convention, $\rho \in Q(H)$. Notice that a positive operator $\rho \in Q(H)$ if
and only if $\|\rho^{1/2} 1\| = 1$ where $\rho^{1/2}$ is the unique positive square root of $\rho$.

A rank 1 element of $Q(H)$ is called a pure $q$-probability operator. Thus
$\rho \in Q(H)$ is pure if and only if $\rho$ has the form $\rho = |\psi\rangle \langle \psi|$ for some $\psi \in H$
satisfying

$$|\langle 1, \psi \rangle| = \left| \int \psi \, d\nu \right| = 1$$

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We then call $\psi$ a \textit{q-probability vector} and we denote the set of q-probability vectors by $\mathcal{V}(H)$ and the set of pure q-probability operators by $\mathcal{Q}_p(H)$. Notice that if $\psi \in \mathcal{V}(H)$, then $\|\psi\| \geq 1$ and $\|\psi\| = 1$ if and only if $\psi = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|c| = 1$. Two operators $\rho_1, \rho_2 \in \mathcal{Q}(H)$ are orthogonal if $\rho_1 \rho_2 = 0$.

**Theorem 2.1.** (i) $\mathcal{Q}(A)$ is a convex set and $\mathcal{Q}_p(H)$ is its set of extreme elements. (ii) $\rho \in \mathcal{Q}(H)$ is of trace class if and only if there exists a sequence of mutually orthogonal $\rho_i \in \mathcal{Q}_p(H)$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$ such that $\rho = \sum \alpha_i \rho_i$ in the strong operator topology. The $\rho_i$ are unique if and only if the corresponding $\alpha_i$ are distinct.

**Proof.** (i) If $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$, then $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ is a positive operator and

$$\langle \rho 1, 1 \rangle = \langle (\lambda \rho + (1 - \lambda) \rho_2)1, 1 \rangle = \lambda \langle \rho_1, 1 \rangle + (1 - \lambda) \langle \rho_2, 1 \rangle = 1$$

Hence, $\rho \in \mathcal{Q}(H)$ so $\mathcal{Q}(H)$ is a convex set. Suppose $\rho \in \mathcal{Q}_p(H)$ and $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ where $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$. If $\rho_1 \neq \rho_2$, then rank$(\rho) \neq 1$ which is a contradiction. Hence, $\rho_1 = \rho_2$ so $\rho$ is an extreme element of $\mathcal{Q}(H)$. Conversely, suppose $\rho \in \mathcal{Q}(H)$ is an extreme element. If the cardinality of the spectrum $|\sigma(\rho)| > 1$, then by the spectral theorem $\rho = \rho_1 + \rho_2$ where $\rho_1, \rho_2 \neq 0$ are positive and $\rho_1 \neq \alpha \rho_2$ for $\alpha \in \mathbb{C}$. If $\rho_1 \rho_2 \neq 0$, then $\langle \rho_1, 1 \rangle, \langle \rho_2, 1 \rangle \neq 0$ and we can write

$$\rho = \frac{\rho_1}{\langle \rho_1, 1 \rangle} + \frac{\rho_2}{\langle \rho_2, 1 \rangle}$$

Now $\langle \rho_1, 1 \rangle^{-1} \rho_1, \langle \rho_2, 1 \rangle^{-1} \rho_2 \in \mathcal{Q}(H)$ and

$$\langle \rho_1, 1 \rangle + \langle \rho_2, 1 \rangle = \langle \rho_1, 1 \rangle = 1$$

which is a contradiction. Hence, $\rho_1 = 0$ or $\rho_2 = 0$. Without loss of generality suppose that $\rho_2 = 0$. We can now write

$$\rho = \frac{1}{2} \rho_1 + \frac{1}{2} (\rho_1 + 2 \rho_2)$$

Now $\rho_1 \neq 0$, $(\rho_1 + 2 \rho_2) \neq 0$ and as before we get a contradiction. We conclude that $|\sigma(\rho)| = 1$. Hence, $\rho = \alpha P$ where $P$ is a projection and $\alpha > 0$. If rank$(P) > 1$, then $P = P_1 + P_2$ where $P_1$ and $P_2$ are orthogonal nonzero projections so $\rho = \alpha P_1 + \alpha P_2$. Proceeding as before we obtain a contradiction. Hence, rank$(P) = 1$ so $\rho = \alpha P$ is pure. (ii) This follows from the spectral theorem. \hfill \square
Let \( \{H_n : n = 1,2,\ldots\} \) be a filtration of \( H \) and let \( \rho_n \in Q(H_n) \), \( n = 1,2,\ldots \). The \( n \)-decoherence functional \( D_n : \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C} \) defined by

\[
D_n(A,B) = \langle \rho_n \chi_B, \chi_A \rangle
\]
gives a measure of the interference between \( A \) and \( B \) when the system is described by \( \rho_n \). It is clear that \( D_n(\Omega_n, \Omega_n) = 1 \), \( D_n(A,B) = D_n(B,A) \) and \( A \mapsto D_n(A,B) \) is a complex measure for all \( B \in \mathcal{A}_n \). It is also well-known that if \( A_1,\ldots,A_r \in \mathcal{A}_n \) then the matrix with entries \( D_n(A_j,A_k) \) is positive semidefinite. We define the map \( \mu_n : \mathcal{A}_n \rightarrow \mathbb{R}^+ \) by

\[
\mu_n(A) = D_n(A,A) = \langle \rho_n \chi_A, \chi_A \rangle
\]

Notice that \( \mu_n(\Omega_n) = 1 \). Although \( \mu_n \) is not additive, it does satisfy the grade-2 additivity condition: if \( A, B, C \in \mathcal{A}_n \) are mutually disjoint, then

\[
\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)
\]

(2.1)

We say that \( \rho_{n+1} \) is consistent with \( \rho_n \) if \( D_{n+1}(A,B) = D_n(A,B) \) for all \( A, B \in \mathcal{A}_n \). We call the sequence \( \rho_n \), \( n = 1,2,\ldots \), consistent if \( \rho_{n+1} \) is consistent with \( \rho_n \) for \( n = 1,2,\ldots \). Of course, if the sequence \( \rho_n \), \( n = 1,2,\ldots \), is consistent, then \( \mu_{n+1}(A) = \mu_n(A) \forall A \in \mathcal{A}_n \), \( n = 1,2,\ldots \). A discrete quantum process (DQP) is a consistent sequence \( \rho_n \in Q(H_n) \) for a filtration \( H_n, n = 1,2,\ldots \). A DQP \( \rho_n \) is pure if \( \rho_n \in Q_p(H_n), n = 1,2,\ldots \).

If \( \rho_n \) is a DQP, then the corresponding maps \( \mu_n : \mathcal{A}_n \rightarrow \mathbb{R}^+ \) have the form

\[
\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \| \rho^{1/2}_n \chi_A \|^2
\]

Now \( A \mapsto \rho^{1/2}_n \chi_A \) is a vector-valued measure on \( \mathcal{A}_n \). We conclude that \( \mu_n \) is the squared norm of a vector-valued measure. In particular, if \( \rho_n = |\psi_n\rangle \langle \psi_n| \) is a pure DQP, then \( \mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2 \) so \( \mu_n \) is the squared modulus of the complex-valued measure \( A \mapsto \langle \psi_n, \chi_A \rangle \).

For a DQP \( \rho_n \in Q(H_n) \), we say that a set \( A \in \mathcal{A} \) is suitable if \( \lim \langle \rho_j \chi_A, \chi_A \rangle \) exists and is finite and in this case we define \( \mu(A) \) to be the limit. We denote the set of suitable sets by \( \mathcal{S}(\rho_n) \). If \( A \in \mathcal{A}_n \) then

\[
\lim \langle \rho_j \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle
\]

so \( A \in \mathcal{S}(\rho_n) \) and \( \mu(A) = \mu_n(A) \). This shows that the algebra \( \mathcal{A}_0 = \cup \mathcal{A}_n \subseteq \mathcal{S}(\rho_n) \). In particular, \( \Omega \in \mathcal{S}(\rho_n) \) and \( \mu(\Omega) = 1 \). In general, \( \mathcal{S}(\rho_n) \neq \mathcal{A} \) and \( \mu \)
may not have a well-behaved extension from \( A_0 \) to all of \( A \) \([2,7]\). A subset \( B \) of \( A \) is a quadratic algebra if \( \emptyset, \Omega \in B \) and whenever \( A,B,C \in B \) are mutually disjoint with \( A \cup B, A \cup C, B \cup C \in B \), we have \( A \cup B \cup C \in B \). For a quadratic algebra \( B \), a q-measure is a map \( \mu_0: B \to \mathbb{R}^+ \) that satisfies the grade-2 additivity condition \((2.1)\). Of course, an algebra of sets is a quadratic algebra and we conclude that \( \mu_n: A_n \to \mathbb{R}^+ \) is a q-measure. It is not hard to show that \( S(\rho_n) \) is a quadratic algebra and \( \mu: S(\rho_n) \to \mathbb{R}^+ \) is a q-measure on \( S(\rho_n) \) \([3]\).

### 3 Classical Sequential Growth Processes

A partially ordered set (poset) is a set \( x \) together with an irreflexive, transitive relation \( < \) on \( x \). In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let \( \mathcal{P}_n \) be the collection of all posets with cardinality \( n \), \( n = 1,2,\ldots \). If \( x \in \mathcal{P}_n \), \( y \in \mathcal{P}_{n+1} \), then \( x \) produces \( y \) if \( y \) is obtained from \( x \) by adjoining a single new element to \( x \) that is maximal in \( y \). We also say that \( x \) is a producer of \( y \) and \( y \) is an offspring of \( x \). If \( x \) produces \( y \) we write \( x \to y \). We denote the set of offspring of \( x \) by \( x \to \) and for \( A \subseteq \mathcal{P}_n \) we use the notation

\[
A \to = \{ y \in \mathcal{P}_{n+1}: x \to y, x \in A \}
\]

The transitive closure of \( \to \) makes the set of all finite posets \( \mathcal{P} = \cup \mathcal{P}_n \) into a poset.

A path in \( \mathcal{P} \) is a string (sequence) \( x_1, x_2, \ldots \) where \( x_i \in \mathcal{P}_i \) and \( x_i \to x_{i+1} \), \( i = 1,2,\ldots \). An n-path in \( \mathcal{P} \) is a finite string \( x_1 x_2 \cdots x_n \) where again \( x_i \in \mathcal{P}_i \) and \( x_i \to x_{i+1} \). We denote the set of paths by \( \Omega \) and the set of n-paths by \( \Omega_n \). The set of paths whose initial n-path is \( \omega \) is \( \omega_0 \in \Omega_n \) is denoted by \( \omega \Rightarrow \). Thus, if \( \omega_0 = x_1 x_2 \cdots x_n \) then

\[
\omega_0 \Rightarrow = \{ \omega \in \Omega: \omega = x_1, x_2 \cdots x_n y_{n+1} y_{n+2} \cdots \}
\]

If \( x \) produces \( y \) in \( r \) isomorphic ways, we say that the multiplicity of \( x \to y \) is \( r \) and write \( m(x \to y) = r \). For example, in Figure 1, \( m(x \to y) = 3 \). (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)
If \( x \in \mathcal{P} \) and \( a, b \in x \) we say that \( a \) is an \textit{ancestor} of \( b \) and \( b \) is a \textit{successor} of \( a \) if \( a < b \). We say that \( a \) is a \textit{parent} of \( b \) and \( b \) is a \textit{child} of \( a \) if \( a < b \) and there is no \( c \in x \) such that \( a < c < b \). Let \( c = (c_0, c_1, \ldots) \) be a sequence of nonnegative numbers called \textit{coupling constants} \([5, 9]\). For \( r, s \in \mathbb{N} \) with \( r \leq s \), we define

\[
\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}
\]

For \( x \in \mathcal{P}_n \), \( y \in \mathcal{P}_{n+1} \) with \( x \to y \) we define the \textit{transition probability}

\[
p_c(x \to y) = m(x \to y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}
\]

where \( \alpha \) is the number of ancestors and \( \pi \) the number of parents of the adjoined maximal element in \( y \) that produces \( y \) from \( x \). It is shown in \([5, 9]\) that \( p_c(x \to y) \) is a probability distribution in that it satisfies the Markov-sum rule

\[
\sum \{ p_c(x \to y) : y \in \mathcal{P}_{n+1}, x \to y \} = 1
\]

In discrete quantum gravity, the elements of \( \mathcal{P} \) are thought of as causal sets and \( a < b \) is interpreted as \( b \) being in the causal future of \( a \). The distribution \( y \mapsto p_c(x \to y) \) is essentially the most general that is consistent with principles of causality and covariance \([5, 9]\). It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take \( c_k = 1/k! \) \([6, 7]\). The case \( c_k = c^k \) for some \( c > 0 \) has been previously studied and is called a \textit{percolation dynamics} \([5, 6, 8]\).

We call an element \( x \in \mathcal{P} \) a \textit{site} and we sometimes call an \( n \)-path an \( n \)-\textit{universe} and a path a \textit{universe} The set \( \mathcal{P} \) together with the set of transition probabilities \( p_c(x \to y) \) forms a \textit{classical sequential growth process} (CSGP)
which we denote by \((\mathcal{P}, p_c)\) \([4, 5, 6, 8, 9]\). It is clear that \((\mathcal{P}, p_c)\) is a Markov chain and as usual we define the probability of an \(n\)-path \(\omega = y_1y_2 \cdots y_n\) by

\[ p^n_c(\omega) = p_c(y_1 \rightarrow y_2)p_c(y_2 \rightarrow y_3) \cdots p_c(y_{n-1} \rightarrow y_n) \]

Denoting the power set of \(\Omega_n\) by \(2^{\Omega_n}\), \((\Omega_n, 2^{\Omega_n}, p^n_c)\) becomes a probability space where

\[ p^n_c(A) = \sum \{p^n_c(\omega): \omega \in A\} \]

for all \(A \in 2^{\Omega_n}\). The probability of a site \(x \in \mathcal{P}_n\) is

\[ p^n_c(x) = \sum \{p^n_c(\omega): \omega \in \Omega_n, \omega \text{ ends at } x\} \]

Of course, \(x \mapsto p^n_c(x)\) is a probability measure on \(\mathcal{P}_n\) and we have

\[ \sum_{x \in \mathcal{P}_n} p^n_c(x) = 1 \]

**Example 1.** Figure 2 illustrates the first two steps of a CSGP where the 2 indicates the multiplicity \(m(x_3 \rightarrow x_6) = 2\). Table 1 lists the probabilities of the various sites for the general coupling constants \(c_k\) and the particular coupling constants \(c'_k = 1/k!\) where \(d = (c_0 + c_1)(c_0 + 2c_1 + c_2)\).
For $A \subseteq \Omega_n$ we use the notation $$A \Rightarrow = \cup \{\omega \Rightarrow : \omega \in A\}$$

Thus, $A \Rightarrow$ is the set of paths whose initial $n$-paths are elements of $A$. We call $A \Rightarrow$ a cylinder set and define

$$\mathcal{A}_n = \{A \Rightarrow : A \subseteq \Omega_n\}$$

In particular, if $\omega \in \Omega_n$ then the elementary cylinder set $\text{cyl}(\omega)$ is given by $\text{cyl}(\omega) = \omega \Rightarrow$. It is easy to check that the $\mathcal{A}_n$ form an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on $\Omega$ and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of $\Omega$. Also for $A \in \mathcal{C}(\Omega)$ of the form $A = A_1 \Rightarrow$, $A_1 \subseteq \Omega_n$, we define $p_c(A) = p_c^n(A_1)$. It is easy to check that $p_c$ is a well-defined probability measure on $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that $p_c$ has a unique extension to a probability measure $\nu_c$ on the $\sigma$-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space, the increasing sequence of subalgebras $\mathcal{A}_n$ generates $\mathcal{A}$ and that the restriction $\nu_c|\mathcal{A}_n = p_c^n$. Hence, the subspaces $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ form a filtration of the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$.

### 4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP $(P, p_c)$ developed in Section 3. We have seen that the $n$-path Hilbert space $H_n = L_2(\Omega, \mathcal{A}_n, p_c^n)$ forms a filtration of the path Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$. In the sequel, we assume that $p_c^n(\omega) \neq 0$ for every $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then the set of vectors

$$e^n_\omega = p_c^n(\omega)^{1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n$$
form an orthonormal basis for \( H_n, n = 1, 2, \ldots \). For \( A \in \mathcal{A}_n \), notice that \( \chi_A \in H \) with \( \| \chi_A \| = p_n(A)^{1/2} \).

We call a DQP \( \rho_n \in \mathcal{Q}(H_n) \) a quantum sequential growth process (QSGP). We call \( \rho_n \) the local operators and \( \mu_n(A) = D_n(A, A) \) the local \( q \)-measures for the process. If \( \rho = \lim \rho_n \) exists in the strong operator topology, then \( \rho \) is a \( q \)-probability operator on \( H \) called the global operator for the process. If the global operator \( \rho \) exists, then \( \hat{\mu}(A) = \langle \rho \chi_A, \chi_A \rangle \) is a (continuous) \( q \)-measure on \( \mathcal{A} \) that extends \( \mu_n, n = 1, 2, \ldots \). Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators [2, 3, 7]. In this case, we still have the \( q \)-measure \( \mu \) on the quadratic algebra \( \mathcal{S}(\rho_n) \subseteq \mathcal{A} \) that extends \( \mu_n, n = 1, 2, \ldots \). We frequently identify a set \( A \subseteq \Omega_n \) with the corresponding cylinder set \( (A \Rightarrow) \in \mathcal{A}_n \). We then have the \( q \)-measure, also denoted by \( \mu_n \), on \( 2^{\Omega_n} \) defined by \( \mu_n(A) = \mu_n(A \Rightarrow) \). Moreover, we define the \( q \)-measure, again denoted by \( \mu_n \), on \( \mathcal{P}_n \) by

\[
\mu_n(A) = \mu_n \left( \{ \omega \in \Omega_n : \omega \text{ end in } A \} \right)
\]

for all \( A \subseteq \mathcal{P}_n \). In particular, for \( x \in \mathcal{P}_n \) we have

\[
\mu_n(\{ x \}) = \mu_n \left( \{ \omega \in \Omega_n : \omega \text{ ends with } x \} \right)
\]

If \( A \in \mathcal{A}_n \) has the form \( A_1 \Rightarrow \) for \( A_1 \subseteq \Omega_n \) then \( A \in \mathcal{A}_{n+1} \) and \( A = (A_1 \rightarrow) \Rightarrow \) where \( A_1 \rightarrow \subseteq \Omega_{n+1} \). Let \( \rho_n \in \mathcal{Q}(H_n), \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) and let \( D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle, D_{n+1}(A, B) = \langle \rho_{n+1} \chi_B, \chi_A \rangle \) be the corresponding decoherence functionals. Then \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for all \( A, B \subseteq \Omega_n \) we have

\[
D_{n+1} [(A \Rightarrow), (B \Rightarrow)] = D_n (A \Rightarrow, B \Rightarrow) \quad (4.1)
\]

**Lemma 4.1.** For \( \rho_n \in \mathcal{Q}(H_n), \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for all \( \omega, \omega' \in \Omega_n \) we have

\[
D_{n+1} [(\omega \Rightarrow), (\omega' \Rightarrow)] = D_n (\omega \Rightarrow, \omega' \Rightarrow) \quad (4.2)
\]

**Proof.** Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every \( A, B \subseteq \Omega_n \) we have

\[
D_{n+1} [(A \Rightarrow), (B \Rightarrow)] = \sum_{\omega \in A} \sum_{\omega' \in B} D_{n+1} D_{n+1} [(\omega \Rightarrow), (\omega' \Rightarrow)] = \sum_{\omega \in A} \sum_{\omega' \in B} D_n (\omega \Rightarrow, \omega' \Rightarrow) = D_n (A \Rightarrow, B \Rightarrow)
\]

and the result follows from (4.1). \( \square \)
For \( \omega = x_1x_2 \cdots x_n \in \Omega_n \) and \( y \in \mathcal{P}_{n+1} \) with \( x_n \rightarrow y \) we use the notation \( \omega y \in \Omega_{n+1} \) where \( \omega y = x_1x_2 \cdots x_n y \). We also define \( p_c(\omega \rightarrow y) = p_c(x_n \rightarrow y) \) and write \( \omega \rightarrow y \) whenever \( x_n \rightarrow y \).

**Theorem 4.2.** For \( \rho_n \in \mathcal{Q}(H_n) \), \( \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for every \( \omega, \omega' \in \Omega_n \) we have

\[
\langle \rho_n e^{n}_{\omega'}, e^{n}_{\omega} \rangle = \sum_{\substack{x \in \mathcal{P}_{n+1} \ y \in \mathcal{P}_{n+1} \\ \omega' \rightarrow x \ y \rightarrow y}} p_c(\omega' \rightarrow x)^{1/2} p_c(\omega \rightarrow y)^{1/2} \langle \rho_{n+1} e^{n+1}_{\omega',x}, e^{n+1}_{\omega,y} \rangle \quad (4.3)
\]

**Proof.** By Lemma 4.1, \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if (4.2) holds. But

\[
D_n(\omega \Rightarrow, \omega' \Rightarrow) = \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle = \langle \rho_n \chi_{cyl(\omega')}, \chi_{cyl(\omega)} \rangle = p^n_c(\omega')^{1/2} p^n_c(\omega)^{1/2} \langle \rho_n e^n_{\omega'}, e^n_{\omega} \rangle
\]

Moreover, we have

\[
D_{n+1}[(\omega \rightarrow \Rightarrow) (\omega' \rightarrow \Rightarrow)] = \langle \rho_{n+1} \chi_{(\omega' \rightarrow \Rightarrow), (\omega \rightarrow \Rightarrow)} \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\omega' \Rightarrow, \omega \Rightarrow} \rangle
\]

\[
= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{cyl(\omega')}, \chi_{cyl(\omega)} \rangle
\]

\[
= p^n_c(\omega')^{1/2} p^n_c(\omega)^{1/2} \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p_c(\omega' \rightarrow x) p_c(\omega \rightarrow y)^{1/2} \langle \rho_{n+1} e^{n+1}_{\omega',x}, e^{n+1}_{\omega,y} \rangle
\]

The result now follows. \( \square \)

Viewing \( H_n \) as \( L_2(\Omega_n, 2^{\Omega_n}, p^n_c) \) we can write (4.3) in the simple form

\[
\langle \rho_n \chi_{(\omega') \Rightarrow}, \chi_{(\omega) \Rightarrow} \rangle = \langle \rho_{n+1} \chi_{\omega' \rightarrow}, \chi_{\omega \rightarrow} \rangle \quad (4.4)
\]

**Corollary 4.3.** A sequence \( \rho_n \in \mathcal{Q}(H_n) \) is a QSGP if and only if (4.3) or (4.4) hold for every \( \omega, \omega' \in \Omega_n, \ n = 1, 2, \ldots \).
We now consider pure $q$-probability operators. In the following results we again view $H_n$ as $L_2(\Omega_n, 2^{\Omega_n}, p^n_c)$.

**Corollary 4.4.** If $\rho_n \in Q_p(H_n)$, $\rho_{n+1} \in Q_p(H_{n+1})$ with $p_n = |\psi_n\rangle\langle\psi_n|$, $\rho_{n+1} = |\psi_{n+1}\rangle\langle\psi_{n+1}|$, then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have

$$\langle\psi_n, \chi_\omega|\langle\psi_n, \chi_\omega'\rangle = \langle\psi_{n+1}, \chi_{\omega} \rightarrow |\psi_{n+1}, \chi_{\omega'} \rightarrow \rangle$$

(4.5)

**Corollary 4.5.** A sequence $|\psi_n\rangle\langle\psi_n| \in Q_p(H_n)$ is a QSGP if and only if (4.5) holds for every $\omega, \omega' \in \Omega_n$.

We say that $|\psi_{n+1}\rangle \in V(H_{n+1})$ is strongly consistent with $|\psi_n\rangle \in V(H_n)$ if for every $\omega \in \Omega_n$ we have

$$\langle\psi_n, \chi_\omega| = \langle\psi_{n+1}, \chi_{\omega} \rightarrow$$

(4.6)

By (4.5) strong consistency implies the consistency of the corresponding $q$-probability operators.

**Corollary 4.6.** If $|\psi_{n+1}\rangle \in V(H_{n+1})$ is strongly consistent with $|\psi_n\rangle \in V(H_n)$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n| \in Q_p(H_n)$ is a QSGP.

**Lemma 4.7.** If $|\psi_n\rangle \in V(H_n)$ and $|\psi_{n+1}\rangle \in H_{n+1}$ satisfies (4.6) for every $\omega \in \Omega_n$, then $|\psi_{n+1}\rangle \in V(H_{n+1})$.

**Proof.** Since $|\psi_n\rangle \in V(H_n)$ we have by (4.6) that

$$|\langle\psi_{n+1}, 1| = \left| \sum_{\omega \in \Omega_n} \langle\psi_{n+1}, \chi_{\omega} \rightarrow \rangle \right| = \left| \sum_{\omega \in \Omega_n} \langle\psi_n, \chi_\omega| \right| = |\langle\psi_n, 1| = 1$$

The result now follows.

**Corollary 4.8.** If $\|\psi_1\| = 1$ and $|\psi_n\rangle \in H_n$ satisfies (4.6) for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n| \in Q_p(H_n)$ is a QSGP.

**Proof.** Since $\|\psi_1\| = 1$, it follows that $|\psi_1\rangle \in V(H_1)$. By Lemma 4.7, $|\psi_n\rangle \in V(H_n)$, $n = 1, 2, \ldots$. Since (4.6) holds, the result follows from Corollary 4.6.

Another way of writing (4.6) is

$$\sum_{\omega \rightarrow x} p^n_{\omega+1}(\omega x) \psi_{n+1}(\omega x) = p^n_\omega(\omega) \psi_n(\omega)$$

(4.7)

for every $\omega \in \Omega_n$. 

11
5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for discrete quantum gravity. The simplest way to construct a QSGP is to form the constant pure DQP $\rho_n = |1\rangle\langle 1|$, $n = 1, 2, \ldots$. To show that $\rho_n$ is indeed consistent, we have for $\omega \in \Omega_n$ that

$$\sum_{\omega \rightarrow x} p_c^{n+1}(\omega x) = \sum_{\omega \rightarrow x} p_c^n(\omega)p_c(\omega \rightarrow x) = p_c^n(\omega) \sum_{\omega \rightarrow x} p_c(\omega \rightarrow x) = p_c^n(\omega)$$

so consistency follows from (4.7). The corresponding $q$-measures are given by

$$\mu_n(A) = |\langle 1, \chi_A \rangle|^2 = p_c^n(A)^2$$

for every $A \in \mathcal{A}_n$. Hence, $\mu_n$ is the square of the classical measure. Of course, $|1\rangle\langle 1|$ is the global $q$-probability operator for this QSGP and in this case $\mathcal{S}(\rho_n) = \mathcal{A}$. Moreover, we have the global $q$-measure $\mu(A) = \nu_c(A)^2$ for $A \in \mathcal{A}$.

Another simple way to construct a QSGP is to employ Corollary 4.8. In this way we can let $\psi_1 = 1$, $\psi_2$ any vector in $L_2(\Omega_2, 2^{\Omega_2}, p_c^2)$ satisfying

$$\langle \psi_2, \chi\{x_1 x_2\}\rangle + \langle \psi_2 \chi\{x_1 x_3\}\rangle = \langle \psi_1, \chi\{x_1\}\rangle = 1$$

and so on, where $x_1, x_2, x_3$ are given in Figure 2. As a concrete example, let $\psi_1 = 1$,

$$\psi_2 = \frac{1}{2} \left[ p_c^2(x_1 x_2)^{-1} \chi\{x_1 x_2\} + p_c^2(x_1 x_3) \chi\{x_1 x_3\} \right]$$

and in general

$$\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p_c^n(\omega)^{-1} \chi(\omega)$$

The $q$-measure $\mu_1$ is $\mu_1(\{x_1\}) = 1$ and $\mu_2$ is given by

$$\mu_2(\{x_1 x_2\}) = |\langle \psi_2, \chi\{x_1 x_2\}\rangle|^2 = \frac{1}{4}$$

$$\mu_2(\{x_1 x_3\}) = |\langle \psi_2, \chi\{x_1 x_3\}\rangle|^2 = \frac{1}{4}$$

$$\mu_2(\Omega_2) = |\langle \psi_2, 1\rangle|^2 = 1$$

In general, we have $\mu_n(A) = |A|^2 / |\Omega_n|^2$ for all $A \in \Omega_n$. Thus $\mu_n$ is the square of the uniform distribution. The global operator does not exist because there is no $q$-measure on $\mathcal{A}$ that extends $\mu_n$ for all $n \in \mathbb{N}$. For $A \in \mathcal{A}$ we have

$$\langle \psi_n, \chi_A \rangle = \int \psi_n \chi_A d\nu_c = \frac{|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}$$
Letting \( \rho_n = |\psi_n\rangle \langle \psi_n| \) we conclude that \( A \in S(\rho_n) \) if and only if

\[
\lim_{n \to \infty} \frac{|A \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}|}{|\Omega_n|}
\]

exists. For example, if \( |A| < \infty \) then for \( n \) sufficiently large we have

\[
|A \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}| = |A|
\]

so \( A \in S(\rho_n) \) and \( \mu(A) = 0 \). In a similar way if \( |A| < \infty \) then for the complement \( A' \), if \( n \) is sufficiently large we have

\[
|A' \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}| = |\Omega_n| - |A|
\]

so \( A' \in S(\rho_n) \) with \( \mu(A') = 1 \).

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let \( \alpha_\omega \in \mathbb{C}, \omega \in \Omega_n \) satisfy

\[
\left| \sum_{\omega \in \Omega_n} \alpha_\omega p^n_c(\omega)^{1/2} \right| = 1 \tag{5.1}
\]

and let \( \rho_n \) be the operator on \( H_n \) satisfying

\[
\langle \rho_n e^n_\omega, e^n_{\omega'} \rangle = \alpha_\omega \overline{\alpha_{\omega'}} \tag{5.2}
\]

Then \( \rho_n \) is a positive operator and by (5.1), (5.2) we have

\[
\langle \rho_n 1, 1 \rangle = \left\langle \rho_n \sum_\omega p^n_c(\omega)^{1/2} e^n_\omega, \sum_{\omega'} p^n_c(\omega')^{1/2} e^n_{\omega'} \right\rangle = \sum_{\omega,\omega'} p^n_c(\omega)^{1/2} p^n_c(\omega')^{1/2} \langle \rho_n e^n_\omega, e^n_{\omega'} \rangle = \left| \sum_{\omega} \alpha_\omega p^n_c(\omega)^{1/2} \alpha_{\omega'} \right|^2 = 1
\]

Hence, \( \rho_n \in Q(H_n) \). Now

\[
\Omega_{n+1} = \{\omega x: \omega \in \Omega_n, x \in P_{n+1}, \omega \to x\}
\]
and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy
\[
\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} p_c^{n+1}(\omega x)^{1/2} \right| = 1
\]
Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying
\[
\langle \rho_{n+1} e_{\omega x}^{n+1}, e_{\omega'x'}^{n+1} \rangle = \beta_{\omega'x'} \overline{\beta}_{\omega x}
\] (5.3)
As before, we have that $\rho_{n+1} \in Q(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** The operator $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have
\[
\alpha_{\omega'} \overline{\alpha}_{\omega} = \sum_{x' \in P_{n+1}} \beta_{\omega'x'} p_c(\omega' \rightarrow x')^{1/2} \sum_{x \in P_{n+1}} \overline{\beta}_{\omega x} p_c(\omega \rightarrow x)^{1/2}
\] (5.4)
A sufficient condition for (5.4) to hold is
\[
\sum_{x \in P_{n+1}} \beta_{\omega x} p_c(\omega \rightarrow x)^{1/2} = \alpha_{\omega}
\] (5.5)

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** Let $\rho_n \in Q(H_n)$ be defined by (5.2) and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by (5.3). If (5.5) holds, then $\rho_{n+1} \in Q(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_n$.

The next result gives the general construction.

**Corollary 5.3.** Let $\rho_1 = I \in Q(H_1)$ and define $\rho_n \in Q(H_n)$ inductively by (5.3). Then $\rho_n$ is a QSGP.

**References**


