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MODELS FOR DISCRETE QUANTUM GRAVITY

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Abstract

We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of \( q \)-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is “quantized” to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

1 Introduction

In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) \( \rho_n, n = 1, 2, \ldots \). Instead of requiring that \( \rho_n \) be a state on a Hilbert space \( H_n \), we require that \( \rho_n \) be a \( q \)-probability operator on \( H_n \). This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring \( \text{tr}(\rho_n) = 1 \), this condition normalizes the
corresponding quantum measure. By superimposing a concrete DQP on a
classical sequential growth process we obtain a model for discrete quantum
gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of 
$q$-probability operators is a convex set and find its set of extreme elements.
We discuss the property of consistency for a DQP and introduce the so-
called quadratic algebra of suitable sets. The suitable sets are those on
which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process
(CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets
are discussed. In Section 4 we show how to “quantize” a CSGP to obtain
a quantum sequential growth process (QSGP). Some results concerning the
consistency of a DQP are given. Finally, Section 5 provides two methods for
constructing examples of QSGP.

2 Discrete Quantum Processes

Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and let

$$H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f: \Omega \rightarrow \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\}$$

be the corresponding Hilbert space. Let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$ be an increasing
sequence of sub $\sigma$-algebras of $\mathcal{A}$ that generate $\mathcal{A}$ and let $\nu_n = \nu \mid \mathcal{A}_n$ be the
restriction of $\nu$ to $\mathcal{A}_n$, $n = 1, 2, \ldots$. Then $H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)$ forms an
increasing sequence of closed subspaces of $H$ called a filtration of $H$. A
bounded operator $T$ on $H_n$ will also be considered as a bounded operator
on $H$ by defining $Tf = 0$ for all $f \in H_n^\perp$. We denote the characteristic
function $\chi_{\Omega}$ of $\Omega$ by 1. Of course, $\|1\| = 1$ and $\langle 1, f \rangle = \int f \, d\nu$ for every
$f \in H$. A $q$-probability operator is a bounded positive operator $\rho$ on $H$ that
satisfies $\langle \rho 1, 1 \rangle = 1$. Denote the set of $q$-probability operators on $H$ and
$H_n$ by $\mathcal{Q}(H)$ and $\mathcal{Q}(H_n)$, respectively. Since $1 \in H_n$, if $\rho \in \mathcal{Q}(H_n)$ by our
previous convention, $\rho \in \mathcal{Q}(H)$. Notice that a positive operator $\rho \in \mathcal{Q}(H)$ if
and only if $\|\rho^{1/2}1\| = 1$ where $\rho^{1/2}$ is the unique positive square root of $\rho$.

A rank 1 element of $\mathcal{Q}(H)$ is called a pure $q$-probability operator. Thus
$\rho \in \mathcal{Q}(H)$ is pure if and only if $\rho$ has the form $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in H$
satisfying

$$\|\langle 1, \psi \rangle\| = \left| \int \psi d\nu \right| = 1$$
We then call $\psi$ a $q$-probability vector and we denote the set of $q$-probability vectors by $\mathcal{V}(H)$ and the set of pure $q$-probability operators by $\mathcal{Q}_p(H)$. Notice that if $\psi \in \mathcal{V}(H)$, then $\|\psi\| \geq 1$ and $\|\psi\| = 1$ if and only if $\psi = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Two operators $\rho_1, \rho_2 \in \mathcal{Q}(H)$ are orthogonal if $\rho_1 \rho_2 = 0$.

**Theorem 2.1.** (i) $\mathcal{Q}(A)$ is a convex set and $\mathcal{Q}_p(H)$ is its set of extreme elements. (ii) $\rho \in \mathcal{Q}(H)$ is of trace class if and only if there exists a sequence of mutually orthogonal $\rho_i \in \mathcal{Q}_p(H)$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$ such that $\rho = \sum \alpha_i \rho_i$ in the strong operator topology. The $\rho_i$ are unique if and only if the corresponding $\alpha_i$ are distinct.

**Proof.** (i) If $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$, then $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ is a positive operator and

$$\langle \rho_1, 1 \rangle = \langle (\lambda \rho + (1 - \lambda) \rho_2)1, 1 \rangle = \lambda \langle \rho_1, 1 \rangle + (1 - \lambda) \langle \rho_2, 1 \rangle = 1$$

Hence, $\rho \in \mathcal{Q}(H)$ so $\mathcal{Q}(H)$ is a convex set. Suppose $\rho \in \mathcal{Q}_p(H)$ and $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ where $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$. If $\rho_1 \neq \rho_2$, then rank($\rho$) $\neq 1$ which is a contradiction. Hence, $\rho_1 = \rho_2$ so $\rho$ is an extreme element of $\mathcal{Q}(H)$. Conversely, suppose $\rho \in \mathcal{Q}(H)$ is an extreme element. If the cardinality of the spectrum $|\sigma(\rho)| > 1$, then by the spectral theorem $\rho = \rho_1 + \rho_2$ where $\rho_1, \rho_2 \neq 0$ are positive and $\rho_1 \neq \alpha \rho_2$ for $\alpha \in \mathbb{C}$. If $\rho_1, \rho_2 \neq 1$, then $\langle \rho_1, 1 \rangle, \langle \rho_2, 1 \rangle \neq 0$ and we can write

$$\rho = \langle \rho_1, 1 \rangle \frac{\rho_1}{\langle \rho_1, 1 \rangle} + \langle \rho_2, 1 \rangle \frac{\rho_2}{\langle \rho_2, 1 \rangle}$$

Now $\langle \rho_1, 1 \rangle^{-1} \rho_1, \langle \rho_2, 1 \rangle^{-1} \rho_2 \in \mathcal{Q}(H)$ and

$$\langle \rho_1, 1 \rangle + \langle \rho_2, 1 \rangle = \langle \rho_1, 1 \rangle = 1$$

which is a contradiction. Hence, $\rho_1, 1 = 0$ or $\rho_2, 1 = 0$. Without loss of generality suppose that $\rho_2, 1 = 0$. We can now write

$$\rho = \frac{1}{2} \rho_1 + \frac{1}{2} (\rho_1 + 2 \rho_2)$$

Now $\rho_1, 1 \neq 0$, $(\rho_1 + 2 \rho_2), 1 \neq 0$ and as before we get a contradiction. We conclude that $|\sigma(\rho)| = 1$. Hence, $\rho = \alpha P$ where $P$ is a projection and $\alpha > 0$. If rank($P$) $> 1$, then $P = P_1 + P_2$ where $P_1$ and $P_2$ are orthogonal nonzero projections so $\rho = \alpha P_1 + \alpha P_2$. Proceeding as before we obtain a contradiction. Hence, rank($P$) $= 1$ so $\rho = \alpha P$ is pure. (ii) This follows from the spectral theorem. \qed
Let \( \{H_n : n = 1, 2, \ldots \} \) be a filtration of \( H \) and let \( \rho_n \in \mathcal{Q}(H_n) \), \( n = 1, 2, \ldots \). The \textit{n-decoherence functional} \( D_n : \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C} \) defined by

\[
D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle
\]

gives a measure of the interference between \( A \) and \( B \) when the system is described by \( \rho_n \). It is clear that \( D_n(\Omega_n, \Omega_n) = 1 \), \( D_n(A, B) = D_n(B, A) \) and \( A \mapsto D_n(A, B) \) is a complex measure for all \( B \in \mathcal{A}_n \). It is also well-known that if \( A_1, \ldots, A_r \in \mathcal{A}_n \) then the matrix with entries \( D_n(A_j, A_k) \) is positive semidefinite. We define the map \( \mu_n : \mathcal{A}_n \rightarrow \mathbb{R}^+ \) by

\[
\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle
\]

Notice that \( \mu_n(\Omega_n) = 1 \). Although \( \mu_n \) is not additive, it does satisfy the \textit{grade-2 additivity condition}: if \( A, B, C \in \mathcal{A}_n \) are mutually disjoint, then

\[
\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)
\]  

(2.1)

We say that \( \rho_{n+1} \) is \textit{consistent} with \( \rho_n \) if \( D_{n+1}(A, B) = D_n(A, B) \) for all \( A, B \in \mathcal{A}_n \). We call the sequence \( \rho_n, n = 1, 2, \ldots, \) \textit{consistent} if \( \rho_{n+1} \) is consistent with \( \rho_n \) for \( n = 1, 2, \ldots \). Of course, if the sequence \( \rho_n, n = 1, 2, \ldots, \) is consistent, then \( \mu_{n+1}(A) = \mu_n(A) \) \( \forall A \in \mathcal{A}_n, n = 1, 2, \ldots \). A \textit{discrete quantum process} (DQP) is a consistent sequence \( \rho_n \in \mathcal{Q}(H_n) \) for a filtration \( H_n, n = 1, 2, \ldots \). A DQP \( \rho_n \) is \textit{pure} if \( \rho_n \in \mathcal{Q}_p(H_n), n = 1, 2, \ldots \).

If \( \rho_n \) is a DQP, then the corresponding maps \( \mu_n : \mathcal{A}_n \rightarrow \mathbb{R}^+ \) have the form

\[
\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \|\rho_n^{1/2} \chi_A\|^2
\]

Now \( A \mapsto \rho_n^{1/2} \chi_A \) is a vector-valued measure on \( \mathcal{A}_n \). We conclude that \( \mu_n \) is the squared norm of a vector-valued measure. In particular, if \( \rho_n = |\psi_n\rangle \langle \psi_n| \) is a pure DQP, then \( \mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2 \) so \( \mu_n \) is the squared modulus of the complex-valued measure \( A \mapsto \langle \psi_n, \chi_A \rangle \).

For a DQP \( \rho_n \in \mathcal{Q}(H_n) \), we say that a set \( A \in \mathcal{A} \) is \textit{suitable} if \( \lim \langle \rho_j \chi_A, \chi_A \rangle \) exists and is finite and in this case we define \( \mu(A) \) to be the limit. We denote the set of suitable sets by \( \mathcal{S}(\rho_n) \). If \( A \in \mathcal{A}_n \) then

\[
\lim \langle \rho_j \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle
\]

so \( A \in \mathcal{S}(\rho_n) \) and \( \mu(A) = \mu_n(A) \). This shows that the algebra \( \mathcal{A}_0 = \cup \mathcal{A}_n \subseteq \mathcal{S}(\rho_n) \). In particular, \( \Omega \in \mathcal{S}(\rho_n) \) and \( \mu(\Omega) = 1 \). In general, \( \mathcal{S}(\rho_n) \neq \mathcal{A} \) and \( \mu \)
may not have a well-behaved extension from $\mathcal{A}_0$ to all of $\mathcal{A}$ [2, 7]. A subset $\mathcal{B}$ of $\mathcal{A}$ is a quadratic algebra if $\emptyset, \Omega \in \mathcal{B}$ and whenever $A, B, C \in \mathcal{B}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{B}$, we have $A \cup B \cup C \in \mathcal{B}$. For a quadratic algebra $\mathcal{B}$, a q-measure is a map $\mu_0 : \mathcal{B} \to \mathbb{R}^+$ that satisfies the grade-2 additivity condition (2.1). Of course, an algebra of sets is a quadratic algebra and we conclude that $\mu_n : \mathcal{A}_n \to \mathbb{R}^+$ is a q-measure. It is not hard to show that $\mathcal{S}(\rho_n)$ is a quadratic algebra and $\mu : \mathcal{S}(\rho_n) \to \mathbb{R}^+$ is a q-measure on $\mathcal{S}(\rho_n)$ [3].

3 Classical Sequential Growth Processes

A partially ordered set (poset) is a set $x$ together with an irreflexive, transitive relation $<$ on $x$. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $\mathcal{P}_n$ be the collection of all posets with cardinality $n$, $n = 1, 2, \ldots$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then $x$ produces $y$ if $y$ is obtained from $x$ by adjoining a single new element to $x$ that is maximal in $y$. We also say that $x$ is a producer of $y$ and $y$ is an offspring of $x$. If $x$ produces $y$ we write $x \rightarrow y$. We denote the set of offspring of $x$ by $x \rightarrow$ and for $A \subseteq \mathcal{P}_n$ we use the notation

$$A \rightarrow = \{ y \in \mathcal{P}_{n+1} : x \rightarrow y, x \in A \}$$

The transitive closure of $\rightarrow$ makes the set of all finite posets $\mathcal{P} = \bigcup \mathcal{P}_n$ into a poset.

A path in $\mathcal{P}$ is a string (sequence) $x_1, x_2, \ldots$ where $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$, $i = 1, 2, \ldots$. An $n$-path in $\mathcal{P}$ is a finite string $x_1 x_2 \cdots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$. We denote the set of paths by $\Omega$ and the set of $n$-paths by $\Omega_n$. The set of paths whose initial $n$-path is $\omega_0 \in \Omega_n$ is denoted by $\omega_0 \Rightarrow$. Thus, if $\omega_0 = x_1 x_2 \cdots x_n$ then

$$\omega_0 \Rightarrow = \{ \omega \in \Omega : \omega = x_1, x_2 \cdots x_n y_{n+1} y_{n+2} \cdots \}$$

If $x$ produces $y$ in $r$ isomorphic ways, we say that the multiplicity of $x \rightarrow y$ is $r$ and write $m(x \rightarrow y) = r$. For example, in Figure 1, $m(x \rightarrow y) = 3$. (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)
If \( x \in \mathcal{P} \) and \( a, b \in x \) we say that \( a \) is an ancestor of \( b \) and \( b \) is a successor of \( a \) if \( a < b \). We say that \( a \) is a parent of \( b \) and \( b \) is a child of \( a \) if \( a < b \) and there is no \( c \in x \) such that \( a < c < b \). Let \( c = (c_0, c_1, \ldots) \) be a sequence of nonnegative numbers called coupling constants [5, 9]. For \( r, s \in \mathbb{N} \) with \( r \leq s \), we define

\[
\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}
\]

For \( x \in \mathcal{P}_n, y \in \mathcal{P}_{n+1} \) with \( x \rightarrow y \) we define the transition probability

\[
p_c(x \rightarrow y) = m(x \rightarrow y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}
\]

where \( \alpha \) is the number of ancestors and \( \pi \) the number of parents of the adjoined maximal element in \( y \) that produces \( y \) from \( x \). It is shown in [5, 9] that \( p_c(x \rightarrow y) \) is a probability distribution in that it satisfies the Markov-sum rule

\[
\sum \{p_c(x \rightarrow y) : y \in \mathcal{P}_{n+1}, x \rightarrow y \} = 1
\]

In discrete quantum gravity, the elements of \( \mathcal{P} \) are thought of as causal sets and \( a < b \) is interpreted as \( b \) being in the causal future of \( a \). The distribution \( y \mapsto p_c(x \rightarrow y) \) is essentially the most general that is consistent with principles of causality and covariance [5, 9]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take \( c_k = 1/k! \) [6, 7]. The case \( c_k = c^k \) for some \( c > 0 \) has been previously studied and is called a percolation dynamics [5, 6, 8].

We call an element \( x \in \mathcal{P} \) a site and we sometimes call an \( n \)-path an \( n \)-universe and a path a universe. The set \( \mathcal{P} \) together with the set of transition probabilities \( p_c(x \rightarrow y) \) forms a classical sequential growth process (CSGP).
which we denote by \((\mathcal{P}, p_c)\) \([4, 5, 6, 8, 9]\). It is clear that \((\mathcal{P}, p_c)\) is a Markov chain and as usual we define the probability of an \(n\)-path \(\omega = y_1y_2 \cdots y_n\) by

\[
p^n_c(\omega) = p_c(y_1 \rightarrow y_2)p_c(y_2 \rightarrow y_3) \cdots p_c(y_{n-1} \rightarrow y_n)
\]

Denoting the power set of \(\Omega_n\) by \(2^{\Omega_n}\), \((\Omega_n, 2^{\Omega_n}, p^n_c)\) becomes a probability space where

\[
p^n_c(A) = \sum \{p^n_c(\omega) : \omega \in A\}
\]

for all \(A \in 2^{\Omega_n}\). The probability of a site \(x \in \mathcal{P}_n\) is

\[
p^n_c(x) = \sum \{p^n_c(\omega) : \omega \in \Omega_n, \omega \text{ ends at } x\}
\]

Of course, \(x \mapsto p^n_c(x)\) is a probability measure on \(\mathcal{P}_n\) and we have

\[
\sum_{x \in \mathcal{P}_n} p^n_c(x) = 1
\]

**Example 1.** Figure 2 illustrates the first two steps of a CSGP where the 2 indicates the multiplicity \(m(x_3 \rightarrow x_6) = 2\). Table 1 lists the probabilities of the various sites for the general coupling constants \(c_k\) and the particular coupling constants \(c'_k = 1/k!\) where \(d = (c_0 + c_1)(c_0 + 2c_1 + c_2)\).

![Figure 2](image-url)
For $A \subseteq \Omega_n$ we use the notation

$$A \Rightarrow = \cup \{\omega \Rightarrow: \omega \in A\}$$

Thus, $A \Rightarrow$ is the set of paths whose initial $n$-paths are elements of $A$. We call $A \Rightarrow$ a cylinder set and define

$$\mathcal{A}_n = \{A \Rightarrow: A \subseteq \Omega_n\}$$

In particular, if $\omega \in \Omega_n$ then the elementary cylinder set $\text{cyl}(\omega)$ is given by $\text{cyl}(\omega) = \omega \Rightarrow$. It is easy to check that the $\mathcal{A}_n$ form an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on $\Omega$ and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of $\Omega$. Also for $A \in \mathcal{C}(\Omega)$ of the form $A = A_1 \Rightarrow$, $A_1 \subseteq \Omega_n$, we define $p_c(A) = p^n_c(A_1)$. It is easy to check that $p_c$ is a well-defined probability measure on $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that $p_c$ has a unique extension to a probability measure $\nu_c$ on the $\sigma$-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space, the increasing sequence of subalgebras $\mathcal{A}_n$ generates $\mathcal{A}$ and that the restriction $\nu_c | \mathcal{A}_n = p^n_c$. Hence, the subspaces $H_n = L^2(\Omega, \mathcal{A}_n, p^n_c)$ form a filtration of the Hilbert space $H = L^2(\Omega, \mathcal{A}, \nu_c)$.

### 4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP $(\mathcal{P}, p_c)$ developed in Section 3. We have seen that the $n$-path Hilbert space $H_n = L^2(\Omega, \mathcal{A}_n, p^n_c)$ forms a filtration of the path Hilbert space $H = L^2(\Omega, \mathcal{A}, \nu_c)$. In the sequel, we assume that $p^n_c(\omega) \neq 0$ for every $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then the set of vectors

$$e^n_\omega = p^n_c(\omega)^{1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n$$

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^{(n)}_c(x_i)$</td>
<td>1</td>
<td>$\frac{c_1}{c_0+c_1}$</td>
<td>$\frac{c_0}{c_0+c_1}$</td>
<td>$\frac{c_1(c_1+c_2)}{d}$</td>
<td>$\frac{c_1}{d}$</td>
<td>$\frac{3c_0c_1}{d}$</td>
<td>$\frac{c_0c_2}{d}$</td>
<td>$\frac{c_1^2}{d}$</td>
</tr>
<tr>
<td>$p^n_c(x_i)$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{14}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{1}{14}$</td>
<td>$\frac{1}{7}$</td>
</tr>
</tbody>
</table>

Table 1
form an orthonormal basis for $H_n$, $n = 1, 2, \ldots$. For $A \in \mathcal{A}_n$, notice that
\[ \chi_A \in H \quad \text{with} \quad \|\chi_A\| = p^n(A)^{1/2}. \]

We call a DQP $\rho_n \in \mathcal{Q}(H_n)$ a quantum sequential growth process (QSGP). We call $\rho_n$ the local operators and $\mu_n(A) = D_n(A, A)$ the local $q$-measures for the process. If $\rho = \lim \rho_n$ exists in the strong operator topology, then $\rho$ is a $q$-probability operator on $H$ called the global operator for the process. If the global operator $\rho$ exists, then $\hat{\mu}(A) = \langle \rho \chi_A, \chi_A \rangle$ is a (continuous) $q$-measure on $\mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators $[2, 3, 7]$. In this case, we still have the $q$-measure $\mu$ on the quadratic algebra $\mathcal{S}(\rho_n) \subseteq \mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. We frequently identify a set $A \subseteq \Omega_n$ with the corresponding cylinder set $(A \Rightarrow) \in \mathcal{A}_n$. We then have the $q$-measure, also denoted by $\mu_n$, on $2^{\Omega_n}$ defined by $\mu_n(A) = \mu_n(A \Rightarrow)$. Moreover, we define the $q$-measure, again denoted by $\mu_n$, on $\mathcal{P}_n$ by
\[ \mu_n(A) = \mu_n\{\omega \in \Omega_n: \omega \text{ end in } A\} \]
for all $A \subseteq \mathcal{P}_n$. In particular, for $x \in \mathcal{P}_n$ we have
\[ \mu_n\{x\} = \mu_n\{\omega \in \Omega_n: \omega \text{ ends with } x\} \]

If $A \in \mathcal{A}_n$ has the form $A_1 \Rightarrow$ for $A_1 \subseteq \Omega_n$ then $A \in \mathcal{A}_{n+1}$ and $A = (A_1 \Rightarrow) \Rightarrow$ where $A_1 \Rightarrow \subseteq \Omega_{n+1}$. Let $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and let
\[ D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle, \quad D_{n+1}(A, B) = \langle \rho_{n+1} \chi_B, \chi_A \rangle \]
be the corresponding decoherence functionals. Then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $A, B \subseteq \Omega_n$ we have
\[ D_{n+1}[\omega \Rightarrow, (\omega' \Rightarrow)] = D_n(A \Rightarrow, B \Rightarrow) \quad (4.1) \]

**Lemma 4.1.** For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $\omega, \omega' \in \Omega_n$ we have
\[ D_{n+1}[\omega \Rightarrow, (\omega' \Rightarrow)] = D_n(\omega \Rightarrow, \omega' \Rightarrow) \quad (4.2) \]

**Proof.** Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every $A, B \subseteq \Omega_n$ we have
\[ D_{n+1}[\omega \Rightarrow, (\omega' \Rightarrow)] = \sum_{\omega \in A} \sum_{\omega' \in B} D_n(A \Rightarrow, B \Rightarrow) \]
and the result follows from (4.1). \(\square\)
For \( \omega = x_1 x_2 \cdots x_n \in \Omega_n \) and \( y \in P_{n+1} \) with \( x_n \rightarrow y \) we use the notation \( \omega y \in \Omega_{n+1} \) where \( \omega y = x_1 x_2 \cdots x_n y \). We also define \( \rho_c(\omega \rightarrow y) = \rho_c(x_n \rightarrow y) \) and write \( \omega \rightarrow y \) whenever \( x_n \rightarrow y \).

**Theorem 4.2.** For \( \rho_n \in Q(H_n) \), \( \rho_{n+1} \in Q(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for every \( \omega, \omega' \in \Omega_n \) we have

\[
\langle \rho_n e^\omega_n, e^\omega_n \rangle = \sum_{x \in P_{n+1}} \sum_{y \in P_{n+1}} p_c(\omega' \rightarrow x)^{1/2} p_c(\omega \rightarrow y)^{1/2} \left\langle \rho_{n+1} e^{\omega' x}_n, e^{\omega y}_n \right\rangle \quad (4.3)
\]

**Proof.** By Lemma 4.1, \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if (4.2) holds. But

\[
D_n(\omega \Rightarrow, \omega' \Rightarrow) = \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle = \langle \rho_n \chi_{\text{cyl}(\omega')}, \chi_{\text{cyl}(\omega)} \rangle
\]

\[
= p^\omega_n(\omega')^{1/2} p^\omega_n(\omega)^{1/2} \langle \rho_{n+1} e^\omega_{n+1}_n, e^\omega_n \rangle
\]

Moreover, we have

\[
D_{n+1} [(\omega \rightarrow) \Rightarrow, (\omega' \rightarrow) \Rightarrow] = \langle \rho_{n+1} \chi(\omega' \rightarrow), \chi(\omega \rightarrow) \rangle
\]

\[
= \sum_{x \in P_{n+1}} \sum_{y \in P_{n+1}} \langle \rho_{n+1} \chi_{\omega' x \rightarrow}, \chi_{\omega y \Rightarrow} \rangle
\]

\[
= \sum_{x \in P_{n+1}} \sum_{y \in P_{n+1}} \langle \rho_{n+1} \chi_{\text{cyl}(\omega' x)}, \chi_{\text{cyl}(\omega y)} \rangle
\]

\[
= \sum_{x \in P_{n+1}} \sum_{y \in P_{n+1}} p^\omega_n(\omega' x)^{1/2} p^\omega_n(\omega y)^{1/2} \left\langle \rho_{n+1} e^{\omega' x}_{n+1}_n, e^{\omega y}_n \right\rangle
\]

\[
= p^\omega_n(\omega')^{1/2} p^\omega_n(\omega)^{1/2} \sum_{x \in P_{n+1}} \sum_{y \in P_{n+1}} p_c(\omega' \rightarrow x) p_c(\omega \rightarrow y)^{1/2} \left\langle \rho_{n+1} e^{\omega' x}_{n+1}_n, e^{\omega y}_{n+1} \right\rangle
\]

The result now follows. \( \square \)

Viewing \( H_n \) as \( L_2(\Omega_n, 2^{\Omega_n}, p^\omega_c) \) we can write (4.3) in the simple form

\[
\langle \rho_n \chi(\omega'), \chi(\omega) \rangle = \langle \rho_{n+1} \chi(\omega' \rightarrow), \chi(\omega \rightarrow) \rangle \quad (4.4)
\]

**Corollary 4.3.** A sequence \( \rho_n \in Q(H_n) \) is a QSGP if and only if (4.3) or (4.4) hold for every \( \omega, \omega' \in \Omega_n, n = 1, 2, \ldots \).
We now consider pure $q$-probability operators. In the following results we again view $H_n$ as $L^2(\Omega_n, \sigma^\Omega_n, p_c^n)$.

**Corollary 4.4.** If $\rho_n \in \mathcal{Q}_p(H_n)$, $\rho_{n+1} \in \mathcal{Q}_p(H_{n+1})$ with $p_n = |\psi_n\rangle\langle\psi_n|$, $\rho_{n+1} = |\psi_{n+1}\rangle\langle\psi_{n+1}|$, then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have

$$\langle\psi_n, \chi_{\{\omega\}}\rangle\langle\chi_{\{\omega'\}}, \psi_n\rangle = \langle\psi_{n+1}, \chi_{\omega\rightarrow}\rangle\langle\chi_{\omega'\rightarrow}, \psi_{n+1}\rangle \quad (4.5)$$

**Corollary 4.5.** A sequence $|\psi_n\rangle\langle\psi_n| \in \mathcal{Q}_p(H_n)$ is a QSGP if and only if (4.5) holds for every $\omega, \omega' \in \Omega_n$.

We say that $\psi_{n+1} \in \mathcal{V}(H_{n+1})$ is strongly consistent with $\psi_n \in \mathcal{V}(H_n)$ if for every $\omega \in \Omega_n$ we have

$$\langle\psi_n, \chi_{\{\omega\}}\rangle = \langle\psi_{n+1}, \chi_{\omega\rightarrow}\rangle \quad (4.6)$$

By (4.5) strong consistency implies the consistency of the corresponding $q$-probability operators.

**Corollary 4.6.** If $\psi_{n+1} \in \mathcal{V}(H_{n+1})$ is strongly consistent with $\psi_n \in \mathcal{V}(H_n)$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n| \in \mathcal{Q}_p(H_n)$ is a QSGP.

**Lemma 4.7.** If $\psi_n \in \mathcal{V}(H_n)$ and $\psi_{n+1} \in H_{n+1}$ satisfies (4.6) for every $\omega \in \Omega_n$, then $|\psi_{n+1}\rangle \in \mathcal{V}(H_{n+1})$.

**Proof.** Since $\psi_n \in \mathcal{V}(H_n)$ we have by (4.6) that

$$|\langle\psi_{n+1}, 1\rangle| = \left| \sum_{\omega \in \Omega_n} \langle\psi_{n+1}, \chi_{\omega\rightarrow}\rangle \right| = \left| \sum_{\omega \in \Omega_n} \langle\psi_n, \chi_{\{\omega\}}\rangle \right| = |\langle\psi_n, 1\rangle| = 1$$

The result now follows.

**Corollary 4.8.** If $||\psi_1|| = 1$ and $\psi_n \in H_n$ satisfies (4.6) for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle\psi_n|$ is a QSGP.

**Proof.** Since $||\psi_1|| = 1$, it follows that $\psi_1 \in \mathcal{V}(H_1)$. By Lemma 4.7, $\psi_n \in \mathcal{V}(H_n)$, $n = 1, 2, \ldots$. Since (4.6) holds, the result follows from Corollary 4.6.

Another way of writing (4.6) is

$$\sum_{\omega \rightarrow x} p_c^{n+1}(\omega x)\psi_{n+1}(\omega x) = p_c^n(\omega)\psi_n(x) \quad (4.7)$$

for every $\omega \in \Omega_n$.  

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5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for
discrete quantum gravity. The simplest way to construct a QSGP is to form
the constant pure DQP $\rho_n = |1\rangle\langle 1|$, $n = 1, 2, \ldots$. To show that $\rho_n$ is indeed
consistent, we have for $\omega \in \Omega_n$ that

$$\sum_{\omega \to x} p_c^{n+1}(\omega x) = \sum_{\omega \to x} p_c^n(\omega)p_c(\omega \to x) = p_c^n(\omega) \sum_{\omega \to x} p_c(\omega \to x) = p_c^n(\omega)$$

so consistency follows from (4.7). The corresponding $q$-measures are given
by

$$\mu_n(A) = |\langle 1, \chi_A \rangle|^2 = p_c^n(A)^2$$

for every $A \in A_n$. Hence, $\mu_n$ is the square of the classical measure. Of
course, $|1\rangle\langle 1|$ is the global $q$-probability operator for this QSGP and in this
case $S(\rho_n) = A$. Moreover, we have the global $q$-measure $\mu(A) = \nu_c(A)^2$ for
$A \in A$.

Another simple way to construct a QSGP is to employ Corollary 4.8. In
this way we can let $\psi_1 = 1$, $\psi_2$ any vector in $L_2(\Omega_2, 2^{\Omega_2}, p_c^2)$ satisfying

$$\langle \psi_1, \chi_{\{x_1x_2\}} \rangle + \langle \psi_2, \chi_{\{x_1x_3\}} \rangle = \langle \psi_1, \chi_{\{x_1\}} \rangle = 1$$

and so on, where $x_1, x_2, x_3$ are given in Figure 2. As a concrete example, let $\psi_1 = 1$,

$$\psi_2 = \frac{1}{2} \left[ p_c^2(x_1x_2)^{-1}\chi_{\{x_1x_2\}} + p_c^2(x_1x_3)\chi_{\{x_1x_3\}} \right]$$

and in general

$$\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p_c^n(\omega)^{-1}\chi_{\{\omega\}}$$

The $q$-measure $\mu_1$ is $\mu_1(\{x_1\}) = 1$ and $\mu_2$ is given by

$$\mu_2(\{x_1x_2\}) = |\langle \psi_2, \chi_{\{x_1x_2\}} \rangle|^2 = \frac{1}{4}$$
$$\mu_2(\{x_1x_3\}) = |\langle \psi_2, \chi_{\{x_1x_3\}} \rangle|^2 = \frac{1}{4}$$
$$\mu_2(\Omega_2) = |\langle \psi_2, 1 \rangle|^2 = 1$$

In general, we have $\mu_n(A) = |A|^2 / |\Omega_n|^2$ for all $A \in \Omega_n$. Thus $\mu_n$ is the square
of the uniform distribution. The global operator does not exist because there
is no $q$-measure on $\mathcal{A}$ that extends $\mu_n$ for all $n \in \mathbb{N}$. For $A \in \mathcal{A}$ we have

$$\langle \psi_n, \chi_A \rangle = \int \psi_n(\chi_A d\nu_c = \frac{|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}$$
Letting $\rho_n = |\psi_n\rangle\langle\psi_n|$ we conclude that $A \in S(\rho_n)$ if and only if
\[
\lim_{n \to \infty} \frac{|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}
\]
events. For example, if $|A| < \infty$ then for $n$ sufficiently large we have
\[
|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |A|
\]
so $A \in S(\rho_n)$ and $\mu(A) = 0$. In a similar way if $|A| < \infty$ then for the complement $A'$, if $n$ is sufficiently large we have
\[
|A' \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |\Omega_n| - |A|
\]
so $A' \in S(\rho_n)$ with $\mu(A') = 1$.

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let $\alpha_\omega \in \mathbb{C}$, $\omega \in \Omega_n$ satisfy
\[
\left| \sum_{\omega \in \Omega_n} \alpha_\omega p^n_c(\omega)^{1/2} \right| = 1 \quad (5.1)
\]
and let $\rho_n$ be the operator on $H_n$ satisfying
\[
\langle \rho_n e^n_\omega, e^n_{\omega'} \rangle = \alpha_\omega \overline{\alpha_{\omega'}} \quad (5.2)
\]
Then $\rho_n$ is a positive operator and by (5.1), (5.2) we have
\[
\langle \rho_n 1, 1 \rangle = \left\langle \rho_n \sum_{\omega} p^n_c(\omega)^{1/2} e^n_\omega, \sum_{\omega'} p^n_c(\omega')^{1/2} e^n_{\omega'} \right\rangle = \sum_{\omega \omega'} p^n_c(\omega)^{1/2} p^n_c(\omega')^{1/2} \langle \rho_n e^n_\omega, e^n_{\omega'} \rangle = \left| \sum_{\omega} p^n_c(\omega)^{1/2} \alpha_\omega \right|^2 = 1
\]
Hence, $\rho_n \in \mathcal{Q}(H_n)$. Now
\[
\Omega_{n+1} = \{\omega x : \omega \in \Omega_n, x \in \mathcal{P}_{n+1}, \omega \rightarrow x\}
\]
and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy
\[
\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} p_c^{n+1}(\omega x)^{1/2} \right| = 1
\]

Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying
\[
\langle \rho_{n+1} e^{n+1}_{\omega x}, e^{n+1}_{\omega' x'} \rangle = \beta_{\omega' x'} \beta_{\omega x} \quad (5.3)
\]

As before, we have that $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** The operator $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have
\[
\alpha_{\omega'} \overline{\alpha_{\omega}} = \sum_{x' \in P_{n+1}} \beta_{\omega' x'} p_c(\omega' \rightarrow x')^{1/2} \sum_{x \in P_{n+1}} \overline{\beta_{\omega x}} p_c(\omega \rightarrow x)^{1/2} \quad (5.4)
\]

A sufficient condition for (5.4) to hold is
\[
\sum_{x \in P_{n+1}} \beta_{\omega x} p_c(\omega \rightarrow x)^{1/2} = \alpha_{\omega} \quad (5.5)
\]

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** Let $\rho_n \in \mathcal{Q}(H_n)$ be defined by (5.2) and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by (5.3). If (5.5) holds, then $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_n$.

The next result gives the general construction.

**Corollary 5.3.** Let $\rho_1 = I \in \mathcal{Q}(H_1)$ and define $\rho_n \in \mathcal{Q}(H_n)$ inductively by (5.3). Then $\rho_n$ is a QSGP.

**References**


