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The final version of this article published in Reports on Mathematical Physics is available online at:
https://doi.org/10.1016/S0034-4877(13)60010-5

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MODELS FOR
DISCRETE QUANTUM GRAVITY

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Abstract

We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of $q$-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is "quantized" to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

1 Introduction

In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) $\rho_n$, $n = 1, 2, \ldots$. Instead of requiring that $\rho_n$ be a state on a Hilbert space $H_n$, we require that $\rho_n$ be a $q$-probability operator on $H_n$. This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring $\text{tr}(\rho_n) = 1$, this condition normalizes the
corresponding quantum measure. By superimposing a concrete DQP on a classical sequential growth process we obtain a model for discrete quantum gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of q-probability operators is a convex set and find its set of extreme elements. We discuss the property of consistency for a DQP and introduce the so-called quadratic algebra of suitable sets. The suitable sets are those on which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process (CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets are discussed. In Section 4 we show how to “quantize” a CSGP to obtain a quantum sequential growth process (QSGP). Some results concerning the consistency of a DQP are given. Finally, Section 5 provides two methods for constructing examples of QSGP.

2 Discrete Quantum Processes

Let \((\Omega, \mathcal{A}, \nu)\) be a probability space and let

\[ H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f : \Omega \to \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\} \]

be the corresponding Hilbert space. Let \(\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}\) be an increasing sequence of sub \(\sigma\)-algebras of \(\mathcal{A}\) that generate \(\mathcal{A}\) and let \(\nu_n = \nu | \mathcal{A}_n\) be the restriction of \(\nu\) to \(\mathcal{A}_n\), \(n = 1, 2, \ldots\). Then \(H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)\) forms an increasing sequence of closed subspaces of \(H\) called a filtration of \(H\). A bounded operator \(T\) on \(H_n\) will also be considered as a bounded operator on \(H\) by defining \(Tf = 0\) for all \(f \in H_n^\perp\). We denote the characteristic function \(\chi_\Omega\) of \(\Omega\) by \(1\). Of course, \(\|1\| = 1\) and \(\langle 1, f \rangle = \int f \, d\nu\) for every \(f \in H\). A q-probability operator is a bounded positive operator \(\rho\) on \(H\) that satisfies \(\langle 1, 1 \rangle = 1\). Denote the set of q-probability operators on \(H\) and \(H_n\) by \(\mathcal{Q}(H)\) and \(\mathcal{Q}(H_n)\), respectively. Since \(1 \in H_n\), if \(\rho \in \mathcal{Q}(H_n)\) by our previous convention, \(\rho \in \mathcal{Q}(H)\). Notice that a positive operator \(\rho \in \mathcal{Q}(H)\) if and only if \(\|\rho^{1/2}1\| = 1\) where \(\rho^{1/2}\) is the unique positive square root of \(\rho\).

A rank 1 element of \(\mathcal{Q}(H)\) is called a pure q-probability operator. Thus \(\rho \in \mathcal{Q}(H)\) is pure if and only if \(\rho\) has the form \(\rho = |\psi\rangle \langle \psi|\) for some \(\psi \in H\) satisfying

\[ \|\langle 1, \psi\rangle\| = \left| \int \psi \, d\nu \right| = 1 \]
We then call $\psi$ a *q-probability vector* and we denote the set of q-probability vectors by $V(H)$ and the set of pure q-probability operators by $Q_p(H)$. Notice that if $\psi \in V(H)$, then $\|\psi\| \geq 1$ and $\|\psi\| = 1$ if and only if $\psi = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Two operators $\rho_1, \rho_2 \in Q(H)$ are orthogonal if $\rho_1 \rho_2 = 0$.

**Theorem 2.1.** (i) $Q(A)$ is a convex set and $Q_p(H)$ is its set of extreme elements. (ii) $\rho \in Q(H)$ is of trace class if and only if there exists a sequence of mutually orthogonal $\rho_i \in Q_p(H)$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$ such that $\rho = \sum \alpha_i \rho_i$ in the strong operator topology. The $\rho_i$ are unique if and only if the corresponding $\alpha_i$ are distinct.

**Proof.** (i) If $0 < \lambda < 1$ and $\rho_1, \rho_2 \in Q(H)$, then $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ is a positive operator and

$$\langle \rho_1, 1 \rangle = \langle (\lambda \rho + (1 - \lambda) \rho_2) 1, 1 \rangle = \lambda \langle \rho_1, 1 \rangle + (1 - \lambda) \langle \rho_2, 1 \rangle = 1$$

Hence, $\rho \in Q(H)$ so $Q(H)$ is a convex set. Suppose $\rho \in Q_p(H)$ and $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ where $0 < \lambda < 1$ and $\rho_1, \rho_2 \in Q\langle 1 \rangle$. If $\rho_1 \neq \rho_2$, then $\text{rank}(\rho) \neq 1$ which is a contradiction. Hence, $\rho_1 = \rho_2$ so $\rho$ is an extreme element of $Q(H)$. Conversely, suppose $\rho \in Q(H)$ is an extreme element. If the cardinality of the spectrum $|\sigma(\rho)| > 1$, then by the spectral theorem $\rho = \rho_1 + \rho_2$ where $\rho_1, \rho_2 \neq 0$ are positive and $\rho_1 \neq \alpha \rho_2$ for $\alpha \in \mathbb{C}$. If $\rho_1, \rho_2 \neq 0$, then $\langle \rho_1, 1 \rangle, \langle \rho_2, 1 \rangle \neq 0$ and we can write

$$\rho = \frac{\rho_1}{\langle \rho_1, 1 \rangle} \langle \rho_1, 1 \rangle + \frac{\rho_2}{\langle \rho_2, 1 \rangle} \langle \rho_2, 1 \rangle$$

Now $\langle \rho_1, 1 \rangle^{-1} \rho_1, \langle \rho_2, 1 \rangle^{-1} \rho_2 \in Q(H)$ and

$$\langle \rho_1, 1 \rangle + \langle \rho_2, 1 \rangle = \langle \rho_1, 1 \rangle = 1$$

which is a contradiction. Hence, $\rho_1 = 0$ or $\rho_2 = 0$. Without loss of generality suppose that $\rho_2 = 0$. We can now write

$$\rho = \frac{1}{2} \rho_1 + \frac{1}{2} (\rho_1 + 2 \rho_2)$$

Now $\rho_1 \neq 0$, $(\rho_1 + 2 \rho_2) \neq 0$ and as before we get a contradiction. We conclude that $|\sigma(\rho)| = 1$. Hence, $\rho = \alpha P$ where $P$ is a projection and $\alpha > 0$. If $\text{rank}(P) > 1$, then $P = P_1 + P_2$ where $P_1$ and $P_2$ are orthogonal nonzero projections so $\rho = \alpha P_1 + \alpha P_2$. Proceeding as before we obtain a contradiction. Hence, $\text{rank}(P) = 1$ so $\rho = \alpha P$ is pure. (ii) This follows from the spectral theorem. \qed
Let \( \{H_n: n = 1, 2, \ldots \} \) be a filtration of \( H \) and let \( \rho_n \in \mathcal{Q}(H_n) \), \( n = 1, 2, \ldots \). The \( n \)-decoherence functional \( D_n: A_n \times A_n \to \mathbb{C} \) defined by

\[
D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle
\]

gives a measure of the interference between \( A \) and \( B \) when the system is described by \( \rho_n \). It is clear that \( D_n(\Omega_n, \Omega_n) = 1 \), \( D_n(A, B) = D_n(B, A) \) and \( A \mapsto D_n(A, B) \) is a complex measure for all \( B \in A_n \). It is also well-known that if \( A_1, \ldots, A_r \in A_n \), then the matrix with entries \( D_n(A_j, A_k) \) is positive semidefinite. We define the map \( \mu_n: A_n \to \mathbb{R}^+ \) by

\[
\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle
\]

Notice that \( \mu_n(\Omega_n) = 1 \). Although \( \mu_n \) is not additive, it does satisfy the grade-2 additivity condition: if \( A, B, C \in A_n \) are mutually disjoint, then

\[
\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C) \tag{2.1}
\]

We say that \( \rho_{n+1} \) is consistent with \( \rho_n \) if \( D_{n+1}(A, B) = D_n(A, B) \) for all \( A, B \in A_n \). We call the sequence \( \rho_n \), \( n = 1, 2, \ldots \), consistent if \( \rho_{n+1} \) is consistent with \( \rho_n \) for \( n = 1, 2, \ldots \). Of course, if the sequence \( \rho_n \), \( n = 1, 2, \ldots \), is consistent, then \( \mu_{n+1}(A) = \mu_n(A) \) \( \forall A \in A_n \), \( n = 1, 2, \ldots \). A discrete quantum process (DQP) is a consistent sequence \( \rho_n \in \mathcal{Q}(H_n) \) for a filtration \( H_n, n = 1, 2, \ldots \). A DQP \( \rho_n \) is pure if \( \rho_n \in \mathcal{Q}_p(H_n) \), \( n = 1, 2, \ldots \).

If \( \rho_n \) is a DQP, then the corresponding maps \( \mu_n: A_n \to \mathbb{R}^+ \) have the form

\[
\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \| \rho_n^{1/2} \chi_A \|^2
\]

Now \( A \mapsto \rho_n^{1/2} \chi_A \) is a vector-valued measure on \( A_n \). We conclude that \( \mu_n \) is the squared norm of a vector-valued measure. In particular, if \( \rho_n = |\psi_n\rangle \langle \psi_n| \) is a pure DQP, then \( \mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2 \) so \( \mu_n \) is the squared modulus of the complex-valued measure \( A \mapsto \langle \psi_n, \chi_A \rangle \).

For a DQP \( \rho_n \in \mathcal{Q}(H_n) \), we say that a set \( A \in \mathcal{A} \) is suitable if \( \lim \langle \rho_n \chi_A, \chi_A \rangle \) exists and is finite and in this case we define \( \mu(A) \) to be the limit. We denote the set of suitable sets by \( \mathcal{S}(\rho_n) \). If \( A \in A_n \) then

\[
\lim \langle \rho_n \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle
\]

so \( A \in \mathcal{S}(\rho_n) \) and \( \mu(A) = \mu_n(A) \). This shows that the algebra \( \mathcal{A}_0 = \bigcup A_n \subseteq \mathcal{S}(\rho_n) \). In particular, \( \Omega \in \mathcal{S}(\rho_n) \) and \( \mu(\Omega) = 1 \). In general, \( \mathcal{S}(\rho_n) \neq \mathcal{A} \) and \( \mu \)
may not have a well-behaved extension from $A_0$ to all of $A$ [2, 7]. A subset $B$ of $A$ is a quadratic algebra if $\emptyset, \Omega \in B$ and whenever $A, B, C \in B$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in B$, we have $A \cup B \cup C \in B$. For a quadratic algebra $B$, a $q$-measure is a map $\mu: B \to \mathbb{R}^+$ that satisfies the grade-2 additivity condition (2.1). Of course, an algebra of sets is a quadratic algebra and we conclude that $\mu_n: A_n \to \mathbb{R}^+$ is a $q$-measure. It is not hard to show that $S(\rho_n)$ is a quadratic algebra and $\mu: S(\rho_n) \to \mathbb{R}^+$ is a $q$-measure on $S(\rho_n)$ [3].

3 Classical Sequential Growth Processes

A partially ordered set (poset) is a set $x$ together with an irreflexive, transitive relation $<$ on $x$. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $P_n$ be the collection of all posets with cardinality $n$, $n = 1, 2, \ldots$. If $x \in P_n$, $y \in P_{n+1}$, then $x$ produces $y$ if $y$ is obtained from $x$ by adjoining a single new element to $x$ that is maximal in $y$. We also say that $x$ is a producer of $y$ and $y$ is an offspring of $x$. If $x$ produces $y$ we write $x \rightarrow y$. We denote the set of offspring of $x$ by $x \rightarrow$ and for $A \subseteq P_n$ we use the notation

$$A \rightarrow = \{ y \in P_{n+1}: x \rightarrow y, x \in A \}$$

The transitive closure of $\rightarrow$ makes the set of all finite posets $P = \bigcup P_n$ into a poset.

A path in $P$ is a string (sequence) $x_1, x_2, \ldots$ where $x_i \in P_i$ and $x_i \rightarrow x_{i+1}$, $i = 1, 2, \ldots$. An $n$-path in $P$ is a finite string $x_1 x_2 \cdots x_n$ where again $x_i \in P_i$ and $x_i \rightarrow x_{i+1}$. We denote the set of paths by $\Omega$ and the set of $n$-paths by $\Omega_n$. The set of paths whose initial $n$-path is $\omega_0 \in \Omega_n$ is denoted by $\omega_0 \Rightarrow$. Thus, if $\omega_0 = x_1 x_2 \cdots x_n$ then

$$\omega_0 \Rightarrow = \{ \omega \in \Omega: \omega = x_1, x_2 \cdots x_n y_{n+1} y_{n+2} \cdots \}$$

If $x$ produces $y$ in $r$ isomorphic ways, we say that the multiplicity of $x \rightarrow y$ is $r$ and write $m(x \rightarrow y) = r$. For example, in Figure 1, $m(x \rightarrow y) = 3$. (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)
If \( x \in \mathcal{P} \) and \( a, b \in x \) we say that \( a \) is an ancestor of \( b \) and \( b \) is a successor of \( a \) if \( a < b \). We say that \( a \) is a parent of \( b \) and \( b \) is a child of \( a \) if \( a < b \) and there is no \( c \in x \) such that \( a < c < b \). Let \( c = (c_0, c_1, \ldots) \) be a sequence of nonnegative numbers called coupling constants [5, 9]. For \( r, s \in \mathbb{N} \) with \( r \leq s \), we define

\[
\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}
\]

For \( x \in \mathcal{P}_n \), \( y \in \mathcal{P}_{n+1} \) with \( x \rightarrow y \) we define the transition probability

\[
p_c(x \rightarrow y) = m(x \rightarrow y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}
\]

where \( \alpha \) is the number of ancestors and \( \pi \) the number of parents of the adjoined maximal element in \( y \) that produces \( y \) from \( x \). It is shown in [5, 9] that \( p_c(x \rightarrow y) \) is a probability distribution in that it satisfies the Markov-sum rule

\[
\sum \{ p_c(x \rightarrow y) : y \in \mathcal{P}_{n+1}, x \rightarrow y \} = 1
\]

In discrete quantum gravity, the elements of \( \mathcal{P} \) are thought of as causal sets and \( a < b \) is interpreted as \( b \) being in the causal future of \( a \). The distribution \( y \mapsto p_c(x \rightarrow y) \) is essentially the most general that is consistent with principles of causality and covariance [5, 9]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take \( c_k = 1/k! \) [6, 7]. The case \( c_k = c^k \) for some \( c > 0 \) has been previously studied and is called a percolation dynamics [5, 6, 8].

We call an element \( x \in \mathcal{P} \) a site and we sometimes call an \( n \)-path an \( n \)-universe and a path a universe. The set \( \mathcal{P} \) together with the set of transition probabilities \( p_c(x \rightarrow y) \) forms a classical sequential growth process (CSGP).
which we denote by \((\mathcal{P}, p_c)\) \([4, 5, 6, 8, 9]\). It is clear that \((\mathcal{P}, p_c)\) is a Markov chain and as usual we define the probability of an \(n\)-path \(\omega = y_1y_2\cdots y_n\) by

\[
p^n_c(\omega) = p_c(y_1 \rightarrow y_2)p_c(y_2 \rightarrow y_3)\cdots p_c(y_{n-1} \rightarrow y_n)
\]

Denoting the power set of \(\Omega_n\) by \(2^{\Omega_n}\), \((\Omega_n, 2^{\Omega_n}, p^n_c)\) becomes a probability space where

\[
p^n_c(A) = \sum \{p^n_c(\omega) : \omega \in A\}
\]

for all \(A \in 2^{\Omega_n}\). The probability of a site \(x \in \mathcal{P}_n\) is

\[
p^n_c(x) = \sum \{p^n_c(\omega) : \omega \in \Omega_n, \omega \text{ ends at } x\}
\]

Of course, \(x \mapsto p^n_c(x)\) is a probability measure on \(\mathcal{P}_n\) and we have

\[
\sum_{x \in \mathcal{P}_n} p^n_c(x) = 1
\]

**Example 1.** Figure 2 illustrates the first two steps of a CSGP where the 2 indicates the multiplicity \(m(x_3 \rightarrow x_6) = 2\). Table 1 lists the probabilities of the various sites for the general coupling constants \(c_k\) and the particular coupling constants \(c'_k = 1/k!\) where \(d = (c_0 + c_1)(c_0 + 2c_1 + c_2)\).

\[
\begin{aligned}
  &x_1 & &x_2 & &x_3 \\
  & & & & & & 2 \\
  &x_4 & &x_5 & &x_6 \\
  & & & & & & \\
  & & & & & & \\
  &x_7 & &x_8 \\
  & & & & & & \\
  & & & & & & \\
\end{aligned}
\]

**Figure 2**
For \( A \subseteq \Omega_n \) we use the notation
\[
A \Rightarrow = \cup \{ \omega \Rightarrow: \omega \in A \}
\]
Thus, \( A \Rightarrow \) is the set of paths whose initial \( n \)-paths are elements of \( A \). We call \( A \Rightarrow \) a cylinder set and define
\[
A_n = \{ A \Rightarrow: A \subseteq \Omega_n \}
\]
In particular, if \( \omega \in \Omega_n \) then the elementary cylinder set \( \text{cyl}(\omega) \) is given by \( \text{cyl}(\omega) = \omega \Rightarrow \). It is easy to check that the \( A_n \) form an increasing sequence \( A_1 \subseteq A_2 \subseteq \cdots \) of algebras on \( \Omega \) and hence \( \mathcal{C}(\Omega) = \cup A_n \) is an algebra of subsets of \( \Omega \). Also for \( A \in \mathcal{C}(\Omega) \) of the form \( A = A_1 \Rightarrow \), \( A_1 \subseteq \Omega_n \), we define \( p_c(A) = p^n_c(A_1) \). It is easy to check that \( p_c \) is a well-defined probability measure on \( \mathcal{C}(\Omega) \). It follows from the Kolmogorov extension theorem that \( p_c \) has a unique extension to a probability measure \( \nu_c \) on the \( \sigma \)-algebra \( \mathcal{A} \) generated by \( \mathcal{C}(\Omega) \). We conclude that \((\Omega, \mathcal{A}, \nu_c)\) is a probability space, the increasing sequence of subalgebras \( A_n \) generates \( \mathcal{A} \) and that the restriction \( \nu_c | A_n = p^n_c \). Hence, the subspaces \( H_n = L_2(\Omega, A_n, p^n_c) \) form a filtration of the Hilbert space \( H = L_2(\Omega, \mathcal{A}, \nu_c) \).

### 4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP \((\mathcal{P}, p_c)\) developed in Section 3. We have seen that the \( n \)-path Hilbert space \( H_n = L_2(\Omega, A_n, p^n_c) \) forms a filtration of the path Hilbert space \( H = L_2(\Omega, \mathcal{A}, \nu_c) \). In the sequel, we assume that \( p^n_c(\omega) \neq 0 \) for every \( \omega \in \Omega_n, n = 1, 2, \ldots \). Then the set of vectors
\[
e^n_c = p^n_c(\omega)^{1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n
\]
form an orthonormal basis for $H_n$, $n = 1, 2, \ldots$. For $A \in \mathcal{A}_n$, notice that $\chi_A \in H$ with $\|\chi_A\| = p_n(A)^{1/2}$.

We call a DQP $\rho_n \in \mathcal{Q}(H_n)$ a quantum sequential growth process (QSGP). We call $\rho_n$ the local operators and $\mu_n(A) = D_n(A, A)$ the local $q$-measures for the process. If $\rho = \lim \rho_n$ exists in the strong operator topology, then $\rho$ is a $q$-probability operator on $H$ called the global operator for the process. If the global operator $\rho$ exists, then $\hat{\mu}(A) = \langle \rho \chi_A, \chi_A \rangle$ is a (continuous) $q$-measure on $\mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators $[2, 3, 7]$. In this case, we still have the $q$-measure $\mu$ on the quadratic algebra $\mathcal{S}(\rho_n) \subseteq \mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. We frequently identify a set $A \subseteq \Omega_n$ with the corresponding cylinder set $(A \Rightarrow) \in \mathcal{A}_n$. We then have the $q$-measure, also denoted by $\mu_n$, on $2^{\Omega_n}$ defined by $\mu_n(A) = \mu_n(A \Rightarrow)$. Moreover, we define the $q$-measure, again denoted by $\mu_n$, on $\mathcal{P}_n$ by

$$\mu_n(A) = \mu_n(\{\omega \in \Omega_n: \omega \text{ end in } A\})$$

for all $A \subseteq \mathcal{P}_n$. In particular, for $x \in \mathcal{P}_n$ we have

$$\mu_n(\{x\}) = \mu_n(\{\omega \in \Omega_n: \omega \text{ ends with } x\})$$

If $A \in \mathcal{A}_n$ has the form $A_1 \Rightarrow$ for $A_1 \subseteq \Omega_n$ then $A \in \mathcal{A}_{n+1}$ and $A = (A_1 \Rightarrow) \Rightarrow$ where $A_1 \Rightarrow \subseteq \Omega_{n+1}$. Let $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and let $D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$, $D_{n+1}(A, B) = \langle \rho_{n+1} \chi_B, \chi_A \rangle$ be the corresponding decoherence functionals. Then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $A, B \subseteq \Omega_n$ we have

$$D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = D_n(A \Rightarrow, B \Rightarrow) \tag{4.1}$$

**Lemma 4.1.** For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $\omega, \omega' \in \Omega_n$ we have

$$D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow] = D_n(\omega \Rightarrow, \omega' \Rightarrow) \tag{4.2}$$

**Proof.** Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every $A, B \subseteq \Omega_n$ we have

$$D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = \sum_{\omega \in A} \sum_{\omega' \in B} D_{n+1}D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow]$$

$$= \sum_{\omega \in A} \sum_{\omega' \in B} D_n(\omega \Rightarrow, \omega' \Rightarrow) = D_n(A \Rightarrow, B \Rightarrow)$$

and the result follows from (4.1). \hfill \Box
For \( \omega = x_1 x_2 \cdots x_n \in \Omega_n \) and \( y \in \mathcal{P}_{n+1} \) with \( x_n \to y \) we use the notation \( \omega y \in \Omega_{n+1} \) where \( \omega y = x_1 x_2 \cdots x_n y \). We also define \( p_c(\omega \to y) = p_c(x_n \to y) \) and write \( \omega \to y \) whenever \( x_n \to y \).

**Theorem 4.2.** For \( \rho_n \in \mathcal{Q}(H_n) \), \( \rho_{n+1} \in \mathcal{Q}(H_{n+1}) \) we have that \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for every \( \omega, \omega' \in \Omega_n \) we have

\[
\langle \rho_n e^n_{\omega'}, e^n_{\omega} \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p_c(\omega' \to x) \frac{1}{2} p_c(\omega \to y) \frac{1}{2} \left\langle \rho_{n+1} e^{n+1}_{\omega' x}, e^{n+1}_{\omega y} \right\rangle \quad (4.3)
\]

**Proof.** By Lemma 4.1, \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if (4.2) holds. But

\[
D_n(\omega \Rightarrow, \omega' \Rightarrow) = \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle = \langle \rho_n \chi_{\text{cyl} (\omega')}, \chi_{\text{cyl} (\omega)} \rangle = p^n_c(\omega') \frac{1}{2} p^n_c(\omega) \frac{1}{2} \langle \rho_n e^n_{\omega'}, e^n_{\omega} \rangle
\]

Moreover, we have

\[
D_{n+1}[(\omega \Rightarrow) \cdot (\omega' \Rightarrow) \Rightarrow] = \langle \rho_{n+1} \chi_{(\omega' \Rightarrow) \Rightarrow}, \chi_{(\omega \Rightarrow) \Rightarrow} \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\omega' x \Rightarrow}, \chi_{\omega y \Rightarrow} \rangle
\]

\[
= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\text{cyl} (\omega' x)}, \chi_{\text{cyl} (\omega y)} \rangle
\]

\[
= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p^n_c(\omega' x) \frac{1}{2} p^n_c(\omega y) \frac{1}{2} \langle \rho_{n+1} e^{n+1}_{\omega' x}, e^{n+1}_{\omega y} \rangle
\]

\[
= p^n_c(\omega') \frac{1}{2} p^n_c(\omega) \frac{1}{2} \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p_c(\omega' \to x) p_c(\omega \to y) \frac{1}{2} \langle \rho_{n+1} e^{n+1}_{\omega' x}, e^{n+1}_{\omega y} \rangle
\]

The result now follows.  

Viewing \( H_n \) as \( L_2(\Omega_n, 2^{\Omega_n}, p^0_c) \) we can write (4.3) in the simple form

\[
\langle \rho_n \chi(\omega'), \chi(\omega) \rangle = \langle \rho_{n+1} \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle \quad (4.4)
\]

**Corollary 4.3.** A sequence \( \rho_n \in \mathcal{Q}(H_n) \) is a QSGP if and only if (4.3) or (4.4) hold for every \( \omega, \omega' \in \Omega_n, n = 1, 2, \ldots \).
We now consider pure $q$-probability operators. In the following results we again view $H_n$ as $L_2(\Omega_n, 2^{\Omega_n}, p_n^c)$.

**Corollary 4.4.** If $\rho_n \in Q_p(H_n), \rho_{n+1} \in Q_p(H_{n+1})$ with $p_n = |\psi_n\rangle\langle \psi_n|$, $\rho_{n+1} = |\psi_{n+1}\rangle\langle \psi_{n+1}|$, then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have

$$\langle \psi_n, \chi_{\{\omega}\} \rangle \langle \chi_{\{\omega'\}}, \psi_n \rangle = \langle \psi_{n+1}, \chi_{\omega \rightarrow} \rangle \langle \chi_{\omega' \rightarrow}, \psi_{n+1} \rangle$$  \hspace{1cm} (4.5)

**Corollary 4.5.** A sequence $|\psi_n\rangle\langle \psi_n| \in Q_p(H_n)$ is a QSGP if and only if (4.5) holds for every $\omega, \omega' \in \Omega_n$.

We say that $\psi_{n+1} \in V(H_{n+1})$ is strongly consistent with $\psi_n \in V(H_n)$ if for every $\omega \in \Omega_n$ we have

$$\langle \psi_n, \chi_{\{\omega}\} \rangle = \langle \psi_{n+1}, \chi_{\omega} \rangle$$  \hspace{1cm} (4.6)

By (4.5) strong consistency implies the consistency of the corresponding $q$-probability operators.

**Corollary 4.6.** If $\psi_{n+1} \in V(H_{n+1})$ is strongly consistent with $\psi_n \in V(H_n)$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle \psi_n| \in Q_p(H_n)$ is a QSGP.

**Lemma 4.7.** If $\psi_n \in V(H_n)$ and $\psi_{n+1} \in H_{n+1}$ satisfies (4.6) for every $\omega \in \Omega_n$, then $\psi_{n+1} \in V(H_{n+1})$.

**Proof.** Since $\psi_n \in V(H_n)$ we have by (4.6) that

$$|\langle \psi_{n+1}, 1 \rangle| = \left| \sum_{\omega \in \Omega_n} \langle \psi_{n+1}, \chi_{\omega} \rangle \right| = \left| \sum_{\omega \in \Omega_n} \langle \psi_n, \chi_{\{\omega}\} \rangle \right| = |\langle \psi_n, 1 \rangle| = 1$$

The result now follows. \hfill $\Box$

**Corollary 4.8.** If $\|\psi_1\| = 1$ and $\psi_n \in H_n$ satisfies (4.6) for all $\omega \in \Omega_n$, $n = 1, 2, \ldots$, then $|\psi_n\rangle\langle \psi_n|$ is a QSGP.

**Proof.** Since $\|\psi_1\| = 1$, it follows that $\psi_1 \in V(H_1)$. By Lemma 4.7, $\psi_n \in V(H_n), n = 1, 2, \ldots$. Since (4.6) holds, the result follows from Corollary 4.6. \hfill $\Box$

Another way of writing (4.6) is

$$\sum_{\omega \rightarrow x} p_{\psi_{n+1}}^o(\omega x) \psi_{n+1}(\omega x) = p_{\psi_n}^o(\omega) \psi_n(x)$$ \hspace{1cm} (4.7)

for every $\omega \in \Omega_n$. 

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5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for discrete quantum gravity. The simplest way to construct a QSGP is to form the constant pure DQP \( \rho_n = |1\rangle\langle 1| \), \( n = 1, 2, \ldots \). To show that \( \rho_n \) is indeed consistent, we have for \( \omega \in \Omega_n \) that

\[
\sum_{\omega \to x} p_{c}^{n+1}(\omega x) = \sum_{\omega \to x} p_{c}^{n}(\omega) p_{c}(\omega \to x) = p_{c}^{n}(\omega) \sum_{\omega \to x} p_{c}(\omega \to x) = p_{c}^{n}(\omega)
\]

so consistency follows from (4.7). The corresponding q-measures are given by

\[
\mu_n(A) = |\langle 1, \chi_A \rangle|^2 = p_{c}^{n}(A)^2
\]

for every \( A \in A_n \). Hence, \( \mu_n \) is the square of the classical measure. Of course, \( |1\rangle\langle 1| \) is the global q-probability operator for this QSGP and in this case \( S(\rho_n) = A \). Moreover, we have the global q-measure \( \mu(A) = \nu_{c}(A)^2 \) for \( A \in A \).

Another simple way to construct a QSGP is to employ Corollary 4.8. In this way we can let \( \psi_1 = 1 \), \( \psi_2 \) any vector in \( L_{2}(\Omega_n, 2^{\Omega_n}, p_{c}^{2}) \) satisfying

\[
\langle \psi_2, \chi_{\{x_1x_2\}} \rangle + \langle \psi_2, \chi_{\{x_1x_3\}} \rangle = \langle \psi_1, \chi_{\{x_1\}} \rangle = 1
\]

and so on, where \( x_1, x_2, x_3 \) are given in Figure 2. As a concrete example, let \( \psi_1 = 1 \),

\[
\psi_2 = \frac{1}{2} \left[ p_{c}^{2}(x_1x_2)^{-1}\chi_{\{x_1x_2\}} + p_{c}^{2}(x_1x_3)\chi_{\{x_1x_3\}} \right]
\]

and in general

\[
\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p_{c}^{n}(\omega)^{-1}\chi_{\omega}
\]

The q-measure \( \mu_1 \left( \{x_1\} \right) = 1 \) and \( \mu_2 \) is given by

\[
\mu_2 \left( \{x_1x_2\} \right) = |\langle \psi_2, \chi_{\{x_1x_2\}} \rangle|^2 = \frac{1}{4}
\]

\[
\mu_2 \left( \{x_1x_3\} \right) = |\langle \psi_2, \chi_{\{x_1x_3\}} \rangle|^2 = \frac{1}{4}
\]

\[
\mu_2(\Omega_2) = |\langle \psi_2, 1 \rangle|^2 = 1
\]

In general, we have \( \mu_n(A) = |A|^2 / |\Omega_n|^2 \) for all \( A \in \Omega_n \). Thus \( \mu_n \) is the square of the uniform distribution. The global operator does not exist because there is no q-measure on \( A \) that extends \( \mu_n \) for all \( n \in \mathbb{N} \). For \( A \in A \) we have

\[
\langle \psi_n, \chi_A \rangle = \int \psi_n \chi_A d\nu_c = \frac{|A \cap \{cyl(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}
\]

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Letting $\rho_n = |\psi_n\rangle\langle\psi_n|$ we conclude that $A \in S(\rho_n)$ if and only if

$$\lim_{n \to \infty} \frac{|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}|}{|\Omega_n|}$$

exists. For example, if $|A| < \infty$ then for $n$ sufficiently large we have

$$|A \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |A|$$

so $A \in S(\rho_n)$ and $\mu(A) = 0$. In a similar way if $|A| < \infty$ then for the complement $A'$, if $n$ is sufficiently large we have

$$|A' \cap \{\text{cyl}(\omega) : \omega \in \Omega_n\}| = |\Omega_n| - |A|$$

so $A' \in S(\rho_n)$ with $\mu(A') = 1$.

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let $\alpha_\omega \in \mathbb{C}$, $\omega \in \Omega_n$ satisfy

$$\left| \sum_{\omega \in \Omega_n} \alpha_\omega p_c^n(\omega)^{1/2} \right|^2 = 1 \quad (5.1)$$

and let $\rho_n$ be the operator on $H_n$ satisfying

$$\langle \rho_n e_n^\omega, e_n^{\omega'} \rangle = \alpha_\omega \overline{\alpha_\omega} \quad (5.2)$$

Then $\rho_n$ is a positive operator and by (5.1), (5.2) we have

$$\langle \rho_n 1, 1 \rangle = \left\langle \rho_n \sum_\omega p_c^n(\omega)^{1/2} e_\omega, \sum_{\omega'} p_c^n(\omega')^{1/2} e_{\omega'} \right\rangle$$

$$= \sum_{\omega, \omega'} p_c^n(\omega)^{1/2} p_c^n(\omega')^{1/2} \langle \rho_n e_\omega, e_{\omega'} \rangle$$

$$= \left| \sum_{\omega} p_c^n(\omega)^{1/2} \alpha_\omega \right|^2 = 1$$

Hence, $\rho_n \in Q(H_n)$. Now

$$\Omega_{n+1} = \{\omega x : \omega \in \Omega_n, x \in P_{n+1}, \omega \to x\}$$
and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy
\[
\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} p_{c}^{n+1}(\omega x)^{1/2} \right| = 1
\]

Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying
\[
\langle \rho_{n+1} e_{\omega x}^{n+1}, e_{\omega x'}^{n+1} \rangle = \beta_{\omega x'} \beta_{\omega x} \tag{5.3}
\]

As before, we have that $\rho_{n+1} \in Q(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** The operator $\rho_{n+1}$ is consistent with $\rho_{n}$ if and only if for every $\omega, \omega' \in \Omega_{n}$ we have
\[
\alpha_{\omega'} \alpha_{\omega} = \sum_{x' \in P_{n+1}^{\omega'}} \beta_{\omega' x'} p_{c}^{n+1}(\omega' x')^{1/2} \sum_{x \in P_{n+1}^{\omega x}} \overline{\beta_{\omega x}} p_{c}^{n+1}(\omega x)^{1/2} \tag{5.4}
\]

A sufficient condition for (5.4) to hold is
\[
\sum_{x \in P_{n+1}^{\omega x}} \beta_{\omega x} p_{c}^{n+1}(\omega x)^{1/2} = \alpha_{\omega} \tag{5.5}
\]

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** Let $\rho_{n} \in Q(H_{n})$ be defined by (5.2) and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by (5.3). If (5.5) holds, then $\rho_{n+1} \in Q(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_{n}$.

The next result gives the general construction.

**Corollary 5.3.** Let $\rho_{1} = I \in Q(H_{1})$ and define $\rho_{n} \in Q(H_{n})$ inductively by (5.3). Then $\rho_{n}$ is a QSGP.

**References**


