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MODELS FOR
DISCRETE QUANTUM GRAVITY

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Abstract

We first discuss a framework for discrete quantum processes (DQP). It is shown that the set of $q$-probability operators is convex and its set of extreme elements is found. The property of consistency for a DQP is studied and the quadratic algebra of suitable sets is introduced. A classical sequential growth process is “quantized” to obtain a model for discrete quantum gravity called a quantum sequential growth process (QSGP). Two methods for constructing concrete examples of QSGP are provided.

1 Introduction

In a previous article, the author introduced a general framework for a discrete quantum gravity [3]. However, we did not include any concrete examples or models for this framework. In particular, we did not consider the problem of whether nontrivial models for a discrete quantum gravity actually exist. In this paper we provide a method for constructing an infinite number of such models. We first make a slight modification of our definition of a discrete quantum process (DQP) $\rho_n$, $n = 1, 2, \ldots$. Instead of requiring that $\rho_n$ be a state on a Hilbert space $H_n$, we require that $\rho_n$ be a $q$-probability operator on $H_n$. This latter condition seems more appropriate from a probabilistic viewpoint and instead of requiring $\text{tr}(\rho_n) = 1$, this condition normalizes the
corresponding quantum measure. By superimposing a concrete DQP on a classical sequential growth process we obtain a model for discrete quantum gravity that we call a quantum sequential growth process.

Section 2 considers the DQP formalism. We show that the set of \( q \)-probability operators is a convex set and find its set of extreme elements. We discuss the property of consistency for a DQP and introduce the so-called quadratic algebra of suitable sets. The suitable sets are those on which well-defined quantum measures (or quantum probabilities) exist.

Section 3 reviews the concept of a classical sequential growth process (CSGP) [1, 4, 5, 6, 8, 9]. The important notions of paths and cylinder sets are discussed. In Section 4 we show how to “quantize” a CSGP to obtain a quantum sequential growth process (QSGP). Some results concerning the consistency of a DQP are given. Finally, Section 5 provides two methods for constructing examples of QSGP.

2 Discrete Quantum Processes

Let \((\Omega, \mathcal{A}, \nu)\) be a probability space and let

\[
H = L_2(\Omega, \mathcal{A}, \nu) = \left\{ f : \Omega \to \mathbb{C}, \int |f|^2 \, d\nu < \infty \right\}
\]

be the corresponding Hilbert space. Let \(\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}\) be an increasing sequence of sub \(\sigma\)-algebras of \(\mathcal{A}\) that generate \(\mathcal{A}\) and let \(\nu_n = \nu | \mathcal{A}_n\) be the restriction of \(\nu\) to \(\mathcal{A}_n\), \(n = 1, 2, \ldots\). Then \(H_n = L_2(\Omega, \mathcal{A}_n, \nu_n)\) forms an increasing sequence of closed subspaces of \(H\) called a filtration of \(H\). A bounded operator \(T\) on \(H_n\) will also be considered as a bounded operator on \(H\) by defining \(Tf = 0\) for all \(f \in H_n^\perp\). We denote the characteristic function \(\chi_\Omega\) of \(\Omega\) by \(1\). Of course, \(\|1\| = 1\) and \(\langle 1, f \rangle = \int f \, d\nu\) for every \(f \in H\). A q-probability operator is a bounded positive operator \(\rho\) on \(H\) that satisfies \(\langle \rho 1, 1 \rangle = 1\). Denote the set of q-probability operators on \(H\) and \(H_n\) by \(\mathcal{Q}(H)\) and \(\mathcal{Q}(H_n)\), respectively. Since \(1 \in H_n\), if \(\rho \in \mathcal{Q}(H_n)\) by our previous convention, \(\rho \in \mathcal{Q}(H)\). Notice that a positive operator \(\rho \in \mathcal{Q}(H)\) if and only if \(\|\rho^{1/2}1\| = 1\) where \(\rho^{1/2}\) is the unique positive square root of \(\rho\).

A rank 1 element of \(\mathcal{Q}(H)\) is called a pure q-probability operator. Thus \(\rho \in \mathcal{Q}(H)\) is pure if and only if \(\rho\) has the form \(\rho = |\psi\rangle\langle\psi|\) for some \(\psi \in H\) satisfying

\[
|\langle 1, \psi \rangle| = \left| \int \psi \, d\nu \right| = 1
\]
We then call $\psi$ a \emph{q-probability vector} and we denote the set of q-probability vectors by $\mathcal{V}(H)$ and the set of pure q-probability operators by $\mathcal{Q}_p(H)$. Notice that if $\psi \in \mathcal{V}(H)$, then $\|\psi\| \geq 1$ and $\|\psi\| = 1$ if and only if $\psi = \alpha 1$ for some $\alpha \in \mathbb{C}$ with $|c| = 1$. Two operators $\rho_1, \rho_2 \in \mathcal{Q}(H)$ are orthogonal if $\rho_1 \rho_2 = 0$.

\textbf{Theorem 2.1.} (i) $\mathcal{Q}(A)$ is a convex set and $\mathcal{Q}_p(H)$ is its set of extreme elements. (ii) $\rho \in \mathcal{Q}(H)$ is of trace class if and only if there exists a sequence of mutually orthogonal $\rho_i \in \mathcal{Q}_p(H)$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$ such that $\rho = \sum \alpha_i \rho_i$ in the strong operator topology. The $\rho_i$ are unique if and only if the corresponding $\alpha_i$ are distinct.

\textbf{Proof.} (i) If $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$, then $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ is a positive operator and

$$\langle \rho_1, 1 \rangle = \langle (\lambda \rho + (1 - \lambda) \rho_2)1, 1 \rangle = \lambda \langle \rho_1, 1 \rangle + (1 - \lambda) \langle \rho_2, 1 \rangle = 1$$

Hence, $\rho \in \mathcal{Q}(H)$ so $\mathcal{Q}(H)$ is a convex set. Suppose $\rho \in \mathcal{Q}_p(H)$ and $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$ where $0 < \lambda < 1$ and $\rho_1, \rho_2 \in \mathcal{Q}(H)$. If $\rho_1 \neq \rho_2$, then $\text{rank}(\rho) \neq 1$ which is a contradiction. Hence, $\rho_1 = \rho_2$ so $\rho$ is an extreme element of $\mathcal{Q}(H)$. Conversely, suppose $\rho \in \mathcal{Q}(H)$ is an extreme element. If the cardinality of the spectrum $|\sigma(\rho)| > 1$, then by the spectral theorem $\rho = \rho_1 + \rho_2$ where $\rho_1, \rho_2 \neq 0$ are positive and $\rho_1 \neq \alpha \rho_2$ for $\alpha \in \mathbb{C}$. If $\rho_1, \rho_2 > 0$, then $\langle \rho_1, 1 \rangle, \langle \rho_2, 1 \rangle \neq 0$ and we can write

$$\rho = \rho_1 \frac{\rho_1}{\langle \rho_1, 1 \rangle} + \rho_2 \frac{\rho_2}{\langle \rho_2, 1 \rangle}$$

Now $\langle \rho_1, 1 \rangle^{-1} \rho_1, \langle \rho_2, 1 \rangle^{-1} \rho_2 \in \mathcal{Q}(H)$ and

$$\langle \rho_1, 1 \rangle + \langle \rho_2, 1 \rangle = \langle \rho_1, 1 \rangle = 1$$

which is a contradiction. Hence, $\rho_1 = 0$ or $\rho_2 = 0$. Without loss of generality suppose that $\rho_2 = 0$. We can now write

$$\rho = \frac{1}{2} \rho_1 + \frac{1}{2} (\rho_1 + 2 \rho_2)$$

Now $\rho_1 \neq 0$, $(\rho_1 + 2 \rho_2)1 \neq 0$ and as before we get a contradiction. We conclude that $|\sigma(\rho)| = 1$. Hence, $\rho = \alpha P$ where $P$ is a projection and $\alpha > 0$. If $\text{rank}(P) > 1$, then $P = P_1 + P_2$ where $P_1$ and $P_2$ are orthogonal nonzero projections so $\rho = \alpha P_1 + \alpha P_2$. Proceeding as before we obtain a contradiction. Hence, $\text{rank}(P) = 1$ so $\rho = \alpha P$ is pure. (ii) This follows from the spectral theorem. \hfill $\Box$
Let \( \{H_n : n = 1, 2, \ldots \} \) be a filtration of \( H \) and let \( \rho_n \in \mathcal{Q}(H_n) \), \( n = 1, 2, \ldots \). The \( n \)-decoherence functional \( D_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C} \) defined by
\[
D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle
\]
gives a measure of the interference between \( A \) and \( B \) when the system is described by \( \rho_n \). It is clear that \( D_n(\Omega_n, \Omega_n) = 1 \), \( D_n(A, B) = D_n(B, A) \) and \( A \mapsto D_n(A, B) \) is a complex measure for all \( B \in \mathcal{A}_n \). It is also well-known that if \( A_1, \ldots, A_r \in \mathcal{A}_n \) then the matrix with entries \( D_n(A_j, A_k) \) is positive semidefinite. We define the map \( \mu_n : \mathcal{A}_n \to \mathbb{R}^+ \) by
\[
\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle
\]
Notice that \( \mu_n(\Omega_n) = 1 \). Although \( \mu_n \) is not additive, it does satisfy the \textit{grade-2 additivity condition}: if \( A, B, C \in \mathcal{A}_n \) are mutually disjoint, then
\[
\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)
\]
We say that \( \rho_{n+1} \) is consistent with \( \rho_n \) if \( D_{n+1}(A, B) = D_n(A, B) \) for all \( A, B \in \mathcal{A}_n \). We call the sequence \( \rho_n, n = 1, 2, \ldots, \) consistent if \( \rho_{n+1} \) is consistent with \( \rho_n \) for \( n = 1, 2, \ldots \). Of course, if the sequence \( \rho_n, n = 1, 2, \ldots \), is consistent, then \( \mu_{n+1}(A) = \mu_n(A) \) \( \forall A \in \mathcal{A}_n \), \( n = 1, 2, \ldots \). A \textit{discrete quantum process} (DQP) is a consistent sequence \( \rho_n \in \mathcal{Q}(H_n) \) for a filtration \( H_n, n = 1, 2, \ldots \). A DQP \( \rho_n \) is pure if \( \rho_n \in \mathcal{Q}_p(H_n), n = 1, 2, \ldots \).

If \( \rho_n \) is a DQP, then the corresponding maps \( \mu_n : \mathcal{A}_n \to \mathbb{R}^+ \) have the form
\[
\mu_n(A) = \langle \rho_n \chi_A, \chi_A \rangle = \| \rho_n^{1/2} \chi_A \|^2
\]
Now \( A \mapsto \rho_n^{1/2} \chi_A \) is a vector-valued measure on \( \mathcal{A}_n \). We conclude that \( \mu_n \) is the squared norm of a vector-valued measure. In particular, if \( \rho_n = |\psi_n\rangle\langle \psi_n| \) is a pure DQP, then \( \mu_n(A) = |\langle \psi_n, \chi_A \rangle|^2 \) so \( \mu_n \) is the squared modulus of the complex-valued measure \( A \mapsto \langle \psi_n, \chi_A \rangle \).

For a DQP \( \rho_n \in \mathcal{Q}(H_n) \), we say that a set \( A \in \mathcal{A} \) is \textit{suitable} if \( \lim \langle \rho_j \chi_A, \chi_A \rangle \) exists and is finite and in this case we define \( \mu(A) \) to be the limit. We denote the set of suitable sets by \( \mathcal{S}(\rho_n) \). If \( A \in \mathcal{A}_n \) then
\[
\lim \langle \rho_j \chi_A, \chi_A \rangle = \langle \rho_n \chi_A, \chi_A \rangle
\]
so \( A \in \mathcal{S}(\rho_n) \) and \( \mu(A) = \mu_n(A) \). This shows that the algebra \( \mathcal{A}_0 = \cup \mathcal{A}_n \subseteq \mathcal{S}(\rho_n) \). In particular, \( \Omega \in \mathcal{S}(\rho_n) \) and \( \mu(\Omega) = 1 \). In general, \( \mathcal{S}(\rho_n) \neq \mathcal{A} \) and \( \mu \)
may not have a well-behaved extension from $\mathcal{A}_0$ to all of $\mathcal{A}$ [2, 7]. A subset $\mathcal{B}$ of $\mathcal{A}$ is a *quadratic algebra* if $\emptyset, \Omega \in \mathcal{B}$ and whenever $A, B, C \in \mathcal{B}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{B}$, we have $A \cup B \cup C \in \mathcal{B}$. For a quadratic algebra $\mathcal{B}$, a *q-measure* is a map $\mu_0: \mathcal{B} \rightarrow \mathbb{R}^+$ that satisfies the grade-2 additivity condition (2.1). Of course, an algebra of sets is a quadratic algebra and we conclude that $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$ is a q-measure. It is not hard to show that $\mathcal{S}(\rho_n)$ is a quadratic algebra and $\mu: \mathcal{S}(\rho_n) \rightarrow \mathbb{R}^+$ is a q-measure on $\mathcal{S}(\rho_n)$ [3].

### 3 Classical Sequential Growth Processes

A *partially ordered set* (poset) is a set $x$ together with an irreflexive, transitive relation $<$ on $x$. In this work we only consider unlabeled posets and isomorphic posets are considered to be identical. Let $\mathcal{P}_n$ be the collection of all posets with cardinality $n$, $n = 1, 2, \ldots$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then $x$ *produces* $y$ if $y$ is obtained from $x$ by adjoining a single new element to $x$ that is maximal in $y$. We also say that $x$ is a *producer* of $y$ and $y$ is an *offspring* of $x$. If $x$ produces $y$ we write $x \rightarrow y$. We denote the set of offspring of $x$ by $x\rightarrow$ and for $A \subseteq \mathcal{P}_n$ we use the notation

$$\mathcal{A} \rightarrow = \{y \in \mathcal{P}_{n+1}; x \rightarrow y, x \in A\}$$

The transitive closure of $\rightarrow$ makes the set of all finite posets $\mathcal{P} = \bigcup \mathcal{P}_n$ into a poset.

A *path* in $\mathcal{P}$ is a string (sequence) $x_1, x_2, \ldots$ where $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$, $i = 1, 2, \ldots$. An *n-path* in $\mathcal{P}$ is a finite string $x_1x_2\cdots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$. We denote the set of paths by $\Omega$ and the set of n-paths by $\Omega_n$. The set of paths whose initial n-path is $\omega_0 \in \Omega_n$ is denoted by $\omega_0 \Rightarrow$. Thus, if $\omega_0 = x_1x_2\cdots x_n$ then

$$\omega_0 \Rightarrow = \{\omega \in \Omega; \omega = x_1, x_2\cdots x_ny_{n+1}y_{n+2}\cdots\}$$

If $x$ produces $y$ in $r$ isomorphic ways, we say that the *multiplicity* of $x \rightarrow y$ is $r$ and write $m(x \rightarrow y) = r$. For example, in Figure 1, $m(x \rightarrow y) = 3$. (To be precise, these different isomorphic ways require a labeling of the posets and this is the only place that labeling needs to be mentioned.)
Figure 1

If \( x \in \mathcal{P} \) and \( a, b \in x \) we say that \( a \) is an ancestor of \( b \) and \( b \) is a successor of \( a \) if \( a < b \). We say that \( a \) is a parent of \( b \) and \( b \) is a child of \( a \) if \( a < b \) and there is no \( c \in x \) such that \( a < c < b \). Let \( c = (c_0, c_1, \ldots) \) be a sequence of nonnegative numbers called coupling constants [5, 9]. For \( r, s \in \mathbb{N} \) with \( r \leq s \), we define

\[
\lambda_c(s, r) = \sum_{k=r}^{s} \binom{s-r}{k-r} c_k = \sum_{k=0}^{s-r} \binom{s-r}{k} c_{r+k}
\]

For \( x \in \mathcal{P}_n, y \in \mathcal{P}_{n+1} \) with \( x \rightarrow y \) we define the transition probability

\[
p_c(x \rightarrow y) = m(x \rightarrow y) \frac{\lambda_c(\alpha, \pi)}{\lambda_c(n, 0)}
\]

where \( \alpha \) is the number of ancestors and \( \pi \) the number of parents of the adjoined maximal element in \( y \) that produces \( y \) from \( x \). It is shown in [5, 9] that \( p_c(x \rightarrow y) \) is a probability distribution in that it satisfies the Markov-sum rule

\[
\sum \{ p_c(x \rightarrow y) : y \in \mathcal{P}_{n+1}, x \rightarrow y \} = 1
\]

In discrete quantum gravity, the elements of \( \mathcal{P} \) are thought of as causal sets and \( a < b \) is interpreted as \( b \) being in the causal future of \( a \). The distribution \( y \mapsto p_c(x \rightarrow y) \) is essentially the most general that is consistent with principles of causality and covariance [5, 9]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take \( c_k = 1/k! \) [6, 7]. The case \( c_k = c^k \) for some \( c > 0 \) has been previously studied and is called a percolation dynamics [5, 6, 8].

We call an element \( x \in \mathcal{P} \) a site and we sometimes call an \( n \)-path an \( n \)-universe and a path a universe. The set \( \mathcal{P} \) together with the set of transition probabilities \( p_c(x \rightarrow y) \) forms a classical sequential growth process (CSGP)
which we denote by \((P, p_c)\) [4, 5, 6, 8, 9]. It is clear that \((P, p_c)\) is a Markov
chain and as usual we define the probability of an \(n\)-path \(\omega = y_1y_2\cdots y_n\) by

\[
p^n_c(\omega) = p_c(y_1 \rightarrow y_2)p_c(y_2 \rightarrow y_3)\cdots p_c(y_{n-1} \rightarrow y_n)
\]

Denoting the power set of \(\Omega_n\) by \(2^{\Omega_n}\), \((\Omega_n, 2^{\Omega_n}, p^n_c)\) becomes a probability
space where

\[
p^n_c(A) = \sum \{p^n_c(\omega) : \omega \in A\}
\]

for all \(A \in 2^{\Omega_n}\). The probability of a site \(x \in P_n\) is

\[
p^n_c(x) = \sum \{p^n_c(\omega) : \omega \in \Omega_n, \omega \text{ ends at } x\}
\]

Of course, \(x \mapsto p^n_c(x)\) is a probability measure on \(P_n\) and we have

\[
\sum_{x \in P_n} p^n_c(x) = 1
\]

**Example 1.** Figure 2 illustrates the first two steps of a CSGP where the
2 indicates the multiplicity \(m(x_3 \rightarrow x_6) = 2\). Table 1 lists the probabilities
of the various sites for the general coupling constants \(c_k\) and the particular
coupling constants \(c'_k = 1/k!\) where \(d = (c_0 + c_1)(c_0 + 2c_1 + c_2)\).
For $A \subseteq \Omega_n$ we use the notation

$$A \Rightarrow = \cup \{ \omega \Rightarrow : \omega \in A \}$$

Thus, $A \Rightarrow$ is the set of paths whose initial $n$-paths are elements of $A$. We call $A \Rightarrow$ a cylinder set and define

$$\mathcal{A}_n = \{ A \Rightarrow : A \subseteq \Omega_n \}$$

In particular, if $\omega \in \Omega_n$ then the elementary cylinder set $\text{cyl}(\omega)$ is given by $\text{cyl}(\omega) = \omega \Rightarrow$. It is easy to check that the $\mathcal{A}_n$ form an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of algebras on $\Omega$ and hence $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$ is an algebra of subsets of $\Omega$. Also for $A \in \mathcal{C}(\Omega)$ of the form $A = A_1 \Rightarrow$, $A_1 \subseteq \Omega_n$, we define $p_c(A) = p_c^n(A_1)$. It is easy to check that $p_c$ is a well-defined probability measure on $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that $p_c$ has a unique extension to a probability measure $\nu_c$ on the $\sigma$-algebra $\mathcal{A}$ generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_c)$ is a probability space, the increasing sequence of subalgebras $\mathcal{A}_n$ generates $\mathcal{A}$ and that the restriction $\nu_c | \mathcal{A}_n = p^n_c$. Hence, the subspaces $H_n = L_2(\Omega, \mathcal{A}, p^n_c)$ form a filtration of the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$.

### 4 Quantum Sequential Growth Processes

This section employs the framework of Section 2 to obtain a quantum sequential growth process (QSGP) from the CSGP $(\mathcal{P}, p_c)$ developed in Section 3. We have seen that the $n$-path Hilbert space $H_n = L_2(\Omega, \mathcal{A}_n, p^n_c)$ forms a filtration of the path Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu_c)$. In the sequel, we assume that $p^n_c(\omega) \neq 0$ for every $\omega \in \Omega_n$, $n = 1, 2, \ldots$. Then the set of vectors

$$e^n_\omega = p^n_c(\omega)^{1/2} \chi_{\text{cyl}(\omega)}, \omega \in \Omega_n$$

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_c^n(x_i)$</td>
<td>1</td>
<td>$\frac{c_1}{c_0+c_1}$</td>
<td>$\frac{c_0}{c_0+c_1}$</td>
<td>$\frac{c_1(c_1+c_2)}{d}$</td>
<td>$\frac{c_1^2}{d}$</td>
<td>$\frac{3c_0c_1}{d}$</td>
<td>$\frac{c_0c_2}{d}$</td>
<td>$\frac{c_1^2}{d}$</td>
</tr>
<tr>
<td>$p^n_c(x_i)$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{14}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{1}{14}$</td>
<td>$\frac{1}{7}$</td>
</tr>
</tbody>
</table>

**Table 1**
form an orthonormal basis for $H_n$, $n = 1, 2, \ldots$. For $A \in \mathcal{A}_n$, notice that $\chi_A \in H$ with $\|\chi_A\| = p^n(A)^{1/2}$.

We call a DQP $\rho_n \in \mathcal{Q}(H_n)$ a quantum sequential growth process (QSGP). We call $\rho_n$ the local operators and $\mu_n(A) = D_n(A, A)$ the local $q$-measures for the process. If $\rho = \lim \rho_n$ exists in the strong operator topology, then $\rho$ is a $q$-probability operator on $H$ called the global operator for the process. If the global operator $\rho$ exists, then $\hat{\mu}(A) = \langle \rho \chi_A, \chi_A \rangle$ is a (continuous) $q$-measure on $\mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. Unfortunately, the global operator does not exist, in general, so we must be content to work with the local operators [2, 3, 7]. In this case, we still have the $q$-measure $\mu$ on the quadratic algebra $\mathcal{S}(\rho_n) \subseteq \mathcal{A}$ that extends $\mu_n$, $n = 1, 2, \ldots$. We frequently identify a set $A \subseteq \Omega_n$ with the corresponding cylinder set $(A \Rightarrow) \in \mathcal{A}_n$. We then have the $q$-measure, also denoted by $\mu_n$, on $2^{\Omega_n}$ defined by $\mu_n(A) = \mu_n(A \Rightarrow)$. Moreover, we define the $q$-measure, again denoted by $\mu_n$, on $\mathcal{P}_n$ by

$$\mu_n(A) = \mu_n(\{\omega \in \Omega_n: \omega \text{ end in } A\})$$

for all $A \subseteq \mathcal{P}_n$. In particular, for $x \in \mathcal{P}_n$ we have

$$\mu_n(\{x\}) = \mu_n(\{\omega \in \Omega_n: \omega \text{ ends with } x\})$$

If $A \in \mathcal{A}_n$ has the form $A_1 \Rightarrow$ for $A_1 \subseteq \Omega_n$ then $A \in \mathcal{A}_{n+1}$ and $A = (A_1 \Rightarrow) \Rightarrow$ where $A_1 \Rightarrow \subseteq \Omega_{n+1}$. Let $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and let $D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$, $D_{n+1}(A, B) = \langle \rho_{n+1} \chi_B, \chi_A \rangle$ be the corresponding decoherence functionals. Then $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $A, B \subseteq \Omega_n$ we have

$$D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = D_n(A \Rightarrow, B \Rightarrow) \tag{4.1}$$

**Lemma 4.1.** For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for all $\omega, \omega' \in \Omega_n$ we have

$$D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow] = D_n(\omega \Rightarrow, \omega' \Rightarrow) \tag{4.2}$$

**Proof.** Necessity is clear. For sufficiency, suppose (4.2) holds. Then for every $A, B \subseteq \Omega_n$ we have

$$D_{n+1}[(A \Rightarrow) \Rightarrow, (B \Rightarrow) \Rightarrow] = \sum_{\omega \in A} \sum_{\omega' \in B} D_{n+1}D_{n+1}[(\omega \Rightarrow) \Rightarrow, (\omega' \Rightarrow) \Rightarrow]$$

$$= \sum_{\omega \in A} \sum_{\omega' \in B} D_n(\omega \Rightarrow, \omega' \Rightarrow) = D_n(A \Rightarrow, B \Rightarrow)$$

and the result follows from (4.1). \(\square\)
For $\omega = x_1 x_2 \cdots x_n \in \Omega_n$ and $y \in \mathcal{P}_{n+1}$ with $x_n \to y$ we use the notation $\omega y \in \Omega_{n+1}$ where $\omega y = x_1 x_2 \cdots x_n y$. We also define $p_c(\omega \to y) = p_c(x_n \to y)$ and write $\omega \to y$ whenever $x_n \to y$.

**Theorem 4.2.** For $\rho_n \in \mathcal{Q}(H_n)$, $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ we have that $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have

$$\langle \rho_n e^n_{\omega'}, e^n_{\omega} \rangle = \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \frac{p_c(\omega' \to x)^{1/2} p_c(\omega \to y)^{1/2}}{\rho_{n+1} e^{n+1}_{\omega' x} e^{n+1}_{\omega y}}$$

(4.3)

**Proof.** By Lemma 4.1, $\rho_{n+1}$ is consistent with $\rho_n$ if and only if (4.2) holds. But

$$D_n(\omega \Rightarrow, \omega' \Rightarrow) = \langle \rho_n \chi_{\omega' \Rightarrow}, \chi_{\omega\Rightarrow} \rangle = \langle \rho_n \chi_{\text{cyl}(\omega')}, \chi_{\text{cyl}(\omega)} \rangle$$

$$= p^n_c(\omega')^{1/2} p^n_c(\omega)^{1/2} \langle \rho_n e^n_{\omega'}, e^n_{\omega} \rangle$$

Moreover, we have

$$D_{n+1}[(\omega \to) \Rightarrow, (\omega' \to) \Rightarrow] = \langle \rho_{n+1} \chi_{\omega' \to}, \chi_{\omega \to} \rangle$$

$$= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\omega' x \to}, \chi_{\omega y \to} \rangle$$

$$= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \langle \rho_{n+1} \chi_{\text{cyl}(\omega' x)}, \chi_{\text{cyl}(\omega y)} \rangle$$

$$= \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} p^n_c(\omega' x)^{1/2} p^n_c(\omega y)^{1/2} \langle \rho_{n+1} e^{n+1}_{\omega' x} e^{n+1}_{\omega y} \rangle$$

$$= p^n_c(\omega')^{1/2} p^n_c(\omega)^{1/2} \sum_{x \in \mathcal{P}_{n+1}} \sum_{y \in \mathcal{P}_{n+1}} \frac{p_c(\omega' \to x)^{1/2} p_c(\omega \to y)^{1/2}}{\rho_{n+1} e^{n+1}_{\omega' x} e^{n+1}_{\omega y}}$$

The result now follows. \(\square\)

Viewing $H_n$ as $L_2(\Omega_n, 2^{\Omega_n}, p^n_c)$ we can write (4.3) in the simple form

$$\langle \rho_n \chi(\omega'), \chi(\omega) \rangle = \langle \rho_{n+1} \chi_{\omega' \Rightarrow}, \chi_{\omega \Rightarrow} \rangle$$

(4.4)

**Corollary 4.3.** A sequence $\rho_n \in \mathcal{Q}(H_n)$ is a QSGP if and only if (4.3) or (4.4) hold for every $\omega, \omega' \in \Omega_n$, $n = 1, 2, \ldots$. 

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We now consider pure \( q \)-probability operators. In the following results we again view \( H_n \) as \( L_2(\Omega_n, 2^\Omega_n, p^n_c) \).

**Corollary 4.4.** If \( \rho_n \in Q_p(H_n), \rho_{n+1} \in Q_p(H_{n+1}) \) with \( p_n = |\psi_n\rangle\langle\psi_n|, \rho_{n+1} = |\psi_{n+1}\rangle\langle\psi_{n+1}| \), then \( \rho_{n+1} \) is consistent with \( \rho_n \) if and only if for every \( \omega, \omega' \in \Omega_n \) we have

\[
\langle \psi_n, \chi(\omega) \rangle \langle \chi(\omega'), \psi_n \rangle = \langle \psi_{n+1}, \chi_{\omega' \rightarrow} \rangle \langle \chi_{\omega \rightarrow}, \psi_{n+1} \rangle \quad (4.5)
\]

**Corollary 4.5.** A sequence \( |\psi_n\rangle\langle\psi_n| \in Q_p(H_n) \) is a QSGP if and only if (4.5) holds for every \( \omega, \omega' \in \Omega_n \).

We say that \( \psi_{n+1} \in V(H_{n+1}) \) is strongly consistent with \( \psi_n \in V(H_n) \) if for every \( \omega \in \Omega_n \) we have

\[
\langle \psi_n, \chi(\omega) \rangle = \langle \psi_{n+1}, \chi_{\omega \rightarrow} \rangle \quad (4.6)
\]

By (4.5) strong consistency implies the consistency of the corresponding \( q \)-probability operators.

**Corollary 4.6.** If \( \psi_{n+1} \in V(H_{n+1}) \) is strongly consistent with \( \psi_n \in V(H_n) \), \( n = 1, 2, \ldots, \) then \( |\psi_n\rangle\langle\psi_n| \in Q_p(H_n) \) is a QSGP.

**Lemma 4.7.** If \( \psi_n \in V(H_n) \) and \( \psi_{n+1} \in H_{n+1} \) satisfies (4.6) for every \( \omega \in \Omega_n \), then \( \psi_{n+1} \in V(H_{n+1}) \).

**Proof.** Since \( \psi_n \in V(H_n) \) we have by (4.6) that

\[
|\langle \psi_{n+1}, 1 \rangle| = \left| \sum_{\omega \in \Omega_n} \langle \psi_{n+1}, \chi_{\omega \rightarrow} \rangle \right| = \left| \sum_{\omega \in \Omega_n} \langle \psi_n, \chi(\omega) \rangle \right| = |\langle \psi_n, 1 \rangle| = 1
\]

The result now follows. \( \square \)

**Corollary 4.8.** If \( \|\psi_1\| = 1 \) and \( \psi_n \in H_n \) satisfies (4.6) for all \( \omega \in \Omega_n \), \( n = 1, 2, \ldots, \) then \( |\psi_n\rangle\langle\psi_n| \) is a QSGP.

**Proof.** Since \( \|\psi_1\| = 1 \), it follows that \( \psi_1 \in V(H_1) \). By Lemma 4.7, \( \psi_n \in V(H_n), n = 1, 2, \ldots. \) Since (4.6) holds, the result follows from Corollary 4.6. \( \square \)

Another way of writing (4.6) is

\[
\sum_{\omega \rightarrow x} p^{n+1}_c(\omega x) \psi_{n+1}(\omega x) = p^n_c(\omega) \psi_n(x) \quad (4.7)
\]

for every \( \omega \in \Omega_n \).
5 Discrete Quantum Gravity Models

This section gives some examples of QSGP that can serve as models for discrete quantum gravity. The simplest way to construct a QSGP is to form the constant pure DQP \( \rho_n = \lvert 1 \rangle \langle 1 \rvert \), \( n = 1, 2, \ldots \). To show that \( \rho_n \) is indeed consistent, we have for \( \omega \in \Omega_n \) that

\[
\sum_{\omega \rightarrow x} p_c^{n+1}(\omega x) = \sum_{\omega \rightarrow x} p_c^n(\omega) p_c(\omega \rightarrow x) = p_c^n(\omega) \sum_{\omega \rightarrow x} p_c(\omega \rightarrow x) = p_c^n(\omega)
\]

so consistency follows from (4.7). The corresponding \( q \)-measures are given by

\[
\mu_n(A) = \lvert \langle 1, \chi_A \rangle \rvert^2 = p_c^n(A)^2
\]

for every \( A \in A_n \). Hence, \( \mu_n \) is the square of the classical measure. Of course, \( \lvert 1 \rangle \langle 1 \rvert \) is the global \( q \)-probability operator for this QSGP and in this case \( S(\rho_n) = A \). Moreover, we have the global \( q \)-measure \( \mu(A) = \nu(A)^2 \) for \( A \in A \).

Another simple way to construct a QSGP is to employ Corollary 4.8. In this way we can let \( \psi_1 = 1 \), \( \psi_2 \) any vector in \( L_2(\Omega_2, 2^{\Omega_2}, p_c^2) \) satisfying

\[
\langle \psi_2, \chi_{\{x_1 x_2\}} \rangle + \langle \psi_2, \chi_{\{x_1 x_3\}} \rangle = \langle \psi_1, \chi_{\{x_1\}} \rangle = 1
\]

and so on, where \( x_1, x_2, x_3 \) are given in Figure 2. As a concrete example, let \( \psi_1 = 1 \),

\[
\psi_2 = \frac{1}{2} \left[ p_c^2(x_1 x_2)^{-1} \chi_{\{x_1 x_2\}} + p_c^2(x_1 x_3)^{-1} \chi_{\{x_1 x_3\}} \right]
\]

and in general

\[
\psi_n = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} p_c^n(\omega)^{-1} \chi(\omega)
\]

The \( q \)-measure \( \mu_1(\{x_1\}) = 1 \) and \( \mu_2 \) is given by

\[
\mu_2(\{x_1 x_2\}) = \lvert \langle \psi_2, \chi_{\{x_1 x_2\}} \rangle \rvert^2 = \frac{1}{4}
\]

\[
\mu_2(\{x_1 x_3\}) = \lvert \langle \psi_2, \chi_{\{x_1 x_3\}} \rangle \rvert^2 = \frac{1}{4}
\]

\[
\mu_2(\Omega_2) = \lvert \langle \psi_2, 1 \rangle \rvert^2 = 1
\]

In general, we have \( \mu_n(A) = |A|^2 / |\Omega_n|^2 \) for all \( A \in \Omega_n \). Thus \( \mu_n \) is the square of the uniform distribution. The global operator does not exist because there is no \( q \)-measure on \( A \) that extends \( \mu_n \) for all \( n \in \mathbb{N} \). For \( A \in A \) we have

\[
\langle \psi_n, \chi_A \rangle = \int \psi_n \chi_A d\nu_c = \frac{|A \cap \{\text{cyl(}\omega): \omega \in \Omega_n\}|}{|\Omega_n|}
\]
Letting $\rho_n = |\psi_n\rangle\langle\psi_n|$ we conclude that $A \in \mathcal{S}(\rho_n)$ if and only if

$$\lim_{n \to \infty} \frac{|A \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}|}{|\Omega_n|}$$

exists. For example, if $|A| < \infty$ then for $n$ sufficiently large we have

$$|A \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}| = |A|$$

so $A \in \mathcal{S}(\rho_n)$ and $\mu(A) = 0$. In a similar way if $|A| < \infty$ then for the complement $A'$, if $n$ is sufficiently large we have

$$|A' \cap \{\text{cyl}(\omega): \omega \in \Omega_n\}| = |\Omega_n| - |A|$$

so $A' \in \mathcal{S}(\rho_n)$ with $\mu(A') = 1$.

We now present another method for constructing a QSGP. Unlike the previous method this DQP is not pure. Let $\alpha_\omega \in \mathbb{C}$, $\omega \in \Omega_n$ satisfy

$$\left| \sum_{\omega \in \Omega_n} \alpha_\omega p^n_c(\omega)^{1/2} \right| = 1$$

and let $\rho_n$ be the operator on $H_n$ satisfying

$$\langle \rho_n e^n_n, e^n_{\omega'} \rangle = \alpha_\omega \overline{\alpha_\omega}$$

Then $\rho_n$ is a positive operator and by (5.1), (5.2) we have

$$\langle \rho_n 1, 1 \rangle = \left( \rho_n \sum_\omega p^n_c(\omega)^{1/2} e^n_\omega, \sum_{\omega'} p^n_c(\omega')^{1/2} e^n_{\omega'} \right)$$

$$= \sum_{\omega, \omega'} p^n_c(\omega)^{1/2} p^n_c(\omega')^{1/2} \langle \rho_n e^n_\omega, e^n_{\omega'} \rangle$$

$$= \left| \sum_{\omega} p^n_c(\omega)^{1/2} \alpha_\omega \right|^2 = 1$$

Hence, $\rho_n \in \mathcal{Q}(H_n)$. Now

$$\Omega_{n+1} = \{\omega x: \omega \in \Omega_n, x \in \mathcal{P}_{n+1}, \omega \to x\}$$
and for each $\omega x \in \Omega_{n+1}$, let $\beta_{\omega x} \in \mathbb{C}$ satisfy
\[
\left| \sum_{\omega x \in \Omega_{n+1}} \beta_{\omega x} \rho_{n+1}^c(\omega x)^{1/2} \right| = 1
\]

Let $\rho_{n+1}$ be the operator on $H_{n+1}$ satisfying
\[
\langle \rho_{n+1}^e_{\omega x}, e_{\omega x}^{n+1} \rangle = \beta_{\omega'x'} \beta_{\omega x} (5.3)
\]

As before, we have that $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$. The next result follows from Theorem 4.2.

**Theorem 5.1.** The operator $\rho_{n+1}$ is consistent with $\rho_n$ if and only if for every $\omega, \omega' \in \Omega_n$ we have
\[
\alpha_{\omega'} \alpha_{\omega} = \sum_{x' \in \mathcal{P}_{n+1}} \beta_{\omega'x'} \rho_c(\omega' \rightarrow x')^{1/2} \sum_{x \in \mathcal{P}_{n+1}} \beta_{\omega x} \rho_c(\omega \rightarrow x)^{1/2} (5.4)
\]

A sufficient condition for (5.4) to hold is
\[
\sum_{x \in \mathcal{P}_{n+1}} \beta_{\omega x} \rho_c(\omega \rightarrow x)^{1/2} = \alpha_{\omega} (5.5)
\]

The proof of the next result is similar to the proof of Lemma 4.7.

**Lemma 5.2.** Let $\rho_n \in \mathcal{Q}(H_n)$ be defined by (5.2) and let $\rho_{n+1}$ be the operator on $H_{n+1}$ defined by (5.3). If (5.5) holds, then $\rho_{n+1} \in \mathcal{Q}(H_{n+1})$ and $\rho_{n+1}$ is consistent with $\rho_n$.

The next result gives the general construction.

**Corollary 5.3.** Let $\rho_1 = I \in \mathcal{Q}(H_1)$ and define $\rho_n \in \mathcal{Q}(H_n)$ inductively by (5.3). Then $\rho_n$ is a QSGP.

**References**


