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Banach Spaces on Infinitely Branching Trees

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BANACH SPACES ON INFINITELY BRANCHING TREES

A Dissertation

Presented to

the Faculty of Natural Sciences and Mathematics

University of Denver

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Annette B. Locke

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Advisor: James N. Hagler

Abstract

An example of a Banach space, X_∞ , with a nonseparable dual such that l_1 does not imbed in X_∞ is investigated. Not every weakly null sequence has a subsequence equivalent to the usual basis of c_0 , but c_0 imbeds in many subspaces of X_∞ . The space l_1 does imbed in X_∞^* , the dual space of X_∞ , yet weakly converging sequences in X_∞^* need not converge in norm.

Acknowledgements

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This research would not have been possible without the generosity, insight, and tenacity of my advisor and mentor, James Hagler. As an undergraduate, he gave me the confidence I lacked to keep going. As a graduate student, my respect for him only grew and has been a continual source of strength. I once told him, many years ago, that if someone like James Hagler could believe in me, then I must believe in myself. I will be forever grateful.

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Introduction

We construct a Banach space X_∞ comprised of real-valued functions defined on an infinitely branching tree \mathfrak{S}_∞ . The actual construction of the space and the results that follow were greatly influenced by the Hagler Tree space, defined and studied in [4], which we denote by X throughout this paper. The space X is comprised of functions on a dyadic tree \mathfrak{S} . The space X_∞ is another example of a separable Banach space with a nonseparable dual such that l_1 does not imbed in X_∞ . The space X_∞ shares some other important properties with X , although the two spaces are not isomorphic.

The author would like to thank Professor Haskell Rosenthal for suggesting that constructing a space on an infinitely branching tree as a variation of X would be worthwhile and for pointing out that if the nodes in the tree are taken in a particular order, they give rise to a basis. We prove this result in Proposition 2.2.4 of Chapter 2.

After defining a partial order on the nodes in the tree \mathfrak{S}_∞ , we show that the infinite branching of \mathfrak{S}_∞ introduces some properties and complications in X_∞ that are not present in X . In particular, there are sequences of nodes in the tree \mathfrak{S}_∞ that have a property we call "strongly rooted", a concept which is defined in Definition 1.3.4 on page 14. There are no such sequences of nodes in the tree \mathfrak{S} used in [4]. In Section 1.3, we recall the definition of "strongly incomparable nodes" and nodes that uniquely determine a branch in the tree from [4]. We discuss in detail how sequences of nodes in \mathfrak{S}_∞ interact with one

another, and in Lemma 1.3.5 we extend the methods used in [4] to prove that any sequence of nodes in \mathfrak{S}_∞ has a subsequence that determines a unique branch, is a strongly rooted sequence, or is a strongly incomparable sequence.

While the tree \mathfrak{S} used by Hagler in [4] contains sequences of nodes that either determine a unique branch or are strongly incomparable, the appearance of strongly rooted sequences in \mathfrak{S}_∞ is a result of the infinite branching in the tree. Many of the main results we prove about the space X_∞ and X_∞^* are due to the presence of strongly rooted sequences in \mathfrak{S}_∞ . Particularly, certain sequences in X_∞ whose terms are functions defined on nodes forming a strongly rooted sequence are shown to contain no c_0 subsequence. This is one distinguishing characteristic between the spaces X_∞ and X . Another characteristic distinguishing the spaces X_∞ and X is that the dual X_∞^* of X_∞ contains sequences which converge weakly to zero but not in norm. We prove this by looking at linear functionals on elements in X_∞ that are nonzero on nodes of a strongly rooted sequence in \mathfrak{S}_∞ . These and other results regarding the space X_∞ are outlined below.

We prove the following results regarding the space X_∞ :

1. There exist basic sequences in X_∞ which converge weakly but not in norm to zero that have do not contain a c_0 subsequence. The first example of such a sequence is given in Section 3.3.

2. The space c_0 imbeds in X_∞ . In Section 3.2, we give several examples of basic sequences that are equivalent to the usual c_0 basis.
3. The closed linear span of many sequences in X_∞ that do not contain a c_0 subsequence is hereditarily c_0 . In Chapter 3, we show that if (x_k) satisfies one of two cases, then we can apply a blocking method to obtain a normalized block basic sequence of (x_k) that is a c_0 sequence. However, in Section 3.3 we show that these two cases are not enough as we give an example of a sequence that does not satisfy either of the cases we have handled earlier in this chapter. Our hope is to generalize this example in the future to show that X_∞ is hereditarily c_0 .

A basic sequence in X_∞ contains a subsequence satisfying one of three mutually exclusive cases. To discuss sequences in X_∞ , we need a few definitions which will be discussed in more detail in the body of the paper. If (x_k) is a basic sequence in X_∞ , then we define "the support of x_k " to be the subset $G_k \subseteq \mathfrak{S}_\infty$, $G_k = \{\psi \in \mathfrak{S}_\infty : x_k(\psi) \neq 0\}$. We assume that G_k is finite and if $k \neq j$, then $G_k \cap G_j = \emptyset$. In Section 1.1 we define a partial order, \leq , on the nodes of the tree \mathfrak{S}_∞ . If (x_k) satisfies case (i) or (ii) and it does not contain a c_0 subsequence, then in Sections 3.5 and 3.6 we show that (x_k) can be blocked in such a way that the block basic sequence we construct is a c_0 sequence. Case (iii) is the case that is left for future work.

- (i) For every $k \neq j$, every element in the support of x_k is incomparable, with respect to the partial order \leq , to every element in the support x_j ;

- (ii) The sequence (x_k) is a weakly null sequence, for every k , the support of x_k is higher in the tree than the support of x_{k+1} , and if $k \neq j$, there exists at least one element in the support of x_k that is comparable to an element in x_j ;
- (iii) The sequence (x_k) is a weakly null sequence, the support of all x_k begin at the same height in the tree, and if $k \neq j$, there exists at least one element in the support of x_k that is comparable to an element in x_j .

An example of a sequence satisfying (iii) is given in Section 3.3. In this section, we use a blocking method similar to that used by Casazza and Shura in [2] which we use throughout our analysis of the space X_∞ . The blocking method enables us to build a normalized block basic sequence of (x_k) that is a c_0 sequence. The hope is that a similar blocking method can be applied to the general case.

4. The dual space X_∞^* of X_∞ is nonseparable. This is an immediate consequence of Theorem 4.2.5 which shows that there exists a separable subspace F of X_∞^* such that $X_\infty^*/F \cong c_0(\Gamma)$ where Γ has cardinality \mathfrak{c} .
5. The space l_1 does not imbed in X_∞ . In Section 4.2, we first show that $X_\infty^{**} \approx F^* \oplus l_1(\Gamma)$ which in turn shows that X_∞^{**} has cardinality \mathfrak{c} . Then by Rosenthal's l_1 Theorem, we have that $l_1 \not\hookrightarrow X_\infty$.
6. The space l_1 imbeds in X_∞^* . In Section 4.3, we give an example of a sequence in X_∞^* that does not contain an l_1 subsequence. Once again, we apply a blocking method to construct a normalized block basic sequence in X_∞^* that is an l_1 sequence.

7. The space X_∞^* does not have the Schur property; i.e. there exists a sequence in X_∞^* that converges weakly but not in norm. One example of this is given in Section 4.3.

Although the spaces X_∞ and X are not isomorphic, some known results apply to both spaces. The following is a list of known results that the two spaces share.

- The space c_0 imbeds in both X and X_∞ . In fact, X is hereditarily c_0 . It is believed, but not yet known, that X_∞ is hereditarily c_0 .
- The space l_1 does not imbed in X or in X_∞ .
- The dual space of X is nonseparable as is the dual space of X_∞ .
- The space l_1 imbeds in both X^* and X_∞^* . In fact, X^* is hereditarily l_1 . It is not yet known if X_∞^* is hereditarily l_1 .

The spaces X_∞ and X also have some important differences. The following is a list of known results that the two spaces do not share.

- Every sequence in X which converges weakly but not in norm to zero has a c_0 subsequence. Proposition 3.3.1 on page 45 shows that this is not the case in X_∞ . It follows immediately that X_∞ and X are not isomorphic. In fact, X_∞ is not isomorphic to any subspace of X .

- X^* has the Schur property. Theorem 4.3.1 on page 96 shows that X_∞^* does not. It follows immediately that X_∞^* and X^* are not isomorphic. In fact, X_∞^* is not isomorphic to any subspace of X^* .

Let us briefly outline the organization of the paper. Chapter 1 gives the basic definitions concerning the tree and the nodes within the tree. The key definitions in this chapter are those of an "admissible family of segments" and "strongly rooted sequence of nodes" which can be found in Section 1.2 and Section 1.3 respectively. Also in Section 1.3, Lemma 1.3.5 shows that a sequence of nodes in the tree must contain a subsequence that falls into one of three mutually exclusive categories.

Chapter 2 defines the Banach space X_∞ . In particular, the norm is defined in Section 2.2 as well as several examples illustrating the norm. Definitions that are used throughout the paper to describe sequences are also given in this section.

Chapter 3 contains some of our main results. In Section 3.2 we look at several sequences that are c_0 sequences, while in Sections 3.3 and 3.4 we examine some sequences that are not c_0 sequences. The key results in the remainder of this chapter are Lemma 3.5.2, Theorem 3.5.3 and Theorem 3.6.1.

Chapter 4 investigates properties of the dual space X_∞^* . The key results in this chapter are Theorem 4.2.5, from which it follows that X_∞^* is nonseparable, Theorem 4.2.7, from which it follows that l_1 does not imbed in X_∞ , and Theorem 4.3.1, from which it follows that X_∞^* does not have the Schur property.

Chapter 5 discusses some open problems and direction for future work. The reader may wish to visit the Appendix for standard notation and definitions which are used throughout this paper.

Chapter 1

Tree Structure

We begin by defining the tree on which our Banach space is built. In Section 1.1 the tree itself is defined and some standard definitions are given. In Section 1.2 we define the structural components of the tree while Section 1.3 gives some combinatorial properties involving the nodes of the tree. Many of the definitions in this chapter can also be found in [4]. The key new combinatorial idea, that of a strongly rooted sequence, is defined on page 14 in Definition 1.3.4. The main result of this chapter is Lemma 1.3.5 on page 16 which shows that a sequence of nodes in the tree contains a subsequence that satisfies one of three mutually exclusive properties.

1.1 Tree Definition

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ be the set of nonnegative integers. Let $\theta_0 = (0)$. For each $k \in \mathbb{N}_0, k > 0$, let $\theta_k = (\varepsilon_{m_k} \cdots \varepsilon_0)$ be the binary representation of k with $\varepsilon_{m_k} = 1$. Define \leq on the set $\bigcup_{k=0}^{\infty} \{\theta_k\}$ by $\theta_k \leq \theta_j$ if $\theta_k = (\varepsilon_{m_k} \cdots \varepsilon_0)$ and $\theta_j = (\varepsilon_{m_j} \cdots \varepsilon_{m_k+1} \varepsilon_{m_k} \cdots \varepsilon_0)$.

In other words, $\theta_0 \leq \theta_k$ for all k , and for $k > 0$, $\theta_k \leq \theta_j$ if the binary expansion of k agrees with the binary expansion of j on the first m_k terms. It is easy to check that \leq is a partial order. If $\theta_k \leq \theta_j$ and $\theta_k \neq \theta_j$, we write $\theta_k < \theta_j$.

Define the infinitely branching tree \mathfrak{S}_∞ to be the set $\mathfrak{S}_\infty = \bigcup_{k=0}^\infty \{\theta_k\}$ with the partial order \leq . The elements of \mathfrak{S}_∞ are called *nodes*. Figure 1 illustrates the order of the nodes of \mathfrak{S}_∞ . If a node θ_k is above a node θ_j and there is a line connecting θ_k and θ_j , then $\theta_k < \theta_j$.

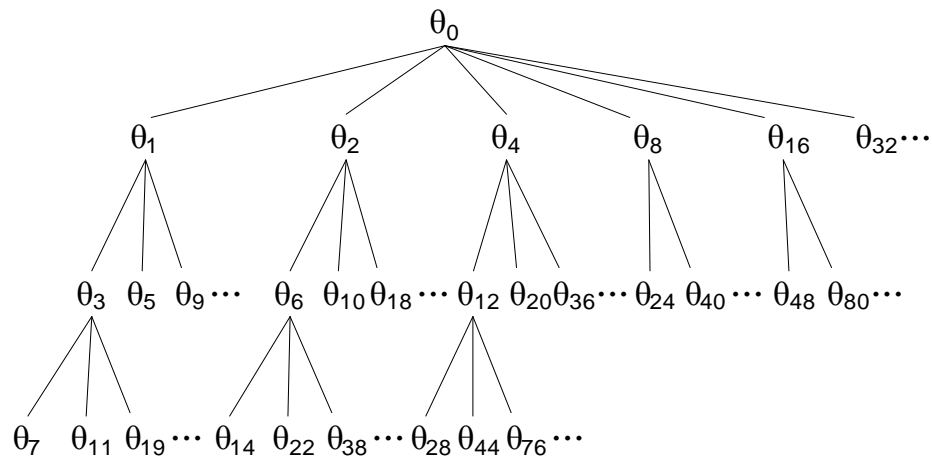


Figure 1 Structure of \mathfrak{S}_∞

Many times we want to describe where in the tree structure a particular node lives and how it interacts with other nodes in the tree. One notion that is useful in this area is that of levels.

Definition 1.1.1 Let $\theta_k \in \mathfrak{S}_\infty$. The *level* of the node θ_k , written $lev(\theta_k)$, is the number of 1's in the binary representation of k . For example, $lev(\theta_0) = 0$, $lev(\theta_1) = 1$, $lev(\theta_2) = 1$, $lev(\theta_4) = 1$. In fact, $lev(\theta_{2^j}) = 1$ for all $j \geq 0$. Similarly, $lev(\theta_3) = 2$, $lev(\theta_5) = 2$, etc. In other words, $lev(\theta_{2^{j_1+2^{j_2}}}) = 2$ for all $j_1 \geq 0$, $j_2 \geq 1$. In general, $lev(\theta_k) = n_k$ where $k = \sum_{i=1}^{n_k} 2^{j_i}$ and $j_{i+1} > j_i \geq i - 1$, as shown in Figure 2.

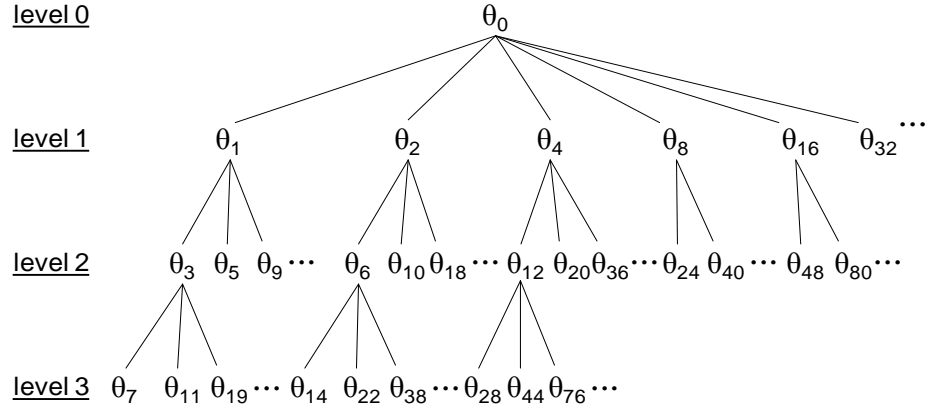


Figure 2 Levels of \mathfrak{S}_∞

Definition 1.1.2 A node ψ_k is the *predecessor* of a node ψ_j if $\psi_k < \psi_j$ and $lev(\psi_k) = lev(\psi_j) - 1$. In this case, we write $\psi_k = pred(\psi_j)$.

1.2 Segments in \mathfrak{S}_∞

In order to define the Banach space X_∞ , we need to define structure on subsets of \mathfrak{S}_∞ .

Definition 1.2.1 Let $S \subseteq \mathfrak{S}_\infty$ be a finite set.

1. The set S is a *segment* if $S = \{\psi_1, \dots, \psi_k\}$ where for each $i = 1, \dots, k-1$,
 $\psi_i = \text{pred}(\psi_{i+1})$.
2. The segment S is an *m-segment* if $\text{lev}(\psi_1) = m$, that is, S begins at level m .
3. The segment S is an *m-n segment* if $\text{lev}(\psi_1) = m$ and $\text{lev}(\psi_k) = n$, that is, S begins at level m and ends at level n .
4. The *length* of S is the cardinality of S .
5. The segment S *passes through a node* ψ if $\psi \in S$.

Observe that if $S = \{\theta_{k_1}, \dots, \theta_{k_n}\}$ is an m -segment, then $k_1 < \dots < k_n$ and for any j , $1 \leq j < n$, $\{\theta_{k_1}, \dots, \theta_{k_j}\}$ is also an m -segment.

A key idea when defining the norm associated with X_∞ is that of an admissible family of segments.

Definition 1.2.2 A family of segments, F , is *admissible* if there exists $m \in \mathbb{N}$ such that

1. for each segment $S \in F$, S is an m -segment;
2. the cardinality of F is less than or equal to $m + 1$;
3. if $S, R \in F$, $R \neq S$, then $R \cap S = \emptyset$.

So, an admissible family of segments is a collection of pairwise disjoint segments that begin at the same level. If every segment in the admissible family begins at level m , then there are no more than $m + 1$ segments in the family. Note that the segments in an admissible family of segments may end at different levels. We let \mathcal{F} denote the class of all admissible families of segments. Figure 3 shows an admissible family of 5-segments.

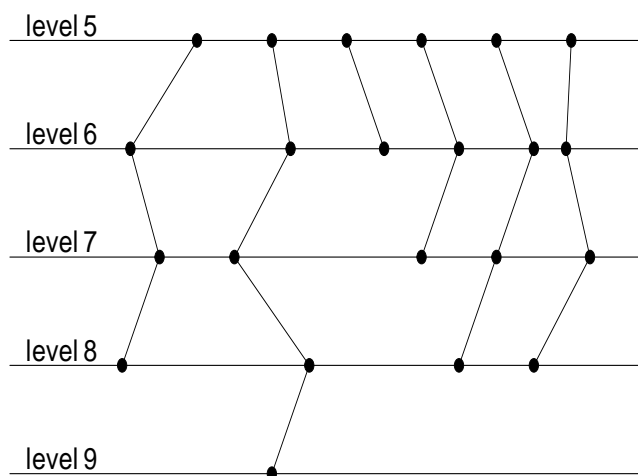


Figure 3 An admissible family of 5-segments

Definition 1.2.3 A *branch* B of \mathfrak{S}_∞ is an infinite sequence of nodes $(\psi_0, \psi_1, \psi_2, \dots)$ such that

1. $\psi_0 = \theta_0$;
2. $\psi_k = \text{pred}(\psi_{k+1})$ for each k .

1.3 Interaction of Nodes and Comparability

Another useful notion regarding the interaction of nodes is that of comparability.

Definition 1.3.1 Let ψ_k and ψ_j be distinct nodes in \mathfrak{S}_∞ .

1. If $\psi_k \leq \psi_j$, we say ψ_k and ψ_j are *comparable*.
2. If $\psi_k \not\leq \psi_j$ and $\psi_j \not\leq \psi_k$, then we say ψ_k and ψ_j are *incomparable*. We write $\psi_k \parallel \psi_j$ if ψ_k and ψ_j are incomparable.

Definition 1.3.2 Let $A, B \subseteq \mathfrak{S}_\infty$.

1. The set A is *pairwise incomparable* if for all $\psi_k, \psi_j \in A$, $k \neq j$, we have $\psi_k \parallel \psi_j$.
2. The pair of subsets A, B is *pairwise incomparable* if $A \cup B$ is a pairwise incomparable set. If there is a $\psi_k \in A$ and $\psi_j \in B$ such that ψ_k and ψ_j are comparable, then A and B are *comparable sets*.

The behavior of infinite sequences of nodes in \mathfrak{S}_∞ is of particular interest. Note that if (ψ_k) is a pairwise incomparable sequence in \mathfrak{S}_∞ and F is an admissible family of segments, then for each segment $S \in F$, there is at most one integer k such that S passes through ψ_k .

Definition 1.3.3 Let (ψ_k) be a pairwise incomparable sequence of nodes in \mathfrak{S}_∞ .

The sequence (ψ_k) is *strongly incomparable* if for any admissible family of segments

S_1, \dots, S_{l+1} , then $\text{card} \{k : \psi_k \in S_i \text{ for some } i = 1, \dots, l + 1\} \leq 2$. In other words, a sequence of nodes is strongly incomparable if any admissible family of segments passes through at most two of the nodes in the sequence, as shown in Figure 4.

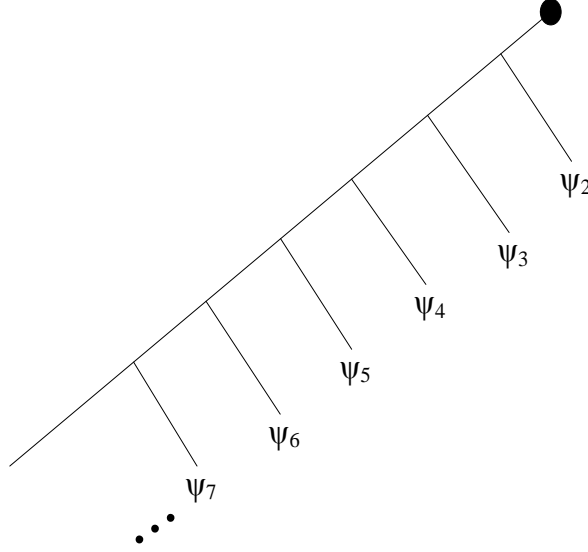


Figure 4 Strongly Incomparable Sequence

Definition 1.3.4 Let (ψ_k) be a pairwise incomparable sequence of nodes in \mathfrak{S}_∞ . The sequence (ψ_k) is *strongly rooted* if there exists $\psi \in \mathfrak{S}_\infty$ such that $\psi < \psi_k$ for all k and for all $\psi' > \psi$, we have $\psi' \leq \psi_k$ for at most one ψ_k . In this case, we say that the node ψ is a *strong root*.

Figure 5 shows a strongly rooted sequence (ψ_k) in \mathfrak{S}_∞ where $\text{lev}(\psi_k) < \text{lev}(\psi_{k+1})$ for all k .

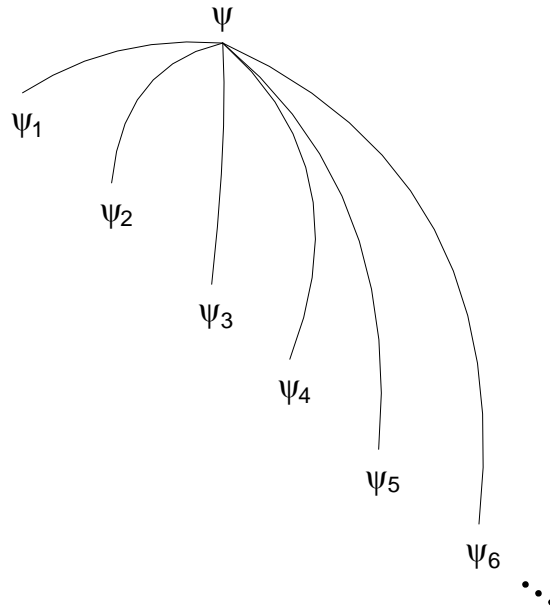


Figure 5 Strongly Rooted Sequence with Increasing Levels

Figure 6 shows a strongly rooted sequence (ψ_k) in \mathfrak{S}_∞ where $lev(\psi_k) = lev(\psi_{k+1})$ for all k .

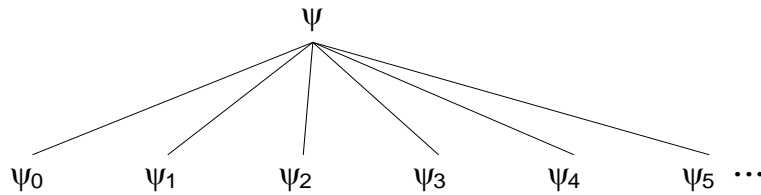


Figure 6 Strongly Rooted Sequence On the Same Level

The next lemma shows that given a sequence of nodes in \mathfrak{S}_∞ , there is a subsequence that adheres to one of the three mutually exclusive structures of sequences of nodes; it determines a branch, is strongly incomparable, or is strongly rooted. This result is similar to Lemma 2 in [4] with the added notion of strongly rooted sequences. In general, when referring to a subsequence of a sequence, we do not reindex unless it is necessary. For example, we may say that there is a subsequence (ψ_k) of (ψ_k) to mean there is a subsequence (ψ_{k_n}) of (ψ_k) .

Lemma 1.3.5 Let (ψ_k) be a sequence of nodes in \mathfrak{S}_∞ with $lev(\psi_k) \leq lev(\psi_{k+1})$.

Then there is a subsequence (ψ_k) of (ψ_k) that is either

1. a sequence that determines a unique branch;
2. a strongly rooted sequence;
3. a strongly incomparable sequence.

Proof If $k > j$, then either $\psi_k > \psi_j$ or $\psi_k \parallel \psi_j$. By Ramsey's Theorem, found in [5] for example, we can pick a subsequence (ψ_k) of (ψ_k) that is either a sequence that determines a unique branch or a pairwise incomparable sequence. Now we must show that if we picked a subsequence that is pairwise incomparable, then a further subsequence of (ψ_k) is either strongly rooted or strongly incomparable.

Let

$$A = \{\varphi \in \mathfrak{S}_\infty : \{k : \psi_k > \varphi\} \text{ is infinite}\}.$$

If A has a maximal element φ with respect to the ordering on the tree, then infinitely many of the ψ_k 's lie below φ as shown in Figure 7. It is easy to see that φ is a strong root for this subsequence.

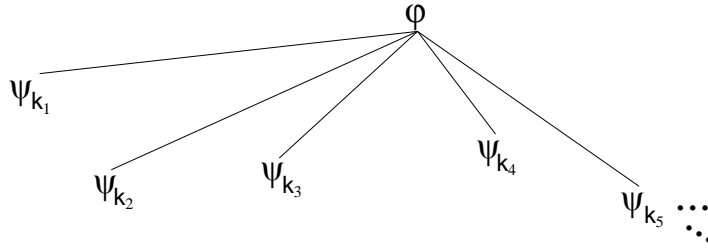


Figure 7 φ is maximal in A

If A has no maximal element, then for all $\varphi \in A$, there is $\varphi' \in A$ with $\varphi' > \varphi$. We use this to build a strongly incomparable subsequence (ψ_k) of (ψ_k) . Let $\varphi_1 = \min A$. Pick $\psi_1 > \varphi_1$. Let $\varphi_2 \in A$ be such that $\varphi_2 > \varphi_1$ and $lev(\varphi_2) \geq lev(\psi_1)$. See Figure 8. Note that since (ψ_k) is pairwise incomparable and $\varphi_2 \in A$, then $\psi_1 \parallel \varphi_2$.

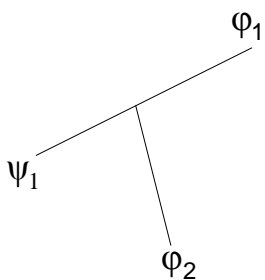


Figure 8

Pick $\psi_2 > \varphi_2$. Continue inductively to obtain a sequence (ψ_k) of (ψ_k) and an increasing sequence (φ_k) in A such that $\psi_k > \varphi_k$ and $lev(\varphi_{k+1}) > lev(\psi_k)$.

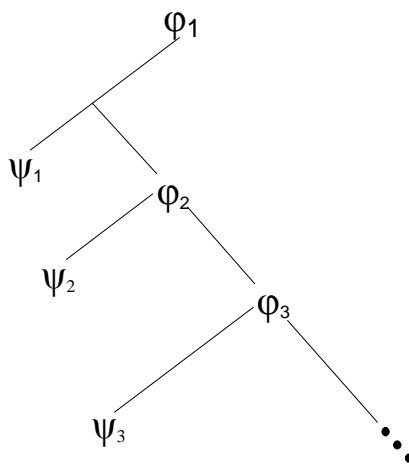


Figure 9 (ψ_k) is strongly incomparable

To show that (ψ_k) is a strongly incomparable sequence, let F be an admissible family of segments. Let

$$k_0 = \min \{k : \text{there is a segment } S \in F \text{ passing through } \psi_k\}.$$

For any $k > k_0$, $\psi_k > \varphi_k \geq \varphi_{k_0+1}$ and $lev(\varphi_{k_0+1}) \geq lev(\psi_{k_0})$. So any segment in F passing through ψ_k must pass through φ_{k_0+1} . Clearly, only one $S \in F$ can pass through φ_{k_0+1} . ■

Chapter 2

The Banach Space X_∞

In this chapter, the Banach space X_∞ is defined. We make some definitions and examine several examples which we use frequently in our discussion of X_∞ . The Hagler Tree space, which we denote by X throughout this paper, defined in [4] provides the inspiration for the definition and the study of the space X_∞ . Consequently, many of the definitions found in this chapter can also be found in [4]. Recall that X is a space of functions defined on a dyadic tree. The space X_∞ is a variation of X in that it is a space of functions defined on the infinitely branching tree \mathfrak{S}_∞ . The infinite branching of the tree creates properties of X_∞ that are not characteristic of X . We describe properties that X_∞ and X have in common, as well as properties for which the two spaces differ. It is these differences that provide the motivation for our analysis of X_∞ .

2.1 Finitely Nonzero Functions on \mathfrak{S}_∞

Before discussing either of the two Banach spaces X and X_∞ , we need to look at the space of finitely nonzero functions on the infinitely branching tree \mathfrak{S}_∞ . The Banach space X_∞ is ultimately defined as the completion of this space with a particular norm.

Let $x : \mathfrak{S}_\infty \rightarrow \mathbb{R}$ be a finitely nonzero function, i.e. the set $\{\psi \in \mathfrak{S}_\infty : x(\psi) \neq 0\}$ is finite. We denote $x = \{a_\psi : \psi \in \mathfrak{S}_\infty\}$ where $a_\psi = x(\psi)$.

Let's give the space of finitely nonzero functions on the infinitely branching tree \mathfrak{S}_∞ a name which we use throughout our discussion. The set Λ is defined to be

$$\Lambda = \{x : \mathfrak{S}_\infty \rightarrow \mathbb{R}, x \text{ is finitely nonzero}\}.$$

Before we proceed, we need a few definitions regarding this space. For each segment and each branch in \mathfrak{S}_∞ , we associate a linear functional as given by the following definition.

Definition 2.1.1 Let $x \in \Lambda$, $x = \{a_\psi : \psi \in \mathfrak{S}_\infty\}$.

1. Let S be a segment in \mathfrak{S}_∞ . The *linear functional*, S^* , generated by the segment S , is defined by

$$S^*(x) = \sum_{\psi \in S} a_\psi$$

If $S = \{\psi\}$, we write ψ^* instead of $\{\psi\}^*$.

2. Let B be a branch in \mathfrak{S}_∞ . The *linear functional*, B^* , generated by the branch B , is defined by

$$B^*(x) = \sum_{\psi \in B} a_\psi.$$

We now define some projections on a vector $x \in \Lambda$. We begin by defining projections in terms of a single node in the tree, then projections in terms of a fixed level in the tree, and finally, projections in terms of a fixed branch in the tree. Once the norm is defined on X_∞ , it is immediate that these projections are defined on all $x \in \mathfrak{S}_\infty$.

Definition 2.1.2 Let $x \in \Lambda$, $x = \{a_\psi : \psi \in \mathfrak{S}_\infty\}$.

1. Let $\varphi \in \mathfrak{S}_\infty$. The projection $P_\varphi : \Lambda \rightarrow \Lambda$ defined by

$$P_\varphi x(\psi) = \begin{cases} a_\psi & \text{if } \psi \geq \varphi \\ 0 & \text{if } \psi < \varphi \end{cases}$$

is a *single-node projection* by φ .

2. Let $n \geq 0$ be an integer. The projection $P_n : \Lambda \rightarrow \Lambda$ is defined by

$$P_n x(\psi) = \begin{cases} a_\psi & \text{if } \text{lev}(\psi) \geq n \\ 0 & \text{if } \text{lev}(\psi) < n \end{cases}$$

3. Let B be a branch in \mathfrak{S}_∞ . The projection $P_B : \Lambda \rightarrow \Lambda$ is defined by

$$P_B x(\psi) = \begin{cases} a_\psi & \text{if } \psi \in B \\ 0 & \text{if } \psi \notin B \end{cases}$$

2.2 Definition of the Banach space X_∞

We now define the Banach space X_∞ . We begin with the definition of the norm on a vector $x \in \Lambda$ in terms of admissible families of segments. Let $x : \mathfrak{S}_\infty \rightarrow \mathbb{R}$ be a finitely

nonzero function. Define

$$\|x\| = \max \sum_{j=1}^{\ell+1} |S_j^*(x)|$$

where the max is taken over all admissible families of segments $S_1, \dots, S_{\ell+1}$. Since it is immediate that this is a norm, we won't check it here. The Banach space X_∞ is the completion of the space of finitely nonzero functions on \mathfrak{S}_∞ with the above norm.

Remark 2.2.1 Many of the results that follow are proved on the space Λ of finitely nonzero functions on \mathfrak{S}_∞ . Since Λ is dense subspace of X_∞ , it is clear, by standard perturbation arguments, found in [1] and [11] for example, that the results extend to the entire space X_∞ .

Some examples illustrating the use of the norm in X_∞ may be useful at this time.

Examples 2.2.2 In the illustrations that follow, a vector x is assumed to have the value of 0 at all unlabeled nodes. In Figure 10, $\|x\| = 2$ even though there are segments at level 1 which yield a sum larger than 2. Beginning at level 1, there are only two admissible segments, and we check that any two segments at level 1 give a smaller value than the one admissible segment at level 0. We simply take $\{\theta_0\}$ as our one and only admissible segment to attain the norm.

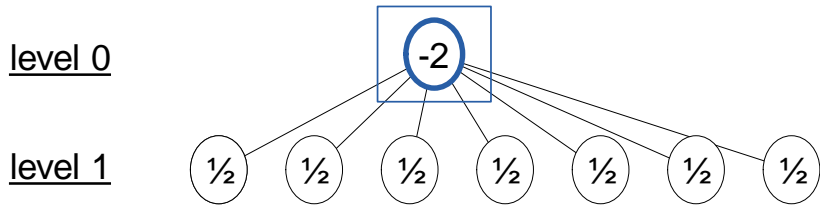


Figure 10 The vector x , $\|x\| = 2$

In Figure 11, we are reminded that admissible segments begin at the same level, but they may end at any level.

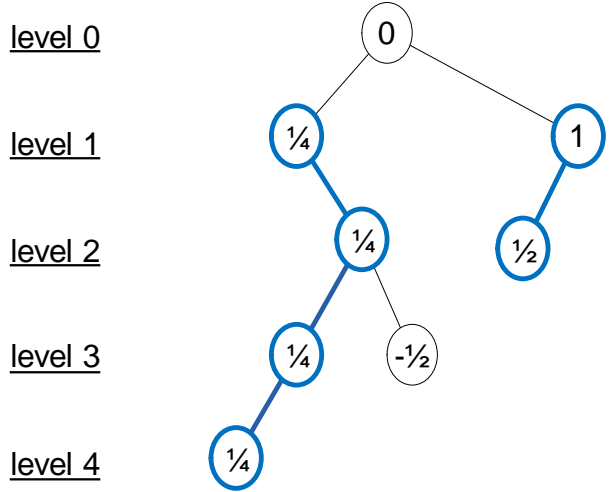


Figure 11 The vector x , $\|x\| = \frac{5}{2}$

The next example is that of a vector that is the sum of images of single-node projections where the nodes are a strongly incomparable sequence. Any admissible family of segments passes through at most two nodes.

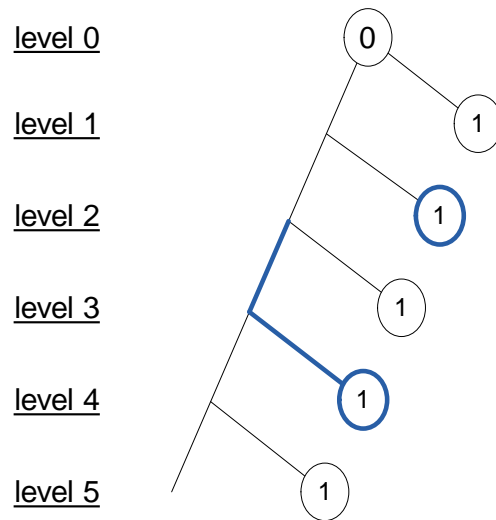


Figure 12 The vector x , $\|x\| = 2$

The next example is that of a vector that is nonzero on nodes at the same level.

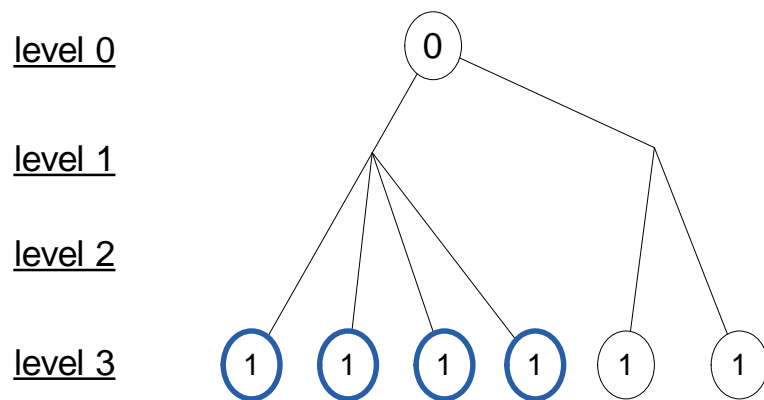


Figure 13 The vector x , $\|x\| = 4$

This last example is critical as it is not present in a finitely branching tree. It is that of a vector that is the sum of images of single-node projections where the nodes are strongly rooted and the levels of the nodes increase.

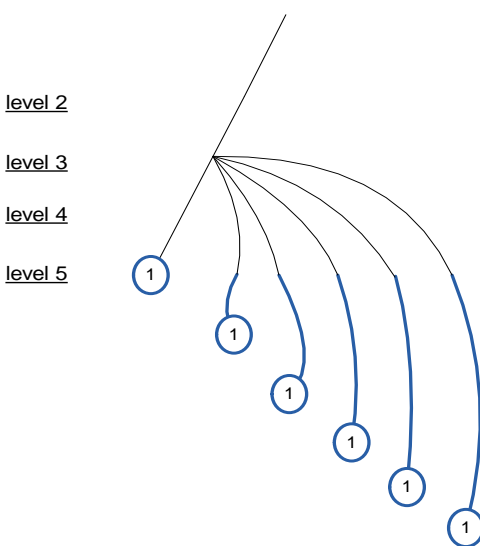


Figure 14 The vector x , $\|x\| = 6$

The vector illustrated in Figure 14 has norm 6. To see this, look at the admissible family of segments beginning at level 5 such that for each segment S in the family, $|S^*(x)| = 1$. There are six such segments so $\|x\| = 6$.

The next definition describes the set of vectors which, when taken in order, are the standard unit vector basis for the space X_∞ .

Definition 2.2.3 Let $\psi \in \mathfrak{S}_\infty$, and let $e_\psi \in X_\infty$ be defined by

$$e_\psi(\varphi) = \begin{cases} 1 & \text{if } \varphi = \psi \\ 0 & \text{if } \varphi \neq \psi \end{cases}.$$

Proposition 2.2.4 Let $(\theta_0, \theta_1, \theta_2, \dots)$ be the sequence in \mathfrak{S}_∞ such that for each k , $\theta_k = (\varepsilon_{m_k} \cdots \varepsilon_0)$ is the binary representation of k with $\varepsilon_{m_k} = 1$. Then the sequence $(e_{\theta_k})_{k=0}^\infty$ forms a Schauder basis for the space X_∞ .

Proof Clearly, $[(e_{\theta_k})] = X_\infty$. Let $k_2 \geq k_1 \geq 0$ be integers, and let $\alpha_0, \dots, \alpha_{k_1}, \dots, \alpha_{k_2}$ be scalars. Let

$$\mathcal{F}_1 = \left\{ F : \begin{array}{l} F \text{ is an admissible family of segments,} \\ S \in F \implies S \cap \mathfrak{S}_\infty \subseteq \{\theta_0, \dots, \theta_{k_1}\} \end{array} \right\}.$$

Then

$$\left\| \sum_{k=0}^{k_1} \alpha_k e_{\theta_k} \right\| = \max_{F \in \mathcal{F}_1} \sum_{S \in F} \left| S^* \left(\sum_{k=0}^{k_1} \alpha_k e_{\theta_k} \right) \right|.$$

Now if we let

$$\mathcal{F}_2 = \left\{ F : \begin{array}{l} F \text{ is an admissible family of segments,} \\ S \in F \implies S \cap \mathfrak{S}_\infty \subseteq \{\theta_0, \dots, \theta_{k_1}, \dots, \theta_{k_2}\} \end{array} \right\},$$

then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and

$$\begin{aligned} \left\| \sum_{k=0}^{k_2} \alpha_k e_{\theta_k} \right\| &= \max_{F \in \mathcal{F}_2} \sum_{S \in F} \left| S^* \left(\sum_{k=0}^{k_2} \alpha_k e_{\theta_k} \right) \right| \geq \max_{F \in \mathcal{F}_1} \sum_{S \in F} \left| S^* \left(\sum_{k=0}^{k_2} \alpha_k e_{\theta_k} \right) \right| \\ &= \max_{F \in \mathcal{F}_1} \sum_{S \in F} \left| S^* \left(\sum_{k=0}^{k_1} \alpha_k e_{\theta_k} \right) \right| = \left\| \sum_{k=0}^{k_1} \alpha_k e_{\theta_k} \right\|. \end{aligned}$$

■

In our analysis, we work with elements of Λ , that is, finitely nonzero functions on \mathfrak{S}_∞ . Here we make a few definitions characterizing the elements of X_∞ which we use most often.

Definition 2.2.5 Let $x \in X_\infty$.

1. The vector x is *finitely supported* if the set $G = \{\psi \in \mathfrak{S}_\infty : x(\psi) \neq 0\}$ is finite. The set G is called *the support of x* , and we say x is *finitely supported on G* ;
2. The vector x is *finitely supported on (n, l)* if x is finitely supported and $x = (P_n - P_l)(x)$ where $n \leq l$;
3. The vector x is *finitely supported on (n, l, F)* if x is finitely supported on (n, l) and $x = \sum_{\psi \in F} P_\psi(x)$ where F is a pairwise incomparable finite subset of \mathfrak{S}_∞ .
4. We say a sequence (x_k) is *finitely supported on (n_k, l_k)* (or (n_k, l_k, F_k)) if each element x_k in the sequence is finitely supported on (n_k, l_k) (or (n_k, l_k, F_k)).

We may assume that if (x_k) is a basic sequence and G_k is the support of x_k , then $G_k \cap G_j = \emptyset$ if $k \neq j$.

The next definition, which can also be found in [4], describes how segments in \mathfrak{S}_∞ interact with elements of X_∞ .

Definition 2.2.6 Let $x \in X_\infty$ be finitely supported on (n, l) . Let S be a p - q segment in \mathfrak{S}_∞ .

1. The segment S *meets the support of x* if $S \cap G \neq \emptyset$ where G is the support of x ;
2. The segment S *begins in the support of x* if S meets the support of x and $n \leq p \leq l$;
3. The segment S *ends in the support of x* if S meets the support of x and $n \leq q \leq l$;
4. The segment S *passes through the support of x* if S meets the support of x and $p \leq n \leq l \leq q$;

Chapter 3

Properties of the Space X_∞

In this chapter, we prove the main results regarding the space X_∞ . In particular, we are interested in basic sequences that are c_0 sequences and basic sequences that can be blocked into c_0 sequences. The ultimate goal is to show that any infinite dimensional subspace of X_∞ contains a c_0 sequence. We handle two cases and demonstrate that there exists a third case. The central ideas in this chapter are Proposition 3.3.1, Theorem 3.5.3 and Theorem 3.6.1.

If (x_k) is a basic sequence that is finitely supported on (n_k, l_k, F_k) , then the sequence (x_k) satisfies one of the following cases:

- (i) For every $k \neq j$, $\bigcup_k F_k$ is a pairwise incomparable set;
- (ii) The sequence (x_k) is a weakly null sequence, for every k , $l_k < n_{k+1}$, and if $k \neq j$, the sets F_k and F_j are comparable;
- (iii) The sequence (x_k) is a weakly null sequence, there exists a positive integer n such that for every k , $n_k = n$, and if $k \neq j$, the sets F_k and F_j are comparable.

If (x_k) satisfies case (i) or (ii) and it does not contain a c_0 sequence, then in Theorem 3.5.3 and Theorem 3.6.1, we show that (x_k) can be blocked in such a way that the block

basic sequence we construct is a c_0 sequence. Case (iii) is illustrated in Proposition 3.3.1. The techniques developed in this chapter to construct c_0 sequences, do not directly apply to the sequences fitting case (iii). We believe, however, that similar methods to those used in 3.3.1 can be applied to the general case. If so, then we can show that X_∞ is hereditarily c_0 . We leave this generalization for future work.

We begin with Section 3.1 which describes some basic properties of sequences whose terms are single-node projections. Many such sequences are already c_0 sequences, and so in Section 3.2 we begin our discussion of sequences whose terms are single-node projections. Recall that by Lemma 1.3.5, a sequence of nodes contains a subsequence that is pairwise incomparable or determines a unique branch. Further, a pairwise incomparable sequence of nodes has a subsequence that is either strongly incomparable or strongly rooted.

Two examples get us started when we consider certain sequences of single-node projection vectors such that the nodes of projection are pairwise incomparable. Lemma 3.2.2 on page 39 shows that if the nodes are strongly incomparable, then the sequence of single-node projections is a c_0 sequence. Secondly, Lemma 3.2.3 shows that if the nodes of projection are strongly rooted and the levels of support are bounded vertically in the tree, then the sequence of single-node projections is a c_0 sequence.

Lemma 3.2.4 on page 41 shows that if the nodes determine a branch such that the terms of the sequence in X_∞ have somewhat "small" values on this branch, then the sequence of single-node projections is a c_0 sequence. Lastly, in Lemma 3.2.5 we look at basic sequences in X_∞ not necessarily made up of single-node projections where the terms of the sequence have "small" values on all branches in \mathfrak{S}_∞ . In an effort to build c_0 sequences in every subspace of X_∞ , we then decompose certain sequences that have no c_0 subsequences. The decomposed parts that fall into the categories of sequences studied in this section are c_0 sequences even after being blocked. This is an important concept for the remainder of this chapter.

Once we have a good foundation of sequences that are c_0 sequences, we show that there are weakly null basic sequences in X_∞ that are not c_0 sequences nor do they contain a c_0 subsequence. It was shown in [4] that X has the property that every weakly null sequence has a c_0 subsequence. Once it is established that X_∞ does not have this property, we see that two spaces X_∞ and X are not isomorphic.

In Section 3.3, we begin our study of sequences that are not c_0 sequences with a key example. We analyze a particular sequence in X_∞ such that the support of each term in the sequence begins at the same level but is not bounded vertically in the tree. By calculating a lower estimate on the norm of finite linear combinations of the terms of the sequence, we show that the sequence is not a c_0 sequence. However, using blocking methods similar to those used in the analysis of Schreier's space by Casazza and Shura in [2], we show c_0 does imbed in the closed linear span of the sequence. The blocking

methods found in [2] are crucial in our analysis to follow. Section 3.4 gives a general description of another type of basic sequence does not contain a c_0 subsequence. It is that of a sequence whose terms are single node projections where the nodes form a strongly rooted sequence in the tree \mathfrak{S}_∞ . The levels of support increase causing the norms of finite linear combinations of the terms of the sequence to also increase without bound. Lemma 3.4.1 on page 50 proves this claim showing that any sequence with these properties is not a c_0 sequence. We are reminded again that the dyadic tree used by Hagler in [4] does not contain strongly rooted sequences. It is the infinite branching in \mathfrak{S}_∞ which creates new obstacles to overcome.

In Section 3.5, we continue to use the blocking methods in Casazza and Shura [2] to show that it is possible to obtain c_0 sequences from many other sequences that are not c_0 sequences themselves. Our ultimate goal is to show that c_0 imbeds into every subspace of X_∞ , i.e. that X_∞ is hereditarily c_0 . After showing that there are indeed basic sequences in X_∞ with no c_0 subsequence, we can distinguish the three cases outlined above.

The main results of Sections 3.5 and 3.6 show that the closed linear span of a sequence satisfying case (i) or (ii) contains a c_0 sequence. A blocking method similar to that found in [2] and a decomposition method similar to that found in [4] are used to achieve this end.

3.1 Basic Properties of Single Node Projections

We begin by making some useful observations about basic sequences with the property that each element in the sequence is the image of a single-node projection. In other words, we consider a basic sequence (x_k) such that for each k , there is $\psi_k \in \mathfrak{S}_\infty$ with $x_k = P_{\psi_k}(x_k)$ as shown in Figure 15.

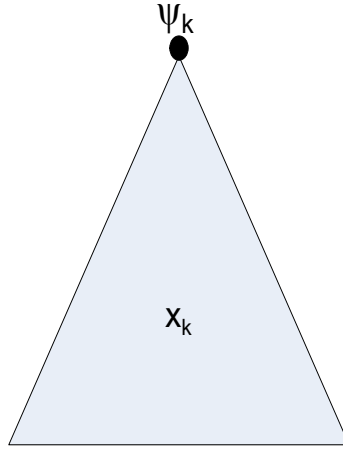


Figure 15 $x_k = P_{\psi_k}(x_k)$

Since (x_k) is basic, we may assume that if $k \neq j$, then $\psi_k \neq \psi_j$ and $\text{supp}(x_k) \cap \text{supp}(x_j) = \emptyset$.

Remark 3.1.1 Suppose $x_k \in X_\infty$ is finitely supported on $(n_k, l_k, \{\psi_k\})$. Then if S is a segment passing through ψ_k , there is a branch B in \mathfrak{S}_∞ such that $S^*(x_k) = B^*(x_k)$.

Figure 16 illustrates this situation with x_k finitely supported on $(n_k, n_{k+4}, \{\psi_k\})$.

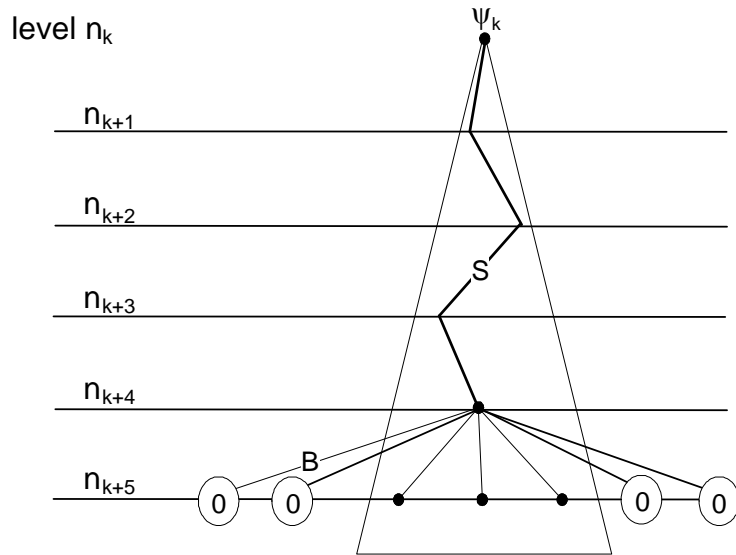


Figure 16 Branch B in \mathfrak{S}_∞ , $B^*(x_k) = S^*(x_k)$

Remark 3.1.2 Let (ψ_k) be a pairwise incomparable sequence, and let (x_k) be a basic sequence in X_∞ finitely supported on $(n_k, l_k, \{\psi_k\})$. Figure 17 is a possible representation of the first few elements of the sequence.

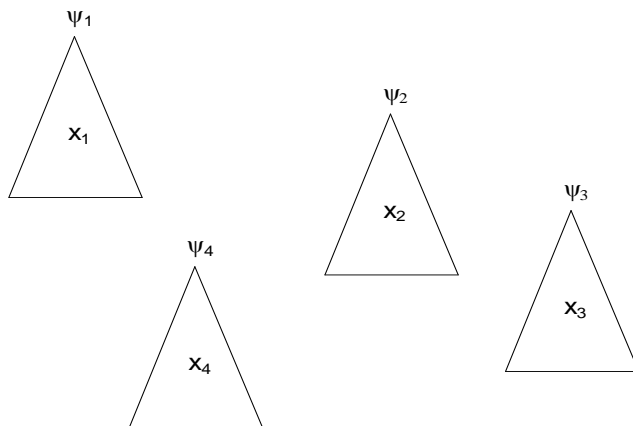


Figure 17 The vectors x_1, x_2, x_3, x_4

Observe that if S_1, \dots, S_l is an admissible family of segments, then each segment meets the support of at most one x_k . For example, consider Figure 18 which shows an admissible family of segments S_1, \dots, S_5 .

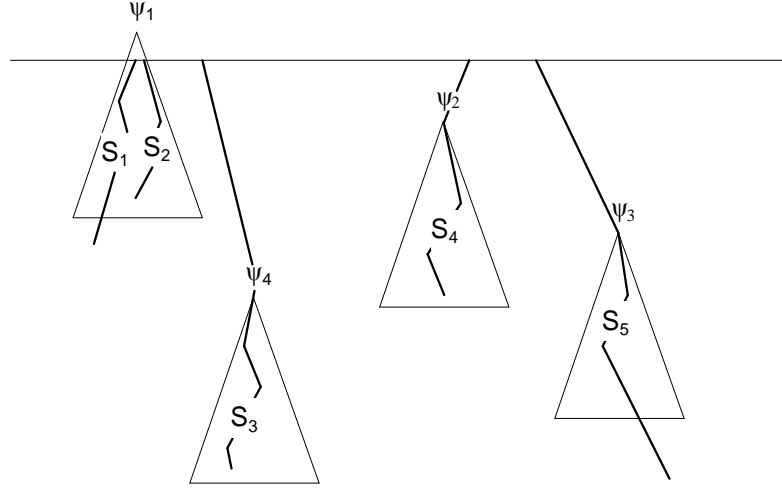


Figure 18 An admissible family of segments S_1, S_2, S_3, S_4, S_5

The following easy proposition gives an example of an unconditional basic sequence in X_∞ . Such a sequence proves to be quite useful later on.

Proposition 3.1.3 *Let (ψ_k) be a pairwise incomparable sequence, and let (x_k) be a basic sequence in X_∞ finitely supported on $(n_k, l_k, \{\psi_k\})$. If S_1, \dots, S_{l+1} is an admissible family of l -segments with $l \leq n_c$, then*

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| \leq \max_k |t_k| \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d x_k \right) \right|$$

for all c, d , and scalars t_c, \dots, t_d .

Proof Since (ψ_k) is pairwise incomparable, by Remark 3.1.2, we see that any segment meets the support of at most one x_k . For each $j = 1, \dots, l+1$, let S_j be a segment which meets the support of x_j . So we have that

$$\left| S_j^* \left(\sum_{k=c}^d t_k y_k \right) \right| = |S_j^*(t_j y_j)| = |t_j| |S_j^*(y_j)|$$

and

$$\begin{aligned} \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k y_k \right) \right| &= \sum_{j=1}^{l+1} |S_j^*(t_j y_j)| = \sum_{j=1}^{l+1} |t_j| |S_j^*(y_j)| \\ &\leq \max_k |t_k| \sum_{j=1}^{l+1} |S_j^*(y_j)| = \max_k |t_k| \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d y_k \right) \right|. \end{aligned}$$

■

3.2 Basic Sequences in X_∞ Equivalent to the Standard c_0 Basis

Many sequences in X_∞ are c_0 sequences from the start. We identify some of those sequences in this section.

Remark 3.2.1 Recall the following standard results that can be found in [10].

- If (x_k) is a basic sequence with $\inf_k \|x_k\| > 0$, then the upper c_0 estimate is necessary and sufficient to show that (x_k) is equivalent to the standard c_0 basis.
- If (x_k) is a basic sequence that is a c_0 sequence, then any normalized block basic sequence of (x_k) is also a c_0 sequence.

We begin by considering a basic sequence whose terms are single-node projections. That is, a sequence (x_k) such that for each k , there is $\psi_k \in \mathfrak{S}_\infty$ with $x_k = P_{\psi_k}(x_k)$. If such sequences have certain properties, we can build other sequences that are also c_0 sequences.

The first of these properties is that the nodes of projection form a strongly incomparable sequence in \mathfrak{S}_∞ . Since sequences such as this can also be found in the Hagler Tree space in [4], the following argument can be found in [4] as well.

Lemma 3.2.2 *Let (ψ_k) be a strongly incomparable sequence in \mathfrak{S}_∞ , and let (x_k) be a normalized basic sequence in X_∞ finitely supported on $(n_k, l_k, \{\psi_k\})$. Suppose there is a $\delta > 0$ such that for every branch B , $|B^*(x_k)| \leq \delta$. If S_1, \dots, S_{l+1} is an admissible family of l -segments with $l \leq n_c$, then*

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| \leq 2\delta \max_k |t_k|$$

for all c, d and scalars t_c, \dots, t_d .

Proof Since (ψ_k) is a pairwise incomparable sequence, each segment can pass through at most one of the ψ_k 's. Furthermore, since (ψ_k) is a strongly incomparable sequence, there are at most two segments which can pass through any of the ψ_k 's. So if S_n, S_m , $1 \leq n, m \leq l+1$, pass through ψ_n and ψ_m respectively, then

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| = |S_n^*(t_n x_n)| + |S_m^*(t_m x_m)| \leq 2\delta \max_k |t_k|.$$

■

Our next example of a c_0 sequence in X_∞ is a sequence (x_k) that is finitely supported on $(n_k, l_k, \{\psi_k\})$ where (ψ_k) is a pairwise incomparable sequence in \mathfrak{S}_∞ and there is $n, m \in \mathbb{N}$ such that $n_k = n, l_k = m$ for all k . In other words, (x_k) is finitely supported on $(n, m, \{\psi_k\})$, as shown in Figure 19.

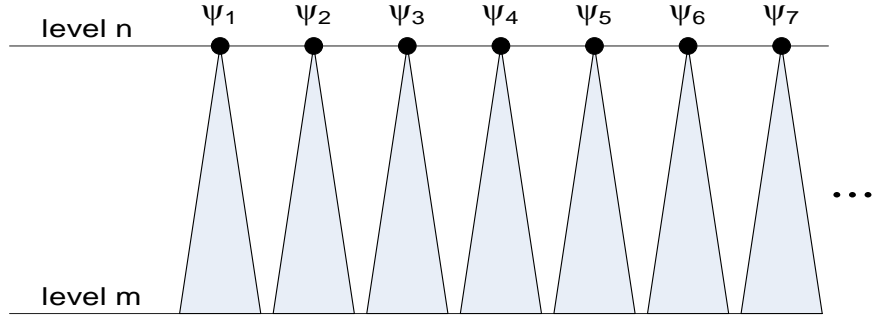


Figure 19 The sequence (x_k) supported on $(n, m, \{\psi_k\})$

Lemma 3.2.3 *Let $n, m > 0$ be fixed. Let (ψ_k) be a pairwise incomparable sequence in \mathfrak{S}_∞ such that $\text{lev}(\psi_k) = n$ for all k , and let (x_k) be a normalized basic sequence in X_∞ that is finitely supported on $(n, m, \{\psi_k\})$. If S_1, \dots, S_{l+1} is an admissible family of l -segments with $l \leq m$, then*

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| \leq (m+1) \max_k |t_k|$$

for all c, d and scalars t_c, \dots, t_d .

Proof Since (ψ_k) is a pairwise incomparable sequence, for each $j = 1, \dots, l+1$, the segment S_j meets the support of at most one x_j . So for a single segment S_j which meets

the support of x_j , we have

$$\left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| = |S_j^*(t_j x_j)| = |t_j| |S_j^*(x_j)| \leq |t_j| \|x_j\| = |t_j|.$$

It follows that

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| = \sum_{j=1}^{l+1} |S_j^*(t_j x_j)| \leq (l+1) \max_k |t_k| \leq (m+1) \max_k |t_k|.$$

■

Next we look at sequences in X_∞ which are made up of projections of nodes determining a branch. If each of the x_k 's has small values on nodes in the determined branch, then we can control the norm of finite linear combinations of the x_k 's.

Lemma 3.2.4 *Let (x_k) be a normalized basic sequence in X_∞ , (δ_k) a positive sequence of real numbers, and $\lambda > 0$ satisfy the following:*

1. *The sequence (x_k) is finitely supported on $(n_k, l_k, \{\psi_k\})$ where (ψ_k) uniquely determines a branch B_ψ ;*
2. *For each k , $|B_\psi^*(x_k)| \leq \delta_k$ and for all c, d , $\sum_{k=c}^d \delta_k \leq 2\delta_c$;*
3. *For every branch B , $|B^*(x_k)| \leq \lambda$.*

If S_1, \dots, S_{l+1} is an admissible family of l -segments, then

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| \leq (2\delta_c + \lambda) \max_k |t_k|$$

for all c, d and scalars t_c, \dots, t_d .

Proof Since (ψ_k) determines a branch, given an admissible family of segments S_1, \dots, S_{l+1} , there is at most one j , $1 \leq j \leq l+1$, for which S_j passes through ψ_c, \dots, ψ_d . Let S_j , $1 \leq j \leq l+1$, be a segment passing through ψ_c, \dots, ψ_d which ends in the support of x_d . In this case it should be noted that for all $k \leq d-1$ we have $|S^*(x_k)| = |B_\psi^*(x_k)|$.

By Remark 3.1.1, there is a branch B such that $|S_j^*(x_k)| = |B^*(x_k)|$ for all k . So we have

$$\begin{aligned} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| &\leq \left| B_\psi^* \left(\sum_{k=c}^{d-1} t_k x_k \right) \right| + |S_j^*(t_d x_d)| \leq \sum_{k=c}^{d-1} |B_\psi^*(t_k x_k)| + |S_j^*(t_d x_d)| \\ &\leq \left(\sum_{k=c}^{d-1} \delta_k + \lambda \right) \max_k |t_k| \leq (2\delta_c + \lambda) \max_k |t_k|. \end{aligned}$$

■

The last type of sequence in X_∞ we look at in this section are those for which every linear functional B^* corresponding to a branch B is small on elements of the sequence.

Lemma 3.2.5 *Let (x_k) be a normalized basic sequence in X_∞ finitely supported on (n_k, l_k) . For each k , let $\delta_k > 0$ and $M > 0$ be such that for every branch B , $|B^*(x_k)| \leq \delta_k$ and for all c, d , $(n_c + 1) \sum_{k=c}^d \delta_k \leq M$. If S_1, \dots, S_{l+1} is an admissible family of l -segments with $l \leq n_c$, then*

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| \leq M \max_k |t_k|$$

for all c, d , and scalars t_c, \dots, t_d .

Proof An application of the triangle inequality gives us

$$\begin{aligned} \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k x_k \right) \right| &\leq \max_k |t_k| \sum_{j=1}^{l+1} \left(\sum_{k=c}^d |S_j^*(t_k x_k)| \right) \\ &\leq \max_k |t_k| (n_c + 1) \sum_{k=c}^d \delta_k \leq M \max_k |t_k|. \end{aligned}$$

■

3.3 A Basic Sequence with No c_0 Subsequence

There are basic sequences in X_∞ which do not contain a c_0 subsequence. This provides the motivation for constructing block basic sequences in X_∞ that are c_0 sequences. In this section, we look at a specific example of a sequence that does not contain a c_0 subsequence. The support for each vector, x_k , begins at level 1 and ends at level k . Each vector has norm 1 with an evenly balanced load on each node for which x_k is nonzero. In other words, if the set G_k is the support of x_k and $\text{card}(G_k) = 5$, for example, then $x_k(\psi) = \frac{1}{5}$ for all $\psi \in G_k$.

The sequence is defined by $x_k = \frac{1}{k} \sum_{i=2^{k-1}}^{2^k-1} e_{\theta_i}$ where the sequence of nodes (θ_i) is the standard unit vector basis of X_∞ defined in Section 1.1. Figure 20 shows the sum of first four vectors in the sequence. In this figure, $x_1 = 1$ on the single node labeled 1 and $x_1 = 0$ elsewhere. $x_2 = \frac{1}{2}$ on the two nodes labeled $\frac{1}{2}$ and $x_2 = 0$ elsewhere. $x_3 = \frac{1}{3}$ on the four nodes labeled $\frac{1}{3}$ and $x_3 = 0$ elsewhere. $x_4 = \frac{1}{4}$ on the eight nodes labeled $\frac{1}{4}$ and $x_4 = 0$ elsewhere.

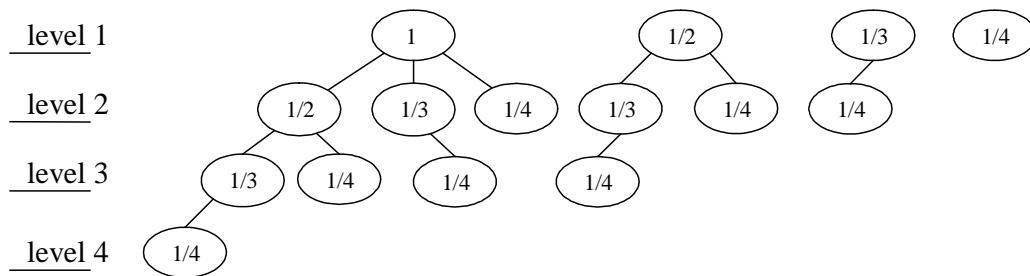


Figure 20 $x_1 + x_2 + x_3 + x_4$

To help us see that $\|x_k\| = 1$ for all k , let's take a look at x_4 . By taking S_1, \dots, S_4 to be the 3-segments passing through the support of x_4 , we have that $\sum_{j=1}^4 |S_j^*(x_4)| = 1$ as shown in Figure 21. Any admissible family of segments beginning above level 3, yields a sum less than 1.

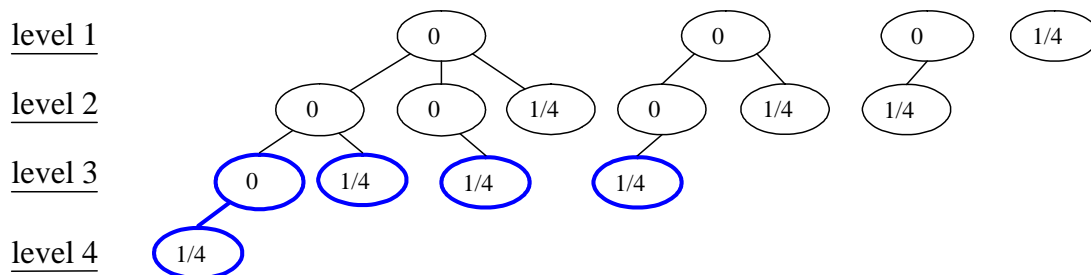


Figure 21 $\|x_4\| = 1$

After showing that this sequence does not contain a c_0 sequence, we use a variation of the blocking method used by Casazza and Shura in their analysis of Schreier's space in [2]. The blocking method in [2], or some variation of it, is used throughout this paper and had a considerable influence on obtaining some of our main results.

Proposition 3.3.1 *Let $x_k = \frac{1}{k} \sum_{i=2^{k-1}}^{2^k-1} e_{\theta_i}$. Then*

- (i) *the sequence (x_k) is an unconditional basic sequence;*
- (ii) *$\|x_k\| = 1$ for all k ;*
- (iii) *for all $m, n, m < n$, $\|\sum_{k=m}^n x_k\| \geq \frac{1}{m} + (m+1) \sum_{k=m+1}^n \frac{1}{k}$;*
- (iv) *the sequence (x_k) does not contain a c_0 subsequence.*

Proof To prove (i), for all k , let G_k be the support of x_k . If $j < k$, then there is $\psi_j \in G_j, \psi_k \in G_k$ such that $\psi_j = \text{pred}(\psi_k)$. So for a finite subset $A \subset \mathbb{N}$ and a sequence of scalars (t_k) , if S is an m -segment with $S^*(\sum_{k \in A} t_k x_k) \neq 0$, then there is an m -segment R , and $\varphi \in S, \text{lev}(\varphi) = m$, such that $\varphi \in R$, $R^*(\sum_{k \in A} t_k x_k) = S^*(\sum_{k \in A} t_k x_k)$, and $R \subseteq \bigcup_{k \in A} G_k$. So for a finite subset $B \subset \mathbb{N}$ such that $A \subseteq B$, we have $S^*(\sum_{k \in A} t_k x_k) = R^*(\sum_{k \in A} t_k x_k) = R^*(\sum_{k \in B} t_k x_k)$. It follows that $\|\sum_{k \in A} t_k x_k\| \leq \|\sum_{k \in B} t_k x_k\|$, so the sequence (x_k) is an unconditional basic sequence. For $m \leq k$, let $F(k, m) = \{\psi \in \mathfrak{S}_\infty : \text{lev}(\psi) = m, x_k(\psi) \neq 0\}$. It is

not difficult to see that the levels in the tree and $\text{card}(F(k, m))$ have the same structure as Pascal's Triangle. That is,

$$\text{card}(F(k, m)) = \binom{k-1}{m-1} = \frac{(k-1)!}{(m-1)!(k-m)!}.$$

So letting $m = k - 1$, we have $\text{card}(F(k, k - 1)) = \binom{k-1}{k-2} = k - 1$, and for $m = k$, $\text{card}(F(k, k)) = \binom{k-1}{k-1} = 1$. Suppose that $F(k, k) = \{\psi_1\}$ and $F(k, k - 1) = \{\psi_2, \dots, \psi_k\}$. By letting $S_0 = \{\text{pred}(\psi_1), \psi_1\}$ and $S_i = \{\psi_i\}$ for $i = 2, \dots, k$, we have that S_1, \dots, S_k is an admissible family of $(k - 1)$ -segments. It follows that $\|x_k\| \geq \sum_{j=1}^k |S_j^*(x_k)| = 1$. But since the support of x_k ends at level k , if \mathcal{F} is an admissible family of segments such that every segment in \mathcal{F} meets the support of x_k , then $\text{card}(\mathcal{F}) \leq k$ and $S^*(x_k) = \frac{1}{k}$ for all $S \in \mathcal{F}$. Now we have that $\|x_k\| = 1$, which concludes the proof of (ii).

To prove (iii), we use a similar argument. Let S_1 be the m -segment passing through the support of x_m, \dots, x_n and let S_2, \dots, S_m be the m -segments passing through the support of x_{m+1}, \dots, x_n . Then S_1, \dots, S_m is an admissible family of segments. It follows that

$$\left\| \sum_{k=m}^n x_k \right\| \geq \sum_{j=1}^m \left| S_j^* \left(\sum_{k=m}^n x_k \right) \right| = \sum_{k=m}^n \frac{1}{k} + m \sum_{k=m+1}^n \frac{1}{k} = \frac{1}{m} + (m+1) \sum_{k=m+1}^n \frac{1}{k}.$$

Now (iv) follows immediately since for a fixed m ,

$$\sup_n \left\| \sum_{k=m}^n x_k \right\| \geq \sup_n \left(\frac{1}{m} + (m+1) \sum_{k=m+1}^n \frac{1}{k} \right) = \infty.$$

So (x_k) does not contain a c_0 subsequence. ■

Let's take another look at the example given in Proposition 3.3.1. We can employ a blocking method to show that the span of the sequence (x_k) is hereditarily c_0 .

Proposition 3.3.2 Let $x_k = \frac{1}{k} \sum_{i=2^{k-1}}^{2^k-1} e_i$. If (u_n) is a normalized block basic sequence of (x_k) , then $c_0 \hookrightarrow [(u_n)]$.

Proof Let $u_n = \sum_{j=p_n+1}^{p_{n+1}} a_j x_j$ where $0 = p_1 < p_2 < \dots$. Since (u_n) is an unconditional basic sequence, we may assume that $a_i \geq 0$ for all i . To see this, for each i , let $\varepsilon_i = \text{sgn}(a_i)$. Then by the proposition on page 112 of the Appendix, for all m , $\|\sum_{i=1}^m \varepsilon_i |a_i| x_i\| \leq K \|\sum_{i=1}^m |a_i| x_i\|$ where K is the unconditional constant for (x_k) . Note also that since $\|u_n\| = 1$ for all n , we must have $a_i \leq 1$ for all i .

Suppose there exists $N > 0$ such that $\sup_m \|\sum_{n=N}^m u_n\| < \infty$, then by the lemma on page 112 of the Appendix and the unconditionality of (u_n) , we are done.

Otherwise, suppose that for all $N > 0$ and all $M > 0$, there exists $m > N$ such that $\|\sum_{n=N}^m u_n\| > M$.

Let $n_1 = 1$. Pick $n_2 > n_1$ so that

$$\left\| \sum_{i=n_1+1}^{n_2} u_i \right\| > p_{n_1+1} + 1 \text{ and } \frac{p_{n_1+1}}{p_{n_2+1}} < \frac{1}{2}.$$

Put $w_1 = \frac{\sum_{i=n_1+1}^{n_2} u_i}{\|\sum_{i=n_1+1}^{n_2} u_i\|}$. For the sake of notation, let $q_1 = p_{n_1+1}$ and $q_2 = p_{n_2+1}$. Then we have

$$\frac{q_1}{q_2} < \frac{1}{2} \text{ and } w_1 = \sum_{i=n_1+1}^{n_2} \frac{1}{\|\sum_{i=n_1+1}^{n_2} u_i\|} \sum_{j=p_i+1}^{p_{i+1}} a_j x_j = \sum_{i=q_1+1}^{q_2} \gamma_i x_i$$

where $\gamma_i = \frac{a_i}{\|\sum_{i=n_1+1}^{n_2} u_i\|}$ for $q_1 + 1 \leq i \leq q_2$. Observe that for $q_1 + 1 \leq i \leq q_2$, $\gamma_i < \frac{1}{q_1+1}$.

Now pick $n_3 > n_2$ so that

$$\left\| \sum_{i=n_2+1}^{n_3} u_i \right\| > p_{n_2+1} + 1 \text{ and } \frac{p_{n_1+1}}{p_{n_3+1}} < \frac{1}{4}, \frac{p_{n_2+1}}{p_{n_3+1}} < \frac{1}{2}.$$

Put $w_2 = \frac{\sum_{i=n_2+1}^{n_3} u_i}{\left\| \sum_{i=n_2+1}^{n_3} u_i \right\|}$. Again, let $q_3 = p_{n_3+1}$ so that we have

$$\frac{q_1}{q_3} < \frac{1}{4}, \frac{q_2}{q_3} < \frac{1}{2} \text{ and } w_2 = \sum_{i=n_2+1}^{n_3} \frac{1}{\left\| \sum_{i=n_2+1}^{n_3} u_i \right\|} \sum_{j=p_i+1}^{p_{i+1}} a_j x_j = \sum_{i=q_2+1}^{q_3} \gamma_i x_i$$

where $\gamma_i = \frac{a_i}{\left\| \sum_{i=n_2+1}^{n_3} u_i \right\|}$ for $q_2 + 1 \leq i \leq q_3$. Observe that for $q_2 + 1 \leq i \leq q_3$, $\gamma_i < \frac{1}{q_2+1}$.

Continue inductively to obtain an increasing sequence of integers (q_k) and a normalized block basic sequence (w_k) of (u_n) such that if $0 < j < k$, then $\frac{q_j}{q_k} < \frac{1}{2^{k-j}}$ and

$$w_k = \sum_{i=q_k+1}^{q_{k+1}} \gamma_i x_i$$

where $|\gamma_i| < \frac{1}{q_{k+1}}$ for $q_k + 1 \leq i \leq q_{k+1}$.

By Proposition 3.3.1, for all k ,

$$1 = \|w_k\| \geq \frac{\gamma_{q_k+1}}{q_k + 1} + (q_k + 2) \sum_{i=q_k+2}^{q_{k+1}} \frac{\gamma_i}{i}$$

from which it follows that

$$\sum_{i=q_k+2}^{q_{k+1}} \frac{\gamma_i}{i} \leq \frac{q_k + 1 - \gamma_{q_k+1}}{(q_k + 1)(q_k + 2)}.$$

Now let $m \geq 1$. Suppose $\left\| \sum_{k=1}^m w_k \right\| = \sum_{j=1}^{l+1} |S_j^* (\sum_{k=1}^m w_k)|$ for an admissible family of segment S_1, \dots, S_{l+1} where $q_c + 1 \leq l \leq q_{c+1}$ for some c . Then for all $k < c$, $S_j^*(w_k) = 0$ for all j , so

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=1}^m w_k \right) \right| &\leq \|w_c\| + \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^m w_k \right) \right| \\
&\leq 1 + \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^m \sum_{i=q_k+1}^{q_{k+1}} \gamma_i x_i \right) \right| \\
&\leq 1 + \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^m \gamma_{q_k+1} x_{q_k+1} \right) \right| \\
&\quad + \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^m \sum_{i=q_k+2}^{q_{k+1}} \gamma_i x_i \right) \right| \\
&\leq 1 + (q_{c+1} + 1) \sum_{k=c+1}^m \frac{\gamma_{q_k+1}}{q_k+1} + (q_{c+1} + 1) \sum_{k=c+1}^m \sum_{i=q_k+2}^{q_{k+1}} \frac{\gamma_i}{i} \\
&\leq 1 + (q_{c+1} + 1) \max_{c+1 \leq k \leq m} \gamma_{q_k+1} \sum_{k=c+1}^m \frac{1}{q_k+1} \\
&\quad + (q_{c+1} + 1) \sum_{k=c+1}^m \frac{q_k + 1 - \gamma_{q_k+1}}{(q_k + 1)(q_k + 2)} \\
&\leq 2 + \sum_{k=c+1}^m \frac{q_{c+1} + 1}{q_k} \\
&= 3 + \sum_{k=c+2}^m \frac{q_{c+1}}{q_k} + \sum_{k=c+1}^m \frac{1}{q_k} \\
&\leq 4 + \sum_{k=c+2}^m \frac{1}{2^{k-(c+1)}} \leq 5.
\end{aligned}$$

Now we have that $\sup_m \left\| \sum_{k=1}^m w_k \right\| < \infty$, and we are done. ■

3.4 Basic Sequences in X_∞ Not Equivalent to the Standard c_0 Basis

In this section we examine more generally a type of sequence that is not a c_0 sequence. Again, the infinite branching of \mathfrak{X}_∞ is responsible for the presence of such sequences in X_∞ , while the space X contains no such sequences.

The next lemma shows that if a sequence (x_k) in X_∞ is finitely supported on $(n_k, l_k, \{\psi_k\})$ where (ψ_k) is a strongly rooted sequence and the levels of support of (x_k) increase without bound, then (x_k) has no c_0 subsequence. In general, when referring to a subsequence of a sequence, we do not reindex unless it is necessary. For example, we may say that there is a subsequence (x_k) of (x_k) to mean there is a subsequence (x_{k_n}) of (x_k) .

Lemma 3.4.1 *Let (ψ_k) be a strongly rooted sequence, and let (x_k) be a normalized basic sequence in X_∞ that is finitely supported on $(n_k, l_k, \{\psi_k\})$ such that for all k , $l_k < n_k + 1$. Suppose there is a $\delta > 0$ such that, for each k , there is a branch B_k through ψ_k with $|B_k^*(x_k)| > \delta$. Then the following conditions are satisfied:*

- (i) *for all $m \geq 1$, $\|\sum_{k=m}^{n_m+m} x_k\| > (n_m + 1)\delta$;*
- (ii) *the sequence (x_k) does not contain a c_0 subsequence.*

Proof To prove (i), let's look at the case when $m = 1$ before considering the general case. Let $S_1 \subset B_1$. For each $j = 2, \dots, n_1 + 1$, let S_j be an n_1 -segment through ψ_j such that $|S_j^*(x_j)| = |B_j^*(x_j)| > \delta$. Then S_1, \dots, S_{n_1+1} is an admissible family of segments

and each S_j meets the support of exactly one x_j . It follows that

$$\left\| \sum_{k=1}^{n_1+1} x_k \right\| \geq \sum_{j=1}^{n_1+1} \left| S_j \left(\sum_{k=1}^{n_1+1} x_k \right) \right| = \sum_{j=1}^{n_1+1} |S_j(x_j)| > (n_1 + 1) \delta.$$

Now in the general case, let $S_m \subset B_m$. For $j = m + 1, \dots, n_m + m$, let S_j be an n_m -segment through ψ_j such that $|S_j^*(x_j)| = |B_j^*(x_j)| > \delta$. Then S_m, \dots, S_{n_m+m} is an admissible family of segments, and each S_j meets the support of exactly one x_j . It follows that

$$\left\| \sum_{k=m}^{n_m+m} x_k \right\| \geq \sum_{j=m}^{n_m+m} \left| S_j \left(\sum_{k=m}^{n_m+m} x_k \right) \right| = \sum_{j=m}^{n_m+m} |S_j(x_j)| > (n_m + 1) \delta.$$

This concludes the proof of (i). To prove (ii), let (x_k) be a subsequence of (x_k) . By Part (i), for all $m \geq 1$, $\left\| \sum_{k=m}^{n_m+m} x_k \right\| > (n_m + 1) \delta$. Since the sequence (x_k) is a basic sequence with basis constant 1, then

$$\left\| \sum_{k=1}^{n_m+m} x_k \right\| \geq \left\| \sum_{k=m}^{n_m+m} x_k \right\| > (n_m + 1) \delta$$

and $n_m \rightarrow \infty$ as $m \rightarrow \infty$. It follows that

$$\sup_n \left\| \sum_{k=1}^n x_k \right\| = \infty.$$

So (x_k) does not contain a c_0 subsequence. ■

The next corollary follows immediately.

Corollary 3.4.2 *Let (ψ_k) be a strongly rooted sequence in \mathfrak{S}_∞ such that*

$lev(\psi_k) < lev(\psi_{k+1})$ for all k . Then $\sup_n \left\| \sum_{k=1}^n e_{\psi_k} \right\| = \infty$.

In some instances we can build block basic sequences from sequences of the type given in Lemma 3.4.1 that are c_0 sequences. We discuss this construction in the next section.

3.5 Building Block Basic Sequences in X_∞

It turns out that the sequences in Proposition 3.3.1 and Lemma 3.4.1 characterize, in some sense, those basic sequences in X_∞ which do not contain a c_0 subsequence. However, in many known instances we can construct a block basic sequence from such sequences that is a c_0 sequence. As in Proposition 3.3.1 on page 45, we use the blocking method found in [2]. The block basic sequences that result have a common structure so we define that structure here.

Definition 3.5.1 Let (x_k) be a basic sequence in X_∞ . The sequence (w_k) in X_∞ is a $[p_k, \gamma_k]$ -*block basic sequence* of (x_k) if there are sequences (p_k) and (γ_k) such that (p_k) , (γ_k) , and (w_k) satisfy the following properties:

1. The sequence (p_k) is a sequence of nonnegative integers with

$$0 = p_1 < p_2 < p_3 < \cdots;$$

2. The sequence (γ_k) is a sequence of positive scalars;

3. For each k ,

$$w_k = \gamma_k \sum_{i=p_k+1}^{p_{k+1}} x_i.$$

We use the notation $[p_k, \gamma_k]$ with square brackets to distinguish a $[p_k, \gamma_k]$ -block basic sequence from a sequence that is finitely supported on (n_k, l_k) .

The next lemma provides the method we implement when building c_0 sequences. This method was motivated in large part by the analysis of Schreier's space found in [2]. The importance of the analysis of Schreier's space cannot be minimized as the following lemma is a cornerstone for the majority of the results that follow.

Lemma 3.5.2 *Let (x_k) be a normalized basic sequence in X_∞ finitely supported on (n_k, l_k) with the property that*

$$\sup_n \left\| \sum_{k=1}^n x_k \right\| = \infty.$$

Then there is a strictly increasing sequence of (L_k) of positive integers and a normalized $[p_k, \gamma_k]$ -block basic sequence (w_k) of (x_k) satisfying the following conditions:

1. $\|P_{L_{k+1}}(w_k)\| = 0$ for all k ;
2. The sequence (γ_k) is strictly decreasing and $\gamma_k < \frac{1}{L_{k+1}}$ for all k .

Proof We use an induction process to construct the normalized block basic sequence (w_k) . Let $p_1 = 0$ and $L_1 > 0$ be such that $\|P_{L_1}(x_1)\| = 0$. Since $\sup_n \left\| \sum_{k=1}^n x_k \right\| = \infty$,

we can pick $p_2 > p_1$ such that $\left\| \sum_{i=p_1+1}^{p_2} x_i \right\| > L_1 + 1$. Let $\gamma_1 = \frac{1}{\left\| \sum_{i=p_1+1}^{p_2} x_i \right\|}$ and put

$$w_1 = \gamma_1 \sum_{i=p_1+1}^{p_2} x_i.$$

Note that $\gamma_1 < \frac{1}{L_1+1}$ and $\|w_1\| = 1$. Now let $L_2 \geq \frac{1}{\gamma_1} > L_1$ be such that $\|P_{L_2}(w_1)\| = 0$.

Pick $p_3 > p_2$ such that

$$\left\| \sum_{i=p_2+1}^{p_3} x_i \right\| > L_2 + 1.$$

Let $\gamma_2 = \frac{1}{\left\| \sum_{i=p_2+1}^{p_3} x_i \right\|}$ and put

$$w_2 = \gamma_2 \sum_{i=p_2+1}^{p_3} x_i.$$

Note that $\gamma_2 < \frac{1}{L_2+1} < \gamma_1$ and $\|w_2\| = 1$. Continue inductively to obtain a sequence of positive scalars $\gamma_1 > \gamma_2 > \gamma_3 \cdots$, a sequence of positive integers $L_1 < L_2 < L_3 < \cdots$

and a normalized block basic sequence (w_k) of (y_k) such that

$$w_k = \gamma_k \sum_{i=p_k+1}^{p_{k+1}} y_i;$$

$$\gamma_k < \frac{1}{L_k + 1};$$

$$\|P_{L_{k+1}}(w_k)\| = 0.$$

■

We have already seen that if (x_k) is a sequence in X_∞ with $x_k = P_{\psi_k}(x_k)$ where (ψ_k) is a strongly rooted sequence with increasing levels of support, then (x_k) does not contain a c_0 subsequence; in particular, $\sup_n \left\| \sum_{k=1}^n x_k \right\| = \infty$. Now by applying Lemma 3.5.2 to such a sequence, we have a c_0 sequence, as the following theorem demonstrates.

In fact, the theorem applies to any bounded sequence such that the union of the nodes in the support of each term is a pairwise incomparable set. This is a critical result.

Theorem 3.5.3 *Let (x_k) be a bounded basic sequence in X_∞ , (w_k) a normalized $[p_k, \gamma_k]$ -block basic sequence (w_k) of (x_k) , (L_k) a strictly increasing sequence of positive integers and $\lambda > 0$ satisfy the following:*

1. *The sequence (x_k) is finitely supported on (n_k, l_k, F_k) ;*
2. *$\bigcup_{k=1}^{\infty} F_k$ is a pairwise incomparable set;*
3. *For every branch B , $|B^*(x_k)| \leq \lambda$;*
4. *$\|P_{L_{k+1}}(w_k)\| = 0$ for all k ;*
5. *The sequence (γ_k) is strictly decreasing and $\gamma_k < \frac{1}{L_{k+1}}$ for all k .*

If S_1, \dots, S_{l+1} is an admissible family of l -segments with $l \leq L_c$, then

$$\sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d t_k w_k \right) \right| \leq \lambda \max_k |t_k|$$

for all c, d and scalars t_c, \dots, t_d .

Proof Recall that since $\bigcup_{k=1}^{\infty} F_k$ is a pairwise incomparable set, each segment S_j meets the support of at most one x_j . Now $l \leq L_c \leq L_{c+1}$ so we have

$$\begin{aligned} \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d w_k \right) \right| &= \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d \left(\gamma_k \sum_{i=p_k+1}^{p_{k+1}} x_i \right) \right) \right| \leq \max_k \gamma_k \sum_{j=1}^{l+1} \left| S_j \left(\sum_{i=p_c+1}^{p_{d+1}} x_i \right) \right| \\ &= \gamma_c \sum_{j=1}^{l+1} |S_j(x_j)| \leq \gamma_c (L_{c+1} + 1) \lambda \leq \lambda. \end{aligned}$$

By Lemma 3.1.3,

$$\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k w_k \right) \right| \leq \lambda \max_k |t_k|.$$

■

The next corollary follows immediately from Lemma 3.5.3 and Corollary 3.4.2.

Corollary 3.5.4 *Let (ψ_k) be a strongly rooted sequence in \mathfrak{S}_{∞} such that $\text{lev}(\psi_k) < \text{lev}(\psi_{k+1})$ for all k . Then there is a normalized $[p_k, \gamma_k]$ -block basic sequence (w_k) of (e_{ψ_k}) that is a c_0 sequence.*

3.6 Constructing c_0 Sequences in X_{∞} by Decomposing and Blocking

The next theorem is the one of the main ingredients in proving that certain subspaces of X_{∞} are hereditarily c_0 . We are given a weakly null sequence (x_k) in X_{∞} that does not contain a c_0 subsequence. We apply a decomposition method to each element similar to that found in Hagler [4], but with the added complexity arising from the infinite branching of the tree. The blocking method from Lemma 3.5.2 is then applied to obtain a c_0

sequence. This theorem is key and applies to a large class of basic sequences in X_∞ . Our future work includes the one class of sequences for which Theorem 3.5.3 and Theorem 3.6.1 do not apply, namely, the generalization of the example given in Proposition 3.3.1. The reader may wish to revisit Definition 2.2.5 on page 28 as a reminder of the notation used for finitely supported sequences at this time.

Theorem 3.6.1 *Let (x_k) be a normalized basic sequence in X_∞ satisfying the following:*

- *(x_k) is finitely supported on (n_k, l_k, F_k) where $l_k < n_{k+1}$ for all k ;*
- *For all $k, j, k \neq j$, the sets F_k and F_j are comparable sets;*
- *(x_k) is a weakly null sequence; that is, $x_k \xrightarrow{w} 0$;*
- *No subsequence of (x_k) is a c_0 sequence.*

Then there is a normalized block basic sequence of (x_k) which is a c_0 sequence.

Proof We inductively pick a subsequence and decompose each element of the subsequence in such a way that the previous lemmas may be applied to each part. By picking a subsequence and reindexing if necessary, we may assume that $l_k < 2l_k + 1 \leq n_{k+1}$ for all k . We now begin the induction process to construct a subsequence of (x_k) from which we build a block basic sequence. For each k , let

$$F(k, 1) = \left\{ \begin{array}{l} \psi \in F_k : \text{there exists a branch } B_\psi \text{ through } \psi \text{ with } |B(x_k)| > \frac{1}{n_1+1}, \\ \text{for every branch } B \text{ through } \psi, |B(x_k)| \leq 1 \end{array} \right\}.$$

Observe that $\text{card}(F(k, 1)) < n_1 + 1$ for all k . To see this, fix k , and for each $\psi \in F(k, 1)$, let S_ψ be an n_k -segment such that $S_\psi(x_k) = B_\psi(x_k) > \frac{1}{n_1 + 1}$. Then $\{S_\psi : \psi \in F(k, 1)\}$ is an admissible family of segments and

$$1 = \|x_k\| \geq \sum_{\psi \in F(k, 1)} |S_\psi(x_k)| > \frac{\text{card}(F(k, 1))}{n_1 + 1}.$$

We can now pick a subsequence (x_k) of (x_k) such that for some $b_1 < n_1 + 1$, $\text{card}(F(k, 1)) = b_1$ for all k . If $b_1 = 0$, let $k_1 = 1$. If $b_1 > 0$, enumerate $F(k, 1) = \{\psi(k, 1; 1), \dots, \psi(k, 1; b_1)\}$, and construct subsets $F_S(k, 1)$, $F_R(k, 1)$, $F_B(k, 1)$, of $F(k, 1)$, in the following manner. For each $n = 1, \dots, b_1$, apply Lemma 1.3.5 to the sequence $(\psi(k, 1; n))_{k=1}^\infty$ to obtain a subsequence $(\psi(k, 1; n))_{k=1}^\infty$ that is either

1. a strongly incomparable sequence, in which case $\psi(k, 1; n) \in F_S(k, 1)$;
2. a strongly rooted sequence, in which case $\psi(k, 1; n) \in F_R(k, 1)$;
3. a sequence which uniquely determines a branch, in which case $\psi(k, 1; n) \in F_B(k, 1)$.

In case 3, by passing to a further subsequence if necessary, we may assume that if $k < j$ and n is fixed, then $\psi(k, 1; n) < \psi(j, 1; n)$.

Now define

$$\begin{aligned} s(k, 1) &= \sum_{\psi \in F_S(k, 1)} P_\psi(x_k) \\ r(k, 1) &= \sum_{\psi \in F_R(k, 1)} P_\psi(x_k) \\ b(k, 1) &= \sum_{\psi \in F_B(k, 1)} P_\psi(x_k) \end{aligned}$$

and

$$x(k, 1) = s(k, 1) + r(k, 1) + b(k, 1).$$

Let

$$\mathcal{B}_1 = \left\{ B \subseteq \mathfrak{S}_\infty : B \text{ is a branch uniquely determined by a sequence in } \bigcup_{k=1}^{\infty} F_B(k, 1) \right\}.$$

If $\mathcal{B}_1 = \emptyset$, put $k_1 = 1$. Recall that $x_k \xrightarrow{w} 0$ and \mathcal{B}_1 is finite, so if $\mathcal{B}_1 \neq \emptyset$, pick k_1 so that for all $B \in \mathcal{B}_1$,

$$|B^*(x_{k_1})| < \frac{1}{n_1 + 1}.$$

Note that $b_1 < n_1 + 1 \leq n_{k_1} + 1$.

We are nearly done with the decomposition of the first element, x_{k_1} , as we consider the part of x_{k_1} that is excluded from $x(k_1, 1)$. Let

$$F'_{k_1} = \left\{ \psi \in F_{k_1} : \text{for every branch } B \text{ through } \psi, |B^*(x_{k_1})| < \frac{1}{n_1 + 1} \right\}$$

and

$$x'_{k_1} = \sum_{\psi \in F'_{k_1}} P_{\psi(x_{k_1})}.$$

So we have

$$x_{k_1} = x(k_1, 1) + x'_{k_1}$$

which completes the decomposition of x_{k_1} . Now let $A_1 = \{k \in \mathbb{N} : k > k_1\}$.

For the sake of clarity, let's demonstrate the next step in the induction process before we move on to the general setting. For each $k \in A_1$, let

$$F(k, 2) = \left\{ \begin{array}{l} \psi \in F_k : \text{there exists a branch } B_\psi \text{ through } \psi \text{ with } |B(x_k)| > \frac{1}{n_{k_1+1}+1}, \\ \text{for every branch } B \text{ through } \psi, |B(x_k)| \leq \frac{1}{n_1+1} \end{array} \right\}.$$

As before, observe that $\text{card}(F(k, 2)) < n_{k_1+1} + 1$ for all k . We can now pick a subsequence $(x_k)_{k \in A_1}$ of $(x_k)_{k \in A_1}$ such that for some $b_2 < n_{k_1+1} + 1$, $\text{card}(F(k, 2)) = b_2$ for all k . If $b_2 = 0$, let $k_2 = k_1 + 1$. If $b_2 > 0$, enumerate $F(k, 2) = \{\psi(k, 2; 1), \dots, \psi(k, 2; b_2)\}$, and construct $F_S(k, 2)$, $F_R(k, 2)$, $F_B(k, 2)$, subsets of $F(k, 2)$, in the following manner. For each $n = 1, \dots, b_2$, apply Lemma 1.3.5 to the sequence $(\psi(k, 2; n))_{k \in A_1}$ to obtain a subsequence $(\psi(k, 2; n))_{k \in A_1}$ that is either

1. a strongly incomparable sequence, in which case $\psi(k, 2; n) \in F_S(k, 2)$;
2. a strongly rooted sequence, in which case $\psi(k, 2; n) \in F_R(k, 2)$;
3. a sequence which uniquely determines a branch, in which case $\psi(k, 2; n) \in F_B(k, 2)$.

Again, in case 3, by passing to a further subsequence if necessary, we may assume that if $k < j$ and n is fixed, then $\psi(k, 2; n) < \psi(j, 2; n)$.

Now define

$$\begin{aligned} s(k, 2) &= \sum_{\psi \in F_S(k, 2)} P_\psi(x_k) \\ r(k, 2) &= \sum_{\psi \in F_R(k, 2)} P_\psi(x_k) \\ b(k, 2) &= \sum_{\psi \in F_B(k, 2)} P_\psi(x_k) \end{aligned}$$

and

$$x(k, 2) = s(k, 2) + r(k, 2) + b(k, 2).$$

Let

$$\mathcal{B}_2 = \left\{ B \subseteq \mathfrak{S}_\infty : B \text{ is a branch uniquely determined by a sequence in } \bigcup_{k \in A_1} F_B(k, 2) \right\}.$$

If $\mathcal{B}_2 = \emptyset$, put $k_2 = k_1 + 1$. If $\mathcal{B}_2 \neq \emptyset$, since $x_k \xrightarrow{w} 0$ and \mathcal{B}_2 is finite, we can pick k_2 so that for all $B \in \bigcup_{n=1}^2 \mathcal{B}_n$,

$$|B^*(x_{k_2})| < \frac{1}{n_{k_1+1} + 1}$$

and

$$\sum_{n=1}^2 b_n \leq n_{k_2} + 1.$$

We are nearly done with the decomposition of the second element, x_{k_2} , as we consider the part of x_{k_2} that is excluded from $x(k_2, 1) + x(k_2, 2)$. Let

$$F'_{k_2} = \left\{ \psi \in F_{k_2} : \text{for every branch } B \text{ through } \psi, |B^*(x_{k_2})| < \frac{1}{n_{k_1+1} + 1} \right\}$$

and

$$x'_{k_2} = \sum_{\psi \in F'_{k_2}} P_{\psi(x_{k_2})}.$$

So we have

$$x_{k_2} = x(k_2, 1) + x(k_2, 2) + x'_{k_2}$$

which completes the decomposition of x_{k_2} . Now let $A_2 = \{k \in \mathbb{N} : k > k_2\}$.

Suppose x_{k_1}, \dots, x_{k_m} have been chosen. For each $k \in A_m$, let

$$F(k, m+1) = \left\{ \begin{array}{l} \psi \in F_k : \text{there exists a branch } B_\psi \text{ through } \psi \\ \text{with } |B(x_k)| > \frac{1}{n_{k_{m+1}+1}}, \\ \text{for every branch } B \text{ through } \psi, |B(x_k)| \leq \frac{1}{n_{k_{m-1}+1}+1} \end{array} \right\}.$$

As before, observe that $\text{card}(F(k, m+1)) < n_{k_{m+1}} + 1$ for all k . We can now pick a subsequence $(x_k)_{k \in A_m}$ of $(x_k)_{k \in A_m}$ such that for some $b_{m+1} < n_{k_{m+1}} + 1$, $\text{card}(F(k, m+1)) = b_{m+1}$ for all k . If $b_{m+1} = 0$, let $k_{m+1} = k_m + 1$. If $b_{m+1} > 0$, enumerate $F(k, m+1) = \{\psi(k, m+1; 1), \dots, \psi(k, m+1; b_{m+1})\}$, and construct $F_S(k, m+1)$, $F_R(k, m+1)$, $F_B(k, m+1)$, subsets of $F(k, m+1)$, in the following manner. For each $n = 1, \dots, b_{m+1}$, apply Lemma 1.3.5 to the sequence $(\psi(k, m+1; n))_{k \in A_m}$ to obtain a subsequence $(\psi(k, m+1; n))_{k \in A_m}$ that is either

1. a strongly incomparable sequence, in which case $\psi(k, m+1; n) \in F_S(k, m+1)$;
2. a strongly rooted sequence, in which case $\psi(k, m+1; n) \in F_R(k, m+1)$;
3. a sequence which uniquely determines a branch, in which case $\psi(k, m+1; n) \in F_B(k, m+1)$.

As before, in case 3, by passing to a further subsequence if necessary, we may assume that if $k < j$ and n is fixed, then $\psi(k, m+1; n) < \psi(j, m+1; n)$.

Now define

$$\begin{aligned} s(k, m+1) &= \sum_{\psi \in F_S(k, m+1)} P_\psi(x_k) \\ r(k, m+1) &= \sum_{\psi \in F_R(k, m+1)} P_\psi(x_k) \\ b(k, m+1) &= \sum_{\psi \in F_B(k, m+1)} P_\psi(x_k) \end{aligned}$$

and

$$x(k, m+1) = s(k, m+1) + r(k, m+1) + b(k, m+1).$$

Let

$$\mathcal{B}_{m+1} = \left\{ B \subseteq \mathfrak{S}_\infty : B \text{ is a branch uniquely determined by a sequence in } \bigcup_{k \in A_m} F_B(k, m+1) \right\}.$$

If $\mathcal{B}_{m+1} = \emptyset$, put $k_{m+1} = k_m + 1$. If $\mathcal{B}_{m+1} \neq \emptyset$, since $x_k \xrightarrow{w} 0$ and \mathcal{B}_{m+1} is finite, we can pick k_{m+1} so that for all $B \in \bigcup_{n=1}^{m+1} \mathcal{B}_n$,

$$|B^*(x_{k_{m+1}})| < \frac{1}{n_{k_{m+1}} + 1}$$

and

$$\sum_{n=1}^{m+1} b_n \leq n_{k_{m+1}} + 1.$$

We are nearly done with the decomposition of the $m+1$ st element, $x_{k_{m+1}}$, as we consider the part of $x_{k_{m+1}}$ that is excluded from $x(k_{m+1}, 1) + \cdots + x(k_{m+1}, m+1)$. Let

$$F'_{k_{m+1}} = \left\{ \psi \in F_{k_{m+1}} : \text{for every branch } B \text{ through } \psi, |B^*(x_{k_{m+1}})| < \frac{1}{n_{k_{m+1}} + 1} \right\}$$

and

$$x'_{k_{m+1}} = \sum_{\psi \in F'_{k_{m+1}}} P_{\psi}(x_{k_{m+1}}).$$

So we have

$$x_{k_{m+1}} = x(k_{m+1}, 1) + \cdots + x(k_{m+1}, m+1) + x'_{k_{m+1}}$$

which completes the decomposition of $x_{k_{m+1}}$. Now let $A_{m+1} = \{k \in \mathbb{N} : k > k_{m+1}\}$.

With the induction process complete, reindex (x_{k_i}) to obtain a sequence (x_i) which is finitely supported on (n_{k_i}, l_{k_i}, F_i) . Note that we do not reindex the sequences (n_{k_i}) or (l_{k_i}) since these sequences represent levels in \mathfrak{S}_{∞} which are dependent on the original subsequence (x_{k_i}) . The sequence (x_i) has a structure that can be visualized as in Figure 29.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$i = 1$	$x(1,1) + x'_1$			
$i = 2$	$x(2,1) + x(2,2) + x'_2$			
$i = 3$	$x(3,1) + x(3,2) + x(3,3) + x'_3$			
$i = 4$	$x(4,1) + x(4,2) + x(4,3) + x(4,4) + x'_4$			

Figure 29 Decomposition of x_1, x_2, x_3, x_4

For the remainder of the proof, let $\delta_0 = 1, \delta_m = \frac{1}{n_{k_{m-1}+1}+1}$ for $m \geq 1$. So for all i , if B is a branch, $B \in \bigcup_{n=1}^i \mathcal{B}_n$ then

$$|B^*(x_i)| \leq \delta_i,$$

for every branch B through $\psi \in F(i, m)$

$$|B^*(x(i, m))| \leq \delta_{m-1}$$

for all $m \leq i$, and for every branch B through $\psi \in F'_i$

$$|B^*(x'_i)| \leq \delta_i.$$

Since no subsequence of (x_k) is a c_0 sequence, for all $M > 0$ there exists scalars a_1, \dots, a_n such that

$$\left\| \sum_{i=1}^n a_i x_i \right\| > M \max_k |a_k|$$

We may assume $\max_i |a_i| = 1$ from which it follows that the sequence $(a_i x_i)$ is a bounded basic sequence with $\|a_i x_i\| \leq 1$ and

$$\sup_n \left\| \sum_{i=1}^n a_i x_i \right\| = \infty.$$

Applying Lemma 3.5.2, we obtain a normalized $[p_k, \gamma_k]$ -block basic sequence (w_k) of (x_k) where (γ_k) is a strictly decreasing sequence of positive real numbers. For each k , let $N_k = n_{k_{p_k+1}}$ so that the sequence (w_k) is finitely supported on (N_k, L_{k+1}) and $\gamma_k < \frac{1}{L_{k+1}}$. We now use the decomposition of each x_i to form a similar decomposition of each w_k . For each $m \leq k$, let $I(k, m), R(k, m), B(k, m)$ be $[p_k, \gamma_k]$ -block basic sequence of $s(k, m)$,

$r(k, m)$, and $b(k, m)$ respectively. That is,

$$\begin{aligned} I(k, m) &= \gamma_k \sum_{i=p_k+1}^{p_{k+1}} s(k, m) \\ R(k, m) &= \gamma_k \sum_{i=p_k+1}^{p_{k+1}} r(k, m) \\ B(k, m) &= \gamma_k \sum_{i=p_k+1}^{p_{k+1}} b(k, m) \end{aligned}$$

so that

$$w(k, m) = I(k, m) + R(k, m) + B(k, m).$$

Now we consider the part of w_k that is excluded from $w(k, 1) + \dots + w(k, k)$ by putting

$$w'_k = \gamma_k \left(\sum_{i=p_k+1}^{p_{k+1}} x'_i + \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right).$$

Each w_k has a unique decomposition of the form

$$w_k = w(k, 1) + \dots + w(k, p_{k+1}) + w'_k.$$

We claim that the sequence (w_k) is a c_0 sequence. Let S_1, \dots, S_{l+1} be an admissible family of l -segments with $l \leq N_c$ and t_c, \dots, t_d be any given scalars.

We first look at the sequence (w'_k) . Recall that for any branch B , $|B^*(x'_i)| \leq \delta_i$ and

$\sum_{i=p_k+1}^{p_{k+1}} \delta_i \leq 2\delta_{p_k+1}$. It follows that

$$\left| B^* \left(\gamma_k \sum_{i=p_k+1}^{p_{k+1}} x'_i \right) \right| \leq 2\gamma_k \delta_{p_k+1}$$

and for any c, d ,

$$\begin{aligned}
(N_c + 1) \sum_{k=c}^d 2\gamma_k \delta_{p_k+1} &\leq 2(N_c + 1) \left(\max_k \gamma_k \right) \sum_{k=c}^d \delta_{p_k+1} \\
&\leq 2(N_c + 1) \left(\frac{1}{L_c + 1} \right) (2\delta_{p_c+1}) \\
&\leq 4.
\end{aligned}$$

Now letting $M = 4$ in Lemma 3.2.5, we have

$$\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+1}^{p_{k+1}} x'_i \right) \right| \leq 4 \max_k |t_k|.$$

We claim that for any branch B ,

$$\left| B^* \left(\gamma_k \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right) \right| \leq 3\gamma_k \delta_{p_k+1}.$$

To see this, first assume that for some $m, \alpha, \beta, p_k + 2 \leq m \leq \alpha \leq \beta \leq p_{k+1}$, B meets the support of $b(\alpha, m), \dots, b(\beta, m)$. Then there is a branch $B_0 \in \mathcal{B}_m$ such that

$|B^*(x(i, m))| = |B_0^*(x(i, m))| \leq \delta_i$ for all $i = \alpha, \dots, \beta - 1$. It follows that

$$\left| B^* \left(\gamma_k \sum_{i=p_k+2}^{\beta-1} \sum_{m=p_k+2}^i x(i, m) \right) \right| = \left| B_0 \left(\gamma_k \sum_{i=\alpha}^{\beta-1} x(i, m) \right) \right|.$$

Recall that $|B^*(x(j, m))| \leq \delta_{m-1}$. Letting $\lambda = \delta_{m-1}$ in Lemma 3.2.4, we have

$$\left| B^* \left(\gamma_k \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right) \right| \leq \gamma_k (2\delta_{p_\alpha} + \delta_{m-1}) \leq 3\gamma_k \delta_{p_k+1}.$$

If for all m, α , B does not meet the support of any $b(\alpha, m)$, then for each m , B meets the support of at most one $x(i, m)$. In this case we have

$$\left| B^* \left(\gamma_k \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right) \right| \leq \gamma_k \sum_{m=p_k+2}^{p_{k+1}} \delta_{m-1} \leq 2\gamma_k \delta_{p_k+1}$$

which proves the claim. We also have that for any c, d ,

$$\begin{aligned}
(N_c + 1) \sum_{k=c}^d 3\gamma_k \delta_{p_{k+1}} &\leq 3(N_c + 1) \left(\max_k \gamma_k \right) \sum_{k=c}^d \delta_{p_{k+1}} \\
&\leq 3(N_c + 1) \left(\frac{1}{L_c + 1} \right) (2\delta_{p_c+1}) \\
&\leq 6.
\end{aligned}$$

So letting $M = 6$ in Lemma 3.2.5, we have

$$\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right) \right| \leq 6 \max_k |t_k|.$$

Now we have

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k w'_k \right) \right| &\leq \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+1}^{p_{k+1}} x'_i \right) \right| \\
&\quad + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+2}^{p_{k+1}} \sum_{m=p_k+2}^i x(i, m) \right) \right| \\
&\leq 10 \max_k |t_k|.
\end{aligned}$$

Following a similar argument,

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c+1}^d t_k \sum_{m=p_c+2}^k w(k, m) \right) \right| &= \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c+1}^d t_k \gamma_k \sum_{m=p_c+2}^k \sum_{i=p_k+1}^{p_{k+1}} x(i, m) \right) \right| \\
&\leq 6 \max_k |t_k|.
\end{aligned}$$

Now we look at the part of the decomposition of (w_k) that is made up of projections on nodes that uniquely determine a branch, namely, the sequences $(B(k, m))$. For a fixed

$m \leq p_c + 1$, letting $\lambda = \delta_{m-1}$ in Lemma 3.2.4 gives us

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k B(k, m) \right) \right| &= \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+1}^{p_{k+1}} b(i, m) \right) \right| \\
&\leq \max_k |t_k| \max_k |\gamma_k| \sum_{j=1}^{l+1} \left| S_j \left(\sum_{i=p_c+1}^{p_{d+1}} b(i, m) \right) \right| \\
&\leq \max_k |t_k| \frac{b_m}{L_c + 1} (2\delta_{p_c+1} + \delta_{m-1})
\end{aligned}$$

and since $\sum_{m=1}^{p_c+1} b_m \leq n_{k_{p_c+1}} + 1 = N_c + 1$,

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \left(\sum_{m=1}^{p_c+1} B(k, m) \right) \right) \right| &\leq \sum_{m=1}^{p_c+1} \left(\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k B(k, m) \right) \right| \right) \\
&\leq \max_k |t_k| \sum_{m=1}^{p_c+1} \frac{b_m}{L_c + 1} (2\delta_{p_c+1} + \delta_{m-1}) \\
&= \max_k |t_k| \frac{2\delta_{p_c+1}}{L_c + 1} \sum_{m=1}^{p_c+1} b_m (1 + \delta_{m-1}) \\
&\leq \max_k |t_k| \frac{4\delta_{p_c+1}}{L_c + 1} \sum_{m=1}^{p_c+1} b_m \\
&\leq \max_k |t_k| \frac{4\delta_{p_c+1}}{L_c + 1} (N_c + 1) \\
&\leq 4 \max_k |t_k|.
\end{aligned}$$

For a fixed $m \leq p_c + 1$, letting $\lambda = \delta_{m-1}$ in Lemma 3.2.2, gives us

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k I(k, m) \right) \right| &= \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \gamma_k \sum_{i=p_k+1}^{p_{k+1}} i(k, m) \right) \right| \\
&\leq 2 \max_k |t_k| \max_k |\gamma_k| b_m \delta_{m-1} \\
&\leq 2 \max_k |t_k| \frac{b_m \delta_{m-1}}{L_c + 1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \left(\sum_{m=1}^{p_{c+1}} I(k, m) \right) \right) \right| &\leq \sum_{m=1}^{p_{c+1}} \left(\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k I(k, m) \right) \right| \right) \\
&\leq 2 \max_k |t_k| \left(\sum_{m=1}^{p_{c+1}} \frac{b_m \delta_{m-1}}{L_c + 1} \right) \\
&\leq 2 \max_k |t_k| \left(\frac{1}{L_c + 1} \right) \sum_{m=1}^{p_{c+1}} b_m \\
&\leq 2 \max_k |t_k| \left(\frac{1}{L_c + 1} \right) (N_c + 1) \\
&\leq 2 \max_k |t_k|.
\end{aligned}$$

At last, we come to our strongly rooted sequences in $(R(k, m))$. For a fixed $m \leq p_c + 1$, put $\lambda = \delta_{m-1}$ in Lemma 3.5.3 gives us

$$\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k R(k, m) \right) \right| \leq \max_k |t_k| \delta_{m-1}$$

and

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \left(\sum_{m=1}^{p_{c+1}} R(k, m) \right) \right) \right| &\leq \sum_{m=1}^{p_{c+1}} \left(\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k R(k, m) \right) \right| \right) \\
&\leq \max_k |t_k| \sum_{m=1}^{p_{c+1}} \delta_{m-1} \\
&\leq 2 \max_k |t_k|.
\end{aligned}$$

Finally, we are ready to estimate $\left\| \sum_{k=1}^d t_k w_k \right\|$. Assume that the segments

S_1, \dots, S_{l+1} begin in the support of w_{c-1} . Then

$$\begin{aligned}
\sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=1}^n t_k w_k \right) \right| &\leq \|t_{c-1} w_{c-1}\| + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k w_k \right) \right| \\
&\leq |t_{c-1}| \|w_{c-1}\| \\
&\quad + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \left(\sum_{m=1}^{p_c+1} w(k, m) \right) \right) \right| \\
&\quad + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k \left(\sum_{m=p_c+2}^k w(k, m) \right) \right) \right| \\
&\quad + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k w'_k \right) \right| \\
&\leq 28 \max_k |t_k|.
\end{aligned}$$

So, (w_k) is a c_0 sequence. ■

In order to prove that subspaces spanned by sequences satisfying the hypotheses of Theorem 3.5.3 and Theorem 3.6.1 are hereditarily c_0 , it remains to be shown that every sequence of this type contains a weak-Cauchy subsequence. To this end, we prove in the next chapter that the space l_1 does not imbed in X_∞ . We then use Rosenthal's l_1 Theorem [12] which gives us that every sequence in X_∞ has a weak Cauchy subsequence. From there, we can easily construct a weakly null sequence and apply the previous results to produce a c_0 sequence.

Theorem 3.6.2 *Let (y_k) be a normalized basic sequence in X_∞ that is finitely supported on (n_k, l_k, F_k) . Suppose that there is a subsequence (y_k) of (y_k) such that either of the following conditions is satisfied:*

1. $\bigcup_{k=1}^\infty F_k$ is a pairwise incomparable set;
2. For all k we have $l_k < n_{k+1}$, and if $k \neq j$, the sets F_k and F_j are comparable sets.

Then there is a normalized block basic sequence (w_k) of (y_k) that is a c_0 sequence.

Proof If (y_k) does not contain a c_0 subsequence, then for all $M > 0$ there exist scalars a_1, \dots, a_n with $\max |a_k| = 1$ such that

$$\left\| \sum_{k=1}^n a_k y_k \right\| > M.$$

It follows that

$$\sup_n \left\| \sum_{k=1}^n a_k y_k \right\| = \infty.$$

For each k , let $x_k = a_k y_k$. Then (x_k) is supported on (n_k, l_k, F_k) , and there is a subsequence (x_k) of (x_k) such that $\bigcup_k F_k$ is a pairwise incomparable set. The sequence (x_k) is a bounded basic sequence so we may assume that $\|x_k\| = 1$ for all k . Then for any branch B in \mathfrak{S}_∞ , $|B^*(x_k)| \leq 1$, so we let $\lambda = 1$ in Lemma 3.5.3. By Lemma 3.5.2, there is a normalized $[p_k, \gamma_k]$ -block basic sequence (w_k) of (x_k) and a strictly increasing sequence of positive integers (L_k) such that (γ_k) is a strictly decreasing sequence and $\gamma_k < \frac{1}{L_{k+1}}$ for all k . Then for any admissible family of N_{c-1} -segments S_1, \dots, S_{l+1} and

any scalars t_{c-1}, \dots, t_d we have

$$\begin{aligned} \left\| \sum_{k=c-1}^d t_k w_k \right\| &\leq \|t_{c-1} w_{c-1}\| + \sum_{j=1}^{l+1} \left| S_j \left(\sum_{k=c}^d t_k w_k \right) \right| \\ &\leq 2 \max_k |t_k|. \end{aligned}$$

Thus, (w_k) is a c_0 subsequence. Now suppose every subsequence (x_k) of (x_k) yields k, j , $k < j$, such that $l_k < n_j$. Since l_1 does not imbed in X_∞ , by Rosenthal's l_1 Theorem [12], (x_k) has a weak Cauchy subsequence (x_k) . Put

$$z_k = \frac{x_{2k} - x_{2k-1}}{\|x_{2k} - x_{2k-1}\|}$$

for all k . Then (z_k) is a normalized, weakly null basic sequence. By Theorem 3.6.1, there is a normalized block basic sequence of (z_k) that is a c_0 sequence. ■

Chapter 4

Properties of the Dual Space, X_∞^*

In this chapter, we investigate some properties of the dual space X_∞^* of X_∞ . We show that there exists a separable subspace F of X_∞^* such that X_∞^*/F is isometrically isomorphic to $c_0(\Gamma)$, where Γ has the cardinality of the continuum, \mathfrak{c} . This will show that X_∞^{**} has cardinality \mathfrak{c} , which in turn shows that X_∞ does not contain a copy of l_1 . The argument is similar to that of Hagler [4], but additional care must be taken to handle the infinite branching at each node of the tree.

We look at the set of linear functionals generated by branches in \mathfrak{S}_∞ , as well as functions on the set of branches themselves.

4.1 Linear Functionals Generated by Branches in \mathfrak{S}_∞

We first look at the w^* -closure of the set of linear functionals generated by branches in \mathfrak{S}_∞ . We denote by Γ the set of branches in the tree \mathfrak{S}_∞ and \mathcal{S}_0 the set of segments in \mathfrak{S}_∞ beginning at level 0. That is, let $\Gamma = \{B \subseteq \mathfrak{S}_\infty : B \text{ is a branch}\}$ and $\mathcal{S}_0 = \{S \subseteq \mathfrak{S}_\infty : S \text{ is a 0-segment}\}$. The linear functionals associated to Γ and \mathcal{S}_0 are also denoted by $\Gamma = \{B^* \in X_\infty^* : B \in \Gamma\}$ and $\mathcal{S}_0 = \{S^* \in X_\infty^* : S \in \mathcal{S}_0\}$. The meaning will be clear from the context. By an abuse of language, we say " f is a branch" or " f is a

segment" when we mean " $f = B^*$ where B is a branch" or " $f = S^*$ where S is a segment" respectively.

Lemma 4.1.1 $\bar{\Gamma}^{w^*} = \Gamma \cup \mathcal{S}_0$.

Proof Let $S \in \mathcal{S}_0$ be a 0- m segment, and let (B_n) be a sequence of distinct branches in Γ such that $S \subseteq B_n$ for all n and if $\psi_i \in B_i$, $\text{lev}(\psi_i) = m + 1$, then $\psi_i \notin B_j$ if $j \neq i$. Let $x \in X_\infty$ be finitely supported on (n_x, l_x) . Observe that the set

$$F = \{n \in \mathbb{N} : B_n \text{ passes through the support of } P_m x\}$$

is finite, while the set

$$I = \{n \in \mathbb{N} : B_n \text{ passes through the support of } x - P_m x\}$$

is infinite.

If $n_x > m$, then $S^*(x) = 0$. Since the set F is finite, $B_n^*(x) = 0$ for all sufficiently large n so $B_n^*(x) \rightarrow S^*(x)$. If $n_x \leq m$, then at least part of the support of x lies above level m so for all $n \in I$, $B_n^*(x) = S^*(x)$. Since F is finite, $B_n^*(x) \rightarrow S^*(x)$. Thus we have that $B_n \xrightarrow{w^*} S^*$, implying that $S^* \in \bar{\Gamma}^{w^*}$ and $\Gamma \cup \mathcal{S}_0 \subseteq \bar{\Gamma}^{w^*}$.

Now to show that $\Gamma \cup \mathcal{S}_0 = \bar{\Gamma}^{w^*}$, we show that $\Gamma \cup \mathcal{S}_0$ is w^* -closed. Suppose that (f_n) is a sequence in $\Gamma \cup \mathcal{S}_0$, $f_n \xrightarrow{w^*} f$, and $f \notin \Gamma \cup \mathcal{S}_0$. Note that since $f_n \in \Gamma \cup \mathcal{S}_0$ for all

n , then for all $\psi \in \mathfrak{S}_\infty$,

$$f_n(e_\psi) = \begin{cases} 0 & \text{if } \psi \in f_n \\ 1 & \text{if } \psi \notin f_n \end{cases}.$$

So if $f_n \xrightarrow{w^*} f$, then $f(e_\psi) \in \{0, 1\}$. Since f is not a branch or a 0-segment, either

1. there exists nodes $\psi_1, \psi_2 \in f$ such that $\psi_1 \parallel \psi_2$ or
2. f is an m -segment with $m > 0$.

In the first case, since $f_n \xrightarrow{w^*} f$, then $f_n(e_{\psi_1}) = f_n(e_{\psi_2}) = 1$ for all sufficiently large n . But since $f_n \in \Gamma \cup \mathcal{S}_0$ and $\psi_1 \parallel \psi_2$, this is a contradiction.

So f must be an m -segment with $m > 0$. Let $l \geq m > 0$, $f = \{\psi_m, \dots, \psi_l\}$ where $\text{lev}(\psi_i) = i$ for all $i = m, \dots, l$. If the set $\{n \in \mathbb{N} : \psi_m \in f_n\}$ is finite, then $f_n \not\xrightarrow{w^*} f$.

So we must have that the set $\{n \in \mathbb{N} : \psi_m \in f_n\}$ is infinite. Let $\psi < \psi_m$. Then the set $A_\psi = \{n \in \mathbb{N} : \psi \in f_n\}$ is infinite as well. Let $x = e_\psi + \sum_{k=m}^l e_{\psi_k}$. Then for all $n \in A_\psi$, $|f_n(x) - f(x)| = |f_n(e_\psi)| = 1$ which contradicts $f_n \xrightarrow{w^*} f$. So we must have $f \in \Gamma \cup \mathcal{S}_0$ from which it follows that $\Gamma \cup \mathcal{S}$ is w^* -closed and $\Gamma \cup \mathcal{S}_0 = \overline{\Gamma}^{w^*}$. ■

4.2 Spaces of Functions on the Branches in \mathfrak{S}_∞

Our main goal in this section is to show that there exists a separable subspace F of X_∞^* such that $X_\infty^*/F \cong c_0(\Gamma)$ where Γ has cardinality \mathfrak{c} . We then show that $X_\infty^{**} \approx l_1(\Gamma) \oplus F^*$ which implies that $\text{card}(X_\infty^{**})$ is \mathfrak{c} . From here, it follows that $l_1 \not\hookrightarrow X_\infty$.

Recall that

$$l_\infty(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{R}, \sup_{B \in \Gamma} |f(B)| < \infty \right\}.$$

The norm of $f \in l_\infty(\Gamma)$ is the sup norm, $\|f\| = \sup_{B \in \Gamma} |f(B)|$. Following the analysis in Hagler [4], we define an operator $Q : X_\infty^* \rightarrow l_\infty(\Gamma)$ by

$$Qx^*(B) = \lim_n x^*(e_{\psi_n})$$

where $B = (\psi_0, \psi_1, \psi_2, \dots)$. To simplify notation, we write $Qx^*(B) = \lim_{\psi \in B} x^*(e_\psi)$.

To show that Q is well-defined, we first look at the space E which we define to be the completion of the normed space of all finitely non-zero sequences $(\alpha_1, \dots, \alpha_s, \bar{0})$ with

$$\|(\alpha_n)\| = \max_{k \leq m} \left| \sum_{i=k}^m \alpha_i \right|.$$

As in Hagler [4], we have the following lemma regarding the space E . The proof is included for completeness.

Lemma 4.2.1

1. The space E is isomorphic to c_0 .
2. Let $(e_n)_{n=1}^\infty$ be the unit vector basis for E , $(f_n)_{n=1}^\infty$ the sequence in E^* such that $f_n(e_m) = \delta_{nm}$. If $f \in E^*$, then $\lim_n f(e_n)$ exists. Furthermore, $\lim_n f(e_n) = 0$ if and only if $f \in [(f_n)]$.

Proof Let (x_n) be the usual unit vector basis for c_0 . Put $x_0 = e_0 = 0$, and let $T : c_0 \rightarrow E$ be the linear operator defined by $Tx_n = e_n - e_{n-1}$ for all $n \geq 1$. Let $x = (\alpha_1, \dots, \alpha_n, \bar{0}) \in c_0$. Then $(\alpha_1, \dots, \alpha_n, \bar{0}) \mapsto (\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_n - \alpha_{n-1}, \bar{0})$.

From the definition of the norm in E , we have

$$\|Tx\| = \max_{k \leq m} \left| \sum_{i=k}^m (\alpha_i - \alpha_{i-1}) \right|.$$

For any $k \leq m$,

$$\sum_{i=k}^m (\alpha_i - \alpha_{i-1}) = \alpha_m - \alpha_{k-1}$$

so

$$\|Tx\| = \max_{k \leq m} \left| \sum_{i=k}^m (\alpha_i - \alpha_{i-1}) \right| = \max_{k \leq m} |\alpha_m - \alpha_{k-1}| \leq 2 \max_k |\alpha_k| = 2 \|x\|.$$

Note that for all $m > n$, $\alpha_m = 0$ and

$$\|x\| = \max_k |\alpha_k| = \max_{k \leq n} |\alpha_k - \alpha_{n+1}| \leq \|Tx\|.$$

We conclude that T is an isomorphism.

To show that T is onto, note that $e_1 = Tx_1$ and, in general, $e_n = T(\sum_{i=1}^n x_i)$. This implies that T maps c_0 onto E . This concludes the proof of 1.

To prove 2, let $f \in E^*$. Note that $T^* : E^* \rightarrow c_0^* \cong l_1$ where T^* is the adjoint operator of T . Since $c_0^* \cong l_1$, we may regard T^*f as an element of l_1 . Now we have that

$$f(e_n) = f\left(T\left(\sum_{i=1}^n x_i\right)\right) = (T^*f)\left(\sum_{i=1}^n x_i\right)$$

which converges. Therefore, $\lim_n f(e_n)$ exists.

Clearly, if $f \in [(f_n)]$, then $\lim_n f(e_n) = 0$ since (e_n, f_n) is a biorthogonal system. Suppose that $\lim_n f(e_n) = 0$ and $f \in [(f_n)]^{w^*} \setminus [(f_n)]$. Then by the Hahn-Banach Theorem, there is $x \in E$ such that $f(x) = 1$ and $f_n(x) = 0$ for all n . Now $f \in [(f_n)]^{w^*}$ so there is a unique sequence of scalars (β_n) such that

$$f = w^* \text{-} \lim_m \sum_{n=1}^m \beta_n f_n.$$

Then we have

$$f(x) = \sum_{n=1}^m \beta_n f_n(x) = 1$$

which contradicts $f_n(x) = 0$ for all x . ■

It is clear that for any branch B in \mathfrak{S}_∞ , $P_B(X_\infty) \cong E$, and so $P_B(X_\infty) \cong c_0$.

Lemma 4.2.1 further implies that if $x^* \in X_\infty^*$, then $Qx^*(B) = \lim_{\psi \in B} x^*(e_\psi) = \lim_{\psi \in B} P_B^* x^*(e_\psi)$ exists and is equal to zero if and only if $P_B^* x^* \in [(\psi^* : \psi \in B)]$. Thus, Q is well-defined.

The next lemma is a slight generalization of Lemma 7 in Hagler [4] and shows that we may regard Q as an operator from X_∞^* into $c_0(\Gamma)$. Recall that

$$c_0(\Gamma) = \{ f : \Gamma \rightarrow \mathbb{R}, \text{ for all } \varepsilon > 0, |f(B)| > \varepsilon \text{ for finitely many } B \}.$$

Lemma 4.2.2 Let $Q : X_\infty^* \rightarrow l_\infty(\Gamma)$ be the operator defined by $Qx^*(B) = \lim_{\psi \in B} x^*(e_\psi)$, and let $x^* \in X_\infty^*$ and $\varepsilon > 0$ be given. Then $\{B \in \Gamma : |Qx^*(B)| > \varepsilon\}$ is finite.

Proof Suppose (B_n) is a sequence of distinct branches in \mathfrak{S}_∞ with $|Qx^*(B_n)| > \varepsilon$ for all n . Since $\bar{\Gamma}^{w^*}$ is w^* -compact, by passing to a subsequence and reindexing if necessary, we

may assume $B_n^* \xrightarrow{w^*} f$ for some $f \in \bar{\Gamma}^{w^*} = \{B^* \in X_\infty^* : B \in \Gamma\} \cup \{S^* \in X_\infty^* : S \in \mathcal{S}_0\}$.

Suppose $f = B^*$ for some $B \in \Gamma$, and let $n_1 = 1$. Pick $\psi_1 \in B_1 \setminus B$ with $x^*(e_{\psi_1}) > \varepsilon$.

Pick $\varphi_1 \in B \setminus B_1$ such that $lev(\varphi_1) \geq lev(\psi_1)$. See Figure 23.

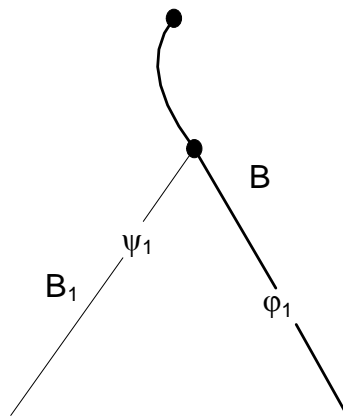


Figure 23

Then since $B_n^* \xrightarrow{w^*} B^*$, there is an N_1 such that for all $n \geq N_1$, B_n passes through φ_1 . So we can pick $n_2 > N_1$ and $\psi_2 \in B_{n_2} \setminus B$ such that $x^*(e_{\psi_2}) > \varepsilon$. Pick $\varphi_2 \in B \setminus (B_{n_1} \cup B_{n_2})$ such that $lev(\varphi_2) \geq lev(\psi_1)$. Figure 24 shows such a choice.

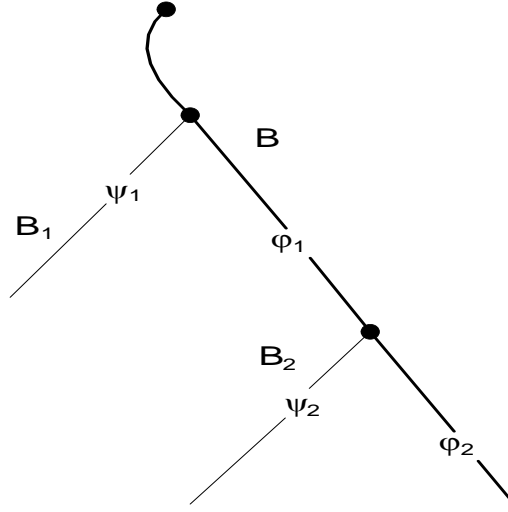


Figure 24

Continue inductively to obtain sequences (n_k) , (ψ_k) where (ψ_k) is a strongly incomparable sequence, $\psi_k \in B_{n_k}$ and $x^*(e_{\psi_k}) > \varepsilon$ for all k . Letting $\delta = 1$ in Lemma 3.2.2, we have that (e_{ψ_k}) is a c_0 sequence so $e_{\psi_k} \xrightarrow{w} 0$. Thus $x^*(e_{\psi_k}) \rightarrow 0$ which is a contradiction.

So we must have $f = S^*$ for some $S \in \mathcal{S}_0$. Assume S is a 0 - m segment for some $m \geq 0$, and let $\varphi \in S$, $lev(\varphi) = m$. Since $B_n^* \xrightarrow{w^*} S^*$, the set $\{n \in \mathbb{N} : \varphi \in B_n\}$ is infinite. By passing to a subsequence and reindexing if necessary, we may assume that for all n , $\varphi \in B_n$ and if $\psi \in B_n$, $\psi < \varphi$, $\psi \notin B_j$ for all $j \neq n$ as shown in Figure 25.

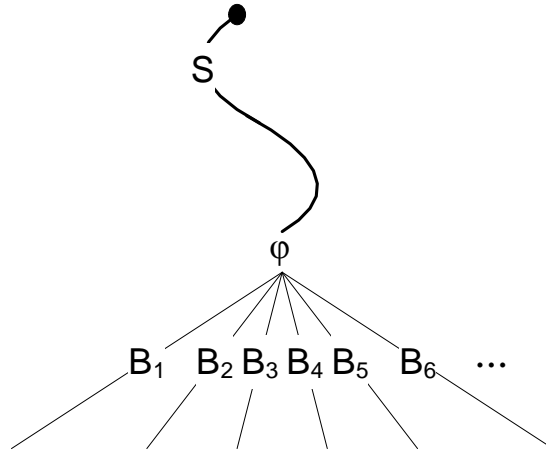


Figure 25

Pick $\psi_1 \in B_1$, $\psi_1 < \varphi$ such that $x^*(e_{\psi_1}) > \varepsilon$. Now pick $\psi_2 \in B_2 \setminus B_1$, $lev(\psi_2) > lev(\psi_1)$ such that $x^*(e_{\psi_2}) > \varepsilon$. Continue inductively to obtain a sequence (ψ_k) such that $lev(\psi_{k+1}) > lev(\psi_k)$. It is easy to see that the sequence (ψ_k) is a strongly rooted sequence. By Corollary 3.5.4, there is a normalized $[p_k, \gamma_k]$ -block basic sequence, (w_k) , of (e_{ψ_k}) that is a c_0 sequence. This means that $\gamma_k = \frac{1}{\|\sum_{i=p_k+1}^{p_{k+1}} e_{\psi_i}\|} \geq \frac{1}{p_{k+1} - p_k}$. Now for each k ,

$$x^*(w_k) = x^*\left(\gamma_k \sum_{i=p_k+1}^{p_{k+1}} e_{\psi_i}\right) = \gamma_k \sum_{i=p_k+1}^{p_{k+1}} x^*(e_{\psi_i}) > \gamma_k (p_{k+1} - p_k) \varepsilon \geq \varepsilon$$

so $w_k \xrightarrow{w} 0$. Therefore, (w_k) cannot be a c_0 sequence which is a contradiction. ■

In light of Lemma 4.2.2, we may regard the operator $Q : X_\infty^* \rightarrow c_0(\Gamma)$. Now we show that $\|Q\| = 1$ and is a quotient map, i.e. $\overline{QB_{X_\infty^*}} = B_{c_0(\Gamma)}$. Once established, it follows that $X_\infty^*/\ker Q \approx Q(X_\infty^*) = c_0(\Gamma)$.

Lemma 4.2.3 Let $Q : X_\infty^* \rightarrow c_0(\Gamma)$ be defined by $Qx^*(B) = \lim_{\psi \in B} x^*(e_\psi)$. Then $\|Q\| = 1$, and Q is a quotient map.

Proof To show that $\|Q\| = 1$, fix a branch B and let $x^* \in X_\infty^*$. Since $x^*(e_\psi) \leq \|x^*\|$ for all $\psi \in B$, then $|Qx^*(B)| = \lim_{\psi \in B} x^*(e_\psi) \leq \|x^*\|$, and $\|Q\| \leq 1$. But $|QB^*(B)| = \lim_{\psi \in B} B^*(e_\psi) = 1$, so we have that $\|Q\| = 1$. To show that Q is a quotient map, let $f \in c_0(\Gamma)$ be such that $|f(B)| > 0$ for finitely many $B \in \Gamma$. Then there are scalars $\alpha_1, \dots, \alpha_r$ with $\max_j |\alpha_j| = 1$ and distinct branches B_1, \dots, B_r such that

$$f(B) = \begin{cases} \alpha_j & \text{if } B = B_j \\ 0 & \text{if } B \neq B_j \end{cases}$$

for $j = 1, \dots, r$.

Pick $m > r$ such that if $\psi_i \in B_i$, $lev(\psi_i) = m$, then $\psi_i \notin B_j$ if $i \neq j$. Figure 26 illustrates this situation.

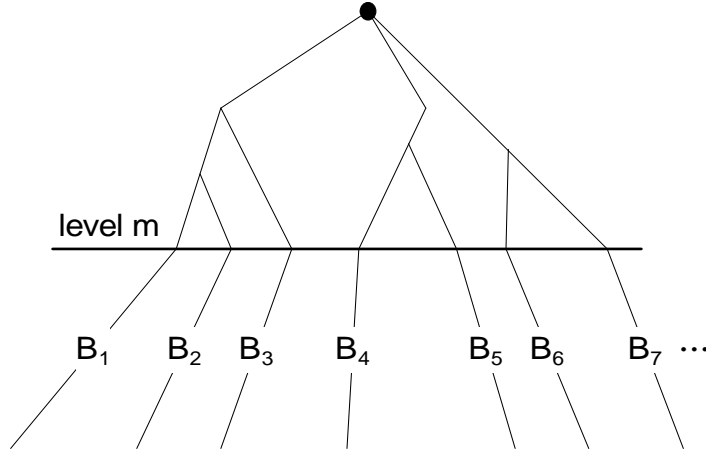


Figure 26

Define

$$x^* = \sum_{j=1}^r \alpha_j P_m^* B_j^*.$$

Let $x \in X_\infty$, $\|x\| = 1$. For each $j = 1, \dots, r$, if $S_j \subseteq B_j$ is an m -segment, then since $r < m$, S_1, \dots, S_r is an admissible family of segments. It follows that

$$\left| \sum_{j=1}^r \alpha_j S_j^*(x) \right| \leq \max_j |\alpha_j| \sum_{j=1}^r |S_j^*(x)| \leq \|x\|. \quad (4.1)$$

Since Equation 4.1 holds for any admissible family of m -segments S_1, \dots, S_r with $S_j \subseteq B_j$, we have that

$$|x^*(x)| = \left| \sum_{j=1}^r \alpha_j P_m^* B_j^*(x) \right| \leq \|x\|$$

and $\|x^*\| \leq 1$. Suppose $1 \leq j \leq r$, $|\alpha_j| = 1$, and $\psi_j \in B_j$. Then $|x^*(e_{\psi_j})| = |\alpha_j| = 1$ so $\|x^*\| = 1$.

For all $\psi \in \mathfrak{S}_\infty$, if $\psi \notin \bigcup_{j=1}^r B_j$, then $x^*(\psi) = 0$. So we have

$$Qx^*(B) = \begin{cases} \alpha_j & \text{if } B = B_j \\ 0 & \text{if } B \neq B_j \end{cases} = f(B).$$

It follows that $QB_{X_\infty} = \{f \in c_0(\Gamma) : |f(B)| > 0 \text{ for finitely many } B \in \Gamma\}$ which is dense in $c_0(\Gamma)$. ■

Let $F = [\{\psi^* : \psi \in \mathfrak{S}_\infty\}]$. For any branch B , $\lim_{\psi \in B} \psi^*(B) = 0$ so $F \subseteq \ker Q$.

We claim that, in fact, $F = \ker Q$. To see this, we need the next result which is proved

in the same manner as Lemma 1 of [8]. It involves looking at the $\lim_n \max_{lev(\psi)=n} \|P_\psi^* x^*\|$

for $x^* \in \ker Q$. Let's look at an example before we begin the proof. Suppose for

$j = 1, 2, 3, x_j \in X_\infty, \|x_j\| = 1$, and for $n > 0, P_n(x_j) = P_{\psi_j}(x_j)$ where $lev(\psi_j) = n$.

Figure 27 shows such x_j .

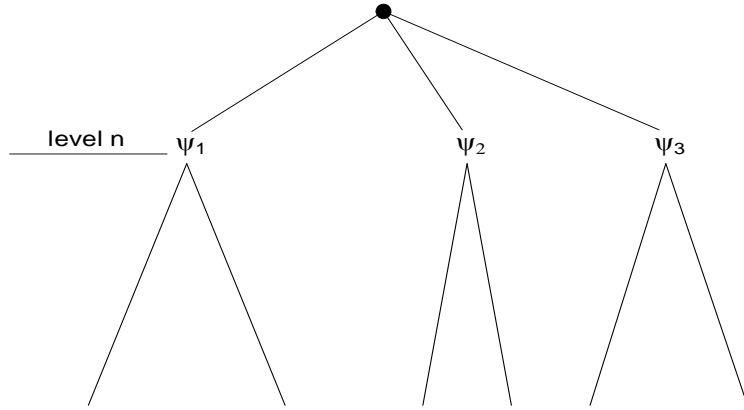


Figure 27 $P_n(x_j) = P_{\psi_j}(x_j), j = 1, 2, 3$

Assume $\|P_{\psi_j}^* x^*\| = |P_{\psi_j}^* x^*(x_j)| = |x^*(P_{\psi_j} x_j)|$ for each $j = 1, 2, 3$, and $P_{\psi}^* x^* = 0$ for all ψ such that $lev(\psi) = n$, $\psi \notin \{\psi_1, \psi_2, \psi_3\}$. If $\|P_{\psi_1}^* x^*\| > \|P_{\psi_2}^* x^*\|, \|P_{\psi_3}^* x^*\|$, then $\max_{lev(\psi)=n} \|P_{\psi}^* x^*\| = \|P_{\psi_1}^* x^*\|$. Now we look at the limit of such norm values as the level n increases.

Lemma 4.2.4 Let $Q : X_{\infty}^* \rightarrow c_0(\Gamma)$ be defined by $Qx^*(B) = \lim_{\psi \in B} x^*(e_{\psi})$. If $x^* \in \ker Q$, then $\lim_n \max_{lev(\psi)=n} \|P_{\psi}^* x^*\| = 0$.

Proof Let $x^* \in \ker Q$. Suppose there is an $\varepsilon > 0$ and a sequence of nodes, (ψ_k) with $lev(\psi_k) < lev(\psi_{k+1})$ such that $\|P_{\psi_k}^* x^*\| > \varepsilon$ for all k . Then for all k , let $x_k \in X_{\infty}$, $\|x_k\| = 1$, such that $|P_{\psi_k}^* x^*(x_k)| > \varepsilon$. We may assume that $x_k = P_{\psi_k}(x_k)$ and $x^*(x_k) > \varepsilon$ for all k . By passing to a subsequence and reindexing if necessary, we may also assume that if $k \neq j$, then the $supp(x_k) \cap supp(x_j) = \emptyset$. By Lemma 1.3.5, there is a subsequence (ψ_k) of (ψ_k) that is either

1. a strongly incomparable sequence;
2. a strongly rooted sequence;
3. a sequence that determines a unique branch.

If Case 1 holds, then (x_k) is a c_0 sequence, and so $x_k \xrightarrow{w} 0$ contradicting $x^*(x_k) > \varepsilon$ for all k . If Case 2 holds, then since $lev(\psi_k) < lev(\psi_{k+1})$, there exists a normalized

$[p_k, \gamma_k]$ -block basic sequence, (w_k) , of (x_k) that is a c_0 sequence. Hence, $w_k \xrightarrow{w} 0$. Since $\|w_k\| = 1$ for all k , $x^*(w_k) > \varepsilon$ for all k , and again we arrive at a contradiction.

So we must have that the sequence (ψ_k) determines a unique branch B . We may assume that $B^*(x_k) \geq 0$ for all k . Since $x^* \in \ker Q$, $\lim_{\psi \in B} P_B^* x^*(x_k) = 0$ for all k . To see this, let $\delta > 0$, and let $\varphi \in B$ such that if $\psi \in B$ and $\psi > \varphi$, then $x^*(e_\psi) < \delta$. Now for all k , let $F_k = B \cap \text{supp}(x_k)$ and $P_B x_k = \sum_{\psi \in F_k} a_\psi e_\psi$ for some scalars (a_ψ) . Then for all k such that $\psi_k > \varphi$, we have

$$\begin{aligned} P_B^* x^*(x_k) &= x^*(P_B x_k) = x^*\left(\sum_{\psi \in F_k} a_\psi e_\psi\right) = \sum_{\psi \in F_k} a_\psi x^*(e_\psi) \\ &\leq \delta \sum_{\psi \in F_k} a_\psi = \delta B^*(x_k) \leq \delta \|x_k\| = \delta. \end{aligned}$$

Letting $\delta < \frac{\varepsilon}{2}$ and $y_k = x_k - P_B x_k$, we have $\|y_k\| > \frac{\varepsilon}{2}$. Let $y_k = \sum_{\psi \in G_k} P_\psi(y_k)$ for some finite subset $G_k \subseteq \mathfrak{S}_\infty$. See Figure 28.

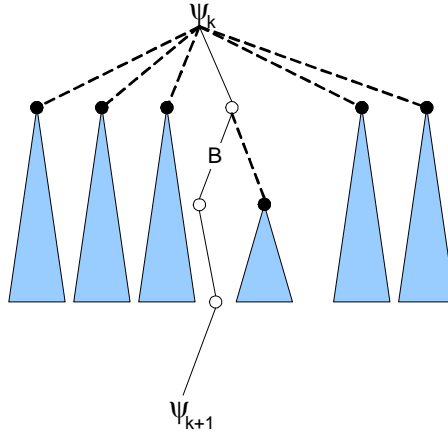


Figure 28 The vector y_k is represented by the shaded regions

The set $\bigcup_{k=1}^{\infty} G_k$ is a pairwise incomparable set so either (y_k) contains a c_0 subsequence or there is a normalized block basic sequence of (y_k) that is a c_0 sequence. In either case, following the same arguments as Case 1 or Case 2, we obtain a contradiction. ■

The next result is proved in the same way as Lemma 8 in [4]. The proof is included for completeness.

Theorem 4.2.5 $X_{\infty}^*/F \approx c_0(\Gamma)$.

Proof Let $Q : X_{\infty}^* \rightarrow c_0(\Gamma)$ be the operator defined by $Qx^*(B) = \lim_{\psi \in B} x^*(e_{\psi})$.

We show that $\ker Q = F$. Let $0 < \delta < \frac{1}{8}$. Note that $2 + 4\delta < 3 - 4\delta$. Assume $x^* \in \ker Q$, $\|x^*\| = 1$, and

$$\inf \{\|x^* - f\| : f \in F\} > 1 - \delta.$$

Let $x_1 \in X_{\infty}$, $\|x_1\| = 1$, be such that for some n_1, l_1 , x_1 is finitely supported on (n_1, l_1) and $x^*(x_1) > 1 - \delta$. Let $\varepsilon > 0$ be such that $(n_1 + 1)\varepsilon < \delta$. By Lemma 4.2.4, there is an $n_2 > 2(l_1 + 1)$ such that $\|P_{\psi}^* x^*\| < \varepsilon$ for all ψ with $\text{lev}(\psi) = n_2$. Pick $x_2 \in X_{\infty}$, $\|x_2\| = 1$, such that for some l_2 , x_2 is finitely supported on (n_2, l_2) and $x^*(x_2) > 1 - \delta$. Pick $x_3 \in X_{\infty}$, $\|x_3\| = 1$, such that $P_{l_2}(x_3) = x_3$ and $x^*(x_3) > 1 - \delta$. Observe that

$$\|x_1 + x_2 + x_3\| > 3(1 - \delta).$$

We can distinguish two mutually exclusive cases.

1. For any admissible family of segments S_1, \dots, S_{l+1} , passing through the support of x_2 with $l \leq l_1$, we have

$$\sum_{j=1}^{l+1} |S_j^*(x_2)| \leq 1 - 4\delta.$$

Then $\|x_1 + x_2 + x_3\| \leq \|x_1\| + \|x_2\| + \|x_3\| \leq 3 - 4\delta$ which contradicts

$$\|x_1 + x_2 + x_3\| > 3(1 - \delta).$$

2. There exists an admissible family of segments S_1, \dots, S_{l+1} , passing through the support of x_2 with $l \leq l_1$ such that

$$\sum_{j=1}^{l+1} |S_j^*(x_2)| > 1 - 4\delta.$$

For each $j = 1, \dots, l + 1$, let $\psi_j \in S_j$ such that $lev(\psi_j) = n_2$, as shown in Figure 29.

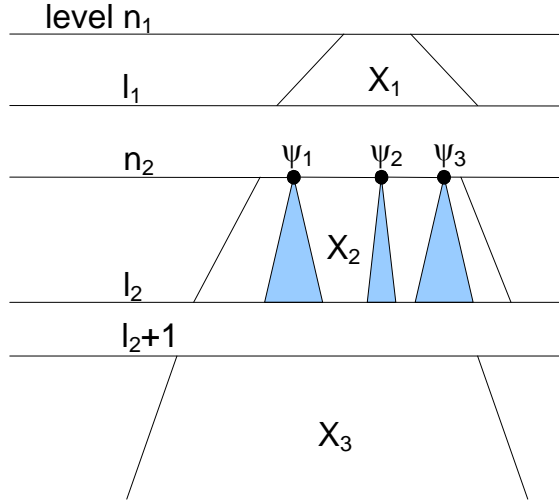


Figure 29 $x(2, 1)$ shaded, $x(2, 2)$ non-shaded regions in x_2

Then for any admissible family of segments R_1, \dots, R_{l+1} passing through the support of x_2 that are disjoint from $S_1, \dots, l + 1$, we have

$$\sum_{j=1}^{l+1} |R_j^*(x_2)| = \sum_{j=1}^{l+1} |R_j^*(x(2, 2))| < 4\delta$$

since $2(l_1 + 1) < n_2$ and $S_1, \dots, S_{l+1}, R_1, \dots, l + 1$ are an admissible family of segments.

So for any admissible family of segments R_1, \dots, R_{l+1} passing through the support of $x(2, 2)$, we have

$$\sum_{j=1}^{l+1} |R_j^*(x(2, 2))| < 4\delta. \tag{4.2}$$

Recall that for each $j = 1, \dots, l + 1$, $\|P_{\psi_j}^* x^*\| < \varepsilon$ which implies that

$$x^*(x(2, 1)) = x^*\left(\sum_{j=1}^{l+1} P_{\psi_j}^* x^*(x_2)\right) \leq (l + 1)\varepsilon < \delta.$$

So now

$$x^*(x(2, 2)) = x^*(x_2) - x^*(x(2, 1)) \geq (1 - \delta) - \delta = 1 - 2\delta$$

and

$$x^*(x_1 + x(2, 2) + x_3) > (1 - \delta) + (1 - 2\delta) + (1 - \delta) = 3 - 4\delta.$$

It follows that

$$\|x_1 + x(2, 2) + x_3\| \geq 3 - 4\delta. \quad (4.3)$$

Then for any admissible family of segments R_1, \dots, R_{l+1} , $l \leq l_1$, through the support of $x(2, 2)$, by Equation 4.2, we have

$$\begin{aligned} \sum_{j=1}^{l+1} |R_j(x_1 + x(2, 2) + x_3)| &\leq \|x_1\| + \sum_{j=1}^{l+1} |R_j(x(2, 2))| + \|x_3\| < 2 + 4\delta \\ \implies \|x_1 + x(2, 2) + x_3\| &< 2 + 4\delta < 3 - 4\delta \end{aligned}$$

contradicting Equation 4.3.

So we must have

$$\inf \{\|x^* - f\| : f \in F\} \leq 1 - \delta < 1$$

if $\|x^*\| = 1$. It follows that there is an $f_1 \in F$ such that

$$\|x^* - f_1\| < 1 - \delta.$$

Since $\left\| \frac{x^* - f_1}{\|x^* - f_1\|} \right\| = 1$, it follows that there is an $f_2 \in F$ such that

$$\left\| \frac{x^* - f_1}{\|x^* - f_1\|} - f_2 \right\| < 1 - \delta \implies \|(x^* - f_1) - (\|x^* - f_1\| f_2)\| < (1 - \delta)^2.$$

Once again since $\left\| \frac{(x^* - f_1) - (\|x^* - f_1\| f_2)}{\|(x^* - f_1) - (\|x^* - f_1\| f_2)\|} \right\| = 1$, there is an $f_3 \in F$ such that

$$\left\| \frac{x^* - f_1 - (\|x^* - f_1\| f_2)}{\|(x^* - f_1) - (\|x^* - f_1\| f_2)\|} - f_3 \right\| < 1 - \delta$$

$$\implies \|(x^* - f_1) - (\|x^* - f_1\| f_2) - (\|(x^* - f_1) - (\|x^* - f_1\| f_2)\| f_3)\| < (1 - \delta)^3.$$

Continue inductively to obtain a sequence in F converging to x^* . It follows that $x^* \in F$ and $\ker Q \subseteq F$. Since we already know $F \subseteq \ker Q$, we have that $\ker Q = F$.

We conclude that $X_\infty^*/F \approx Q(X_\infty^*) = c_0(\Gamma)$. ■

The next result, due to Johnson, is used to show that $l_1(\Gamma)$ is isomorphic to a complemented subspace of X_∞^{**} . For completeness, we quote the result verbatim without proof.

Theorem 4.2.6 (Johnson [7]) Suppose $T : X \rightarrow Y$ is an operator and $\{F_\alpha\}$ is a net, directed by inclusion, of subspaces of Y with $\bigcup F_\alpha$ dense in Y . Assume that for all α there is an operator $L_\alpha : F_\alpha \rightarrow X$ such that $TL_\alpha = I_{F_\alpha}$ where I_{F_α} is the identity operator on F_α , and $\limsup_\alpha \|L_\alpha\| \leq \lambda < \infty$ for some $\lambda > 0$. Then T^* , the adjoint of T , is an isomorphism of Y^* into X^* and there is a projection $P : X^* \rightarrow X^*$ such that $P(X^*) = T^*Y^*$ and $\|P\| \leq \lambda \|T\|$.

Theorem 4.2.7 There exists a complemented subspace W of X_∞^{**} such that

$$l_1(\Gamma) \approx W.$$

Proof For distinct branches B_1, \dots, B_n in Γ , pick $m > n$ such that if $\psi_i \in B_i$, $lev(\psi_i) = m$, then $\psi_i \neq \psi_j$ if $i \neq j$. See Figure 26. Let

$$G_{(B_1, \dots, B_n; m)} = [\{P_m^* B_j^* : j = 1, \dots, n\}]$$

and let

$$A = \left\{ (B_1, \dots, B_n; m) : m, n \in \mathbb{N}, m > n, B_1, \dots, B_n \text{ are distinct branches in } \Gamma \right. \\ \left. \text{such that if } \psi_i \in B_i, lev(\psi_i) = m, \text{ then } \psi_i \neq \psi_j \text{ if } i \neq j \right\}.$$

For each $\alpha \in A$, $\alpha = (B_1, \dots, B_n; m)$, let $T_\alpha : G_\alpha \rightarrow l_\infty^n$ be the operator defined by

$$T_\alpha(P_m^* B_j^*) = f_j$$

where the sequence (f_j) is the usual unit vector basis for l_∞^n . We claim that T_α is an isometry. To see this, let $x^* \in G_\alpha$, $x^* = \sum_{j=1}^n \alpha_j P_m^* B_j^*$. Then for $x \in X_\infty$, $\|x\| = 1$,

$$|x^*(x)| = \left| \sum_{j=1}^n \alpha_j P_m^* B_j^*(x) \right| = \left| \sum_{j=1}^n \alpha_j B_j^*(P_m x) \right| \leq \sum_{j=1}^n |\alpha_j| |B_j^*(x)| \leq \max_j |\alpha_j|$$

since $\sum_{j=1}^n |B_j^*(x)| \leq \|x\| = 1$. So $\|x^*\| \leq \max_j |\alpha_j|$. Now let $1 \leq i \leq n$ be such that

$$|\alpha_i| = \max_j |\alpha_j|. \text{ Then } |x^*(e_{\psi_i})| = |\alpha_i| = \max_j |\alpha_j|, \text{ and } \|x^*\| = \max_j |\alpha_j| = \|T_\alpha x^*\|.$$

Thus, T_α is an isometry.

Now define a subspace, F_α , of l_∞^n by

$$F_\alpha = \{f \in c_0(\Gamma) : f(B) = 0 \text{ if } B \neq B_j \text{ for any } j = 1, \dots, n\}.$$

Then $T_\alpha(G_\alpha) = F_\alpha$ so we may regard $T_\alpha : G_\alpha \rightarrow F_\alpha$ as an isometry onto F_α . Since $c_0(\Gamma)$ is the completion of the space of finitely nonzero functions on Γ , $\bigcup F_\alpha$ is dense in $c_0(\Gamma)$.

Now let $L_\alpha = T_\alpha^{-1} : F_\alpha \rightarrow G_\alpha \subseteq X_\infty^*$, and let $Q : X_\infty^* \rightarrow c_0(\Gamma)$ be the operator defined by

$$Qx^*(B) = \lim_{\psi \in B} x^*(e_\psi).$$

Then for $f \in F_\alpha$, $L_\alpha f = x^* = \sum_{j=1}^n \alpha_j P_m^* B_j^* \in G_\alpha$ and

$$Q(L_\alpha f)(B) = \lim_{\psi \in B} x^*(e_\psi) = \begin{cases} \alpha_j & \text{if } B = B_j \text{ for some } j = 1, \dots, n \\ 0 & \text{if } B \neq B_j \text{ for any } j = 1, \dots, n \end{cases} = f(B).$$

So we have $L_\alpha f = I_{F_\alpha} f$ where I_{F_α} is the identity operator on F_α . Clearly, $\|L_\alpha\| = 1$ for all α , so by Theorem 4.2.6, $Q^* : l_1(\Gamma) \rightarrow X_\infty^{**}$ is an isomorphism, and $Q^*(l_1(\Gamma)) \xrightarrow{c} X_\infty^{**}$. ■

For the next theorem, we recall a standard definition. See [10], for example.

Definition 4.2.8 Let X be a Banach space. For subset $A \subseteq X$, define the subspace A^\perp of X^* by

$$A^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in A\};$$

Then A^\perp (pronounced "A perp") is the *annihilator* of A in X^* . Two standard results involving the annihilator A^\perp of a subset $A \subseteq X$ are that

$$A^\perp \cong (Y/A)^*, \quad A^* \cong Y^*/A^\perp.$$

Again, the interested reader may wish to refer to [10] for the proof. We use these results in the following theorem.

Theorem 4.2.9 $X_\infty^{**} \approx F^* \oplus l_1(\Gamma)$.

Proof By Theorem 4.2.5, $X_\infty^*/F \approx c_0(\Gamma)$. Since the annihilator of F , F^\perp , is isometrically isomorphic to $(X_\infty^*/F)^*$, we have

$$F^\perp \cong (X_\infty^*/F)^* \approx (c_0(\Gamma))^* \cong l_1(\Gamma).$$

We also have that since F is a subspace of X_∞^* , then

$$F^* \cong X_\infty^{**}/F^\perp \approx X_\infty^{**}/l_1(\Gamma).$$

Now by Theorem 4.2.7, there is a subspace Z of X_∞^{**} such that

$$l_1(\Gamma) \approx Z \xhookrightarrow{c} X_\infty^{**}.$$

It follows that

$$X_\infty^{**} \approx X_\infty^{**}/l_1(\Gamma) \oplus l_1(\Gamma) \approx F^* \oplus l_1(\Gamma).$$

■

Corollary 4.2.10 l_1 does not imbed in X_∞ .

Proof If $l_1 \hookrightarrow X_\infty$, then $l_\infty^* \hookrightarrow X_\infty^{**}$, so we must have $\text{card}(X_\infty^{**}) \geq 2^{\mathfrak{c}}$. However, F^* is the dual of a separable space, and so $\text{card}(F^*)$ is \mathfrak{c} . $\text{card}(\Gamma)$ is also \mathfrak{c} . Since $X_\infty^{**} \approx F^* \oplus l_1(\Gamma)$, it follows that $\text{card}(X_\infty^{**})$ is \mathfrak{c} , and $l_1 \not\hookrightarrow X_\infty$. ■

4.3 X_∞^* Does Not Have the Schur Property

In this section, we show that, in contrast to the space X^* , there exists a basic sequence in X_∞^* that converges weakly to zero but not in norm. In other words, X^* has the Schur property, while X_∞^* does not. It follows immediately that X and X_∞ are not isomorphic. We then show that this sequence is not an l_1 sequence, but l_1 imbeds into the closed linear span of the sequence.

Theorem 4.3.1 *Let (ψ_k) be a strongly rooted sequence in \mathfrak{S}_∞ such that $\text{lev}(\psi_k) < \text{lev}(\psi_{k+1})$ for all k . Let (e_{θ_k}) be the usual basis for X_∞ and (ψ_k^*) the sequence in X_∞^* such that*

$$\psi_k^*(e_{\theta_j}) = \begin{cases} 1 & \text{if } \theta_j = \psi_k \\ 0 & \text{if } \theta_j \neq \psi_k \end{cases}.$$

Then the sequence (ψ_k^) has the following properties:*

1. $\|\sum_{k=m}^{2m} \psi_k^*\| \leq 2$ for all m ;
2. $\psi_k^* \xrightarrow{w} 0$;
3. *the sequence (ψ_k^*) does not contain an l_1 subsequence;*
4. $l_1 \hookrightarrow [(\psi_k^*)]$.

Proof For all k , let $n_k = \text{lev}(\psi_k)$. Note that $n_k \geq k$.

1. Let's look at the case when $m = 1$, and then look at the general case. Let

$$x = \sum_{k=1}^N a_k e_{\varphi_k}, \|x\| = 1.$$

If $m = 1$, let $S_1 = \{\psi_1\}$ and let S_2 be the $n_1 - n_2$ segment containing ψ_2 . Then

S_1, S_2 is an admissible family of segments. It follows that

$$|S_1^*(x)| + |S_2^*(x)| = |a_{\psi_1}| + |S_2^*(x)| \leq \|x\| = 1.$$

Now let $R_2 = S_2 \setminus \{\psi_2\}$. Observe that R_2 is a $n_1 - (n_2 - 1)$ segment and

$|a_{\psi_2} + R_2^*(x)| = |S_2^*(x)|$. Since S_1, R_2 is an admissible family of segments, we have

$$|S_1^*(x)| + |R_2^*(x)| = |a_{\psi_1}| + |S_2^*(x)| \leq \|x\| = 1$$

$$|R_2^*(x)| \leq 1 - |a_{\psi_1}|.$$

Now by the triangle inequality we can see that

$$\begin{aligned} |a_{\psi_1}| + |a_{\psi_2}| &\leq |a_{\psi_1}| + |a_{\psi_2} + S_2^*(x)| + |R_2^*(x)| \\ &= |a_{\psi_1}| + |S_2^*(x)| + |R_2^*(x)| \\ &\leq 2. \end{aligned}$$

This implies that

$$|(\psi_1^* + \psi_2^*)(x)| = |a_{\psi_1} + a_{\psi_2}| \leq |a_{\psi_1}| + |a_{\psi_2}| \leq 2.$$

Since x is an arbitrary norm 1 vector in X_∞ , it follows that

$$\|\psi_1^* + \psi_2^*\| \leq 2.$$

For an m , let $S_m = \{\psi_m\}$ and for each $k = m + 1, \dots, 2m$, let S_k be the $n_m - n_k$ segment containing ψ_k . Then S_m, \dots, S_{2m} is an admissible family of segments so we have

$$\sum_{k=m}^{2m} |S_k^*(x)| = |a_{\psi_m}| + \sum_{k=m+1}^{2m} |S_k^*(x)| \leq \|x\| = 1.$$

Now for each $k = m + 1, \dots, 2m$, let $R_k = S_k \setminus \{\psi_k\}$. Observe that R_k is a $n_m - (n_k - 1)$ segment and $|a_{\psi_k} + R_k^*(x)| = |S_k^*(x)|$. Since $S_m, R_{m+1}, \dots, R_{2m}$ is an admissible family of segments it follows that

$$\begin{aligned} |S_m^*(x)| + \sum_{k=m+1}^{2m} |R_k^*(x)| &= |a_{\psi_m}| + \sum_{k=m+1}^{2m} |R_k^*(x)| \leq \|x\| = 1 \\ \sum_{k=m+1}^{2m} |R_k^*(x)| &\leq 1 - |a_{\psi_m}|. \end{aligned}$$

By the triangle inequality we see that

$$\begin{aligned} \sum_{k=m}^{2m} |a_{\psi_k}| &\leq |a_{\psi_m}| + \sum_{k=m+1}^{2m} |a_{\psi_k} + R_k^*(x)| + \sum_{k=m+1}^{2m} |R_k^*(x)| \\ &= |a_{\psi_m}| + \sum_{k=m+1}^{2m} |S_k^*(x)| + \sum_{k=m+1}^{2m} |R_k^*(x)| \\ &\leq 2. \end{aligned}$$

Now we have

$$\left| \sum_{k=m}^{2m} \psi_k^*(x) \right| = \left| \sum_{k=m}^{2m} a_{\psi_k} \right| \leq \sum_{k=m}^{2m} |a_{\psi_k}| \leq 2$$

and so

$$\left\| \sum_{k=m}^{2m} \psi_k^* \right\| \leq 2.$$

2. For any $x = \sum_{k=1}^N a_k e_{\varphi_k}$, $\psi_k^*(x) = a_{\psi_k} \rightarrow 0$ as $k \rightarrow \infty$ so $\psi_k^* \xrightarrow{w^*} 0$. Suppose $x^{**} \in X_{\infty}^{**} \setminus X_{\infty}$ is such that $x^{**}(\psi_k^*)$ does not converge to zero. Then there is a subsequence (ψ_k^*) of (ψ_k^*) and $\varepsilon > 0$ such that $|x^{**}(\psi_k^*)| > \varepsilon$ for all k . Since $\psi_k^* \xrightarrow{w} 0$ if and only if $-\psi_k^* \xrightarrow{w} 0$, by passing to a further subsequence if necessary, we may assume $x^{**}(\psi_k^*) > \varepsilon$ for all k . Letting $m > \frac{2\|x^{**}\|}{\varepsilon}$, we have

$$(m+1)\varepsilon < \sum_{k=m}^{2m} x^{**}(\psi_k^*) = x^{**}\left(\sum_{k=m}^{2m} \psi_k^*\right) \leq \|x^{**}\| \left\| \sum_{k=m}^{2m} \psi_k^* \right\| \leq 2\|x^{**}\|.$$

This implies that $m+1 < \frac{2\|x^{**}\|}{\varepsilon}$ which is a contradiction. So we must have that $x^{**}(\psi_k^*) \rightarrow 0$.

3. $\|\psi_k^*\| = 1$ for all k , so (ψ_k^*) does not converge to zero in norm. Since l_1 has the Schur property, i.e. weak sequential convergence and norm convergence coincide, (ψ_k^*) cannot contain a subsequence equivalent to the l_1 basis.
4. Pick a subsequence (ψ_k) of (ψ_k) so that $2n_k < n_{k+1}$ for all k . Let $L_1 = n_1$ and put

$$w_1 = \frac{1}{L_1 + 1} \sum_{i=L_1}^{2L_1} e_{\psi_i}.$$

Note that $\|w_1\| = 1$. Let $L_2 = \text{lev}(\psi_{2L_1}) = n_{2L_1}$ so that $\|P_{L_2}(w_1)\| = 0$. Put

$$w_2 = \frac{1}{L_2 + 1} \sum_{i=L_2}^{2L_2} e_{\psi_i}.$$

Note that $\|w_2\| = 1$. As before, let $L_3 = \text{lev}(\psi_{2L_2}) = n_{2L_2}$ so that $\|P_{L_3}(w_2)\| = 0$.

Continue inductively to construct a normalized block basic sequence (w_k) of (e_{ψ_k})

where

$$w_k = \frac{1}{L_k + 1} \sum_{i=L_k}^{2L_k} e_{\psi_i}$$

with $1 \leq L_1 < L_2 < \dots$ and $\|P_{L_{k+1}}(w_k)\| = 0$. The sequence (w_k) is a c_0 sequence.

To see this, first observe that any segment in \mathfrak{S}_∞ passes through at most one e_{ψ_k} . Let

b_c, \dots, b_d be scalars and S_1, \dots, S_{l+1} an admissible family of segments beginning at level $l \leq L_c$ such that

$$\left\| \sum_{k=c}^d b_k w_k \right\| = \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c}^d b_k w_k \right) \right|.$$

Then $l + 1 \leq L_{c+1}$ and

$$\begin{aligned} \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^d b_k w_k \right) \right| &= \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^d \frac{b_k}{L_k + 1} \sum_{i=L_k}^{2L_k} e_{\psi_i} \right) \right| \\ &\leq \max |b_k| \frac{l+1}{L_{c+1}} \leq \max |b_k|. \end{aligned}$$

Now we have

$$\left\| \sum_{k=c}^d b_k w_k \right\| \leq \|b_c w_c\| + \sum_{j=1}^{l+1} \left| S_j^* \left(\sum_{k=c+1}^d b_k w_k \right) \right| \leq 2 \max |b_k|.$$

For $k = 1, 2, \dots$, put

$$w_k^* = \sum_{j=L_k}^{2L_k} \psi_j^*.$$

By Part 1, $\|w_k^*\| \leq 2$ for all k . Observe that $w_k^*(w_j) = \delta_{kj}$. Let scalars t_c, \dots, t_d be

given and put $\varepsilon_j = \text{sgn}(t_j)$. Then $\frac{1}{2} \left\| \sum_{j=c}^d \varepsilon_j w_j \right\| \leq 1$, and

$$\left\| \sum_{k=c}^d t_k w_k^* \right\| \geq \frac{1}{2} \sum_{k=c}^d t_k w_k^* \left(\sum_{j=c}^d \varepsilon_j w_j \right) = \frac{1}{2} \sum_{k=c}^d |t_k|.$$

So the sequence (w_k^*) is an l_1 sequence. ■

The following corollary follows immediately.

Corollary 4.3.2 X_∞^* does not have the Schur property.

Chapter 5

Future Work

First and foremost in our future, we wish to determine if X_∞ is hereditarily c_0 . As mentioned earlier, the one remaining case is such that the support of each x_k in a basic sequence begins at the same level, ends at different levels and is comparable to every other support set. We believe that a similar but simpler decomposition method than that of Theorem 3.6.1 along with a blocking method much like that in Theorem 3.5.2 will show that X_∞ is hereditarily c_0 . We also wish to determine whether X_∞^* is hereditarily l_1 .

We conclude by stating some open problems that are related to our analysis of X_∞ and X .

1. Consider finitely nonzero functions z defined on the *dyadic* tree \mathfrak{S} with the norm defined by $\|z\| = \max \sum_{j=1}^l |S_j^*(z)|$ where the \max is taken over all families of segments which are pairwise disjoint and *begin* at the same level. Let Z be the completion of the finitely nonzero functions with this norm. We can show that $Z \xhookrightarrow{c} X \xhookrightarrow{c} X_\infty$.
 - (a) Is $X \approx Z$?
 - (b) Does X imbed in Z ?

- (c) We have that $X_\infty \approx Z \oplus W$ for some subspace W of X_∞ . Since $X_\infty \not\approx X$, it is obvious that $X_\infty \not\approx Z$. Is $X_\infty \approx W$?
2. Recall that a Banach space Y is *primary* if whenever $Y \approx Y_1 \oplus Y_2$, then $Y \approx Y_1$ or $Y \approx Y_2$. Are either of the spaces X or X_∞ primary?

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Appendix

Standard notation and terminology used can be found in references [9] or [10] unless otherwise noted. For the sake of completeness, we introduce those that we use most frequently here. We state well-known results, proofs of which can also be found in [9] or [10]. Not all original sources for these results are given.

Linear Operators Between Normed Spaces

Suppose T is a linear operator from a normed space X into a normed space Y . The kernel of T , denoted $\ker T$, is $\{x \in X : T(x) = 0\}$. The *norm* or *operator norm* $\|T\|$ of T is the nonnegative real number

$$\|T\| = \sup \{\|Tx\| : x \in X, \|x\| \leq 1\}.$$

If $X \neq 0$, this is equivalent to saying

$$\|T\| = \sup \{\|Tx\| : x \in X, \|x\| = 1\}.$$

The operator T is *bounded* if $\|Tx\| \leq \|T\| \|x\|$ for all $x \in X$. Equivalently, T is bounded if and only if T is continuous. T is an *isomorphism* if it is bounded, one-to-one and its inverse, T^{-1} , is bounded on the range of T . T is an *isometric isomorphism* or *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. Clearly, if T is an isometry, then T is an isomorphism. The

operator T is an isomorphism if and only if there are positive constants s and t such that

$$s \|x\| \leq \|Tx\| \leq t \|x\|$$

whenever $x \in X$.

The spaces X and Y are *isomorphic* if there is an isomorphism from X onto Y and are *isometrically isomorphic* or *isometric* if there is an isometry from X onto Y . If X and Y are isomorphic, we write $X \approx Y$, and if X and Y are isometric, we write $X \cong Y$. We say that a normed space X is *imbedded* in Y if there is an isomorphism from X into Y .

Henceforth, the term "operator" means bounded linear operator unless specified otherwise.

Banach Spaces and Subspaces

A *Banach space* is a complete normed space, i.e. a normed space in which every Cauchy sequence converges. A *linear functional* f on a Banach space X , is a real valued function $f : X \rightarrow \mathbb{R}$. If X is a Banach space, then the dual space of X , denoted X^* , is the space of continuous linear functionals on X . A subspace of a Banach space X is a closed linear submanifold of X . Whenever we consider a Banach space X as a subspace of its second dual, X^{**} , we assume that X is imbedded in X^{**} by the *natural map* or *canonical*

embedding $\pi : X \rightarrow X^{**}$ defined by

$$(\pi(x))(x^*) = x^*(x)$$

for all $x^* \in X^*$.

A linear operator $P : X \rightarrow X$ is a *projection* if $P^2 = P$. A subspace Y of a Banach space X is said to be a *complemented subspace* of X , written $Y \overset{c}{\hookrightarrow} X$, if there is a bounded linear projection P from X onto Y . Equivalently, $Y \overset{c}{\hookrightarrow} X$ if there exists a closed linear subspace Z of X so that X is the direct sum of Y and Z , written $X = Y \oplus Z$. In such a case, Y and Z are said to be *complementary subspaces* of X .

Given a vector space X and a subspace Z , define an equivalence relation \sim on X by setting $x \sim y$ if $x - y \in Z$. Let $\tilde{x} = x + Z$ denote the equivalence class of x . If X is a normed space and Z is a closed subspace of X , define the *quotient norm* on X/Z by

$$\|\tilde{x}\| = \inf \{\|x + z\| : z \in Z\}.$$

A Banach space Y is isomorphic to a quotient space of a space X if and only if there exists an operator T from X onto Y . If such a T exists, then we call T a *quotient map*, and $Y \approx X/\ker T$ while $Y^* \approx (\ker T)^\perp$. Similarly, if Z is a subspace of X , then $Z^* \approx X^*/Z^\perp$. It follows that if Y and Z are complementary subspaces of a Banach space X , then $Y \approx X/Z$.

Sequences

For a sequence $(x_k)_{k=1}^{\infty}$ of elements of X , *the closed span of $(x_k)_{k=1}^{\infty}$* is the closure of the set of finite linear combinations of $(x_k)_{k=1}^{\infty}$ and is denoted by $[(x_k)_{k=1}^{\infty}]$. If it is clear from the context, we write (x_k) for the sequence and $[(x_k)]$ for the closed span of (x_k) . In general, when referring to a subsequence of a sequence, we do not reindex unless it is necessary. For example, we may say that there is a subsequence (x_k) of (x_k) to mean there is a subsequence (x_{k_n}) of (x_k) . We say that a sequence (x_k) is a *normalized sequence* if $\|x_k\| = 1$ for all k . If a sequence (x_k) converges to an element x in the norm topology, we write $x_k \rightarrow x$. If (x_k) converges weakly to an element x in X , we write $x_k \xrightarrow{w} x$ or $x = w \lim x_k$. Similarly, if (x_k^*) is a sequence in X^* which converges weak* to an element x^* in X^* , we write $x_k^* \xrightarrow{w^*} x^*$ or $x^* = w^* \lim x_k^*$.

Schauder Bases

A sequence (x_k) in a Banach space X is a *Schauder basis* for X if for each $x \in X$ there is a unique sequence (a_k) of scalars such that

$$x = \sum_{k=1}^{\infty} a_k x_k.$$

A sequence (x_k) in a Banach space is a *Schauder basic sequence* if it is a Schauder basis for $[(x_k)]$. Henceforth, whenever reference is made to a basis for a Banach space or a basic sequence in a Banach space, we assume the reference is to a Schauder basis or Schauder basic sequence unless stated otherwise.

Proposition A sequence (x_k) in a Banach space X is a basis for X if and only if

- (i) each x_k is nonzero;
- (ii) there is a real number M such that

$$\left\| \sum_{k=1}^{m_1} a_k x_k \right\| \leq M \left\| \sum_{k=1}^{m_2} a_k x_k \right\|$$

whenever $m_1, m_2 \in \mathbb{N}$, $m_1 \leq m_2$, and a_1, \dots, a_{m_2} are scalars;

- (iii) $[(x_k)] = X$.

A sequence (x_k) in a Banach space is a basic sequence if and only if conditions (i) and (ii) in Proposition 1 are satisfied. If (x_k) is a basis (or basic sequence) and K is the smallest real number M such that condition (ii) in Proposition 1 is satisfied, then K is called the *basis constant* for (x_k) .

Once it is known that two Banach spaces each have a basis, the notion of equivalent bases is enough to establish that the two Banach spaces are isomorphic. Two bases, (x_k) of a Banach space X and (y_k) of a Banach space Y , are *equivalent* if there is an isomorphism T from X onto Y for which $Tx_k = y_k$ for all k . If the isomorphism T is into Y , then X is imbedded in Y .

We can construct a basic sequence which is equivalent to an existing basic sequence by exploiting certain stability properties of Schauder bases. The idea is that if we "perturb", or as Diestel says in [3], if we "nudge" each element of a basic sequence by just a little bit, the resulting sequence is a basic sequence equivalent to the original one.

Proposition *Let (x_k) be normalized basis (or basic sequence) of a Banach space X with basis constant K . Let (y_k) be a sequence of vectors in X with*

$$\sum_{k=1}^{\infty} \|x_k - y_k\| < \frac{1}{2}K.$$

Then (y_k) is a basis (or basic sequence) of X which is equivalent to (x_k) .

A very useful method which is frequently used to obtain basic sequences, starting from a given basis or basic sequence, is to consider block bases. Let (x_k) be a basic sequence in a Banach space X . A sequence of non-zero vectors (u_k) in X of the form

$$u_k = \sum_{i=p_k+1}^{p_{k+1}} a_k x_i$$

with (a_k) scalars and $p_1 < p_2 < p_3 < \dots$ an increasing sequence of integers, is called a *block basic sequence* or *block basis* of (x_k) . Clearly a block basis (u_k) of (x_k) is a basic sequence whose basis constant is less than or equal to the basis constant of (x_k) . Block basic sequences can be found in infinite dimensional subspaces of a Banach space with a basis as given in the following proposition.

Proposition *Let X be a Banach space with a basis (x_k) , and let Y_0 be an infinite dimensional subspace of X . Then there is a subspace Y of Y_0 which has a basis which is equivalent to a block basis of (x_k) .*

The next result suggests a specific location in a Banach space where one might look for a basic sequence.

Proposition *Suppose (x_k) is a sequence in a Banach space such that (x_k) converges weakly to zero but does not converge to zero with respect to the norm topology. Then some subsequence of (x_k) is a basic sequence.*

The series $\sum_{k=1}^{\infty} x_k$ is *unconditionally convergent* if $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges for each permutation π of \mathbb{N} . A basis (x_k) for a Banach space X is *unconditional* if, for every $x \in X$, the expansion $\sum_{k=1}^{\infty} a_k x_k$ for x in terms of the basis is unconditionally convergent. A basis for a Banach space is *conditional* if it is not unconditional.

Proposition *A sequence (x_k) in a Banach space X is an unconditional basis for X if and only if*

- (i) *each x_k is nonzero;*
- (ii) *there is a real number M such that*

$$\left\| \sum_{k \in A} a_k x_k \right\| \leq M \left\| \sum_{k \in B} a_k x_k \right\|$$

for each pair A and B of finite subsets of \mathbb{N} such that $A \subseteq B$ and each collection $\{a_k\}$ of scalars; and

- (iii) $[(x_k)] = X$.

A sequence (x_k) in a Banach space is an unconditional basic sequence if and only if conditions (i) and (ii) in Proposition 5 are satisfied. If (x_k) is a basis (or basic sequence) and K_u is the smallest real number M such that condition (ii) 5 is satisfied, then K_u is called the *unconditional basis constant* for (x_k) . Any subsequence or block basic sequence of an unconditional basic sequence is an unconditional basic sequence.

Proposition *Let (x_k) be an unconditional basic sequence with an unconditional constant K . Then for every choice of scalars (a_k) such that $\sum_{k=1}^{\infty} a_k x_k$ converges and every choice of bounded scalars (t_k) , we have*

$$\left\| \sum_{k=1}^{\infty} t_k a_k x_k \right\| \leq 2K \sup_k |t_k| \left\| \sum_{k=1}^{\infty} a_k x_k \right\|.$$

If we have a normalized basic sequence such that the norms of finite linear combinations of the terms in the sequence are bounded, then by the preceding proposition and an application of the triangle inequality, we can see that the sequence is equivalent to the usual c_0 basis. This idea is stated precisely in the following lemma.

Lemma *Let (x_k) be an unconditional basic sequence with an unconditional constant K . Suppose there exist an $M > 0$ such that $\sup_n \left\| \sum_{k=1}^n x_k \right\| = M < \infty$. Then for all n*

and every choice of bounded scalars (t_k) , we have

$$\left\| \sum_{k=1}^n t_k x_k \right\| \leq 2KM \max_k |t_k|.$$

Bases and Duality

Let X be a Banach space with a basis (x_k) . For every integer k the linear functional x_k^* on X defined by

$$x_k^* \left(\sum_{i=1}^{\infty} a_i x_i \right) = a_k$$

is a bounded linear functional. In fact, $\|x_k^*\| \leq \frac{2K}{\|x_k\|}$ where K is the basis constant of (x_k) . These linear functionals (x_k^*) are called the *biorthogonal functionals* or *coefficient functionals* associated with the basis (x_k) . Together, the system (x_k, x_k^*) forms a *biorthogonal system* which is characterized by the relation $x_k^*(x_j) = \delta_{kj}$.

If (x_k^*) is a sequence in X^* , we denote the closed linear span of (x_k^*) in the weak* topology by $[(x_k^*)]^{w^*}$. A basic sequence (x_k^*) in X^* is a *weak* basic sequence*, written *w*-basic*, if there is a sequence (x_k) in X for which $x_k^*(x_j) = \delta_{kj}$ and such that, for every x^* in $[(x_k^*)]^{w^*}$, we have

$$x^* = w^* \lim_N \sum_{k=1}^N x^*(x_k) x_k^*.$$

If (x_k) is a basis for X , then the sequence of biorthogonal functionals (x_k^*) associated with (x_k) is a *w*-basic sequence*. These definitions were introduced by Johnson and Rosenthal in [6].