Strong Orbit Equivalence and Residuality

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Strong Orbit Equivalence
and
Residuality

A Dissertation
Presented to the Faculty
of Natural Sciences and Mathematics
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of Doctor of Philosophy

by
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Advisor: Dr. Nicholas S. Ormes
Abstract

In this dissertation, we consider notions of equivalence between minimal Cantor systems, in particular strong orbit equivalence. By constructing the systems, we show that there exist two nonisomorphic substitution systems that are both Kakutani equivalent and strongly orbit equivalent. We go on to define a metric on a strong orbit equivalence class of minimal Cantor systems and prove several properties about the metric space. If the strong orbit equivalence class contains a finite rank system, we show that the set of finite rank systems is residual in the metric space. The last result shown is that set of systems with zero entropy is residual in the strong orbit equivalence class of any minimal Cantor system.
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Chapter 1

Introduction

There are two main parts of this dissertation. The first main part, found in Chapter 3, is essentially the content of [16]. The main theorem of [16] was motivated by [4] in which Dartnell, Durand, and Maass posed the following question: If two minimal Cantor systems are orbit equivalent and Kakutani equivalent, are they necessarily conjugate? In their paper, they showed that this is true for Sturmian systems. In [11], Kosek, Ormes, and Rudolph answered this question negatively by finding two orbit equivalent and Kakutani equivalent substitution systems that are not conjugate. The question under consideration in [16] is the following: If the orbit equivalence condition is strengthened to strong orbit equivalence, is the statement then true? We answer this question negatively by finding two Kakutani equivalent and strongly orbit equivalent substitution systems that are not conjugate.

The second main part of this dissertation is found in Chapter 4. In this chapter, we consider strong orbit equivalence classes of minimal Cantor systems. In the measure-theoretic category, Dye’s Theorem states that any two ergodic measure-preserving transformations on nonatomic probability spaces are orbit equivalent. In [13], Rudolph introduced the idea of restricted orbit equivalence. By defining a notion of the size of an orbit equivalence, Rudolph gave a natural way to more pre-
cisely distinguish between measure-theoretic systems. In the topological category, even within the category of minimal Cantor systems, there are several nontrivial systems which are not orbit equivalent. However, serving the same purpose as Rudolph’s restricted orbit equivalence in the measure-theoretic setting, strong orbit equivalence provides a more precise way to distinguish between topological systems. Strong orbit equivalence was first introduced by Giordano, Putnam, and Skau in [7] where they proved the following theorem:

**Theorem 1.0.1.** Two minimal Cantor systems are strongly orbit equivalent if and only if their associated dimension groups are order isomorphic by an order isomorphism preserving the distinguished order unit.

In [10], Hochman considered a metric on the space of homeomorphisms of the Cantor set and proved several genericity results about the metric space. In particular, Hochman showed that the universal odometer is residual in the space of transitive systems. Along the same lines, we define a metric on a strong orbit equivalence class of minimal homeomorphisms of a Cantor space. We prove several properties about the resulting metric space including that it is complete and separable but not compact. These results are also related to the work done in [2] where Bezuglyi, Dooley, and Kwiatkowski considered several different topologies on the space of homeomorphisms of the Cantor set. We go on to show that finite rank systems, as defined in [5] by Downarowicz and Maass, are residual in any strong orbit equivalence class containing a finite rank system. In particular, we show that odometers are residual in any class containing an odometer. Finally, we show that systems with zero entropy are residual in the strong orbit equivalence class of any minimal Cantor system. These residuality results are related to the measure-theoretic results of Rudolph found in [14]. To help the reader understand the results in this dissertation, we begin by introducing much of the needed background information in Chapter 2.
Chapter 2

Minimal Cantor Systems

A Cantor space is a nonempty topological space that is perfect, compact, totally disconnected, and metrizable. It is well known that any two such spaces are homeomorphic. A minimal Cantor system is an ordered pair $(X, T)$ where $X$ is a Cantor space and $T : X \to X$ is a minimal homeomorphism. The minimality of $T$ means that every orbit under $T$ is dense in $X$, i.e. if for $x \in X$ we define $O_T(x) = \{T^k x \mid k \in \mathbb{Z}\}$, then for all $x \in X$, $O_T(x)$ is dense in $X$. Because $X$ is metrizable, we can define a metric on $X$ that induces the topology of $X$. We will denote this metric by $d_X$.

2.1 Notions of Equivalence in Minimal Cantor Systems

There are several notions of equivalence in minimal Cantor systems that we will consider. The strongest notion of equivalence is conjugacy. Two minimal Cantor systems $(X, T)$ and $(Y, S)$ are conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$. A weaker notion of equivalence is orbit equivalence. Two systems $(X, T)$ and $(Y, S)$ are orbit equivalent if there exists a homeomorphism $h : X \to Y$ that preserves orbits between the systems. Stated more explicitly, a homeomorphism $h : X \to Y$ is an orbit equivalence if there exist functions $a, b : X \to \mathbb{Z}$ such that for all $x \in X$, $h \circ T(x) = S^{a(x)} \circ h(x)$ and $h \circ T^{b(x)}(x) = S \circ h(x)$.
We call \(a\) and \(b\) the orbit cocycles associated to \(h\). If the orbit cocycles associated to \(h\) each have at most one point of discontinuity, we say the systems \((X,T)\) and \((Y,S)\) are strongly orbit equivalent.

The last notion of equivalence we will consider is Kakutani equivalence. Let \((X,T)\) be a minimal Cantor system and let \(A \subset X\) be clopen. Then because \(T\) is minimal, each \(a \in A\) returns to \(A\) in a finite number of \(T\)-iterations. This allows us to define a function \(r_A : A \to \mathbb{N}^+\) where \(r_A(a) = \min\{n \geq 1 \mid T^n a \in A\}\). It is easily verified that \(r_A\) is a continuous function, and we say that \(r_A(a)\) is the return time of \(a\) to \(A\). If we define the map \(T_A : A \to A\) by \(T_A(a) = T^{r_A(a)}(a)\), then the system \((A,T_A)\) is another minimal Cantor system. We say that \((A,T_A)\) is an induced system of \((X,T)\). Two systems are Kakutani equivalent if they have conjugate induced systems.

### 2.2 Tower Partitions

Tower partitions provide a visual representation of minimal Cantor systems. Let \((X,T)\) be a minimal Cantor system and let \(A \subset X\) be clopen. As discussed when defining Kakutani equivalence, the return time map \(r_A : A \to \mathbb{N}^+\) is continuous. Because \(A\) is compact, \(r_A\) takes on only finitely many values. Therefore, we can partition \(A\) into finitely many clopen sets \(A_1, A_2, \ldots, A_k\) such that the return time to \(A\) is constant on each \(A_j\). For \(j = 1, \ldots, k\), let \(r_j\) denote the return time of \(A_j\) to \(A\). For each \(j\), we construct a tower over \(A_j\) by vertically stacking the sets \(A_j, TA_j, \ldots, T^{r_j-1}A_j\), which we will call the floors of the tower over \(A_j\). An example with \(A\) partitioned into three sets \(A_1, A_2, A_3\) with return times of 4, 3, and 5, respectively, is shown in Figure 2.1. We define the height of the tower over \(A_j\) to be \(r_j\), the return time of \(A_j\) to \(A\). If \(0 \leq i \leq r_j-1\), we will say that the height of the tower floor \(T^i(A_j)\) is \(i\). The floors of these towers create a clopen partition of \(X\), and we will call this a tower partition of \((X,T)\) over \(A\). If \(P\) is a tower partition
of \((X, T)\) over \(A\), notice that the bottom floors of \(P\) partition \(A\). We will denote this partition of \(A\) by \(P(A)\). Also notice that the top floors of \(P\) partition the set \(T^{-1}(A)\). An important property of a tower partition that we will consider is the minimum height of a tower in the partition. If \(P\) is a tower partition, we will let \(H(P)\) denote the minimum height of a tower in \(P\). For example, if \(P\) is the tower partition shown in Figure 2.1, then \(H(P) = 3\).

Let \(\{A_n\}\) be a sequence of clopen sets in \(X\) such that \(A_{n+1} \subset A_n\) for all \(n\). For every \(n\), let \(P_n\) be a tower partition of \((X, T)\) over \(A_n\) such that for all \(n \geq 1\) the tower partition \(P_{n+1}\) is a refinement of \(P_n\). We say that the tower partition sequence \(\{P_n\}\) generates the topology of \(X\) if for any clopen set \(C \subset X\), there exists an \(N > 0\) such that if \(n \geq N\), then \(C\) can be written as a finite union of sets in \(P_n\). Suppose the sequence \(\{P_n\}\) generates the topology of \(X\) and in addition diam\((A_n)\) \(\to 0\). Then \(\bigcap A_n = \{x_1\}\) for some \(x_1 \in X\), so we will say that \(\{P_n\}\) is a generating sequence of tower partitions over \(x_1\).

**Proposition 2.2.1.** If \(\{P_n\}\) is a sequence of finite clopen partitions of a Cantor space \(X\), then \(\{P_n\}\) generates the topology of \(X\) if and only if \(\lim_{n \to \infty} \text{diam}(P_n) = 0\).

**Proof.** Since there are clopen sets of arbitrarily small diameter contained in \(X\), if \(\{P_n\}\) generates the topology of \(X\), then clearly diam\((P_n)\) \(\to 0\). Conversely, assume diam\((P_n)\) \(\to 0\) and let \(C\) be a clopen set in \(X\). Because \(X \setminus C\) is also clopen, there
exists an $\epsilon > 0$ such that $d_X(C, X \setminus C) > \epsilon$. Pick $N$ such that if $n \geq N$, then $\text{diam}(\mathcal{P}_n) < \epsilon$. Fix $n \geq N$ and suppose $P \in \mathcal{P}_n$. We will show that either $P \subset C$ or $P \subset X \setminus C$. Assume $P \cap C \neq \emptyset$, so there exists some $x \in P \cap C$. Now suppose $y \in P$. Since $\text{diam}(P) \leq \text{diam}(\mathcal{P}_n) < \epsilon$, this means $d_X(x, y) < \epsilon$ and thus $y \in C$. So $P \subset C$. Therefore, either $P \subset C$ or $P \subset X \setminus C$. Since $P$ was chosen arbitrarily, we can conclude that each set of $\mathcal{P}_n$ is either contained in $C$ or contained in $X \setminus C$. Since $\mathcal{P}_n$ has finitely many sets and covers $X$, $C$ can be written as a finite union of sets in $\mathcal{P}_n$. 

**Proposition 2.2.2.** Let $(X, T)$ be a minimal Cantor system and let $x_1 \in X$. If $\{\mathcal{P}_n\}$ is a generating sequence of tower partitions over $x_1$, then $\lim_{n \to \infty} \mathcal{H}(\mathcal{P}_n) = \infty$.

*Proof.* For all $n$, let $A_n$ be the clopen set in $X$ such that $\mathcal{P}_n$ is a tower partition over $A_n$, so $\bigcap A_n = \{x_1\}$. Fix $k \in \mathbb{N}^+$ and let $B$ be the clopen set in $\mathcal{P}_k$ with $x_1 \in B$. Since $\text{diam}(A_n) \to 0$, there exists an $N > 0$ such that if $n \geq N$, then $A_n \subset B$. Then for $n \geq N$, the tower height of every tower in $\mathcal{P}_n$ is greater than or equal to the tower height of the tower over $B$ in $\mathcal{P}_k$. Therefore, if we let $\mathcal{P}_n(x_1)$ denote the tower of $\mathcal{P}_n$ that contains $x_1$, it suffices to show that the height of $\mathcal{P}_n(x_1)$ grows arbitrarily large as $n \to \infty$. The height of the tower $\mathcal{P}_n(x_1)$ is the return time of $x_1$ to $A_n$, which we will denote $r_n$. Because $r_n \leq r_m$ for all $n \leq m$, it suffices to show that for all $n \in \mathbb{N}^+$, there exists an $m > n$ such that $r_m > r_n$. Fix $n \in \mathbb{N}^+$ and let $T^{r_n}(x_1) = y_1 \in A_n$. Let $d_X(x_1, y_1) = p > 0$. Then pick $m > n$ such that $\text{diam}(A_m) < p$. Because $d_X(x_1, y_1) = p$, $T^{r_m}(x_1) = y_1 \not\in A_m$, so $r_m \neq r_n$. We must have that $r_m > r_n$ finishing the proof. 

**2.3 Bratteli Diagrams**

Bratteli diagrams give us another way to visually represent minimal Cantor systems. We refer the reader to [9] for a complete discussion of this topic. A Bratteli
diagram $B = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where $V$ and $E$ can be written as the countable union of finite disjoint sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \ldots \quad \text{and} \quad E = E_1 \cup E_2 \cup \ldots.$$  

The set $V_k$ represents the vertices at level $k$ and $E_k$ represents the set of edges between the vertices at level $k-1$ and level $k$. Furthermore, the following properties hold.

1. $V_0 = \{v_0\}$ is a one point set;
2. there is a range map $r$ and a source map $s$ with $r, s : E \to V$ such that $r(E_k) \subset V_k$ and $s(E_k) \subset V_{k-1}$. We also require that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.

### 2.3.1 Ordered Bratteli Diagrams

An ordered Bratteli diagram $B = (V, E, \leq)$ is a Bratteli diagram along with a partial order $\leq$ on $E$ such that two edges are comparable if and only if they have the same range. The first three levels of an ordered Bratteli diagram are shown in Figure 2.2. For $k, l \in \mathbb{N}$, $k < l$, we denote the set of all edge paths from $V_k$ to $V_l$ by $E[k, l]$. There are natural extensions of the range and source maps to $E[k, l]$ by defining $s(e_{k+1}, \ldots, e_l) = s(e_k)$ and $r(e_{k+1}, \ldots, e_l) = r(e_l)$. We can extend the partial order on the edges to a partial order on $E[k, l]$ by ordering paths that begin at the same level and have the same range. The partial order $\leq'$ induced on $E[k, l]$ is a reverse lexicographical ordering given by $(e_{k+1}, \ldots, e_l) \leq' (f_{k+1}, \ldots, f_l)$ if and only if $r(e_i) = r(f_i)$ and there exists a $j$ with $k + 1 \leq j \leq l$ such that $e_j = f_j$ for $j < i \leq l$ and $e_j < f_j$. 

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2.3.2 Telescoping

Given a Bratteli diagram, we can create a new Bratteli diagram by a process called telescoping. Let \( B = (V, E, \leq) \) be an ordered Bratteli Diagram and remove \( E[k, l] \) and \( V_{k+1}, V_{k+2}, \ldots, V_{l-1} \). We then reconnect \( V_k \) and \( V_l \) by single edges, one edge for each of the paths in \( E[k, l] \), beginning and ending at their corresponding source and range, respectively. Ordering these edges by the partial order \( \leq' \) described above, we call this new diagram a telescoping between levels \( k \) and \( l \). A telescoping between two levels of a Bratteli diagram is shown in Figure 2.3. Let \( \{n_k\}_{k=0}^{\infty} \) be a sequence in \( \mathbb{N} \) with \( n_0 = 0 \) and \( n_k < n_{k+1} \) for all \( k \). If we telescope \( B \) between levels \( n_k \) and \( n_{k+1} \) for all \( k \) ordering the edges according to \( \leq' \), we have a new ordered Bratteli diagram \( B' = (V', E', \leq') \). We say that \( B' \) is a telescoping of \( B \). If the telescoping is done by telescoping a finite number of levels, i.e. there exists \( K \in \mathbb{N} \) such that for all \( j \in \mathbb{N} \), \( n_{K+j} = n_K + j \), we say that \( B' \) is a finite telescoping of \( B \).
2.3.3 Dimension Groups

For a Bratteli diagram $B = (V, E)$, let $V_k = \{v(k, j) \mid 1 \leq j \leq |V_k|\}$. For each $k$, we define the incidence matrix $M_k = [m_{ij}]$, $i = 1, \ldots, |V_k|$, $j = 1, \ldots, |V_{k+1}|$, where $m_{ij}$ is the number of edges between the vertices $v(k, i)$ and $v(k+1, j)$. We can associate a dimension group $K_0(V, E)$ to the Bratteli diagram by taking the inductive limit of groups $\varprojlim \mathbb{Z}^{|V_k|}, M_k)$. This can be made into an ordered group by declaring that $[v] \in K_0(V, E)^+$ if there is a $w \in [v]$ such that each coordinate of $w$ is non-negative. We distinguish an order unit in $K_0(V, E)$ as the element associated to $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$.

2.3.4 Bratteli Diagrams to Minimal Cantor Systems

**Definition 2.3.1.** An ordered Bratteli diagram $B = (V, E, \leq)$ is properly ordered if

1. there is a telescoping (not necessarily finite) $B'$ of $B$ such that any two vertices at consecutive levels in $B'$ are connected by an edge;

2. there are unique infinite edge paths $x_{\text{max}}$ and $x_{\text{min}}$ in $B$ such that each edge of $x_{\text{max}}$ is maximal in $\leq$ and each edge of $x_{\text{min}}$ is minimal in $\leq$.

Given a properly ordered Bratteli diagram $B = (V, E, \leq)$, we let $X_B$ be the set of all infinite paths in $B$. We topologize $X_B$ by letting the family of cylinder sets be a basis for the topology. A cylinder set is the set of paths that begin
with a given finite edge path. We will let \([e_1, \ldots, e_k]\) represent the cylinder set 
\(\{(x_1, x_2, \ldots) \in X_B \mid x_i = e_i \ \forall \ i \leq k\}\). The space \(X_B\) along with this topology is 
a Cantor space. We define the \textit{Vershik map} \(V_B : X_B \rightarrow X_B\) in the following way. 
If \(x = (x_1, x_2, \ldots) \in X_B \setminus \{x_{\text{max}}\}\), there is smallest \(k\) such that \(x_k\) is not maximal.
If we let \(y_k\) be the successor of \(x_k\) and let \((y_1, \ldots, y_{k-1})\) be the minimal path from
\(v_0\) to \(s(y_k)\), we define \(V_B(x) = (y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots)\). The tails of \(x\) and
\(V_B(x)\) agree past level \(k\), so we say they are \textit{cofinal}. We define \(V_B(x_{\text{max}}) = x_{\text{min}}\).
The system \((X_B, V_B)\) is a minimal Cantor system and we refer to it as a \textit{Bratteli-Vershik system}. It is shown in \cite{9} that any minimal Cantor system is conjugate to
a Bratteli-Vershik system.

\subsection{Minimal Cantor Systems to Bratteli Diagrams}

We will now describe the construction of a Bratteli-Vershik system from a minimal
Cantor system. Let \(\{P_k\}\) be a generating sequence of tower partitions of a minimal
Cantor system \((X, T)\). For all \(k\), let \(A_k\) be the clopen set contained in \(X\) such that
\(P_k\) is a tower partition over \(A_k\). Let \(P_k(A_k) = \{A(k, 1), \ldots, A(k, n_k)\}\), where each
\(A(k, j)\) is a bottom floor of \(P_k\). For \(j = 1, \ldots, n_k\), denote the return time of \(A(k, j)\)
to \(A_k\) by \(r(k, j)\).

We insert one vertex \(v_0\) at the top level (level 0) of the diagram. For \(k \geq 1\), at
level \(k\) we insert \(n_k\) vertices, one corresponding to each set in \(P_k(A_k)\). If \(1 \leq j \leq n_k\),
we will let \(v(k, j)\) denote the vertex in \(V_k\) corresponding to the set \(A(k, j)\). If
\(1 \leq j \leq n_1\), the number of edges in \(E_1\) from \(v_0\) to the vertex \(v(1, j)\) at level 1 is
\(r(1, j)\). We will now describe how to construct and order the edges in \(E_k\) for \(k \geq 2\).
For \(k \geq 2\), fix \(1 \leq j \leq n_k\). Fix \(x_0 \in A(k, j)\) and find \(i_1\) such that \(1 \leq i_1 \leq n_{k-1}\)
and \(x_0 \in A(k - 1, i_1)\). Because \(P_k\) is a refinement of \(P_{k-1}\), \(A(k, j) \subset A(k - 1, i_1)\),
so \(i_1\) is not dependent on the choice of \(x_0\). Then the minimal edge (order 1) with
range \(v(k, j)\) has source \(v(k - 1, i_1)\). Set \(x_1 = T^{r(k-1, i_1)}(x_0) \in A_{k-1}\). In general
for $m \geq 1$, we define $i_m$ and $x_m$ recursively such that $x_{m-1} \in A(k-1,i_m)$ and $x_m = Tr^{(k-1,i_m)}(x_{m-1}) \in A_{k-1}$ until we reach an $l \geq 1$ such that $x_l \in A_k$. This will happen after finitely many steps because the return time of $x_0$ to $A_k$ is finite. Then we insert $l$ edges in $E_k$ with range $v(k,j)$, and for $1 \leq p \leq l$, the edge of order $p$ with range $v(k,j)$ has source $v(k-1,i_p)$. We apply this same procedure for every vertex in $V_k$ to construct and order $E_k$. Applying this construction of $E_k$ for all $k \geq 2$ completes the construction of the Bratteli diagram. Under this construction, each vertex in $V_k$ corresponds to exactly one tower in $\mathcal{P}_k$, and each edge path in $E[0,k]$ corresponds to exactly one tower floor in $\mathcal{P}_k$. Because $T$ is minimal and diam($A_n$) $\to 0$, this Bratteli diagram will be properly ordered. It is shown in [9] that the Bratteli-Vershik system associated to this diagram is conjugate to the original system $(X,T)$. 
Chapter 3

A Counterexample

In this chapter, we define two substitution systems that are strongly orbit equivalent and Kakutani equivalent but not conjugate. We begin with a brief introduction to substitution systems; we refer the reader to [6] for more details on this topic.

3.1 Substitution Systems

We start with a finite nonempty alphabet \( \mathcal{A} = \{a_1, \ldots, a_d\} \). If we let \( \mathcal{A}^* \) be the set of finite nonempty words in \( \mathcal{A} \), a substitution is a map \( \sigma : \mathcal{A} \rightarrow \mathcal{A}^* \). There is a natural extension of \( \sigma \) to \( \mathcal{A}^* \) by concatenation that allows us to define iterations of \( \sigma \). For example, suppose \( \sigma \) is the following substitution on the alphabet \( \mathcal{A} = \{a, b\} \):

\[
\sigma : \begin{cases} 
    a &\rightarrow ab \\
    b &\rightarrow abb.
\end{cases}
\]  

Then we have the following:

\[
\sigma^2 (a) = \sigma(ab) = \sigma(a)\sigma(b) = ababb \quad \text{&} \quad \sigma^2 (b) = \sigma(abb) = \sigma(a)\sigma(b)\sigma(b) = ababbab.
\]
We say that a substitution $\sigma$ is primitive if there is a $k > 0$ such that for each $i, j \in A$, $j$ appears in $\sigma^k(i)$, and there is some $i \in A$ such that $\lim_{n \to \infty} |\sigma^n(i)| = \infty$ where $|w|$ represents the length of a word $w$. We say $\sigma$ is proper if there exists $p > 0$ and two letters $r, l \in A$ such that

1. $\forall i \in A$, $r$ is the last letter of $\sigma^p(i)$;
2. $\forall i \in A$, $l$ is the first letter of $\sigma^p(i)$.

The substitution $\sigma$ defined in Equation 3.1.1 is primitive and proper.

We say that a word $w$ (not necessarily finite) is $\sigma$-allowed if and only if each finite subword of $w$ is a subword of $\sigma^n(i)$ for some $n \in \mathbb{N}$ and some $i \in A$. We define $X_\sigma$ to be the set of all $\sigma$-allowed bi-infinite words in $A$. There are substitutions $\sigma$ for which $X_\sigma$ will be finite. However, we are only interested in substitutions where $X_\sigma$ is infinite, in which case we will say that $\sigma$ is aperiodic.

If we take $X_\sigma$ with the shift map $S_\sigma$, i.e. if $x = (\ldots x_{-2}x_{-1}x_0x_1x_2\ldots)$, then $S_\sigma(x) = (\ldots x_{-2}x_{-1}x_0x_1x_2\ldots)$, we say the $(X_\sigma, S_\sigma)$ is the substitution system associated to $\sigma$. For $x \in X_\sigma$, we let $[x] = \mathcal{O}_{S_\sigma}(x)$, the set of all backward and forward shifts of $x$. We say that an orbit $[x]$ is left asymptotic if there is another orbit $[x']$ with $[x] \cap [x'] = \emptyset$ and $y \in [x]$, $y' \in [x']$, $k \in \mathbb{Z}$ such that for all $i \leq k$, $y_i = y'_i$. Right asymptotic orbits are defined analogously, and we say an orbit is asymptotic if it is either left or right asymptotic.

### 3.1.1 Substitution Systems to Bratteli Diagrams

If we let $(X_\sigma, S_\sigma)$ be a substitution system associated to a primitive, aperiodic substitution $\sigma$, it is a minimal Cantor system and has a natural representation as a Bratteli-Vershik system as shown in [6]. In the case that $\sigma$ is proper, which is what we are concerned with, the Bratteli diagram is constructed by setting $V_0 = \{v_0\}$ and $|V_k| = |A|$ for all $k \geq 1$. For $k \geq 1$, we associate each vertex at level $k$ to a symbol in $A$, so we will denote vertices at level $k$ by $\{v(k, a) \mid a \in A\}$. For each
Figure 3.1: X_σ as a Bratteli diagram

a ∈ A, v(1, a) is connected by a single edge to v_0. For all a, b ∈ A and for all k ≥ 2, E_k is constructed by connecting v(k, a) to v(k − 1, b) with one edge for each time b appears in σ(a). Furthermore, if σ(a) = a_1...a_n where each a_i ∈ A, then the edge of order i, 1 ≤ i ≤ n, with range v(k, a) has source v(k − 1, a_i). Because E_k is constructed in the same way for all k ≥ 2, the diagram repeats after level 1, so we refer to this as a stationary Bratteli diagram. Figure 3.1 illustrates the Bratteli diagram for the substitution σ defined in Equation 3.1.1.

We will now describe the correspondence between each bi-infinite word in X_σ and infinite paths in the Bratteli diagram. Let x ∈ X_σ and let x' be the corresponding infinite path in the Bratteli diagram. For each k ≥ 0, there is a word in x around the origin, say w = x_{−n}...x_{−1}.x_0...x_m such that for some a ∈ A, σ^k(a) = w. Then the path that x' follows from v_0 down to level k of the diagram is the path of order n + 1 in the set of paths in E[0, k] with range v(k, a).
3.2 The Counterexample

We will now define two substitution systems that are Kakutani equivalent and strong orbit equivalent but not conjugate. The substitutions for these two systems are defined accordingly. First, we define two substitutions $\sigma_1$ and $\sigma_2$ on an alphabet $\mathcal{A} = \{a, b\}$ as follows:

\[
\begin{align*}
\sigma_1 & : \\
& \begin{cases} 
  a \rightarrow aabb \\
  b \rightarrow abb
\end{cases} \\
\sigma_2 & : \\
& \begin{cases} 
  a \rightarrow abab \\
  b \rightarrow abb.
\end{cases}
\end{align*}
\]

We define $\sigma = \sigma_1 \circ \sigma_2$ and $\tau = \sigma_2 \circ \sigma_1$. So, we have

\[
\begin{align*}
\sigma & : \\
& \begin{cases} 
  a \rightarrow aabbabbaabbabb \\
  b \rightarrow aabbaabbabb
\end{cases} \\
\tau & : \\
& \begin{cases} 
  a \rightarrow ababaabbabbabb \\
  b \rightarrow abababbabb.
\end{cases}
\end{align*}
\]

We let $(X, T)$ be the substitution system associated to $\sigma$ and $(Y, S)$ be the substitution system associated to $\tau$. The Bratteli diagrams associated to these systems are shown in Figure 3.2. Telescoping these diagrams between odd levels, we obtain the stationary Bratteli diagrams associated to the substitution systems described previously. However, since the substitutions here are given by the composition of two substitutions, it is more convenient to look at them in their untelescoped form.
Figure 3.2: \((X,T)\) and \((Y,S)\) as Bratteli diagrams

**Theorem 3.2.1.** The systems \((X,T)\) and \((Y,S)\) defined above are Kakutani equivalent and strong orbit equivalent but not conjugate.

In order to prove this theorem, we need the following theorems.

**Theorem 3.2.2** (Durand, Host, and Skau from [6]). Two Bratteli-Vershik systems associated to properly ordered Bratteli diagrams are Kakutani equivalent if and only if one diagram can be obtained from the other by a finite change, i.e. doing a finite number of finite telescopings and adding and/or removing a finite number of edges.

**Theorem 3.2.3** (Barge, Diamond, and Holton from [1]). A primitive, aperiodic, substitution \(\sigma\) on \(d\) letters has at most \(d^2\) asymptotic orbits.

**Theorem 3.2.4** (Gottschalk and Hedlund from [8]). Any infinite minimal substitution system must have at least one pair each of left and right asymptotic orbits.
We will prove Theorem 3.2.1 by a series of propositions.

**Proposition 3.2.5.** The systems \((X, T)\) and \((Y, S)\) defined above are Kakutani equivalent.

**Proof.** By Theorem 3.2.2, two Bratteli-Vershik systems are Kakutani equivalent to one another if one can be obtained from the other by doing a finite change. Looking at the diagrams in Figure 3.2, if we telescope between the top vertex and level 2 of \((X, T)\) and remove all edges except one between the top vertex and each of the two vertices at the new level 2, we get precisely the ordered Bratteli diagram representing \((Y, S)\). Hence, by Theorem 3.2.2 the systems are Kakutani equivalent. 

**Proposition 3.2.6.** The systems \((X, T)\) and \((Y, S)\) defined above are strongly orbit equivalent.

**Proof.** To see that the substitution systems are strongly orbit equivalent, we again refer to the diagrams in Figure 3.2. If we consider the diagrams as being unordered, they are identical. Since the associated ordered dimension groups are independent of the ordering on the diagram, we have that the systems are strongly orbit equivalent by Theorem 1.0.1. 

Showing that these two systems are not conjugate is a more subtle problem as almost any invariants of the two systems are the same. By Theorem 3.2.3, since our substitution systems are primitive and aperiodic on two symbols, they can have at most four asymptotic orbits. Furthermore, from Theorem 3.2.4, we know that each of our systems has at least one pair each of left and right asymptotic orbits, so each of our systems must have exactly two left asymptotic orbits and exactly two right asymptotic orbits. As shown in [1], left asymptotic orbits can arise in only one of two ways. It turns out in our systems, the left asymptotic orbits in \((X, T)\) are the
orbits of

$$\alpha = \ldots \sigma^2(u)\sigma(u)u.ax\sigma(x)\sigma^2(x)\ldots$$
and

$$A = \ldots \sigma^2(u)\sigma(u)u.bb\sigma(b)\sigma^2(b)\ldots$$

where $u = aabbabba$ and $x = bbabb$. The left asymptotic orbits in $(Y, S)$ are the orbits of

$$\beta = \ldots \tau^2(v)\tau(v)v.\tau(z)\tau^2(z)\ldots$$
and

$$B = \ldots \tau^2(v)\tau(v)v.b\tau(w)\tau^2(w)\ldots$$

where $v = ababab$, $z = babbabb$, and $w = abb$.

To see that these are allowable sequences in the systems, notice that for all $n \in \mathbb{N}$,

$$\sigma^n(u) \ldots \sigma^2(u)\sigma(u)u.ax\sigma(x)\sigma^2(x)\ldots \sigma^n(x) = \sigma^{n+1}(a),$$

$$\sigma^n(u) \ldots \sigma^2(u)\sigma(u)u.bb\sigma(b)\sigma^2(b)\ldots \sigma^n(x) = \sigma^{n+1}(b),$$

$$\tau^n(v) \ldots \tau^2(v)\tau(v)v.a\tau(z)\tau^2(z)\ldots \tau^n(z) = \tau^{n+1}(a),$$
and

$$\tau^n(v) \ldots \tau^2(v)\tau(v)v.b\tau(w)\tau^2(w)\ldots \tau^n(w) = \tau^{n+1}(b).$$

So $\alpha$ and $A$ are allowable in $(X, T)$, and $\beta$ and $B$ are allowable sequences in $(Y, S)$. The representations of these points in the Bratteli diagrams are shown in Figure 3.3.

To see that $\alpha$ and $A$ correspond to the paths as shown in Figure 3.3, we first introduce some notation. If $x = (x_1, x_2, \ldots)$ is an infinite path in a Bratteli diagram and $k < l$, let $x|k, l]$ denote the path $(x_{k+1}, \ldots, x_l)$, i.e. the edge path that $x$ follows from level $k$ to level $l$. Also, we will denote the vertices in the Bratteli diagram for $(X, T)$ in the following way: $L_k$ and $R_k$ will represent the vertices on the left and right side, respectively, at level $k$ of the diagram. Furthermore, $P(v)$ will represent the set of paths whose range is $v$ and whose source is $v_0$, i.e. the set of paths that
Figure 3.3: Left asymptotic points shown in bold

start from the top vertex and terminate at $v$. Given a path in $P(v)$, if it is the $n$th path in the ordering, we will refer to $n$ as its order index in $P(v)$.

By the characterization of $\alpha$ above, for all $k \geq 1$, $\alpha$ passes through $L_{2k+1}$ and the order index of $\alpha[0,2k+1]$ in $P(L_{2k+1})$ is $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$. The path of order index $|u| + 1$ in $P(L_3)$ is the $\alpha[0,3]$ path shown in Figure 3.3, and in general for all $k \geq 1$, the path of order index $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$ in $P(L_{2k+1})$ is the $\alpha[0,2k+1]$ path shown in Figure 3.3. Therefore, the representation of $\alpha$ in the Bratteli diagram is as shown in Figure 3.3. By the characterization of $A$ above, for all $k \geq 1$, $A$ passes through $R_{2k+1}$ and the order index of $A[0,2k+1]$ in $P(R_{2k+1})$ is $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$ which corresponds to the $A[0,2k+1]$ path as shown in Figure 3.3. So, $A$ corresponds to the path shown in Figure 3.3. Similarly, we can conclude that $\beta$ and $B$ also coincide with the paths shown in Figure 3.3.

Now, suppose there is a conjugacy $h$ between $(X,T)$ and $(Y,S)$. The conjugacy must map left (right) asymptotic orbits to left (right) asymptotic orbits. To see this, note that if $[x]$ and $[x']$ are left asymptotic orbits in $X$, for each point $y \in [x]$,
there is unique point \( y' \in [x'] \) such that \( \lim_{k \to \infty} d_X(T^{-k}y, T^{-k}y') = 0 \). Since \( h \) is uniformly continuous, we must have that \( \lim_{k \to \infty} d_Y(h(T^{-k}y), h(T^{-k}y')) = 0 \). Then because \( h \) is a conjugacy, \( h(T^{-k}y) = S^{-k}(h(y)) \) and \( h(T^{-k}y') = S^{-k}(h(y')) \) showing that the orbits of \( h(y) \) and \( h(y') \) are left asymptotic and \( h(y') \) is the unique point in \( Y \) such that \( \lim_{k \to \infty} d_Y(S^{-k}(h(y)), S^{-k}(h(y'))) = 0 \). Therefore, if \( h \) is a conjugacy, it must map \( \alpha \) into the orbit of \( \beta \) and \( A \) into the orbit of \( B \) or vice versa. Since a conjugacy can always be modified to map a point to anything in the orbit of its image, without loss of generality, we can assume that \( h \) maps \( \alpha \) to either \( \beta \) or \( B \). Then, since \( A \) is the unique point in \( X \) such that \( \lim_{k \to \infty} d_X(T^{-k}(\alpha), T^{-k}(A)) = 0 \) and \( B \) is the unique point in \( Y \) such that \( \lim_{k \to \infty} d_Y(S^{-k}(\beta), S^{-k}(B)) = 0 \), it must be true that if \( h(\alpha) = \beta \), then \( h(A) = B \). Similarly if \( h(\alpha) = B \), then \( h(A) = \beta \).

If we can show that neither of these cases are possible, we can conclude that these systems are not conjugate.

Consider the sequence \( \{A_k\} \) in \( X \) where \( A_k \) is the path in the diagram in Figure 3.3 that agrees with \( A \) until level \( k \), crosses over to \( L_{k+1} \) on the order 4 path and agrees with \( \alpha \) past level \( k+1 \). Figure 3.4 illustrates \( A_k \) for an even value of \( k \). Since each \( A_k \) is cofinal with \( \alpha \), for each \( k \) there is an \( n_k \) such that \( T^{n_k}(\alpha) = A_k \). If \( h \) is a conjugacy \( h \) between \( (X,T) \) and \( (Y,S) \), the following must hold:

\[
h(A) = h(\lim_{k \to \infty} T^{n_k}(\alpha)) = \lim_{k \to \infty} h(T^{n_k}(\alpha)) = \lim_{k \to \infty} S^{n_k}(h(\alpha)).
\]

Since we are assuming \( h(A) \) must be either \( \beta \) or \( B \) and \( h(\alpha) \) is the other, then either

\[
\lim_{k \to \infty} S^{n_k}(\beta) = B \quad \text{or} \quad \quad \quad (3.2.1)
\]

\[
\lim_{k \to \infty} S^{n_k}(B) = \beta, \quad \quad \quad (3.2.2)
\]

and if neither equation 3.2.1 nor 3.2.2 holds, \( h \) cannot be a conjugacy.
Proposition 3.2.7. The number $n_k$ such that $T^{n_k}(\alpha) = A_k$ is given by

$$n_k = \begin{cases} 
|P(L_k)| + |P(R_k)| & \text{if } k \text{ is odd} \\
|P(L_k)| & \text{if } k \text{ is even} \end{cases}$$

Proof. We let $\Delta_k$ denote the order index of $\alpha[0, k]$ in $P(L_k)$ and $\Gamma_k$ denote the order index of $A_k[0, k+1]$ in $P(L_{k+1})$. Note that $\Delta_k$ is also the order index of $A[0, k]$ in $P(R_k)$. We have the following:

$$\Delta_1 = 1 \text{ and } \forall k \geq 1, \Delta_{k+1} = \begin{cases} 
|P(L_k)| + \Delta_k & \text{if } k \text{ is odd} \\
|P(L_k)| + |P(R_k)| + \Delta_k & \text{if } k \text{ is even} \end{cases}$$

$$\forall k > 1, \Gamma_k = 2|P(L_k)| + |P(R_k)| + \Delta_k.$$
Since both $\alpha$ and $A_k$ pass through $L_{k+1}$ and they agree past level $k+1$, $n_k$ is given by the difference in the order indices of $A_k[0, k+1]$ and $\alpha[0, k+1]$. So, $n_k = \Gamma_k - \Delta_{k+1}$ proving the proposition.

**Proposition 3.2.8.** For odd values of $k$, $\lim_{k \to \infty} S^{n_k}(\beta) = \beta$ and $\lim_{k \to \infty} S^{n_k}(B) = B$.

Before we begin the proof, we introduce some notation. Denote the left and right vertices at level $k$ of $(Y, S)$, respectively, as $L'_k$ and $R'_k$. Let $\Delta'_k$ denote the order index of $\beta[0, k]$ in $P(L'_k)$ and $\Gamma'_k$ the order index of $B[0, k]$ in $P(R'_k)$. For all $k$, note that the recursion $|P(L'_{k+1})| = 2|P(L'_k)| + 2|P(R'_k)|$ is satisfied.

Now, we need a way to identify paths and edges in the diagram. We will denote the maximal path from the top of the diagram to vertex $v$ by $M(v)$ and the minimal path by $m(v)$. Also, we will denote the edge of order index $j$ that terminates at vertex $v$ by $j_v$. We also need to identify compositions of paths in the diagram, so for example, in our notation $M(R'_k)3_{L'_{k+1}}\beta[k+1, k+3]$ represents the path that is maximal down to $R'_k$, takes the order 3 path to $L'_{k+1}$, and follows $\beta$ from level $k+1$ to $k+3$.

**Proof of Proposition 3.2.8.** Consider $S^{n_k}(\beta) = S^{|P(L'_k)| + |P(R'_k)|}(\beta)$ for a fixed odd value of $k$. We determine what this is by comparing order indices of paths in $P(L'_{k+2})$. We would like to know the path whose order index in $P(L'_{k+2})$ is greater than the order index of $\beta[0, k+2]$ by $|P(L'_k)| + |P(R'_k)|$. We do the computation in a series of steps which can easily be checked.

1. The path $M(L'_k)\beta[k, k+2] > \beta[0, k+2]$ and the difference in the order indices is $|P(L'_k)| - \Delta'_k$.

2. The path $m(R'_k)A_{L'_{k+1}}\beta[k+1, k+2] > M(L'_k)\beta[k, k+2]$ and the difference in order indices is 1.

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(3) The path $M(R_k')A_{L_{k+1}'}\beta[k + 1, k + 2] > m(R_k')A_{L_{k+1}'}\beta[k + 1, k + 2]$ and the difference in order indices is $|P(R_k')| - 1$.

(4) The path $m(R_{k+1}')3L_{k+2}' = m(L_k')1R_{k+1}'3L_{k+2}' > M(R_k')A_{L_{k+1}'}\beta[k + 1, k + 2]$ and the difference in order indices is 1.

(5) The path $\beta[0, k]1R_{k+1}'3L_{k+2}' > M(R_k')A_{L_{k+1}'}\beta[k + 1, k + 2]$ and the difference in order indices is $\Delta_k' - 1$.

The difference in order indices applied above add to $|P(L_k')| + |P(R_k')|$, and the last path in our computation begins with $\beta[0, k]$, so $S^m(\beta)$ agrees with $\beta$ down to level $k$ showing $S^m(\beta) \rightarrow \beta$ for odd values of $k$.

We now consider $S^m(B) = S^{P(L_k')+P(R_k')}(B)$ for an odd value of $k$. We calculate this by comparing order indices of paths in $P(R_{k+3}')$. We would like to know the path whose order index in $P(R_{k+3}')$ is greater than the order index of $B[0, k + 3]$ by $|P(L_k')| + |P(R_k')|$. Again, we compute this is in a series of steps which can easily be checked.

1. The path $B[0, k]3R_{k+1}'B[k + 1, k + 3] > B[0, k + 3]$ and the difference in order indices is $|P(R_k')|$.

2. The path $M(R_{k+2}')B[k + 2, k + 3] > B[0, k]3R_{k+1}'B[k + 1, k + 3]$ and the difference in order indices is $|P(R_{k-1}')| - \Gamma_{k-1}'$.

3. The path $m(R_{k+2}')3R_{k+3}' = m(L_k')1L_{k+1}'1R_{k+2}'3R_{k+3}' > M(R_{k+2}')B[k + 2, k + 3]$ and the difference in order indices is 1.

4. The path $m(R_{k-1}')4L_k'1L_{k+1}'1R_{k+2}'3R_{k+3}' > m(L_k')1L_{k+1}'1R_{k+2}'3R_{k+3}'$ and the difference in order indices is $2|P(L_{k-1}')| + |P(R_{k-1}')|$.

5. The path $B[0, k - 1]4L_k'1L_{k+1}'1R_{k+2}'3R_{k+3}' > m(R_{k-1}')4L_k'1L_{k+1}'1R_{k+2}'3R_{k+3}'$ and the difference in order indices is $\Gamma_{k-1}' - 1$. 

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Using the recursion formula from Proposition 3.2.7, we get that the sum of the differences in order indices above is $|P(L'_k)| + |P(R'_k)|$. The last path in our computation begins with $B[0, k - 1]$, so $S^{n_k}(B)$ agrees with $B$ down to level $k - 1$ finishing the proof.

**Proof of Theorem 3.2.1.** By Proposition 3.2.8, neither Equation 3.2.1 nor 3.2.2 can hold. This along with Propositions 3.2.5 and 3.2.6 proves the theorem.

The system $(Y, S^{-1})$ can be represented with the same Bratteli diagram as $(Y, S)$ by only reversing the ordering on the edges. With this representation of $(Y, S^{-1})$, using similar techniques to those used in Proposition 3.2.8, it can also be shown that $(X, T)$ is not conjugate to $(Y, S^{-1})$. This statement along with Theorem 3.2.1 shows that $(X, T)$ and $(Y, S)$ are not *flip conjugate*, i.e $(X, T)$ is not conjugate to $(Y, S)$ or $(Y, S^{-1})$. 

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Chapter 4

Residuality in Strong Orbit
Equivalence Classes

In this chapter, we will define a class of minimal Cantor systems that up to conjugacy contains every system strongly orbit equivalent to a given system. We will then define a metric on this strong orbit equivalence class and prove several properties about the metric space. In particular, we will prove some results about residuality in this metric space.

4.1 Definition of $S(T, x_0)$

If $(X, T)$ is a minimal Cantor system, we define the future orbit of $x$ under $T$, $O_T^+(x) = \{T^k(x) \mid k \geq 0\}$ and the past orbit of $x$ under $T$, $O_T^-(x) = \{T^{-k}(x) \mid k > 0\}$. It is easily verified that for all $x \in X$, both sets $O_T^+(x)$ and $O_T^-(x)$ are dense in $X$. If $(X, T)$ and $(Y, S)$ are strongly orbit equivalent minimal Cantor systems with $x_0 \in X$ and $y_0 \in Y$, we will say that $h : X \to Y$ is a pointed strong orbit equivalence between $(X, T, x_0)$ and $(Y, S, y_0)$ if it is a strong orbit equivalence satisfying the following conditions:
(1) $h(x_0) = y_0$;

(2) $h(Tx_0) = Sy_0$;

(3) the cocycles of $h$ are continuous on $X \setminus \{x_0\}$;

(4) $h(O_T(x_0)) = O_S(y_0)$;

(5) $h(O_T^{+}(x_0)) = O_S^{+}(y_0)$.

**Proposition 4.1.1.** Let $(X, T)$ and $(Y, S)$ be strongly orbit equivalent minimal Cantor systems. For any points $x_0 \in X$ and $y_0 \in Y$, there exists a pointed strong orbit equivalence between $(X, T, x_0)$ and $(Y, S, y_0)$.

**Proof.** This is a consequence of results from [7]. Theorem 3.6 of [7] states that any minimal Cantor system $(X, T)$ with $x_0 \in X$ can be represented as a Bratteli-Vershik system with $x_0$ being the unique maximal path of the associated ordered Bratteli diagram. In the proof of Theorem 1.0.1, given two strongly orbit equivalent Bratteli-Vershik systems, Giordano, Putnam, and Skau construct a strong orbit equivalence between the systems that preserves the minimal and maximal paths and preserves the cofinality of paths. Moreover, they show that the cocycles of this strong orbit equivalence can be discontinuous only at the maximal path.

So given two strongly orbit equivalent minimal Cantor systems $(X, T)$ and $(Y, S)$, we can find a Bratteli-Vershik representation of $(X, T)$ with maximal path $x_0$ and a representation of $(Y, S)$ with maximal path $y_0$. By the proof of Theorem 1.0.1, we can find a strong orbit equivalence $h : X \to Y$ that preserves the minimal and maximal paths, preserves cofinality, and such that the cocycles of $h$ are discontinuous only at $x_0$. Since $x_0$ and $y_0$ are the maximal paths in the diagrams, the points $Tx_0$ and $Sy_0$ are the minimal paths. Therefore, $h$ satisfies properties (1) and (2). Since the cocycles of $h$ are discontinuous only at the maximal path, property (3) is satisfied. The points in $X$ that are cofinal with $x_0$ other than itself are exactly
$O_T^-(x_0)$ and the points cofinal with $Tx_0$ are exactly $O_T^+(x_0) \setminus \{x_0\}$, and the analogous statement is true for $(Y,S)$ with $y_0$ and $Sy_0$. This along with the fact that $h$ preserves the cofinality of paths guarantees that properties (4) and (5) are satisfied.

Let $(X,T)$ and $(Y,S)$ be strongly orbit equivalent minimal Cantor systems and let $h$ be a pointed strong orbit equivalence between $(X,T,x_0)$ and $(Y,S,y_0)$. If we let $S' = h^{-1} \circ S \circ h$, $(X,S')$ is a minimal Cantor system conjugate to $(Y,S)$. It can easily be checked that the identity map on $X$ is a strong orbit equivalence between $(X,S')$ and $(X,T)$. Furthermore, $S'$ satisfies the following properties:

1. $S'(x_0) = T(x_0)$;
2. $O_{S'}^-(x_0) = O_T^-(x_0)$;
3. $O_{S'}^+(x_0) = O_T^+(x_0)$;
4. the cocycles associated to the identity map are continuous on $X \setminus \{x_0\}$.

We will say that a minimal homeomorphism of $X$ satisfying these four properties is $x_0$-id strongly orbit equivalent to $T$. We define $S(T,x_0) = \{ P : X \to X \mid P$ is $x_0$-id strongly orbit equivalent to $T \}$. The cocycle property (property (2)) can be stated more explicitly in the following terms. If $P \in S(T,x_0)$, there exists functions $a,b : X \to \mathbb{Z}$ continuous on $X \setminus \{x_0\}$ such that for all $x \in X$, $Tx = P^a(x)$ and $Px = T^b(x)$. Since $a$ and $b$ depend only on $P$ and $T$, we will refer to the them as the cocycles of $P$ relative to $T$ or just the cocycles of $P$ if $T$ is clear by the context. By the preceding arguments, any minimal Cantor system strongly orbit equivalent to $(X,T)$ is conjugate to $(X,P)$ for some $P \in S(T,x_0)$.

Let $(X,T)$ be a minimal Cantor system with $x_0 \in X$. We will now define a metric $m_T$ on $S(T,x_0)$. For $S \in S(T,x_0)$ with cocycles $a$ and $b$ and $S' \in S(T,x_0)$ with cocyles $a'$ and $b'$, we define
\[ m_T(S, S') = \tilde{m}_T(S, S') + \sup_{x \in X} d_X(Sx, S'x) \]

where
\[
\tilde{m}_T(S, S') = \inf_{\epsilon > 0} \{ a(x) = a'(x) \text{ and } b(x) = b'(x) \text{ for all } x \in X \setminus B(x_0, \epsilon) \}.
\]

The second term in the sum that defines \( m_T(S, S') \) is the supremum metric. Because the sum of two metrics defines another metric, in order to show that \( m_T \) is a metric on \( S(T, x_0) \), it is sufficient to show that \( \tilde{m}_T \) is a metric on \( S(T, x_0) \). If we can show that \( \tilde{m}_T \) satisfies the triangle inequality, the other metric space properties follow trivially.

For \( S_i \in S(T, x_0), i = 1, 2, 3, \) let \( a_i \) and \( b_i \) be the cocycles of \( S_i \). We will show that \( \tilde{m}_T \) satisfies a stronger form of the triangle inequality, namely \( \tilde{m}_T(S_1, S_3) \leq \max\{ \tilde{m}_T(S_1, S_2), \tilde{m}_T(S_2, S_3) \} \). Assume that \( \tilde{m}_T(S_1, S_3) = p > 0 \) and \( \tilde{m}_T(S_1, S_2) = r < p \). Then, by the definition of \( \tilde{m}_T(S_1, S_3) \), if \( r < q < p \), there exists an \( x_q \in X \) with \( q < d_X(x_0, x_q) \leq p \) such that either \( a_1(x_q) \neq a_3(x_q) \) or \( b_1(x_q) \neq b_3(x_q) \). Since \( \tilde{m}_T(S_1, S_2) = r < q, a_1(x_q) = a_2(x_q) \) and \( b_1(x_q) = b_2(x_q) \). Therefore, either \( a_2(x_q) \neq a_3(x_q) \) or \( b_2(x_q) \neq b_3(x_q) \), and thus \( \tilde{m}_T(S_2, S_3) \geq d_X(x_0, x_q) \geq q \). Because this holds for all \( r < q < p \), we can conclude that \( \tilde{m}_T(S_2, S_3) \geq p \), finishing the proof.

### 4.2 Properties of \( S(T, x_0) \)

Here we establish some properties of \( S(T, x_0) \).

**Proposition 4.2.1.** If \( S \in S(T, x_0) \), then \( T(\mathcal{O}_S^+(x_0)) = \mathcal{O}_S^+(x_0) \setminus \{x_0\} \) and \( T(\mathcal{O}_S^-(x_0)) = \mathcal{O}_S^-(x_0) \cup \{x_0\} \). Furthermore, \( S(\mathcal{O}_T^+(x_0)) = \mathcal{O}_T^+(x_0) \setminus \{x_0\} \) and \( S(\mathcal{O}_T^-(x_0)) = \mathcal{O}_T^-(x_0) \cup \{x_0\} \).
Proof. By the definition of $S(T, x_0)$, if $S \in S(T, x_0)$, then $O_T^{-1}(x_0) \cap \mathcal{O}_S^{-1}(x_0)$. Then we have

$$T(O_S^{-1}(x_0)) = T(O_T^{-1}(x_0)) = O_T^{-1}(x_0) \cup \{x_0\} = O_S^{-1}(x_0) \cup \{x_0\}.$$  

The other statements can be proven by a similar argument. \qed

**Definition 4.2.2.** Let $S \in S(T, x_0)$ and let $C$ be a clopen set in $X$. For $x \in C$, define the set $C_x$ in the following way. If $a(x) < 0$, then $C_x = \{S^{a(x)}(x), \ldots, S^{-1}(x), x\}$; if $a(x) > 0$, then $C_x = \{x, Sx, \ldots, S^{a(x)-1}(x)\}$. We define $C_S = \bigcup_{x \in C} C_x$.

**Proposition 4.2.3.** If $C$ is a clopen set in $X$ with $x_0 \notin C$, then the set $C_S$ defined above is clopen in $X$ and $x_0 \notin C_S$.

Proof. Since $x_0 \notin C$, the function $a|_C : C \to \mathbb{Z}$ is continuous. Then because $C$ is compact, $a|_C$ takes on only finitely many values. Therefore, there exists an integer $M > 0$ such that $a|_C(C) \subset [-M, M]$. For $k \in \mathbb{Z}$, $|k| \leq M$, the set $a_{|C}^{-1}\{k\}$ is clopen in $C$, and because $C$ is clopen in $X$, $a_{|C}^{-1}\{k\}$ is also clopen in $X$. Because $S$ is a homeomorphism, the set $S^j(a_{|C}^{-1}\{k\})$ is clopen in $X$ for all $j \in \mathbb{Z}$. If $0 < k \leq M$, we let $C_k = \bigcup_{j=0}^{k-1} S^j(a_{|C}^{-1}\{k\})$ and if $-M \leq k < 0$, we let $C_k = \bigcup_{j=0}^{|k|} S^{-j}(a_{|C}^{-1}\{k\})$. Each $C_k$ is clopen in $X$, and moreover $C_S = \bigcup_{k=-M}^M C_k$. Since $C_S$ is the finite union of clopen sets, $C_S$ is clopen as claimed.

To show $x_0 \notin C_S$, we will argue by contradiction. Assume $x_0 \in C_S$. Then there exists $x \in C$ such that $x_0 = S^j(x)$ where $0 < j < a(x)$ if $a(x) > 0$ or $0 < j \leq a(x)$ if $a(x) < 0$. If we assume $a(x) > 0$, then $x_0 = S^j(x)$ for $0 < j < a(x)$. Then $x = S^{-j}x_0$ and we have

$$T(S^{-j}x_0) = Tx = S^{a(x)}(x) = S^{a(x)-j}S^j(x) = S^{a(x)-j}(x_0).$$

Since $a(x) - j > 0$, $T$ is mapping a point in $O_S^{-1}(x_0)$ to a point in $O_S^{-1}(x_0) \setminus \{x_0\}$.
contradicting Proposition 4.2.1. If \(a(x) < 0\), then \(x_0 = S^{-j}x\) with \(a(x) \leq -j < 0\), and we have \(S^jx_0 = x\). By an argument similar to the one above, \(T(S^jx_0) = S^{a(x)+j}(x_0)\). Since \(a(x) + j \leq 0\), \(T\) is mapping a point in \(O^+_S(x_0)\) to a point in \(O^+_S(x_0) \cup \{x_0\}\), which again contradicts Proposition 4.2.1. This proves \(x_0 \notin C_S\). □

**Proposition 4.2.4.** Suppose \(S \in \mathcal{S}(T,x_0)\) with cocycles \(a\) and \(b\) and \(C\) is a clopen set in \(X\) with \(x_0 \notin C\). If \(S' \in \mathcal{S}(T,x_0)\) with cocycles \(a'\) and \(b'\) such that \(Sx = S'x\) for all \(x \in C_S\), then \(a(x) = a'(x)\) and \(b(x) = b'(x)\) for all \(x \in C\).

**Proof.** Since \(C \subset C_S\), we have that \(Sx = S'x\) for all \(x \in C\). Then because \(Sx = T^b(x)\) and \(S'x = T^{b'}(x)\) for all \(x \in X\), \(b(x) = b'(x)\) for all \(x \in C\). Fix \(x \in C\). If \(a(x) > 0\), then \(S\) and \(S'\) agree on the set \(\{x,Sx,...,S^{a(x)-1}(x)\}\). In particular, \(S^{a(x)}(x) = S^{a(x)}(x) = Tx\), so \(a'(x) = a(x)\). If \(a(x) < 0\), then \(S\) and \(S'\) agree on the set \(\{S^{a(x)}(x),...,S^{-1}(x),x\}\). Since \(S^{a(x)}(x) = Tx\), we have \(x = S^{a(x)}(Tx) = S^{ia(x)}(Tx)\). So \(S^{a(x)}(x) = S^{a(x)}(x) = Tx\), which again shows that \(a'(x) = a(x)\) finishing the proof. □

**Proposition 4.2.5.** If \(S \in \mathcal{S}(T,x_0)\), then \(S(T,x_0) = S(S,x_0)\).

**Proof.** Let \(a\) and \(b\) be the cocycles of \(S\) relative to \(T\) and suppose \(P \in \mathcal{S}(T,x_0)\) with cocycles \(a'\) and \(b'\) relative to \(T\). It is easily seen that \(P\) satisfies properties (1)-(3) of \(S(S,x_0)\) as \(P_{x_0} = Tx_0 = Sx_0\), \(\mathcal{O}_P(x_0) = \mathcal{O}_{T^{-1}}(x_0) = \mathcal{O}_S^+(x_0)\), and \(\mathcal{O}_P^+(x_0) = \mathcal{O}_T^+(x_0)\). We will now show that \(P\) satisfies property (4).

Let \(x \in X\) with \(x \neq x_0\). If we assume \(b(x) = k > 0\), then we have the following:

\[
Sx = T^k(x)
= T(T^{k-1}(x))
= P^{\alpha(T^{k-1}(x))}(T^{k-1}(x))
\]

If we repeat this process until we get \(x\) as the argument on the right hand side, we
get that $Sx = P^{p(x)}(x)$ where

$$p(x) = \sum_{j=0}^{k-1} a'(T^j x).$$

An argument similar to that in the proof of Proposition 4.2.3 shows that $x_0 \neq T^j x$ for $j = 0, \ldots, k - 1$. Therefore, $p$ is continuous on $X \setminus \{x_0\}$. If $b(x) < 0$, the proof is done similarly. If $b'(x) = k > 0$, we have that $P x = S^{q(x)}(x)$ where

$$q(x) = \sum_{j=0}^{k-1} a(T^j x).$$

As stated above, we have that $x_0 \neq T^j x$ for $j = 0, \ldots, k - 1$, so $q$ is continuous on $X \setminus \{x_0\}$. The proof is done similarly if $b'(x) < 0$. The preceding arguments have shown that the cocyles of $P$ relative to $S$ are the functions $p$ and $q$. Since $p$ and $q$ are continuous on $X \setminus \{x_0\}$, $P$ satisfies property (4) of $S(S, x_0)$. This establishes that $S(T, x_0) \subset S(S, x_0)$. By symmetry, $S(T, x_0) = S(S, x_0)$.

**Theorem 4.2.6.** Suppose $(X, T)$ and $(Y, S)$ are strongly orbit equivalent minimal Cantor systems with $x_0 \in X$ and $y_0 \in Y$. Then $(S(T, x_0), m_T)$ and $(S(S, y_0), m_S)$ are uniformly homeomorphic metric spaces.

**Proof.** By Proposition 4.1.1, there exists a pointed strong orbit equivalence $h$ between $(X, T, x_0)$ and $(Y, S, y_0)$. Define the function $f : S(T, x_0) \rightarrow S(S, y_0)$ by $f(P) = h \circ P \circ h^{-1}$. Throughout this proof, we will use $P'$ to denote $f(P) = h \circ P \circ h^{-1}$.

We will first show that $T' \in S(S, y_0)$. Clearly $T' : Y \rightarrow Y$ is a minimal homeomorphism and

$$T'(y_0) = h \circ T \circ h^{-1}(h(x_0)) = h \circ T(x_0) = S y_0.$$
Furthermore, we have that

\[ O^+_T(y_0) = \{(h \circ T \circ h^{-1})^k(y_0) \mid k \geq 0\} \]
\[ = \{(h \circ T^k \circ h^{-1})(h(x_0)) \mid k \geq 0\} \]
\[ = \{(h \circ T^k)(x_0) \mid k \geq 0\} \]
\[ = h(O^+_T(x_0)) \]
\[ = O^+_S(y_0). \]

With a similar calculation, we can show that \( O^-_T(y_0) = O^-_S(y_0) \). It remains to be shown that \( T' \) satisfies property (4) of \( S(S, y_0) \).

Let \( m \) and \( n \) be the cocycles of \( h \), so for all \( x \in X \),

\[ h \circ T(x) = S^m(x) \circ h(x) \text{ and } h \circ T^n(x) = S \circ h(x), \]

and \( m \) and \( n \) are continuous on \( X \setminus \{x_0\} \). Then for \( y \in Y \), we have

\[ (T')^n(h^{-1}(y))(y) = (h \circ T \circ h^{-1})^n(h^{-1}(y))(y) \]
\[ = h \circ T^n(h^{-1}(y))(h^{-1}(y)) \]
\[ = S \circ h(h^{-1}(y)) \]
\[ = Sy \]

and

\[ S^m(h^{-1}(y))(y) = S^m(h^{-1}(y))(h(h^{-1}(y))) \]
\[ = h \circ T(h^{-1}(y)) \]
\[ = T'y. \]
This shows that the cocycles of $T'$ relative to $S$ are the functions $m \circ h^{-1}$ and $n \circ h^{-1}$. These functions are continuous as long as $h^{-1}(y) \neq x_0$, i.e. if $y \neq h(x_0) = y_0$. Therefore the cocycles of $T'$ relative to $S$ are continuous on $Y \setminus \{y_0\}$. This establishes that $T' \in \mathcal{S}(S, y_0)$. By Proposition 4.2.5, we have that $\mathcal{S}(T', y_0) = \mathcal{S}(S, y_0)$. We will now show that if $P \in \mathcal{S}(T, x_0)$, then $P' \in \mathcal{S}(T', y_0)$.

If $P \in \mathcal{S}(T, x_0)$ with cocycles $a$ and $b$, then for $y \in Y$,

\[(P')^a(h^{-1}(y))(y) = h \circ P^a(h^{-1}(y))(h^{-1}(y))\]
\[= h \circ T \circ h^{-1}(y)\]
\[= T'y\]

and

\[(T')^b(h^{-1}(y))(y) = h \circ T^b(h^{-1}(y))(h^{-1}(y))\]
\[= h \circ R \circ h^{-1}(y)\]
\[= P'y.\]

This shows that the cocycles of $P'$ relative to $T'$ are the functions $a \circ h^{-1}$ and $b \circ h^{-1}$. These functions are continuous on $Y \setminus \{y_0\}$, so by an argument similar to the one above, $P' \in \mathcal{S}(T', y_0) = \mathcal{S}(S, y_0)$. We have established that $f$ is a well-defined map from $\mathcal{S}(T, x_0)$ to $\mathcal{S}(S, y_0)$.

We have left to show that $f$ is a uniformly continuous homeomorphism. First, $f$ is clearly invertible as $f^{-1} : \mathcal{S}(S, y_0) \to \mathcal{S}(T, x_0)$ is defined by $f^{-1}(Q) = h^{-1} \circ Q \circ h$. Moreover, $h^{-1}$ is a pointed strong orbit equivalence between $(Y, S, y_0)$ and $(X, T, x_0)$, so if we show that $f$ is uniformly continuous, by the same argument we will have that $f^{-1}$ is uniformly continuous. We will now show that $f$ is a uniformly continuous function.
Fix $\epsilon > 0$. Because $h$ is uniformly continuous on $X$, there exists a $\delta > 0$ such that if $x, x' \in X$ with $d_X(x, x') < \delta$, then $d_Y(h(x), h(x')) < \epsilon$. Pick $P, R \in S(T, x_0)$ with $\sup_{x \in X}(Px, Rx) \leq m_T(P, R) < \delta$. Then we have

$$\sup_{y \in Y} d_Y(P'y, R'y) = \sup_{y \in Y} d_Y(h \circ P \circ h^{-1}(y), h \circ R \circ h^{-1}(y))$$
$$= \sup_{x \in X} d_Y(h(P(x)), h(R(x)))$$
$$< \epsilon.$$ 

We only have left to show that by making $m_T(P, R)$ small enough, we can make the cocycles of $P'$ and $R'$ agree everywhere on $Y$ except in an $\epsilon$-ball around $y_0$. Since $P, R \in S(T, x_0)$, for all $x \in X$,

$$Tx = P^{a(x)}(x) & Px = T^{b(x)}(x) \text{ and } Tx = R^{c(x)}(x) & Rx = T^{d(x)}(x)$$

where $a, b, c,$ and $d$ are each continuous functions on $X \setminus \{x_0\}$. Since $P', R' \in S(S, y_0)$, for all $y \in Y$,

$$Sy = (P')^{a'(y)}(y) & P'y = S^{b'(y)}(y) \text{ and } Sy = (R')^{c'(y)}(y) & R'y = S^{d'(y)}(y)$$

where $a', b', c',$ and $d'$ are each continuous functions on $Y \setminus \{y_0\}$.

Fix $\epsilon > 0$ and let $C$ be a clopen set containing $Y \setminus B(y_0, \epsilon)$ with $y_0 \notin C$. Since $T' \in S(S, y_0)$, we define the set $C_{T'}$ analogously as done in Definition 4.2.2. By Proposition 4.2.3, $C_{T'}$ is clopen in $Y$ with $y_0 \notin C_{T'}$, so there exists a $\delta' > 0$ such that $B(y_0, \delta') \subset Y \setminus C_{T'}$. Since $h$ is uniformly continuous on $X$, we can find a $\delta > 0$ such that if $x, x' \in X$ with $d_X(x, x') < \delta$, then $d_Y(h(x), h(x')) < \delta'$. Now, suppose $m_T(P, R) < \delta$. Fix $y \in Y \setminus B(y_0, \epsilon)$, so $y \in C_{T'}$ and thus $y \notin B(y_0, \delta')$. Suppose $h^{-1}(y) \in B(x_0, \delta)$. Then $d(y, h(x_0)) < \delta'$, but $h(x_0) = y_0$, so $y \in B(y_0, \delta')$ which
is a contradiction. Therefore, \( h^{-1}(y) \not\in B(x_0, \delta) \). Since \( m_T(P, R) < \delta \), \( b(h^{-1}(y)) = d(h^{-1}(y)) \) and so \( P(h^{-1}(y)) = R(h^{-1}(y)) \). From this, we can conclude \( P'y = R'y \) and thus \( b'(y) = d'(y) \) for all \( y \in y \not\in B(y_0, \epsilon) \). We will now show that the same is true for \( a' \) and \( c' \). Fix \( y \in Y \not\in B(y_0, \epsilon) \), and suppose \( Sy = T^k y, k > 0 \). Using that fact shown above that the cocycles of \( P' \) relative to \( T' \) are \( a \circ h^{-1} \) and \( b \circ h^{-1}, \) we have

\[
Sy = (T')^k(y)
\]

\[
= T'((T')^{k-1}(y))
\]

\[
= (P')^{a \circ h^{-1}((T')^{k-1}(y))}((T')^{k-1}(y))
\]

Repeating this procedure \( k \) times, we get

\[
a'(y) = \sum_{j=0}^{k-1} a(h^{-1}((T')^j(y))).
\]

Similarly we get that

\[
c'(y) = \sum_{j=0}^{k-1} c(h^{-1}((T')^j(y))).
\]

But for each \( j = 0, \ldots, k - 1 \), \( (T')^j(y) \in C_{T'} \), so \( h^{-1}((T')^j(y)) \not\in B(x_0, \delta) \). Since \( m_T(P, R) < \delta \), \( a(h^{-1}((T')^j(y))) = c(h^{-1}((T')^j(y))) \) for \( j = 0, \ldots, k - 1 \), so \( a'(y) = c'(y) \).

Now suppose \( Sy = (T')^{-k}y, k > 0 \). Then, we have

\[
y = (T')^k(Sy)
\]

\[
= T'((T')^{k-1}(Sy))
\]

\[
= (P')^{a \circ h^{-1}((T')^{k-1}(Sy))}((T)^{k-1}(Sy))
\]

\[
= (P')^{a \circ h^{-1}((T')^{-1}(y))}((T')^{k-1}(Sy)).
\]
Repeating this procedure \( k \) times, we get that \( y = (P')^{q(y)}(Sy) \) where

\[
q(y) = \sum_{j=1}^{k} a \circ h^{-1}((T')^{-j}(y)).
\]

Therefore,

\[
a'(y) = -q(y) = -\sum_{j=1}^{k} a \circ h^{-1}((T')^{-j}(y)).
\]

Similarly we get that

\[
c'(y) = -\sum_{j=1}^{k} c \circ h^{-1}((T')^{-j}(y)).
\]

Again, for each \( j = 1, \ldots, k - 1 \), \((T')^{-j}(y) \in C_{T'}\), so by the same argument as above we have that \( a \circ h^{-1}((T')^{-j}(y)) = c \circ h^{-1}((T')^{-j}(y)) \) for each \( j = 1, \ldots, k \). This establishes \( a'(y) = c'(y) \) for all \( y \in Y \setminus B(y_0, \epsilon) \). In both of the preceding arguments, the choice of \( \delta \) was independent of \( P \) and \( R \), so we can conclude that \( f \) is uniformly continuous.

\[\square\]

**Corollary 4.2.7.** For \( S \in \mathcal{S}(T, x_0) \), the identity map from \( \mathcal{S}(T, x_0) \to \mathcal{S}(S, x_0) \) is a uniformly continuous homeomorphism.

*Proof.* It is easily verified that the identity map on \( X \) is a pointed strong orbit equivalence between \((X, T, x_0)\) to \((X, S, x_0)\). Then by Proposition 4.2.5, the identity map from \((\mathcal{S}(T, x_0), m_T)\) to \((\mathcal{S}(S, x_0), m_S)\) is a bijection. By Theorem 4.2.6, the identity map is a uniformly continuous homeomorphism. \(\square\)

Theorem 4.2.6 shows that the resulting metric space is independent of map chosen from the strong orbit equivalence class and independent of the point chosen from the space. From this point forward we will only consider one Cantor space \( X \) and one special point \( x_0 \in X \), and we will let \( \mathcal{S}(T) \) denote \( \mathcal{S}(T, x_0) \). We will now establish some properties of \( (\mathcal{S}(T), m_T) \).
Proposition 4.2.8. $(\mathcal{S}(T), m_T)$ is a complete metric space.

Proof. Let $\{S_n\}$ be an $m_T$-Cauchy sequence in $\mathcal{S}(T)$. For all $n$, let $a_n$ and $b_n$ be the cocycles of $S_n$. For all $x \in X$, each of the sequences $\{S_n(x)\}$, $\{a_n(x)\}$, and $\{b_n(x)\}$ are eventually fixed. This holds for $x \in X \setminus \{x_0\}$ because there exists an $N > 0$ such that if $n, m \geq N$, then $a_n(x) = a_m(x)$ and $b_n(x) = b_m(x)$. Since $b_n(x) = b_m(x)$, for all $n, m \geq N$, this also means $S_n(x) = S_m(x)$ for all $n, m \geq N$. Furthermore, $S_n(x_0) = Tx_0$ for all $n$, so $a_n(x_0) = 1 = b_n(x_0)$ for all $n$. This argument can be generalized to show that for any $j \in \mathbb{Z}$ and $x \in X$, the sequence $\{S^j_n(x)\}$ is eventually fixed. So we can define $Sx = \lim_{n \to \infty} S_n(x)$, $a(x) = \lim_{n \to \infty} a_n(x)$, and $b(x) = \lim_{n \to \infty} b_n(x)$ for all $x \in X$. We will show that $S \in \mathcal{S}(T)$ with cocycles $a$ and $b$ and $\{S_n\}$ is $m_T$-convergent to $S$ proving the proposition.

We begin by showing that $S$ is a homeomorphism. Because for every $x \in X$, the sequence $\{S_n(x)\}$ is eventually fixed, $S$ must be one-to-one and onto since each $S_n$ is one-to-one and onto. Since $\sup_{x \in X} d_X(S_n(x), S_m(x)) \to 0$ and $\{S_n\}$ converges pointwise to $S$, by the Cauchy criterion for uniform convergence $\{S_n\}$ converges uniformly to $S$. Since $S$ is the uniform limit of continuous functions, $S$ is continuous. Furthermore, it is a well known theorem that a continuous bijection between compact metric spaces has a continuous inverse.

We will now show that $S$ satisfies the properties of $\mathcal{S}(T)$. It is easily seen that $S$ satisfies property (1) of $\mathcal{S}(T)$ because for all $n \in \mathbb{N}^+$, $S_n(x_0) = Tx_0$ and thus $Sx_0 = Tx_0$. We will now show that the cocycles of $S$ are the functions $a$ and $b$, and they satisfy property (4) of $\mathcal{S}(T)$. Fix $x \in X$. By the argument above, there exists an $N > 0$ such that if $n \geq N$, $b_n(x) = b(x)$. This also means for $n \geq N$, $S_n(x) = S(x)$. So for $n \geq N$,

$$Sx = S_n(x) = T^{b_n(x)}(x) = T^b(x)(x).$$
To see that $b$ is continuous on $X \setminus \{x_0\}$, we fix $x \neq x_0$ and find a clopen neighbourhood $D$ of $x$ with $x_0 \notin D$. If $N$ is chosen large enough such that for $n \geq N$, $b$ and $b_n$ agree on $D$, since $b_n$ is continuous on $D$, $b$ is also continuous on $D$. Because $x \in D$, $b$ is continuous at $x$.

We will now show that $a$ satisfies the desired properties. Fix $x \in X$ and suppose $a(x) > 0$. Pick $N$ large enough so that for $n \geq N$, $S^j(x) = (S_n)^j(x)$ for all $j = 1, \ldots, a(x)$ and $a_n(x) = a(x)$. Then for $n \geq N$,

$$Tx = (S_n)^{a_n(x)}(x) = (S_n)^{a(x)}(x) = S^{a(x)}(x).$$

We can argue in a similar fashion if $a(x) < 0$. Furthermore, by a similar argument to that above, $a$ is continuous on $X \setminus \{x_0\}$. This shows that $S$ satisfies property (4) of $S(T)$.

To see that $S$ satisfies properties (2) and (3) of $S(T)$, fix $j \in \mathbb{Z}$ and pick $N$ such that if $n \geq N$, then $(S_n)^j(x_0)$ is fixed. Then for $n \geq N$, $S^j(x_0) = (S_n)^j(x_0)$. Since $O^-_{S_n}(x_0) = O^-_{T}(x_0)$ and $O^+_{S_n}(x_0) = O^+_{T}(x_0)$, this means $O^-_{S}(x_0) \subset O^-_{T}(x_0)$ and $O^+_{S}(x_0) \subset O^+_{T}(x_0)$. However, we know that $O_{S}(x_0) = O_{T}(x_0)$ because the functions $a$ and $b$ are the cocycles $S$. So we must have that $O^-_{S}(x_0) = O^-_{T}(x_0)$ and $O^+_{S}(x_0) = O^+_{T}(x_0)$. This establishes that $S \in S(T)$.

It remains to be shown that $\{S_n\}$ is $m_T$-convergent to $S$. Above we argued that $\{S_n\}$ converges uniformly to $S$, so to prove that $\{S_n\}$ is $m_T$-convergent to $S$, we only have left to show $\tilde{m}_T(S, S_n) \to 0$. Let $\epsilon > 0$. Pick a clopen set $C$ with $X \setminus B(x_0, \epsilon) \subset C$ and $x_0 \notin C$. Let $C_S$ the set defined in Definition 4.2.2. Since $x_0 \notin C_S$, there exists a $\delta > 0$ such that $B(x_0, \delta) \subset X \setminus C_S$. Pick $N$ such that if $n, m \geq N$, $m_T(S_n, S_m) < \delta$. Then for $n \geq N$, $S_n(x) = Sx$ for all $x \in C_S$. By Proposition 4.2.4, $a(x)$ and $b(x)$ agree with $a_n(x)$ and $b_n(x)$, respectively, for all $x \in C$, so $\tilde{m}_T(S, S_n) < \epsilon$.

\[\square\]
Because \((S(T), m_T)\) is a complete metric space, the Baire Category Theorem applies. We can now ask questions similar to those addressed by Hochman and Rudolph in [10] and [14], respectively, about what systems are typical in these spaces. We will begin by showing that \(S(T)\) is separable for any minimal Cantor system \((X, T)\). This along with Proposition 4.2.8 shows that \((S(T), m_T)\) is a Polish metric space, i.e. it is complete and separable. Before proving that \(S(T)\) is separable, we need some definitions.

Let \(P\) be a tower partition of a minimal Cantor system \((X, T)\) over a clopen set \(A\) such that \(P\) partitions \(A\) into finitely many clopen sets \(A_1, \ldots, A_k\). For each \(1 \leq j \leq k\), let \(r_j\) denote the return time of \(A_j\) to \(A\) and let \(f_j : \{0, \ldots, r_j - 1\} \rightarrow \{0, \ldots, r_j - 1\}\) be a permutation with the properties that \(f_j(0) = 0\) and \(f_j(r_j - 1) = r_j - 1\). Then each \(f_j\) defines a reordering of the tower over \(A_j\) that fixes the top and bottom floors of the tower. Define \(\phi : X \rightarrow X\) in the following way. If \(x \in T^i(A_j)\) for \(1 \leq j \leq k\) and \(0 \leq i \leq r_j - 1\), we define \(\phi(x) = T^{f_j(i) - i}(x)\). We will say that \(\phi\) is a tower permutation of \(P\) with corresponding permutations \(f_1, \ldots, f_k\). We will denote the set of all tower permutations of \(P\) by \(\Pi(P)\). If \(\{P_n\}\) is a sequence of tower partitions of \((X, T)\), we let \(\Pi(P_n) = \bigcup \Pi(P_n)\).

If \(P\) is a tower permutation of a minimal Cantor system \((X, T)\) and \(\phi \in \Pi(P)\), then the map \(\phi T \phi^{-1} : X \rightarrow X\) moves points of \(X\) through the towers of \(P\) according to the corresponding permutations of \(\phi\). For example, suppose \(B \subset X\) is a bottom tower floor of \(P\) and the height of the tower over \(B\) is 5. Let \(\phi \in \Pi(P)\) be a tower permutation whose corresponding permutation \(f\) on the tower over \(B\) is given by the following:

\[
\begin{align*}
    f : & \quad 0 \rightarrow 0 \quad 1 \rightarrow 3 \quad 2 \rightarrow 1 \\
    & \quad 3 \rightarrow 2 \quad 4 \rightarrow 4.
\end{align*}
\]

Then the maps \(\phi\) and \(\phi T \phi^{-1}\) are as shown in Figure 4.1.
Figure 4.1: $T$-tower to $\phi T \phi^{-1}$-tower

**Definition 4.2.9.** For $S \in S(T)$, let $C(S) = \{ P \in S(T) \mid (X, P) \text{ is conjugate to } (X, S) \}$.

**Theorem 4.2.10.** $S(T)$ is separable. In fact, for all $S \in S(T)$, there exists a countable subset of $C(S)$ that is dense in $S(T)$.

Before we proving this theorem, we need a lemma.

**Lemma 4.2.11.** Suppose $S \in S(T)$ and $C$ is a clopen set in $X$ with $x_0 \notin C$. If $\{P_n\}$ is generating sequence of tower partitions over $T x_0$, there exists $\phi \in \Pi \{P_n\}$ such that the cocycles of $\phi T \phi^{-1}$ agree with the cocycles of $S$ for all $x \in C$.

**Proof.** Let $a, b$ be the cocycles of $S$ such that $T x = S^a(x)$ and $S x = T^b(x)$ for all $x \in X$ and let $C_S$ be as in Definition 4.2.2. There exists an $M > 0$ such that $b(C_S) \subset [-M, M]$. Since every forward orbit is dense in $X$, there exists a $K > 0$ such that $S^K(T x_0) \in X \setminus C_S$. Since $S^K$ is continuous, there is a clopen neighbourhood $D$ of $T x_0$ with $S^K(D) \subset X \setminus C_S$. Let $\{P_n\}$ be a sequence of generating tower partitions over $T x_0$, and for all $n$, let $A_n$ be the clopen set such that $P_n$ is a tower partition over $A_n$. By Proposition 2.2.2, $\mathcal{H}(P_n)$ grows arbitrarily large, so we can pick $N'$ large enough such that $P_{N'}$ satisfies the following:
(1’) $C_S$ is the finite union of tower floors in $\mathcal{P}_{N'}$;

(2’) $a$ and $b$ are constant on each of the $C_S$ tower floors;

(3’) the towers of $\mathcal{P}_{N'}$ that contain $x_0$ and $Tx_0$ each have height greater than $M$.

Now, we pick $N > N'$ such that $\mathcal{P}_N$ has the following properties:

(1) $A_N$ is contained in the tower floor of $\mathcal{P}_{N'}$ that contains $Tx_0$ and $T^{−1}(A_N)$ is contained in the tower floor of $\mathcal{P}_{N'}$ that contains $x_0$;

(2) $H(\mathcal{P}_N) > KM$;

(3) $A_N \subset D$;

(4) $T^{−1}(A_N) \cap C_S = \emptyset$.

We will find $\phi \in \Pi(\mathcal{P}_N) \subset \Pi(\mathcal{P}_n)$ so that $\phi T \phi^{-1}$ agrees with $S$ on $C_S$. By Proposition 4.2.4, this will prove the lemma. We consider a fixed tower in $\mathcal{P}_N$ whose bottom floor we will denote by $F$. Suppose the height of the tower over $F$ is $L + 1$. Then the floors of the tower over $F$ are the sets $F, TF, \ldots, T^L(F)$. Fix $i \in \{0, \ldots, L - 1\}$ such that $T^i(F) \subset C_S$. We claim that $S(T^i(F))$ is another floor in the tower over $F$ other than $F$. By condition (2’), $b$ is constant on $T^i(F)$, so for all $x \in T^i(F)$, let $b(x) = m \in [-M, M]$. Then $S(T^i(F)) = T^{i+m}(F)$, and therefore if $0 < i + m \leq L$, $S(T^i(F))$ is another tower floor in the tower over $F$ other than $F$.

We have three cases to consider.

Case 1: If $0 \leq i \leq M$, by conditions (3’) and (1), there exists a tower floor $\tilde{P} \in \mathcal{P}_{N'}$ with height $i$ such that $T^i(F) \subset \tilde{P}$ and $\tilde{P}$ is in the same tower of $\mathcal{P}_{N'}$ that contains $Tx_0$. So there exists an $x \in \tilde{P}$ such that $x = T^i(Tx_0)$. By property (1’), $\tilde{P} \subset C_S$. Therefore $b$ is constant on $\tilde{P}$, so $b(x') = m$ for all $x' \in \tilde{P}$. By Proposition 4.2.1 because $x \in \mathcal{O}_{T}(x_0)$, $m > -i$. Because $m > -i$ and $0 \leq i \leq M$, we have $0 < i + m \leq 2M \leq L$. The last inequality holds by properties (3’) and (1).
Case 2: If $M < i \leq L - M$, then because $-M \leq m \leq M$, we have $0 < m + i \leq L$.

Case 3: If $L - M < i < L$, the argument is similar to that in Case 1. By conditions (3') and (1), there exists a tower floor $\tilde{P} \in \mathcal{P}_{N'}$ with height $i$ such that $T^i(F) \subset \tilde{P}$ and $\tilde{P}$ is in the same tower of $\mathcal{P}_{N'}$ that contains $x_0$. So there exists an $x \in \tilde{P}$ such that $x_0 = T^{L-i}(x)$ or equivalently $T^{-(L-i)}(x_0) = x$. Since $b$ is constant on $\tilde{P}$, $b(x') = m$ for all $x' \in \tilde{P}$. Because $x \in \mathcal{O}_\tilde{T}(x_0)$, by Proposition 4.2.1 $m \leq L - i$. Because $m \leq L - i$ and $L - M < i < L$, we have $0 \leq L - 2M < i + m \leq L$.

Because this tower was chosen arbitrarily, we have shown that for every tower floor of $\mathcal{P}_N$ that is a subset of $C_S$, there is a unique tower floor other than the bottom floor in the same tower that is its image under $S$. We will now show how to permute the tower floors of the tower over $F$ so that if $\phi \in \Pi(\mathcal{P}_N)$ is a map that corresponds to this permutation, then $\phi T \phi^{-1}$ agrees with $S$ on $C_S$. Because the height of the tower over $F$ is $L + 1$, we need to define a permutation $f$ on the set $\{0, \ldots, L\}$ such that $f(0) = 0$ and $f(L) = L$. We define $f$ in the following way.

First, we let $f(0) = i_0 = 0$. If $F \subset C_S$, $S(F) = T^{i_1}(F)$ for some $0 < i_1 < L$, and we define $f(1) = i_1$. If $T^{i_1}(F) \subset C_S$, then $S(T^{i_1}(F)) = T^{i_2}(F)$ for some $0 < i_2 < L$, $i_2 \neq i_1$. We define $f(2) = i_2$. For $j > 2$, we continue defining $f(j) = i_j$ recursively so that $S(T^{i_{j-1}}(F)) = T^{i_j}(F)$ until we reach a $k \geq 0$ such that $T^{i_k}(F) \notin C_S$. From conditions (2) and (3) above, we have that $T^{i_j}(F) \neq T^L(F)$ for any $j = 1, \ldots, k$.

Now we define $f(L) = i_L = L$. If $T^L(F) \subset S(C_S)$, there exists a $0 < i_{L-1} < L$ such that $S(T^{i_{L-1}}(F)) = T^L(F)$. We define $f(L - 1) = i_{L-1}$. We continue defining $f(j) = i_j$ recursively so that $S(T^{i_j}(F)) = T^{i_{j+1}}(F)$ until we reach an $l \geq 0$ such that $T^{i_{L-1}}(F)$ that is not a subset of $S(C_S)$.

We have defined $f$ on two disjoint subsets $\{0, 1, \ldots, k\}$ and $\{L - l, \ldots, L\}$ where $k < L - 1$. If $k + 1 = L - l$, we have completely defined $f$ on the set $\{0, \ldots, L\}$. However, if $k + 1 < L - l$, we need to define $f$ on $\{k + 1, \ldots, L - l - 1\}$. Because $T^{i_k}(F)$ is not a subset of $C_S$, as long as $f(k + 1)$ is the height of a tower floor that
is not a subset of $S(C_S)$, it will not affect whether this rearrangement is an $S$-tower on $C_S$. Let $I = \{1, \ldots, L\} \setminus \{i_1, \ldots, i_k, l_{L-1}, \ldots, l_L\}$ and let $B = \bigcup_{i \in I} T^i(F)$. We want to find $i_{k+1} \in I$ such that $T^{i_{k+1}}(F)$ is not a subset of $S(C_S)$. Suppose no such $i_{k+1}$ exists. This means that every tower floor contained in $B$ is a subset of $S(C_S)$. Every tower floor in the tower over $F$ that is not a subset of $B$ is either not a subset of $C_S$ or has an image under $S$ that is a tower floor not contained in $B$. So for every $i' \in I$, we must have that $(T)^{i'}(F) = S(T^i(F))$ for some $i \in I$, $i \neq i'$. However, this means that $S(B) = B$ contradicting the minimality of $S$. Therefore, there must exist $i_{k+1} \in B$ such that $T^{i_{k+1}}(F)$ is not a subset of $S(C_S)$, and we define $f(k+1) = i_{k+1}$.

In general for $k + 1 < j < L - l$, we defined $f(j) = i_j$ recursively in the following way. If $T^{i_j-1}(F) \subset C_S$, then $S(T^{i_j-1}(F)) = T^{i_j}(F)$ for some $i_j \in \{1, \ldots, L - 1\}$ and we define $f(j) = i_j$. If $T^{i_j-1}(F)$ is not a subset of $C_S$, using the minimality argument as above, we can find $i_j \in \{1, \ldots, L\} \setminus \{i_1, \ldots, i_{j-1}, i_{L-1}, \ldots, L\}$ such that $T^{i_j}(F)$ is not a subset of $S(C_S)$ and we define $f(j) = i_j$. We continue to define $f$ recursively in this manner until it is defined on all of $\{0, \ldots, L\}$.

For each $j \in \{0, \ldots, L\}$, we have defined $f(j) = i_j$ where $i_j$ is defined so that if $i_j \in \{0, \ldots, L - 1\}$ and $T^{i_j}(F) \subset C_S$, then $S(T^{i_j}(F)) = T^{i_j+1}(F) = T^{f(j+1)}(F)$. Furthermore, note that $f(0) = 0$ and $f(L) = L$. Let $\phi \in \Pi(P_n)$ be a map that corresponds to the permutation $f$ on the tower over $F$, so if $0 \leq i \leq L$ and $x \in T^i(F)$, $\phi(x) = T^{f(i)-i}(x)$. Note that for $x \in T^i(F)$, $\phi^{-1}(x) = T^{f^{-1}(i)-i}(x)$. Fix $i \in \{0, \ldots, L - 1\}$ such that $T^i(F) \subset C_S$. We claim that $\phi \phi^{-1}(x) = Sx$ for all $x \in T^i(F)$. Find $j$ with $0 \leq j \leq L - 1$ such that $i = i_j$. Fix $x \in T^i(F) = T^{i_j}(F)$, so $x = T^{i_j}(x')$ for some $x' \in F$. Because $T^{i_j}(F) \subset C_S$, $S(T^{i_j}(F)) = T^{f(j+1)}(F)$ and so $Sx = S(T^{i_j}(x')) = T^{f(j+1)}(x')$. Then we have the following:
\[
\phi T \phi^{-1}(x) = \phi T \phi^{-1}(T^j(x'))
\]
\[
= \phi(T(T^{j-1)}(x'))
\]
\[
= \phi T^{j-1}(x')
\]
\[
= \phi T^j(x')
\]
\[
= T^{j+1}(x')
\]
\[
= T^{j+1}(x')
\]
\[
= Sx.
\]

Therefore, we have shown that there exists \( \phi \in \Pi(\mathcal{P}_N) \) such that \( \phi T \phi^{-1} \) agrees with \( S \) on every tower floor of the tower over \( F \) that is a subset of \( C_S \). If we repeat the construction of the permutation \( f \) for every tower in \( \mathcal{P}_N \) and let \( \phi \in \Pi(\mathcal{P}_N) \) be the map associated to this set of permutations, then \( \phi T \phi^{-1} \) will agree with \( S \) on every tower floor of \( \mathcal{P}_N \) that is a subset of \( C_S \). By Proposition 4.2.4, this finishes the proof.

Proof of Theorem 4.2.10. Let \( \{\mathcal{P}_n\} \) be a sequence of generating tower partitions over \( T x_0 \). Because there are only finitely many ways to permute tower floors in each \( \mathcal{P}_n \), \( \Pi(\mathcal{P}_n) \) is countable. Therefore, the set \( \mathcal{D}(T, \{\mathcal{P}_n\}) = \{\phi T \phi^{-1} | \phi \in \Pi(\mathcal{P}_n)\} \) is a countable subset of \( \mathcal{S}(T) \). If we can show \( \mathcal{D}(T, \mathcal{P}_n) \) is dense, the theorem is proven. Let \( S \in \mathcal{S}(T) \) and fix \( \epsilon > 0 \). Since \( S \) is continuous at \( x_0 \), there is a \( \delta' > 0 \) such that \( S(B(x_0, \delta')) \subset B(S x_0, \epsilon/4) \). Let \( \delta = \min\{\delta', \epsilon/2\} \) and find a clopen set \( C \) such that \( X \setminus B(x_0, \delta) \subset C \) and \( x_0 \notin C \). By the previous lemma, we can find a \( \phi T \phi^{-1} \in \mathcal{D}(T, \mathcal{P}_n) \) whose cocycles agree with the cocycles of \( S \) on \( C \). Now, we will show that \( m_T(\phi T \phi^{-1}, S) < \epsilon \) proving the theorem. Since the cocycles of these two maps agree on \( C \), clearly \( \tilde{m}_T(\phi T \phi^{-1}, S) < \epsilon/2 \). Thus, we only need to show that \( \sup_{x \in X} d_X(\phi T \phi^{-1}(x), S x) < \epsilon/2 \). Since the cocycles of \( \phi T \phi^{-1} \) and \( S \) agree on \( X \setminus B(x_0, \delta) \), \( \phi T \phi^{-1}(x) = S x \) for all \( x \in X \setminus B(x_0, \delta) \). Fix
\( x \in B(x_0, \delta) \) and assume \( y = \phi T \phi^{-1}(x) \notin B(Sx_0, \epsilon/4) \). Then \( S^{-1}(y) \notin B(x_0, \delta) \), so \( \phi T \phi^{-1}(S^{-1}(y)) = S(S^{-1}y) = y \). Since \( \phi T \phi^{-1} \) is a homeomorphism, \( S^{-1}y = x \). This means \( x \notin B(x_0, \delta) \), which is a contradiction. So, for \( x \in B(x_0, \delta) \) we have

\[
d_X(\phi T \phi^{-1}(x), Sx) \leq d_X(\phi T \phi^{-1}(x), Sx_0) + d_X(Sx_0, Sx) < \epsilon/2.
\]

If \( \{P_n\} \) is a sequence of generating tower partitions over \( Tx_0 \), \( D(T, \{P_n\}) \) is a countable dense subset of \( S(T) \) and clearly \( D(T, \{P_n\}) \subset C(T) \). By the preceding arguments, for any \( S \in S(T) \) there exists a countable dense subset \( D(S) \) of \( S(S) \) with \( D(S) \subset C(S) \). By Corollary 4.2.7, the identity map from \( S(T) \) to \( S(S) \) is a uniformly continuous homeomorphism. Because \( D(S) \) is dense in \( S(S) \), it must also be dense in \( S(T) \).

\begin{corollary}
For any \( S \in S(T) \), \( C(S) \) is dense in \( S(T) \).
\end{corollary}

\begin{proposition}
\( (S(T), m_T) \) is not compact.
\end{proposition}

\begin{proof}
Let \( \{P_n\} \) be a sequence of generating partitions over \( Tx_0 \). By Proposition 2.2.2, \( H(P_n) \) grows arbitrarily large as \( n \to \infty \), so we can find a subsequence \( \{P_{n_k}\} \) such that for all \( k \), \( H(P_{n_k}) \geq k + 3 \). For all \( k \), let \( B_k \) be the tower floor in \( P_{n_k} \) such that \( Tx_0 \in B_k \). We define a sequence \( \{\phi_k\} \) in \( \Pi \{P_{n_k}\} \) by

\[
\phi_k(x) = \begin{cases} 
T^k x & \text{if } x \in T(B_k) \\
T^{-k} x & \text{if } x \in T^{k+1}(B_k) \\
x & \text{otherwise.}
\end{cases}
\]

Then for all \( k \), we have

\[
\phi_k T \phi_k^{-1}(Tx_0) = \phi_k(Tx_0) = T^k(Tx_0).
\]

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If \( b_k \) is the cocycle of \( \phi_k T \phi_k^{-1} \) such that \( \phi_k T \phi_k^{-1}(x) = T^{b_k(x)}(x) \) for all \( x \in X \), by the equation above, \( b_k(Tx_0) = k \) for all \( k \). For a sequence to converge in \( \mathcal{S}(T) \), its cocycles values at \( Tx_0 \) need to stabilize to a fixed integer. Therefore the sequence \( \{ \phi_k T \phi_k^{-1} \} \subset \mathcal{S}(T) \) has no converging subsequence proving the proposition. \( \square \)

### 4.3 Residuality and Finite Rank Systems

As defined in [5], a minimal Cantor system \((X,T)\) has finite rank if there exists a \( K > 0 \) such that \((X,T)\) can be represented as a Bratteli-Vershik system with \( K \) or fewer vertices at each level. If \( K \) is the smallest such integer, we say that \((X,T)\) has rank \( K \). We will let \( \mathcal{F}(T) \) denote the set of maps in \( \mathcal{S}(T) \) that have finite rank.

An odometer is a system that has rank 1. We say that \((X,T)\) has \( x_0 \)-finite rank if there exists a \( K > 0 \) such that \((X,T)\) can be represented as a Bratteli-Vershik system with fewer than \( K \) vertices at each level and \( x_0 \) is the maximal path in the diagram. If \( K \) is the smallest such integer, we will say that \((X,T)\) has \( x_0 \)-rank \( K \).

**Definition 4.3.1.** Let \( \epsilon > 0 \) and let \( K \in \mathbb{N}^+ \). We will say that \((X,T)\) satisfies the \((x_0,\epsilon)\)-rank \( K \) condition if there exists a clopen set \( A \subset X \) with the following properties:

1. \( Tx_0 \in A \);
2. \( A \) partitions into \( L \leq K \) clopen sets \( A_1, \ldots, A_L \) each with constant return time \( r_j \) to \( A \);
3. for each \( j = 1, \ldots, L \), \( \text{diam}(T^iA_j) < \epsilon \) for \( i = 0, \ldots, r_j - 1 \);
4. \( \text{diam}(A) < \epsilon \).

**Proposition 4.3.2.** A minimal Cantor system \((X,T)\) has \( x_0 \)-rank less than or equal to \( K \) if and only if it satisfies the \((x_0,\epsilon)\)-rank \( K \) condition for all \( \epsilon > 0 \).
Proof. Suppose \((X, T)\) has \(x_0\)-rank less than or equal to \(K\). Then it can be represented as a Bratteli-Vershik system with fewer than \(K\) vertices at each level and so that \(x_0\) is the maximal path in the diagram, i.e. \(x_{\text{max}} = x_0\). For all \(n\), let \(\mathcal{P}_n\) denote the partition of \(X\) into the cylinder sets of paths that begin with a particular path down to level \(n \) and let \(A_n\) denote the union of cylinders sets in \(\mathcal{P}_n\) that correspond to minimal paths down to level \(n\). Since \(\{\mathcal{P}_n\}\) generates the topology of \(X\), we have that \(diam(\mathcal{P}_n) \to 0\). Because \((X, T)\) has a unique minimal path in its Bratteli diagram, we also have that \(diam(A_n) \to 0\). Fix \(\epsilon > 0\) and pick an \(N > 0\) such that if \(n \geq N\), then \(diam(\mathcal{P}_n) < \epsilon\) and \(diam(A_n) < \epsilon\). Fix \(n \geq N\) and let \(A = A_n\). Since \(Tx_0 = x_{\text{min}}, Tx_0 \in A\). We partition \(A\) the same way it is partitioned in \(\mathcal{P}_n\), and we denote this partition by \(\mathcal{P}_n(A)\). This partition of \(A\) will have fewer than \(K\) sets since the number of sets in \(\mathcal{P}_n(A)\) is equal to the number of vertices at level \(n\) in the Bratteli diagram. Each set in \(\mathcal{P}_n(A)\) will have a constant return time to \(A\) since each set corresponds to a minimal path cylinder set in the diagram. Condition (3) is satisfied because \(diam(\mathcal{P}_n) < \epsilon\) and condition (4) is satisfied because \(diam(A) = diam(A_n) < \epsilon\).

Conversely for \(n \in \mathbb{N}^+\), pick a sequence of sets \(A_n \subset X\) such that \(A_n\) satisfies the \((x_0, 1/n)\)-rank \(K\) condition and so that \(A_{n+1} \subset A_n\) for all \(n\). We then consider the tower partitions of \((X, T)\) over each \(A_n\). Because each \(A_n\) can be partitioned into fewer than \(K\) clopen sets each with constant return time to \(A_n\), we can construct a Bratteli-Vershik representation of \((X, T)\) with fewer than \(K\) vertices at each level. Because \(Tx_0 \in A_n\) for all \(n\), \(Tx_0\) is the minimal path in the diagram; therefore, \(x_0\) is the maximal path in the diagram. Therefore \((X, T)\) has \(x_0\)-rank less than or equal to \(K\).

**Proposition 4.3.3.** A minimal Cantor system \((X, T)\) has finite rank if and only if it has \(x_0\)-finite rank. Moreover, if \((X, T)\) has rank \(K\), then \((X, T)\) has \(x_0\)-rank less than or equal to \(K^2\).
Proof. If \((X, T)\) has \(x_0\)-finite rank, then by definition \((X, T)\) has finite rank. Conversely, if \((X, T)\) has rank \(K\), then it must have \(x_1\)-rank \(K\) for some \(x_1 \in X\). For \(\epsilon > 0\), we will find a set \(B\) containing \(Tx_0\) satisfying the \((x_0, \epsilon)\)-rank \(K^2\) condition. Because \(O^+_T(Tx_1)\) is dense in \(X\), there exists an \(m \geq 0\) such that \(T^m(Tx_1) \in B(Tx_0, \epsilon/4)\). Since \(T^m\) is continuous at \(Tx_1\), there exists a \(\delta' > 0\) such that if \(d_X(x, Tx_1) < \delta'\), then \(T^m(x) \in B(T^m(Tx_1), \epsilon/4)\). Set \(\delta = \min\{\delta', \epsilon/4\}\).

Since \((X, T)\) has \(x_1\)-rank \(K\), by Proposition 4.3.2, there exists a clopen set \(A\) satisfying the \((x_1, \delta)\)-rank \(K^2\) condition. Furthermore, if we let \(P\) denote the tower partition over \(A\) given by the definition of the \((x_1, \delta)\)-rank \(K\) condition, then by Proposition 2.2.2, \(A\) can be chosen so that \(H(P) > m\).

Let \(A\) and \(P\) be as described in the preceding paragraph with \(H(P) > m\). We pick one tower floor from each tower of \(P\) in the following way. Suppose that \(P\) partitions \(A\) into \(L \leq K\) clopen sets \(A_1, \ldots, A_L\). For each \(i \in \{1, \ldots, L\}\), let the tower over \(A_i\) in \(P\) have height \(r_i > 0\). Let \(i_0 \in \{1, \ldots, L\}\) such that \(Tx_0\) is in the same tower as \(A_{i_0}\). Let \(B_{i_0}\) be the tower floor in the tower over \(A_{i_0}\) that contains \(Tx_0\). For all \(i \in \{1, \ldots, L\}, \ i \neq i_0\), let \(B_i\) be the tower floor of height \(m + 1\) in the tower over \(A_i\).

Set \(B = \bigcup_{i=1}^L B_i\). For each \(i = 1, \ldots, L\), we will partition \(B_i\) into \(L\) subsets determined by which \(A_j\) it intersects when it first returns to \(A\) under \(T\), i.e. for a fixed \(i \leq L\), set \(B_{ij} = \{x \in B_i \mid \text{the first time } x \text{ returns to } A \text{ under } T, \text{it returns to } A_j\}\) with \(j \in \{1, \ldots, L\}\). This partitions \(B\) into \(L^2 \leq K^2\) clopen sets. We will now show that \(B\) satisfies the properties desired. By the definition of the \(B_{ij}\) sets, clearly each one has a constant \(T\)-return time to \(B\). Each iteration of a \(B_{ij}\) set under \(T\) before returning to \(B\) is a subset of some \(T^l(A_k) \in P\) with \(k \leq L\) and \(l \leq r_k - 1\). Because \(\text{diam}(P) < \delta < \epsilon\), for all \(i, j \in \{1, \ldots, L\}\), \(\text{diam}(B_{ij}) < \epsilon\). We only have left to show that \(\text{diam}(B) < \epsilon\). Fix \(x, y \in B\). There are three cases that need to consider.
Case 1: Suppose $x, y \in B_{i_0}$. Since $B_{i_0}$ a tower floor in $\mathcal{P}$ and $\operatorname{diam}(\mathcal{P}) < \delta \leq \epsilon/4$, we have $d_X(x, y) < \epsilon/4$.

Case 2: Suppose $x \in B_i$ and $y \in B_j$ where $i, j \neq i_0$. Then $B_i, B_j \subset T^m(A)$, so there exist $x', y' \in A$ such that $T^m(x') = x$ and $T^m(y') = y$. Since $\operatorname{diam}(A) < \delta$, we have

$$d_X(x, y) = d_X(T^m(x'), T^m(y')) \leq d_X(T^m(x'), T^m(Tx_1)) + d_X(T^m(Tx_1), T^m(y')) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$ 

Case 3: Suppose $x \in B_{i_0}$ and $y \in B_j$ with $j \neq i_0$. Since $B_j \subset T^m(A)$, there exists a $y' \in A$ such that $T^m(y') = y$. Then, we have that

$$d_X(x, y) = d_X(x, T^m(y')) \leq d_X(x, T^m(Tx_1)) + d_X(T^m(Tx_1), T^m(y')) \leq d_X(x, Tx_0) + d_X(Tx_0, T^m(Tx_1)) + d_X(T^m(Tx_1), T^m(y')) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4}.$$ 

This shows that $\operatorname{diam}(B) < \epsilon$ and thus $(X, T)$ satisfies the $(x_0, \epsilon)$-rank $K^2$ property. Since $\epsilon$ was chosen arbitrarily, by Proposition 4.3.2 $(X, T)$ has $x_0$-rank less than or equal to $K^2$.

**Theorem 4.3.4.** If $(X, T)$ has finite rank, then the set of finite rank systems $\mathcal{F}(T)$ is residual in $\mathcal{S}(T)$, i.e. $\mathcal{F}(T)$ contains a dense $G_\delta$.

Before we prove this theorem, we need a lemma.
Lemma 4.3.5. Let $S \in S(T)$ and let $P = \{P_1, \ldots, P_n\}$ be a clopen partition of $X$. There exists an $\epsilon > 0$ such that if $m_T(S', S) < \epsilon$, then $S'(P_i) = S(P_i)$ for $i = 1, \ldots, n$.

Proof. Since $S$ is a homeomorphism the set $\{S(P_1), \ldots, S(P_n)\}$ is a clopen partition of $X$, so for $i \neq j$, $d_X(S(P_i), S(P_j)) > 0$. Define $\epsilon = \min_{i \neq j} d_X(S(P_i), S(P_j))$. If $m_T(S', S) < \epsilon$, then $\sup_{x \in X} d_X(S'x, Sx) < \epsilon$, and so $S'(P_i) \subset S(P_i)$ for $i = 1, \ldots, n$. Since $S'$ is a homeomorphism, we have $S'(P_i) = S(P_i)$ for $i = 1, \ldots, n$ finishing the proof.

Proof of Theorem 4.3.4. Let $F_K(T, \epsilon)$ denote the systems that satisfy the $(x_0, \epsilon)$-rank $K$ condition. By Proposition 4.3.2,

$$\bigcap_{n=1}^{\infty} F_K(T, 1/n) = F_K(T, x_0)$$

where $F_K(T, x_0)$ is the set of systems that have $x_0$-rank less than or equal to $K$. Since $F_K(T, x_0) \subset F(T)$, if we can show that each $F_K(T, 1/n)$ is an open dense set in $S(T)$, by the Baire Category Theorem, we will have that $F(T)$ is residual in $S(T)$.

We will show that for all $\epsilon > 0$, the set $F_K(T, \epsilon)$ is dense in $S(T)$. By Proposition 4.3.3, $(X, T)$ has $x_0$-finite rank. Therefore, there exists a $K > 0$ such that $(X, T)$ can be has a Bratteli diagram representation $B$ with $K$ or fewer vertices at each level and with maximal path $x_0$. For all $n$, let $P_n$ denote the tower partition of $X$ over the union of minimal path cylinders sets in $B$ down to level $n$. Then $\{P_n\}$ is a generating sequence of tower partitions, so by Theorem 4.2.10, $D(T, \{P_n\})$ is dense in $S(T)$.

We claim that $D(T, \{P_n\}) \subset F_K(T, x_0)$. If $\phi T \phi^{-1} \in D(T, \{P_n\})$, then there exists some $k \in \mathbb{N}^+$ such that the map $\phi T \phi^{-1}$ is created by rearranging the tower floors of $P_k$ (excluding the top and bottom floors of $P_k$). But a rearrangement of the tower floors of $P_k$ is equivalent to reordering paths of $B$ down to level $k$ (excluding the
minimal and maximal paths). Therefore, by reordering paths of $B$ down to level $k$, we can obtain a Bratteli diagram representation $B'$ of $(X, \phi T \phi^{-1})$. Since the number of vertices at each level of $B'$ is equal to the number of vertices at each corresponding level of $B$ and the maximal path of $B'$ is $x_0$ (since no minimal or maximal paths were reordered), we have that $(X, \phi T \phi^{-1})$ has $x_0$-rank less than or equal to $K$. Therefore, $\phi T \phi^{-1} \in \mathcal{F}(T, x_0)$ proving the claim. Since for all $\epsilon > 0$, $\mathcal{F}_K(T, x_0) \subset \mathcal{F}_K(T, \epsilon)$, we have that $\mathcal{F}_K(T, \epsilon)$ is dense in $S(T)$.

It remains to be shown that for all $\epsilon > 0$, the set $\mathcal{F}_K(T, \epsilon)$ is open in $S(T)$. Let $S \in \mathcal{F}_K(T, \epsilon)$. Let $\mathcal{P}$ be the tower partition of $(X, S)$ given by the definition of the $(x_0, \epsilon)$-rank $K$ condition. By Lemma 4.3.5, there exists an $\epsilon' > 0$ such that if $m_T(S, S') < \epsilon'$, then $S(P) = S'(P)$ for all $P \in \mathcal{P}$. Therefore if $m_T(S, S') < \epsilon$, then $S'$ also satisfies the $(x_0, \epsilon)$-rank $K$ condition with the same partition $\mathcal{P}$. This shows that $\mathcal{F}_K(T, \epsilon)$ is open in $S(T)$ finishing the proof.

**Corollary 4.3.6.** If $(X, T)$ is an odometer, then odometers are residual in $S(T)$.

**Proof.** In the proof of Theorem 4.3.4, it was shown that if $(X, T)$ has rank $K$, then the systems with $x_0$-rank less than or equal to $K$ are residual in $S(T)$. If $(X, T)$ is an odometer, it has rank 1 and thus odometers are residual in $S(T)$.

### 4.4 Residuality and Entropy

#### 4.4.1 Entropy

We will define entropy as done in [15]. Let $(X, T)$ be a minimal Cantor system (this definition is the same for any topological dynamical system). If $\alpha$ and $\beta$ are open covers of $X$, their join $\alpha \vee \beta$ is the open cover containing sets of the form $A \cap B$ where $A \in \alpha$ and $B \in \beta$. The join of any finite number of open covers $\bigvee_{i=1}^n \alpha_i$ is defined similarly. If $\alpha$ is an open cover of $X$, $T^{-1} \alpha$ will denote the open cover of $X$
containing sets of the form $T^{-1}A$ where $A \in \alpha$. Let $N(\alpha)$ denote the number of sets in a subcover of $\alpha$ with minimal cardinality. If we let $H(\alpha) = \log N(\alpha)$, the entropy of $(X,T)$ relative to $\alpha$ is given by

$$h(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i} \alpha).$$

In [15], it is shown that this limit exists and $h(T, \alpha) \leq H(\alpha)$. The topological entropy of $(X,T)$ is defined as

$$h(T) = \sup_{\alpha} h(T, \alpha)$$

where $\alpha$ ranges over all open covers of $X$. Topological entropy is an invariant under conjugacy.

If $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a clopen partition of $X$, then $N(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{P})$ is the number of $T$-itineraries of length $n$ through $\mathcal{P}$. Let $\pi_T(\mathcal{P})$ denote the shift space of itineraries through $\mathcal{P}$, i.e. for $x \in X$ and $i \in \mathbb{Z}$, set $x_i = j \in \{1, \ldots, n\}$ where $T^i x \in P_j$. Then $\pi_T(\mathcal{P})$ is a system consisting of the space $\{\ldots x_{-2}x_{-1}x_0x_1x_2 \ldots | x \in X\}$ along with the shift map. Theorem 7.13 of [15] shows that if $(Y,S)$ is a shift space, then

$$h(S) = \lim_{n \to \infty} \frac{\log |W_n(Y)|}{n}$$

where $W_n(Y)$ is the set of words of length $n$ in $Y$. By the preceding statements, we have that $h(T, \mathcal{P}) = h(\pi_T(\mathcal{P}))$, or equivalently

$$h(T, \mathcal{P}) = \lim_{n \to \infty} \frac{\log |W_n(\pi_T(\mathcal{P}))|}{n}. \quad (4.4.1)$$

### 4.4.2 Zero Entropy Systems are Residual

Fix a sequence of clopen sets $\{A_k\}$ contained in $X$ such that $A_{k+1} \subset A_k$ and $\text{diam}(A_k) \to 0$. Let $\{\mathcal{P}_l\}$ be a sequence of clopen partitions (not necessarily tower
partitions) of $X$ that generates the topology of $X$. It follows from Theorem 7.6 of [15] that $\lim_{l \to \infty} h(S, \mathcal{P}_l) = h(S)$ for any $S \in \mathcal{S}(T)$. For each pair $k, l \in \mathbb{N}^+$ and $S \in \mathcal{S}(T)$, we will define a shift space that describes how points of $A_k$ move through the partition $\mathcal{P}_l$. Fix $k, l \in \mathbb{N}^+$ and let $x \in A_k$ with $T$-return time $r > 0$ to $A_k$. If $\mathcal{P}_l = \{P_1, \ldots, P_n\}$, we define $w_S(k, l)(x) = x_0 \ldots x_{r-1}$ where $x_i = j \in \{1, \ldots, n\}$ if and only if $T^i x \in P_j$. Let $W_S(k, l) = \{w_S(k, l)(x) \mid x \in A_k\}$, and we define $\pi_S(k, l)$ to be the shift space of all bi-infinite words that can be formed by concatenating words in $W_S(k, l)$.

**Proposition 4.4.1.** Let $S \in \mathcal{S}(T)$. For all $k > 0$, there exists an $\epsilon > 0$ such that if $m_T(S', S) < \epsilon$, then $W_S(k, l) = W_{S'}(k, l)$.

**Proof.** This follows directly from Lemma 4.3.5.

**Theorem 4.4.2.** (Lind and Marcus from [12]) Let $\pi_1 \supset \pi_2 \supset \pi_3$ be shift spaces whose intersection is $\pi$. Then $\lim_{k \to \infty} h(\pi_k) = h(\pi)$.

**Lemma 4.4.3.** The sequence $\{h(\pi_S(k, l))\}_{k=1}^\infty$ is decreasing and $\lim_{k \to \infty} h(\pi_S(k, l)) = h(S, \mathcal{P}_l)$.

**Proof.** If $k' > k$, the words in $W_S(k', l)$ are concatenations of the words in $W_S(k, l)$, so $\pi_S(k', l) \subset \pi_S(k, l)$. Therefore, $h(\pi_S(k', l)) \leq h(\pi_S(k, l))$. Since $h(S, \mathcal{P}_l) = h(\pi_S(\mathcal{P}_l))$, if we can show that $\bigcap_k \pi_S(k, l) = \pi_S(\mathcal{P}_l)$, the limit statement holds by Theorem 4.4.2.

If $\mathcal{P}_l = \{P_1, \ldots, P_n\}$, then $\bigcap_k \pi_S(k, l)$ and $\pi_S(\mathcal{P}_l)$ are both closed subspaces of the full shift $\{1, \ldots, n\}^\mathbb{Z}$. Therefore, in order to show that $\bigcap_k \pi_S(k, l) = \pi_S(\mathcal{P}_l)$, it suffices show that any finite word appearing in one space also appears in the other. It is clear that any finite word appearing in $\pi_S(\mathcal{P}_l)$ also appears $\bigcap_k \pi_S(k, l)$ because if some point in $X$ follows a particular $S$-itinerary through $\mathcal{P}_l$, then that same point follows the same $S$-itinerary through every tower partition of $(X, S)$.
We will now show that any finite word appearing in $\bigcap_k \pi_S(k,l)$ also appears in $\pi_S(\mathcal{P}_t)$. Let $w = w_0 \ldots w_{n-1}$ be a finite word that appears in $\bigcap_k \pi_S(k,l)$. Pick $K > 0$ such that if $k \geq K$, then each of the sets $A_k, S(A_k), \ldots, S^{n-1}(A_k)$ is contained in only one element of the partition $\mathcal{P}_t$. For $j = 0, \ldots, n-1$, say $S^j(A_k) \subset P_{i_j} \in \mathcal{P}_t$.

Because of the way $K$ was chosen, every word in $W_S(K,l)$ must begin with the subword $i_0 i_1 \ldots i_{n-1}$. Since $w$ appears in $\bigcap_k \pi_S(k,l)$, in particular it is a subword of some concatenation of words in $W_S(K,l)$. If $w$ is a subword of a single word in $W_S(K,l)$, then clearly $w$ appears in $\pi_S(\mathcal{P}_t)$. If $w$ is a subword of the concatenation of multiple words in $W_S(K,l)$, let $m$ be the minimal positive integer such that $w_m$ is the first symbol of a new word in $W_S(K,l)$. Because $w_m$ is the first symbol of a word in $W_S(K,l)$, we have that $w_j = i_{j-m}$ for $j = m, \ldots, n-1$. Since $w_0 \ldots w_{m-1}$ is a subword of a single word in $W_S(K,l)$, there exists $x \in X$ with $S^j(x) \in P_{i_j}$ for $j = 0, \ldots, m-1$. Because $w_{m-1}$ is the last symbol of a word in $W_S(K,l)$, we also have that $S^m(x) \in A_K$. Then for $j = m, \ldots, n-1$, $S^j(x) \in S^{j-m}(A_K) \subset P_{i_{j-m}} = P_{i_j}$. Therefore, $x$ has exactly the $S$-itinerary $w_0 \ldots w_{n-1}$ through the partition $\mathcal{P}_t$ showing that $w$ does appear in $\pi_S(\mathcal{P}_t)$ and finishing the proof.

\textbf{Lemma 4.4.4.} Let $l \in \mathbb{N}^+$ and $p > 0$, then the set $S(p,l) = \{ S \in S(T) \mid h(S, \mathcal{P}_t) < p \}$ is open in $S(T)$.

\textit{Proof.} Let $S \in S(T)$ with $h(S, \mathcal{P}_t) < p$. By Lemma 4.4.3, there exists a $K$ such that if $k \geq K$, then $h(\pi_S(k,l)) < p$. By Proposition 4.4.1, there exists $\epsilon > 0$ such that if $m_T(S', S) < \epsilon$, then $W_S(K,l) = W_{S'}(K,l)$. Then $h(\pi_{S'}(K,l)) = h(\pi_S(K,l)) < p$. Since $\{ h(\pi_{S'}(k,l)) \}_{k=1}^\infty$ is decreasing and converges to $h(S', \mathcal{P}_t)$, $h(S', \mathcal{P}_t) < p$. \qed

\textbf{Theorem 4.4.5.} (Boyle and Handelman from [3]) Any minimal Cantor system is strongly orbit equivalent to a system with zero entropy.

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**Theorem 4.4.6.** For any minimal Cantor system \((X, T)\), the set of maps in \(S(T)\) with zero entropy is residual.

*Proof.* By Theorem 4.4.5, \(S(T)\) contains a system with zero entropy. By Corollary 4.2.12, the conjugacy class of this zero entropy dense is dense in \(S(T)\). Since entropy is invariant under conjugacy, the systems with zero entropy are dense in \(S(T)\). It follows from the definition of entropy that if \(S \in S(T)\) with \(h(S) = 0\), then \(h(S, \mathcal{P}) = 0\) for any clopen partition \(\mathcal{P}\) of \(X\). Therefore, if \(l\) is a positive integer and \(p > 0\), \(S(p, l)\) contains all systems in \(S(T)\) with zero entropy; therefore, \(S(p, l)\) is dense in \(S(T)\). Define

\[
S(l) = \bigcap_{n=1}^{\infty} S(n^{-1}, l).
\]

From the previous statement and Lemma 4.4.4, we can conclude that \(S(l)\) is residual in \(S(T)\). Furthermore, \(S(l) = \{S \in S(T) \mid h(S, \mathcal{P}_l) = 0\}\). Since \(\lim_{l \to \infty} h(S, \mathcal{P}_l) = h(S)\) for all \(S \in S(T)\), we have that \(\bigcap_{l=1}^{\infty} S(l) = \{S \in S(T) \mid h(S) = 0\}\). Because the countable intersection of residual sets is residual, the theorem is proven. \(\Box\)
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